WHITEHEAD
&
RUSSELL

Principia Mathematica

VOL. I

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VOLUME I

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PRINCIPIA MATHEMATICA

BY

A. N. WHITEHEAD

AND

BERTRAND RUSSELL

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The *Principia* has long been recognized as one of the intellectual landmarks of the century. It was the first book to show clearly the close relationship between mathematics and formal logic. Starting from a minimal number of axioms, Whitehead and Russell display the structure of both kinds of thought. No other book has had such an influence on the subsequent history of mathematical philosophy.

PRINCIPIA MATHEMATICA

BY

ALFRED NORTH WHITEHEAD

AND

BERTRAND RUSSELL, F.R.S.

VOLUME I

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PREFACE

THE mathematical treatment of the principles of mathematics, which is the subject of the present work, has arisen from the conjunction of two different studies, both in the main very modern. On the one hand we have the work of analysts and geometers, in the way of formulating and systematising their axioms, and the work of Cantor and others on such matters as the theory of aggregates. On the other hand we have symbolic logic, which, after a necessary period of growth, has now, thanks to Peano and his followers, acquired the technical adaptability and the logical comprehensiveness that are essential to a mathematical instrument for dealing with what have hitherto been the beginnings of mathematics. From the combination of these two studies two results emerge, namely (1) that what were formerly taken, tacitly or explicitly, as axioms, are either unnecessary or demonstrable; (2) that the same methods by which supposed axioms are demonstrated will give valuable results in regions, such as infinite number, which had formerly been regarded as inaccessible to human knowledge. Hence the scope of mathematics is enlarged both by the addition of new subjects and by a backward extension into provinces hitherto abandoned to philosophy.

The present work was originally intended by us to be comprised in a second volume of *The Principles of Mathematics*. With that object in view, the writing of it was begun in 1900. But as we advanced, it became increasingly evident that the subject is a very much larger one than we had supposed; moreover on many fundamental questions which had been left obscure and doubtful in the former work, we have now arrived at what we believe to be satisfactory solutions. It therefore became necessary to make our book independent of *The Principles of Mathematics*. We have, however, avoided both controversy and general philosophy, and made our statements dogmatic in form. The justification for this is that the chief reason in favour of any theory on the principles of mathematics must always be inductive, *i.e.* it must lie in the fact that the theory in question enables us to deduce ordinary mathematics. In mathematics, the greatest degree of self-evidence is usually not to be found quite at the beginning, but at some later point; hence the early deductions, until they reach this point, give reasons rather for believing the premisses because true consequences follow from them, than for believing the consequences because they follow from the premisses.

In constructing a deductive system such as that contained in the present work, there are two opposite tasks which have to be concurrently performed. On the one hand, we have to analyse existing mathematics, with a view to discovering what premisses are employed, whether these premisses are mutually consistent, and whether they are capable of reduction to more fundamental premisses. On the other hand, when we have decided upon our premisses, we have to build up again as much as may seem necessary of the data previously analysed, and as many other consequences of our premisses as are of sufficient general interest to deserve statement. The preliminary labour of analysis does not appear in the final presentation, which merely sets forth the outcome of the analysis in certain undefined ideas and

undemonstrated propositions. It is not claimed that the analysis could not have been carried farther: we have no reason to suppose that it is impossible to find simpler ideas and axioms by means of which those with which we start could be defined and demonstrated. All that is affirmed is that the ideas and axioms with which we start are sufficient, not that they are necessary.

In making deductions from our premisses, we have considered it essential to carry them up to the point where we have proved as much as is true in whatever would ordinarily be taken for granted. But we have not thought it desirable to limit ourselves too strictly to this task. It is customary to consider only particular cases, even when, with our apparatus, it is just as easy to deal with the general case. For example, cardinal arithmetic is usually conceived in connection with *finite* numbers, but its general laws hold equally for infinite numbers, and are most easily proved without any mention of the distinction between finite and infinite. Again, many of the properties commonly associated with series hold of arrangements which are not strictly serial, but have only some of the distinguishing properties of serial arrangements. In such cases, it is a defect in logical style to prove for a particular class of arrangements what might just as well have been proved more generally. An analogous process of generalization is involved, to a greater or less degree, in all our work. We have sought always the most general reasonably simple hypothesis from which any given conclusion could be reached. For this reason, especially in the later parts of the book, the importance of a proposition usually lies in its hypothesis. The conclusion will often be something which, in a certain class of cases, is familiar, but the hypothesis will, whenever possible, be wide enough to admit many cases besides those in which the conclusion is familiar.

We have found it necessary to give very full proofs, because otherwise it is scarcely possible to see what hypotheses are really required, or whether our results follow from our explicit premisses. (It must be remembered that we are not affirming merely that such and such propositions are true, but also that the axioms stated by us are sufficient to prove them.) At the same time, though full proofs are necessary for the avoidance of errors, and for convincing those who may feel doubtful as to our correctness, yet the proofs of propositions may usually be omitted by a reader who is not specially interested in that part of the subject concerned, and who feels no doubt of our substantial accuracy on the matter in hand. The reader who is specially interested in some particular portion of the book will probably find it sufficient, as regards earlier portions, to read the summaries of previous parts, sections, and numbers, since these give explanations of the ideas involved and statements of the principal propositions proved. The proofs in Part I, Section A, however, are necessary, since in the course of them the manner of stating proofs is explained. The proofs of the earliest propositions are given without the omission of any step, but as the work proceeds the proofs are gradually compressed, retaining however sufficient detail to enable the reader by the help of the references to reconstruct proofs in which no step is omitted.

The order adopted is to some extent optional. For example, we have treated cardinal arithmetic and relation-arithmetic before series, but we might have treated series first. To a great extent, however, the order is determined by logical necessities.

A very large part of the labour involved in writing the present work has been expended on the contradictions and paradoxes which have infected logic and the theory of aggregates. We have examined a great number of hypotheses for dealing with these contradictions; many such hypotheses have been advanced by others, and about as many have been invented by ourselves. Sometimes it has cost us several months' work to convince ourselves that a hypothesis was untenable. In the course of such a prolonged study, we have been led, as was to be expected, to modify our views from time to time; but it gradually became evident to us that some form of the doctrine of types must be adopted if the contradictions were to be avoided. The particular form of the doctrine of types advocated in the present work is not logically indispensable, and there are various other forms equally compatible with the truth of our deductions. We have particularized, both because the form of the doctrine which we advocate appears to us the most probable, and because it was necessary to give at least one perfectly definite theory which avoids the contradictions. But hardly anything in our book would be changed by the adoption of a different form of the doctrine of types. In fact, we may go farther, and say that, supposing some other way of avoiding the contradictions to exist, not very much of our book, except what explicitly deals with types, is dependent upon the adoption of the doctrine of types in any form, so soon as it has been shown (as we claim that we have shown) that it is *possible* to construct a mathematical logic which does not lead to contradictions. It should be observed that the whole effect of the doctrine of types is negative: it forbids certain inferences which would otherwise be valid, but does not permit any which would otherwise be invalid. Hence we may reasonably expect that the inferences which the doctrine of types permits would remain valid even if the doctrine should be found to be invalid.

Our logical system is wholly contained in the numbered propositions, which are independent of the Introduction and the Summaries. The Introduction and the Summaries are wholly explanatory, and form no part of the chain of deductions. The explanation of the hierarchy of types in the Introduction differs slightly from that given in ∗12 of the body of the work. The latter explanation is stricter and is that which is assumed throughout the rest of the book.

The symbolic form of the work has been forced upon us by necessity: without its help we should have been unable to perform the requisite reasoning. It has been developed as the result of actual practice, and is not an excrescence introduced for the mere purpose of exposition. The general method which guides our handling of logical symbols is due to Peano. His great merit consists not so much in his definite logical discoveries nor in the details of his notations (excellent as both are), as in the fact that he first showed how symbolic logic was to be freed from its undue obsession with the forms of ordinary algebra, and thereby made it a suitable instrument for research. Guided by our study of his methods, we have used great freedom in constructing, or reconstructing, a symbolism which shall be adequate to deal with all parts of the subject. No symbol has been introduced except on the ground of its practical utility for the immediate purposes of our reasoning.

A certain number of forward references will be found in the notes and explanations. Although we have taken every reasonable precaution to secure

the accuracy of these forward references, we cannot of course guarantee their accuracy with the same confidence as is possible in the case of backward references.

Detailed acknowledgments of obligations to previous writers have not very often been possible, as we have had to transform whatever we have borrowed, in order to adapt it to our system and our notation. Our chief obligations will be obvious to every reader who is familiar with the literature of the subject. In the matter of notation, we have as far as possible followed Peano, supplementing his notation, when necessary, by that of Frege or by that of Schröder. A great deal of the symbolism, however, has had to be new, not so much through dissatisfaction with the symbolism of others, as through the fact that we deal with ideas not previously symbolised. In all questions of logical analysis, our chief debt is to Frege. Where we differ from him, it is largely because the contradictions showed that he, in common with all other logicians ancient and modern, had allowed some error to creep into his premisses; but apart from the contradictions, it would have been almost impossible to detect this error. In Arithmetic and the theory of series, our whole work is based on that of Georg Cantor. In Geometry we have had continually before us the writings of v. Staudt, Pasch, Peano, Pieri, and Veblen.

We have derived assistance at various stages from the criticisms of friends, notably Mr G. G. Berry of the Bodleian Library and Mr R. G. Hawtrey.

We have to thank the Council of the Royal Society for a grant towards the expenses of printing of £200 from the Government Publication Fund, and also the Syndics of the University Press who have liberally undertaken the greater portion of the expense incurred in the production of the work. The technical excellence, in all departments, of the University Press, and the zeal and courtesy of its officials, have materially lightened the task of proof-correction.

The second volume is already in the press, and both it and the third will appear as soon as the printing can be completed.

A. N. W.

B. R.

CAMBRIDGE,

November, 1910.

CONTENTS OF VOLUME I

ALPHABETICAL LIST OF PROPOSITIONS REFERRED TO BY NAMES

Name	Number	
Abs	*2·01.	$\vdash : p \supset \sim p . \supset . \sim p$
Add	*1·3.	$\vdash : q . \supset . p \lor q$
Ass	*3·35.	$\vdash : p . p \supset q . \supset . q$
Assoc	*1·5.	$\vdash : p \lor (q \lor r) . \supset . q \lor (p \lor r)$
Comm	*2·04.	$\vdash :. p . \supset . q \supset r : \supset : q . \supset . p \supset r$
Comp	*3·43.	$\vdash :. p \supset q . p \supset r . \supset : p . \supset . q . r$
Exp	*3·3.	$\vdash :. p . q . \supset . r : \supset : p . \supset . q \supset r$
Fact	*3·45.	$\vdash :. p \supset q . \supset : p . r . \supset . q . r$
Id	*2·08.	$\vdash . p \supset p$
Imp	*3·31.	$\vdash :. p . \supset . q \supset r : \supset : p . q . \supset . r$
Perm	*1·4.	$\vdash : p \lor q . \supset . q \lor p$
Simp	*2·02.	$\vdash : q . \supset . p \supset q$
”	*3·26.	$\vdash : p . q . \supset . p$
”	*3·27.	$\vdash : p . q . \supset . q$
Sum	*1·6.	$\vdash :. q \supset r . \supset : p \lor q . \supset . p \lor r$
Syll	*2·05.	$\vdash :. q \supset r . \supset : p \supset q . \supset . p \supset r$
”	*2·06.	$\vdash :. p \supset q . \supset : q \supset r . \supset . p \supset r$
”	*3·33.	$\vdash : p \supset q . q \supset r . \supset . p \supset r$
”	*3·34.	$\vdash : q \supset r . p \supset q . \supset . p \supset r$
Taut	*1·2.	$\vdash : p \lor p . \supset . p$
Transp	*2·03.	$\vdash : p \supset \sim q . \supset . q \supset \sim p$
”	*2·15.	$\vdash : \sim p \supset q . \supset . \sim q \supset p$
”	*2·16.	$\vdash : p \supset q . \supset . \sim q \supset \sim p$
”	*2·17.	$\vdash : \sim q \supset \sim p . \supset . p \supset q$
”	*3·37.	$\vdash :. p . q . \supset . r : \supset : p . \sim r . \supset . \sim q$
”	*4·1.	$\vdash : p \supset q . \equiv . \sim q \supset \sim p$
”	*4·11.	$\vdash : p \equiv q . \equiv . \sim p \equiv \sim q$

INTRODUCTION TO THE SECOND EDITION*

In preparing this new edition of *Principia Mathematica*, the authors have thought it best to leave the text unchanged, except as regards misprints and minor errors†, even where they were aware of possible improvements. The chief reason for this decision is that any alteration of the propositions would have entailed alteration of the references, which would have meant a very great labour. It seemed preferable, therefore, to state in an introduction the main improvements which appear desirable. Some of these are scarcely open to question; others are, as yet, a matter of opinion.

The most definite improvement resulting from work in mathematical logic during the past fourteen years is the substitution, in Part I, Section A, of the one indefinable "p and q are incompatible" (or, alternatively, "p and q are both false") for the two indefinables "not-p" and "p or q." This is due to Dr H. M. Sheffer‡. Consequentially, M. Jean Nicod§ showed that one primitive proposition could replace the five primitive propositions $*1{\cdot}2{\cdot}3{\cdot}4{\cdot}5{\cdot}6$.

From this there follows a great simplification in the building up of molecular propositions and matrices; $*9$ is replaced by a new chapter, $*8$, given in Appendix A to this Volume.

Another point about which there can be no doubt is that there is no need of the distinction between real and apparent variables, nor of the primitive idea "assertion of a propositional function." On all occasions where, in *Principia Mathematica*, we have an asserted proposition of the form "$\vdash . fx$" or "$\vdash . fp$," this is to be taken as meaning "$\vdash . (x) . fx$" or "$\vdash . (p) . fp$." Consequently the primitive proposition $*1{\cdot}11$ is no longer required. All that is necessary, in order to adapt the propositions as printed to this change, is the convention that, when the scope of an apparent variable is the whole of the asserted proposition in which it occurs, this fact will not be explicitly indicated unless "some" is involved instead of "all." That is to say, "$\vdash . \phi x$" is to mean "$\vdash . (x) . \phi x$"; but in "$\vdash . (\exists x) . \phi x$" it is still necessary to indicate explicitly the fact that "some" x (not "all" x's) is involved.

It is possible to indicate more clearly than was done formerly what are the novelties introduced in Part I, Section B as compared with Section A.

* In this introduction, as well as in the Appendices, the authors are under great obligations to Mr F. P. Ramsey of King's College, Cambridge, who has read the whole in MS. and contributed valuable criticisms and suggestions.

† In regard to these we are indebted to many readers, but especially to Drs Behmann and Boscovitch, of Göttingen.

‡ *Trans. Amer. Math. Soc.* Vol. XIV. pp. 481—488.

§ "A reduction in the number of the primitive propositions of logic," *Proc. Camb. Phil. Soc.* Vol. XIX.

They are three in number, two being essential logical novelties, and the third merely notational.

(1) For the "p" of Section A, we substitute "ϕx," so that in place of "$\vdash . (p) . fp$" we have "$\vdash . (\phi, x) . f(\phi x)$." Also, if we have "$\vdash . f(p, q, r, \ldots)$," we may substitute ϕx, ϕy, ϕz, ... for p, q, r, ... or ϕx, ϕy for p, q, and ψz, ... for r, ..., and so on. We thus obtain a number of new general propositions different from those of Section A.

(2) We introduce in Section B the new primitive idea "$(\exists x) . \phi x$," *i.e.* existence-propositions, which do not occur in Section A. In virtue of the abolition of the real variable, general propositions of the form "$(p) . fp$" do occur in Section A, but "$(\exists p) . fp$" does not occur.

(3) By means of definitions, we introduce in Section B general propositions which are molecular constituents of other propositions; thus "$(x) . \phi x . \vee . p$" is to mean "$(x) . \phi x \vee p$."

It is these three novelties which distinguish Section B from Section A.

One point in regard to which improvement is obviously desirable is the axiom of reducibility ($*12 \cdot 1 \cdot 11$). This axiom has a purely pragmatic justification: it leads to the desired results, and to no others. But clearly it is not the sort of axiom with which we can rest content. On this subject, however, it cannot be said that a satisfactory solution is as yet obtainable. Dr Leon Chwistek* took the heroic course of dispensing with the axiom without adopting any substitute; from his work, it is clear that this course compels us to sacrifice a great deal of ordinary mathematics. There is another course, recommended by Wittgenstein† for philosophical reasons. This is to assume that functions of propositions are always truth-functions, and that a function can only occur in a proposition through its values. There are difficulties in the way of this view, but perhaps they are not insurmountable‡. It involves the consequence that all functions of functions are extensional. It requires us to maintain that "A believes p" is not a function of p. How this is possible is shown in *Tractatus Logico-Philosophicus* (*loc. cit.* and pp. 19—21). We are not prepared to assert that this theory is certainly right, but it has seemed worth while to work out its consequences in the following pages. It appears that everything in Vol. I remains true (though often new proofs are required); the theory of inductive cardinals and ordinals survives; but it seems that the theory of infinite Dedekindian and well-ordered series largely collapses, so that irrationals, and real numbers generally, can no longer be adequately dealt with. Also Cantor's proof that $2^n > n$ breaks down unless n is finite. Perhaps some further axiom, less objectionable than the axiom of reducibility, might give these results, but we have not succeeded in finding such an axiom.

* In his "Theory of Constructive Types." See references at the end of this Introduction.

† *Tractatus Logico-Philosophicus*, $*5 \cdot 54$ ff.

‡ See Appendix C.

It should be stated that a new and very powerful method in mathematical logic has been invented by Dr H. M. Sheffer. This method, however, would demand a complete re-writing of *Principia Mathematica.* We recommend this task to Dr Sheffer, since what has so far been published by him is scarcely sufficient to enable others to undertake the necessary reconstruction.

We now proceed to the detailed development of the above general sketch.

I. ATOMIC AND MOLECULAR PROPOSITIONS

Our system begins with "atomic propositions." We accept these as a datum, because the problems which arise concerning them belong to the philosophical part of logic, and are not amenable (at any rate at present) to mathematical treatment.

Atomic propositions may be defined negatively as propositions containing no parts that are propositions, and not containing the notions "all" or "some." Thus "this is red," "this is earlier than that," are atomic propositions.

Atomic propositions may also be defined positively—and this is the better course—as propositions of the following sorts:

$R_1(x)$, meaning "x has the predicate R_1";

$R_2(x, y)$ [or xR_2y], meaning "x has the relation R_2 (in intension) to y";

$R_3(x, y, z)$, meaning "x, y, z have the triadic relation R_3 (in intension)";

$R_4(x, y, z, w)$, meaning "x, y, z, w have the tetradic relation R_4 (in intension)";

and so on *ad infinitum*, or at any rate as long as possible. Logic does not know whether there are in fact n-adic relations (in intension); this is an empirical question. We know as an empirical fact that there are at least dyadic relations (in intension), because without them series would be impossible. But logic is not interested in this fact; it is concerned solely with the *hypothesis* of there being propositions of such-and-such a form. In certain cases, this hypothesis is itself of the form in question, or contains a part which is of the form in question; in these cases, the fact that the hypothesis can be framed proves that it is true. But even when a hypothesis occurs in logic, the fact that it can be framed does not itself belong to logic.

Given all true atomic propositions, together with the fact that they are all, every other true proposition can theoretically be deduced by logical methods. That is to say, the apparatus of crude fact required in proofs can all be condensed into the true atomic propositions together with the fact that every true atomic proposition is one of the following: (here the list should follow). If used, this method would presumably involve an infinite enumeration, since it seems natural to suppose that the number of true atomic propositions is infinite, though this should not be regarded as certain. In practice, generality is not obtained by the method of complete enumeration, because this method requires more knowledge than we possess.

We must now advance to molecular propositions. Let p, q, r, s, t denote, to begin with, atomic propositions. We introduce the primitive idea

$$p \mid q,$$

which may be read "p is incompatible with q,"* and is to be true whenever either or both are false. Thus it may also be read "p is false or q is false"; or again, "p implies not-q." But as we are going to define disjunction, implication, and negation in terms of $p \mid q$, these ways of reading $p \mid q$ are better avoided to begin with. The symbol "$p \mid q$" is pronounced: "p stroke q." We now put

$$\begin{aligned} \sim p \,.=.\, p \mid p \quad & \text{Df,} \\ p \supset q \,.=.\, p \mid \sim q \quad & \text{Df,} \\ p \vee q \,.=.\, \sim p \mid \sim q \quad & \text{Df,} \\ p \,.\, q \,.=.\, \sim (p \mid q) \quad & \text{Df.} \end{aligned}$$

Thus all the usual truth-functions can be constructed by means of the stroke. Note that by the above,

$$p \supset q \,.=.\, p \mid (q \mid q) \quad \text{Df.}$$

We find that

$$p \,.\supset.\, q \,.\, r \,.\equiv.\, p \mid (q \mid r).$$

Thus $p \supset q$ is a degenerate case of a function of *three* propositions.

We can construct new propositions indefinitely by means of the stroke; for example, $(p \mid q) \mid r$, $p \mid (q \mid r)$, $(p \mid q) \mid (r \mid s)$, and so on. Note that the stroke obeys the permutative law $(p \mid q) \equiv (q \mid p)$ but not the associative law $(p \mid q) \mid r \equiv p \mid (q \mid r)$. (These of course are results to be proved later.) Note also that, when we construct a new proposition by means of the stroke, we cannot know its truth or falsehood unless either (a) we know the truth or falsehood of some of its constituents, or (b) at least one of its constituents occurs several times in a suitable manner. The case (a) interests logic as giving rise to the *rule of inference*, viz.

Given p and $p \mid (q \mid r)$, we can infer r.

This or some variant must be taken as a primitive proposition. For the moment, we are applying it only when p, q, r are atomic propositions, but we shall extend it later. We shall consider (b) in a moment.

In constructing new propositions by means of the stroke, we assume that the stroke can have on either side of it any proposition so constructed, and need not have an atomic proposition on either side. Thus given three atomic propositions p, q, r, we can form, first, $p \mid q$ and $q \mid r$, and thence $(p \mid q) \mid r$ and $p \mid (q \mid r)$. Given four, p, q, r, s, we can form

$$\{(p \mid q) \mid r\} \mid s, \quad (p \mid q) \mid (r \mid s), \quad p \mid \{q \mid (r \mid s)\}$$

and of course others by permuting p, q, r, s. The above three are substantially

* For what follows, see Nicod, "A reduction in the number of the primitive propositions of logic," *Proc. Camb. Phil. Soc.* Vol. XIX. pp. 32—41.

different propositions. We have in fact

$$\{(p \mid q) \mid r\} \mid s \,.\equiv :. \sim p \vee \sim q \,.\, r : \vee : \sim s,$$
$$(p \mid q) \mid (r \mid s) \,.\equiv : p \,.\, q \,.\vee.\, r \,.\, s,$$
$$p \mid \{q \mid (r \mid s)\} \,.\equiv :. \sim p : \vee : q \,.\sim r \vee \sim s.$$

All the propositions obtained by this method follow from one rule: in "$p \mid q$," substitute, for p or q or both, propositions already constructed by means of the stroke. This rule generates a definite assemblage of new propositions out of the original assemblage of atomic propositions. All the propositions so generated (excluding the original atomic propositions) will be called "molecular propositions." Thus molecular propositions are all of the form $p \mid q$, but the p and q may now themselves be molecular propositions. If p is $p_1 \mid p_2$, p_1 and p_2 may be molecular; suppose $p_1 = p_{11} \mid p_{12}$. p_{11} may be of the form $p_{111} \mid p_{112}$, and so on; but after a finite number of steps of this kind, we are to arrive at atomic constituents. In a proposition $p \mid q$, the stroke between p and q is called the "principal" stroke; if $p = p_1 \mid p_2$, the stroke between p_1 and p_2 is a secondary stroke; so is the stroke between q_1 and q_2 if $q = q_1 \mid q_2$. If $p_1 = p_{11} \mid p_{12}$, the stroke between p_{11} and p_{12} is a tertiary stroke, and so on.

Atomic and molecular propositions together are "elementary propositions." Thus elementary propositions are atomic propositions together with all that can be generated from them by means of the stroke applied any finite number of times. This is a definite assemblage of propositions. We shall now, until further notice, use the letters p, q, r, s, t to denote elementary propositions, not necessarily atomic propositions. The rule of inference stated above is to hold still; *i.e.*

If p, q, r are elementary propositions, given p and $p \mid (q \mid r)$, we can infer r.

This is a primitive proposition.

We can now take up the point (*b*) mentioned above. When a molecular proposition contains repetitions of a constituent proposition in a suitable manner, it can be known to be true without our having to know the truth or falsehood of any constituent. The simplest instance is

$$p \mid (p \mid p),$$

which is always true. It means "p is incompatible with the incompatibility of p with itself," which is obvious. Again, take "$p \,.\, q \,.\supset.\, p$." This is

$$\{(p \mid q) \mid (p \mid q)\} \mid (p \mid p).$$

Again, take "$\sim p \,.\supset.\, \sim p \vee \sim q$." This is

$$(p \mid p) \mid \{(p \mid q) \mid (p \mid q)\}.$$

Again, "$p \,.\supset.\, p \vee q$" is

$$p \mid [\{(p \mid p) \mid (q \mid q)\} \mid \{(p \mid p) \mid (q \mid q)\}].$$

All these are true however p and q may be chosen. It is the fact that we can build up invariable truths of this sort that makes molecular propositions important to logic. Logic is helpless with atomic propositions, because their

truth or falsehood can only be known empirically. But the truth of molecular propositions of suitable form can be known universally without empirical evidence.

The laws of logic, so far as elementary propositions are concerned, are all assertions to the effect that, whatever elementary propositions $p, q, r, \ldots$ may be, a certain function

$$F(p, q, r, \ldots),$$

whose values are molecular propositions, built up by means of the stroke, is always true. The proposition "$F(p)$ is true, whatever elementary proposition p may be" is denoted by

$$(p) . F(p).$$

Similarly the proposition "$F(p, q, r, \ldots)$ is true, whatever elementary propositions $p, q, r, \ldots$ may be" is denoted by

$$(p, q, r, \ldots) . F(p, q, r, \ldots).$$

When such a proposition is *asserted*, we shall omit the "$(p, q, r, \ldots)$" at the beginning. Thus

$$\text{"}\vdash . F(p, q, r, \ldots)\text{"}$$

denotes the assertion (as opposed to the hypothesis) that $F(p, q, r, \ldots)$ is true whatever elementary propositions $p, q, r, \ldots$ may be.

(The distinction between real and apparent variables, which occurs in Frege and in *Principia Mathematica*, is unnecessary. Whatever appears as a real variable in *Principia Mathematica* is to be taken as an apparent variable whose scope is the whole of the asserted proposition in which it occurs.)

The rule of inference, in the form given above, is never required within logic, but only when logic is applied. Within logic, the rule required is different. In the logic of propositions, which is what concerns us at present, the rule used is:

Given, whatever elementary propositions p, q, r may be, both

"$\vdash . F(p, q, r, \ldots)$" and "$\vdash . F(p, q, r, \ldots) \mid \{G(p, q, r, \ldots) \mid H(p, q, r, \ldots)\}$,"

we can infer "$\vdash . H(p, q, r, \ldots)$."

Other forms of the rule of inference will meet us later. For the present, the above is the form we shall use.

Nicod has shown that the logic of propositions (*1—*5) can be deduced, by the help of the rule of inference, from two primitive propositions

$$\vdash . p \mid (p \mid p)$$

and $$\vdash : p \supset q . \supset . s \mid q \supset p \mid s.$$

The first of these may be interpreted as "p is incompatible with not-p," or as "p or not-p," or as "not (p and not-p)," or as "p implies p." The second may be interpreted as

$$p \supset q . \supset : q \supset \sim s . \supset . p \supset \sim s,$$

which is a form of the principle of the syllogism. Written wholly in terms of the stroke, the principle becomes

$$\{p \mid (q \mid q)\} \mid [\{(s \mid q) \mid ((p \mid s) \mid (p \mid s))\} \mid \{(s \mid q) \mid ((p \mid s) \mid (p \mid s))\}].$$

Nicod has shown further that these two principles may be replaced by one. Written wholly in terms of the stroke, this one principle is

$$\{p \mid (q \mid r)\} \mid [\{t \mid (t \mid t)\} \mid \{(s \mid q) \mid ((p \mid s) \mid (p \mid s))\}].$$

It will be seen that, written in this form, the principle is less complex than the second of the above principles written wholly in terms of the stroke. When interpreted into the language of implication, Nicod's one principle becomes

$$p \,.\supset.\, q \,.\, r : \supset . \, t \supset t \,.\, s \mid q \supset p \mid s.$$

In this form, it looks more complex than

$$p \supset q \,.\supset.\, s \mid q \supset p \mid s,$$

but in itself it is less complex.

From the above primitive proposition, together with the rule of inference, everything that logic can ascertain about elementary propositions can be proved, provided we add one other primitive proposition, viz. that, given a proposition $(p, q, r, \ldots) \,.\, F(p, q, r, \ldots)$, we may substitute for $p, q, r, \ldots$ functions of the form

$$f_1(p, q, r, \ldots),\quad f_2(p, q, r, \ldots),\quad f_3(p, q, r, \ldots)$$

and assert

$$(p, q, r, \ldots) \,.\, F\{f_1(p, q, r, \ldots), f_2(p, q, r, \ldots), f_3(p, q, r, \ldots), \ldots\},$$

where $f_1, f_2, f_3, \ldots$ are functions constructed by means of the stroke. Since the former assertion applied to all elementary propositions, while the latter applies only to some, it is obvious that the former implies the latter.

A more general form of this principle will concern us later.

II. ELEMENTARY FUNCTIONS OF INDIVIDUALS

1. *Definition of "individual"*

We saw that atomic propositions are of one of the series of forms:

$$R_1(x),\ R_2(x, y),\ R_3(x, y, z),\ R_4(x, y, z, w),\ \ldots .$$

Here $R_1, R_2, R_3, R_4, \ldots$ are each characteristic of the special form in which they are found: that is to say, R_n cannot occur in an atomic proposition $R_m(x_1, x_2, \ldots x_m)$ unless $n = m$, and then can only occur as R_m occurs, not as $x_1, x_2, \ldots x_m$ occur. On the other hand, any term which can occur as the x's occur in $R_n(x_1, x_2, \ldots x_n)$ can also occur as one of the x's in $R_m(x_1, x_2, \ldots x_m)$ even if m is not equal to n. Terms which can occur in any form of atomic proposition are called "individuals" or "particulars"; terms which occur as the R's occur are called "universals."

We might state our definition compendiously as follows: An "individual" is anything that can be the subject of an atomic proposition.

Given an atomic proposition $R_n(x_1, x_2, \ldots x_n)$, we shall call any of the x's a "constituent" of the proposition, and R_n a "component" of the proposition*. We shall say the same as regards any molecular proposition in which $R_n(x_1, x_2, \ldots x_n)$ occurs. Given an elementary proposition $p \mid q$, where p and q may be atomic or molecular, we shall call p and q "parts" of $p \mid q$; and any parts of p or q will in turn be called parts of $p \mid q$, and so on until we reach the atomic parts of $p \mid q$. Thus to say that a proposition r "occurs in" $p \mid q$ and to say that r is a "part" of $p \mid q$ will be synonymous.

2. *Definition of an elementary function of an individual*

Given any elementary proposition which contains a part of which an individual a is a constituent, other propositions can be obtained by replacing a by other individuals in succession. We thus obtain a certain assemblage of elementary propositions. We may call the original proposition ϕa, and then the propositional function obtained by putting a variable x in the place of a will be called ϕx. Thus ϕx is a function of which the argument is x and the values are elementary propositions. The essential use of "ϕx" is that it collects together a certain set of propositions, namely all those that are its values with different arguments.

We have already had various special functions of propositions. If p is a part of some molecular proposition, we may consider the set of propositions resulting from the substitution of other propositions for p. If we call the original molecular proposition fp, the result of substituting q is called fq.

When an individual or a proposition occurs twice in a proposition, three functions can be obtained, by varying only one, or only another, or both, of the occurrences. For example, $p \mid p$ is a value of any one of the three functions $p \mid q$, $q \mid p$, $q \mid q$, where q is the argument. Similar considerations apply when an argument occurs more than twice. Thus $p \mid (p \mid p)$ is a value of $q \mid (r \mid s)$, or $q \mid (r \mid q)$, or $q \mid (q \mid r)$, or $q \mid (r \mid r)$, or $q \mid (q \mid q)$. When we assert a proposition "$\vdash . (p) . Fp$," the p is to be varied whenever it occurs. We may similarly assert a proposition of the form "$(x) . \phi x$," meaning "all propositions of the assemblage indicated by ϕx are true"; here also, every occurrence of x is to be varied.

3. *"Always true" and "sometimes true"*

Given any function, it may happen that all its values are true; again, it may happen that at least one of its values is true. The proposition that all the values of a function $\phi(x, y, z, \ldots)$ are true is expressed by the symbol

$$\text{"}(x, y, z, \ldots) . \phi(x, y, z, \ldots)\text{"}$$

unless we wish to assert it, in which case the assertion is written

$$\text{"}\vdash . \phi(x, y, z, \ldots).\text{"}$$

* This terminology is taken from Wittgenstein.

We have already had assertions of this kind where the variables were elementary propositions. We want now to consider the case where the variables are individuals and the function is elementary, *i.e.* all its values are elementary propositions. We no longer wish to confine ourselves to the case in which it is *asserted* that all the values of $\phi(x, y, z, \ldots)$ are true; we desire to be able to make the proposition

$$(x, y, z, \ldots) \,.\, \phi(x, y, z, \ldots)$$

a part of a stroke function. For the present, however, we will ignore this desideratum, which will occupy us in Section III of this Introduction.

In addition to the proposition that a function ϕx is "always true" (*i.e.* $(x) \,.\, \phi x$), we need also the proposition that ϕx is "sometimes true," *i.e.* is true for at least one value of x. This we denote by

$$\text{“}(\exists x) \,.\, \phi x.\text{”}$$

Similarly the proposition that $\phi(x, y, z, \ldots)$ is "sometimes true" is denoted by

$$\text{“}(\exists x, y, z, \ldots) \,.\, \phi(x, y, z, \ldots).\text{”}$$

We need, in addition to $(x, y, z, \ldots) \,.\, \phi(x, y, z, \ldots)$ and $(\exists x, y, z, \ldots) \,.\, \phi(x, y, z, \ldots)$, various other propositions of an analogous kind. Consider first a function of two variables. We can form

$$(\exists x) : (y) \,.\, \phi(x, y), (x) : (\exists y) \,.\, \phi(x, y), (\exists y) : (x) \,.\, \phi(x, y), (y) : (\exists x) \,.\, \phi(x, y).$$

These are substantially different propositions, of which no two are always equivalent. It would seem natural, in forming these propositions, to regard the function $\phi(x, y)$ as formed in two stages. Given $\phi(a, b)$, where a and b are constants, we can first form a function $\phi(a, y)$, containing the one variable y; we can then form

$$(y) \,.\, \phi(a, y) \text{ and } (\exists y) \,.\, \phi(a, y).$$

We can now vary a, obtaining again a function of one variable, and leading to the four propositions

$$(x) : (y) \,.\, \phi(x, y), (\exists x) : (y) \,.\, \phi(x, y), (x) : (\exists y) \,.\, \phi(x, y), (\exists x) : (\exists y) \,.\, \phi(x, y).$$

On the other hand, we might have gone from $\phi(a, b)$ to $\phi(x, b)$, thence to $(x) \,.\, \phi(x, b)$ and $(\exists x) \,.\, \phi(x, b)$, and thence to

$$(y) : (x) \,.\, \phi(x, y), (\exists y) : (x) \,.\, \phi(x, y), (y) : (\exists x) \,.\, \phi(x, y), (\exists y) : (\exists x) \,.\, \phi(x, y).$$

All of these will be called "general propositions"; thus eight general propositions can be derived from the function $\phi(x, y)$. We have

$$(x) : (y) \,.\, \phi(x, y) : \equiv : (y) : (x) \,.\, \phi(x, y),$$
$$(\exists x) : (\exists y) \,.\, \phi(x, y) : \equiv : (\exists y) : (\exists x) \,.\, \phi(x, y).$$

But there are no other equivalences that always hold. For example, the distinction between "$(x) : (\exists y) \,.\, \phi(x, y)$" and "$(\exists y) : (x) \,.\, \phi(x, y)$" is the same as the distinction in analysis between "For every ϵ, however small, there is a δ such that..." and "There is a δ such that, for every ϵ, however small, ..."

Although it might seem easier, in view of the above considerations, to regard every function of several variables as obtained by successive steps, each involving only a function of one variable, yet there are powerful considerations on the other side. There are two grounds in favour of the step-by-step method; first, that only functions of *one* variable need be taken as a primitive idea; secondly, that such definitions as the above seem to require *either* that we should first vary x, keeping y constant, *or* that we should first vary y, keeping x constant. The former seems to be involved when "(y)" or "$(\exists y)$" appears to the left of "(x)" or "$(\exists x)$," the latter in the converse case. The grounds against the step-by-step method are that it interferes with the method of matrices, which brings order into the successive generation of types of propositions and functions demanded by the theory of types, and that it requires us, from the start, to deal with such propositions as $(y) . \phi(a, y)$, which are not elementary. Take, for example, the proposition "$\vdash : q . \supset . p \vee q$." This will be

$$\vdash :. (p) :. (q) : q . \supset . p \vee q,$$

or

$$\vdash :. (q) :. (p) : q . \supset . p \vee q,$$

and will thus involve all values of either

$$(q) : q . \supset . p \vee q \text{ considered as a function of } p,$$

or

$$(p) : q . \supset . p \vee q \text{ considered as a function of } q.$$

This makes it impossible to start our logic with elementary propositions, as we wish to do. It is useless to enlarge the definition of elementary propositions, since that only increases the values of q or p in the above functions. Hence it seems necessary to start with an elementary function

$$\phi(x_1, x_2, x_3, \ldots x_n),$$

before which we write, for each x_r, either "(x_r)" or "$(\exists x_r)$," the variables in this process being taken in any order we like. Here $\phi(x_1, x_2, x_3, \ldots x_n)$ is called the "matrix," and what comes before it is called the "prefix." Thus in

$$(\exists x) : (y) . \phi(x, y)$$

"$\phi(x, y)$" is the matrix and "$(\exists x) : (y)$" is the prefix. It thus appears that a matrix containing n variables gives rise to $n!\,2^n$ propositions by taking its variables in all possible orders and distinguishing "(x_r)" and "$(\exists x_r)$" in each case. (Some of these, however, are equivalent.) The process of obtaining such propositions from a matrix will be called "generalization," whether we take "all values" or "some value," and the propositions which result will be called "general propositions."

We shall later have occasion to consider matrices containing variables that are not individuals; we may therefore say:

A "matrix" is a function of any number of variables (which may or may not be individuals), which has elementary propositions as its values, and is used for the purpose of generalization.

A "general proposition" is one derived from a matrix by generalization. We shall add one further definition at this stage:

A "first-order proposition" is one derived by generalization from a matrix in which all the variables are individuals.

4. *Methods of proving general propositions*

There are two fundamental methods of proving general propositions, one for universal propositions, the other for such as assert existence. The method of proving universal propositions is as follows. Given a proposition

$$\text{“}\vdash . F(p, q, r, \ldots),\text{”}$$

where F is built up by the stroke, and $p, q, r, \ldots$ are elementary, we may replace them by elementary functions of individuals in any way we like, putting

$$p = f_1(x_1, x_2, \ldots x_n),$$
$$q = f_2(x_1, x_2, \ldots x_n),$$

and so on, and then assert the result for all values of $x_1, x_2, \ldots x_n$. What we thus assert is less than the original assertion, since $p, q, r, \ldots$ could originally take all values that are elementary propositions, whereas now they can only take such as are values of $f_1, f_2, f_3, \ldots$. (Any two or more of $f_1, f_2, f_3, \ldots$ may be identical.)

For proving existence-theorems we have two primitive propositions, namely

***8·1.** $\vdash . (\exists x, y) . \phi a \mid (\phi x \mid \phi y)$ and

***8·11.** $\vdash . (\exists x) . \phi x \mid (\phi a \mid \phi b)$

Applying the definitions to be given shortly, these assert respectively

$$\phi a . \supset . (\exists x) . \phi x$$

and $$(x) . \phi x . \supset . \phi a . \phi b.$$

These two primitive propositions are to be assumed, not only for one variable, but for any number. Thus we assume

$$\phi(a_1, a_2, \ldots a_n) . \supset . (\exists x_1, x_2, \ldots x_n) . \phi(x_1, x_2, \ldots x_n),$$
$$(x_1, x_2, \ldots x_n) . \phi(x_1, x_2, \ldots x_n) . \supset . \phi(a_1, a_2, \ldots a_n) . \phi(b_1, b_2, \ldots b_n).$$

The proposition $(x) . \phi x . \supset . \phi a . \phi b$, in this form, does not look suitable for proving existence-theorems. But it may be written

$$(\exists x) . \sim \phi x . \vee . \phi a . \phi b$$

or $$\sim \phi a \vee \sim \phi b . \supset . (\exists x) . \sim \phi x,$$

in which form it is identical with *9·11, writing ϕ for $\sim \phi$. Thus our two primitive propositions are the same as *9·1 and *9·11.

For purposes of inference, we still assume that from $(x) . \phi x$ and $(x) . \phi x \supset \psi x$ we can infer $(x) . \psi x$, and from p and $p \supset q$ we can infer q, even when the functions or propositions involved are not elementary.

Existence-theorems are very often obtained from the above primitive propositions in the following manner. Suppose we know a proposition

$$\vdash . f(x, x).$$

Since $\phi x . \supset . (\exists y) . \phi y$, we can infer

$$\vdash . (\exists y) . f(x, y),$$

i.e. $\vdash : (x) : (\exists y) . f(x, y)$.

Similarly $\vdash : (y) : (\exists x) . f(x, y)$.

Again, since $\phi(x, y) . \supset . (\exists z, w) . \phi(z, w)$, we can infer

$$\vdash . (\exists x, y) . f(x, y)$$

and $\vdash . (\exists y, x) . f(x, y)$.

We may illustrate the proofs both of universal and of existence propositions by a simple example. We have

$$\vdash . (p) . p \supset p.$$

Hence, substituting ϕx for p,

$$\vdash . (x) . \phi x \supset \phi x.$$

Hence, as in the case of $f(x, x)$ above,

$$\vdash : (x) : (\exists y) . \phi x \supset \phi y,$$

$$\vdash : (y) : (\exists x) . \phi x \supset \phi y,$$

$$\vdash . (\exists x, y) . \phi x \supset \phi y.$$

Apart from special axioms asserting existence-theorems (such as the axiom of reducibility, the multiplicative axiom, and the axiom of infinity), the above two primitive propositions give the sole method of proving existence-theorems in logic. They are, in fact, always derived from general propositions of the form $(x) . f(x, x)$ or $(x) . f(x, x, x)$ or etc., by substituting other variables for some of the occurrences of x.

III. GENERAL PROPOSITIONS OF LIMITED SCOPE

In virtue of a primitive proposition, given $(x) . \phi x$ and $(x) . \phi x \supset \psi x$, we can infer $(x) . \psi x$. So far, however, we have introduced no notation which would enable us to state the corresponding *implication* (as opposed to *inference*). Again, $(\exists x) . \phi x$ and $(x, y) . \phi x \supset \psi y$ enable us to infer $(y) . \psi y$; here again, we wish to be able to state the corresponding implication. So far, we have only defined occurrences of general propositions as complete asserted propositions. Theoretically, this is their only use, and there is no need to define any other. But practically, it is highly convenient to be able to treat them as parts of stroke-functions. This is entirely a matter of definition. By introducing suitable definitions, first-order propositions can be shown to satisfy all the propositions of *1—*5. Hence in using the propositions of *1—*5, it will no longer be necessary to assume that p, q, r, ... are elementary.

The fundamental definitions are given below.

When a general proposition occurs as part of another, it is said to have limited scope. If it contains an apparent variable x, the scope of x is said to be limited to the general proposition in question. Thus in $p \mid \{(x) \,.\, \phi x\}$, the scope of x is limited to $(x) \,.\, \phi x$, whereas in $(x) \,.\, p \mid \phi x$ the scope of x extends to the whole proposition. Scope is indicated by dots.

The new chapter ∗8 (given in Appendix A) should replace ∗9 in *Principia Mathematica.* Its general procedure will, however, be explained now.

The occurrence of a general proposition as part of a stroke-function is defined by means of the following definitions:

$$\{(x) \,.\, \phi x\} \mid q \,.\!=.\, (\exists x) \,.\, \phi x \mid q \quad \text{Df,}$$
$$\{(\exists x) \,.\, \phi x\} \mid q \,.\!=.\, (x) \,.\, \phi x \mid q \quad \text{Df,}$$
$$p \mid \{(y) \,.\, \psi y\} \,.\!=.\, (\exists y) \,.\, p \mid \psi y \quad \text{Df,}$$
$$p \mid \{(\exists y) \,.\, \psi y\} \,.\!=.\, (y) \,.\, p \mid \psi y \quad \text{Df.}$$

These define, in the first place, only what is meant by the stroke when it occurs between two propositions of which one is elementary while the other is of the first order. When the stroke occurs between two propositions which are both of the first order, we shall adopt the convention that the one on the left is to be eliminated first, treating the one on the right as if it were elementary; then the one on the right is to be eliminated, in each case, in accordance with the above definitions. Thus

$$\begin{aligned}
\{(x) \,.\, \phi x\} \mid \{(y) \,.\, \psi y\} \,.\!=:\, & (\exists x) : \phi x \mid \{(y) \,.\, \psi y\} : \\
=: \, & (\exists x) : (\exists y) \,.\, \phi x \mid \psi y, \\
\{(x) \,.\, \phi x\} \mid \{(\exists y) \,.\, \psi y\} \,.\!=:\, & (\exists x) : \phi x \mid \{(\exists y) \,.\, \psi y\} : \\
=: \, & (\exists x) : (y) \,.\, \phi x \mid \psi y, \\
\{(\exists x) \,.\, \phi x\} \mid \{(y) \,.\, \psi y\} \,.\!=:\, & (x) : (\exists y) \,.\, \phi x \mid \psi y.
\end{aligned}$$

The rule about the order of elimination is only required for the sake of definiteness, since the two orders give equivalent results. For example, in the last of the above instances, if we had eliminated y first we should have obtained

$$(\exists y) : (x) \,.\, \phi x \mid \psi y,$$

which requires either $(x) \,.\, \sim\phi x$ or $(\exists y) \,.\, \sim\psi y$, and is then true.

And $$(x) : (\exists y) \,.\, \phi x \mid \psi y$$

is true in the same circumstances. This possibility of changing the order of the variables in the prefix is only due to the way in which they occur, *i.e.* to the fact that x only occurs on one side of the stroke and y only on the other. The order of the variables in the prefix is indifferent whenever the occurrences of one variable are all on one side of a certain stroke, while those of the other are all on the other side of it. We do not have in general

$$(\exists x) : (y) \,.\, \chi(x, y) : \equiv : (y) : (\exists x) \,.\, \chi(x, y);$$

here the right-hand side is more often true than the left-hand side. But we do have

$$(\exists x) : (y) \,.\, \phi x \mid \psi y : \equiv : (y) : (\exists x) \,.\, \phi x \mid \psi y.$$

The possibility of altering the order of the variables in the prefix when they are separated by a stroke is a primitive proposition. In general it is convenient to put on the left the variables of which "all" are involved, and on the right those of which "some" are involved, after the elimination has been finished, always assuming that the variables occur in a way to which our primitive proposition is applicable.

It is not necessary for the above primitive proposition that the stroke separating x and y should be the principal stroke, *e.g.*

$$\begin{aligned} p \mid [\{(\exists x) \,.\, \phi x\} \mid \{(y) \,.\, \psi y\}] \,.\, = \,.\, & p \mid [(x) : (\exists y) \,.\, \phi x \mid \psi y] \,. \\ & = : (\exists x) : (y) \,.\, p \mid (\phi x \mid \psi y) : \\ & \equiv : (y) : (\exists x) \,.\, p \mid (\phi x \mid \psi y). \end{aligned}$$

All that is necessary is that there should be *some* stroke which separates x from y. When this is not the case, the order cannot in general be changed. Take *e.g.* the matrix

$$\phi x \lor \psi y \,.\, \sim \phi x \lor \sim \psi y.$$

This may be written $\qquad (\phi x \supset \psi y) \mid (\psi y \supset \phi x)$

or $\qquad \{\phi x \mid (\psi y \mid \psi y)\} \mid \{\psi y \mid (\phi x \mid \phi x)\}.$

Here there is no stroke which separates all the occurrences of x from all those of y, and in fact the two propositions

$$(y) : (\exists x) \,.\, \phi x \lor \psi y \,.\, \sim \phi x \lor \sim \psi y$$

and $\qquad (\exists x) : (y) \,.\, \phi x \lor \psi y \,.\, \sim \phi x \lor \sim \psi y$

are not equivalent except for special values of ϕ and ψ.

By means of the above definitions, we are able to derive all propositions, of whatever order, from a matrix of elementary propositions combined by means of the stroke. Given any such matrix, containing a part p, we may replace p by ϕx or $\phi(x, y)$ or etc., and proceed to add the prefix (x) or $(\exists x)$ or (x, y) or $(x) : (\exists y)$ or $(y) : (\exists x)$ or etc. If p and q both occur, we may replace p by ϕx and q by ψy, or we may replace both by ϕx, or one by ϕx and another by some stroke-function of ϕx.

In the case of a proposition such as

$$p \mid \{(x) : (\exists y) \,.\, \psi(x, y)\}$$

we must treat it as a case of $p \mid \{(x) \,.\, \phi x\}$, and first eliminate x. Thus

$$p \mid \{(x) : (\exists y) \,.\, \psi(x, y)\} \,.\, = : (\exists x) : (y) \,.\, p \mid \psi(x, y).$$

That is to say, the definitions of $\{(x) \,.\, \phi x)\} \mid q$ etc. are to be applicable unchanged when ϕx is not an elementary function.

The definitions of $\sim p$, $p \vee q$, $p \,.\, q$, $p \supset q$ are to be taken over unchanged. Thus

$$
\begin{aligned}
\sim\{(x) \,.\, \phi x\} \,.=:\, & \{(x) \,.\, \phi x\} \mid \{(x) \,.\, \phi x\}: \\
=: & (\exists x) : \phi x \mid \{(x) \,.\, \phi x\}: \\
=: & (\exists x) : (\exists y) \,.\, (\phi x \mid \phi y), \\
\sim\{(\exists x) \,.\, \phi x\} \,.=:\, & (x) : (y) \,.\, (\phi x \mid \phi y), \\
p \,.\supset.\, (x) \,.\, \phi x :=:\, & p \mid [\{(x) \,.\, \phi x\} \mid \{(x) \,.\, \phi x\}]: \\
=: & p \mid \{(\exists x) : (\exists y) \,.\, (\phi x \mid \phi y)\}: \\
=: & (x) : (y) \,.\, p \mid (\phi x \mid \phi y), \\
(x) \,.\, \phi x \,.\supset.\, p :=:\, & \{(x) \,.\, \phi x\} \mid (p \mid p): \\
=: & (\exists x) \,.\, \phi x \mid (p \mid p) :=: (\exists x) \,.\, \phi x \supset p, \\
(x) \,.\, \phi x \,.\vee.\, p :=:\, & [\sim\{(x) \,.\, \phi x\}] \mid \sim p: \\
=: & \{(\exists x) : (\exists y) \,.\, (\phi x \mid \phi y)\} \mid (p \mid p): \\
=: & (x) \,.\, \{(\exists y) \,.\, (\phi x \mid \phi y)\} \mid (p \mid p): \\
=: & (x) : (y) \,.\, (\phi x \mid \phi y) \mid (p \mid p), \\
p \,.\vee.\, (x) \,.\, \phi x :=:\, & (x) : (y) \,.\, (p \mid p) \mid (\phi x \mid \phi y).
\end{aligned}
$$

It will be seen that in the above two variables appear where only one might have been expected. We shall find, before long, that the two variables can be reduced to one; *i.e.* we shall have

$$
\begin{aligned}
(\exists x) : (\exists y) \,.\, \phi x \mid \phi y :\equiv.\, & (\exists x) \,.\, \phi x \mid \phi x, \\
(x) : (y) \,.\, \phi x \mid \phi y :\equiv.\, & (x) \,.\, \phi x \mid \phi x.
\end{aligned}
$$

These lead to

$$
\begin{aligned}
\sim\{(x) \,.\, \phi x\} \,.\equiv.\, & (\exists x) \,.\sim \phi x, \\
\sim\{(\exists x) \,.\, \phi x\} \,.\equiv.\, & (x) \,.\sim \phi x.
\end{aligned}
$$

But we cannot prove these propositions at our present stage; nor, if we could, would they be of much use to us, since we do not yet know that, when two general propositions are equivalent, either may be substituted for the other as part of a stroke-proposition without changing the truth-value.

For the present, therefore, suppose we have a stroke-function in which p occurs several times, say $p \mid (p \mid p)$, and we wish to replace p by $(x) \,.\, \phi x$, we shall have to write the second occurrence of p "$(y) \,.\, \phi y$," and the third "$(z) \,.\, \phi z$." Thus the resulting proposition will contain as many separate variables as there are occurrences of p.

The primitive propositions required, which have been already mentioned, are four in number. They are as follows:

(1) $\vdash .\, (\exists x, y) \,.\, \phi a \mid (\phi x \mid \phi y)$, *i.e.* $\vdash : \phi a \,.\supset.\, (\exists x) \,.\, \phi x$.

(2) $\vdash .\, (\exists x) \,.\, \phi x \mid (\phi a \mid \phi b)$, *i.e.* $\vdash : (x) \,.\, \phi x \,.\supset.\, \phi a \,.\, \phi b$.

(3) The extended rule of inference, *i.e.* from $(x) \,.\, \phi x$ and $(x) \,.\, \phi x \supset \psi x$ we can infer $(x) \,.\, \psi x$, even when ϕ and ψ are not elementary.

(4) If all the occurrences of x are separated from all the occurrences of y by a certain stroke, the order of x and y can be changed in the prefix; *i.e.*

For $(\exists x):(y).\phi x \mid \psi y$ we can substitute $(y):(\exists x).\phi x \mid \psi y$, and *vice versa*, even when this is only a part of the whole asserted proposition.

The above primitive propositions are to be assumed, not only for one variable, but for any number.

By means of the above primitive propositions it can be proved that all the propositions of ∗1—∗5 apply equally when one or more of the propositions $p, q, r, \ldots$ involved are not elementary. For this purpose, we make use of the work of Nicod, who proved that the primitive propositions of ∗1 can all be deduced from

$$\vdash . p \supset p$$

and $$\vdash . p \supset q . \supset . s \mid q \supset p \mid s$$

together with the rule of inference: "Given p and $p \mid (q \mid r)$, we can infer r."

Thus all we have to do is to show that the above propositions remain true when p, q, s, or some of them, are not elementary. This is done in ∗8 in Appendix A.

IV. FUNCTIONS AS VARIABLES

The essential use of a variable is to pick out a certain assemblage of elementary propositions, and enable us to assert that all members of this assemblage are true, or that at least one member is true. We have already used functions of individuals, by substituting ϕx for p in the propositions of ∗1—∗5, and by the primitive propositions of ∗8. But hitherto we have always supposed that the function is kept constant while the individual is varied, and we have not considered cases where we have "$\exists\phi$," or where the scope of "ϕ" is less than the whole asserted proposition. It is necessary now to consider such cases.

Suppose a is a constant. Then "ϕa" will denote, for the various values of ϕ, all the various elementary propositions of which a is a constituent. This is a different assemblage of elementary propositions from any that can be obtained by variation of individuals; consequently it gives rise to new general propositions. The values of the function are still elementary propositions, just as when the argument is an individual; but they are a new assemblage of elementary propositions, different from previous assemblages.

As we shall have occasion later to consider functions whose values are not elementary propositions, we will distinguish those that have elementary propositions for their values by a note of exclamation between the letter denoting the function and the letter denoting the argument. Thus "$\phi ! x$" is a function of two variables, x and $\phi ! \hat{z}$. It is a matrix, since it contains no apparent variable and has elementary propositions for its values. We shall henceforth write "$\phi ! x$" where we have hitherto written ϕx.

If we replace x by a constant a, we can form such propositions as

$$(\phi) . \phi ! a, \ (\exists\phi) . \phi ! a.$$

These are not elementary propositions, and are therefore not of the form $\phi ! a$. The assertion of such propositions is derived from matrices by the method of ∗8. The primitive propositions of ∗8 are to apply when the variables, or some of them, are elementary functions as well as when they are all individuals.

*A function can only appear in a matrix through its values**. To obtain a matrix, proceed, as before, by writing $\phi ! x$, $\psi ! y$, $\chi ! z$, ... in place of p, q, r,... in some molecular proposition built up by means of the stroke. We can then apply the rules of ∗8 to ϕ, ψ, χ, ... as well as to x, y, z, ... The difference between a function of an individual and a function of an elementary function of individuals is that, in the former, the passage from one value to another is effected by making the same statement about a different individual, while in the latter it is effected by making a different statement about the same individual. Thus the passage from "Socrates is mortal" to "Plato is mortal" is a passage from $f ! x$ to $f ! y$, but the passage from "Socrates is mortal" to "Socrates is wise" is a passage from $\phi ! a$ to $\psi ! a$. Functional variation is involved in such a proposition as: "Napoleon had all the characteristics of a great general."

Taking the collection of elementary propositions, every matrix has values all of which belong to this collection. Every general proposition results from some matrix by generalization†. Every matrix intrinsically determines a certain classification of elementary propositions, which in turn determines the scope of the generalization of that matrix. Thus "x loves Socrates" picks out a certain collection of propositions, generalized in "$(x) . x$ loves Socrates" and "$(\exists x) . x$ loves Socrates." But "$\phi !$ Socrates" picks out those, among elementary propositions, which mention Socrates. The generalizations "$(\phi) . \phi !$ Socrates" and "$(\exists\phi) . \phi !$ Socrates" involve a class of elementary propositions which cannot be obtained from an individual-variable. But any value of "$\phi !$Socrates" is an ordinary elementary proposition; the novelty introduced by the variable ϕ is a novelty of classification, not of material classified. On the other hand, $(x) . x$ loves Socrates, $(\phi) . \phi !$ Socrates, etc. are new propositions, not contained among elementary propositions. It is the business of ∗8 to show that these propositions obey the same rules as elementary propositions. The method of proof makes it irrelevant what the variables are, so long as all the functions concerned have values which are elementary propositions. The variables may themselves be elementary propositions, as they are in ∗1—∗5.

A variable function which has values that are not elementary propositions starts a new set. But variables of this sort seem unnecessary. Every elementary proposition is a value of $\phi ! \hat{x}$; therefore

$$(p) . fp . \equiv . (\phi, x) . f(\phi ! x) : (\exists p) . fp . \equiv . (\exists\phi, x) . f(\phi ! x).$$

* This assumption is fundamental in the following theory. It has its difficulties, but for the moment we ignore them. It takes the place (not quite adequately) of the axiom of reducibility. It is discussed in Appendix C.

† In a proposition of logic, all the variables in the matrix must be generalized. In other general propositions, such as "all men are mortal," some of the variables in the matrix are replaced by constants.

Hence all second-order propositions in which the variable is an elementary proposition can be derived from elementary matrices. The question of other second-order propositions will be dealt with in the next section. A function of two variables, say $\phi(x, y)$, picks out a certain class of classes of propositions. We shall have the class $\phi(a, y)$, for given a and variable y; then the class of all classes $\phi(a, y)$ as a varies. Whether we are to regard our function as giving classes $\phi(a, y)$ or $\phi(x, b)$ depends upon the order of generalization adopted. Thus "$(\exists x):(y)$" involves $\phi(a, y)$, but "$(y):(\exists x)$" involves $\phi(x, b)$.

Consider now the matrix $\phi!x$, as a function of two variables. If we first vary x, keeping ϕ fixed (which seems the more natural order), we form a class of propositions $\phi!x$, $\phi!y$, $\phi!z$, ... which differ solely by the substitution of one individual for another. Having made one such class, we make another, and so on, until we have done so in all possible ways. But now suppose we vary ϕ first, keeping x fixed and equal to a. We then first form the class of all propositions of the form $\phi!a$, *i.e.* all elementary propositions of which a is a constituent; we next form the class $\phi!b$; and so on. The set of propositions which are values of $\phi!a$ is a set not obtainable by variation of individuals, *i.e.* not of the form fx [for constant f and variable x]. This is what makes ϕ a new sort of variable, different from x. This also is why generalization of the form $(\phi).F!(\phi!\hat{z}, x)$ gives a function not of the form $f!x$ [for constant f]. Observe also that whereas a is a constituent of $f!a$, f is not; thus the matrix $\phi!x$ has the peculiarity that, when a value is assigned to x, this value is a constituent of the result, but when a value is assigned to ϕ, this value is absorbed in the resulting proposition, and completely disappears. We may define a function $\phi!\hat{x}$ as that kind of similarity between propositions which exists when one results from the other by the substitution of one individual for another.

We have seen that there are matrices containing, as variables, functions of individuals. We may denote any such matrix by

$$f!(\phi!\hat{z}, \psi!\hat{z}, \chi!\hat{z}, \ldots x, y, z, \ldots).$$

Since a function can only occur through its values, $\phi!\hat{z}$ (*e.g.*) can only occur in the above matrix through the occurrence of $\phi!x$, $\phi!y$, $\phi!z$, ... or of $\phi!a$, $\phi!b$, $\phi!c$, ..., where a, b, c are constants. Constants do not occur in logic, that is to say, the a, b, c which we have been supposing constant are to be regarded as obtained by an extra-logical assignment of values to variables. They may therefore be absorbed into the $x, y, z, \ldots$. Now x, y, z themselves will only occur in logic as arguments to variable functions. Hence any matrix which contains the variables $\phi!\hat{z}$, $\psi!\hat{z}$, $\chi!\hat{z}$, x, y, z and no others, if it is of the sort that can occur explicitly in logic, will result from substituting $\phi!x$, $\phi!y$, $\phi!z$, $\psi!x$, $\psi!y$, $\psi!z$, $\chi!x$, $\chi!y$, $\chi!z$, or some of them, for elementary propositions in some stroke-function.

It is necessary here to explain what is meant when we speak of a "matrix that can occur explicitly in logic," or, as we may call it, a "logical matrix." A logical matrix is one that contains no constants. Thus $p \mid q$ is a logical matrix; so is $\phi ! x$, where ϕ and x are both variable. Taking any elementary proposition, we shall obtain a logical matrix if we replace all its components and constituents by variables. Other matrices result from logical matrices by assigning values to some of their variables. There are, however, various ways of analysing a proposition, and therefore various logical matrices can be derived from a given proposition. Thus a proposition which is a value of $p \mid q$ will also be a value of $(\phi ! x) \mid (\psi ! y)$ and of $\chi ! (x, y)$. Different forms are required for different purposes; but all the forms of matrices required explicitly in logic are logical matrices as above defined. This is merely an illustration of the fact that logic aims always at complete generality. The test of a logical matrix is that it can be expressed without introducing any symbols other than those of logic, *e.g.* we must not require the symbol "Socrates." Consider the expression

$$f ! (\phi ! \hat{z}, \psi ! \hat{z}, \chi ! \hat{z}, \ldots x, y, z).$$

When a value is assigned to f, this represents a matrix containing the variables $\phi, \psi, \chi, \ldots x, y, z, \ldots$. But while f remains unassigned, it is a matrix of a new sort, containing the new variable f. We call f a "second-order function," because it takes functions among its arguments. When a value is assigned, not only to f, but also to $\phi, \psi, \chi, \ldots x, y, z, \ldots$, we obtain an elementary proposition; but when a value is assigned to f alone, we obtain a matrix containing as variables only first-order functions and individuals. This is analogous to what happens when we consider the matrix $\phi ! x$. If we give values to both ϕ and x, we obtain an elementary proposition; but if we give a value to ϕ alone, we obtain a matrix containing only an individual as variable.

There is no logical matrix of the form $f ! (\phi ! \hat{z})$. The only matrices in which $\phi ! \hat{z}$ is the only argument are those containing $\phi ! a, \phi ! b, \phi ! c, \ldots$, where $a, b, c, \ldots$ are constants; but these are not logical matrices, being derived from the logical matrix $\phi ! x$. Since ϕ can only appear through its values, it must appear, in a logical matrix, with one or more variable arguments. The simplest logical functions of ϕ alone are $(x) . \phi ! x$ and $(\exists x) . \phi ! x$, but these are not matrices. A logical matrix

$$f ! (\phi ! \hat{z}, x_1, x_2, \ldots x_n)$$

is always derived from a stroke-function

$$F(p_1, p_2, p_3, \ldots p_n)$$

by substituting $\phi ! x_1, \phi ! x_2, \ldots \phi ! x_n$ for $p_1, p_2, \ldots p_n$. This is the sole method of constructing such matrices. (We may however have $x_r = x_s$ for some values of r and s.)

Second-order functions have two connected properties which first-order functions do not have. The first of these is that, when a value is assigned to

f, the result may be a logical matrix; the second is that certain constant values of f can be assigned without going outside logic.

To take the first point first: $f ! (\phi ! \hat{z}, x)$, for example, is a matrix containing three variables, f, ϕ, and x. The following logical matrices (among an infinite number) result from the above by assigning a value to f: $\phi ! x$, $(\phi ! x) | (\phi ! x)$, $\phi ! x \supset \phi ! x$, etc. Similarly $\phi ! x \supset \phi ! y$, which is a logical matrix, results from assigning a value to f in $f ! (\phi ! \hat{z}, x, y)$. In all these cases, the constant value assigned to f is one which can be expressed in logical symbols alone (which was the second property of f). This is not the case with $\phi ! x$: in order to assign a value to ϕ, we must introduce what we may call "empirical constants," such as "Socrates" and "mortality" and "being Greek." The functions of x that can be formed without going outside logic must involve a function as a generalized variable; they are (in the simplest case) such as $(\phi) . \phi ! x$ and $(\exists \phi) . \phi ! x$.

To some extent, however, the above peculiarity of functions of the second and higher orders is arbitrary. We might have adopted in logic the symbols

$$R_1 (x),\ R_2 (x, y),\ R_3 (x, y, z),\ \ldots,$$

where R_1 represents a variable predicate, R_2 a variable dyadic relation (in intension), and so on. Each of the symbols $R_1 (x)$, $R_2 (x, y)$, $R_3 (x, y, z)$, ... is a logical matrix, so that, if we used them, we should have logical matrices not containing variable functions. It is perhaps worth while to remind ourselves of the meaning of "$\phi ! a$," where a is a constant. The meaning is as follows. Take any finite number of propositions of the various forms $R_1 (x)$, $R_2 (x, y)$, ... and combine them by means of the stroke in any way desired, allowing any one of them to be repeated any finite number of times. If at least one of them has a as a constituent, *i.e.* is of the form

$$R_n (a, b_1, b_2, \ldots b_{n-1}),$$

then the molecular proposition we have constructed is of the form $\phi ! a$, *i.e.* is a value of "$\phi ! a$" with a suitable ϕ. This of course also holds of the proposition $R_n (a, b_1, b_2, \ldots b_{n-1})$ itself. It is clear that the logic of propositions, and still more of general propositions concerning a given argument, would be intolerably complicated if we abstained from the use of variable functions; but it can hardly be said that it would be impossible. As for the question of matrices, we could form a matrix $f ! (R_1, x)$, of which $R_1 (x)$ would be a value. That is to say, the properties of second-order matrices which we have been discussing would also belong to matrices containing variable universals. They cannot belong to matrices containing only variable individuals.

By assigning $\phi ! \hat{z}$ and x in $f ! (\phi ! \hat{z}, x)$, while leaving f variable, we obtain an assemblage of elementary propositions not to be obtained by means of variables representing individuals and first-order functions. This is why the new variable f is useful.

We can proceed in like manner to matrices

$$F ! \{f ! (\hat{\phi} ! \hat{z}, \hat{x}), g ! (\hat{\phi} ! \hat{z}, \hat{x}), \ldots \psi ! \hat{z}, \chi ! \hat{z}, \ldots x, y, \ldots\}$$

and so on indefinitely. These merely represent new ways of grouping elementary propositions, leading to new kinds of generality.

V. FUNCTIONS OTHER THAN MATRICES

When a matrix contains several variables, functions of some of them can be obtained by turning the others into apparent variables. Functions obtained in this way are not matrices, and their values are not elementary propositions. The simplest examples are

$$(y) . \phi ! (x, y) \text{ and } (\exists y) . \phi ! (x, y).$$

When we have a general proposition $(\phi) . F\{\phi ! \hat{z}, x, y, \ldots\}$, the only values ϕ can take are matrices, so that functions containing apparent variables are not included. We can, if we like, introduce a new variable, to denote not only functions such as $\phi ! \hat{x}$, but also such as

$$(y) . \phi ! (\hat{x}, y), (y, z) . \phi ! (\hat{x}, y, z), \ldots (\exists y) . \phi ! (\hat{x}, y), \ldots;$$

in a word, all such functions of one variable as can be derived by generalization from matrices containing only individual-variables. Let us denote any such function by $\phi_1 x$, or $\psi_1 x$, or $\chi_1 x$, or etc. Here the suffix 1 is intended to indicate that the values of the functions may be first-order propositions, resulting from generalization in respect of individuals. In virtue of $*8$, no harm can come from including such functions along with matrices as values of single variables.

Theoretically, it is unnecessary to introduce such variables as ϕ_1, because they can be replaced by an infinite conjunction or disjunction. Thus *e.g.*

$$(\phi_1) . \phi_1 x . \equiv : (\phi) . \phi ! x : (\phi, y) . \phi ! (x, y) : (\phi) : (\exists y) . \phi ! (x, y) : \text{etc.},$$

$$(\exists \phi_1) . \phi_1 x . \equiv : (\exists \phi) . \phi ! x : \mathbf{v} : (\exists \phi) : (y) . \phi ! (x, y) : \mathbf{v} : (\exists \phi, y) . \phi ! (x, y) : \mathbf{v} : \text{etc.},$$

and generally, given any matrix $f ! (\phi ! \hat{z}, x)$, we shall have the following process for interpreting $(\phi_1) . f ! (\phi_1 \hat{z}, x)$ and $(\exists \phi_1) . f ! (\phi_1 \hat{z}, x)$. Put

$$(\phi^1) . f ! (\phi^1 \hat{z}, x) . = : (\phi) . f ! \{(y) . \phi ! (\hat{z}, y), x\} : (\phi) . f ! \{(\exists y) . \phi ! (\hat{z}, y), x\},$$

where $f ! \{(y) . \phi ! (\hat{z}, y), x\}$ is constructed as follows: wherever, in $f ! \{\phi ! \hat{z}, x\}$, a value of ϕ, say $\phi ! a$, occurs, substitute $(y) . \phi ! (a, y)$, and develop by the definitions at the beginning of $*8$. $f ! \{(\exists y) . \phi ! (\hat{z}, y), x\}$ is similarly constructed. Similarly put

$$(\phi^2) . f ! (\phi^2 ! \hat{z}, x) . = : (\phi) . f ! \{(y, w) . \phi ! (\hat{z}, y, w), x\} :$$
$$(\phi) . f ! \{(y) : (\exists w) . \phi ! (\hat{z}, y, w), x\} : \text{etc.},$$

where "etc." covers the prefixes $(\exists y) : (w) .$, $(\exists y, w) .$, $(w) : (\exists y)$. We define ϕ^3, ϕ^4, ... similarly. Then

$$(\phi_1) . f ! (\phi_1 \hat{z}, x) . = : (\phi^1) . f ! (\phi^1 \hat{z}, x) : (\phi^2) . f ! (\phi^2 \hat{z}, x) : \text{etc.}$$

This process depends upon the fact that $f ! (\phi ! \hat{z}, x)$, for each value of ϕ and x, is a proposition constructed out of elementary propositions by the stroke, and

that ∗8 enables us to replace any of these by a proposition which is not elementary. $(\exists\phi_1) . f!(\phi_1\hat{z}, x)$ is defined by an exactly analogous *disjunction.*

It is obvious that, in practice, an infinite conjunction or disjunction such as the above cannot be manipulated without assumptions *ad hoc.* We can work out results for any segment of the infinite conjunction or disjunction, and we can "see" that these results hold throughout. But we cannot prove this, because mathematical induction is not applicable. We therefore adopt certain primitive propositions, which assert only that what we can prove in each case holds generally. By means of these it becomes possible to manipulate such variables as ϕ_1.

In like manner we can introduce $f_1(\phi_1\hat{z}, \hat{x})$, where any number of individuals and functions $\psi_1, \chi_1, \ldots$ may appear as apparent variables.

No essential difficulty arises in this process so long as the apparent variables involved in a function are not of higher order than the argument to the function. For example, $x \in \text{D}‘R$, which is $(\exists y) . xRy$, may be treated without danger as if it were of the form $\phi!x$. In virtue of ∗8, $\phi_1 x$ may be substituted for $\phi!x$ without interfering with the truth of any logical proposition which $\phi!x$ is a part. Similarly whatever logical proposition holds concerning $f!(\phi_1\hat{z}, x)$ will hold concerning $f_1(\phi_1\hat{z}, x)$.

But when the apparent variable is of higher order than the argument, a new situation arises. The simplest cases are

$$(\phi) . f!(\phi!\hat{z}, x), \quad (\exists\phi) . f!(\phi!\hat{z}, x).$$

These are functions of x, but are obviously not included among the values for $\phi!x$ (where ϕ is the argument). If we adopt a new variable ϕ_2 which is to include functions in which $\phi!\hat{z}$ can be an apparent variable, we shall obtain other new functions

$$(\phi_2) . f!(\phi_2\hat{z}, x), \quad (\exists\phi_2) . f!(\phi_2\hat{z}, x),$$

which are again not among values for $\phi_2 x$ (where ϕ_2 is the argument), because the totality of values of $\phi_2\hat{z}$, which is now involved, is different from the totality of values of $\phi!\hat{z}$, which was formerly involved. However much we may enlarge the meaning of ϕ, a function of x in which ϕ occurs as apparent variable has a correspondingly enlarged meaning, so that, however ϕ may be defined,

$$(\phi) . f!(\phi\hat{z}, x) \text{ and } (\exists\phi) . f!(\phi\hat{z}, x)$$

can never be values for ϕx. To attempt to make them so is like attempting to catch one's own shadow. It is impossible to obtain one variable which embraces among its values all possible functions of individuals.

We denote by $\phi_2 x$ a function of x in which ϕ_1 is an apparent variable, but there is no variable of higher order. Similarly $\phi_3 x$ will contain ϕ_2 as apparent variable, and so on.

The essence of the matter is that a variable may travel through any well-defined totality of values, provided these values are all such that any one can replace any other significantly in any context. In constructing $\phi_1 x$, the only totality involved is that of individuals, which is already presupposed. But when we allow ϕ to be an apparent variable in a function of x, we enlarge the totality of functions of x, however ϕ may have been defined. It is therefore always necessary to specify what sort of ϕ is involved, whenever ϕ appears as an apparent variable.

The other condition, that of significance, is fully provided for by the definitions of $*8$, together with the principle that a function can only occur through its values. In virtue of the principle, a function of a function is a stroke-function of values of the function. And in virtue of the definitions in $*8$, a value of any function can significantly replace any proposition in a stroke-function, because propositions containing any number of apparent variables can always be substituted for elementary propositions and for each other in any stroke-function. What is necessary for significance is that every complete asserted proposition should be derived from a matrix by generalization, and that, in the matrix, the substitution of constant values for the variables should always result, ultimately, in a stroke-function of atomic propositions. We say "ultimately," because, when such variables as $\phi_2 \hat{z}$ are admitted, the substitution of a value for ϕ_2 may yield a proposition still containing apparent variables, and in this proposition the apparent variables must be replaced by constants before we arrive at a stroke-function of atomic propositions. We may introduce variables requiring several such stages, but the end must always be the same: a stroke-function of atomic propositions.

It seems, however, though it might be difficult to prove formally, that the functions ϕ_1, f_1 introduce no propositions that cannot be expressed without them. Let us take first a very simple illustration. Consider the proposition

$$(\exists \phi_1) . \phi_1 x . \phi_1 a, \text{ which we will call } f(x, a).$$

Since ϕ_1 includes all possible values of $\phi\,!$ and also a great many other values in its range, $f(x, a)$ might seem to make a smaller assertion than would be made by

$$(\exists \phi) . \phi\,! x . \phi\,! a, \text{ which we will call } f_0(x, a).$$

But in fact $f(x, a) . \supset . f_0(x, a)$. This may be seen as follows: $\phi_1 x$ has one of the various sets of forms:

$$(y) . \phi\,! (x, y),\ (y, z) . \phi\,! (x, y, z),\ \ldots,$$
$$(\exists y) . \phi\,! (x, y),\ (\exists y, z) . \phi\,! (x, y, z),\ \ldots,$$
$$(y) : (\exists z) . \phi\,! (x, y, z),\ (\exists y) : (z) . \phi\,! (x, y, z),\ \ldots.$$

Suppose first that $\phi_1 x . = . (y) . \phi\,! (x, y)$. Then

$$\begin{aligned} \phi_1 x . \phi_1 a . \equiv &: (y) . \phi\,! (x, y) : (y) . \phi\,! (a, y) : \\ \supset &: \phi\,! (x, b) . \phi\,! (a, b) : \\ \supset &: (\exists \phi) . \phi\,! x . \phi\,! a. \end{aligned}$$

Next suppose $\phi_1 x . = . (\exists y) . \phi ! (x, y)$. Then

$$\phi_1 x . \phi_1 a . \equiv : (\exists y) . \phi ! (x, y) : (\exists z) . \phi ! (a, z) :$$
$$\supset : (\exists y, z) : \phi ! (x, y) \vee \phi ! (x, z) . \phi ! (a, y) \vee \phi ! (a, z) :$$
$$\supset : (\exists \phi) . \phi ! x . \phi ! a,$$

because $\phi ! (x, y) \vee \phi ! (x, z)$ is of the form $\phi ! x$, when y and z are fixed. It is obvious that this method of proof applies to the other cases mentioned above. Hence

$$(\exists \phi_1) . \phi_1 x . \phi_1 a . \equiv . (\exists \phi) . \phi ! x . \phi ! a.$$

We can satisfy ourselves that the same result holds in the general form

$$(\exists \phi_1) . f ! (\phi_1 \hat{z}, x) . \equiv . (\exists \phi) . f ! (\phi ! \hat{z}, x)$$

by a similar argument. We know that $f ! (\phi ! \hat{z}, x)$ is derived from some stroke-function

$$F(p, q, r, \ldots)$$

by substituting $\phi ! x$, $\phi ! a$, $\phi ! b$, ... (where a, b, ... are constants) for some of the propositions p, q, r, ... and $g_1 ! x$, $g_2 ! x$, $g_3 ! x$, ... (where g_1, g_2, g_3, ... are constants) for others of p, q, r, ..., while replacing any remaining propositions p, q, r, ... by constant propositions. Take a typical case; suppose

$$f ! (\phi ! \hat{z}, x) . = . (\phi ! a) \mid \{(\phi ! x) \mid (\phi ! b)\}.$$

We then have to prove

$$\phi_1 a \mid (\phi_1 x \mid \phi_1 b) . \supset . (\exists \phi) . \phi ! a \mid (\phi ! x \mid \phi ! b),$$

where $\phi_1 x$ may have any of the forms enumerated above.

Suppose first that $\phi_1 x . = . (y) . \phi ! (x, y)$. Then

$$\phi_1 a \mid (\phi_1 x \mid \phi_1 b) . = : (\exists y) : (z, w) . \phi ! (a, y) \mid \{\phi ! (x, z) \mid \phi ! (b, w)\} :$$
$$\supset : (\exists y) . \phi ! (a, y) \mid \{\phi ! (x, y) \mid \phi ! (b, y)\} :$$
$$\supset : (\exists \phi) . \phi ! a \mid (\phi ! x \mid \phi ! b)$$

because, for a given y, $\phi ! (x, y)$ is of the form $\phi ! x$.

Suppose next that $\phi_1 x . = . (\exists y) . \phi ! (x, y)$. Then

$$\phi_1 a \mid (\phi_1 x \mid \phi_1 b) . = : (y) : (\exists z, w) . \phi ! (a, y) \mid \{\phi ! (x, z) \mid \phi ! (b, w)\} :$$
$$\supset : (\exists \psi) . \psi ! a \mid (\psi ! x \mid \psi ! b),$$

putting $\psi ! x . = . \phi ! (x, z) \vee \phi ! (x, w)$. Similarly the other cases can be dealt with. Hence the result follows.

Consider next the correlative proposition

$$(\phi_1) . f ! (\phi_1 \hat{z}, x) . \equiv . (\phi) . f ! (\phi ! \hat{z}, x).$$

Here it is the converse implication that needs proving, *i.e.*

$$(\phi) . f ! (\phi ! \hat{z}, x) . \supset . (\phi_1) . f ! (\phi_1 \hat{z}, x).$$

This follows from the previous case by transposition. It can also be seen independently as follows. Suppose, as before, that

$$f ! (\phi_1 \hat{z}, x) . = . (\phi_1 a) \mid (\phi_1 x \mid \phi_1 b),$$

and put first $\quad \phi_1 x . = . (y) . \phi ! (x, y)$.

Then $\quad (\phi_1 a) \mid (\phi_1 x \mid \phi_1 b) . = : (\exists y) : (z, w) . \phi ! (a, y) \mid \{\phi ! (x, z) \mid \phi ! (b, w)\}$.

Thus we require that, given

$$(\psi) . (\psi ! a) \mid (\psi ! x \mid \psi ! b),$$

we should have $(\exists y) : (z, w) . \phi ! (a, y) \mid \{\phi ! (x, z) \mid \phi ! (b, w)\}$.

Now

$$\begin{aligned}(\psi) . \psi ! a \mid (\psi ! x \mid \psi ! b) . \supset :. \phi ! (a, z) . \supset . \phi ! (x, z) . \phi ! (b, z) : \\ \phi ! (a, w) . \supset . \phi ! (x, w) . \phi ! (b, w) :. \\ \supset :. \phi ! (a, z) . \phi ! (a, w) . \supset . \phi ! (x, z) . \phi ! (b, w) :. \\ \supset :. \phi ! (a, w) . \supset : \phi ! (a, z) . \supset . \phi ! (x, z) . \phi ! (b, w) \quad (1)\end{aligned}$$

Also $\qquad \sim \phi ! (a, w) . \supset : \phi ! (a, w) . \supset . \phi ! (x, z) . \phi ! (b, w) \quad (2)$

$$(1) . (2) . \supset :. (\psi) . \psi ! a \mid (\psi ! x \mid \psi ! b) : \supset :. (\exists y) : \phi ! (a, y) . \supset . \phi ! (x, z) . \phi ! (b, w)$$

which was to be proved.

Put next $\qquad \phi_1 x . = . (\exists y) . \phi ! (x, y)$.

Then $\quad (\phi_1 a) \mid (\phi_1 x \mid \phi_1 b) . = : (y) : (\exists z, w) . \phi ! (a, y) \mid \{\phi ! (x, z) \mid \phi ! (b, w)\}$.

In this case we merely put $z = w = y$ and the result follows.

The method will be the same in any other case. Hence generally:

$$(\phi_1) . f ! (\phi_1 \hat{z}, x) . \equiv . (\phi) . f ! (\phi ! \hat{z}, x).$$

Although the above arguments do not amount to formal proofs, they suffice to make it clear that, in fact, any general propositions about $\phi ! \hat{z}$ are also true about $\phi_1 \hat{z}$. This gives us, so far as such functions are concerned, all that could have been got from the axiom of reducibility.

Since the proof can only be conducted in each separate case, it is necessary to introduce a primitive proposition stating that the result holds always. This primitive proposition is

$$\vdash : (\phi) . f ! (\phi ! \hat{z}, x) . \supset . f ! (\phi_1 \hat{z}, x) \quad \text{Pp.}$$

As an illustration: suppose we have proved some property of all classes defined by functions of the form $\phi ! \hat{z}$, the above primitive proposition enables us to substitute the class D'R, where R is the relation defined by $\phi ! (\hat{x}, \hat{y})$, or by $(\exists z) . \phi ! (\hat{x}, \hat{y}, z)$, or etc. Wherever a class or relation is defined by a function containing no apparent variables except individuals, the above primitive proposition enables us to treat it as if it were defined by a matrix.

We have now to consider functions of the form $\phi_2 x$, where

$$\phi_2 x . = . (\phi) . f ! (\phi ! \hat{z}, x) \text{ or } \phi_2 x . = . (\exists \phi) . f ! (\phi ! \hat{z}, x).$$

We want to discover whether, or under what circumstances, we have

$$(\phi) . g ! (\phi ! \hat{z}, x) . \supset . g ! (\phi_2 \hat{z}, x). \qquad \text{(A)}$$

Let us begin with an important particular case. Put

$$g ! (\phi ! \hat{z}, x) . = . \phi ! a \supset \phi ! x.$$

Then $(\phi) . g ! (\phi ! \hat{z}, x) . = . x = a$, according to $*13 \cdot 1$.

We want to prove

$$(\phi) . \phi ! a \supset \phi ! x . \supset . \phi_2 a \supset \phi_2 x,$$

i.e.

$$(\phi) . \phi ! a \supset \phi ! x . \supset : (\phi) . f ! (\phi ! \hat{z}, a) . \supset . (\phi) . f ! (\phi ! \hat{z}, x) :$$

$$(\exists \phi) . f ! (\phi ! \hat{z}, a) . \supset . (\exists \phi) . f ! (\phi ! \hat{z}, x).$$

Now $f ! (\phi ! \hat{z}, x)$ must be derived from some stroke-function

$$F(p, q, r, \ldots)$$

by substituting for some of $p, q, r, \ldots$ the values $\phi ! x, \phi ! b, \phi ! c, \ldots$ where $b, c, \ldots$ are constants. As soon as ϕ is assigned, this is of the form $\psi ! x$. Hence

$$(\phi) . \phi ! a \supset \phi ! x . \supset : (\phi) : f ! (\phi ! \hat{z}, a) . \supset . f ! (\phi ! \hat{z}, x) :$$

$$\supset : (\phi) . f ! (\phi ! \hat{z}, a) . \supset . (\phi) . f ! (\phi ! \hat{z}, x) :$$

$$(\exists \phi) . f ! (\phi ! \hat{z}, a) . \supset . (\exists \phi) . f ! (\phi ! \hat{z}, x).$$

Thus generally $(\phi) . \phi ! a \supset \phi ! x . \supset . (\phi_2) . \phi_2 a \supset \phi_2 x$ without the need of any axiom of reducibility.

It must not, however, be assumed that (A) is always true. The procedure is as follows: $f ! (\phi ! \hat{z}, x)$ results from some stroke-function

$$F(p, q, r, \ldots)$$

by substituting for some of $p, q, r, \ldots$ the values $\phi ! x, \phi ! a, \phi ! b, \ldots$ ($a, b, \ldots$ being constants). We assume that, *e.g.*

$$\phi_2 x . = . (\phi) . f ! (\phi ! \hat{z}, x).$$

Thus

$$\phi_2 x . = . (\phi) . F(\phi ! x, \phi ! a, \phi ! b, \ldots). \qquad \text{(B)}$$

What we want to discover is whether

$$(\phi) . g ! (\phi ! \hat{z}, x) . \supset . g ! (\phi_2 \hat{z}, x).$$

Now $g ! (\phi ! \hat{z}, x)$ will be derived from a stroke-function

$$G(p, q, r, \ldots)$$

by substituting $\phi ! x, \phi ! a', \phi ! b', \ldots$ for some of $p, q, r, \ldots$. To obtain $g ! (\phi_2 \hat{z}, x)$, we have to put $\phi_2 x, \phi_2 a', \phi_2 b', \ldots$ in $G(p, q, r, \ldots)$, instead of $\phi ! x, \phi ! a', \phi ! b', \ldots$. We shall thus obtain a new matrix.

If $(\phi) . g ! (\phi ! \hat{z}, x)$ is known to be true because $G(p, q, r, \ldots)$ is always true, then $g ! (\phi_2 \hat{z}, x)$ is true in virtue of $*8$, because it is obtained from $G(p, q, r, \ldots)$ by substituting for some of $p, q, r, \ldots$ the propositions $\phi_2 x$, $\phi_2 a', \phi_2 b', \ldots$ which contain apparent variables. Thus in this case an inference is warranted.

We have thus the following important proposition:

Whenever $(\phi) . g ! (\phi ! \hat{z}, x)$ is known to be true because $g ! (\phi ! \hat{z}, x)$ is always a value of a stroke-function

$$G(p, q, r, \ldots),$$

which is true for all values of $p, q, r, \ldots$, then $g ! (\phi_2 \hat{z}, x)$ is also true, and so (of course) is $(\phi_2) . g ! (\phi_2 \hat{z}, x)$.

This, however, does not cover the case where $(\phi) . g ! (\phi ! \hat{z}, x)$ is not a truth of logic, but a hypothesis, which may be true for some values of x and false for others. When this is the case, the inference to $g ! (\phi_2 \hat{z}, x)$ is sometimes legitimate and sometimes not; the various cases must be investigated separately. We shall have an important illustration of the failure of the inference in connection with mathematical induction.

VI. CLASSES

The theory of classes is at once simplified in one direction and complicated in another by the assumption that functions only occur through their values and by the abandonment of the axiom of reducibility.

According to our present theory, all functions of functions are extensional, *i.e.*

$$\phi x \equiv_x \psi x . \supset . f(\phi \hat{z}) \equiv f(\psi \hat{z}).$$

This is obvious, since ϕ can only occur in $f(\phi \hat{z})$ by the substitution of values of ϕ for $p, q, r, \ldots$ in a stroke-function, and, if $\phi x \equiv \psi x$, the substitution of ϕx for p in a stroke-function gives the same truth-value to the truth-function as the substitution of ψx. Consequently there is no longer any reason to distinguish between functions and classes, for we have, in virtue of the above,

$$\phi x \equiv_x \psi x . \supset . \phi \hat{x} = \psi \hat{x}.$$

We shall continue to use the notation $\hat{x}(\phi x)$, which is often more convenient than $\phi \hat{x}$; but there will no longer be any difference between the meanings of the two symbols. Thus classes, as distinct from functions, lose even that shadowy being which they retain in $*20$. The same, of course, applies to relations in extension. This, so far, is a simplification.

On the other hand, we now have to distinguish classes of different orders composed of members of the same order. Taking classes of individuals as the simplest case, $\hat{x}(\phi ! x)$ must be distinguished from $\hat{x}(\phi_2 x)$ and so on. In virtue of the proposition at the end of the last section, the general logical properties of classes will be the same for classes of all orders. Thus *e.g.*

$$\alpha \subset \beta . \beta \subset \gamma . \supset . \alpha \subset \gamma$$

will hold whatever may be the orders of α, β, γ respectively. In other kinds of cases, however, trouble arises. Take, as a first instance, $p`\kappa$ and $s`\kappa$. We have

$$x \in p`\kappa . \equiv : \alpha \in \kappa . \supset_\alpha . x \in \alpha.$$

Thus $p`\kappa$ is a class of higher order than any of the members of $\kappa$. Hence the hypothesis $(\alpha) . f\alpha$ may not imply $f(p`\kappa)$, if α is of the order of the members of κ. There is a kind of proof invented by Zermelo, of which the simplest example is his second proof of the Schröder-Bernstein theorem (given in $*73$). This kind of proof consists in defining a certain class of classes κ, and then showing that $p`\kappa \in \kappa$. On the face of it, "$p`\kappa \in \kappa$" is impossible, since $p`\kappa$ is

not of the same order as members of κ. This, however, is not all that is to be said. A class of classes κ is always defined by some function of the form

$$(x_1, x_2, \ldots) : (\exists y_1, y_2, \ldots) . F(x_1 \epsilon \alpha, x_2 \epsilon \alpha, \ldots y_1 \epsilon \alpha, y_2 \epsilon \alpha, \ldots),$$

where F is a stroke-function, and "$\alpha \epsilon \kappa$" means that the above function is true. It may well happen that the above function is true when $p`\kappa$ is substituted for $\alpha$, and the result is interpreted by $*8$. Does this justify us in asserting $p`\kappa \epsilon \kappa$?

Let us take an illustration which is important in connection with mathematical induction. Put

$$\kappa = \hat{\alpha}(\breve{R}``\alpha \subset \alpha . a \epsilon \alpha).$$

Then $$\breve{R}``p`\kappa \subset p`\kappa . a \epsilon p`\kappa \quad (\text{see } *40 \cdot 81)$$

so that, in a sense, $p`\kappa \epsilon \kappa$. That is to say, if we substitute $p`\kappa$ for α in the defining function of κ, and apply $*8$, we obtain a true proposition. By the definition of $*90$,

$$\overleftarrow{R_*}`a = p`\kappa.$$

Thus $\overleftarrow{R_*}`a$ is a second-order class. Consequently, if we have a hypothesis $(\alpha) . f\alpha$, where α is a first-order class, we cannot assume

$$(\alpha) . f\alpha . \supset . f(\overleftarrow{R_*}`a). \qquad \text{(A)}$$

By the proposition at the end of the previous section, if $(\alpha) . f\alpha$ is deduced by logic from a universally-true stroke-function of elementary propositions, $f(\overleftarrow{R_*}`a)$ will also be true. Thus we may substitute $\overleftarrow{R_*}`a$ for α in any asserted proposition "$\vdash . f\alpha$" which occurs in *Principia Mathematica*. But when $(\alpha) . f\alpha$ is a hypothesis, not a universal truth, the implication (A) is not, *prima facie*, necessarily true.

For example, if $\kappa = \hat{\alpha}(\breve{R}``\alpha \subset \alpha . a \epsilon \alpha)$, we have

$$\alpha \epsilon \kappa . \supset : \alpha \cap \beta \epsilon \kappa . \equiv . \breve{R}``(\alpha \cap \beta) \subset \beta . a \epsilon \beta.$$

Hence $$\alpha \epsilon \kappa . \breve{R}``(\alpha \cap \beta) \subset \beta . a \epsilon \beta . \supset . p`\kappa \subset \beta \qquad (1)$$

In many of the propositions of $*90$, as hitherto proved, we substitute $p`\kappa$ for α, whence we obtain

$$\breve{R}``(\beta \cap p`\kappa) \subset \beta . a \epsilon \beta . \supset . p`\kappa \subset \beta \qquad (2)$$

i.e.

$$z \epsilon \beta . aR_*z . \supset_{z,w} . w \epsilon \beta : a \epsilon \beta . aR_*x : \supset . x \epsilon \beta$$

or $$aR_*x . \supset :. z \epsilon \beta . aR_*z . \supset_{z,w} . w \epsilon \beta : a \epsilon \beta : \supset . x \epsilon \beta.$$

This is a more powerful form of induction than that used in the definition of aR_*x. But the proof is not valid, because we have no right to substitute $p`\kappa$ for α in passing from (1) to (2). Therefore the proofs which use this form of induction have to be reconstructed.

It will be found that the form to which we can reduce most of the fallacious inferences that seem plausible is the following:

Given "$\vdash . (x) . f(x, x)$" we can infer "$\vdash : (x) : (\exists y) . f(x, y)$." Thus given "$\vdash . (\alpha) . f(\alpha, \alpha)$" we can infer "$\vdash : (\alpha) : (\exists \beta) . f(\alpha, \beta)$." But this depends upon the possibility of $\alpha = \beta$. If, now, α is of one order and β of another, we do not know that $\alpha = \beta$ is possible. Thus suppose we have

$$\alpha \epsilon \kappa . \supset_\alpha . g\alpha$$

and we wish to infer $g\beta$, where β is a class of higher order satisfying $\beta \epsilon \kappa$. The proposition

$$(\beta) :. \alpha \epsilon \kappa . \supset_\alpha . g\alpha : \supset : \beta \epsilon \kappa . \supset . g\beta$$

becomes, when developed by $*8$,

$$(\beta) :: (\exists \alpha) :. \alpha \epsilon \kappa . \supset . g\alpha : \supset : \beta \epsilon \kappa . \supset . g\beta.$$

This is only valid if $\alpha = \beta$ is possible. Hence the inference is fallacious if β is of higher order than α.

Let us apply these considerations to Zermelo's proof of the Schröder-Bernstein theorem, given in $*73{\cdot}8$ ff. We have a class of classes

$$\kappa = \hat{\alpha}(\alpha \subset \text{D}`R . \beta - \text{Cl}`R \subset \alpha . \breve{R}``\alpha \subset \alpha)$$

and we prove $p`\kappa \epsilon \kappa$ ($*73{\cdot}81$), which is admissible in the limited sense explained above. We then add the hypothesis

$$x \sim\epsilon (\beta - \text{Cl}`R) \cup \breve{R}``p`\kappa$$

and proceed to prove $p`\kappa - \iota`x \epsilon \kappa$ (in the fourth line of the proof of $*73{\cdot}82$). This also is admissible in the limited sense. But in the next line of the same proof we make a use of it which is not admissible, arguing from $p`\kappa - \iota`x \epsilon \kappa$ to $p`\kappa \subset p`\kappa - \iota`x$, because

$$\alpha \epsilon \kappa . \supset_\alpha . p`\kappa \subset \alpha.$$

The inference from

$$\alpha \epsilon \kappa . \supset_\alpha . p`\kappa \subset \alpha \quad \text{to} \quad p`\kappa - \iota`x \epsilon \kappa . \supset . p`\kappa \subset p`\kappa - \iota`x$$

is only valid if $p`\kappa - \iota`x$ is a class of the same order as the members of κ. For, when $\alpha \epsilon \kappa . \supset_\alpha . p`\kappa \subset \alpha$ is written out it becomes

$$(\alpha) ::: (\exists \beta) ::. (x) :: \alpha \epsilon \kappa . \supset :. \beta \epsilon \kappa . \supset . x \epsilon \beta : \supset . x \epsilon \alpha.$$

This is deduced from

$$\alpha \epsilon \kappa . \supset :. \alpha \epsilon \kappa . \supset . x \epsilon \alpha : \supset . x \epsilon \alpha$$

by the principle that $f(\alpha, \alpha)$ implies $(\exists \beta) . f(\alpha, \beta)$. But here the β must be of the same order as the α, while in our case α and β are not of the same order, if $\alpha = p`\kappa - \iota`x$ and β is an ordinary member of κ. At this point, therefore, where we infer $p`\kappa \subset p`\kappa - \iota`x$, the proof breaks down.

It is easy, however, to remedy this defect in the proof. All we need is

$$x \sim\epsilon (\beta - \text{Cl}`R) \cup \breve{R}``p`\kappa . \supset . x \sim\epsilon p`\kappa$$

or, conversely,

$$x \epsilon p`\kappa . \supset . x \epsilon (\beta - \text{Cl}`R) \cup \breve{R}``p`\kappa.$$

Now

$$\begin{aligned} x \in p'\kappa \,.\supset:.\, \alpha \in \kappa \,.\supset_\alpha:\, \alpha - \iota'x \sim\in \kappa :\\ \supset_\alpha : \sim(\beta - \text{Ɑ}'R \subset \alpha - \iota'x) \,.\vee.\, \sim\{\breve{R}''(\alpha - \iota'x) \subset \alpha - \iota'x\} :\\ \supset_\alpha : x \in \beta - \text{Ɑ}'R \,.\vee.\, x \in \breve{R}''(\alpha - \iota'x)\\ \supset:.\, x \in \beta - \text{Ɑ}'R :\vee:\, \alpha \in \kappa \,.\supset_\alpha.\, x \in \breve{R}''\alpha. \end{aligned}$$

Hence, by $*72{\cdot}341$,

$$x \in p'\kappa \,.\supset.\, x \in (\beta - \text{Ɑ}'R) \cup \breve{R}''p'\kappa$$

which gives the required result.

We assume that $\alpha - \iota'x$ is of no higher order than α; this can be secured by taking α to be of at least the second order, since $\iota'x$, and therefore $-\iota'x$, is of the second order. We may always assume our classes raised to a given order, but not raised indefinitely.

Thus the Schröder-Bernstein theorem survives.

Another difficulty arises in regard to sub-classes. We put

$$\text{Cl}'\alpha = \hat{\beta}\,(\beta \subset \alpha) \quad \text{Df.}$$

Now "$\beta \subset \alpha$" is significant when β is of higher order than α, provided its members are of the same type as those of α. But when we have

$$\beta \subset \alpha \,.\supset_\beta.\, f\beta,$$

the β must be of some definite type. As a rule, we shall be able to show that a proposition of this sort holds whatever the type of β, if we can show that it holds when β is of the same type as α. Consequently no difficulty arises until we come to Cantor's proposition $2^n > n$, which results from the proposition

$$\sim\{(\text{Cl}'\alpha) \text{ sm } \alpha\}$$

which is proved in $*102$. The proof is as follows:

$$\begin{aligned} R \in 1 \to 1 \,.\, \text{D}'R = \alpha \,.\, \text{Ɑ}'R \subset \text{Cl}'\alpha \,.\, \xi = \hat{x}\,\{x \in \alpha - \breve{R}'x\} \,.\supset:\\ y \in \alpha \,.\, y \in \breve{R}'y \,.\supset_y.\, y \sim\in \xi : y \in \alpha \,.\, y \sim\in \breve{R}'y \,.\supset_y.\, y \in \xi :\supset: y \in \alpha \,.\supset_y.\, \xi \neq \breve{R}'y :\\ \supset: \xi \sim\in \text{Ɑ}'R. \end{aligned}$$

As this proposition is crucial, we shall enter into it somewhat minutely.

Let $\alpha = \hat{x}\,(\text{A} \,!\, x)$, and let

$$xR\,\{\hat{z}\,(\phi \,!\, z)\} \,.=.\, f\,!\,(\phi \,!\, \hat{z}, x).$$

Then by our data,

$$\begin{aligned} &\text{A} \,!\, x \,.\supset.\, (\exists\phi) \,.\, f\,!\,(\phi \,!\, \hat{z}, x),\\ &f\,!\,(\phi \,!\, \hat{z}, x) \,.\supset.\, \text{A} \,!\, x \,.\, \phi \,!\, y \supset_y \text{A} \,!\, y,\\ &f\,!\,(\phi \,!\, \hat{z}, x) \,.\, f\,!\,(\phi \,!\, \hat{z}, y) \,.\supset.\, x = y,\\ &f\,!\,(\phi \,!\, \hat{z}, x) \,.\, f\,!\,(\psi \,!\, \hat{z}, x) \,.\supset.\, \phi \,!\, y \equiv_y \psi \,!\, y. \end{aligned}$$

With these data,

$$x \in \alpha - \breve{R}'x \,.\equiv:\, \text{A} \,!\, x : f\,!\,(\phi \,!\, \hat{z}, x) \,.\supset_\phi.\, \sim\phi \,!\, x.$$

Thus

$$\xi = \hat{x}\,\{(\phi) : \text{A} \,!\, x : f\,!\,(\phi \,!\, \hat{z}, x) \,.\supset.\, \sim\phi \,!\, x\}.$$

Thus ξ is defined by a function in which ϕ appears as apparent variable. If we enlarge the initial range of ϕ, we shall enlarge the range of values involved in the definition of ξ. There is therefore no way of escaping from the result that ξ is of higher order than the sub-classes of α contemplated in the definition of Cl'α. Consequently the proof of $2^n > n$ collapses when the axiom of reducibility is not assumed. We shall find, however, that the proposition remains true when n is finite.

With regard to relations, exactly similar questions arise as with regard to classes. A relation is no longer to be distinguished from a function of two variables, and we have

$$\phi(\hat{x}, \hat{y}) = \psi(\hat{x}, \hat{y}) \,.\equiv:\, \phi(x, y) \,.\equiv_{x,y}.\, \psi(x, y).$$

The difficulties as regards $\dot{p}$'λ and Rl'P are less important than those concerning p'κ and Cl'α, because $\dot{p}$'λ and Rl'P are less used. But a very serious difficulty occurs as regards similarity. We have

$$\alpha \operatorname{sm} \beta \,.\equiv.\, (\exists R) \,.\, R \in 1 \rightarrow 1 \,.\, \alpha = \mathrm{D}\text{'}R \,.\, \beta = \mathrm{Cl}\text{'}R.$$

Here R must be confined within some type; but whatever type we choose, there may be a correlator of higher type by which α and β can be correlated. Thus we can never prove $\sim(\alpha \operatorname{sm} \beta)$, except in such special cases as when either α or β is finite. This difficulty was illustrated by Cantor's theorem $2^n > n$, which we have just examined. Almost all our propositions are concerned in proving that two classes *are* similar, and these can all be interpreted so as to remain valid. But the few propositions which are concerned with proving that two classes are *not* similar collapse, except where one at least of the two is finite.

VII. MATHEMATICAL INDUCTION

All the propositions on mathematical induction in Part II, Section E and Part III, Section C remain valid, when suitably interpreted. But the proofs of many of them become fallacious when the axiom of reducibility is not assumed, and in some cases new proofs can only be obtained with considerable labour. The difficulty becomes at once apparent on observing the definition of "xR_*y" in $*90$. Omitting the factor "$x \in C\text{'}R$," which is irrelevant for our purposes, the definition of "xR_*y" may be written

$$zRw \,.\supset_{z,w}.\, \phi!z \supset \phi!w :\supset_\phi.\, \phi!x \supset \phi!y, \qquad \text{(A)}$$

i.e. "y has every elementary hereditary property possessed by x." We may, instead of elementary properties, take any other order of properties; as we shall see later, it is advantageous to take third-order properties when R is one-many or many-one, and fifth-order properties in other cases. But for preliminary purposes it makes no difference what order of properties we take, and therefore for the sake of definiteness we take elementary properties to begin with. The difficulty is that, if ϕ_2 is any second-order property, we cannot deduce from (A)

$$zRw \,.\supset_{z,w}.\, \phi_2 z \supset \phi_2 w :\supset.\, \phi_2 x \supset \phi_2 y. \qquad \text{(B)}$$

Suppose, for example, that $\phi_2 z . = . (\phi) . f ! (\phi ! \hat{z}, z)$; then from (A) we can deduce

$$zRw . \supset_{z,w} . f ! (\phi ! \hat{z}, z) \supset_\phi f ! (\phi ! \hat{z}, w) : \supset : f ! (\phi ! \hat{z}, x) . \supset_\phi . f ! (\phi ! \hat{z}, y) : \supset : \phi_2 x . \supset . \phi_2 y. \qquad \text{(C)}$$

But in general our hypothesis here is not implied by the hypothesis of (B). If we put $\phi_2 z . = . (\exists \phi) . f ! (\phi ! \hat{z}, z)$, we get exactly analogous results.

Hence in order to apply mathematical induction to a second-order property, it is not sufficient that it should be itself hereditary, but it must be composed of hereditary elementary properties. That is to say, if the property in question is $\phi_2 z$, where $\phi_2 z$ is either

$$(\phi) . f ! (\phi ! \hat{z}, z) \text{ or } (\exists \phi) . f ! (\phi ! \hat{z}, z),$$

it is not enough to have

$$zRw . \supset_{z,w} . \phi_2 z \supset \phi_2 w,$$

but we must have, for each elementary ϕ,

$$zRw . \supset_{z,w} . f ! (\phi ! \hat{z}, z) \supset f ! (\phi ! \hat{z}, w).$$

One inconvenient consequence is that, *primâ facie*, an inductive property must not be of the form

$$xR_* z . \phi ! z$$

or

$$S \in \text{Potid}`R . \phi ! S$$

or

$$\alpha \in \text{NC induct} . \phi ! \alpha.$$

This is inconvenient, because often such properties are hereditary when ϕ alone is not, *i.e.* we may have

$$xR_* z . \phi ! z . zRw . \supset_{z,w} . xR_* w . \phi ! w$$

when we do not have

$$\phi ! z . zRw . \supset_{z,w} . \phi ! w,$$

and similarly in the other cases.

These considerations make it necessary to re-examine all inductive proofs. In some cases they are still valid, in others they are easily rectified; in still others, the rectification is laborious, but it is always possible. The method of rectification is explained in Appendix B to this volume.

There is, however, so far as we can discover, no way by which our present primitive propositions can be made adequate to Dedekindian and well-ordered relations. The practical uses of Dedekindian relations depend upon *211·63—·692, which lead to *214·3—·34, showing that the series of segments of a series is Dedekindian. It is upon this that the theory of real numbers rests, real numbers being defined as segments of the series of rationals. This subject is dealt with in *310. If we were to regard as doubtful the proposition that the series of real numbers is Dedekindian, analysis would collapse.

The proofs of this proposition in *Principia Mathematica* depend upon the axiom of reducibility, since they depend upon *211·64, which asserts

$$\lambda \subset \text{D}`P_\epsilon . \supset . s`\lambda \in \text{D}`P_\epsilon.$$

For reasons explained above, if α is of the order of members of λ, $(\alpha) . f\alpha$ may not imply $f(s`\lambda)$, because $s`\lambda$ is a class of higher order than the members of λ. Thus although we have

$$\mathrm{D}`P_\epsilon = \hat{\alpha}\{(\exists\beta) . \alpha = P``\beta\},$$
$$s`\lambda = P``s`\breve{P}_\epsilon``\lambda,$$

yet we cannot infer $s`\lambda \, \epsilon \, \mathrm{D}`P_\epsilon$ except when $s`\lambda$ or $s`\breve{P}_\epsilon``\lambda$ is, for some special reason, of the same order as the members of λ. This will be the case when λ is finite, but not necessarily otherwise. Hence the theory of irrationals will require reconstruction.

Exactly similar difficulties arise in regard to well-ordered series. The theory of well-ordered series rests on the definition *250·01:

$$\mathrm{Bord} = \hat{P}(\mathrm{Cl\,ex}`C`P \subset \mathrm{G}`\mathrm{min}_P) \quad \mathrm{Df},$$

whence $P \, \epsilon \, \mathrm{Bord} . \equiv : \alpha \subset C`P . \exists ! \alpha . \supset_\alpha . \exists ! \alpha - \breve{P}``\alpha.$

In making deductions, we constantly substitute for α some constructed class of higher order than $C`P$. For instance, in *250·122 we substitute for $\alpha$ the class $C`P \cap p`\overleftarrow{P}``(\alpha \cap C`P)$, which is in general of higher order than $\alpha$. If this substitution is illegitimate, we cannot prove that a class contained in $C`P$ and having successors must have an immediate successor, without which the theory of well-ordered series becomes impossible. This particular difficulty might be overcome, but it is obvious that many important propositions must collapse.

It might be possible to sacrifice infinite well-ordered series to logical rigour, but the theory of real numbers is an integral part of ordinary mathematics, and can hardly be the object of a reasonable doubt. We are therefore justified in supposing that some logical axiom which is true will justify it. The axiom required may be more restricted than the axiom of reducibility, but, if so, it remains to be discovered.

The following are among the contributions to mathematical logic since the publication of the first edition of *Principia Mathematica.*

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P. Bernays. Ueber Hilbert's Gedanken zur Grundlegung der Arithmetik, *Jahresbericht der deutschen Mathematiker-Vereinigung*, Vol. 31.

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L. Chwistek. Ueber die Antinomien der Prinzipien der Mathematik, *Mathematische Zeitschrift*, Vol. 14. The Theory of Constructive Types. *Annales de la Société Mathématique de Pologne*, 1923. (Dr Chwistek has kindly allowed us to read in MS. a longer work with the same title.)

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A. Tajtelbaum-Tarski. Sur le terme primitif de la logistique, *Fundamenta Mathematicae*, Tom. IV. Sur les "truth-functions" au sens de MM. Russell et Whitehead, *ib.* Tom. V. Sur quelques théorèmes qui équivalent à l'axiome du choix, *ib.*

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INTRODUCTION

THE mathematical logic which occupies Part I of the present work has been constructed under the guidance of three different purposes. In the first place, it aims at effecting the greatest possible analysis of the ideas with which it deals and of the processes by which it conducts demonstrations, and at diminishing to the utmost the number of the undefined ideas and undemonstrated propositions (called respectively *primitive* ideas and *primitive* propositions) from which it starts. In the second place, it is framed with a view to the perfectly precise expression, in its symbols, of mathematical propositions: to secure such expression, and to secure it in the simplest and most convenient notation possible, is the chief motive in the choice of topics. In the third place, the system is specially framed to solve the paradoxes which, in recent years, have troubled students of symbolic logic and the theory of aggregates; it is believed that the theory of types, as set forth in what follows, leads both to the avoidance of contradictions, and to the detection of the precise fallacy which has given rise to them.

Of the above three purposes, the first and third often compel us to adopt methods, definitions, and notations which are more complicated or more difficult than they would be if we had the second object alone in view. This applies especially to the theory of descriptive expressions (∗14 and ∗30) and to the theory of classes and relations (∗20 and ∗21). On these two points, and to a lesser degree on others, it has been found necessary to make some sacrifice of lucidity to correctness. The sacrifice is, however, in the main only temporary: in each case, the notation ultimately adopted, though its real meaning is very complicated, has an apparently simple meaning which, except at certain crucial points, can without danger be substituted in thought for the real meaning. It is therefore convenient, in a preliminary explanation of the notation, to treat these apparently simple meanings as primitive ideas, *i.e.* as ideas introduced without definition. When the notation has grown more or less familiar, it is easier to follow the more complicated explanations which we believe to be more correct. In the body of the work, where it is necessary to adhere rigidly to the strict logical order, the easier order of development could not be adopted; it is therefore given in the Introduction. The explanations given in Chapter I of the Introduction are such as place lucidity before correctness; the full explanations are partly supplied in succeeding Chapters of the Introduction, partly given in the body of the work.

The use of a symbolism, other than that of words, in all parts of the book which aim at embodying strictly accurate demonstrative reasoning, has been

forced on us by the consistent pursuit of the above three purposes. The reasons for this extension of symbolism beyond the familiar regions of number and allied ideas are many:

(1). The ideas here employed are more abstract than those familiarly considered in language. Accordingly there are no words which are used mainly in the exact consistent senses which are required here. Any use of words would require unnatural limitations to their ordinary meanings, which would be in fact more difficult to remember consistently than are the definitions of entirely new symbols.

(2) The grammatical structure of language is adapted to a wide variety of usages. Thus it possesses no unique simplicity in representing the few simple, though highly abstract, processes and ideas arising in the deductive trains of reasoning employed here. In fact the very abstract simplicity of the ideas of this work defeats language. Language can represent complex ideas more easily. The proposition "a whale is big" represents language at its best, giving terse expression to a complicated fact; while the true analysis of "one is a number" leads, in language, to an intolerable prolixity. Accordingly terseness is gained by using a symbolism especially designed to represent the ideas and processes of deduction which occur in this work.

(3) The adaptation of the rules of the symbolism to the processes of deduction aids the intuition in regions too abstract for the imagination readily to present to the mind the true relation between the ideas employed. For various collocations of symbols become familiar as representing important collocations of ideas; and in turn the possible relations—according to the rules of the symbolism—between these collocations of symbols become familiar, and these further collocations represent still more complicated relations between the abstract ideas. And thus the mind is finally led to construct trains of reasoning in regions of thought in which the imagination would be entirely unable to sustain itself without symbolic help. Ordinary language yields no such help. Its grammatical structure does not represent uniquely the relations between the ideas involved. Thus, "a whale is big" and "one is a number" both look alike, so that the eye gives no help to the imagination.

(4) The terseness of the symbolism enables a whole proposition to be represented to the eyesight as one whole, or at most in two or three parts divided where the natural breaks, represented in the symbolism, occur. This is a humble property, but is in fact very important in connection with the advantages enumerated under the heading (3).

(5) The attainment of the first-mentioned object of this work, namely the complete enumeration of all the ideas and steps in reasoning employed

in mathematics, necessitates both terseness and the presentation of each proposition with the maximum of formality in a form as characteristic of itself as possible.

Further light on the methods and symbolism of this book is thrown by a slight consideration of the limits to their useful employment:

(α) Most mathematical investigation is concerned not with the analysis of the complete process of reasoning, but with the presentation of such an abstract of the proof as is sufficient to convince a properly instructed mind. For such investigations the detailed presentation of the steps in reasoning is of course unnecessary, provided that the detail is carried far enough to guard against error. In this connection it may be remembered that the investigations of Weierstrass and others of the same school have shown that, even in the common topics of mathematical thought, much more detail is necessary than previous generations of mathematicians had anticipated.

(β) In proportion as the imagination works easily in any region of thought, symbolism (except for the express purpose of analysis) becomes only necessary as a convenient shorthand writing to register results obtained without its help. It is a subsidiary object of this work to show that, with the aid of symbolism, deductive reasoning can be extended to regions of thought not usually supposed amenable to mathematical treatment. And until the ideas of such branches of knowledge have become more familiar, the detailed type of reasoning, which is also required for the analysis of the steps, is appropriate to the investigation of the general truths concerning these subjects.

CHAPTER I

PRELIMINARY EXPLANATIONS OF IDEAS AND NOTATIONS

THE notation adopted in the present work is based upon that of Peano, and the following explanations are to some extent modelled on those which he prefixes to his *Formulario Mathematico.* His use of dots as brackets is adopted, and so are many of his symbols.

Variables. The idea of a variable, as it occurs in the present work, is more general than that which is explicitly used in ordinary mathematics. In ordinary mathematics, a variable generally stands for an undetermined number or quantity. In mathematical logic, any symbol whose meaning is not determinate is called a *variable,* and the various determinations of which its meaning is susceptible are called the *values* of the variable. The values may be any set of entities, propositions, functions, classes or relations, according to circumstances. If a statement is made about "Mr A and Mr B," "Mr A" and "Mr B" are variables whose values are confined to men. A variable may either have a conventionally-assigned range of values, or may (in the absence of any indication of the range of values) have as the range of its values all determinations which render the statement in which it occurs significant. Thus when a text-book of logic asserts that "A is A," without any indication as to what A may be, what is meant is that *any* statement of the form "A is A" is true. We may call a variable *restricted* when its values are confined to some only of those of which it is capable; otherwise, we shall call it *unrestricted.* Thus when an unrestricted variable occurs, it represents any object such that the statement concerned can be made significantly (*i.e.* either truly or falsely) concerning that object. For the purposes of logic, the unrestricted variable is more convenient than the restricted variable, and we shall always employ it. We shall find that the unrestricted variable is still subject to limitations imposed by the manner of its occurrence, *i.e.* things which can be said significantly concerning a proposition cannot be said significantly concerning a class or a relation, and so on. But the limitations to which the unrestricted variable is subject do not need to be explicitly indicated, since they are the limits of significance of the statement in which the variable occurs, and are therefore intrinsically determined by this statement. This will be more fully explained later*.

To sum up, the three salient facts connected with the use of the variable are: (1) that a variable is ambiguous in its denotation and accordingly undefined; (2) that a variable preserves a recognizable identity in various occurrences throughout the same context, so that many variables can occur together in the

* Cf. Chapter II of the Introduction.

same context each with its separate identity; and (3) that either the range of possible determinations of two variables may be the same, so that a possible determination of one variable is also a possible determination of the other, or the ranges of two variables may be different, so that, if a possible determination of one variable is given to the other, the resulting complete phrase is meaningless instead of becoming a complete unambiguous proposition (true or false) as would be the case if all variables in it had been given any *suitable* determinations.

The uses of various letters. Variables will be denoted by single letters, and so will certain constants; but a letter which has once been assigned to a constant by a definition must not afterwards be used to denote a variable. The small letters of the ordinary alphabet will all be used for variables, except p and s after $*40$, in which constant meanings are assigned to these two letters. The following capital letters will receive constant meanings: B, C, D, E, F, I and J. Among small Greek letters, we shall give constant meanings to ϵ, ι and (at a later stage) to η, θ and ω. Certain Greek capitals will from time to time be introduced for constants, but Greek capitals will not be used for variables. Of the remaining letters, p, q, r will be called *propositional letters*, and will stand for variable propositions (except that, from $*40$ onwards, p must not be used for a variable); f, g, ϕ, ψ, χ, θ and (until $*33$) F will be called *functional letters*, and will be used for variable functions.

The small Greek letters not already mentioned will be used for variables whose values are classes, and will be referred to simply as *Greek letters*. Ordinary capital letters not already mentioned will be used for variables whose values are relations, and will be referred to simply as *capital letters*. Ordinary small letters other than p, q, r, s, f, g will be used for variables whose values are not known to be functions, classes, or relations; these letters will be referred to simply as *small Latin letters*.

After the early part of the work, variable propositions and variable functions will hardly ever occur. We shall then have three main kinds of variables: variable classes, denoted by small Greek letters; variable relations, denoted by capitals; and variables not given as necessarily classes or relations, which will be denoted by small Latin letters.

In addition to this usage of small Greek letters for variable classes, capital letters for variable relations, small Latin letters for variables of type wholly undetermined by the context (these arise from the possibility of "systematic ambiguity," explained later in the explanations of the theory of types), the reader need only remember that all letters represent variables, unless they have been defined as constants in some previous place in the book. In general the structure of the context determines the scope of the variables contained in it; but the special indication of the nature of the variables employed, as here proposed, saves considerable labour of thought.

The fundamental functions of propositions. An aggregation of propositions, considered as wholes not necessarily unambiguously determined, into a single proposition more complex than its constituents, is a function *with propositions as arguments.* The general idea of such an aggregation of propositions, or of variables representing propositions, will not be employed in this work. But there are four special cases which are of fundamental importance, since all the aggregations of subordinate propositions into one complex proposition which occur in the sequel are formed out of them step by step.

They are (1) the Contradictory Function, (2) the Logical Sum, or Disjunctive Function, (3) the Logical Product, or Conjunctive Function, (4) the Implicative Function. These functions in the sense in which they are required in this work are not all independent; and if two of them are taken as primitive undefined ideas, the other two can be defined in terms of them. It is to some extent—though not entirely—arbitrary as to which functions are taken as primitive. Simplicity of primitive ideas and symmetry of treatment seem to be gained by taking the first two functions as primitive ideas.

The Contradictory Function with argument p, where p is any proposition, is the proposition which is the contradictory of p, that is, the proposition asserting that p is not true. This is denoted by $\sim p$. Thus $\sim p$ is the contradictory function with p as argument and means the negation of the proposition p. It will also be referred to as the proposition not-p. Thus $\sim p$ means not-p, which means the negation of p.

The Logical Sum is a propositional function with two arguments p and q, and is the proposition asserting p or q disjunctively, that is, asserting that at least one of the two p and q is true. This is denoted by $p \vee q$. Thus $p \vee q$ is the logical sum with p and q as arguments. It is also called the logical sum of p and q. Accordingly $p \vee q$ means that at least p or q is true, not excluding the case in which both are true.

The Logical Product is a propositional function with two arguments p and q, and is the proposition asserting p and q conjunctively, that is, asserting that both p and q are true. This is denoted by $p \,.\, q$, or—in order to make the dots act as brackets in a way to be explained immediately—by $p : q$, or by $p :. q$, or by $p :: q$. Thus $p \,.\, q$ is the logical product with p and q as arguments. It is also called the logical product of p and q. Accordingly $p \,.\, q$ means that both p and q are true. It is easily seen that this function can be defined in terms of the two preceding functions. For when p and q are both true it must be false that either $\sim p$ or $\sim q$ is true. Hence in this book $p \,.\, q$ is merely a shortened form of symbolism for

$$\sim(\sim p \vee \sim q).$$

If any further idea attaches to the proposition "both p and q are true," it is not required here.

The Implicative Function is a propositional function with two arguments p and q, and is the proposition that either not-p or q is true, that is, it is the proposition $\sim p \vee q$. Thus if p is true, $\sim p$ is false, and accordingly the only alternative left by the proposition $\sim p \vee q$ is that q is true. In other words if p and $\sim p \vee q$ are both true, then q is true. In this sense the proposition $\sim p \vee q$ will be quoted as stating that p implies q. The idea contained in this propositional function is so important that it requires a symbolism which with direct simplicity represents the proposition as connecting p and q without the intervention of $\sim p$. But "implies" as used here expresses nothing else than the connection between p and q also expressed by the disjunction "not-p or q." The symbol employed for "p implies q," *i.e.* for "$\sim p \vee q$," is "$p \supset q$." This symbol may also be read "if p, then q." The association of implication with the use of an apparent variable produces an extension called "formal implication." This is explained later: it is an idea derivative from "implication" as here defined. When it is necessary explicitly to discriminate "implication" from "formal implication," it is called "material implication." Thus "material implication" is simply "implication" as here defined. The process of inference, which in common usage is often confused with implication, is explained immediately.

These four functions of propositions are the fundamental constant (*i.e.* definite) propositional functions with *propositions as arguments*, and all other constant propositional functions with propositions as arguments, so far as they are required in the present work, are formed out of them by successive steps. No *variable* propositional functions of this kind occur in this work.

Equivalence. The simplest example of the formation of a more complex function of propositions by the use of these four fundamental forms is furnished by "equivalence." Two propositions p and q are said to be "equivalent" when p implies q and q implies p. This relation between p and q is denoted by "$p \equiv q$." Thus "$p \equiv q$" stands for "$(p \supset q) . (q \supset p)$." It is easily seen that two propositions are equivalent when, and only when, they are both true or are both false. Equivalence rises in the scale of importance when we come to "formal implication" and thus to "formal equivalence." It must not be supposed that two propositions which are equivalent are in any sense identical or even remotely concerned with the same topic. Thus "Newton was a man" and "the sun is hot" are equivalent as being both true, and "Newton was not a man" and "the sun is cold" are equivalent as being both false. But here we have anticipated deductions which follow later from our formal reasoning. Equivalence in its origin is merely mutual implication as stated above.

Truth-values. The "truth-value" of a proposition is *truth* if it is true, and *falsehood* if it is false*. It will be observed that the truth-values of

* This phrase is due to Frege.

$p \vee q$, $p \,.\, q$, $p \supset q$, $\sim p$, $p \equiv q$ depend only upon those of p and q, namely the truth-value of "$p \vee q$" is truth if the truth-value of either p or q is truth, and is falsehood otherwise; that of "$p \,.\, q$" is truth if that of both p and q is truth, and is falsehood otherwise; that of "$p \supset q$" is truth if either that of p is falsehood or that of q is truth; that of "$\sim p$" is the opposite of that of p; and that of "$p \equiv q$" is truth if p and q have the same truth-value, and is falsehood otherwise. Now the only ways in which propositions will occur in the present work are ways derived from the above by combinations and repetitions. Hence it is easy to see (though it cannot be formally proved except in each particular case) that if a proposition p occurs in any proposition $f(p)$ which we shall ever have occasion to deal with, the truth-value of $f(p)$ will depend, not upon the particular proposition p, but only upon its truth-value; *i.e.* if $p \equiv q$, we shall have $f(p) \equiv f(q)$. Thus whenever two propositions are known to be equivalent, either may be substituted for the other in any formula with which we shall have occasion to deal.

We may call a function $f(p)$ a "truth-function" when its argument p is a proposition, and the truth-value of $f(p)$ depends only upon the truth-value of p. Such functions are by no means the only common functions of propositions. For example, "A believes p" is a function of p which will vary its truth-value for different arguments having the same truth-value: A may believe one true proposition without believing another, and may believe one false proposition without believing another. Such functions are not excluded from our consideration, and are included in the scope of any general propositions we may make about functions; but the particular functions of propositions which we shall have occasion to construct or to consider explicitly are all truth-functions. This fact is closely connected with a characteristic of mathematics, namely, that mathematics is always concerned with extensions rather than intensions. The connection, if not now obvious, will become more so when we have considered the theory of classes and relations.

Assertion-sign. The sign "$\vdash$," called the "assertion-sign," means that what follows is asserted. It is required for distinguishing a complete proposition, which we assert, from any subordinate propositions contained in it but not asserted. In ordinary written language a sentence contained between full stops denotes an asserted proposition, and if it is false the book is in error. The sign "$\vdash$" prefixed to a proposition serves this same purpose in our symbolism. For example, if "$\vdash (p \supset p)$" occurs, it is to be taken as a complete assertion convicting the authors of error unless the proposition "$p \supset p$" is true (as it is). Also a proposition stated in symbols without this sign "$\vdash$" prefixed is not asserted, and is merely put forward for consideration, or as a subordinate part of an asserted proposition.

Inference. The process of inference is as follows: a proposition "p" is asserted, and a proposition "p implies q" is asserted, and then as a sequel

the proposition "q" is asserted. The trust in inference is the belief that if the two former assertions are not in error, the final assertion is not in error. Accordingly whenever, in symbols, where p and q have of course special determinations,

$$\text{“}\vdash p\text{” and “}\vdash (p \supset q)\text{”}$$

have occurred, then "$\vdash q$" will occur if it is desired to put it on record. The process of the inference cannot be reduced to symbols. Its sole record is the occurrence of "$\vdash q$." It is of course convenient, even at the risk of repetition, to write "$\vdash p$" and "$\vdash (p \supset q)$" in close juxtaposition before proceeding to "$\vdash q$" as the result of an inference. When this is to be done, for the sake of drawing attention to the inference which is being made, we shall write instead

$$\text{“}\vdash p \supset \vdash q\text{,”}$$

which is to be considered as a mere abbreviation of the threefold statement

$$\text{“}\vdash p\text{” and “}\vdash (p \supset q)\text{” and “}\vdash q\text{.”}$$

Thus "$\vdash p \supset \vdash q$" may be read "$p$, therefore q," being in fact the same abbreviation, essentially, as this is; for "p, therefore q" does not explicitly state, what is part of its meaning, that p implies q. An inference is the dropping of a true premiss; it is the dissolution of an implication.

The use of dots. Dots on the line of the symbols have two uses, one to bracket off propositions, the other to indicate the logical product of two propositions. Dots immediately preceded or followed by "$\vee$" or "$\supset$" or "$\equiv$" or "$\vdash$," or by "(x)," "(x, y)," "(x, y, z)"... or "$(\exists x)$," "$(\exists x, y)$," "$(\exists x, y, z)$"... or "$[(\imath x)(\phi x)]$" or "$[R`y]$" or analogous expressions, serve to bracket off a proposition; dots occurring otherwise serve to mark a logical product. The general principle is that a larger number of dots indicates an outside bracket, a smaller number indicates an inside bracket. The exact rule as to the scope of the bracket indicated by dots is arrived at by dividing the occurrences of dots into three groups which we will name I, II, and III. Group I consists of dots adjoining a sign of implication ($\supset$) or of equivalence ($\equiv$) or of disjunction ($\vee$) or of equality by definition ($=$Df). Group II consists of dots following brackets indicative of an apparent variable, such as (x) or (x, y) or $(\exists x)$ or $(\exists x, y)$ or $[(\imath x)(\phi x)]$ or analogous expressions*. Group III consists of dots which stand between propositions in order to indicate a logical product. Group I is of greater force than Group II, and Group II than Group III. The scope of the bracket indicated by any collection of dots extends backwards or forwards beyond any *smaller* number of dots, or any *equal* number from a group of less force, until we reach either the end of the asserted proposition or a *greater* number of dots or an *equal* number belonging to a group of equal or superior force. Dots indicating a logical product have a scope which works both backwards and forwards; other dots only work away from the

* The meaning of these expressions will be explained later, and examples of the use of dots in connection with them will be given on pp. 16, 17.

adjacent sign of disjunction, implication, or equivalence, or forward from the adjacent symbol of one of the other kinds enumerated in Group II.

Some examples will serve to illustrate the use of dots.

"$p \vee q . \supset . q \vee p$" means the proposition "'p or q' implies 'q or p.'" When we *assert* this proposition, instead of merely considering it, we write

$$\text{“} \vdash : p \vee q . \supset . q \vee p, \text{”}$$

where the two dots after the assertion-sign show that what is asserted is the whole of what follows the assertion-sign, since there are not as many as two dots anywhere else. If we had written "$p : \vee : q . \supset . q \vee p$," that would mean the proposition "either p is true, or q implies 'q or p.'" If we wished to assert this, we should have to put three dots after the assertion-sign. If we had written "$p \vee q . \supset . q : \vee : p$," that would mean the proposition "either 'p or q' implies q, or p is true." The forms "$p . \vee . q . \supset . q \vee p$" and "$p \vee q . \supset . q . \vee . p$" have no meaning.

"$p \supset q . \supset : q \supset r . \supset . p \supset r$" will mean "if p implies q, then if q implies r, p implies r." If we wish to assert this (which is true) we write

$$\text{“} \vdash :. p \supset q . \supset : q \supset r . \supset . p \supset r. \text{”}$$

Again "$p \supset q . \supset . q \supset r : \supset . p \supset r$" will mean "if '$p$ implies q' implies 'q implies r,' then p implies r." This is in general untrue. (Observe that "$p \supset q$" is sometimes most conveniently read as "p implies q," and sometimes as "if p, then q.") "$p \supset q . q \supset r . \supset . p \supset r$" will mean "if p implies q, and q implies r, then p implies r." In this formula, the first dot indicates a logical product; hence the scope of the second dot extends backwards to the beginning of the proposition. "$p \supset q : q \supset r . \supset . p \supset r$" will mean "$p$ implies q; and if q implies r, then p implies r." (This is not true in general.) Here the two dots indicate a logical product; since two dots do not occur anywhere else, the scope of these two dots extends backwards to the beginning of the proposition, and forwards to the end.

"$p \vee q . \supset :. p . \vee . q \supset r : \supset . p \vee r$" will mean "if either p or q is true, then if either p or 'q implies r' is true, it follows that either p or r is true." If this is to be asserted, we must put four dots after the assertion-sign, thus:

$$\text{“} \vdash :: p \vee q . \supset :. p . \vee . q \supset r : \supset . p \vee r. \text{”}$$

(This proposition is proved in the body of the work; it is $*2{\cdot}75$.) If we wish to assert (what is equivalent to the above) the proposition: "if either p or q is true, and either p or 'q implies r' is true, then either p or r is true," we write

$$\text{“} \vdash :. p \vee q : p . \vee . q \supset r : \supset . p \vee r. \text{”}$$

Here the first pair of dots indicates a logical product, while the second pair does not. Thus the scope of the second pair of dots passes over the first pair, and back until we reach the three dots after the assertion-sign.

Other uses of dots follow the same principles, and will be explained as they are introduced. In reading a proposition, the dots should be noticed

first, as they show its structure. In a proposition containing several signs of implication or equivalence, the one with the greatest number of dots before or after it is the *principal* one: everything that goes before this one is stated by the proposition to imply or be equivalent to everything that comes after it.

Definitions. A definition is a declaration that a certain newly-introduced symbol or combination of symbols is to mean the same as a certain other combination of symbols of which the meaning is already known. Or, if the defining combination of symbols is one which only acquires meaning when combined in a suitable manner with other symbols*, what is meant is that any combination of symbols in which the newly-defined symbol or combination of symbols occurs is to have that meaning (if any) which results from substituting the defining combination of symbols for the newly-defined symbol or combination of symbols wherever the latter occurs. We will give the names of *definiendum* and *definiens* respectively to what is defined and to that which it is defined as meaning. We express a definition by putting the *definiendum* to the left and the *definiens* to the right, with the sign "=" between, and the letters "Df" to the right of the *definiens*. It is to be understood that the sign "=" and the letters "Df" are to be regarded as together forming one symbol. The sign "=" without the letters "Df" will have a different meaning, to be explained shortly.

An example of a definition is

$$p \supset q \,.=.\, \sim p \vee q \quad \text{Df.}$$

It is to be observed that a definition is, strictly speaking, no part of the subject in which it occurs. For a definition is concerned wholly with the symbols, not with what they symbolise. Moreover it is not true or false, being the expression of a volition, not of a proposition. (For this reason, definitions are not preceded by the assertion-sign.) Theoretically, it is unnecessary ever to give a definition: we might always use the *definiens* instead, and thus wholly dispense with the *definiendum*. Thus although we employ definitions and do not define "definition," yet "definition" does not appear among our primitive ideas, because the definitions are no part of our subject, but are, strictly speaking, mere typographical conveniences. Practically, of course, if we introduced no definitions, our formulae would very soon become so lengthy as to be unmanageable; but theoretically, all definitions are superfluous.

In spite of the fact that definitions are theoretically superfluous, it is nevertheless true that they often convey more important information than is contained in the propositions in which they are used. This arises from two causes. First, a definition usually implies that the *definiens* is worthy of careful consideration. Hence the collection of definitions embodies our choice

* This case will be fully considered in Chapter III of the Introduction. It need not further concern us at present.

of subjects and our judgment as to what is most important. Secondly, when what is defined is (as often occurs) something already familiar, such as cardinal or ordinal numbers, the definition contains an analysis of a common idea, and may therefore express a notable advance. Cantor's definition of the continuum illustrates this: his definition amounts to the statement that what he is defining is the object which has the properties commonly associated with the word "continuum," though what precisely constitutes these properties had not before been known. In such cases, a definition is a "making definite": it gives definiteness to an idea which had previously been more or less vague.

For these reasons, it will be found, in what follows, that the definitions are what is most important, and what most deserves the reader's prolonged attention.

Some important remarks must be made respecting the variables occurring in the *definiens* and the *definiendum*. But these will be deferred till the notion of an "apparent variable" has been introduced, when the subject can be considered as a whole.

Summary of preceding statements. There are, in the above, three primitive ideas which are not "defined" but only descriptively explained. Their primitiveness is only relative to our exposition of logical connection and is not absolute; though of course such an exposition gains in importance according to the simplicity of its primitive ideas. These ideas are symbolised by "$\sim p$" and "$p \vee q$," and by "$\vdash$" prefixed to a proposition.

Three definitions have been introduced:

$$p \,.\, q \,.\!=\!.\, \sim(\sim p \vee \sim q) \quad \mathrm{Df},$$
$$p \supset q \,.\!=\!.\, \sim p \vee q \quad \mathrm{Df},$$
$$p \equiv q \,.\!=\!.\, p \supset q \,.\, q \supset p \quad \mathrm{Df}.$$

Primitive propositions. Some propositions must be assumed without proof, since all inference proceeds from propositions previously asserted. These, as far as they concern the functions of propositions mentioned above, will be found stated in ∗1, where the formal and continuous exposition of the subject commences. Such propositions will be called "primitive propositions." These, like the primitive ideas, are to some extent a matter of arbitrary choice; though, as in the previous case, a logical system grows in importance according as the primitive propositions are few and simple. It will be found that owing to the weakness of the imagination in dealing with simple abstract ideas no very great stress can be laid upon their obviousness. They are obvious to the instructed mind, but then so are many propositions which cannot be quite true, as being disproved by their contradictory consequences. The proof of a logical system is its adequacy and its coherence. That is: (1) the system must embrace among its deductions all those propositions which we believe to be true and capable of deduction from logical premisses alone, though possibly they may

require some slight limitation in the form of an increased stringency of enunciation; and (2) the system must lead to no contradictions, namely in pursuing our inferences we must never be led to assert both p and not-p, *i.e.* both "$\vdash . p$" and "$\vdash . \sim p$" cannot legitimately appear.

The following are the primitive propositions employed in the calculus of propositions. The letters "Pp" stand for "primitive proposition."

(1) Anything implied by a true premiss is true Pp.

This is the rule which justifies inference.

(2) $\vdash : p \vee p . \supset . p$ Pp,

i.e. if p or p is true, then p is true.

(3) $\vdash : q . \supset . p \vee q$ Pp,

i.e. if q is true, then p or q is true.

(4) $\vdash : p \vee q . \supset . q \vee p$ Pp,

i.e. if p or q is true, then q or p is true.

(5) $\vdash : p \vee (q \vee r) . \supset . q \vee (p \vee r)$ Pp,

i.e. if either p is true or "q or r" is true, then either q is true or "p or r" is true.

(6) $\vdash :. q \supset r . \supset : p \vee q . \supset . p \vee r$ Pp,

i.e. if q implies r, then "p or q" implies "p or r."

(7) Besides the above primitive propositions, we require a primitive proposition called "the axiom of identification of real variables." When we have separately asserted two different functions of x, where x is undetermined, it is often important to know whether we can identify the x in one assertion with the x in the other. This will be the case—so our axiom allows us to infer—if both assertions present x as the argument to some one function, that is to say, if ϕx is a constituent in both assertions (whatever propositional function ϕ may be), or, more generally, if $\phi(x, y, z, \ldots)$ is a constituent in one assertion, and $\phi(x, u, v, \ldots)$ is a constituent in the other. This axiom introduces notions which have not yet been explained; for a fuller account, see the remarks accompanying ∗3·03, ∗1·7, ∗1·71, and ∗1·72 (which is the statement of this axiom) in the body of the work, as well as the explanation of propositional functions and ambiguous assertion to be given shortly.

Some simple propositions. In addition to the primitive propositions we have already mentioned, the following are among the most important of the elementary properties of propositions appearing among the deductions.

The law of excluded middle:

$$\vdash . p \vee \sim p.$$

This is ∗2·11 below. We shall indicate in brackets the numbers given to the following propositions in the body of the work.

The law of contradiction (∗3·24):

$$\vdash . \sim (p . \sim p).$$

The law of double negation (∗4·13):

$$\vdash . p \equiv \sim(\sim p).$$

The principle of *transposition*, *i.e.* "if p implies q, then not-q implies not-p," and vice versa: this principle has various forms, namely

$$(∗4·1)\quad \vdash : p \supset q . \equiv . \sim q \supset \sim p,$$

$$(∗4·11)\quad \vdash : p \equiv q . \equiv . \sim p \equiv \sim q,$$

$$(∗4·14)\quad \vdash :. p . q . \supset . r : \equiv : p . \sim r . \supset . \sim q,$$

as well as others which are variants of these.

The law of tautology, in the two forms:

$$(∗4·24)\quad \vdash : p . \equiv . p . p,$$

$$(∗4·25)\quad \vdash : p . \equiv . p \vee p,$$

i.e. "p is true" is equivalent to "p is true and p is true," as well as to "p is true or p is true." From a formal point of view, it is through the law of tautology and its consequences that the algebra of logic is chiefly distinguished from ordinary algebra.

The law of absorption:

$$(∗4·71)\quad \vdash :. p \supset q . \equiv : p . \equiv . p . q,$$

i.e. "p implies q" is equivalent to "p is equivalent to $p . q$." This is called the law of absorption because it shows that the factor q in the product is absorbed by the factor p, if p implies q. This principle enables us to replace an implication ($p \supset q$) by an equivalence ($p . \equiv . p . q$) whenever it is convenient to do so.

An analogous and very important principle is the following:

$$(∗4·73)\quad \vdash :. q . \supset : p . \equiv . p . q.$$

Logical addition and multiplication of propositions obey the associative and commutative laws, and the distributive law in two forms, namely

$$(∗4·4)\quad \vdash :. p . q \vee r . \equiv : p . q . \vee . p . r,$$

$$(∗4·41)\quad \vdash :. p . \vee . q . r : \equiv : p \vee q . p \vee r.$$

The second of these distinguishes the relations of logical addition and multiplication from those of arithmetical addition and multiplication.

Propositional functions. Let ϕx be a statement containing a variable x and such that it becomes a proposition when x is given any fixed determined meaning. Then ϕx is called a "propositional function"; it is not a proposition, since owing to the ambiguity of x it really makes no assertion at all. Thus "x is hurt" really makes no assertion at all, till we have settled who x is. Yet owing to the individuality retained by the ambiguous variable x, it is an ambiguous example from the collection of propositions arrived at by giving all possible determinations to x in "x is hurt" which yield a proposition, true or false. Also if "x is hurt" and "y is hurt" occur *in the same context*, where y is

another variable, then according to the determinations given to x and y, they can be settled to be (possibly) the same proposition or (possibly) different propositions. But apart from some determination given to x and y, they retain in that context their ambiguous differentiation. Thus "x is hurt" is an ambiguous "value" of a propositional function. When we wish to speak of the propositional function corresponding to "x is hurt," we shall write "$\hat{x}$ is hurt." Thus "$\hat{x}$ is hurt" is the propositional function and "x is hurt" is an ambiguous value of that function. Accordingly though "x is hurt" and "y is hurt" *occurring in the same context* can be distinguished, "$\hat{x}$ is hurt" and "$\hat{y}$ is hurt" convey no distinction of meaning at all. More generally, ϕx is an ambiguous value of the propositional function $\phi\hat{x}$, and when a definite signification a is substituted for x, ϕa is an unambiguous value of $\phi\hat{x}$.

Propositional functions are the fundamental kind from which the more usual kinds of function, such as "$\sin x$" or "$\log x$" or "the father of x," are derived. These derivative functions are considered later, and are called "descriptive functions." The functions of propositions considered above are a particular case of propositional functions.

The range of values and total variation. Thus corresponding to any propositional function $\phi\hat{x}$, there is a range, or collection, of values, consisting of all the propositions (true or false) which can be obtained by giving every possible determination to x in ϕx. A value of x for which ϕx is true will be said to "satisfy" $\phi\hat{x}$. Now in respect to the truth or falsehood of propositions of this range three important cases must be noted and symbolised. These cases are given by three propositions of which one at least must be true. Either (1) all propositions of the range are true, or (2) some propositions of the range are true, or (3) no proposition of the range is true. The statement (1) is symbolised by "$(x) . \phi x$," and (2) is symbolised by "$(\exists x) . \phi x$." No definition is given of these two symbols, which accordingly embody two new primitive ideas in our system. The symbol "$(x) . \phi x$" may be read "ϕx always," or "ϕx is always true," or "ϕx is true for all possible values of x." The symbol "$(\exists x) . \phi x$" may be read "there exists an x for which ϕx is true," or "there exists an x satisfying $\phi\hat{x}$," and thus conforms to the natural form of the expression of thought.

Proposition (3) can be expressed in terms of the fundamental ideas now on hand. In order to do this, note that "$\sim \phi x$" stands for the contradictory of ϕx. Accordingly $\sim \phi\hat{x}$ is another propositional function such that each value of $\phi\hat{x}$ contradicts a value of $\sim \phi\hat{x}$, and vice versa. Hence "$(x) . \sim \phi x$" symbolises the proposition that every value of $\phi\hat{x}$ is untrue. This is number (3) as stated above.

It is an obvious error, though one easy to commit, to assume that cases (1) and (3) are each other's contradictories. The symbolism exposes this fallacy at once, for (1) is $(x) . \phi x$, and (3) is $(x) . \sim \phi x$, while the contradictory of (1) is $\sim \{(x) . \phi x\}$. For the sake of brevity of symbolism a definition is made, namely

$$\sim (x) . \phi x . = . \sim \{(x) . \phi x\} \quad \text{Df.}$$

Definitions of which the object is to gain some trivial advantage in brevity by a slight adjustment of symbols will be said to be of "merely symbolic import," in contradistinction to those definitions which invite consideration of an important idea.

The proposition $(x) \centerdot \phi x$ is called the "total variation" of the function $\phi \hat{x}$.

For reasons which will be explained in Chapter II, we do not take negation as a primitive idea when propositions of the forms $(x) \centerdot \phi x$ and $(\exists x) \centerdot \phi x$ are concerned, but we *define* the negation of $(x) \centerdot \phi x$, *i.e.* of "ϕx is always true," as being "ϕx is sometimes false," *i.e.* "$(\exists x) \centerdot \sim \phi x$," and similarly we *define* the negation of $(\exists x) \centerdot \phi x$ as being $(x) \centerdot \sim \phi x$. Thus we put

$$\sim \{(x) \centerdot \phi x\} \centerdot = \centerdot (\exists x) \centerdot \sim \phi x \quad \text{Df},$$

$$\sim \{(\exists x) \centerdot \phi x\} \centerdot = \centerdot (x) \centerdot \sim \phi x \quad \text{Df}.$$

In like manner we define a disjunction in which one of the propositions is of the form "$(x) \centerdot \phi x$" or "$(\exists x) \centerdot \phi x$" in terms of a disjunction of propositions not of this form, putting

$$(x) \centerdot \phi x \centerdot \vee \centerdot p : = \centerdot (x) \centerdot \phi x \vee p \quad \text{Df},$$

i.e. "either ϕx is always true, or p is true" is to mean "'ϕx or p' is always true," with similar definitions in other cases. This subject is resumed in Chapter II, and in $*9$ in the body of the work.

Apparent variables. The symbol "$(x) \centerdot \phi x$" denotes one definite proposition, and there is no distinction in meaning between "$(x) \centerdot \phi x$" and "$(y) \centerdot \phi y$" when they occur in the same context. Thus the "x" in "$(x) \centerdot \phi x$" is not an ambiguous constituent of any expression in which "$(x) \centerdot \phi x$" occurs; and such an expression does not cease to convey a determinate meaning by reason of the ambiguity of the x in the "ϕx." The symbol "$(x) \centerdot \phi x$" has some analogy to the symbol

$$\text{“}\int_a^b \phi(x)\, dx\text{”}$$

for definite integration, since in neither case is the expression a function of x.

The range of x in "$(x) \centerdot \phi x$" or "$(\exists x) \centerdot \phi x$" extends over the complete field of the values of x for which "ϕx" has meaning, and accordingly the meaning of "$(x) \centerdot \phi x$" or "$(\exists x) \centerdot \phi x$" involves the supposition that such a field is determinate. The x which occurs in "$(x) \centerdot \phi x$" or "$(\exists x) \centerdot \phi x$" is called (following Peano) an "apparent variable." It follows from the meaning of "$(\exists x) \centerdot \phi x$" that the x in this expression is also an apparent variable. A proposition in which x occurs as an apparent variable is not a function of x. Thus *e.g.* "$(x) \centerdot x = x$" will mean "everything is equal to itself." This is an absolute constant, not a function of a variable x. This is why the x is called an *apparent* variable in such cases.

Besides the "*range*" of x in "$(x) \centerdot \phi x$" or "$(\exists x) \centerdot \phi x$," which is the field of the values that x may have, we shall speak of the "*scope*" of x, meaning

the function of which all values or some value are being affirmed. If we are asserting all values (or some value) of "ϕx," "ϕx" is the scope of x; if we are asserting all values (or some value) of "$\phi x \supset p$," "$\phi x \supset p$" is the scope of x; if we are asserting all values (or some value) of "$\phi x \supset \psi x$," "$\phi x \supset \psi x$" will be the scope of x, and so on. The scope of x is indicated by the number of dots after the "(x)" or "$(\exists x)$"; that is to say, the scope extends forwards until we reach an equal number of dots not indicating a logical product, or a greater number indicating a logical product, or the end of the asserted proposition in which the "(x)" or "$(\exists x)$" occurs, whichever of these happens first*. Thus *e.g.*

$$``(x) : \phi x \,.\supset.\, \psi x"$$

will mean "ϕx always implies ψx," but

$$``(x) \,.\, \phi x \,.\supset.\, \psi x"$$

will mean "if ϕx is always true, then ψx is true for the argument x."

Note that in the proposition

$$(x) \,.\, \phi x \,.\supset.\, \psi x$$

the two x's have no connection with each other. Since only one dot follows the x in brackets, the scope of the first x is limited to the "ϕx" immediately following the x in brackets. It usually conduces to clearness to write

$$(x) \,.\, \phi x \,.\supset.\, \psi y$$

rather than $(x) \,.\, \phi x \,.\supset.\, \psi x$,

since the use of different letters emphasises the absence of connection between the two variables; but there is no logical necessity to use different letters, and it is *sometimes* convenient to use the same letter.

Ambiguous assertion and the real variable. Any value "ϕx" of the function $\phi \hat{x}$ can be asserted. Such an assertion of an ambiguous member of the values of $\phi \hat{x}$ is symbolised by

$$``\vdash . \phi x."$$

Ambiguous assertion of this kind is a primitive idea, which cannot be defined in terms of the assertion of propositions. This primitive idea is the one which embodies the use of the variable. Apart from ambiguous assertion, the consideration of "ϕx," which is an ambiguous member of the values of $\phi \hat{x}$, would be of little consequence. When we are considering or asserting "ϕx," the variable x is called a "real variable." Take, for example, the law of excluded middle in the form which it has in traditional formal logic:

"a is either b or not b."

Here a and b are real variables: as they vary, different propositions are expressed, though all of them are true. While a and b are undetermined, as in the above enunciation, no one definite proposition is asserted, but what is asserted is *any* value of the propositional function in question. This can only

* This agrees with the rules for the occurrences of dots of the type of Group II as explained above, pp. 9 and 10.

be legitimately asserted if, whatever value may be chosen, that value is true, *i.e.* if all the values are true. Thus the above form of the law of excluded middle is equivalent to

$$\text{"}(a, b) \,.\, a \text{ is either } b \text{ or not } b,\text{"}$$

i.e. to "it is always true that a is either b or not b." But these two, though equivalent, are not identical, and we shall find it necessary to keep them distinguished.

When we assert something containing a real variable, as in *e.g.*

$$\text{"}\vdash . \, x = x,\text{"}$$

we are asserting *any* value of a propositional function. When we assert something containing an apparent variable, as in

$$\text{"}\vdash . \, (x) \,.\, x = x\text{"}$$

or

$$\text{"}\vdash . \, (\exists x) \,.\, x = x,\text{"}$$

we are asserting, in the first case *all* values, in the second case *some* value (undetermined), of the propositional function in question. It is plain that we can only legitimately assert "*any* value" if *all* values are true; for otherwise, since the value of the variable remains to be determined, it might be so determined as to give a false proposition. Thus in the above instance, since we have

$$\vdash . \, x = x$$

we may infer

$$\vdash . \, (x) \,.\, x = x.$$

And generally, given an assertion containing a real variable x, we may transform the real variable into an apparent one by placing the x in brackets at the beginning, followed by as many dots as there are after the assertion-sign.

When we assert something containing a real variable, we cannot strictly be said to be asserting a *proposition*, for we only obtain a definite proposition by assigning a value to the variable, and then our assertion only applies to one definite case, so that it has not at all the same force as before. When what we assert contains a real variable, we are asserting a wholly undetermined one of all the propositions that result from giving various values to the variable. It will be convenient to speak of such assertions as *asserting a propositional function*. The ordinary formulae of mathematics contain such assertions; for example

$$\text{"}\sin^2 x + \cos^2 x = 1\text{"}$$

does not assert this or that particular case of the formula, nor does it assert that the formula holds for *all* possible values of x, though it is equivalent to this latter assertion; it simply asserts that the formula holds, leaving x wholly undetermined; and it is able to do this legitimately, because, however x may be determined, a true proposition results.

Although an assertion containing a real variable does not, in strictness, assert a proposition, yet it will be spoken of as asserting a proposition except when the nature of the ambiguous assertion involved is under discussion.

Definition and real variables. When the *definiens* contains one or more real variables, the *definiendum* must also contain them. For in this case we have a function of the real variables, and the *definiendum* must have the same meaning as the *definiens* for all values of these variables, which requires that the symbol which is the *definiendum* should contain the letters representing the real variables. This rule is not always observed by mathematicians, and its infringement has sometimes caused important confusions of thought, notably in geometry and the philosophy of space.

In the definitions given above of "$p \,.\, q$" and "$p \supset q$" and "$p \equiv q$," p and q are real variables, and therefore appear on both sides of the definition. In the definition of "$\sim \{(x) \,.\, \phi x\}$" only the function considered, namely $\phi \hat{z}$, is a real variable; thus so far as concerns the rule in question, x need not appear on the left. But when a real variable is a function, it is necessary to indicate how the argument is to be supplied, and therefore there are objections to omitting an apparent variable where (as in the case before us) this is the argument to the function which is the real variable. This appears more plainly if, instead of a general function $\phi \hat{x}$, we take some particular function, say "$\hat{x} = a$," and consider the definition of $\sim \{(x) \,.\, x = a\}$. Our definition gives

$$\sim \{(x) \,.\, x = a\} \,.\!=\!.\, (\exists x) \,.\, \sim (x = a) \quad \text{Df.}$$

But if we had adopted a notation in which the ambiguous value "$x = a$," containing the apparent variable x, did not occur in the *definiendum*, we should have had to construct a notation employing the function itself, namely "$\hat{x} = a$." This does not involve an apparent variable, but would be clumsy in practice. In fact we have found it convenient and possible—except in the explanatory portions—to keep the explicit use of symbols of the type "$\phi \hat{x}$," either as constants [*e.g.* $\hat{x} = a$] or as real variables, almost entirely out of this work.

Propositions connecting real and apparent variables. The most important propositions connecting real and apparent variables are the following:

(1) "When a propositional function can be asserted, so can the proposition that all values of the function are true." More briefly, if less exactly, "what holds of any, however chosen, holds of all." This translates itself into the rule that when a real variable occurs in an assertion, we may turn it into an apparent variable by putting the letter representing it in brackets immediately after the assertion-sign.

(2) "What holds of all, holds of any," *i.e.*

$$\vdash : (x) \,.\, \phi x \,.\, \supset .\, \phi y.$$

This states "if ϕx is always true, then ϕy is true."

(3) "If ϕy is true, then ϕx is sometimes true," *i.e.*

$$\vdash : \phi y \,.\, \supset .\, (\exists x) \,.\, \phi x.$$

An asserted proposition of the form "$(\exists x) . \phi x$" expresses an "existence-theorem," namely "there exists an x for which ϕx is true." The above proposition gives what is in practice the only way of proving existence-theorems: we always have to find some particular y for which ϕy holds, and thence to infer "$(\exists x) . \phi x$." If we were to assume what is called the multiplicative axiom, or the equivalent axiom enunciated by Zermelo, that would, in an important class of cases, give an existence-theorem where no particular instance of its truth can be found.

In virtue of "$\vdash : (x) . \phi x . \supset . \phi y$" and "$\vdash : \phi y . \supset . (\exists x) . \phi x$," we have "$\vdash : (x) . \phi x . \supset . (\exists x) . \phi x$," *i.e.* "what is always true is sometimes true." This would not be the case if nothing existed; thus our assumptions contain the assumption that there is something. This is involved in the principle that what holds of all, holds of any; for this would not be true if there were no "any."

(4) "If ϕx is always true, and ψx is always true, then '$\phi x . \psi x$' is always true," *i.e.*

$$\vdash :. (x) . \phi x : (x) . \psi x : \supset . (x) . \phi x . \psi x.$$

(This requires that ϕ and ψ should be functions which take arguments of the same *type*. We shall explain this requirement at a later stage.) The converse also holds; *i.e.* we have

$$\vdash :. (x) . \phi x . \psi x . \supset : (x) . \phi x : (x) . \psi x.$$

It is to some extent optional which of the propositions connecting real and apparent variables are taken as primitive propositions. The primitive propositions assumed, on this subject, in the body of the work (*9), are the following:

(1) $$\vdash : \phi x . \supset . (\exists z) . \phi z.$$

(2) $$\vdash : \phi x \vee \phi y . \supset . (\exists z) . \phi z,$$

i.e. if either ϕx is true, or ϕy is true, then $(\exists z) . \phi z$ is true. (On the necessity for this primitive proposition, see remarks on *9·11 in the body of the work.)

(3) If we can assert ϕy, where y is a real variable, then we can assert $(x) . \phi x$; *i.e.* what holds of any, however chosen, holds of all.

Formal implication and formal equivalence. When an implication, say $\phi x . \supset . \psi x$, is said to hold always, *i.e.* when $(x) : \phi x . \supset . \psi x$, we shall say that ϕx *formally implies* ψx; and propositions of the form "$(x) : \phi x . \supset . \psi x$" will be said to state *formal implications*. In the usual instances of implication, such as "'Socrates is a man' implies 'Socrates is mortal,'" we have a proposition of the form "$\phi x . \supset . \psi x$" in a case in which "$(x) : \phi x . \supset . \psi x$" is true. In such a case, we feel the implication as a particular case of a formal implication. Thus it has come about that implications which are not particular cases of formal implications have not been regarded as implications at all. There is also a practical ground for the neglect of such implications, for, speaking

generally, they can only be *known* when it is already known either that their hypothesis is false or that their conclusion is true; and in neither of these cases do they serve to make us know the conclusion, since in the first case the conclusion need not be true, and in the second it is known already. Thus such implications do not serve the purpose for which implications are chiefly useful, namely that of making us know, by deduction, conclusions of which we were previously ignorant. *Formal* implications, on the contrary, do serve this purpose, owing to the psychological fact that we often know "$(x):\phi x.\supset.\psi x$" and ϕy, in cases where ψy (which follows from these premisses) cannot easily be known directly.

These reasons, though they do not warrant the complete neglect of implications that are not instances of formal implications, are reasons which make formal implication very important. A formal implication states that, for all possible values of x, if the hypothesis ϕx is true, the conclusion ψx is true. Since "$\phi x.\supset.\psi x$" will always be true when ϕx is false, it is only the values of x that make ϕx true that are *important* in a formal implication; what is effectively stated is that, for all these values, ψx is true. Thus propositions of the form "all α is β," "no α is β" state formal implications, since the first (as appears by what has just been said) states

$$(x):x \text{ is an } \alpha.\supset.x \text{ is a } \beta,$$

while the second states

$$(x):x \text{ is an } \alpha.\supset.x \text{ is not a } \beta.$$

And any formal implication "$(x):\phi x.\supset.\psi x$" may be interpreted as: "All values of x which satisfy* ϕx satisfy ψx," while the formal implication "$(x):\phi x.\supset.\sim\psi x$" may be interpreted as: "No values of x which satisfy ϕx satisfy ψx."

We have similarly for "some α is β" the formula

$$(\exists x).x \text{ is an } \alpha.x \text{ is a } \beta,$$

and for "some α is not β" the formula

$$(\exists x).x \text{ is an } \alpha.x \text{ is not a } \beta.$$

Two functions ϕx, ψx are called *formally equivalent* when each always implies the other, *i.e.* when

$$(x):\phi x.\equiv.\psi x,$$

and a proposition of this form is called a *formal equivalence*. In virtue of what was said about truth-values, if ϕx and ψx are formally equivalent, either may replace the other in any truth-function. Hence for all the purposes of mathematics or of the present work, $\phi\hat{z}$ may replace $\psi\hat{z}$ or vice versa in any proposition with which we shall be concerned. Now to say that ϕx and ψx are formally equivalent is the same thing as to say that $\phi\hat{z}$ and $\psi\hat{z}$ have the same *extension*, *i.e.* that any value of x which satisfies either satisfies the other.

* A value of x is said to *satisfy* ϕx or $\phi\hat{x}$ when ϕx is true for that value of x.

Thus whenever a constant function occurs in our work, the truth-value of the proposition in which it occurs depends only upon the extension of the function. A proposition containing a function $\phi\hat{z}$ and having this property (*i.e.* that its truth-value depends only upon the extension of $\phi\hat{z}$) will be called an *extensional* function of $\phi\hat{z}$. Thus the functions of functions with which we shall be specially concerned will all be extensional functions of functions.

What has just been said explains the connection (noted above) between the fact that the functions of propositions with which mathematics is specially concerned are all truth-functions and the fact that mathematics is concerned with extensions rather than intensions.

Convenient abbreviation. The following definitions give alternative and often more convenient notations:

$$\phi x \,.\, \supset_x .\, \psi x : = : (x) : \phi x \,.\, \supset .\, \psi x \quad \text{Df},$$
$$\phi x \,.\, \equiv_x .\, \psi x : = : (x) : \phi x \,.\, \equiv .\, \psi x \quad \text{Df}.$$

This notation "$\phi x \,.\, \supset_x .\, \psi x$" is due to Peano, who, however, has no notation for the general idea "$(x) \,.\, \phi x$." It may be noticed as an exercise in the use of dots as brackets that we might have written

$$\phi x \supset_x \psi x \,.\, = .\, (x) \,.\, \phi x \supset \psi x \quad \text{Df},$$
$$\phi x \equiv_x \psi x \,.\, = .\, (x) \,.\, \phi x \equiv \psi x \,.\, \text{Df}.$$

In practice however, when $\phi\hat{x}$ and $\psi\hat{x}$ are special functions, it is not possible to employ fewer dots than in the first form, and often more are required.

The following definitions give abbreviated notations for functions of two or more variables:

$$(x, y) \,.\, \phi(x, y) \,.\, = : (x) : (y) \,.\, \phi(x, y) \quad \text{Df},$$

and so on for any number of variables;

$$\phi(x, y) \,.\, \supset_{x,y} .\, \psi(x, y) : = : (x, y) : \phi(x, y) \,.\, \supset .\, \psi(x, y) \quad \text{Df},$$

and so on for any number of variables.

Identity. The propositional function "x is identical with y" is expressed by

$$x = y.$$

This will be defined (cf. ∗13·01), but, owing to certain difficult points involved in the definition, we shall here omit it (cf. Chapter II). We have, of course,

$$\vdash .\, x = x \quad \text{(the law of identity)},$$
$$\vdash : x = y \,.\, \equiv .\, y = x,$$
$$\vdash : x = y \,.\, y = z \,.\, \supset .\, x = z.$$

The first of these expresses the *reflexive* property of identity: a relation is called *reflexive* when it holds between a term and itself, either universally, or whenever it holds between that term and some term. The second of the above propositions expresses that identity is a *symmetrical* relation: a relation is called *symmetrical* if, whenever it holds between x and y, it also holds

between y and x. The third proposition expresses that identity is a *transitive* relation: a relation is called *transitive* if, whenever it holds between x and y and between y and z, it holds also between x and z.

We shall find that no new definition of the sign of equality is required in mathematics: all mathematical equations in which the sign of equality is used in the ordinary way express some identity, and thus use the sign of equality in the above sense.

If x and y are identical, either can replace the other in any proposition without altering the truth-value of the proposition; thus we have

$$\vdash : x = y \,.\, \supset \,.\, \phi x \equiv \phi y.$$

This is a fundamental property of identity, from which the remaining properties mostly follow.

It might be thought that identity would not have much importance, since it can only hold between x and y if x and y are different symbols for the same object. This view, however, does not apply to what we shall call "descriptive phrases," *i.e.* "the so-and-so." It is in regard to such phrases that identity is important, as we shall shortly explain. A proposition such as "Scott was the author of Waverley" expresses an identity in which there is a descriptive phrase (namely "the author of Waverley"); this illustrates how, in such cases, the assertion of identity may be important. It is essentially the same case when the newspapers say "the identity of the criminal has not transpired." In such a case, the criminal is known by a descriptive phrase, namely "the man who did the deed," and we wish to find an x of whom it is true that "x=the man who did the deed." When such an x has been found, the identity of the criminal has transpired.

Classes and relations. A *class* (which is the same as a *manifold* or *aggregate*) is all the objects satisfying some propositional function. If α is the class composed of the objects satisfying $\phi\hat{x}$, we shall say that α is the class *determined* by $\phi\hat{x}$. Every propositional function thus determines a class, though if the propositional function is one which is always false, the class will be *null*, *i.e.* will have no members. The class determined by the function $\phi\hat{x}$ will be represented by $\hat{z}(\phi z)$*. Thus for example if ϕx is an equation, $\hat{z}(\phi z)$ will be the class of its roots; if ϕx is "x has two legs and no feathers," $\hat{z}(\phi z)$ will be the class of men; if ϕx is "$0 < x < 1$," $\hat{z}(\phi z)$ will be the class of proper fractions, and so on.

It is obvious that the same class of objects will have many determining functions. When it is not necessary to specify a determining function of a class, the class may be conveniently represented by a single Greek letter. Thus Greek letters, other than those to which some constant meaning is assigned, will be exclusively used for classes.

* Any other letter may be used instead of z.

There are two kinds of difficulties which arise in formal logic; one kind arises in connection with classes and relations and the other in connection with descriptive functions. The point of the difficulty for classes and relations, so far as it concerns classes, is that a class cannot be an object suitable as an argument to any of its determining functions. If α represents a class and $\phi\hat{x}$ one of its determining functions [so that $\alpha = \hat{z}(\phi z)$], it is not sufficient that $\phi\alpha$ be a false proposition, it must be nonsense. Thus a certain classification of what appear to be objects into things of essentially different types seems to be rendered necessary. This whole question is discussed in Chapter II, on the theory of types, and the formal treatment in the systematic exposition, which forms the main body of this work, is guided by this discussion. The part of the systematic exposition which is specially concerned with the theory of classes is $*20$, and in this Introduction it is discussed in Chapter III. It is sufficient to note here that, in the complete treatment of $*20$, we have avoided the decision as to whether a class of things has in any sense an existence as one object. A decision of this question in either way is indifferent to our logic, though perhaps, if we had regarded some solution which held classes and relations to be in some real sense objects as both true and likely to be universally received, we might have simplified one or two definitions and a few preliminary propositions. Our symbols, such as "$\hat{x}(\phi x)$" and α and others, which represent classes and relations, are merely defined in their use, just as ∇^2, standing for

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2},$$

has no meaning apart from a suitable function of x, y, z on which to operate. The result of our definitions is that the way in which we use classes corresponds in general to their use in ordinary thought and speech; and whatever may be the ultimate interpretation of the one is also the interpretation of the other. Thus in fact our classification of types in Chapter II really performs the single, though essential, service of justifying us in refraining from entering on trains of reasoning which lead to contradictory conclusions. The justification is that what seem to be propositions are really nonsense.

The definitions which occur in the theory of classes, by which the idea of a class (at least in use) is based on the other ideas assumed as primitive, cannot be understood without a fuller discussion than can be given now (cf. Chapter II of this Introduction and also $*20$). Accordingly, in this preliminary survey, we proceed to state the more important simple propositions which result from those definitions, leaving the reader to employ in his mind the ordinary unanalysed idea of a class of things. Our symbols in their usage conform to the ordinary usage of this idea in language. It is to be noticed that in the systematic exposition our treatment of classes and relations requires no new primitive ideas and only two new primitive propositions, namely the two forms of the "Axiom of Reducibility" (cf. next Chapter) for one and two variables respectively.

The propositional function "x is a member of the class α" will be expressed, following Peano, by the notation

$$x \epsilon \alpha.$$

Here ϵ is chosen as the initial of the word ἐστί. "$x \epsilon \alpha$" may be read "x is an α." Thus "$x \epsilon$ man" will mean "x is a man," and so on. For typographical convenience we shall put

$$x \sim\epsilon \alpha \,.=.\, \sim(x \epsilon \alpha) \quad \text{Df},$$

$$x, y \epsilon \alpha \,.=.\, x \epsilon \alpha \,.\, y \epsilon \alpha \quad \text{Df}.$$

For "class" we shall write "Cls"; thus "$\alpha \epsilon$ Cls" means "α is a class."

We have

$$\vdash : x \epsilon \hat{z}(\phi z) \,.\equiv.\, \phi x,$$

i.e. "'x is a member of the class determined by $\phi\hat{z}$' is equivalent to 'x satisfies $\phi\hat{z}$,' or to 'ϕx is true.'"

A class is wholly determinate when its membership is known, that is, there cannot be two different classes having the same membership. Thus if ϕx, ψx are formally equivalent functions, they determine the same class; for in that case, if x is a member of the class determined by $\phi\hat{x}$, and therefore satisfies ϕx, it also satisfies ψx, and is therefore a member of the class determined by $\psi\hat{x}$. Thus we have

$$\vdash :. \hat{z}(\phi z) = \hat{z}(\psi z) \,.\equiv:\, \phi x \,.\equiv_x.\, \psi x.$$

The following propositions are obvious and important:

$$\vdash :. \alpha = \hat{z}(\phi z) \,.\equiv:\, x \epsilon \alpha \,.\equiv_x.\, \phi x,$$

i.e. α is identical with the class determined by $\phi\hat{z}$ when, and only when, "x is an α" is formally equivalent to ϕx;

$$\vdash :. \alpha = \beta \,.\equiv:\, x \epsilon \alpha \,.\equiv_x.\, x \epsilon \beta,$$

i.e. two classes α and β are identical when, and only when, they have the same membership;

$$\vdash .\, \hat{x}(x \epsilon \alpha) = \alpha,$$

i.e. the class whose determining function is "x is an α" is α, in other words, α is the class of objects which are members of α;

$$\vdash .\, \hat{z}(\phi z) \epsilon \text{Cls},$$

i.e. the class determined by the function $\phi\hat{z}$ is a class.

It will be seen that, according to the above, any function of one variable can be replaced by an equivalent function of the form "$x \epsilon \alpha$." Hence any extensional function of functions which holds when its argument is a function of the form "$\hat{z} \epsilon \alpha$," whatever possible value α may have, will hold also when its argument is any function $\phi\hat{z}$. Thus variation of classes can replace variation of functions of one variable in all the propositions of the sort with which we are concerned.

In an exactly analogous manner we introduce dual or dyadic relations, *i.e.* relations between two terms. Such relations will be called simply "relations"; relations between more than two terms will be distinguished as *multiple* relations, or (when the number of their terms is specified) as triple, quadruple,... relations, or as triadic, tetradic,... relations. Such relations will not concern us until we come to Geometry. For the present, the only relations we are concerned with are *dual* relations.

Relations, like classes, are to be taken in *extension*, *i.e.* if R and S are relations which hold between the same pairs of terms, R and S are to be identical. We may regard a relation, in the sense in which it is required for our purposes, as a class of couples; *i.e.* the couple (x, y) is to be one of the class of couples constituting the relation R if x has the relation R to y*. This view of relations as classes of couples will not, however, be introduced into our symbolic treatment, and is only mentioned in order to show that it is possible so to understand the meaning of the word *relation* that a relation shall be determined by its extension.

Any function $\phi(x, y)$ determines a relation R between x and y. If we regard a relation as a class of couples, the relation determined by $\phi(x, y)$ is the class of couples (x, y) for which $\phi(x, y)$ is true. The relation determined by the function $\phi(x, y)$ will be denoted by

$$\hat{x}\hat{y}\phi(x, y).$$

We shall use a capital letter for a relation when it is not necessary to specify the determining function. Thus whenever a capital letter occurs, it is to be understood that it stands for a relation.

The propositional function "x has the relation R to y" will be expressed by the notation

$$xRy.$$

This notation is designed to keep as near as possible to common language, which, when it has to express a relation, generally mentions it between its terms, as in "x loves y," "x equals y," "x is greater than y," and so on. For "relation" we shall write "Rel"; thus "$R \epsilon \text{Rel}$" means "R is a relation."

Owing to our taking relations in extension, we shall have

$$\vdash :. \hat{x}\hat{y}\phi(x, y) = \hat{x}\hat{y}\psi(x, y) . \equiv : \phi(x, y) . \equiv_{x,y} . \psi(x, y),$$

i.e. two functions of two variables determine the same relation when, and only when, the two functions are formally equivalent.

We have $\qquad \vdash . z\{\hat{x}\hat{y}\phi(x, y)\}w . \equiv . \phi(z, w),$

* Such a couple has a *sense*, *i.e.* the couple (x, y) is different from the couple (y, x), unless $x = y$. We shall call it a "couple with sense," to distinguish it from the class consisting of x and y. It may also be called an *ordered* couple.

i.e. "z has to w the relation determined by the function $\phi(x, y)$" is equivalent to $\phi(z, w)$;

$$\vdash :. R = \hat{x}\hat{y}\phi(x, y) . \equiv : xRy . \equiv_{x,y} . \phi(x, y),$$

$$\vdash :. R = S . \equiv : xRy . \equiv_{x,y} . xSy,$$

$$\vdash . \hat{x}\hat{y}(xRy) = R,$$

$$\vdash . \{\hat{x}\hat{y}\phi(x, y)\} \epsilon \text{Rel}.$$

These propositions are analogous to those previously given for classes. It results from them that any function of two variables is formally equivalent to some function of the form xRy; hence, in extensional functions of two variables, variation of relations can replace variation of functions of two variables.

Both classes and relations have properties analogous to most of those of propositions that result from negation and the logical sum. The *logical product* of two classes α and β is their common part, *i.e.* the class of terms which are members of both. This is represented by $\alpha \cap \beta$. Thus we put

$$\alpha \cap \beta = \hat{x}(x \epsilon \alpha . x \epsilon \beta) \quad \text{Df.}$$

This gives us $\quad \vdash : x \epsilon \alpha \cap \beta . \equiv . x \epsilon \alpha . x \epsilon \beta,$

i.e. "x is a member of the logical product of α and β" is equivalent to the logical product of "x is a member of α" and "x is a member of β."

Similarly the *logical sum* of two classes α and β is the class of terms which are members of either; we denote it by $\alpha \cup \beta$. The definition is

$$\alpha \cup \beta = \hat{x}(x \epsilon \alpha . \vee . x \epsilon \beta) \quad \text{Df,}$$

and the connection with the logical sum of propositions is given by

$$\vdash :. x \epsilon \alpha \cup \beta . \equiv : x \epsilon \alpha . \vee . x \epsilon \beta.$$

The *negation* of a class α consists of those terms x for which "$x \epsilon \alpha$" can be *significantly and truly* denied. We shall find that there are terms of other types for which "$x \epsilon \alpha$" is neither true nor false, but nonsense. These terms are not members of the negation of α.

Thus the *negation* of a class α is the class of terms of suitable type which are not members of it, *i.e.* the class $\hat{x}(x \sim \epsilon \alpha)$. We call this class "$-\alpha$" (read "not-$\alpha$"); thus the definition is

$$-\alpha = \hat{x}(x \sim \epsilon \alpha) \quad \text{Df,}$$

and the connection with the negation of propositions is given by

$$\vdash : x \epsilon -\alpha . \equiv . x \sim \epsilon \alpha.$$

In place of implication we have the relation of *inclusion*. A class α is said to be included or contained in a class β if all members of α are members of β, *i.e.* if $x \epsilon \alpha . \supset_x . x \epsilon \beta$. We write "$\alpha \subset \beta$" for "$\alpha$ is contained in β." Thus we put

$$\alpha \subset \beta . = : x \epsilon \alpha . \supset_x . x \epsilon \beta \quad \text{Df.}$$

Most of the formulae concerning $p \,.\, q$, $p \vee q$, $\sim p$, $p \supset q$ remain true if we substitute $\alpha \cap \beta$, $\alpha \cup \beta$, $-\alpha$, $\alpha \subset \beta$. In place of equivalence, we substitute identity; for "$p \equiv q$" was defined as "$p \supset q \,.\, q \supset p$," but "$\alpha \subset \beta \,.\, \beta \subset \alpha$" gives "$x \epsilon \alpha \,.\, \equiv_x \,.\, x \epsilon \beta$," whence $\alpha = \beta$.

The following are some propositions concerning classes which are analogues of propositions previously given concerning propositions:

$$\vdash \,.\, \alpha \cap \beta = -(-\alpha \cup -\beta),$$

i.e. the common part of α and β is the negation of "not-α or not-β";

$$\vdash \,.\, x \epsilon (\alpha \cup -\alpha),$$

i.e. "x is a member of α or not-α";

$$\vdash \,.\, x \sim\epsilon\, (\alpha \cap -\alpha),$$

i.e. "x is not a member of both α and not-α";

$$\vdash \,.\, \alpha = -(-\alpha),$$
$$\vdash : \alpha \subset \beta \,.\, \equiv \,.\, -\beta \subset -\alpha,$$
$$\vdash : \alpha = \beta \,.\, \equiv \,.\, -\alpha = -\beta,$$
$$\vdash : \alpha = \alpha \cap \alpha,$$
$$\vdash : \alpha = \alpha \cup \alpha.$$

The two last are the two forms of the law of tautology.

The law of absorption holds in the form

$$\vdash : \alpha \subset \beta \,.\, \equiv \,.\, \alpha = \alpha \cap \beta.$$

Thus for example "all Cretans are liars" is equivalent to "Cretans are identical with lying Cretans."

Just as we have $\vdash : p \supset q \,.\, q \supset r \,.\, \supset \,.\, p \supset r,$

so we have $\vdash : \alpha \subset \beta \,.\, \beta \subset \gamma \,.\, \supset \,.\, \alpha \subset \gamma.$

This expresses the ordinary syllogism in Barbara (with the premisses interchanged); for "$\alpha \subset \beta$" means the same as "all α's are β's," so that the above proposition states: "If all α's are β's, and all β's are γ's, then all α's are γ's." (It should be observed that syllogisms are traditionally expressed with "therefore," as if they asserted both premisses and conclusion. This is, of course, merely a slipshod way of speaking, since what is really asserted is only the connection of premisses with conclusion.)

The syllogism in Barbara when the minor premiss has an individual subject is

$$\vdash : x \epsilon \beta \,.\, \beta \subset \gamma \,.\, \supset \,.\, x \epsilon \gamma,$$

e.g. "if Socrates is a man, and all men are mortals, then Socrates is a mortal." This, as was pointed out by Peano, is not a particular case of "$\alpha \subset \beta \,.\, \beta \subset \gamma \,.\, \supset \,.\, \alpha \subset \gamma$," since "$x \epsilon \beta$" is not a particular case of "$\alpha \subset \beta$." This point is important, since traditional logic is here mistaken. The nature and magnitude of its mistake will become clearer at a later stage.

For relations, we have precisely analogous definitions and propositions. We put

$$\dot{R} \mathbin{\dot{\frown}} S = \hat{x}\hat{y}\,(xRy \,.\, xSy) \quad \text{Df},$$

which leads to $\quad \vdash : x\,(R \mathbin{\dot{\frown}} S)\,y \,.\, \equiv .\, xRy \,.\, xSy.$

Similarly $\quad R \mathbin{\dot{\smile}} S = \hat{x}\hat{y}\,(xRy \,.\, \mathsf{v} \,.\, xSy) \quad \text{Df},$

$$\dot{-}\, R = \hat{x}\hat{y}\,\{\sim(xRy)\} \quad \text{Df},$$

$$R \mathbin{\dot{\subset}} S \,.\, = : xRy \,.\, \supset_{x,y} .\, xSy \quad \text{Df}.$$

Generally, when we require analogous but different symbols for relations and for classes, we shall choose for relations the symbol obtained by adding a dot, in some convenient position, to the corresponding symbol for classes. (The dot must not be put on the line, since that would cause confusion with the use of dots as brackets.) But such symbols require and receive a special definition in each case.

A class is said to *exist* when it has at least one member: "α exists" is denoted by "$\exists\,!\,\alpha$." Thus we put

$$\exists\,!\,\alpha \,.\, = .\, (\exists x) \,.\, x \in \alpha \quad \text{Df}.$$

The class which has no members is called the "null-class," and is denoted by "Λ." Any propositional function which is always false determines the null-class. One such function is known to us already, namely "x is not identical with x," which we denote by "$x \neq x$." Thus we may use this function for defining Λ, and put

$$\Lambda = \hat{x}\,(x \neq x) \quad \text{Df}.$$

The class determined by a function which is always true is called the *universal class*, and is represented by V; thus

$$\mathrm{V} = \hat{x}\,(x = x) \quad \text{Df}.$$

Thus Λ is the negation of V. We have

$$\vdash .\, (x) \,.\, x \in \mathrm{V},$$

i.e. "'x is a member of V' is always true"; and

$$\vdash .\, (x) \,.\, x \sim\in \Lambda,$$

i.e. "'x is a member of Λ' is always false." Also

$$\vdash : \alpha = \Lambda \,.\, \equiv .\, \sim\exists\,!\,\alpha,$$

i.e. "α is the null-class" is equivalent to "α does not exist."

For relations we use similar notations. We put

$$\dot{\exists}\,!\,R \,.\, = .\, (\exists x, y) \,.\, xRy,$$

i.e. "$\dot{\exists}\,!\,R$" means that there is at least one couple x, y between which the relation R holds. $\dot{\Lambda}$ will be the relation which never holds, and $\dot{\mathrm{V}}$ the relation which always holds. $\dot{\mathrm{V}}$ is practically never required; $\dot{\Lambda}$ will be the relation $\hat{x}\hat{y}\,(x \neq x \,.\, y \neq y)$. We have

$$\vdash .\, (x, y) \,.\, \sim(x \,\dot{\Lambda}\, y),$$

and $\quad \vdash : R = \dot{\Lambda} \,.\, \equiv .\, \sim\dot{\exists}\,!\,R.$

There are no classes which contain objects of more than one type. Accordingly there is a universal class and a null-class proper to each type of object. But these symbols need not be distinguished, since it will be found that there is no possibility of confusion. Similar remarks apply to relations.

Descriptions. By a "description" we mean a phrase of the form "*the* so-and-so" or of some equivalent form. For the present, we confine our attention to *the* in the singular. We shall use this word strictly, so as to imply uniqueness; *e.g.* we should not say "A is *the* son of B" if B had other sons besides A. Thus a description of the form "the so-and-so" will only have an application in the event of there being one so-and-so and no more. Hence a description requires some propositional function $\phi\hat{x}$ which is satisfied by one value of x and by no other values; then "the x which satisfies $\phi\hat{x}$" is a description which definitely describes a certain object, though we may not know what object it describes. For example, if y is a man, "x is the father of y" must be true for one, and only one, value of x. Hence "the father of y" is a description of a certain man, though we may not know *what* man it describes. A phrase containing "the" always presupposes some initial propositional function not containing "the"; thus instead of "x is the father of y" we ought to take as our initial function "x begot y"; then "the father of y" means the one value of x which satisfies this propositional function.

If $\phi\hat{x}$ is a propositional function, the symbol "$(\imath x)(\phi x)$" is used in our symbolism in such a way that it can always be read as "the x which satisfies $\phi\hat{x}$." But we do not define "$(\imath x)(\phi x)$" as standing for "the x which satisfies $\phi\hat{x}$," thus treating this last phrase as embodying a primitive idea. Every use of "$(\imath x)(\phi x)$," where it apparently occurs as a constituent of a proposition in the place of an object, is defined in terms of the primitive ideas already on hand. An example of this definition in use is given by the proposition "$E ! (\imath x)(\phi x)$" which is considered immediately. The whole subject is treated more fully in Chapter III.

The symbol should be compared and contrasted with "$\hat{x}(\phi x)$" which in use can always be read as "the x's which satisfy $\phi\hat{x}$." Both symbols are incomplete symbols defined only in use, and as such are discussed in Chapter III. The symbol "$\hat{x}(\phi x)$" always has an application, namely to the class determined by ϕx; but "$(\imath x)(\phi x)$" only has an application when $\phi\hat{x}$ is only satisfied by one value of x, neither more nor less. It should also be observed that the meaning given to the symbol by the definition, given immediately below, of $E ! (\imath x)(\phi x)$ does not presuppose that we know the meaning of "one." This is also characteristic of the definition of any other use of $(\imath x)(\phi x)$.

We now proceed to define "$E ! (\imath x)(\phi x)$" so that it can be read "the x satisfying ϕx exists." (It will be observed that this is a different meaning of existence from that which we express by "$\exists$.") Its definition is

$$E ! (\imath x)(\phi x) . = : (\exists c) : \phi x . \equiv_x . x = c \quad \text{Df},$$

i.e. "the x satisfying $\phi\hat{x}$ exists" is to mean "there is an object c such that ϕx is true when x is c but not otherwise."

The following are equivalent forms:

$$\vdash :. E! (\imath x)(\phi x) . \equiv : (\exists c) : \phi c : \phi x . \supset_x . x = c,$$

$$\vdash :. E! (\imath x)(\phi x) . \equiv : (\exists c) . \phi c : \phi x . \phi y . \supset_{x,y} . x = y,$$

$$\vdash :. E! (\imath x)(\phi x) . \equiv : (\exists c) : \phi c : x \neq c . \supset_x . \sim \phi x.$$

The last of these states that "the x satisfying $\phi\hat{x}$ exists" is equivalent to "there is an object c satisfying $\phi\hat{x}$, and every object other than c does not satisfy $\phi\hat{x}$."

The kind of existence just defined covers a great many cases. Thus for example "the most perfect Being exists" will mean:

$$(\exists c) : x \text{ is most perfect} . \equiv_x . x = c,$$

which, taking the last of the above equivalences, is equivalent to

$$(\exists c) : c \text{ is most perfect} : x \neq c . \supset_x . x \text{ is not most perfect}.$$

A proposition such as "Apollo exists" is really of the same logical form, although it does not explicitly contain the word *the*. For "Apollo" means really "the object having such-and-such properties," say "the object having the properties enumerated in the Classical Dictionary*." If these properties make up the propositional function ϕx, then "Apollo" means "$(\imath x)(\phi x)$," and "Apollo exists" means "$E! (\imath x)(\phi x)$." To take another illustration, "the author of Waverley" means "the man who (or rather, the object which) wrote Waverley." Thus "Scott is the author of Waverley" is

$$\text{Scott} = (\imath x)(x \text{ wrote Waverley}).$$

Here (as we observed before) the importance of *identity* in connection with descriptions plainly appears.

The notation "$(\imath x)(\phi x)$," which is long and inconvenient, is seldom used, being chiefly required to lead up to another notation, namely "$R\text{‘}y$," meaning "the object having the relation R to y." That is, we put

$$R\text{‘}y = (\imath x)(xRy) \quad \text{Df.}$$

The inverted comma may be read "of." Thus "$R\text{‘}y$" is read "the R of y." Thus if R is the relation of father to son, "$R\text{‘}y$" means "the father of y"; if R is the relation of son to father, "$R\text{‘}y$" means "the son of y," which will only "exist" if y has one son and no more. $R\text{‘}y$ is a function of y, but not a propositional function; we shall call it a *descriptive* function. All the ordinary functions of mathematics are of this kind, as will appear more fully in the sequel. Thus in our notation, "sin y" would be written "sin ‘y," and "sin" would stand for the relation which sin ‘y has to y. Instead of a variable descriptive function fy, we put $R\text{‘}y$, where the variable relation R takes the

* The same principle applies to many uses of the proper names of existent objects, *e.g.* to all uses of proper names for objects known to the speaker only by report, and not by personal acquaintance.

place of the variable function f. A descriptive function will in general exist while y belongs to a certain domain, but not outside that domain; thus if we are dealing with positive rationals, $\sqrt{y}$ will be significant if y is a perfect square, but not otherwise; if we are dealing with real numbers, and agree that "$\sqrt{y}$" is to mean the *positive* square root (or, is to mean the negative square root), $\sqrt{y}$ will be significant provided y is positive, but not otherwise; and so on. Thus every descriptive function has what we may call a "domain of definition" or a "domain of existence," which may be thus defined: If the function in question is R‘y, its domain of definition or of existence will be the class of those arguments y for which we have E! R‘y, *i.e.* for which E! $(\imath x)(xRy)$, *i.e.* for which there is one x, and no more, having the relation R to y.

If R is any relation, we will speak of R‘y as the "associated descriptive function." A great many of the constant relations which we shall have occasion to introduce are only or chiefly important on account of their associated descriptive functions. In such cases, it is easier (though less correct) to begin by assigning the meaning of the descriptive function, and to deduce the meaning of the relation from that of the descriptive function. This will be done in the following explanations of notation.

Various descriptive functions of relations. If R is any relation, the *converse* of R is the relation which holds between y and x whenever R holds between x and y. Thus *greater* is the converse of *less*, *before* of *after*, *cause* of *effect* *husband* of *wife*, etc. The converse of R is written* Cnv‘R or $\breve{R}$. The definition is

$$\breve{R} = \hat{x}\hat{y}\,(yRx) \quad \text{Df},$$

$$\text{Cnv}‘R = \breve{R} \qquad \text{Df.}$$

The second of these is not a formally correct definition, since we ought to define "Cnv" and deduce the meaning of Cnv‘R. But it is not worth while to adopt this plan in our present introductory account, which aims at simplicity rather than formal correctness.

A relation is called *symmetrical* if $R = \breve{R}$, *i.e.* if it holds between y and x whenever it holds between x and y (and therefore vice versa). Identity, diversity, agreement or disagreement in any respect, are symmetrical relations. A relation is called *asymmetrical* when it is incompatible with its converse, *i.e.* when $R \mathbin{\dot{\cap}} \breve{R} = \dot{\Lambda}$, or, what is equivalent,

$$xRy \,.\, \supset_{x,y} .\, \sim(yRx).$$

Before and after, greater and less, ancestor and descendant, are asymmetrical, as are all other relations of the sort that lead to *series*. But there are many asymmetrical relations which do not lead to series, for instance, that of

* The second of these notations is taken from Schröder's *Algebra und Logik der Relative*.

wife's brother*. A relation may be neither symmetrical nor asymmetrical; for example, this holds of the relation of inclusion between classes: $\alpha \subset \beta$ and $\beta \subset \alpha$ will both be true if $\alpha = \beta$, but otherwise only one of them, at most, will be true. The relation *brother* is neither symmetrical nor asymmetrical, for if x is the brother of y, y may be either the brother or the sister of x.

In the propositional function xRy, we call x the *referent* and y the *relatum*. The class $\hat{x}(xRy)$, consisting of all the x's which have the relation R to y, is called the class of referents of y with respect to R; the class $\hat{y}(xRy)$, consisting of all the y's to which x has the relation R, is called the class of relata of x with respect to R. These two classes are denoted respectively by $\overrightarrow{R}\text{‘}y$ and $\overleftarrow{R}\text{‘}x$. Thus

$$\overrightarrow{R}\text{‘}y = \hat{x}(xRy) \quad \text{Df},$$
$$\overleftarrow{R}\text{‘}x = \hat{y}(xRy) \quad \text{Df}.$$

The arrow runs towards y in the first case, to show that we are concerned with things having the relation R *to* y; it runs away from x in the second case, to show that the relation R goes *from* x to the members of $\overleftarrow{R}\text{‘}x$. It runs in fact *from* a referent and *towards* a relatum.

The notations $\overrightarrow{R}\text{‘}y$, $\overleftarrow{R}\text{‘}x$ are very important, and are used constantly. If R is the relation of parent to child, $\overrightarrow{R}\text{‘}y$ = the parents of y, $\overleftarrow{R}\text{‘}x$ = the children of x. We have

$$\vdash : x \in \overrightarrow{R}\text{‘}y \,.\equiv.\, xRy$$

and

$$\vdash : y \in \overleftarrow{R}\text{‘}x \,.\equiv.\, xRy.$$

These equivalences are often embodied in common language. For example, we say indiscriminately "x is an inhabitant of London" or "x inhabits London." If we put "R" for "inhabits," "x inhabits London" is "xR London," while "x is an inhabitant of London" is "$x \in \overrightarrow{R}\text{‘}$ London."

Instead of $\overrightarrow{R}$ and $\overleftarrow{R}$ we sometimes use $\text{sg‘}R$, $\text{gs‘}R$, where "sg" stands for "sagitta," and "gs" is "sg" backwards. Thus we put

$$\text{sg‘}R = \overrightarrow{R} \quad \text{Df},$$
$$\text{gs‘}R = \overleftarrow{R} \quad \text{Df}.$$

These notations are sometimes more convenient than an arrow when the relation concerned is represented by a combination of letters, instead of a single letter such as R. Thus *e.g.* we should write $\text{sg‘}(R \mathbin{\dot\cap} S)$, rather than put an arrow over the whole length of $(R \mathbin{\dot\cap} S)$.

The class of all terms that have the relation R to something or other is called the *domain* of R. Thus if R is the relation of parent and child, the

* This relation is not strictly asymmetrical, but is so except when the wife's brother is also the sister's husband. In the Greek Church the relation is strictly asymmetrical.

domain of R will be the class of parents. We represent the domain of R by "D'R." Thus we put

$$\mathrm{D}\text{'}R = \hat{x}\{(\exists y) \,.\, xRy\} \quad \text{Df.}$$

Similarly the class of all terms to which something or other has the relation R is called the *converse domain* of R; it is the same as the domain of the converse of R. The converse domain of R is represented by "Ɑ'R"; thus

$$\text{Ɑ}\text{'}R = \hat{y}\{(\exists x) \,.\, xRy\} \quad \text{Df.}$$

The sum of the domain and the converse domain is called the *field*, and is represented by C'R: thus

$$C\text{'}R = \mathrm{D}\text{'}R \cup \text{Ɑ}\text{'}R \quad \text{Df.}$$

The *field* is chiefly important in connection with series. If R is the ordering relation of a series, C'R will be the class of terms of the series, D'R will be all the terms except the last (if any), and Ɑ'R will be all the terms except the first (if any). The first term, if it exists, is the only member of $\mathrm{D}\text{'}R \cap -\text{Ɑ}\text{'}R$, since it is the only term which is a predecessor but not a follower. Similarly the last term (if any) is the only member of $\text{Ɑ}\text{'}R \cap -\mathrm{D}\text{'}R$. The condition that a series should have no end is $\text{Ɑ}\text{'}R \subset \mathrm{D}\text{'}R$, *i.e.* "every follower is a predecessor"; the condition for no beginning is $\mathrm{D}\text{'}R \subset \text{Ɑ}\text{'}R$. These conditions are equivalent respectively to $\mathrm{D}\text{'}R = C\text{'}R$ and $\text{Ɑ}\text{'}R = C\text{'}R$.

The *relative product* of two relations R and S is the relation which holds between x and z when there is an intermediate term y such that x has the relation R to y and y has the relation S to z. The relative product of R and S is represented by $R \mid S$; thus we put

$$R \mid S = \hat{x}\hat{z}\{(\exists y) \,.\, xRy \,.\, ySz\} \quad \text{Df,}$$

whence $\vdash : x(R \mid S)z \,.\, \equiv \,.\, (\exists y) \,.\, xRy \,.\, ySz.$

Thus "paternal aunt" is the relative product of *sister* and *father*; "paternal grandmother" is the relative product of *mother* and *father*; "maternal grandfather" is the relative product of *father* and *mother*. The relative product is not commutative, but it obeys the associative law, *i.e.*

$$\vdash . (P \mid Q) \mid R = P \mid (Q \mid R).$$

It also obeys the distributive law with regard to the logical addition of relations, *i.e.* we have

$$\vdash . P \mid (Q \,\dot{\cup}\, R) = (P \mid Q) \,\dot{\cup}\, (P \mid R),$$
$$\vdash . (Q \,\dot{\cup}\, R) \mid P = (Q \mid P) \,\dot{\cup}\, (R \mid P).$$

But with regard to the logical *product*, we have only

$$\vdash . P \mid (Q \,\dot{\cap}\, R) \,\dot{\subset}\, (P \mid Q) \,\dot{\cap}\, (P \mid R),$$
$$\vdash . (Q \,\dot{\cap}\, R) \mid P \,\dot{\subset}\, (Q \mid P) \,\dot{\cap}\, (Q \mid R).$$

The relative product does not obey the law of tautology, *i.e.* we do not have in general $R\ \ R = R$. We put

$$R^2 = R \mid R \quad \text{Df.}$$

Thus paternal grandfather = (father)^2,
maternal grandmother = (mother)^2.

A relation is called *transitive* when $R^2 \subset R$, *i.e.* when, if xRy and yRz, we always have xRz, *i.e.* when

$$xRy \,.\, yRz \,.\supset_{x,y,z} .\, xRz.$$

Relations which generate series are always transitive; thus *e.g.*

$$x > y \,.\, y > z \,.\supset_{x,y,z} .\, x > z.$$

If P is a relation which generates a series, P may conveniently be read "precedes"; thus "$xPy \,.\, yPz \,.\supset_{x,y,z} .\, xPz$" becomes "if x precedes y and y precedes z, then x always precedes z." The class of relations which generate series are partially characterized by the fact that they are transitive and asymmetrical, and never relate a term to itself.

If P is a relation which generates a series, and if we have not merely $P^2 \subset P$, but $P^2 = P$, then P generates a series which is *compact* (*überall dicht*), *i.e.* such that there are terms between any two. For in this case we have

$$xPz \,.\supset .\, (\exists y) \,.\, xPy \,.\, yPz,$$

i.e. if x precedes z, there is a term y such that x precedes y and y precedes z, *i.e.* there is a term between x and z. Thus among relations which generate series, those which generate compact series are those for which $P^2 = P$.

Many relations which do not generate series are transitive, for example, identity, or the relation of inclusion between classes. Such cases arise when the relations are not asymmetrical. Relations which are transitive and symmetrical are an important class: they may be regarded as consisting in the possession of some common property.

Plural descriptive functions. The class of terms x which have the relation R to some member of a class α is denoted by $R``\alpha$ or $R_\epsilon`\alpha$. The definition is

$$R``\alpha = \hat{x}\{(\exists y) \,.\, y \in \alpha \,.\, xRy\} \quad \text{Df.}$$

Thus for example let R be the relation of *inhabiting*, and α the class of towns; then $R``\alpha$ = inhabitants of towns. Let $R$ be the relation "less than" among rationals, and $\alpha$ the class of those rationals which are of the form $1 - 2^{-n}$, for integral values of $n$; then $R``\alpha$ will be all rationals less than some member of α, *i.e.* all rationals less than 1. If P is the generating relation of a series, and α is any class of members of the series, $P``\alpha$ will be predecessors of $\alpha$'s, *i.e.* the segment defined by $\alpha$. If $P$ is a relation such that $P`y$ always exists when $y \in \alpha$, $P``\alpha$ will be the class of all terms of the form $P`y$ for values of y which are members of α; *i.e.*

$$P``\alpha = \hat{x}\{(\exists y) \,.\, y \in \alpha \,.\, x = P`y\}.$$

Thus a member of the class "fathers of great men" will be the father of y, where y is some great man. In other cases, this will not hold; for instance, let P be the relation of a number to any number of which it is a factor; then

$P``$(even numbers) = factors of even numbers, but this class is not composed of terms of the form "*the* factor of x," where x is an even number, because numbers do not have only one factor apiece.

Unit classes. The class whose only member is x might be thought to be identical with x, but Peano and Frege have shown that this is not the case. (The reasons why this is not the case will be explained in a preliminary way in Chapter II of the Introduction.) We denote by "$\iota`x$" the class whose only member is x: thus

$$\iota`x = \hat{y}\,(y = x) \quad \mathrm{Df},$$

i.e. "$\iota`x$" means "the class of objects which are identical with x."

The class consisting of x and y will be $\iota`x \cup \iota`y$; the class got by adding x to a class α will be $\alpha \cup \iota`x$; the class got by taking away $x$ from a class $\alpha$ will be $\alpha - \iota`x$. (We write $\alpha - \beta$ as an abbreviation for $\alpha \cap -\beta$.)

It will be observed that unit classes have been defined without reference to the number 1; in fact, we use unit classes to define the number 1. This number is defined as the class of unit classes, *i.e.*

$$1 = \hat{\alpha}\,\{(\exists x)\,.\,\alpha = \iota`x\} \quad \mathrm{Df}.$$

This leads to

$$\vdash :.\ \alpha \in 1\,.\equiv :(\exists x):y \in \alpha\,.\equiv_y .\,y = x:$$

From this it appears further that

$$\vdash :\alpha \in 1\,.\equiv .\,\mathrm{E}\,!\,(\imath x)\,(x \in \alpha),$$

whence

$$\vdash :\hat{z}\,(\phi z) \in 1\,.\equiv .\,\mathrm{E}\,!\,(\imath x)\,(\phi x),$$

i.e. "$\hat{z}\,(\phi z)$ is a unit class" is equivalent to "the x satisfying $\phi\hat{x}$ exists."

If $\alpha \in 1$, $\breve{\iota}`\alpha$ is the only member of $\alpha$, for the only member of $\alpha$ is the only term to which $\alpha$ has the relation $\iota$. Thus "$\breve{\iota}`\alpha$" takes the place of "$(\imath x)\,(\phi x)$," if α stands for $\hat{z}\,(\phi z)$. In practice, "$\breve{\iota}`\alpha$" is a more convenient notation than "$(\imath x)\,(\phi x)$," and is generally used instead of "$(\imath x)\,(\phi x)$."

The above account has explained most of the logical notation employed in the present work. In the applications to various parts of mathematics, other definitions are introduced; but the objects defined by these later definitions belong, for the most part, rather to mathematics than to logic. The reader who has mastered the symbols explained above will find that any later formulae can be deciphered by the help of comparatively few additional definitions.

CHAPTER II

THE THEORY OF LOGICAL TYPES

THE theory of logical types, to be explained in the present Chapter, recommended itself to us in the first instance by its ability to solve certain contradictions, of which the one best known to mathematicians is Burali-Forti's concerning the greatest ordinal. But the theory in question is not wholly dependent upon this indirect recommendation: it has also a certain consonance with common sense which makes it inherently credible. In what follows, we shall therefore first set forth the theory on its own account, and then apply it to the solution of the contradictions.

I. *The Vicious-Circle Principle.*

An analysis of the paradoxes to be avoided shows that they all result from a certain kind of vicious circle*. The vicious circles in question arise from supposing that a collection of objects may contain members which can only be defined by means of the collection as a whole. Thus, for example, the collection of *propositions* will be supposed to contain a proposition stating that "all propositions are either true or false." It would seem, however, that such a statement could not be legitimate unless "all propositions" referred to some already definite collection, which it cannot do if new propositions are created by statements about "all propositions." We shall, therefore, have to say that statements about "all propositions" are meaningless. More generally, given any set of objects such that, if we suppose the set to have a total, it will contain members which presuppose this total, then such a set cannot have a total. By saying that a set has "no total," we mean, primarily, that no significant statement can be made about "all its members." Propositions, as the above illustration shows, must be a set having no total. The same is true, as we shall shortly see, of propositional functions, even when these are restricted to such as can significantly have as argument a given object a. In such cases, it is necessary to break up our set into smaller sets, each of which is capable of a total. This is what the theory of types aims at effecting.

The principle which enables us to avoid illegitimate totalities may be stated as follows: "Whatever involves *all* of a collection must not be one of the collection"; or, conversely: "If, provided a certain collection had a total, it would have members only definable in terms of that total, then the said collection has no total." We shall call this the "vicious-circle principle," because it enables us to avoid the vicious circles involved in the assumption of illegitimate totalities. Arguments which are condemned by the vicious-circle

* See the last section of the present Chapter. Cf. also H. Poincaré, "Les mathématiques et la logique," *Revue de Métaphysique et de Morale*, Mai 1906, p. 307.

principle will be called "vicious-circle fallacies." Such arguments, in certain circumstances, may lead to contradictions, but it often happens that the conclusions to which they lead are in fact true, though the arguments are fallacious. Take, for example, the law of excluded middle, in the form "all propositions are true or false." If from this law we argue that, because the law of excluded middle is a proposition, therefore the law of excluded middle is true or false, we incur a vicious-circle fallacy. "All propositions" must be in some way limited before it becomes a legitimate totality, and any limitation which makes it legitimate must make any statement about the totality fall outside the totality. Similarly, the imaginary sceptic, who asserts that he knows nothing, and is refuted by being asked if he knows that he knows nothing, has asserted nonsense, and has been fallaciously refuted by an argument which involves a vicious-circle fallacy. In order that the sceptic's assertion may become significant, it is necessary to place some limitation upon the things of which he is asserting his ignorance, because the things of which it is possible to be ignorant form an illegitimate totality. But as soon as a suitable limitation has been placed by him upon the collection of propositions of which he is asserting his ignorance, the proposition that he is ignorant of every member of this collection must not itself be one of the collection. Hence any significant scepticism is not open to the above form of refutation.

The paradoxes of symbolic logic concern various sorts of objects: propositions, classes, cardinal and ordinal numbers, etc. All these sorts of objects, as we shall show, represent illegitimate totalities, and are therefore capable of giving rise to vicious-circle fallacies. But by means of the theory (to be explained in Chapter III) which reduces statements that are verbally concerned with classes and relations to statements that are concerned with propositional functions, the paradoxes are reduced to such as are concerned with propositions and propositional functions. The paradoxes that concern propositions are only indirectly relevant to mathematics, while those that more nearly concern the mathematician are all concerned with *propositional functions*. We shall therefore proceed at once to the consideration of propositional functions.

II. *The Nature of Propositional Functions.*

By a "propositional function" we mean something which contains a variable x, and expresses a *proposition* as soon as a value is assigned to x. That is to say, it differs from a proposition solely by the fact that it is ambiguous: it contains a variable of which the value is unassigned. It agrees with the ordinary functions of mathematics in the fact of containing an unassigned variable; where it differs is in the fact that the values of the function are propositions. Thus *e.g.* "x is a man" or "$\sin x = 1$" is a propositional function. We shall find that it is possible to incur a vicious-circle

fallacy at the very outset, by admitting as possible arguments to a propositional function terms which presuppose the function. This form of the fallacy is very instructive, and its avoidance leads, as we shall see, to the hierarchy of types.

The question as to the nature of a function* is by no means an easy one. It would seem, however, that the essential characteristic of a function is *ambiguity*. Take, for example, the law of identity in the form "A is A," which is the form in which it is usually enunciated. It is plain that, regarded psychologically, we have here a single judgment. But what are we to say of the object of the judgment? We are not judging that Socrates is Socrates, nor that Plato is Plato, nor any other of the definite judgments that are instances of the law of identity. Yet each of these judgments is, in a sense, within the scope of our judgment. We are in fact judging an ambiguous instance of the propositional function "A is A." We appear to have a single thought which does not have a definite object, but has as its object an undetermined one of the values of the function "A is A." It is this kind of ambiguity that constitutes the essence of a function. When we speak of "ϕx," where x is not specified, we mean one value of the function, but not a definite one. We may express this by saying that "ϕx" *ambiguously denotes* ϕa, ϕb, ϕc, etc., where ϕa, ϕb, ϕc, etc., are the various values of "ϕx."

When we say that "ϕx" ambiguously denotes ϕa, ϕb, ϕc, etc., we mean that "ϕx" means one of the objects ϕa, ϕb, ϕc, etc., though not a definite one, but an undetermined one. It follows that "ϕx" only has a well-defined meaning (well-defined, that is to say, except in so far as it is of its essence to be ambiguous) if the objects ϕa, ϕb, ϕc, etc., are well-defined. That is to say, a function is not a well-defined function unless all its values are already well-defined. It follows from this that no function can have among its values anything which presupposes the function, for if it had, we could not regard the objects ambiguously denoted by the function as definite until the function was definite, while conversely, as we have just seen, the function cannot be definite until its values are definite. This is a particular case, but perhaps the most fundamental case, of the vicious-circle principle. A function is what ambiguously denotes some one of a certain totality, namely the values of the function; hence this totality cannot contain any members which involve the function, since, if it did, it would contain members involving the totality, which, by the vicious-circle principle, no totality can do.

It will be seen that, according to the above account, the values of a function are presupposed by the function, not vice versa. It is sufficiently obvious, in any particular case, that a value of a function does not presuppose the function. Thus for example the proposition "Socrates is human" can be perfectly apprehended without regarding it as a value of the function "x is human." It is true that, conversely, a function can be apprehended without

* When the word "function" is used in the sequel, "propositional function" is always meant. Other functions will not be in question in the present Chapter.

its being necessary to apprehend its values severally and individually. If this were not the case, no function could be apprehended at all, since the number of values (true and false) of a function is necessarily infinite and there are necessarily possible arguments with which we are unacquainted. What is necessary is not that the values should be given individually and extensionally, but that the totality of the values should be given intensionally, so that, concerning any assigned object, it is at least theoretically determinate whether or not the said object is a value of the function.

It is necessary practically to distinguish the function itself from an undetermined value of the function. We may regard the function itself as that which ambiguously denotes, while an undetermined value of the function is that which is ambiguously denoted. If the undetermined value is written "ϕx," we will write the function itself "$\phi \hat{x}$." (Any other letter may be used in place of x.) Thus we should say "ϕx is a proposition," but "$\phi \hat{x}$ is a propositional function." When we say "ϕx is a proposition," we mean to state something which is true for every possible value of x, though we do not decide what value x is to have. We are making an ambiguous statement about any value of the function. But when we say "$\phi \hat{x}$ is a function," we are not making an ambiguous statement. It would be more correct to say that we are making a statement about an ambiguity, taking the view that a function is an ambiguity. The function itself, $\phi \hat{x}$, is the single thing which ambiguously denotes its many values; while ϕx, where x is not specified, is one of the denoted objects, with the ambiguity belonging to the manner of denoting.

We have seen that, in accordance with the vicious-circle principle, the values of a function cannot contain terms only definable in terms of the function. Now given a function $\phi \hat{x}$, the values for the function* are all propositions of the form ϕx. It follows that there must be no propositions, of the form ϕx, in which x has a value which involves $\phi \hat{x}$. (If this were the case, the values of the function would not all be determinate until the function was determinate, whereas we found that the function is not determinate unless its values are previously determinate.) Hence there must be no such thing as the value for $\phi \hat{x}$ with the argument $\phi \hat{x}$, or with any argument which involves $\phi \hat{x}$. That is to say, the symbol "$\phi (\phi \hat{x})$" must not express a proposition, as "ϕa" does if ϕa is a value for $\phi \hat{x}$. In fact "$\phi (\phi \hat{x})$" must be a symbol which does not express anything: we may therefore say that it is not significant. Thus given any function $\phi \hat{x}$, there are arguments with which the function has no value, as well as arguments with which it has a value. We will call the arguments with which $\phi \hat{x}$ has a value "possible values of x." We will say that $\phi \hat{x}$ is "significant with the argument x" when $\phi \hat{x}$ has a value with the argument x.

* We shall speak in this Chapter of "values *for* $\phi \hat{x}$" and of "values *of* ϕx," meaning in each case the same thing, namely ϕa, ϕb, ϕc, etc. The distinction of phraseology serves to avoid ambiguity where several variables are concerned, especially when one of them is a function.

When it is said that *e.g.* "$\phi(\phi\hat{z})$" is meaningless, and therefore neither true nor false, it is necessary to avoid a misunderstanding. If "$\phi(\phi\hat{z})$" were interpreted as meaning "the value for $\phi\hat{z}$ with the argument $\phi\hat{z}$ is true," that would be not meaningless, but false. It is false for the same reason for which "the King of France is bald" is false, namely because there is no such thing as "the value for $\phi\hat{z}$ with the argument $\phi\hat{z}$." But when, with some argument a, we assert ϕa, we are not meaning to assert "the value for $\phi\hat{x}$ with the argument a is true"; we are meaning to assert the actual proposition which *is* the value for $\phi\hat{x}$ with the argument a. Thus for example if $\phi\hat{x}$ is "$\hat{x}$ is a man," ϕ (Socrates) will be "Socrates is a man," *not* "the value for the function '$\hat{x}$ is a man,' with the argument Socrates, is true." Thus in accordance with our principle that "$\phi(\phi\hat{z})$" is meaningless, we cannot legitimately deny "the function '$\hat{x}$ is a man' is a man," because this is nonsense, but we can legitimately deny "the value for the function '$\hat{x}$ is a man' with the argument '$\hat{x}$ is a man' is true," not on the ground that the value in question is false, but on the ground that there is no such value for the function.

We will denote by the symbol "$(x) . \phi x$" the proposition "ϕx always*," *i.e.* the proposition which asserts *all* the values for $\phi\hat{x}$. This proposition involves the function $\phi\hat{x}$, not merely an ambiguous value of the function. The assertion of ϕx, where x is unspecified, is a different assertion from the one which asserts all values for $\phi\hat{x}$, for the former is an ambiguous assertion, whereas the latter is in no sense ambiguous. It will be observed that "$(x) . \phi x$" does not assert "ϕx with all values of x," because, as we have seen, there must be values of x with which "ϕx" is meaningless. What is asserted by "$(x) . \phi x$" is all propositions which are values for $\phi\hat{x}$; hence it is only with such values of x as make "ϕx" significant, *i.e.* with all *possible* arguments, that ϕx is asserted when we assert "$(x) . \phi x$." Thus a convenient way to read "$(x) . \phi x$" is "ϕx is true with all possible values of x." This is, however, a less accurate reading than "ϕx always," because the notion of *truth* is not part of the content of what is judged. When we judge "all men are mortal," we judge truly, but the notion of truth is not necessarily in our minds, any more than it need be when we judge "Socrates is mortal."

III. *Definition and Systematic Ambiguity of Truth and Falsehood.*

Since "$(x) . \phi x$" involves the function $\phi\hat{x}$, it must, according to our principle, be impossible as an argument to ϕ. That is to say, the symbol "$\phi\{(x) . \phi x\}$" must be meaningless. This principle would seem, at first sight, to have certain exceptions. Take, for example, the function "$\hat{p}$ is false," and consider the proposition "$(p) . p$ is false." This should be a proposition asserting all propositions of the form "p is false." Such a proposition, we

* We use "always" as meaning "in all cases," not "at all times." Similarly "sometimes" will mean "in some cases."

should be inclined to say, must be false, because "p is false" is not always true. Hence we should be led to the proposition

$$\text{"}\{(p) \,.\, p \text{ is false}\} \text{ is false,"}$$

i.e. we should be led to a proposition in which "$(p) \,.\, p$ is false" is the argument to the function "$\hat{p}$ is false," which we had declared to be impossible. Now it will be seen that "$(p) \,.\, p$ is false," in the above, purports to be a proposition about all propositions, and that, by the general form of the vicious-circle principle, there must be no propositions about *all* propositions. Nevertheless, it seems plain that, given any function, there is a proposition (true or false) asserting all its values. Hence we are led to the conclusion that "p is false" and "q is false" must not always be the values, with the arguments p and q, for a single function "$\hat{p}$ is false." This, however, is only possible if the word "false" really has many different meanings, appropriate to propositions of different kinds.

That the words "true" and "false" have many different meanings, according to the kind of proposition to which they are applied, is not difficult to see. Let us take any function $\phi\hat{x}$, and let ϕa be one of its values. Let us call the sort of truth which is applicable to ϕa "*first* truth." (This is not to assume that this would be first truth in another context: it is merely to indicate that it is the first sort of truth in our context.) Consider now the proposition $(x) \,.\, \phi x$. If this has truth of the sort appropriate to it, that will mean that every value ϕx has "first truth." Thus if we call the sort of truth that is appropriate to $(x) \,.\, \phi x$ "*second* truth," we may define "$\{(x) \,.\, \phi x\}$ has second truth" as meaning "every value for $\phi\hat{x}$ has first truth," *i.e.* "$(x) \,.\, (\phi x$ has first truth)." Similarly, if we denote by "$(\exists x) \,.\, \phi x$" the proposition "ϕx sometimes," *i.e.* as we may less accurately express it, "ϕx with some value of x," we find that $(\exists x) \,.\, \phi x$ has second truth if there is an x with which ϕx has first truth; thus we may define "$\{(\exists x) \,.\, \phi x\}$ has second truth" as meaning "some value for $\phi\hat{x}$ has first truth," *i.e.* "$(\exists x) \,.\, (\phi x$ has first truth)." Similar remarks apply to falsehood. Thus "$\{(x) \,.\, \phi x\}$ has second falsehood" will mean "some value for $\phi\hat{x}$ has first falsehood," *i.e.* "$(\exists x) \,.\, (\phi x$ has first falsehood)," while "$\{(\exists x) \,.\, \phi x\}$ has second falsehood" will mean "all values for $\phi\hat{x}$ have first falsehood," *i.e.* "$(x) \,.\, (\phi x$ has first falsehood)." Thus the sort of falsehood that can belong to a general proposition is different from the sort that can belong to a particular proposition.

Applying these considerations to the proposition "$(p) \,.\, p$ is false," we see that the kind of falsehood in question must be specified. If, for example, first falsehood is meant, the function "$\hat{p}$ has first falsehood" is only significant when p is the sort of proposition which has first falsehood or first truth. Hence "$(p) \,.\, p$ is false" will be replaced by a statement which is equivalent to "all propositions having either first truth or first falsehood have first falsehood." This proposition has *second* falsehood, and is not

a possible argument to the function "$\hat{p}$ has *first* falsehood." Thus the apparent exception to the principle that "$\phi\{(x) . \phi x\}$" must be meaningless disappears.

Similar considerations will enable us to deal with "not-p" and with "p or q." It might seem as if these were functions in which *any* proposition might appear as argument. But this is due to a systematic ambiguity in the meanings of "not" and "or," by which they adapt themselves to propositions of any order. To explain fully how this occurs, it will be well to begin with a definition of the simplest kind of *truth* and *falsehood.*

The universe consists of objects having various qualities and standing in various relations. Some of the objects which occur in the universe are complex. When an object is complex, it consists of interrelated parts. Let us consider a complex object composed of two parts a and b standing to each other in the relation R. The complex object "a-in-the-relation-R-to-b" may be capable of being *perceived*; when perceived, it is perceived as one object. Attention may show that it is complex; we then *judge* that a and b stand in the relation R. Such a judgment, being derived from perception by mere attention, may be called a "judgment of perception." This judgment of perception, considered as an actual occurrence, is a relation of four terms, namely a and b and R and the percipient. The perception, on the contrary, is a relation of two terms, namely "a-in-the-relation-R-to-b," and the percipient. Since an object of perception cannot be nothing, we cannot perceive "a-in-the-relation-R-to-b" unless a is in the relation R to b. Hence a judgment of perception, according to the above definition, must be true. This does not mean that, in a judgment which *appears* to us to be one of perception, we are sure of not being in error, since we may err in thinking that our judgment has really been derived merely by analysis of what was perceived. But if our judgment has been so derived, it must be true. In fact, we may define *truth*, where such judgments are concerned, as consisting in the fact that there is a complex *corresponding* to the discursive thought which is the judgment. That is, when we judge "a has the relation R to b," our judgment is said to be *true* when there is a complex "a-in-the-relation-R-to-b," and is said to be *false* when this is not the case. This is a definition of truth and falsehood in relation to judgments of this kind.

It will be seen that, according to the above account, a judgment does not have a single object, namely the proposition, but has several interrelated objects. That is to say, the relation which constitutes judgment is not a relation of two terms, namely the judging mind and the proposition, but is a relation of several terms, namely the mind and what are called the constituents of the proposition. That is, when we judge (say) "this is red," what occurs is a relation of three terms, the mind, and "this," and red. On the other hand, when we *perceive* "the redness of this," there is a relation of two terms, namely

the mind and the complex object "the redness of this." When a judgment occurs, there is a certain complex entity, composed of the mind and the various objects of the judgment. When the judgment is *true*, in the case of the kind of judgments we have been considering, there is a corresponding complex of the *objects* of the judgment alone. Falsehood, in regard to our present class of judgments, consists in the absence of a corresponding complex composed of the objects alone. It follows from the above theory that a "proposition," in the sense in which a proposition is supposed to be *the* object of a judgment, is a false abstraction, because a judgment has several objects, not one. It is the severalness of the objects in judgment (as opposed to perception) which has led people to speak of thought as "discursive," though they do not appear to have realized clearly what was meant by this epithet.

Owing to the plurality of the objects of a single judgment, it follows that what we call a "proposition" (in the sense in which this is distinguished from the phrase expressing it) is not a single entity at all. That is to say, the phrase which expresses a proposition is what we call an "incomplete" symbol*; it does not have meaning in itself, but requires some supplementation in order to acquire a complete meaning. This fact is somewhat concealed by the circumstance that judgment in itself supplies a sufficient supplement, and that judgment in itself makes no *verbal* addition to the proposition. Thus "the proposition 'Socrates is human'" uses "Socrates is human" in a way which requires a supplement of some kind before it acquires a complete meaning; but when I judge "Socrates is human," the meaning is completed by the act of judging, and we no longer have an incomplete symbol. The fact that propositions are "incomplete symbols" is important philosophically, and is relevant at certain points in symbolic logic.

The judgments we have been dealing with hitherto are such as are of the same form as judgments of perception, *i.e.* their subjects are always particular and definite. But there are many judgments which are not of this form. Such are "all men are mortal," "I met a man," "some men are Greeks." Before dealing with such judgments, we will introduce some technical terms.

We will give the name of "a *complex*" to any such object as "a in the relation R to b" or "a having the quality q," or "a and b and c standing in the relation S." Broadly speaking, a *complex* is anything which occurs in the universe and is not simple. We will call a judgment *elementary* when it merely asserts such things as "a has the relation R to b," "a has the quality q" or "a and b and c stand in the relation S." Then an *elementary* judgment is true when there is a corresponding complex, and false when there is no corresponding complex.

But take now such a proposition as "all men are mortal." Here the judgment does not correspond to *one* complex, but to many, namely "Socrates

* See Chapter III.

is mortal," "Plato is mortal," "Aristotle is mortal," etc. (For the moment, it is unnecessary to inquire whether each of these does not require further treatment before we reach the ultimate complexes involved. For purposes of illustration, "Socrates is mortal" is here treated as an elementary judgment, though it is in fact not one, as will be explained later. Truly elementary judgments are not very easily found.) We do not mean to deny that there may be some relation of the concept *man* to the concept *mortal* which may be *equivalent* to "all men are mortal," but in any case this relation is not the same thing as what we affirm when we say that all men are mortal. Our judgment that all men are mortal collects together a number of elementary judgments. It is not, however, composed of these, since (*e.g.*) the fact that Socrates is mortal is no part of what we assert, as may be seen by considering the fact that our assertion can be understood by a person who has never heard of Socrates. In order to understand the judgment "all men are mortal," it is not necessary to know what men there are. We must admit, therefore, as a radically new kind of judgment, such general assertions as "all men are mortal." We assert that, given that x is human, x is always mortal. That is, we assert "x is mortal" of *every* x which is human. Thus we are able to judge (whether truly or falsely) that *all* the objects which have some assigned property also have some other assigned property. That is, given any propositional functions $\phi\hat{x}$ and $\psi\hat{x}$, there is a judgment asserting ψx with every x for which we have ϕx. Such judgments we will call *general judgments*.

It is evident (as explained above) that the definition of *truth* is different in the case of general judgments from what it was in the case of elementary judgments. Let us call the meaning of *truth* which we gave for elementary judgments "elementary truth." Then when we assert that it is true that all men are mortal, we shall mean that all judgments of the form "x is mortal," where x is a man, have elementary truth. We may define this as "truth of the second order" or "second-order truth." Then if we express the proposition "all men are mortal" in the form

"$(x) \,.\, x$ is mortal, where x is a man,"

and call this judgment p, then "p is true" must be taken to mean "p has second-order truth," which in turn means

"$(x) \,.\,$ 'x is mortal' has elementary truth, where x is a man."

In order to avoid the necessity for stating explicitly the limitation to which our variable is subject, it is convenient to replace the above interpretation of "all men are mortal" by a slightly different interpretation. The proposition "all men are mortal" is equivalent to "'x is a man' implies 'x is mortal,' with all possible values of x." Here x is not restricted to such values as are men, but may have any value with which "'x is a man' implies 'x is mortal'" is *significant*, *i.e.* either true or false. Such a proposition is called a "formal implication." The advantage of this form is that the values which the variable may take are given by the function to which it is the argument: the

values which the variable may take are all those with which the function is significant.

We use the symbol "$(x) . \phi x$" to express the general judgment which asserts all judgments of the form "ϕx." Then the judgment "all men are mortal" is equivalent to

"(x) . 'x is a man' implies 'x is a mortal,'"

i.e. (in virtue of the definition of implication) to

"$(x) . x$ is not a man or x is mortal."

As we have just seen, the meaning of *truth* which is applicable to this proposition is not the same as the meaning of *truth* which is applicable to "x is a man" or to "x is mortal." And generally, in any judgment $(x) . \phi x$, the sense in which this judgment is or may be true is not the same as that in which ϕx is or may be true. If ϕx is an elementary judgment, it is true when it *points to* a corresponding complex. But $(x) . \phi x$ does not point to a single corresponding complex: the corresponding complexes are as numerous as the possible values of x.

It follows from the above that such a proposition as "all the judgments made by Epimenides are true" will only be prima facie capable of truth if all his judgments are of the same order. If they are of varying orders, of which the nth is the highest, we may make n assertions of the form "all the judgments of order m made by Epimenides are true," where m has all values up to n. But no such judgment can include itself in its own scope, since such a judgment is always of higher order than the judgments to which it refers.

Let us consider next what is meant by the negation of a proposition of the form "$(x) . \phi x$." We observe, to begin with, that "ϕx in some cases," or "ϕx sometimes," is a judgment which is on a par with "ϕx in all cases," or "ϕx always." The judgment "ϕx sometimes" is true if one or more values of x exist for which ϕx is true. We will express the proposition "ϕx sometimes" by the notation "$(\exists x) . \phi x$," where "$\exists$" stands for "there exists," and the whole symbol may be read "there exists an x such that ϕx." We take the two kinds of judgment expressed by "$(x) . \phi x$" and "$(\exists x) . \phi x$" as primitive ideas. We also take as a primitive idea the negation of an *elementary* proposition. We can then define the negations of $(x) . \phi x$ and $(\exists x) . \phi x$. The negation of any proposition p will be denoted by the symbol "$\sim p$." Then the negation of $(x) . \phi x$ will be *defined* as meaning

"$(\exists x) . \sim \phi x$,"

and the negation of $(\exists x) . \phi x$ will be *defined* as meaning "$(x) . \sim \phi x$." Thus, in the traditional language of formal logic, the negation of a universal affirmative is to be defined as the particular negative, and the negation of the particular affirmative is to be defined as the universal negative. Hence the meaning of negation for such propositions is different from the meaning of negation for elementary propositions.

An analogous explanation will apply to disjunction. Consider the statement "either p, or ϕx always." We will denote the disjunction of two propositions p, q by "$p \vee q$." Then our statement is "$p . \vee . (x) . \phi x$." We will suppose that p is an elementary proposition, and that ϕx is always an elementary proposition. We take the disjunction of two elementary propositions as a primitive idea, and we wish to *define* the disjunction

$$\text{“}p . \vee . (x) . \phi x.\text{”}$$

This may be defined as "$(x) . p \vee \phi x$," *i.e.* "either p is true, or ϕx is always true" is to mean "'p or ϕx' is always true." Similarly we will define

$$\text{“}p . \vee . (\exists x) . \phi x\text{”}$$

as meaning "$(\exists x) . p \vee \phi x$," *i.e.* we define "either p is true or there is an x for which ϕx is true" as meaning "there is an x for which either p or ϕx is true." Similarly we can define a disjunction of two universal propositions: "$(x) . \phi x . \vee . (y) . \psi y$" will be defined as meaning "$(x, y) . \phi x \vee \psi y$," *i.e.* "either ϕx is always true or ψy is always true" is to mean "'ϕx or ψy' is always true." By this method we obtain definitions of disjunctions containing propositions of the form $(x) . \phi x$ or $(\exists x) . \phi x$ in terms of disjunctions of elementary propositions; but the meaning of "disjunction" is not the same for propositions of the forms $(x) . \phi x$, $(\exists x) . \phi x$, as it was for elementary propositions.

Similar explanations could be given for implication and conjunction, but this is unnecessary, since these can be defined in terms of negation and disjunction.

IV. *Why a Given Function requires Arguments of a Certain Type.*

The considerations so far adduced in favour of the view that a function cannot significantly have as argument anything defined in terms of the function itself have been more or less indirect. But a direct consideration of the kinds of functions which have functions as arguments and the kinds of functions which have arguments other than functions will show, if we are not mistaken, that not only is it impossible for a function $\phi \hat{z}$ to have itself or anything derived from it as argument, but that, if $\psi \hat{z}$ is another function such that there are arguments a with which both "ϕa" and "ψa" are significant, then $\psi \hat{z}$ and anything derived from it cannot significantly be argument to $\phi \hat{z}$. This arises from the fact that a function is essentially an ambiguity, and that, if it is to occur in a definite proposition, it must occur in such a way that the ambiguity has disappeared, and a wholly unambiguous statement has resulted. A few illustrations will make this clear. Thus "$(x) . \phi x$," which we have already considered, is a function of $\phi \hat{x}$; as soon as $\phi \hat{x}$ is assigned, we have a definite proposition, wholly free from ambiguity. But it is obvious that we cannot substitute for the function something which is not a function: "$(x) . \phi x$" means "ϕx in all cases," and depends for its significance upon the fact that there are "cases" of ϕx, *i.e.* upon the

ambiguity which is characteristic of a function. This instance illustrates the fact that, when a function can occur significantly as argument, something which is not a function cannot occur significantly as argument. But conversely, when something which is not a function can occur significantly as argument, a function cannot occur significantly. Take, *e.g.* "x is a man," and consider "$\phi\hat{x}$ is a man." Here there is nothing to eliminate the ambiguity which constitutes $\phi\hat{x}$; there is thus nothing definite which is said to be a man. A function, in fact, is not a definite object, which could be or not be a man; it is a mere ambiguity awaiting determination, and in order that it may occur significantly it must receive the necessary determination, which it obviously does not receive if it is merely substituted for something determinate in a proposition*. This argument does not, however, apply directly as against such a statement as "$\{(x) . \phi x\}$ is a man." Common sense would pronounce such a statement to be meaningless, but it cannot be condemned on the ground of ambiguity in its subject. We need here a new objection, namely the following: A proposition is not a single entity, but a relation of several; hence a statement in which a proposition appears as subject will only be significant if it can be reduced to a statement about the terms which appear in the proposition. A proposition, like such phrases as "the so-and-so," where grammatically it appears as subject, must be broken up into its constituents if we are to find the true subject or subjects†. But in such a statement as "p is a man," where p is a proposition, this is not possible. Hence "$\{(x) . \phi x\}$ is a man" is meaningless.

V. *The Hierarchy of Functions and Propositions.*

We are thus led to the conclusion, both from the vicious-circle principle and from direct inspection, that the functions to which a given object a can be an argument are incapable of being arguments to each other, and that they have no term in common with the functions to which they can be arguments. We are thus led to construct a hierarchy. Beginning with a and the other terms which can be arguments to the same functions to which a can be argument, we come next to functions to which a is a possible argument, and then to functions to which such functions are possible arguments, and so on. But the hierarchy which has to be constructed is not so simple as might at first appear. The functions which can take a as argument form an illegitimate totality, and themselves require division into a hierarchy of functions. This is easily seen as follows. Let $f(\phi\hat{z}, x)$ be a function of the two variables $\phi\hat{z}$ and x. Then if, keeping x fixed for the moment, we assert this with all possible values of ϕ, we obtain a proposition:

$$(\phi) . f(\phi\hat{z}, x).$$

* Note that statements concerning the significance of a phrase containing "$\phi\hat{z}$" concern the *symbol* "$\phi\hat{z}$," and therefore do not fall under the rule that the elimination of the functional ambiguity is necessary to significance. Significance is a property of signs. Cf. pp. 40, 41.

† Cf. Chapter III.

Here, if x is variable, we have a function of x; but as this function involves a totality of values of $\phi\hat{z}$*, it cannot itself be one of the values included in the totality, by the vicious-circle principle. It follows that the totality of values of $\phi\hat{z}$ concerned in $(\phi) . f(\phi\hat{z}, x)$ is not the totality of all functions in which x can occur as argument, and that there is no such totality as that of all functions in which x can occur as argument.

It follows from the above that a function in which $\phi\hat{z}$ appears as argument requires that "$\phi\hat{z}$" should not stand for *any* function which is capable of a given argument, but must be restricted in such a way that none of the functions which are possible values of "$\phi\hat{z}$" should involve any reference to the totality of such functions. Let us take as an illustration the definition of identity. We might attempt to define "x is identical with y" as meaning "whatever is true of x is true of y," *i.e.* "ϕx always implies ϕy." But here, since we are concerned to assert all values of "ϕx implies ϕy" regarded as a function of ϕ, we shall be compelled to impose upon ϕ some limitation which will prevent us from including among values of ϕ values in which "all possible values of ϕ" are referred to. Thus for example "x is identical with a" is a function of x; hence, if it is a legitimate value of ϕ in "ϕx always implies ϕy," we shall be able to infer, by means of the above definition, that if x is identical with a, and x is identical with y, then y is identical with a. Although the conclusion is sound, the reasoning embodies a vicious-circle fallacy, since we have taken "$(\phi) . \phi x$ implies ϕa" as a possible value of ϕx, which it cannot be. If, however, we impose any limitation upon ϕ, it may happen, so far as appears at present, that with other values of ϕ we might have ϕx true and ϕy false, so that our proposed definition of identity would plainly be wrong. This difficulty is avoided by the "axiom of reducibility," to be explained later. For the present, it is only mentioned in order to illustrate the necessity and the relevance of the hierarchy of functions of a given argument.

Let us give the name "a-functions" to functions that are significant for a given argument a. Then suppose we take any selection of a-functions, and consider the proposition "a satisfies all the functions belonging to the selection in question." If we here replace a by a variable, we obtain an a-function; but by the vicious-circle principle this a-function cannot be a member of our selection, since it refers to the whole of the selection. Let the selection consist of all those functions which satisfy $f(\phi\hat{z})$. Then our new function is

$$(\phi) . \{f(\phi\hat{z}) \text{ implies } \phi x\},$$

where x is the argument. It thus appears that, whatever selection of a-functions we may make, there will be other a-functions that lie outside our

* When we speak of "values *of* $\phi\hat{z}$" it is ϕ, not z, that is to be assigned. This follows from the explanation in the note on p. 40. When the function itself is the variable, it is possible and simpler to write ϕ rather than $\phi\hat{z}$, except in positions where it is necessary to emphasize that an argument must be supplied to secure significance.

selection. Such α-functions, as the above instance illustrates, will always arise through taking a function of two arguments, $\phi\hat{z}$ and x, and asserting all or some of the values resulting from varying ϕ. What is necessary, therefore, in order to avoid vicious-circle fallacies, is to divide our α-functions into "types," each of which contains no functions which refer to the whole of that type.

When something is asserted or denied about all possible values or about some (undetermined) possible values of a variable, that variable is called *apparent*, after Peano. The presence of the words *all* or *some* in a proposition indicates the presence of an apparent variable; but often an apparent variable is really present where language does not at once indicate its presence. Thus for example "A is mortal" means "there is a time at which A will die." Thus a variable time occurs as apparent variable.

The clearest instances of propositions not containing apparent variables are such as express immediate judgments of perception, such as "this is red" or "this is painful," where "this" is something immediately given. In other judgments, even where at first sight no variable appears to be present, it often happens that there really is one. Take (say) "Socrates is human." To Socrates himself, the word "Socrates" no doubt stood for an object of which he was immediately aware, and the judgment "Socrates is human" contained no apparent variable. But to us, who only know Socrates by description, the word "Socrates" cannot mean what it meant to him; it means rather "the person having such-and-such properties," (say) "the Athenian philosopher who drank the hemlock." Now in all propositions about "the so-and-so" there is an apparent variable, as will be shown in Chapter III. Thus in what *we* have in mind when we say "Socrates is human" there is an apparent variable, though there was no apparent variable in the corresponding judgment as made by Socrates, provided we assume that there is such a thing as immediate awareness of oneself.

Whatever may be the instances of propositions not containing apparent variables, it is obvious that propositional functions whose values do not contain apparent variables are the source of propositions containing apparent variables, in the sense in which the function $\phi\hat{x}$ is the source of the proposition $(x) \,.\, \phi x$. For the values for $\phi\hat{x}$ do not contain the apparent variable x, which appears in $(x) \,.\, \phi x$; if they contain an apparent variable y, this can be similarly eliminated, and so on. This process must come to an end, since no proposition which we can apprehend can contain more than a finite number of apparent variables, on the ground that whatever we can apprehend must be of finite complexity. Thus we must arrive at last at a function of as many variables as there have been stages in reaching it from our original proposition, and this function will be such that its values contain no apparent variables. We may call this function the *matrix* of our original proposition and of any other

propositions and functions to be obtained by turning some of the arguments to the function into apparent variables. Thus for example, if we have a matrix-function whose values are $\phi(x, y)$, we shall derive from it

$(y) . \phi(x, y)$, which is a function of x,

$(x) . \phi(x, y)$, which is a function of y,

$(x, y) . \phi(x, y)$, meaning "$\phi(x, y)$ is true with all possible values of x and y."

This last is a proposition containing no *real* variable, *i.e.* no variable except apparent variables.

It is thus plain that all possible propositions and functions are obtainable from matrices by the process of turning the arguments to the matrices into apparent variables. In order to divide our propositions and functions into types, we shall, therefore, start from matrices, and consider how they are to be divided with a view to the avoidance of vicious-circle fallacies in the definitions of the functions concerned. For this purpose, we will use such letters as a, b, c, x, y, z, w, to denote objects which are neither propositions nor functions. Such objects we shall call *individuals*. Such objects will be constituents of propositions or functions, and will be *genuine* constituents, in the sense that they do not disappear on analysis, as (for example) classes do, or phrases of the form "the so-and-so."

The first matrices that occur are those whose values are of the forms

$$\phi x, \ \psi(x, y), \ \chi(x, y, z \ldots),$$

i.e. where the arguments, however many there may be, are all individuals. The functions ϕ, ψ, $\chi \ldots$, since (by definition) they contain no apparent variables, and have no arguments except individuals, do not presuppose any totality of functions. From the functions ψ, $\chi \ldots$ we may proceed to form other functions of x, such as $(y) . \psi(x, y)$, $(\exists y) . \psi(x, y)$, $(y, z) . \chi(x, y, z)$, $(y) : (\exists z) . \chi(x, y, z)$, and so on. All these presuppose no totality except that of individuals. We thus arrive at a certain collection of functions of x, characterized by the fact that they involve no variables except individuals. Such functions we will call "*first-order* functions."

We may now introduce a notation to express "any first-order function." We will denote any first-order function by "$\phi ! \hat{x}$" and any value for such a function by "$\phi ! x$." Thus "$\phi ! x$" stands for any value for any function which involves no variables except individuals. It will be seen that "$\phi ! x$" is itself a function of *two* variables, namely $\phi ! \hat{z}$ and x. Thus $\phi ! x$ involves a variable which is not an individual, namely $\phi ! \hat{z}$. Similarly "$(x) . \phi ! x$" is a function of the variable $\phi ! \hat{z}$, and thus involves a variable other than an individual. Again, if a is a given individual,

"$\phi ! x$ implies $\phi ! a$ with all possible values of ϕ"

is a function of x, but it is not a function of the form $\phi ! x$, because it involves an (apparent) variable ϕ which is not an individual. Let us give the name "predicate" to any first-order function $\phi ! \hat{x}$. (This use of the word "predicate"

is only proposed for the purposes of the present discussion.) Then the statement "$\phi ! x$ implies $\phi ! a$ with all possible values of ϕ" may be read "all the predicates of x are predicates of a." This makes a statement about x, but does not attribute to x a *predicate* in the special sense just defined.

Owing to the introduction of the variable first-order function $\phi ! \hat{z}$, we now have a new set of matrices. Thus "$\phi ! x$" is a function which contains no apparent variables, but contains the two real variables $\phi ! \hat{z}$ and x. (It should be observed that when ϕ is assigned, we may obtain a function whose values do involve individuals as apparent variables, for example if $\phi ! x$ is $(y) . \psi (x, y)$. But so long as ϕ is variable, $\phi ! x$ contains no apparent variables.) Again, if a is a definite individual, $\phi ! a$ is a function of the one variable $\phi ! \hat{z}$. If a and b are definite individuals, "$\phi ! a$ implies $\psi ! b$" is a function of the two variables $\phi ! \hat{z}$, $\psi ! \hat{z}$, and so on. We are thus led to a whole set of new matrices,

$$f (\phi ! \hat{z}),\ g (\phi ! \hat{z}, \psi ! \hat{z}),\ F (\phi ! \hat{z}, x), \text{ and so on.}$$

These matrices contain individuals and first-order functions as arguments, but (like all matrices) they contain no apparent variables. Any such matrix, if it contains more than one variable, gives rise to new functions of one variable by turning all its arguments except one into apparent variables. Thus we obtain the functions

$(\phi) . g (\phi ! \hat{z}, \psi ! \hat{z})$, which is a function of $\psi ! \hat{z}$.

$(x) . F (\phi ! \hat{z}, x)$, which is a function of $\phi ! \hat{z}$.

$(\phi) . F (\phi ! \hat{z}, x)$, which is a function of x.

We will give the name of *second-order matrices* to such matrices as have first-order functions among their arguments, and have no arguments except first-order functions and individuals. (It is not *necessary* that they should have individuals among their arguments.) We will give the name of *second-order functions* to such as either are second-order matrices or are derived from such matrices by turning some of the arguments into apparent variables. It will be seen that either an individual or a first-order function may appear as argument to a second-order function. Second-order functions are such as contain variables which are first-order functions, but contain no other variables except (possibly) individuals.

We now have various new classes of functions at our command. In the first place, we have second-order functions which have one argument which is a first-order function. We will denote a variable function of this kind by the notation $f ! (\hat{\phi} ! \hat{z})$, and any value of such a function by $f ! (\phi ! \hat{z})$. Like $\phi ! x$, $f ! (\phi ! \hat{z})$ is a function of two variables, namely $f ! (\hat{\phi} ! \hat{z})$ and $\phi ! \hat{z}$. Among possible values of $f ! (\phi ! \hat{z})$ will be $\phi ! a$ (where a is constant), $(x) . \phi ! x$, $(\exists x) . \phi ! x$, and so on. (These result from assigning a value to f, leaving ϕ to be assigned.) We will call such functions "predicative functions of first-order functions."

In the second place, we have second-order functions of two arguments, one of which is a first-order function while the other is an individual. Let us denote undetermined values of such functions by the notation

$$f\,!\,(\phi\,!\,\hat{z}, x).$$

As soon as x is assigned, we shall have a predicative function of $\phi\,!\,\hat{z}$. If our function contains no first-order function as apparent variable, we shall obtain a predicative function of x if we assign a value to $\phi\,!\,\hat{z}$. Thus, to take the simplest possible case, if $f\,!\,(\phi\,!\,\hat{z}, x)$ is $\phi\,!\,x$, the assignment of a value to ϕ gives us a predicative function of x, in virtue of the definition of "$\phi\,!\,x$." But if $f\,!\,(\phi\,!\,\hat{z}, x)$ contains a first-order function as apparent variable, the assignment of a value to $\phi\,!\,\hat{z}$ gives us a second-order function of x.

In the third place, we have second-order functions of individuals. These will all be derived from functions of the form $f\,!\,(\phi\,!\,\hat{z}, x)$ by turning ϕ into an apparent variable. We do not, therefore, need a new notation for them.

We have also second-order functions of two first-order functions, or of two such functions and an individual, and so on.

We may now proceed in exactly the same way to third-order matrices, which will be functions containing second-order functions as arguments, and containing no apparent variables, and no arguments except individuals and first-order functions and second-order functions. Thence we shall proceed, as before, to third-order functions; and so we can proceed indefinitely. If the highest order of variable occurring in a function, whether as argument or as apparent variable, is a function of the nth order, then the function in which it occurs is of the $n+1$th order. We do not arrive at functions of an infinite order, because the number of arguments and of apparent variables in a function must be finite, and therefore every function must be of a finite order. Since the orders of functions are only defined step by step, there can be no process of "proceeding to the limit," and functions of an infinite order cannot occur.

We will define a function of one variable as *predicative* when it is of the next order above that of its argument, *i.e.* of the lowest order compatible with its having that argument. If a function has several arguments, and the highest order of function occurring among the arguments is the nth, we call the function predicative if it is of the $n+1$th order, *i.e.* again, if it is of the lowest order compatible with its having the arguments it has. A function of several arguments is predicative if there is one of its arguments such that, when the other arguments have values assigned to them, we obtain a predicative function of the one undetermined argument.

It is important to observe that all possible functions in the above hierarchy can be obtained by means of predicative functions and apparent variables. Thus, as we saw, second-order functions of an individual x are of the form

$(\phi) \,.\, f\,!\,(\phi\,!\,\hat{z}, x)$ or $(\exists\phi) \,.\, f\,!\,(\phi\,!\,\hat{z}, x)$ or $(\phi, \psi) \,.\, f\,!\,(\phi\,!\,\hat{z}, \psi\,!\,\hat{z}, x)$ or etc.,

where f is a second-order predicative function. And speaking generally, a

non-predicative function of the nth order is obtained from a predicative function of the nth order by turning all the arguments of the $n-1$th order into apparent variables. (Other arguments also may be turned into apparent variables.) Thus we need not introduce as variables any functions except predicative functions. Moreover, to obtain any function of one variable x, we need not go beyond predicative functions of *two* variables. For the function $(\psi) . f ! (\phi ! \hat{z}, \psi ! \hat{z}, x)$, where f is given, is a function of $\phi ! \hat{z}$ and x, and is predicative. Thus it is of the form $F ! (\phi ! \hat{z}, x)$, and therefore $(\phi, \psi) . f ! (\phi ! \hat{z}, \psi ! \hat{z}, x)$ is of the form $(\phi) . F ! (\phi ! \hat{z}, x)$. Thus speaking generally, by a succession of steps we find that, if $\phi ! \hat{u}$ is a predicative function of a sufficiently high order, any assigned non-predicative function of x will be of one of the two forms

$$(\phi) . F ! (\phi ! \hat{u}, x), \ (\exists \phi) . F ! (\phi ! \hat{u}, x),$$

where F is a predicative function of $\phi ! \hat{u}$ and x.

The nature of the above hierarchy of functions may be restated as follows. A function, as we saw at an earlier stage, presupposes as part of its meaning the totality of its values, or, what comes to the same thing, the totality of its possible arguments. The arguments to a function may be functions or propositions or individuals. (It will be remembered that individuals were defined as whatever is neither a proposition nor a function.) For the present we neglect the case in which the argument to a function is a proposition. Consider a function whose argument is an individual. This function presupposes the totality of individuals; but unless it contains functions as apparent variables, it does not presuppose any totality of functions. If, however, it does contain a function as apparent variable, then it cannot be defined until some totality of functions has been defined. It follows that we must first define the totality of those functions that have individuals as arguments and contain no functions as apparent variables. These are the *predicative* functions of individuals. Generally, a predicative function of a variable argument is one which involves no totality except that of the possible values of the argument, and those that are presupposed by any one of the possible arguments. Thus a predicative function of a variable argument is any function which can be specified without introducing new kinds of variables not necessarily presupposed by the variable which is the argument.

A closely analogous treatment can be developed for propositions. Propositions which contain no functions and no apparent variables may be called *elementary propositions*. Propositions which are not elementary, which contain no functions, and no apparent variables except individuals, may be called *first-order propositions*. (It should be observed that no variables except *apparent* variables can occur in a proposition, since whatever contains a *real* variable is a function, not a proposition.) Thus elementary and first-order propositions will be values of first-order functions. (It should be remembered

that a function is not a constituent in one of its values: thus for example the function "$\hat{x}$ is human" is not a constituent of the proposition "Socrates is human.") Elementary and first-order propositions presuppose no totality except (at most) the totality of individuals. They are of one or other of the three forms

$$\phi ! x; \ (x) . \phi ! x; \ (\exists x) . \phi ! x,$$

where $\phi ! x$ is a predicative function of an individual. If follows that, if p represents a variable elementary proposition or a variable first-order proposition, a function fp is either $f(\phi ! x)$ or $f\{(x) . \phi ! x\}$ or $f\{(\exists x) . \phi ! x\}$. Thus a function of an elementary or a first-order proposition may always be reduced to a function of a first-order function. It follows that a proposition involving the totality of first-order propositions may be reduced to one involving the totality of first-order functions; and this obviously applies equally to higher orders. The propositional hierarchy can, therefore, be derived from the functional hierarchy, and we may define a proposition of the nth order as one which involves an apparent variable of the $n-1$th order in the functional hierarchy. The propositional hierarchy is never required in practice, and is only relevant for the solution of paradoxes; hence it is unnecessary to go into further detail as to the types of propositions.

VI. *The Axiom of Reducibility.*

It remains to consider the "axiom of reducibility." It will be seen that, according to the above hierarchy, no statement can be made significantly about "all a-functions," where a is some given object. Thus such a notion as "all properties of a," meaning "all functions which are true with the argument a," will be illegitimate. We shall have to distinguish the order of function concerned. We can speak of "all predicative properties of a," "all second-order properties of a," and so on. (If a is not an individual, but an object of order n, "second-order properties of a" will mean "functions of order $n+2$ satisfied by a.") But we cannot speak of "all properties of a." In some cases, we can see that some statement will hold of "all nth-order properties of a," whatever value n may have. In such cases, no practical harm results from regarding the statement as being about "all properties of a," provided we remember that it is really a number of statements, and not a single statement which could be regarded as assigning another property to a, over and above all properties. Such cases will always involve some systematic ambiguity, such as that involved in the meaning of the word "truth," as explained above. Owing to this systematic ambiguity, it will be possible, sometimes, to combine into a single verbal statement what are really a number of different statements, corresponding to different orders in the hierarchy. This is illustrated in the case of the liar, where the statement "all A's statements are false" should be broken up into different statements referring to his statements of various orders, and attributing to each the appropriate kind of falsehood.

The axiom of reducibility is introduced in order to legitimate a great mass of reasoning, in which, prima facie, we are concerned with such notions as "all properties of a" or "all a-functions," and in which, nevertheless, it seems scarcely possible to suspect any substantial error. In order to state the axiom, we must first define what is meant by "formal equivalence." Two functions $\phi\hat{x}$, $\psi\hat{x}$ are said to be "formally equivalent" when, with every possible argument x, ϕx is equivalent to ψx, *i.e.* ϕx and ψx are either both true or both false. Thus two functions are formally equivalent when they are satisfied by the same set of arguments. The axiom of reducibility is the assumption that, given any function $\phi\hat{x}$, there is a formally equivalent *predicative* function, *i.e.* there is a predicative function which is true when ϕx is true and false when ϕx is false. In symbols, the axiom is:

$$\vdash : (\exists \psi) : \phi x \,.\, \equiv_x \,.\, \psi \,!\, x.$$

For two variables, we require a similar axiom, namely: Given any function $\phi(\hat{x}, \hat{y})$, there is a formally equivalent *predicative* function, *i.e.*

$$\vdash : (\exists \psi) : \phi(x, y) \,.\, \equiv_{x,y} \,.\, \psi \,!\, (x, y).$$

In order to explain the purposes of the axiom of reducibility, and the nature of the grounds for supposing it true, we shall first illustrate it by applying it to some particular cases.

If we call a *predicate* of an object a predicative function which is true of that object, then the predicates of an object are only some among its properties. Take for example such a proposition as "Napoleon had all the qualities that make a great general." We may interpret this as meaning "Napoleon had all the predicates that make a great general." Here there is a predicate which is an apparent variable. If we put "$f(\phi \,!\, \hat{z})$" for "$\phi \,!\, \hat{z}$ is a predicate required in a great general," our proposition is

$$(\phi) : f(\phi \,!\, \hat{z}) \text{ implies } \phi \,!\, (\text{Napoleon}).$$

Since this refers to a totality of predicates, it is not itself a predicate of Napoleon. It by no means follows, however, that there is not some one predicate common and peculiar to great generals. In fact, it is certain that there is such a predicate. For the number of great generals is finite, and each of them certainly possessed some predicate not possessed by any other human being —for example, the exact instant of his birth. The disjunction of such predicates will constitute a predicate common and peculiar to great generals*. If we call this predicate $\psi \,!\, \hat{z}$, the statement we made about Napoleon was equivalent to $\psi \,!\, (\text{Napoleon})$. And this equivalence holds equally if we substitute any other individual for Napoleon. Thus we have arrived at a predicate which is always equivalent to the property we ascribed to Napoleon, *i.e.* it belongs to those objects which have this property, and to no others. The axiom of reducibility states that such a predicate always exists, *i.e.* that any property

* When a (finite) set of predicates is given by actual enumeration, their disjunction is a predicate, because no predicate occurs as apparent variable in the disjunction.

of an object belongs to the same collection of objects as those that possess some predicate.

We may next illustrate our principle by its application to *identity*. In this connection, it has a certain affinity with Leibniz's identity of indiscernibles. It is plain that, if x and y are identical, and ϕx is true, then ϕy is true. Here it cannot matter what sort of function $\phi \hat{x}$ may be: the statement must hold for *any* function. But we cannot say, conversely: "If, with all values of ϕ, ϕx implies ϕy, then x and y are identical"; because "all values of ϕ" is inadmissible. If we wish to speak of "all values of ϕ," we must confine ourselves to functions of one order. We may confine ϕ to predicates, or to second-order functions, or to functions of any order we please. But we must necessarily leave out functions of all but one order. Thus we shall obtain, so to speak, a hierarchy of different degrees of identity. We may say "all the predicates of x belong to y," "all second-order properties of x belong to y," and so on. Each of these statements implies all its predecessors: for example, if all second-order properties of x belong to y, then all predicates of x belong to y, for to have all the predicates of x is a second-order property, and this property belongs to x. But we cannot, without the help of an axiom, argue conversely that if all the predicates of x belong to y, all the second-order properties of x must also belong to y. Thus we cannot, without the help of an axiom, be sure that x and y are identical if they have the same predicates. Leibniz's identity of indiscernibles supplied this axiom. It should be observed that by "indiscernibles" he cannot have meant two objects which agree as to *all* their properties, for one of the properties of x is to be identical with x, and therefore this property would necessarily belong to y if x and y agreed in *all* their properties. Some limitation of the common properties necessary to make things indiscernible is therefore implied by the necessity of an axiom. For purposes of illustration (not of interpreting Leibniz) we may suppose the common properties required for indiscernibility to be limited to predicates. Then the identity of indiscernibles will state that if x and y agree as to all their predicates, they are identical. This can be proved if we assume the axiom of reducibility. For, in that case, every property belongs to the same collection of objects as is defined by some predicate. Hence there is some predicate common and peculiar to the objects which are identical with x. This predicate belongs to x, since x is identical with itself; hence it belongs to y, since y has all the predicates of x; hence y is identical with x. It follows that we may *define* x and y as identical when all the predicates of x belong to y, *i.e.* when $(\phi) : \phi ! x . \supset . \phi ! y$. We therefore adopt the following definition of identity*:

$$x = y . = : (\phi) : \phi ! x . \supset . \phi ! y \quad \text{Df.}$$

* Note that in this definition the second sign of equality is to be regarded as combining with "Df" to form one symbol; what is defined is the sign of equality *not* followed by the letters "Df."

But apart from the axiom of reducibility, or some axiom equivalent in this connection, we should be compelled to regard identity as indefinable, and to admit (what seems impossible) that two objects may agree in all their predicates without being identical.

The axiom of reducibility is even more essential in the theory of classes. It should be observed, in the first place, that if we assume the existence of classes, the axiom of reducibility can be proved. For in that case, given any function $\phi\hat{z}$ of whatever order, there is a class α consisting of just those objects which satisfy $\phi\hat{z}$. Hence "ϕx" is equivalent to "x belongs to α." But "x belongs to α" is a statement containing no apparent variable, and is therefore a predicative function of x. Hence if we assume the existence of classes, the axiom of reducibility becomes unnecessary. The assumption of the axiom of reducibility is therefore a smaller assumption than the assumption that there are classes. This latter assumption has hitherto been made unhesitatingly. However, both on the ground of the contradictions, which require a more complicated treatment if classes are assumed, and on the ground that it is always well to make the smallest assumption required for proving our theorems, we prefer to assume the axiom of reducibility rather than the existence of classes. But in order to explain the use of the axiom in dealing with classes, it is necessary first to explain the theory of classes, which is a topic belonging to Chapter III. We therefore postpone to that Chapter the explanation of the use of our axiom in dealing with classes.

It is worth while to note that all the purposes served by the axiom of reducibility are equally well served if we assume that there is always a function of the nth order (where n is fixed) which is formally equivalent to $\phi\hat{x}$, whatever may be the order of $\phi\hat{x}$. Here we shall mean by "a function of the nth order" a function of the nth order relative to the arguments to $\phi\hat{x}$; thus if these arguments are absolutely of the mth order, we assume the existence of a function formally equivalent to $\phi\hat{x}$ whose absolute order is the $m+n$th. The axiom of reducibility in the form assumed above takes $n=1$, but this is not necessary to the use of the axiom. It is also unnecessary that n should be the same for different values of m; what is necessary is that n should be constant so long as m is constant. What is needed is that, where extensional functions of functions are concerned, we should be able to deal with any a-function by means of some formally equivalent function of a given type, so as to be able to obtain results which would otherwise require the illegitimate notion of "all a-functions"; but it does not matter what the given type is. It does not appear, however, that the axiom of reducibility is rendered appreciably more plausible by being put in the above more general but more complicated form.

The axiom of reducibility is equivalent to the assumption that "any

combination or disjunction of predicates* is equivalent to a single predicate," *i.e.* to the assumption that, if we assert that x has all the predicates that satisfy a function $f(\phi ! \hat{z})$, there is some one predicate which x will have whenever our assertion is true, and will not have whenever it is false, and similarly if we assert that x has some one of the predicates that satisfy a function $f(\phi ! \hat{z})$. For by means of this assumption, the order of a non-predicative function can be lowered by one; hence, after some finite number of steps, we shall be able to get from any non-predicative function to a formally equivalent predicative function. It does not seem probable that the above assumption could be substituted for the axiom of reducibility in symbolic deductions, since its use would require the explicit introduction of the further assumption that by a finite number of downward steps we can pass from any function to a predicative function, and this assumption could not well be made without developments that are scarcely possible at an early stage. But on the above grounds it seems plain that in fact, if the above alternative axiom is true, so is the axiom of reducibility. The converse, which completes the proof of equivalence, is of course evident.

VII. *Reasons for Accepting the Axiom of Reducibility.*

That the axiom of reducibility is self-evident is a proposition which can hardly be maintained. But in fact self-evidence is never more than a part of the reason for accepting an axiom, and is never indispensable. The reason for accepting an axiom, as for accepting any other proposition, is always largely inductive, namely that many propositions which are nearly indubitable can be deduced from it, and that no equally plausible way is known by which these propositions could be true if the axiom were false, and nothing which is probably false can be deduced from it. If the axiom is apparently self-evident, that only means, practically, that it is nearly indubitable; for things have been thought to be self-evident and have yet turned out to be false. And if the axiom itself is nearly indubitable, that merely adds to the inductive evidence derived from the fact that its consequences are nearly indubitable: it does not provide new evidence of a radically different kind. Infallibility is never attainable, and therefore some element of doubt should always attach to every axiom and to all its consequences. In formal logic, the element of doubt is less than in most sciences, but it is not absent, as appears from the fact that the paradoxes followed from premisses which were not previously known to require limitations. In the case of the axiom of reducibility, the inductive evidence in its favour is very strong, since the reasonings which it permits and the results to which it leads are all such as appear valid. But although it seems very improbable that the axiom should turn out to be false,

* Here the combination or disjunction is supposed to be given intensionally. If given extensionally (*i.e.* by enumeration), no assumption is required; but in this case the number of predicates concerned must be finite.

it is by no means improbable that it should be found to be deducible from some other more fundamental and more evident axiom. It is possible that the use of the vicious-circle principle, as embodied in the above hierarchy of types, is more drastic than it need be, and that by a less drastic use the necessity for the axiom might be avoided. Such changes, however, would not render anything false which had been asserted on the basis of the principles explained above: they would merely provide easier proofs of the same theorems. There would seem, therefore, to be but the slenderest ground for fearing that the use of the axiom of reducibility may lead us into error.

VIII. *The Contradictions.*

We are now in a position to show how the theory of types affects the solution of the contradictions which have beset mathematical logic. For this purpose, we shall begin by an enumeration of some of the more important and illustrative of these contradictions, and shall then show how they all embody vicious-circle fallacies, and are therefore all avoided by the theory of types. It will be noticed that these paradoxes do not relate exclusively to the ideas of number and quantity. Accordingly no solution can be adequate which seeks to explain them merely as the result of some illegitimate use of these ideas. The solution must be sought in some such scrutiny of fundamental logical ideas as has been attempted in the foregoing pages.

(1) The oldest contradiction of the kind in question is the *Epimenides.* Epimenides the Cretan said that all Cretans were liars, and all other statements made by Cretans were certainly lies. Was this a lie? The simplest form of this contradiction is afforded by the man who says "I am lying"; if he is lying, he is speaking the truth, and vice versa.

(2) Let w be the class of all those classes which are not members of themselves. Then, whatever class x may be, "x is a w" is equivalent to "x is not an x." Hence, giving to x the value w, "w is a w" is equivalent to "w is not a w."

(3) Let T be the relation which subsists between two relations R and S whenever R does not have the relation R to S. Then, whatever relations R and S may be, "R has the relation T to S" is equivalent to "R does not have the relation R to S." Hence, giving the value T to both R and S, "T has the relation T to T" is equivalent to "T does not have the relation T to T."

(4) Burali-Forti's contradiction* may be stated as follows: It can be shown that every well-ordered series has an ordinal number, that the series of ordinals up to and including any given ordinal exceeds the given ordinal by one, and (on certain very natural assumptions) that the series of all ordinals (in order of magnitude) is well-ordered. It follows that the series of all

* "Una questione sui numeri transfiniti," *Rendiconti del circolo matematico di Palermo*, Vol. xi. (1897). See *256.

ordinals has an ordinal number, Ω say. But in that case the series of all ordinals including Ω has the ordinal number $\Omega+1$, which must be greater than Ω. Hence Ω is not the ordinal number of all ordinals.

(5) The number of syllables in the English names of finite integers tends to increase as the integers grow larger, and must gradually increase indefinitely, since only a finite number of names can be made with a given finite number of syllables. Hence the names of some integers must consist of at least nineteen syllables, and among these there must be a least. Hence "the least integer not nameable in fewer than nineteen syllables" must denote a definite integer; in fact, it denotes 111,777. But "the least integer not nameable in fewer than nineteen syllables" is itself a name consisting of eighteen syllables; hence the least integer not nameable in fewer than nineteen syllables can be named in eighteen syllables, which is a contradiction*.

(6) Among transfinite ordinals some can be defined, while others can not; for the total number of possible definitions is $\aleph_0$†, while the number of transfinite ordinals exceeds $\aleph_0$. Hence there must be indefinable ordinals, and among these there must be a least. But this is defined as "the least indefinable ordinal," which is a contradiction‡.

(7) Richard's paradox§ is akin to that of the least indefinable ordinal. It is as follows: Consider all decimals that can be defined by means of a finite number of words; let E be the class of such decimals. Then E has $\aleph_0$ terms; hence its members can be ordered as the 1st, 2nd, 3rd, ... Let N be a number defined as follows: If the nth figure in the nth decimal is p, let the nth figure in N be $p+1$ (or 0, if $p=9$). Then N is different from all the members of E, since, whatever finite value n may have, the nth figure in N is different from the nth figure in the nth of the decimals composing E, and therefore N is different from the nth decimal. Nevertheless we have defined N in a finite number of words, and therefore N ought to be a member of E. Thus N both is and is not a member of E.

In all the above contradictions (which are merely selections from an indefinite number) there is a common characteristic, which we may describe as self-reference or reflexiveness. The remark of Epimenides must include itself in its own scope. If *all* classes, provided they are not members of themselves, are members of w, this must also apply to w; and similarly for the

* This contradiction was suggested to us by Mr G. G. Berry of the Bodleian Library.

† $\aleph_0$ is the number of finite integers. See *123.

‡ Cf. König, "Ueber die Grundlagen der Mengenlehre und das Kontinuumproblem," *Math. Annalen*, Vol. LXI. (1905); A. C. Dixon, "On 'well-ordered' aggregates," *Proc. London Math. Soc.* Series 2, Vol. IV. Part I. (1906); and E. W. Hobson, "On the Arithmetic Continuum," *ibid.* The solution offered in the last of these papers depends upon the variation of the "apparatus of definition," and is thus in outline in agreement with the solution adopted here. But it does not invalidate the statement in the text, if "definition" is given a constant meaning.

§ Cf. Poincaré, "Les mathématiques et la logique," *Revue de Métaphysique et de Morale*, Mai 1906, especially sections VII. and IX.; also Peano, *Revista de Mathematica*, Vol. VIII. No. 5 (1906), p. 149 ff.

analogous relational contradiction. In the cases of names and definitions, the paradoxes result from considering non-nameability and indefinability as elements in names and definitions. In the case of Burali-Forti's paradox, the series whose ordinal number causes the difficulty is the series of all ordinal numbers. In each contradiction something is said about *all* cases of some kind, and from what is said a new case seems to be generated, which both is and is not of the same kind as the cases of which *all* were concerned in what was said. But this is the characteristic of illegitimate totalities, as we defined them in stating the vicious-circle principle. Hence all our contradictions are illustrations of vicious-circle fallacies. It only remains to show, therefore, that the illegitimate totalities involved are excluded by the hierarchy of types which we have constructed.

(1) When a man says "I am lying," we may interpret his statement as: "There is a proposition which I am affirming and which is false." That is to say, he is asserting the truth of some value of the function "I assert p, and p is false." But we saw that the word "false" is ambiguous, and that, in order to make it unambiguous, we must specify the order of falsehood, or, what comes to the same thing, the order of the proposition to which falsehood is ascribed. We saw also that, if p is a proposition of the nth order, a proposition in which p occurs as an apparent variable is not of the nth order, but of a higher order. Hence the kind of truth or falsehood which can belong to the statement "there is a proposition p which I am affirming and which has falsehood of the nth order" is truth or falsehood of a higher order than the nth. Hence the statement of Epimenides does not fall within its own scope, and therefore no contradiction emerges.

If we regard the statement "I am lying" as a compact way of simultaneously making all the following statements: "I am asserting a false proposition of the first order," "I am asserting a false proposition of the second order," and so on, we find the following curious state of things: As no proposition of the first order is being asserted, the statement "I am asserting a false proposition of the first order" is false. This statement is of the second order, hence the statement "I am making a false statement of the second order" is true. This is a statement of the third order, and is the only statement of the third order which is being made. Hence the statement "I am making a false statement of the third order" is false. Thus we see that the statement "I am making a false statement of order $2n + 1$" is false, while the statement "I am making a false statement of order $2n$" is true. But in this state of things there is no contradiction.

(2) In order to solve the contradiction about the class of classes which are not members of themselves, we shall assume, what will be explained in the next Chapter, that a proposition about a class is always to be reduced to a statement about a function which defines the class, *i.e.* about a function which

is satisfied by the members of the class and by no other arguments. Thus a class is an object derived from a function and presupposing the function, just as, for example, $(x) . \phi x$ presupposes the function $\phi \hat{x}$. Hence a class cannot, by the vicious-circle principle, significantly be the argument to its defining function, that is to say, if we denote by "$\hat{z}(\phi z)$" the class defined by $\phi \hat{z}$, the symbol "$\phi \{\hat{z}(\phi z)\}$" must be meaningless. Hence a class neither satisfies nor does not satisfy its defining function, and therefore (as will appear more fully in Chapter III) is neither a member of itself nor not a member of itself. This is an immediate consequence of the limitation to the possible arguments to a function which was explained at the beginning of the present Chapter. Thus if α is a class, the statement "α is not a member of α" is always meaningless, and there is therefore no sense in the phrase "the class of those classes which are not members of themselves." Hence the contradiction which results from supposing that there is such a class disappears.

(3) Exactly similar remarks apply to "the relation which holds between R and S whenever R does not have the relation R to S." Suppose the relation R is defined by a function $\phi(x, y)$, *i.e.* R holds between x and y whenever $\phi(x, y)$ is true, but not otherwise. Then in order to interpret "R has the relation R to S," we shall have to suppose that R and S can significantly be the arguments to ϕ. But (assuming, as will appear in Chapter III, that R presupposes its defining function) this would require that ϕ should be able to take as argument an object which is defined in terms of ϕ, and this no function can do, as we saw at the beginning of this Chapter. Hence "R has the relation R to S" is meaningless, and the contradiction ceases.

(4) The solution of Burali-Forti's contradiction requires some further developments for its solution. At this stage, it must suffice to observe that a series is a relation, and an ordinal number is a class of series. (These statements are justified in the body of the work.) Hence a series of ordinal numbers is a relation between classes of relations, and is of higher type than any of the series which are members of the ordinal numbers in question. Burali-Forti's "ordinal number of all ordinals" must be the ordinal number of all ordinals of a given type, and must therefore be of higher type than any of these ordinals. Hence it is not one of these ordinals, and there is no contradiction in its being greater than any of them*.

(5) The paradox about "the least integer not nameable in fewer than nineteen syllables" embodies, as is at once obvious, a vicious-circle fallacy. For the word "nameable" refers to the totality of names, and yet is allowed to occur in what professes to be one among names. Hence there can be no such thing as a totality of names, in the sense in which the paradox speaks

* The solution of Burali-Forti's paradox by means of the theory of types is given in detail in *256.

of "names." It is easy to see that, in virtue of the hierarchy of functions, the theory of types renders a totality of "names" impossible. We may, in fact, distinguish names of different orders as follows: (*a*) Elementary names will be such as are true "proper names," *i.e.* conventional appellations not involving any description. (*b*) First-order names will be such as involve a description by means of a first-order function; that is to say, if $\phi ! \hat{x}$ is a first-order function, "the term which satisfies $\phi ! \hat{x}$" will be a first-order name, though there will not always be an object named by this name. (*c*) Second-order names will be such as involve a description by means of a second-order function; among such names will be those involving a reference to the totality of first-order names. And so we can proceed through a whole hierarchy. But at no stage can we give a meaning to the word "nameable" unless we specify the order of names to be employed; and any name in which the phrase "nameable by names of order n" occurs is necessarily of a higher order than the nth. Thus the paradox disappears.

The solutions of the paradox about the least indefinable ordinal and of Richard's paradox are closely analogous to the above. The notion of "definable," which occurs in both, is nearly the same as "nameable," which occurs in our fifth paradox: "definable" is what "nameable" becomes when elementary names are excluded, *i.e.* "definable" means "nameable by a name which is not elementary." But here there is the same ambiguity as to type as there was before, and the same need for the addition of words which specify the type to which the definition is to belong. And however the type may be specified, "the least ordinal not definable by definitions of this type" is a definition of a higher type; and in Richard's paradox, when we confine ourselves, as we must, to decimals that have a definition of a given type, the number N, which causes the paradox, is found to have a definition which belongs to a higher type, and thus not to come within the scope of our previous definitions.

An indefinite number of other contradictions, of similar nature to the above seven, can easily be manufactured. In all of them, the solution is of the same kind. In all of them, the appearance of contradiction is produced by the presence of some word which has systematic ambiguity of type, such as *truth, falsehood, function, property, class, relation, cardinal, ordinal, name, definition*. Any such word, if its typical ambiguity is overlooked, will apparently generate a totality containing members defined in terms of itself, and will thus give rise to vicious-circle fallacies. In most cases, the conclusions of arguments which involve vicious-circle fallacies will not be self-contradictory, but wherever we have an illegitimate totality, a little ingenuity will enable us to construct a vicious-circle fallacy leading to a contradiction, which disappears as soon as the typically ambiguous words are rendered typically definite, *i.e.* are determined as belonging to this or that type.

Thus the appearance of contradiction is always due to the presence of words embodying a concealed typical ambiguity, and the solution of the apparent contradiction lies in bringing the concealed ambiguity to light.

In spite of the contradictions which result from unnoticed typical ambiguity, it is not desirable to avoid words and symbols which have typical ambiguity. Such words and symbols embrace practically all the ideas with which mathematics and mathematical logic are concerned: the systematic ambiguity is the result of a systematic analogy. That is to say, in almost all the reasonings which constitute mathematics and mathematical logic, we are using ideas which may receive any one of an infinite number of different typical determinations, any one of which leaves the reasoning valid. Thus by employing typically ambiguous words and symbols, we are able to make one chain of reasoning applicable to any one of an infinite number of different cases, which would not be possible if we were to forego the use of typically ambiguous words and symbols.

Among propositions wholly expressed in terms of typically ambiguous notions practically the only ones which may differ, in respect of truth or falsehood, according to the typical determination which they receive, are existence-theorems. If we assume that the total number of individuals is n, then the total number of classes of individuals is 2^n, the total number of classes of classes of individuals is 2^{2^n}, and so on. Here n may be either finite or infinite, and in either case $2^n > n$. Thus cardinals greater than n but not greater than 2^n exist as applied to classes of classes, but not as applied to classes of individuals, so that whatever may be supposed to be the number of individuals, there will be existence-theorems which hold for higher types but not for lower types. Even here, however, so long as the number of individuals is not asserted, but is merely assumed hypothetically, we may replace the type of individuals by any other type, provided we make a corresponding change in all the other types occurring in the same context. That is, we may give the name "relative individuals" to the members of an arbitrarily chosen type τ, and the name "relative classes of individuals" to classes of "relative individuals," and so on. Thus so long as only hypotheticals are concerned, in which existence-theorems for one type are shown to be implied by existence-theorems for another, only *relative* types are relevant even in existence-theorems. This applies also to cases where the hypothesis (and therefore the conclusion) is *asserted*, provided the assertion holds for any type, however chosen. For example, any type has at least one member; hence any type which consists of classes, of whatever order, has at least two members. But the further pursuit of these topics must be left to the body of the work.

CHAPTER III

INCOMPLETE SYMBOLS

(1) *Descriptions.* By an "incomplete" symbol we mean a symbol which is not supposed to have any meaning in isolation, but is only defined in certain contexts. In ordinary mathematics, for example, $\frac{d}{dx}$ and $\int_a^b$ are incomplete symbols: something has to be supplied before we have anything significant. Such symbols have what may be called a "definition in use." Thus if we put

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad \mathrm{Df},$$

we define the *use* of ∇^2, but ∇^2 by itself remains without meaning. This distinguishes such symbols from what (in a generalized sense) we may call *proper names*: "Socrates," for example, stands for a certain man, and therefore has a meaning by itself, without the need of any context. If we supply a context, as in "Socrates is mortal," these words express a fact of which Socrates himself is a constituent: there is a certain object, namely Socrates, which does have the property of mortality, and this object is a constituent of the complex fact which we assert when we say "Socrates is mortal." But in other cases, this simple analysis fails us. Suppose we say: "The round square does not exist." It seems plain that this is a true proposition, yet we cannot regard it as denying the existence of a certain object called "the round square." For if there were such an object, it would exist: we cannot first assume that there is a certain object, and then proceed to deny that there is such an object. Whenever the grammatical subject of a proposition can be supposed not to exist without rendering the proposition meaningless, it is plain that the grammatical subject is not a proper name, *i.e.* not a name directly representing some object. Thus in all such cases, the proposition must be capable of being so analysed that what was the grammatical subject shall have disappeared. Thus when we say "the round square does not exist," we may, as a first attempt at such analysis, substitute "it is false that there is an object x which is both round and square." Generally, when "the so-and-so" is said not to exist, we have a proposition of the form*

$$\text{"} \sim \mathrm{E}\,!\,(\imath x)(\phi x), \text{"}$$

i.e.

$$\sim \{(\exists c) : \phi x \,.\equiv_x .\, x = c\},$$

or some equivalent. Here the apparent grammatical subject $(\imath x)(\phi x)$ has completely disappeared; thus in "$\sim \mathrm{E}\,!\,(\imath x)(\phi x)$," $(\imath x)(\phi x)$ is an *incomplete* symbol.

* Cf. pp. 30, 31.

By an extension of the above argument, it can easily be shown that $(\iota x)(\phi x)$ is *always* an incomplete symbol. Take, for example, the following proposition: "Scott is the author of Waverley." [Here "the author of Waverley" is (ιx) (x wrote Waverley).] This proposition expresses an identity; thus if "the author of Waverley" could be taken as a proper name, and supposed to stand for some object c, the proposition would be "Scott is c." But if c is any one except Scott, this proposition is false; while if c *is* Scott, the proposition is "Scott is Scott," which is trivial, and plainly different from "Scott is the author of Waverley." Generalizing, we see that the proposition

$$a = (\iota x)(\phi x)$$

is one which may be true or may be false, but is never merely trivial, like $a = a$; whereas, if $(\iota x)(\phi x)$ were a proper name, $a = (\iota x)(\phi x)$ would necessarily be either false or the same as the trivial proposition $a = a$. We may express this by saying that $a = (\iota x)(\phi x)$ is not a value of the propositional function $a = y$, from which it follows that $(\iota x)(\phi x)$ is not a value of y. But since y may be anything, it follows that $(\iota x)(\phi x)$ is nothing. Hence, since in use it has meaning, it must be an incomplete symbol.

It might be suggested that "Scott is the author of Waverley" asserts that "Scott" and "the author of Waverley" are two names for the same object. But a little reflection will show that this would be a mistake. For if that were the meaning of "Scott is the author of Waverley," what would be required for its truth would be that Scott should have been *called* the author of Waverley: if he had been so called, the proposition would be true, even if some one else had written Waverley; while if no one called him so, the proposition would be false, even if he had written Waverley. But in fact he was the author of Waverley at a time when no one called him so, and he would not have been the author if every one had called him so but some one else had written Waverley. Thus the proposition "Scott is the author of Waverley" is not a proposition about names, like "Napoleon is Bonaparte"; and this illustrates the sense in which "the author of Waverley" differs from a true proper name.

Thus all phrases (other than propositions) containing the word *the* (in the singular) are incomplete symbols: they have a meaning in use, but not in isolation. For "the author of Waverley" cannot mean the same as "Scott," or "Scott is the author of Waverley" would mean the same as "Scott is Scott," which it plainly does not; nor can "the author of Waverley" mean anything other than "Scott," or "Scott is the author of Waverley" would be false. Hence "the author of Waverley" means nothing.

It follows from the above that we must not attempt to define "$(\iota x)(\phi x)$," but must define the *uses* of this symbol, *i.e.* the propositions in whose symbolic expression it occurs. Now in seeking to define the uses of this symbol, it is important to observe the import of propositions in which it occurs. Take as

an illustration: "The author of Waverley was a poet." This implies (1) that Waverley was written, (2) that it was written by one man, and not in collaboration, (3) that the one man who wrote it was a poet. If any one of these fails, the proposition is false. Thus "the author of 'Slawkenburgius on Noses' was a poet" is false, because no such book was ever written; "the author of 'The Maid's Tragedy' was a poet" is false, because this play was written by Beaumont and Fletcher jointly. These two possibilities of falsehood do not arise if we say "Scott was a poet." Thus our interpretation of the uses of $(\imath x)(\phi x)$ must be such as to allow for them. Now taking ϕx to replace "x wrote Waverley," it is plain that any statement apparently about $(\imath x)(\phi x)$ requires (1) $(\exists x) . (\phi x)$ and (2) $\phi x . \phi y . \supset_{x,y} . x = y$; here (1) states that *at least* one object satisfies ϕx, while (2) states that *at most* one object satisfies ϕx. The two together are equivalent to

$$(\exists c) : \phi x . \equiv_x . x = c,$$

which we defined as $E ! (\imath x)(\phi x)$.

Thus "$E ! (\imath x)(\phi x)$" must be part of what is affirmed by any proposition about $(\imath x)(\phi x)$. If our proposition is $f\{(\imath x)(\phi x)\}$, what is further affirmed is fc, if $\phi x . \equiv_x . x = c$. Thus we have

$$f\{(\imath x)(\phi x)\} . = : (\exists c) : \phi x . \equiv_x . x = c : fc \quad \text{Df},$$

i.e. "the x satisfying ϕx satisfies fx" is to mean: "There is an object c such that ϕx is true when, and only when, x is c, and fc is true," or, more exactly: "There is a c such that 'ϕx' is always equivalent to 'x is c,' and fc." In this, "$(\imath x)(\phi x)$" has completely disappeared; thus "$(\imath x)(\phi x)$" is merely symbolic, and does not directly represent an object, as single small Latin letters are assumed to do*.

The proposition "$a = (\imath x)(\phi x)$" is easily shown to be equivalent to "$\phi x . \equiv_x . x = a$." For, by the definition, it is

$$(\exists c) : \phi x . \equiv_x . x = c : a = c,$$

i.e. "there is a c for which $\phi x . \equiv_x . x = c$, and this c is a," which is equivalent to "$\phi x . \equiv_x . x = a$." Thus "Scott is the author of Waverley" is equivalent to:

"'x wrote Waverley' is always equivalent to 'x is Scott,'"

i.e. "x wrote Waverley" is true when x is Scott and false when x is not Scott.

Thus although "$(\imath x)(\phi x)$" has no meaning by itself, it may be substituted for y in any propositional function fy, and we get a significant proposition, though not a value of fy.

When $f\{(\imath x)(\phi x)\}$, as above defined, forms part of some other proposition, we shall say that $(\imath x)(\phi x)$ has a *secondary* occurrence. When $(\imath x)(\phi x)$ has a secondary occurrence, a proposition in which it occurs may be true even when $(\imath x)(\phi x)$ does not exist. This applies, *e.g.* to the proposition: "There

* We shall generally write "$f(\imath x)(\phi x)$" rather than "$f\{(\imath x)(\phi x)\}$" in future.

is no such person as the King of France." We may interpret this as

$$\sim \{E\,!\,(\iota x)(\phi x)\},$$

or as

$$\sim \{(\exists c) \,.\, c = (\iota x)(\phi x)\},$$

if "ϕx" stands for "x is King of France." In either case, what is asserted is that a proposition p in which $(\iota x)(\phi x)$ occurs is false, and this proposition p is thus part of a larger proposition. The same applies to such a proposition as the following: "If France were a monarchy, the King of France would be of the House of Orleans."

It should be observed that such a proposition as

$$\sim f\{(\iota x)(\phi x)\}$$

is ambiguous; it may deny $f\{(\iota x)(\phi x)\}$, in which case it will be true if $(\iota x)(\phi x)$ does not exist, or it may mean

$$(\exists c) : \phi x \,.\, \equiv_x \,.\, x = c : \sim fc,$$

in which case it can only be true if $(\iota x)(\phi x)$ exists. In ordinary language, the latter interpretation would usually be adopted. For example, the proposition "the King of France is not bald" would usually be rejected as false, being held to mean "the King of France exists and is not bald," rather than "it is false that the King of France exists and is bald." When $(\iota x)(\phi x)$ exists, the two interpretations of the ambiguity give equivalent results; but when $(\iota x)(\phi x)$ does not exist, one interpretation is true and one is false. It is necessary to be able to distinguish these in our notation; and generally, if we have such propositions as

$$\psi(\iota x)(\phi x) \,.\, \supset \,.\, p,$$
$$p \,.\, \supset \,.\, \psi(\iota x)(\phi x),$$
$$\psi(\iota x)(\phi x) \,.\, \supset \,.\, \chi(\iota x)(\phi x),$$

and so on, we must be able by our notation to distinguish whether the whole or only part of the proposition concerned is to be treated as the "$f(\iota x)(\phi x)$" of our definition. For this purpose, we will put "$[(\iota x)(\phi x)]$" followed by dots at the beginning of the part (or whole) which is to be taken as $f(\iota x)(\phi x)$, the dots being sufficiently numerous to bracket off the $f(\iota x)(\phi x)$; *i.e.* $f(\iota x)(\phi x)$ is to be everything following the dots until we reach an equal number of dots not signifying a logical product, or a greater number signifying a logical product, or the end of the sentence, or the end of a bracket enclosing "$[(\iota x)(\phi x)]$." Thus

$$[(\iota x)(\phi x)] \,.\, \psi(\iota x)(\phi x) \,.\, \supset \,.\, p$$

will mean

$$(\exists c) : \phi x \,.\, \equiv_x \,.\, x = c : \psi c : \supset \,.\, p,$$

but

$$[(\iota x)(\phi x)] : \psi(\iota x)(\phi x) \,.\, \supset \,.\, p$$

will mean

$$(\exists c) : \phi x \,.\, \equiv_x \,.\, x = c : \psi c \,.\, \supset \,.\, p.$$

It is important to distinguish these two, for if $(\iota x)(\phi x)$ does not exist, the first is true and the second false. Again

$$[(\iota x)(\phi x)] \,.\, \sim \psi(\iota x)(\phi x)$$

will mean $(\exists c) : \phi x \,.\, \equiv_x . \, x = c : \sim \psi c,$

while $\sim \{[(\imath x)(\phi x)] \,.\, \psi(\imath x)(\phi x)\}$

will mean $\sim \{(\exists c) : \phi x \,.\, \equiv_x . \, x = c : \psi c\}.$

Here again, when $(\imath x)(\phi x)$ does not exist, the first is false and the second true.

In order to avoid this ambiguity in propositions containing $(\imath x)(\phi x)$, we amend our definition, or rather our notation, putting

$$[(\imath x)(\phi x)] \,.\, f(\imath x)(\phi x) \,. = : (\exists c) : \phi x \,.\, \equiv_x . \, x = c : fc \quad \text{Df.}$$

By means of this definition, we avoid any doubt as to the portion of our whole asserted proposition which is to be treated as the "$f(\imath x)(\phi x)$" of the definition. This portion will be called the *scope* of $(\imath x)(\phi x)$. Thus in

$$[(\imath x)(\phi x)] \,.\, f(\imath x)(\phi x) \,.\, \supset . \, p$$

the scope of $(\imath x)(\phi x)$ is $f(\imath x)(\phi x)$; but in

$$[(\imath x)(\phi x)] : f(\imath x)(\phi x) \,.\, \supset . \, p$$

the scope is $f(\imath x)(\phi x) \,.\, \supset . \, p;$

in $\sim \{[(\imath x)(\phi x)] \,.\, f(\imath x)(\phi x)\}$

the scope is $f(\imath x)(\phi x)$; but in

$$[(\imath x)(\phi x)] \,.\, \sim f(\imath x)(\phi x)$$

the scope is $\sim f(\imath x)(\phi x).$

It will be seen that when $(\imath x)(\phi x)$ has the whole of the proposition concerned for its scope, the proposition concerned cannot be true unless $E!(\imath x)(\phi x)$; but when $(\imath x)(\phi x)$ has only part of the proposition concerned for its scope, it may often be true even when $(\imath x)(\phi x)$ does not exist. It will be seen further that when $E!(\imath x)(\phi x)$, we may enlarge or diminish the scope of $(\imath x)(\phi x)$ as much as we please without altering the truth-value of any proposition in which it occurs.

If a proposition contains two descriptions, say $(\imath x)(\phi x)$ and $(\imath x)(\psi x)$, we have to distinguish which of them has the larger scope, *i.e.* we have to distinguish

(1) $[(\imath x)(\phi x)] : [(\imath x)(\psi x)] \,.\, f\{(\imath x)(\phi x), (\imath x)(\psi x)\},$

(2) $[(\imath x)(\psi x)] : [(\imath x)(\phi x)] \,.\, f\{(\imath x)(\phi x), (\imath x)(\psi x)\}.$

The first of these, eliminating $(\imath x)(\phi x)$, becomes

(3) $(\exists c) : \phi x \,.\, \equiv_x . \, x = c : [(\imath x)(\psi x)] \,.\, f\{c, (\imath x)(\psi x)\},$

which, eliminating $(\imath x)(\psi x)$, becomes

(4) $(\exists c) :. \phi x \,.\, \equiv_x . \, x = c :. (\exists d) : \psi x \,.\, \equiv_x . \, x = c : f(c, d),$

and the same proposition results if, in (1), we eliminate first $(\imath x)(\psi x)$ and then $(\imath x)(\phi x)$. Similarly (2) becomes, when $(\imath x)(\phi x)$ and $(\imath x)(\psi x)$ are eliminated,

(5) $(\exists d) :. \psi x \,.\, \equiv_x . \, x = d :. (\exists c) : \phi x \,.\, \equiv_x . \, x = c : f(c, d).$

(4) and (5) are equivalent, so that the truth-value of a proposition containing two descriptions is independent of the question which has the larger scope.

It will be found that, in most cases in which descriptions occur, their scope is, in practice, the smallest proposition enclosed in dots or other brackets in which they are contained. Thus for example

$$[(\imath x)(\phi x)] . \psi(\imath x)(\phi x) . \supset . [(\imath x)(\phi x)] . \chi(\imath x)(\phi x)$$

will occur much more frequently than

$$[(\imath x)(\phi x)] : \psi(\imath x)(\phi x) . \supset . \chi(\imath x)(\phi x).$$

For this reason it is convenient to decide that, when the scope of an occurrence of $(\imath x)(\phi x)$ is the smallest proposition, enclosed in dots or other brackets, in which the occurrence in question is contained, the scope need not be indicated by "$[(\imath x)(\phi x)]$." Thus *e.g.*

$$p . \supset . a = (\imath x)(\phi x)$$

will mean $$p . \supset . [(\imath x)(\phi x)] . a = (\imath x)(\phi x);$$

and $$p . \supset . (\exists a) . a = (\imath x)(\phi x)$$

will mean $$p . \supset . (\exists a) . [(\imath x)(\phi x)] . a = (\imath x)(\phi x);$$

and $$p . \supset . a \neq (\imath x)(\phi x)$$

will mean $$p . \supset . [(\imath x)(\phi x)] . \sim \{a = (\imath x)(\phi x)\};$$

but $$p . \supset . \sim \{a = (\imath x)(\phi x)\}$$

will mean $$p . \supset . \sim \{[(\imath x)(\phi x)] . a = (\imath x)(\phi x)\}.$$

This convention enables us, in the vast majority of cases that actually occur, to dispense with the explicit indication of the scope of a descriptive symbol; and it will be found that the convention agrees very closely with the tacit conventions of ordinary language on this subject. Thus for example, if "$(\imath x)(\phi x)$" is "the so-and-so," "$a \neq (\imath x)(\phi x)$" is to be read "$a$ is not the so-and-so," which would ordinarily be regarded as implying that "the so-and-so" exists; but "$\sim \{a = (\imath x)(\phi x)\}$" is to be read "it is not true that a is the so-and-so," which would generally be allowed to hold if "the so-and-so" does not exist. Ordinary language is, of course, rather loose and fluctuating in its implications on this matter; but subject to the requirement of definiteness, our convention seems to keep as near to ordinary language as possible.

In the case when the smallest proposition enclosed in dots or other brackets contains two or more descriptions, we shall assume, in the absence of any indication to the contrary, that one which typographically occurs earlier has a larger scope than one which typographically occurs later. Thus

$$(\imath x)(\phi x) = (\imath x)(\psi x)$$

will mean $$(\exists c) : \phi x . \equiv_x . x = c : [(\imath x)(\psi x)] . c = (\imath x)(\psi x),$$

while $$(\imath x)(\psi x) = (\imath x)(\phi x)$$

will mean $$(\exists d) : \psi x . \equiv_x . x = d : [(\imath x)(\phi x)] . (\imath x)(\phi x) = d.$$

These two propositions are easily shown to be equivalent.

(2) *Classes.* The symbols for classes, like those for descriptions, are, in our system, incomplete symbols: their *uses* are defined, but they themselves are not assumed to mean anything at all. That is to say, the uses of such

symbols are so defined that, when the *definiens* is substituted for the *definiendum*, there no longer remains any symbol which could be supposed to represent a class. Thus classes, so far as we introduce them, are merely symbolic or linguistic conveniences, not genuine objects as their members are if they are individuals.

It is an old dispute whether formal logic should concern itself mainly with intensions or with extensions. In general, logicians whose training was mainly philosophical have decided for intensions, while those whose training was mainly mathematical have decided for extensions. The facts seem to be that, while mathematical logic requires extensions, philosophical logic refuses to supply anything except intensions. Our theory of classes recognizes and reconciles these two apparently opposite facts, by showing that an extension (which is the same as a class) is an incomplete symbol, whose use always acquires its meaning through a reference to intension.

In the case of descriptions, it was possible to *prove* that they are incomplete symbols. In the case of classes, we do not know of any equally definite proof, though arguments of more or less cogency can be elicited from the ancient problem of the One and the Many*. It is not necessary for our purposes, however, to assert dogmatically that there are no such things as classes. It is only necessary for us to show that the incomplete symbols which we introduce as representatives of classes yield all the propositions for the sake of which classes might be thought essential. When this has been shown, the mere principle of economy of primitive ideas leads to the non-introduction of classes except as incomplete symbols.

To explain the theory of classes, it is necessary first to explain the distinction between *extensional* and *intensional* functions. This is effected by the following definitions:

The *truth-value* of a proposition is truth if it is true, and falsehood if it is false. (This expression is due to Frege.)

Two propositions are said to be *equivalent* when they have the same truth-value, *i.e.* when they are both true or both false.

Two propositional functions are said to be *formally equivalent* when they are equivalent with every possible argument, *i.e.* when any argument which satisfies the one satisfies the other, and vice versa. Thus "$\hat{x}$ is a man" is formally equivalent to "$\hat{x}$ is a featherless biped"; "$\hat{x}$ is an even prime" is formally equivalent to "$\hat{x}$ is identical with 2."

A function of a function is called *extensional* when its truth-value with any argument is the same as with any formally equivalent argument. That is to

* Briefly, these arguments reduce to the following: If there is such an object as a class, it must be in some sense *one* object. Yet it is only of classes that *many* can be predicated. Hence, if we admit classes as objects, we must suppose that the same object can be both one and many, which seems impossible.

say, $f(\phi\hat{z})$ is an extensional function of $\phi\hat{z}$ if, provided $\psi\hat{z}$ is formally equivalent to $\phi\hat{z}$, $f(\phi\hat{z})$ is equivalent to $f(\psi\hat{z})$. Here the apparent variables ϕ and ψ are necessarily of the type from which arguments can significantly be supplied to f. We find no need to use as apparent variables any functions of non-predicative types; accordingly in the sequel all extensional functions considered are in fact functions of predicative functions*.

A function of a function is called *intensional* when it is not extensional.

The nature and importance of the distinction between intensional and extensional functions will be made clearer by some illustrations. The proposition "'x is a man' always implies 'x is a mortal'" is an extensional function of the function "$\hat{x}$ is a man," because we may substitute, for "x is a man," "x is a featherless biped," or any other statement which applies to the same objects to which "x is a man" applies, and to no others. But the proposition "A believes that 'x is a man' always implies 'x is a mortal'" is an intensional function of "$\hat{x}$ is a man," because A may never have considered the question whether featherless bipeds are mortal, or may believe wrongly that there are featherless bipeds which are not mortal. Thus even if "x is a featherless biped" is formally equivalent to "x is a man," it by no means follows that a person who believes that all men are mortal must believe that all featherless bipeds are mortal, since he may have never thought about featherless bipeds, or have supposed that featherless bipeds were not always men. Again the proposition "the number of arguments that satisfy the function $\phi ! \hat{z}$ is n" is an extensional function of $\phi ! \hat{z}$, because its truth or falsehood is unchanged if we substitute for $\phi ! \hat{z}$ any other function which is true whenever $\phi ! \hat{z}$ is true, and false whenever $\phi ! \hat{z}$ is false. But the proposition "A asserts that the number of arguments satisfying $\phi ! \hat{z}$ is n" is an intensional function of $\phi ! \hat{z}$, since, if A asserts this concerning $\phi ! \hat{z}$, he certainly cannot assert it concerning all predicative functions that are equivalent to $\phi ! \hat{z}$, because life is too short. Again, consider the proposition "two white men claim to have reached the North Pole." This proposition states "two arguments satisfy the function '$\hat{x}$ is a white man who claims to have reached the North Pole.'" The truth or falsehood of this proposition is unaffected if we substitute for "$\hat{x}$ is a white man who claims to have reached the North Pole" any other statement which holds of the same arguments, and of no others. Hence it is an extensional function. But the proposition "it is a strange coincidence that two white men should claim to have reached the North Pole," which states "it is a strange coincidence that two arguments should satisfy the function '$\hat{x}$ is a white man who claims to have reached the North Pole,'" is not equivalent to "it is a strange coincidence that two arguments should satisfy the function '$\hat{x}$ is Dr Cook or Commander Peary.'" Thus "it is a strange coincidence that $\phi ! \hat{x}$ should be satisfied by two arguments" is an intensional function of $\phi ! \hat{x}$.

* Cf. p. 53.

The above instances illustrate the fact that the functions of functions with which mathematics is specially concerned are extensional, and that intensional functions of functions only occur where non-mathematical ideas are introduced, such as what somebody believes or affirms, or the emotions aroused by some fact. Hence it is natural, in a mathematical logic, to lay special stress on *extensional* functions of functions.

When two functions are formally equivalent, we may say that they *have the same extension.* In this definition, we are in close agreement with usage. We do not assume that there is such a thing as an extension: we merely define the whole phrase "having the same extension." We may now say that an extensional function of a function is one whose truth or falsehood depends only upon the extension of its argument. In such a case, it is convenient to regard the statement concerned as being about the extension. Since extensional functions are many and important, it is natural to regard the extension as an object, called a *class*, which is supposed to be the subject of all the equivalent statements about various formally equivalent functions. Thus *e.g.* if we say "there were twelve Apostles," it is natural to regard this statement as attributing the property of being twelve to a certain collection of men, namely those who were Apostles, rather than as attributing the property of being satisfied by twelve arguments to the function "$\hat{x}$ was an Apostle." This view is encouraged by the feeling that there is something which is identical in the case of two functions which "have the same extension." And if we take such simple problems as "how many combinations can be made of n things?" it seems at first sight necessary that each "combination" should be a single object which can be counted as one. This, however, is certainly not necessary technically, and we see no reason to suppose that it is true philosophically. The technical procedure by which the apparent difficulty is overcome is as follows.

We have seen that an extensional function of a function may be regarded as a function of the class determined by the argument-function, but that an intensional function cannot be so regarded. In order to obviate the necessity of giving different treatment to intensional and extensional functions of functions, we construct an extensional function derived from any function of a predicative function $\psi ! \hat{z}$, and having the property of being equivalent to the function from which it is derived, provided this function is extensional, as well as the property of being significant (by the help of the systematic ambiguity of equivalence) with any argument $\phi\hat{z}$ whose arguments are of the same type as those of $\psi ! \hat{z}$. The derived function, written "$f\{\hat{z}(\phi z)\}$," is defined as follows: Given a function $f(\psi ! \hat{z})$, our derived function is to be "there is a predicative function which is formally equivalent to $\phi\hat{z}$ and satisfies f." If $\phi\hat{z}$ is a predicative function, our derived function will be true whenever $f(\phi\hat{z})$ is true. If $f(\phi\hat{z})$ is an extensional function, and $\phi\hat{z}$ is a predicative

function, our derived function will not be true unless $f(\phi\hat{z})$ is true; thus in this case, our derived function is equivalent to $f(\phi\hat{z})$. If $f(\phi\hat{z})$ is not an extensional function, and if $\phi\hat{z}$ is a predicative function, our derived function may sometimes be true when the original function is false. But in any case the derived function is always extensional.

In order that the derived function should be significant for any function $\phi\hat{z}$, of whatever order, provided it takes arguments of the right type, it is necessary and sufficient that $f(\psi ! \hat{z})$ should be significant, where $\psi ! \hat{z}$ is any *predicative* function. The reason of this is that we only require, concerning an argument $\phi\hat{z}$, the hypothesis that it is formally equivalent to some predicative function $\psi ! \hat{z}$, and formal equivalence has the same kind of systematic ambiguity as to type that belongs to truth and falsehood, and can therefore hold between functions of any two different orders, provided the functions take arguments of the same type. Thus by means of our derived function we have not merely provided extensional functions everywhere in place of intensional functions, but we have *practically* removed the necessity for considering differences of type among functions whose arguments are of the same type. This effects the same kind of simplification in our hierarchy as would result from never considering any but predicative functions.

If $f(\psi ! \hat{z})$ can be built up by means of the primitive ideas of disjunction, negation, $(x) . \phi x$, and $(\exists x) . \phi x$, as is the case with all the functions of functions that explicitly occur in the present work, it will be found that, in virtue of the systematic ambiguity of the above primitive ideas, any function $\phi\hat{z}$ whose arguments are of the same type as those of $\psi ! \hat{z}$ can significantly be substituted for $\psi ! \hat{z}$ in f without any other symbolic change. Thus in such a case what is symbolically, though not really, the same function f can receive as arguments functions of various different types. If, with a given argument $\phi\hat{z}$, the function $f(\phi\hat{z})$, so interpreted, is equivalent to $f(\psi ! \hat{z})$ whenever $\psi ! \hat{z}$ is formally equivalent to $\phi\hat{z}$, then $f\{\hat{z}(\phi z)\}$ is equivalent to $f(\phi\hat{z})$ provided there is any predicative function formally equivalent to $\phi\hat{z}$. At this point, we make use of the axiom of reducibility, according to which there always is a predicative function formally equivalent to $\phi\hat{z}$.

As was explained above, it is convenient to regard an extensional function of a function as having for its argument not the function, but the class determined by the function. Now we have seen that our derived function is always extensional. Hence if our original function was $f(\psi ! \hat{z})$, we write the derived function $f\{\hat{z}(\phi z)\}$, where "$\hat{z}(\phi z)$" may be read "the class of arguments which satisfy $\phi\hat{z}$," or more simply "the class determined by $\phi\hat{z}$." Thus "$f\{\hat{z}(\phi z)\}$" will mean: "There is a predicative function $\psi ! \hat{z}$ which is formally equivalent to $\phi\hat{z}$ and is such that $f(\psi ! \hat{z})$ is true." This is in reality a function of $\phi\hat{z}$, but we treat it symbolically as if it had an argument $\hat{z}(\phi z)$. By the help of the axiom of reducibility, we find that the usual properties of classes

result. For example, two formally equivalent functions determine the same class, and conversely, two functions which determine the same class are formally equivalent. Also to say that x is a member of $\hat{z}(\phi z)$, *i.e.* of the class determined by $\phi\hat{z}$, is true when ϕx is true, and false when ϕx is false. Thus all the mathematical purposes for which classes might seem to be required are fulfilled by the purely symbolic objects $\hat{z}(\phi z)$, provided we assume the axiom of reducibility.

In virtue of the axiom of reducibility, if $\phi\hat{z}$ is any function, there is a formally equivalent predicative function $\psi ! \hat{z}$; then the class $\hat{z}(\phi z)$ is identical with the class $\hat{z}(\psi ! z)$, so that every class can be defined by a *predicative* function. Hence the totality of the *classes* to which a given term can be significantly said to belong or not to belong is a legitimate totality, although the totality of *functions* which a given term can be significantly said to satisfy or not to satisfy is not a legitimate totality. The classes to which a given term a belongs or does not belong are the classes defined by a-functions; they are also the classes defined by *predicative* a-functions. Let us call them a-classes. Then "a-classes" form a legitimate totality, derived from that of predicative a-functions. Hence many kinds of general statements become possible which would otherwise involve vicious-circle paradoxes. These general statements are none of them such as lead to contradictions, and many of them such as it is very hard to suppose illegitimate. The fact that they are rendered possible by the axiom of reducibility, and that they would otherwise be excluded by the vicious-circle principle, is to be regarded as an argument in favour of the axiom of reducibility.

The above definition of "the class defined by the function $\phi\hat{z}$," or rather, of any proposition in which this phrase occurs, is, in symbols, as follows:

$$f\{\hat{z}(\phi z)\} \,.=:\, (\exists\psi) : \phi x \,.\equiv_x .\, \psi ! x : f\{\psi ! \hat{z}\} \quad \text{Df.}$$

In order to recommend this definition, we shall enumerate five requisites which a definition of classes must satisfy, and we shall then show that the above definition satisfies these five requisites.

We require of classes, if they are to serve the purposes for which they are commonly employed, that they shall have certain properties, which may be enumerated as follows. (1) Every propositional function must determine a class, which may be regarded as the collection of all the arguments satisfying the function in question. This principle must hold when the function is satisfied by an infinite number of arguments as well as when it is satisfied by a finite number. It must hold also when no arguments satisfy the function; *i.e.* the "null-class" must be just as good a class as any other. (2) Two propositional functions which are formally equivalent, *i.e.* such that any argument which satisfies either satisfies the other, must determine the same class; that is to say, a class must be something wholly determined by its membership, so that *e.g.* the class "featherless bipeds" is identical with the class "men," and

the class "even primes" is identical with the class "numbers identical with 2." (3) Conversely, two propositional functions which determine the same class must be formally equivalent; in other words, when the class is given, the membership is determinate: two different sets of objects cannot yield the same class. (4) In the same sense in which there are classes (whatever this sense may be), or in some closely analogous sense, there must also be classes of classes. Thus for example "the combinations of n things m at a time," where the n things form a given class, is a class of classes; each combination of m things is a class, and each such class is a member of the specified set of combinations, which set is therefore a class whose members are classes. Again, the class of unit classes, or of couples, is absolutely indispensable; the former is the number 1, the latter the number 2. Thus without classes of classes, arithmetic becomes impossible. (5) It must under all circumstances be meaningless to suppose a class identical with one of its own members. For if such a supposition had any meaning "$\alpha \in \alpha$" would be a significant propositional function*, and so would "$\alpha \sim \in \alpha$." Hence, by (1) and (4), there would be a class of all classes satisfying the function "$\alpha \sim \in \alpha$." If we call this class κ, we shall have

$$\alpha \in \kappa \,.\equiv_\alpha .\, \alpha \sim \in \alpha.$$

Since, by our hypothesis, "$\kappa \in \kappa$" is supposed significant, the above equivalence, which holds with all possible values of α, holds with the value κ, *i.e.*

$$\kappa \in \kappa \,.\equiv .\, \kappa \sim \in \kappa.$$

But this is a contradiction†. Hence "$\alpha \in \alpha$" and "$\alpha \sim \in \alpha$" must always be meaningless. In general, there is nothing surprising about this conclusion, but it has two consequences which deserve special notice. In the first place, a class consisting of only one member must not be identical with that one member, *i.e.* we must not have $\iota\text{‘}x = x$. For we have $x \in \iota\text{‘}x$, and therefore, if $x = \iota\text{‘}x$, we have $\iota\text{‘}x \in \iota\text{‘}x$, which, we saw, must be meaningless. It follows that "$x = \iota\text{‘}x$" must be absolutely meaningless, not simply false. In the second place, it might appear as if the class of all classes were a class, *i.e.* as if (writing "Cls" for "class") "Cls $\in$ Cls" were a true proposition. But this combination of symbols must be meaningless; unless, indeed, an ambiguity exists in the meaning of "Cls," so that, in "Cls $\in$ Cls," the first "Cls" can be supposed to have a different meaning from the second.

As regards the above requisites, it is plain, to begin with, that, in accordance with our definition, every propositional function $\phi\hat{z}$ determines a class $\hat{z}(\phi z)$. Assuming the axiom of reducibility, there must always be true propositions about $\hat{z}(\phi z)$, *i.e.* true propositions of the form $f\{\hat{z}(\phi z)\}$. For suppose $\phi\hat{z}$ is formally equivalent to $\psi ! \hat{z}$, and suppose $\psi ! \hat{z}$ satisfies some function f. Then

* As explained in Chapter I (p. 25), "$x \in \alpha$" means "x is a member of the class α," or, more shortly, "x is an α." The definition of this expression in terms of our theory of classes will be given shortly.

† This is the second of the contradictions discussed at the end of Chapter II.

$\hat{z}(\phi z)$ also satisfies f. Hence, given any function $\phi\hat{z}$, there are true propositions of the form $f\{\hat{z}(\phi z)\}$, *i.e.* true propositions in which "the class determined by $\phi\hat{z}$" is grammatically the subject. This shows that our definition fulfils the first of our five requisites.

The second and third requisites together demand that the classes $\hat{z}(\phi z)$ and $\hat{z}(\psi z)$ should be identical when, and only when, their defining functions are formally equivalent, *i.e.* that we should have

$$\hat{z}(\phi z) = \hat{z}(\psi z) \,.\equiv:\, \phi x \,.\equiv_x.\, \psi x.$$

Here the meaning of "$\hat{z}(\phi z) = \hat{z}(\psi z)$" is to be derived, by means of a two-fold application of the definition of $f\{\hat{z}(\phi z)\}$, from the definition of

$$\text{“}\chi ! \hat{z} = \theta ! \hat{z}\text{,”}$$

which is $\quad \chi ! \hat{z} = \theta ! \hat{z} \,.=:\, (f) : f ! \chi ! \hat{z} \,.\supset.\, f ! \theta ! \hat{z} \quad$ Df

by the general definition of identity.

In interpreting "$\hat{z}(\phi z) = \hat{z}(\psi z)$," we will adopt the convention which we adopted in regard to $(\iota x)(\phi x)$ and $(\iota x)(\psi x)$, namely that the incomplete symbol which occurs first is to have the larger scope. Thus $\hat{z}(\phi z) = \hat{z}(\psi z)$ becomes, by our definition,

$$(\exists\chi) : \phi x \,.\equiv_x.\, \chi ! x : \chi ! \hat{z} = \hat{z}(\psi z),$$

which, by eliminating $\hat{z}(\psi z)$, becomes

$$(\exists\chi) :.\, \phi x \,.\equiv_x.\, \chi ! x :.\, (\exists\theta) : \psi x \,.\equiv_x.\, \theta ! x : \chi ! \hat{z} = \theta ! \hat{z},$$

which is equivalent to

$$(\exists\chi, \theta) : \phi x \,.\equiv_x.\, \chi ! x : \psi x \,.\equiv_x.\, \theta ! x : \chi ! \hat{z} = \theta ! \hat{z},$$

which, again, is equivalent to

$$(\exists\chi) : \phi x \,.\equiv_x.\, \chi ! x : \psi x \,.\equiv_x.\, \chi ! x,$$

which, in virtue of the axiom of reducibility, is equivalent to

$$\phi x \,.\equiv_x.\, \psi x.$$

Thus our definition of the use of $\hat{z}(\phi z)$ is such as to satisfy the conditions (2) and (3) which we laid down for classes, *i.e.* we have

$$\vdash :.\, \hat{z}(\phi z) = \hat{z}(\psi z) \,.\equiv:\, \phi x \,.\equiv_x.\, \psi x.$$

Before considering classes of classes, it will be well to define membership of a class, *i.e.* to define the symbol "$x \in \hat{z}(\phi z)$," which may be read "x is a member of the class determined by $\phi\hat{z}$." Since this is a function of the form $f\{\hat{z}(\phi z)\}$, it must be derived, by means of our general definition of such functions, from the corresponding function $f\{\psi ! \hat{z}\}$. We therefore put

$$x \in \psi ! \hat{z} \,.=.\, \psi ! x \quad \text{Df.}$$

This definition is only needed in order to give a meaning to "$x \in \hat{z}(\phi z)$"; the meaning it gives is, in virtue of the definition of $f\{\hat{z}(\phi z)\}$,

$$(\exists\psi) : \phi y \,.\equiv_y.\, \psi ! y : \psi ! x.$$

It thus appears that "$x \in \hat{z}(\phi z)$" implies ϕx, since it implies $\psi ! x$, and $\psi ! x$ is equivalent to ϕx; also, in virtue of the axiom of reducibility, ϕx implies "$x \in \hat{z}(\phi z)$," since there is a predicative function ψ formally equivalent to ϕ,

and x must satisfy ψ, since x (*ex hypothesi*) satisfies ϕ. Thus in virtue of the axiom of reducibility we have

$$\vdash : x \epsilon \hat{z}(\phi z) . \equiv . \phi x,$$

i.e. x is a member of the class $\hat{z}(\phi z)$ when, and only when, x satisfies the function ϕ which defines the class.

We have next to consider how to interpret a class of classes. As we have defined $f\{\hat{z}(\phi z)\}$, we shall naturally regard a class of classes as consisting of those values of $\hat{z}(\phi z)$ which satisfy $f\{\hat{z}(\phi z)\}$. Let us write α for $\hat{z}(\phi z)$; then we may write $\hat{\alpha}(f\alpha)$ for the class of values of α which satisfy $f\alpha$*. We shall apply the same definition, and put

$$F\{\hat{\alpha}(f\alpha)\} . = : (\exists g) : f\beta . \equiv_\beta . g ! \beta : F\{g ! \hat{\alpha}\} \quad \text{Df},$$

where "β" stands for any expression of the form $\hat{z}(\psi ! z)$.

Let us take "$\gamma \epsilon \hat{\alpha}(f\alpha)$" as an instance of $F\{\hat{\alpha}(f\alpha)\}$. Then

$$\vdash :. \gamma \epsilon \hat{\alpha}(f\alpha) . \equiv : (\exists g) : f\beta . \equiv_\beta . g ! \beta : \gamma \epsilon g ! \hat{\alpha}.$$

Just as we put $\quad x \epsilon \psi ! \hat{z} . = . \psi ! x \quad$ Df,

so we put $\quad \gamma \epsilon g ! \hat{\alpha} . = . g ! \gamma \quad$ Df.

Thus we find

$$\vdash :. \gamma \epsilon \hat{\alpha}(f\alpha) . \equiv : (\exists g) : f\beta . \equiv_\beta . g ! \beta : g ! \gamma.$$

If we now extend the axiom of reducibility so as to apply to functions of functions, *i.e.* if we assume

$$(\exists g) : f(\psi ! \hat{z}) . \equiv_\psi . g ! (\psi ! \hat{z}),$$

we easily deduce

$$\vdash : (\exists g) : f\{\hat{z}(\psi ! z)\} . \equiv_\psi . g ! \{\hat{z}(\psi ! z)\},$$

i.e. $\quad \vdash : (\exists g) : f\beta . \equiv_\beta . g ! \beta.$

Thus $\quad \vdash : \gamma \epsilon \hat{\alpha}(f\alpha) . \equiv . f\gamma.$

Thus every function which can take classes as arguments, *i.e.* every function of functions, determines a class of classes, whose members are those classes which satisfy the determining function. Thus the theory of classes of classes offers no difficulty.

We have next to consider our fifth requisite, namely that "$\hat{z}(\phi z) \epsilon \hat{z}(\phi z)$" is to be meaningless. Applying our definition of $f\{\hat{z}(\phi z)\}$, we find that if this collection of symbols had a meaning, it would mean

$$(\exists \psi) : \phi x . \equiv_x . \psi ! x : \psi ! \hat{z} \epsilon \psi ! \hat{z},$$

i.e. in virtue of the definition

$$x \epsilon \psi ! \hat{z} . = . \psi ! x \quad \text{Df},$$

it would mean $\quad (\exists \psi) : \phi x . \equiv_x . \psi ! x : \psi ! (\psi ! \hat{z}).$

But here the symbol "$\psi ! (\psi ! \hat{z})$" occurs, which assigns a function as argument to itself. Such a symbol is always meaningless, for the reasons explained at the beginning of Chapter II (pp. 38—41). Hence "$\hat{z}(\phi z) \epsilon \hat{z}(\phi z)$" is meaningless, and our fifth and last requisite is fulfilled.

* The use of a single letter, such as α or β, to represent a variable class, will be further explained shortly.

As in the case of $f(\imath x)(\phi x)$, so in that of $f\{\hat{z}(\phi z)\}$, there is an ambiguity as to the scope of $\hat{z}(\phi z)$ if it occurs in a proposition which itself is part of a larger proposition. But in the case of classes, since we always have the axiom of reducibility, namely

$$(\exists\psi) : \phi x . \equiv_x . \psi ! x,$$

which takes the place of $E!(\imath x)(\phi x)$, it follows that the truth-value of any proposition in which $\hat{z}(\phi z)$ occurs is the same whatever scope we may give to $\hat{z}(\phi z)$, provided the proposition is an extensional function of whatever functions it may contain. Hence we may adopt the convention that the scope is to be always the smallest proposition enclosed in dots or brackets in which $\hat{z}(\phi z)$ occurs. If at any time a larger scope is required, we may indicate it by "$[\hat{z}(\phi z)]$" followed by dots, in the same way as we did for $[(\imath x)(\phi x)]$.

Similarly when two class symbols occur, *e.g.* in a proposition of the form $f\{\hat{z}(\phi z), \hat{z}(\psi z)\}$, we need not remember rules for the scopes of the two symbols, since all choices give equivalent results, as it is easy to prove. For the preliminary propositions a rule is desirable, so we can decide that the class symbol which occurs first in the order of writing is to have the larger scope.

The representation of a class by a single letter α can now be understood. For the denotation of α is ambiguous, in so far as it is undecided as to which of the symbols $\hat{z}(\phi z)$, $\hat{z}(\psi z)$, $\hat{z}(\chi z)$, etc. it is to stand for, where $\phi\hat{z}$, $\psi\hat{z}$, $\chi\hat{z}$, etc. are the various determining functions of the class. According to the choice made, different propositions result. But all the resulting propositions are equivalent by virtue of the easily proved proposition:

$$\text{"}\vdash : \phi x \equiv_x \psi x . \supset . f\{\hat{z}(\phi z)\} \equiv f\{\hat{z}(\psi z)\}.\text{"}$$

Hence unless we wish to discuss the determining function itself, so that the notion of a class is really not properly present, the ambiguity in the denotation of α is entirely immaterial, though, as we shall see immediately, we are led to limit ourselves to predicative determining functions. Thus "$f(\alpha)$," where α is a variable class, is really "$f\{\hat{z}(\phi z)\}$," where ϕ is a variable function, that is, it is

$$\text{"}(\exists\psi) . \phi x \equiv_x \psi ! x . f\{\psi ! \hat{z}\},\text{"}$$

where ϕ is a variable function. But here a difficulty arises which is removed by a limitation to our practice and by the axiom of reducibility. For the determining functions $\phi\hat{z}$, $\psi\hat{z}$, etc. will be of different types, though the axiom of reducibility secures that some are predicative functions. Then, in interpreting α as a variable in terms of the variation of any determining function, we shall be led into errors unless we confine ourselves to predicative determining functions. These errors especially arise in the transition to total variation (cf. pp. 15, 16). Accordingly

$$f\alpha = . (\exists\psi) . \phi ! x \equiv_x \psi ! x . f\{\psi ! \hat{z}\} \quad \text{Df.}$$

It is the peculiarity of a definition of the use of a single letter [viz. α] for a variable incomplete symbol that it, though in a sense a real variable, occurs only in the *definiendum*, while "ϕ," though a real variable, occurs only in the *definiens*.

Thus "$f\hat{\alpha}$" stands for

$$\text{“}(\exists\psi) . \hat{\phi} ! x \equiv_x \psi ! x . f\{\psi ! \hat{z}\},\text{”}$$

and "$(\alpha) . f\alpha$" stands for

$$\text{“}(\phi) : (\exists\psi) . \phi ! x \equiv_x \psi ! x . f\{\psi ! \hat{z}\}.\text{”}$$

Accordingly, in mathematical reasoning, we can dismiss the whole apparatus of functions and think only of classes as "quasi-things," capable of immediate representation by a single name. The advantages are two-fold: (1) classes are determined by their membership, so that to one set of members there is one class, (2) the "type" of a class is entirely defined by the type of its members.

Also a predicative function of a class can be defined thus

$$f ! \alpha = . (\exists\psi) . \phi ! x \equiv_x \psi ! x . f ! \{\psi ! \hat{z}\} \quad \mathrm{Df.}$$

Thus a predicative function of a class is always a predicative function of any predicative determining function of the class, though the converse does not hold.

(3) *Relations.* With regard to relations, we have a theory strictly analogous to that which we have just explained as regards classes. Relations in extension, like classes, are incomplete symbols, We require a division of functions of two variables into predicative and non-predicative functions, again for reasons which have been explained in Chapter II. We use the notation "$\phi ! (x, y)$" for a *predicative* function of x and y.

We use "$\phi ! (\hat{x}, \hat{y})$" for the function as opposed to its values; and we use "$\hat{x}\hat{y}\,\phi(x, y)$" for the relation (in extension) determined by $\phi(x, y)$. We put

$$f\{\hat{x}\hat{y}\,\phi(x, y)\} . = : (\exists\psi) : \phi(x, y) . \equiv_{x,y} . \psi ! (x, y) : f\{\psi ! (\hat{x}, \hat{y})\} \quad \mathrm{Df.}$$

Thus even when $f\{\psi ! (\hat{x}, \hat{y})\}$ is not an extensional function of ψ, $f\{\hat{x}\hat{y}\,\phi(x, y)\}$ *is* an extensional function of ϕ. Hence, just as in the case of classes, we deduce

$$\vdash :. \hat{x}\hat{y}\,\phi(x, y) = \hat{x}\hat{y}\,\psi(x, y) . \equiv : \phi(x, y) . \equiv_{x,y} . \psi(x, y),$$

i.e. a relation is determined by its extension, and vice versa.

On the analogy of the definition of "$x \in \psi ! \hat{z}$," we put

$$x\{\psi ! (\hat{x}, \hat{y})\} y . = . \psi ! (x, y) \quad \mathrm{Df}^*.$$

This definition, like that of "$x \in \psi ! \hat{z}$," is not introduced for its own sake, but in order to give a meaning to

$$x\{\hat{x}\hat{y}\,\phi(x, y)\} y.$$

This meaning, in virtue of our definitions, is

$$(\exists\psi) : \phi(x, y) . \equiv_{x,y} . \psi ! (x, y) : x\{\psi ! (\hat{x}, \hat{y})\} y,$$

i.e.

$$(\exists\psi) : \phi(x, y) . \equiv_{x,y} . \psi ! (x, y) : \psi ! (x, y),$$

and this, in virtue of the axiom of reducibility

$$\text{“}(\exists\psi) : \phi(x, y) . \equiv_{x,y} . \psi ! (x, y),\text{”}$$

is equivalent to

$$\phi(x, y).$$

Thus we have always

$$\vdash : x\{\hat{x}\hat{y}\,\phi(x, y)\} y . \equiv . \phi(x, y).$$

* This definition raises certain questions as to the two senses of a relation, which are dealt with in *21.

Whenever the determining function of a relation is not relevant, we may replace $\hat{x}\hat{y}\,\phi(x, y)$ by a single capital letter. In virtue of the propositions given above,

$$\vdash :. R = S . \equiv : xRy . \equiv_{x,y} . xSy,$$

$$\vdash :. R = \hat{x}\hat{y}\,\phi(x, y) . \equiv : xRy . \equiv_{x,y} . \phi(x, y),$$

and

$$\vdash . R = \hat{x}\hat{y}\,(xRy).$$

Classes of relations, and relations of relations, can be dealt with as classes of classes were dealt with above.

Just as a class must not be capable of being or not being a member of itself, so a relation must neither be nor not be referent or relatum with respect to itself. This turns out to be equivalent to the assertion that $\phi!(\hat{x}, \hat{y})$ cannot significantly be either of the arguments x or y in $\phi!(x, y)$. This principle, again, results from the limitation to the possible arguments to a function explained at the beginning of Chapter II.

We may sum up this whole discussion on incomplete symbols as follows.

The use of the symbol "$(\imath x)(\phi x)$" as if in "$f(\imath x)(\phi x)$" it *directly* represented an argument to the function $f\hat{z}$ is rendered possible by the theorems

$$\vdash :. E!(\imath x)(\phi x) . \supset : (x) . fx . \supset . f(\imath x)(\phi x),$$

$$\vdash : (\imath x)(\phi x) = (\imath x)(\psi x) . \supset . f(\imath x)(\phi x) \equiv f(\imath x)(\psi x),$$

$$\vdash : E!(\imath x)(\phi x) . \supset . (\imath x)(\phi x) = (\imath x)(\phi x),$$

$$\vdash : (\imath x)(\phi x) = (\imath x)(\psi x) . \equiv . (\imath x)(\psi x) = (\imath x)(\phi x),$$

$$\vdash : (\imath x)(\phi x) = (\imath x)(\psi x) . (\imath x)(\psi x) = (\imath x)(\chi x) . \supset . (\imath x)(\phi x) = (\imath x)(\chi x).$$

The use of the symbol "$\hat{x}(\phi x)$" (or of a single letter, such as α, to represent such a symbol) as if, in "$f\{\hat{x}(\phi x)\}$," it *directly* represented an argument α to a function $f\hat{\alpha}$, is rendered possible by the theorems

$$\vdash : (\alpha) . f\alpha . \supset . f\{\hat{x}(\phi x)\},$$

$$\vdash : \hat{x}(\phi x) = \hat{x}(\psi x) . \supset . f\{\hat{x}(\phi x)\} \equiv f\{\hat{x}(\psi x)\}$$

$$\vdash . \hat{x}(\phi x) = \hat{x}(\phi x),$$

$$\vdash : \hat{x}(\phi x) = \hat{x}(\psi x) . \equiv . \hat{x}(\psi x) = \hat{x}(\phi x),$$

$$\vdash : \hat{x}(\phi x) = \hat{x}(\psi x) . \hat{x}(\psi x) = \hat{x}(\chi x) . \supset . \hat{x}(\phi x) = \hat{x}(\chi x).$$

Throughout these propositions the types must be supposed to be properly adjusted, where ambiguity is possible.

The use of the symbol "$\hat{x}\hat{y}\{\phi(x, y)\}$" (or of a single letter, such as R, to represent such a symbol) as if, in "$f\{\hat{x}\hat{y}\,\phi(x, y)\}$," it *directly* represented an argument R to a function $f\hat{R}$, is rendered possible by the theorems

$$\vdash : (R) . fR . \supset . f\{\hat{x}\hat{y}\,\phi(x, y)\},$$

$$\vdash : \hat{x}\hat{y}\,\phi(x, y) = \hat{x}\hat{y}\,\psi(x, y) . \supset . f\{\hat{x}\hat{y}\,\phi(x, y)\} \equiv f\{\hat{x}\hat{y}\,\psi(x, y)\},$$

$$\vdash . \hat{x}\hat{y}\,\phi(x, y) = \hat{x}\hat{y}\,\phi(x, y),$$

$$\vdash : \hat{x}\hat{y}\,\phi(x, y) = \hat{x}\hat{y}\,\psi(x, y) . \equiv . \hat{x}\hat{y}\,\psi(x, y) = \hat{x}\hat{y}\,\phi(x, y),$$

$$\vdash : \hat{x}\hat{y}\,\phi(x, y) = \hat{x}\hat{y}\,\psi(x, y) . \hat{x}\hat{y}\,\psi(x, y) = \hat{x}\hat{y}\,\chi(x, y) .$$
$$\supset . \hat{x}\hat{y}\,\phi(x, y) = \hat{x}\hat{y}\,\chi(x, y).$$

Throughout these propositions the types must be supposed to be properly adjusted where ambiguity is possible.

It follows from these three groups of theorems that these incomplete symbols are obedient to the same formal rules of identity as symbols which directly represent objects, so long as we only consider the *equivalence* of the resulting variable (or constant) values of propositional functions and not their identity. This consideration of the *identity* of propositions never enters into our formal reasoning.

Similarly the *limitations* to the use of these symbols can be summed up as follows. In the case of $(\iota x)(\phi x)$, the chief way in which its incompleteness is relevant is that we do not always have

$$(x) . fx . \supset . f(\iota x)(\phi x),$$

i.e. a function which is always true may nevertheless not be true of $(\iota x)(\phi x)$. This is possible because $f(\iota x)(\phi x)$ is not a value of $f\hat{x}$, so that even when all values of $f\hat{x}$ are true, $f(\iota x)(\phi x)$ may not be true. This happens when $(\iota x)(\phi x)$ does not exist. Thus for example we have $(x) . x = x$, but we do not have

the round square = the round square.

The inference $$(x) . fx . \supset . f(\iota x)(\phi x)$$

is only valid when $E ! (\iota x)(\phi x)$. As soon as we know $E ! (\iota x)(\phi x)$, the fact that $(\iota x)(\phi x)$ is an incomplete symbol becomes irrelevant so long as we confine ourselves to truth-functions* of whatever proposition is its scope. But even when $E ! (\iota x)(\phi x)$, the incompleteness of $(\iota x)(\phi x)$ may be relevant when we pass outside truth-functions. For example, George IV wished to know whether Scott was the author of Waverley, *i.e.* he wished to know whether a proposition of the form "$c = (\iota x)(\phi x)$" was true. But there was no proposition of the form "$c = y$" concerning which he wished to know if it was true.

In regard to classes, the relevance of their incompleteness is somewhat different. It may be illustrated by the fact that we may have

$$\hat{z}(\phi z) = \psi ! \hat{z} . \hat{z}(\phi z) = \chi ! \hat{z}$$

without having $$\psi ! \hat{z} = \chi ! \hat{z}.$$

For, by a direct application of the definitions, we find that

$$\vdash : \hat{z}(\phi z) = \psi ! \hat{z} . \equiv . \phi x \equiv_x \psi ! x.$$

Thus we shall have

$$\vdash : \phi x \equiv_x \psi ! x . \phi x \equiv_x \chi ! x . \supset . \hat{z}(\phi z) = \psi ! \hat{z} . \hat{z}(\phi z) = \chi ! \hat{z},$$

but we shall not necessarily have $\psi ! \hat{z} = \chi ! \hat{z}$ under these circumstances, for two functions may well be formally equivalent without being identical; for example,

$$x = \text{Scott} . \equiv_x . x = \text{the author of Waverley},$$

but the function "$\hat{z}$ = the author of Waverley" has the property that George IV wished to know whether its value with the argument "Scott" was true, whereas

* Cf. p. 8.

the function "$\hat{z} = \text{Scott}$" has no such property, and therefore the two functions are not identical. Hence there is a propositional function, namely

$$x = y \,.\, x = z \,.\supset.\, y = z,$$

which holds without any exception, and yet does not hold when for x we substitute a class, and for y and z we substitute functions. This is only possible because a class is an incomplete symbol, and therefore "$\hat{z}(\phi z) = \psi ! \hat{z}$" is not a value of "$x = y$."

It will be observed that "$\theta ! \hat{z} = \psi ! \hat{z}$" is not an extensional function of $\psi ! \hat{z}$. Thus the scope of $\hat{z}(\phi z)$ is relevant in interpreting the product

$$\hat{z}(\phi z) = \psi ! \hat{z} \,.\, \hat{z}(\phi z) = \chi ! \hat{z}.$$

If we take the whole of the product as the scope of $\hat{z}(\phi z)$, the product is equivalent to

$$(\exists \theta) : \phi x \equiv_x \theta ! x \,.\, \theta ! \hat{z} = \psi ! \hat{z} \,.\, \theta ! \hat{z} = \chi ! \hat{z},$$

and this *does* imply $\psi ! \hat{z} = \chi ! \hat{z}$.

We may say generally that the fact that $\hat{z}(\phi z)$ is an incomplete symbol is not relevant so long as we confine ourselves to extensional functions of functions, but is apt to become relevant for other functions of functions.

PART I

MATHEMATICAL LOGIC

SUMMARY OF PART I

In this Part, we shall deal with such topics as belong traditionally to symbolic logic, or deserve to belong to it in virtue of their generality. We shall, that is to say, establish such properties of propositions, propositional functions, classes and relations as are likely to be required in any mathematical reasoning, and not merely in this or that branch of mathematics.

The subjects treated in Part I may be viewed in two aspects: (1) as a deductive chain depending on the primitive propositions, (2) as a formal calculus. Taking the first view first: We begin, in $*1$, with certain axioms as to deduction of one proposition or asserted propositional function from another. From these primitive propositions, in Section A, we deduce various propositions which are all concerned with four ways of obtaining new propositions from given propositions, namely negation, disjunction, joint assertion and implication, of which the last two can be defined in terms of the first two. Throughout this first section, although, as will be shown at the beginning of Section B, our propositions, symbolically unchanged, will apply to any propositions as values of our variables, yet it will be supposed that our variable propositions are all what we shall call *elementary* propositions, *i.e.* such as contain no reference, explicit or implicit, to any totality. This restriction is imposed on account of the distinction between different *types* of propositions, explained in Chapter II of the Introduction. Its importance and purpose, however, are purely philosophical, and so long as only mathematical purposes are considered, it is unnecessary to remember this preliminary restriction to elementary propositions, which is symbolically removed at the beginning of the next section.

Section B deals, to begin with, with the relations of propositions containing apparent variables (*i.e.* involving the notions of "all" or "some") to each other and to propositions not containing apparent variables. We show that, where propositions containing apparent variables are concerned, we can define negation, disjunction, joint assertion and implication in such a way that their properties shall be exactly analogous to the properties of the corresponding ideas as applied to elementary propositions. We show also that *formal implication, i.e.* "$(x) . \phi x \supset \psi x$" considered as a relation of $\phi \hat{x}$ to $\psi \hat{x}$, has many properties analogous to those of *material implication, i.e.* "$p \supset q$" considered as a relation of p and q. We then consider *predicative* functions and the *axiom of reducibility*, which are vital in the employment of *functions* as apparent variables. An example of such employment is afforded by *identity*, which is the next topic considered in Section B. Finally, this section deals with *descriptions, i.e.* phrases of the form "the so-and-so" (in the singular). It is shown that the appearance of a grammatical subject "the so-and-so" is deceptive,

and that such propositions, fully stated, contain no such subject, but contain instead an apparent variable.

Section C deals with classes, and with relations in so far as they are analogous to classes. Classes and relations, like descriptions, are shown to be "incomplete symbols" (cf. Introduction, Chapter III), and it is shown that a proposition which is grammatically about a class is to be regarded as really concerned with a propositional function and an apparent variable whose values are *predicative* propositional functions (with a similar result for relations). The remainder of Section C deals with the calculus of classes, and with the calculus of relations in so far as it is analogous to that of classes.

Section D deals with those properties of relations which have no analogues for classes. In this section, a number of ideas and notations are introduced which are constantly needed throughout the rest of the work. Most of the properties of relations which have analogues in the theory of classes are comparatively unimportant, while those that have no such analogues are of the very greatest utility. It is partly for this reason that emphasis on the calculus-aspect of symbolic logic has proved a hindrance, hitherto, to the proper development of the theory of relations.

Section E, finally, extends the notions of the addition and multiplication of classes or relations to cases where the summands or factors are not individually given, but are given as the members of some class. The advantage obtained by this extension is that it enables us to deal with an infinite number of summands or factors.

Considered as a formal calculus, mathematical logic has three analogous branches, namely (1) the calculus of propositions, (2) the calculus of classes, (3) the calculus of relations. Of these, (1) is dealt with in Section A, while (2) and (3), in so far as they are analogous, are dealt with in Section C. We have, for each of the three, the four analogous ideas of negation, addition, multiplication, and implication or inclusion. Of these, negation is analogous to the negative in ordinary algebra, and implication or inclusion is analogous to the relation "less than or equal to" in ordinary algebra. But the analogy must not be pressed, as it has important limitations. The sum of two propositions is their disjunction, the sum of two classes is the class of terms belonging to one or other, the sum of two relations is the relation consisting in the fact that one or other of the two relations holds. The sum of a class of classes is the class of all terms belonging to some one or other of the classes, and the sum of a class of relations is the relation consisting in the fact that some one relation of the class holds. The product of two propositions is their joint assertion, the product of two classes is their common part, the product of two relations is the relation consisting in the fact that both the relations hold. The product of a class of classes is the part common to all of them, and the product of a class of relations is the relation consisting

in the fact that all relations of the class in question hold. The inclusion of one class in another consists in the fact that all members of the one are members of the other, while the inclusion of one relation in another consists in the fact that every pair of terms which has the one relation also has the other relation. It is then shown that the properties of negation, addition, multiplication and inclusion are exactly analogous for classes and relations, and are, with certain exceptions, analogous to the properties of negation, addition, multiplication and implication for propositions. (The exceptions arise chiefly from the fact that "p implies q" is itself a proposition, and can therefore imply and be implied, while "α is contained in β," where α and β are classes, is not a class, and can therefore neither contain nor be contained in another class γ.) But classes have certain properties not possessed by propositions: these arise from the fact that classes have not a *two-fold* division corresponding to the division of propositions into true and false, but a *three-fold* division, namely into (1) the universal class, which contains the whole of a certain type, (2) the null-class, which has no members, (3) all other classes, which neither contain nothing nor contain everything of the appropriate type. The resulting properties of classes, which are not analogous to properties of propositions, are dealt with in $*24$. And just as classes have properties not analogous to any properties of propositions, so relations have properties not analogous to any properties of classes, though all the properties of classes have analogues among relations. The special properties of relations are much more numerous and important than the properties belonging to classes but not to propositions. These special properties of relations therefore occupy a whole section, namely Section D.

SECTION A

THE THEORY OF DEDUCTION

THE purpose of the present section is to set forth the first stage of the deduction of pure mathematics from its logical foundations. This first stage is necessarily concerned with deduction itself, *i.e.* with the principles by which conclusions are inferred from premisses. If it is our purpose to make all our assumptions explicit, and to effect the deduction of all our other propositions from these assumptions, it is obvious that the first assumptions we need are those that are required to make deduction possible. Symbolic logic is often regarded as consisting of two coordinate parts, the theory of classes and the theory of propositions. But from our point of view these two parts are not coordinate; for in the theory of classes we deduce one proposition from another by means of principles belonging to the theory of propositions, whereas in the theory of propositions we nowhere require the theory of classes. Hence, in a deductive system, the theory of propositions necessarily precedes the theory of classes.

But the subject to be treated in what follows is not quite properly described as the theory of *propositions*. It is in fact the theory of how one proposition can be inferred from another. Now in order that one proposition may be inferred from another, it is necessary that the two should have that relation which makes the one a consequence of the other. When a proposition q is a consequence of a proposition p, we say that p *implies* q. Thus deduction depends upon the relation of *implication*, and every deductive system must contain among its premisses as many of the properties of implication as are necessary to legitimate the ordinary procedure of deduction. In the present section, certain propositions will be stated as premisses, and it will be shown that they are sufficient for all common forms of inference. It will not be shown that they are all *necessary*, and it is possible that the number of them might be diminished. All that is affirmed concerning the premisses is (1) that they are true, (2) that they are sufficient for the theory of deduction, (3) that we do not know how to diminish their number. But with regard to (2), there must always be some element of doubt, since it is hard to be sure that one never uses some principle unconsciously. The habit of being rigidly guided by formal symbolic rules is a safeguard against unconscious assumptions; but even this safeguard is not always adequate.

$*1$. PRIMITIVE IDEAS AND PROPOSITIONS

Since all definitions of terms are effected by means of other terms, every system of definitions which is not circular must start from a certain apparatus of undefined terms. It is to some extent optional what ideas we take as undefined in mathematics; the motives guiding our choice will be (1) to make the number of undefined ideas as small as possible, (2) as between two systems in which the number is equal, to choose the one which seems the simpler and easier. We know no way of proving that such and such a system of undefined ideas contains as few as will give such and such results*. Hence we can only say that such and such ideas are undefined in such and such a system, not that they are indefinable. Following Peano, we shall call the undefined ideas and the undemonstrated propositions *primitive* ideas and *primitive* propositions respectively. The primitive ideas are *explained* by means of descriptions intended to point out to the reader what is meant; but the explanations do not constitute definitions, because they really involve the ideas they explain.

In the present number, we shall first enumerate the primitive ideas required in this section; then we shall define *implication*; and then we shall enunciate the primitive propositions required in this section. Every definition or proposition in the work has a number, for purposes of reference. Following Peano, we use numbers having a decimal as well as an integral part, in order to be able to insert new propositions between any two. A change in the integral part of the number will be used to correspond to a new chapter. Definitions will generally have numbers whose decimal part is less than ·1, and will be usually put at the beginning of chapters. In references, the integral parts of the numbers of propositions will be distinguished by being preceded by a star; thus "$*1·01$" will mean the definition or proposition so numbered, and "$*1$" will mean the chapter in which propositions have numbers whose integral part is 1, *i.e.* the present chapter. Chapters will generally be called "numbers."

PRIMITIVE IDEAS.

(1) *Elementary propositions.* By an "elementary" proposition we mean one which does not involve any variables, or, in other language, one which does not involve such words as "all," "some," "the" or equivalents for such words. A proposition such as "this is red," where "this" is something given in sensation, will be elementary. Any combination of given elementary propositions by means of negation, disjunction or conjunction (see below) will

* The recognized methods of proving independence are not applicable, without reserve, to fundamentals. Cf. *Principles of Mathematics*, § 17. What is there said concerning primitive propositions applies with even greater force to primitive ideas.

be elementary. In the primitive propositions of the present number, and therefore in the deductions from these primitive propositions in $*2$—$*5$, the letters p, q, r, s will be used to denote elementary propositions.

(2) *Elementary propositional functions.* By an "elementary propositional function" we shall mean an expression containing an undetermined constituent, *i.e.* a variable, or several such constituents, and such that, when the undetermined constituent or constituents are determined, *i.e.* when values are assigned to the variable or variables, the resulting value of the expression in question is an elementary proposition. Thus if p is an undetermined elementary proposition, "not-p" is an elementary propositional function.

We shall show in $*9$ how to extend the results of this and the following numbers ($*1$—$*5$) to propositions which are not elementary.

(3) *Assertion.* Any proposition may be either asserted or merely considered. If I say "Caesar died," I assert the proposition "Caesar died," if I say "'Caesar died' is a proposition," I make a different assertion, and "Caesar died" is no longer asserted, but merely considered. Similarly in a hypothetical proposition, *e.g.* "if $a = b$, then $b = a$," we have two unasserted propositions, namely "$a = b$" and "$b = a$," while what *is* asserted is that the first of these implies the second. In language, we indicate when a proposition is merely considered by "*if* so-and-so" or "*that* so-and-so" or merely by inverted commas. In symbols, if p is a proposition, p by itself will stand for the unasserted proposition, while the asserted proposition will be designated by

$$``\vdash . p."$$

The sign "$\vdash$" is called the assertion-sign*; it may be read "it is true that" (although philosophically this is not exactly what it means). The dots after the assertion-sign indicate its range; that is to say, everything following is asserted until we reach either an equal number of dots preceding a sign of implication or the end of the sentence. Thus "$\vdash : p . \supset . q$" means "it is true that p implies q," whereas "$\vdash . p . \supset \vdash . q$" means "$p$ is true; therefore q is true†." The first of these does not necessarily involve the truth either of p or of q, while the second involves the truth of both.

(4) *Assertion of a propositional function.* Besides the assertion of definite propositions, we need what we shall call "assertion of a propositional function." The general notion of asserting *any* propositional function is not used until $*9$, but we use at once the notion of asserting various special *elementary* propositional functions. Let ϕx be a propositional function whose argument is x; then we may assert ϕx without assigning a value to x. This is done, for example, when the law of identity is asserted in the form "A is A." Here A is left undetermined, because, however A may be deter-

* We have adopted both the idea and the symbol of assertion from Frege.

† Cf. *Principles of Mathematics*, § 38.

mined, the result will be true. Thus when we assert ϕx, leaving x undetermined, we are asserting an ambiguous value of our function. This is only legitimate if, however the ambiguity may be determined, the result will be true. Thus take, as an illustration, the primitive proposition $*1\cdot2$ below, namely

$$\text{“} \vdash : p \vee p . \supset . p, \text{”}$$

i.e. "'p or p' implies p." Here p may be *any* elementary proposition: by leaving p undetermined, we obtain an assertion which can be applied to any particular elementary proposition. Such assertions are like the particular enunciations in Euclid: when it is said "let ABC be an isosceles triangle; then the angles at the base will be equal," what is said applies to *any* isosceles triangle; it is stated concerning *one* triangle, but not concerning a definite one. All the assertions in the present work, with a very few exceptions, assert propositional functions, not definite propositions.

As a matter of fact, no constant elementary proposition will occur in the present work, or can occur in any work which employs only logical ideas. The ideas and propositions of logic are all *general*: an assertion (for example) which is true of Socrates but not of Plato, will not belong to logic*, and if an assertion which is true of both is to occur in logic, it must not be made concerning either, but concerning a variable x. In order to obtain, in logic, a definite proposition instead of a propositional function, it is necessary to take some propositional function and assert that it is true always or sometimes, *i.e.* with all possible values of the variable or with some possible value. Thus, giving the name "individual" to whatever there is that is neither a proposition nor a function, the proposition "every individual is identical with itself" or the proposition "there are individuals" will be a proposition belonging to logic. But these propositions are not elementary.

(5) *Negation.* If p is any proposition, the proposition "not-p," or "p is false," will be represented by "$\sim p$." For the present, p must be an *elementary* proposition.

(6) *Disjunction.* If p and q are any propositions, the proposition "p or q," *i.e.* "either p is true or q is true," where the alternatives are to be not mutually exclusive, will be represented by

$$\text{“} p \vee q. \text{”}$$

This is called the *disjunction* or the *logical sum* of p and q. Thus "$\sim p \vee q$" will mean "p is false or q is true"; "$\sim (p \vee q)$" will mean "it is false that either p or q is true," which is equivalent to "p and q are both false"; "$\sim (\sim p \vee \sim q)$" will mean "it is false that either p is false or q is false," which is equivalent to "p and q are both true"; and so on. For the present, p and q must be elementary propositions.

* When we say that a proposition "belongs to logic," we mean that it can be expressed in terms of the primitive ideas of logic. We do not mean that logic *applies* to it, for that would of course be true of any proposition.

The above are all the primitive ideas required in the theory of deduction. Other primitive ideas will be introduced in Section B.

Definition of Implication. When a proposition q follows from a proposition p, so that if p is true, q must also be true, we say that p *implies* q. The idea of implication, in the form in which we require it, can be defined. The meaning to be given to implication in what follows may at first sight appear somewhat artificial; but although there are other legitimate meanings, the one here adopted is very much more convenient for our purposes than any of its rivals. The essential property that we require of implication is this: "What is implied by a true proposition is true." It is in virtue of this property that implication yields proofs. But this property by no means determines whether anything, and if so what, is implied by a false proposition. What it does determine is that, if p implies q, then it cannot be the case that p is true and q is false, *i.e.* it must be the case that either p is false or q is true. The most convenient interpretation of implication is to say, conversely, that if either p is false or q is true, then "p implies q" is to be true. Hence "p implies q" is to be defined to mean: "Either p is false or q is true." Hence we put:

∗1·01. $p \supset q \,.=.\, \sim p \lor q$ Df.

Here the letters "Df" stand for "definition." They and the sign of equality together are to be regarded as forming one symbol, standing for "is defined to mean*." Whatever comes to the left of the sign of equality is defined to mean the same as what comes to the right of it. Definition is not among the primitive ideas, because definitions are concerned solely with the symbolism, not with what is symbolised; they are introduced for practical convenience, and are theoretically unnecessary.

In virtue of the above definition, when "$p \supset q$" holds, then either p is false or q is true; hence if p is true, q must be true. Thus the above definition preserves the essential characteristic of implication; it gives, in fact, the most general meaning compatible with the preservation of this characteristic.

PRIMITIVE PROPOSITIONS.

∗1·1. Anything implied by a true elementary proposition is true. Pp†.

The above principle will be extended in ∗9 to propositions which are not elementary. It is not the same as "*if* p is true, then *if* p implies q, q is true." This is a true proposition, but it holds equally when p is not true and when p does not imply q. It does not, like the principle we are concerned with, enable us to assert q simply, without any hypothesis. We cannot express the principle symbolically, partly because any symbolism in which p is variable only gives the *hypothesis* that p is true, not the *fact* that it is true‡.

* The sign of equality not followed by the letters "Df" will have a different meaning, to be defined later.

† The letters "Pp" stand for "primitive proposition," as with Peano.

‡ For further remarks on this principle, cf. *Principles of Mathematics*, § 38.

The above principle is used whenever we have to deduce a *proposition* from a *proposition.* But the immense majority of the assertions in the present work are assertions of propositional functions, *i.e.* they contain an undetermined variable. Since the assertion of a propositional function is a different primitive idea from the assertion of a proposition, we require a primitive proposition different from $*1\cdot1$, though allied to it, to enable us to deduce the assertion of a propositional function "ψx" from the assertions of the two propositional functions "ϕx" and "$\phi x \supset \psi x$." This primitive proposition is as follows:

$*1\cdot11$. When ϕx can be asserted, where x is a real variable, and $\phi x \supset \psi x$ can be asserted, where x is a real variable, then ψx can be asserted, where x is a real variable. Pp.

This principle is also to be assumed for functions of several variables.

Part of the importance of the above primitive proposition is due to the fact that it expresses in the symbolism a result following from the theory of types, which requires symbolic recognition. Suppose we have the two assertions of *propositional functions* "$\vdash . \phi x$" and "$\vdash . \phi x \supset \psi x$"; then the "$x$" in ϕx is not absolutely anything, but anything for which as argument the function "ϕx" is significant; similarly in "$\phi x \supset \psi x$" the x is anything for which "$\phi x \supset \psi x$" is significant. Apart from some axiom, we do not know that the x's for which "$\phi x \supset \psi x$" is significant are the same as those for which "ϕx" is significant. The primitive proposition $*1\cdot11$, by securing that, as the result of the assertions of the *propositional functions* "ϕx" and "$\phi x \supset \psi x$," the propositional function "ψx" can also be asserted, secures partial symbolic recognition, in the form most useful in actual deductions, of an important principle which follows from the theory of types, namely that, if there is any one argument a for which both "ϕa" and "ψa" are significant, then the range of arguments for which "ϕx" is significant is the same as the range of arguments for which "ψx" is significant. It is obvious that, if the propositional function "$\phi x \supset \psi x$" can be asserted, there must be arguments a for which "$\phi a \supset \psi a$" is significant, and for which, therefore, "ϕa" and "ψa" must be significant. Hence, by our principle, the values of x for which "ϕx" is significant are the same as those for which "ψx" is significant, *i.e.* the type of possible arguments for $\phi \hat{x}$ (cf. p. 15) is the same as that of possible arguments for $\psi \hat{x}$. The primitive proposition $*1\cdot11$, since it states a practically important consequence of this fact, is called the "axiom of identification of type."

Another consequence of the principle that, if there is an argument a for which both ϕa and ψa are significant, then ϕx is significant whenever ψx is significant, and vice versa, will be given in the "axiom of identification of real variables," introduced in $*1\cdot72$. These two propositions, $*1\cdot11$ and $*1\cdot72$, give what is symbolically essential to the conduct of demonstrations in accordance with the theory of types.

The above proposition $*1{\cdot}11$ is used in every inference from one asserted propositional function to another. We will illustrate the use of this proposition by setting forth at length the way in which it is first used, in the proof of $*2{\cdot}06$. That proposition is

$$\text{“}\vdash :. p \supset q . \supset : q \supset r . \supset . p \supset r.\text{”}$$

We have already proved, in $*2{\cdot}05$, the proposition

$$\vdash :. q \supset r . \supset : p \supset q . \supset . p \supset r.$$

It is obvious that $*2{\cdot}06$ results from $*2{\cdot}05$ by means of $*2{\cdot}04$, which is

$$\vdash :. p . \supset . q \supset r : \supset : q . \supset . p \supset r.$$

For if, in this proposition, we replace p by $q \supset r$, q by $p \supset q$, and r by $p \supset r$, we obtain, as an instance of $*2{\cdot}04$, the proposition

$$\vdash :: q \supset r . \supset : p \supset q . \supset . p \supset r :. \supset :. p \supset q . \supset : q \supset r . \supset . p \supset r \qquad (1),$$

and here the hypothesis is asserted by $*2{\cdot}05$. Thus our primitive proposition $*1{\cdot}11$ enables us to assert the conclusion.

$*1{\cdot}2$. $\vdash : p \vee p . \supset . p$ Pp.

This proposition states: "If either p is true or p is true, then p is true." It is called the "principle of tautology," and will be quoted by the abbreviated title of "Taut." It is convenient, for purposes of reference, to give names to a few of the more important propositions; in general, propositions will be referred to by their numbers.

$*1{\cdot}3$. $\vdash : q . \supset . p \vee q$ Pp.

This principle states: "If q is true, then 'p or q' is true." Thus *e.g.* if q is "to-day is Wednesday" and p is "to-day is Tuesday," the principle states: "If to-day is Wednesday, then to-day is either Tuesday or Wednesday." It is called the "principle of addition," because it states that if a proposition is true, any alternative may be added without making it false. The principle will be referred to as "Add."

$*1{\cdot}4$. $\vdash : p \vee q . \supset . q \vee p$ Pp.

This principle states that "p or q" implies "q or p." It states the permutative law for logical addition of propositions, and will be called the "principle of permutation." It will be referred to as "Perm."

$*1{\cdot}5$. $\vdash : p \vee (q \vee r) . \supset . q \vee (p \vee r)$ Pp.

This principle states: "If either p is true, or 'q or r' is true, then either q is true, or 'p or r' is true." It is a form of the associative law for logical addition, and will be called the "associative principle." It will be referred to as "Assoc." The proposition

$$p \vee (q \vee r) . \supset . (p \vee q) \vee r,$$

which would be the natural form for the associative law, has less deductive power, and is therefore not taken as a primitive proposition.

$*1{\cdot}6$. $\vdash : q \supset r . \supset : p \vee q . \supset . p \vee r$ Pp.

This principle states: "If q implies r, then 'p or q' implies 'p or r.'" In other words, in an implication, an alternative may be added to both premiss and conclusion without impairing the truth of the implication. The principle will be called the "principle of summation," and will be referred to as "Sum."

$*1{\cdot}7$. If p is an elementary proposition, $\sim p$ is an elementary proposition. Pp.

$*1{\cdot}71$. If p and q are elementary propositions, $p \vee q$ is an elementary proposition. Pp.

$*1{\cdot}72$. If ϕp and ψp are elementary propositional functions which take elementary propositions as arguments, $\phi p \vee \psi p$ is an elementary propositional function. Pp.

This axiom is to apply also to functions of two or more variables. It is called the "axiom of identification of real variables." It will be observed that if ϕ and ψ are functions which take arguments of different types, there is no such function as "$\phi x \vee \psi x$," because ϕ and ψ cannot significantly have the same argument. A more general form of the above axiom will be given in $*9$.

The use of the above axioms $*1{\cdot}7{\cdot}71{\cdot}72$ will generally be tacit. It is only through them and the axioms of $*9$ that the theory of types explained in the Introduction becomes relevant, and any view of logic which justifies these axioms justifies such subsequent reasoning as employs the theory of types.

This completes the list of primitive propositions required for the theory of deduction as applied to elementary propositions.

$*2$. IMMEDIATE CONSEQUENCES OF THE PRIMITIVE PROPOSITIONS

Summary of $*2$.

The proofs of the earlier of the propositions of this number consist simply in noticing that they are instances of the general rules given in $*1$. In such cases, these rules are not premisses, since they assert any instance of themselves, not something other than their instances. Hence when a general rule is adduced in early proofs, it will be adduced in brackets*, with indications, when required, as to the changes of letters from those given in the rule to those in the case considered. Thus "Taut $\dfrac{\sim p}{p}$" will mean what "Taut" becomes when $\sim p$ is written in place of p. If "Taut $\dfrac{\sim p}{p}$" is enclosed in square brackets before an asserted proposition, that means that, in accordance with "Taut," we are asserting what "Taut" becomes when $\sim p$ is written in place of p. The recognition that a certain proposition is an instance of some general proposition previously proved or assumed is essential to the process of deduction from general rules, but cannot itself be erected into a general rule, since the application required is particular, and no general rule can *explicitly* include a particular application.

Again, when two different sets of symbols express the same proposition in virtue of a definition, say $*1{\cdot}01$, and one of these, which we will call (1), has been asserted, the assertion of the other is made by writing "$[(1).(*1{\cdot}01)]$" before it, meaning that, in virtue of $*1{\cdot}01$, the new set of symbols asserts the same proposition as was asserted in (1). A reference to a definition is distinguished from a reference to a previous proposition by being enclosed in round brackets.

The propositions in this number are all, or nearly all, actually needed in deducing mathematics from our primitive propositions. Although certain abbreviating processes will be gradually introduced, proofs will be given very fully, because the importance of the present subject lies, not in the propositions themselves, but (1) in the fact that they follow from the primitive propositions, (2) in the fact that the subject is the easiest, simplest, and most elementary example of the symbolic method of dealing with the principles of mathematics generally. Later portions—the theories of classes, relations, cardinal numbers, series, ordinal numbers, geometry, etc.—all employ the same method, but with an increasing complexity in the entities and functions considered.

* Later on we shall cease to mark the distinction between a premiss and a rule according to which an inference is conducted. It is only in early proofs that this distinction is important.

The most important propositions proved in the present number are the following:

$*2\cdot02.\ \vdash : q . \supset . p \supset q$

I.e. q implies that p implies q, *i.e.* a true proposition is implied by any proposition. This proposition is called the "principle of simplification" (referred to as "Simp"), because, as will appear later, it enables us to pass from the joint assertion of q and p to the assertion of q simply. When the special meaning which we have given to implication is remembered, it will be seen that this proposition is obvious.

$*2\cdot03.\ \vdash : p \supset \sim q . \supset . q \supset \sim p$

$*2\cdot15.\ \vdash : \sim p \supset q . \supset . \sim q \supset p$

$*2\cdot16.\ \vdash : p \supset q . \supset . \sim q \supset \sim p$

$*2\cdot17.\ \vdash : \sim q \supset \sim p . \supset . p \supset q$

These four analogous propositions constitute the "principle of transposition," referred to as "Transp." They lead to the rule that in an implication the two sides may be interchanged by turning negative into positive and positive into negative. They are thus analogous to the algebraical rule that the two sides of an equation may be interchanged by changing the signs.

$*2\cdot04.\ \vdash :. p . \supset . q \supset r : \supset : q . \supset . p \supset r$

This is called the "commutative principle" and referred to as "Comm." It states that, if r follows from q provided p is true, then r follows from p provided q is true.

$*2\cdot05.\ \vdash :. q \supset r . \supset : p \supset q . \supset . p \supset r$

$*2\cdot06.\ \vdash :. p \supset q . \supset : q \supset r . \supset . p \supset r$

These two propositions are the source of the syllogism in Barbara (as will be shown later) and are therefore called the "principle of the syllogism" (referred to as "Syll"). The first states that, if r follows from q, then if q follows from p, r follows from p. The second states the same thing with the premisses interchanged.

$*2\cdot08.\ \vdash . p \supset p$

I.e. any proposition implies itself. This is called the "principle of identity" and referred to as "Id." It is not the same as the "law of identity" ("x is identical with x"), but the law of identity is inferred from it (cf. $*13\cdot15$).

$*2\cdot21.\ \vdash : \sim p . \supset . p \supset q$

I.e. a false proposition implies any proposition.

The later propositions of the present number are mostly subsumed under propositions in $*3$ or $*4$, which give the same results in more compendious forms. We now proceed to formal deductions.

∗2·01. $\vdash : p \supset \sim p . \supset . \sim p$

This proposition states that, if p implies its own falsehood, then p is false. It is called the "principle of the *reductio ad absurdum*," and will be referred to as 'Abs."* The proof is as follows (where "*Dem.*" is short for "demonstration"):

Dem.

$$\left[\text{Taut}\ \frac{\sim p}{p}\right] \quad \vdash : \sim p \vee \sim p . \supset . \sim p \qquad (1)$$

$$[(1).(∗1{\cdot}01)] \quad \vdash : p \supset \sim p . \supset . \sim p$$

∗2·02. $\vdash : q . \supset . p \supset q$

Dem.

$$\left[\text{Add}\ \frac{\sim p}{p}\right] \quad \vdash : q . \supset . \sim p \vee q \qquad (1)$$

$$[(1).(∗1{\cdot}01)] \quad \vdash : q . \supset . p \supset q$$

∗2·03. $\vdash : p \supset \sim q . \supset . q \supset \sim p$

Dem.

$$\left[\text{Perm}\ \frac{\sim p, \sim q}{p, \quad q}\right] \quad \vdash : \sim p \vee \sim q . \supset . \sim q \vee \sim p \qquad (1)$$

$$[(1).(∗1{\cdot}01)] \quad \vdash : p \supset \sim q . \supset . q \supset \sim p$$

∗2·04. $\vdash :. p . \supset . q \supset r : \supset : q . \supset . p \supset r$

Dem.

$$\left[\text{Assoc}\ \frac{\sim p, \sim q}{p, \quad q}\right] \quad \vdash :. \sim p \vee (\sim q \vee r) . \supset . \sim q \vee (\sim p \vee r) \qquad (1)$$

$$[(1).(∗1{\cdot}01)] \quad \vdash :. p . \supset . q \supset r : \supset : q . \supset . p \supset r$$

∗2·05. $\vdash :. q \supset r . \supset : p \supset q . \supset . p \supset r$

Dem.

$$\left[\text{Sum}\ \frac{\sim p}{p}\right] \quad \vdash :. q \supset r . \supset : \sim p \vee q . \supset . \sim p \vee r \qquad (1)$$

$$[(1).(∗1{\cdot}01)] \quad \vdash :. q \supset r . \supset : p \supset q . \supset . p \supset r$$

∗2·06. $\vdash :. p \supset q . \supset : q \supset r . \supset . p \supset r$

Dem.

$$\left[\text{Comm}\ \frac{q \supset r, p \supset q, p \supset r}{p, \qquad q, \qquad r}\right] \quad \vdash :: q \supset r . \supset : p \supset q . \supset . p \supset r :.$$

$$\supset :. p \supset q . \supset : q \supset r . \supset . p \supset r \qquad (1)$$

$$[∗2{\cdot}05] \quad \vdash :. q \supset r . \supset : p \supset q . \supset . p \supset r \qquad (2)$$

$$[(1).(2).∗1{\cdot}11] \quad \vdash :. p \supset q . \supset : q \supset r . \supset . p \supset r$$

In the last line of this proof, "(1) . (2) . ∗1·11" means that we are inferring in accordance with ∗1·11, having before us a proposition, namely $p \supset q . \supset : q \supset r . \supset . p \supset r$, which, by (1), is implied by $q \supset r . \supset : p \supset q . \supset . p \supset r$, which, by (2), is true. In general, in such cases, we shall omit the reference to ∗1·11.

* There is an interesting historical article on this principle by Vailati, "A proposito d' un passo del Teeteto e di una dimostrazione di Euclide," *Rivista di Filosofia e scienze affine*, 1904.

The above two propositions will both be referred to as the "principle of the syllogism" (shortened to "Syll"), because, as will appear later, the syllogism in Barbara is derived from them.

$*2\cdot07$. $\vdash : p \,.\, \supset . \, p \vee p \quad \left[*1\cdot3\,\frac{p}{q}\right]$

Here we put nothing beyond "$*1\cdot3\,\frac{p}{q}$," because the proposition to be proved is what $*1\cdot3$ becomes when p is written in place of q.

$*2\cdot08$. $\vdash . \, p \supset p$

Dem.

$\left[*2\cdot05\,\frac{p\vee p,\, p}{q,\quad r}\right] \quad \vdash :: p \vee p \,.\, \supset . \, p : \supset :. \, p \,.\, \supset . \, p \vee p : \supset . \, p \supset p \qquad (1)$

$[\text{Taut}] \quad \vdash : p \vee p \,.\, \supset . \, p \qquad (2)$

$[(1).(2).*1\cdot11] \quad \vdash :. \, p \,.\, \supset . \, p \vee p : \supset . \, p \supset p \qquad (3)$

$[*2\cdot07] \quad \vdash : p \,.\, \supset . \, p \vee p \qquad (4)$

$[(3).(4).*1\cdot11] \quad \vdash . \, p \supset p$

$*2\cdot1$. $\vdash . \sim p \vee p \quad [*2\cdot08 \,.\, (*1\cdot01)]$

$*2\cdot11$. $\vdash . \, p \vee \sim p$

Dem.

$\left[\text{Perm}\,\frac{\sim p,\, p}{p,\quad q}\right] \quad \vdash : \sim p \vee p \,.\, \supset . \, p \vee \sim p \qquad (1)$

$[(1).*2\cdot1.*1\cdot11] \quad \vdash . \, p \vee \sim p$

This is the law of excluded middle.

$*2\cdot12$. $\vdash . \, p \supset \sim(\sim p)$

Dem.

$\left[*2\cdot11\,\frac{\sim p}{p}\right] \quad \vdash . \sim p \vee \sim(\sim p) \qquad (1)$

$[(1).(*1\cdot01)] \quad \vdash . \, p \supset \sim(\sim p)$

$*2\cdot13$. $\vdash . \, p \vee \sim\{\sim(\sim p)\}$

This proposition is a lemma for $*2\cdot14$, which, with $*2\cdot12$, constitutes the principle of double negation.

Dem.

$\left[\text{Sum}\,\frac{\sim p,\, \sim\{\sim(\sim p)\}}{q,\qquad r}\right] \quad \vdash :. \sim p \,.\, \supset . \sim\{\sim(\sim p)\} \,.\, \supset :$
$\qquad p \vee \sim p \,.\, \supset . \, p \vee \sim\{\sim(\sim p)\} \qquad (1)$

$\left[*2\cdot12\,\frac{\sim p}{p}\right] \quad \vdash : \sim p \,.\, \supset . \sim\{\sim(\sim p)\} \qquad (2)$

$[(1).(2).*1\cdot11] \quad \vdash : p \vee \sim p \,.\, \supset . \, p \vee \sim\{\sim(\sim p)\} \qquad (3)$

$[(3).*2\cdot11.*1\cdot11] \quad \vdash . \, p \vee \sim\{\sim(\sim p)\}$

***2·14.** $\vdash . \sim(\sim p) \supset p$

Dem.

$$\left[\text{Perm}\,\frac{\sim\{\sim(\sim p)\}}{q}\right] \vdash : p \lor \sim\{\sim(\sim p)\} . \supset . \sim\{\sim(\sim p)\} \lor p \quad (1)$$

$$[(1).*2\cdot13.*1\cdot11] \quad \vdash . \sim\{\sim(\sim p)\} \lor p \quad (2)$$

$$[(2).(*1\cdot01)] \quad \vdash . \sim(\sim p) \supset p$$

***2·15.** $\vdash : \sim p \supset q . \supset . \sim q \supset p$

Dem.

$$\left[*2\cdot05\,\frac{\sim p, \sim(\sim q)}{p, \quad r}\right] \vdash :. q \supset \sim(\sim q) . \supset : \sim p \supset q . \supset . \sim p \supset \sim(\sim q) \quad (1)$$

$$\left[*2\cdot12\,\frac{q}{p}\right] \quad \vdash . q \supset \sim(\sim q) \quad (2)$$

$$[(1).(2).*1\cdot11] \quad \vdash : \sim p \supset q . \supset . \sim p \supset \sim(\sim q) \quad (3)$$

$$\left[*2\cdot03\,\frac{\sim p, \sim q}{p, \quad q}\right] \quad \vdash : \sim p \supset \sim(\sim q) . \supset . \sim q \supset \sim(\sim p) \quad (4)$$

$$\left[*2\cdot05\,\frac{\sim q, \sim(\sim p), p}{p, \quad q, \quad r}\right] \vdash :. \sim(\sim p) \supset p . \supset : \sim q \supset \sim(\sim p) . \supset . \sim q \supset p \quad (5)$$

$$[(5).*2\cdot14.*1\cdot11] \quad \vdash : \sim q \supset \sim(\sim p) . \supset . \sim q \supset p \quad (6)$$

$$\left[*2\cdot05\,\frac{\sim p \supset q, \sim p \supset \sim(\sim q), \sim q \supset \sim(\sim p)}{p, \quad q, \quad r}\right] \vdash ::$$

$$\sim p \supset \sim(\sim q) . \supset . \sim q \supset \sim(\sim p) : \supset :.$$

$$\sim p \supset q . \supset . \sim p \supset \sim(\sim q) : \supset : \sim p \supset q . \supset . \sim q \supset \sim(\sim p) \quad (7)$$

$$[(4).(7).*1\cdot11] \vdash :. \sim p \supset q . \supset . \sim p \supset \sim(\sim q) : \supset :$$

$$\sim p \supset q . \supset . \sim q \supset \sim(\sim p) \quad (8)$$

$$[(3).(8).*1\cdot11] \vdash : \sim p \supset q . \supset . \sim q \supset \sim(\sim p) \quad (9)$$

$$\left[*2\cdot05\,\frac{\sim p \supset q, \sim q \supset \sim(\sim p), \sim q \supset p}{p, \quad q, \quad r}\right] \vdash :: \sim q \supset \sim(\sim p) . \supset . \sim q \supset p :$$

$$\supset :. \sim p \supset q . \supset . \sim q \supset \sim(\sim p) : \supset : \sim p \supset q . \supset . \sim q \supset p \quad (10)$$

$$[(6).(10).*1\cdot11] \vdash :. \sim p \supset q . \supset . \sim q \supset \sim(\sim p) : \supset :$$

$$\sim p \supset q . \supset . \sim q \supset p \quad (11)$$

$$[(9).(11).*1\cdot11] \vdash : \sim p \supset q . \supset . \sim q \supset p$$

Note on the proof of *2·15. In the above proof, it will be seen that (3), (4), (6) are respectively of the forms $p_1 \supset p_2$, $p_2 \supset p_3$, $p_3 \supset p_4$, where $p_1 \supset p_4$ is the proposition to be proved. From $p_1 \supset p_2$, $p_2 \supset p_3$, $p_3 \supset p_4$ the proposition $p_1 \supset p_4$ results by repeated applications of *2·05 or *2·06 (both of which are called "Syll"). It is tedious and unnecessary to repeat this process every time it is used; it will therefore be abbreviated into

"[Syll] $\vdash . (a) . (b) . (c) . \supset \vdash . (d)$,"

where (a) is of the form $p_1 \supset p_2$, (b) of the form $p_2 \supset p_3$, (c) of the form $p_3 \supset p_4$, and (d) of the form $p_1 \supset p_4$. The same abbreviation will be applied to a sorites of any length.

Also where we have "$\vdash . p_1$" and "$\vdash . p_1 \supset p_2$," and p_2 is the proposition to be proved, it is convenient to write simply

$$\text{[etc.]} \qquad \begin{array}{l} \text{"}\vdash . p_1 . \supset \\ \vdash . p_2\text{,"} \end{array}$$

where "etc." will be a reference to the previous propositions in virtue of which the implication "$p_1 \supset p_2$" holds. This form embodies the use of $*1{\cdot}11$ or $*1{\cdot}1$, and makes many proofs at once shorter and easier to follow. It is used in the first two lines of the following proof.

$*2{\cdot}16$. $\vdash : p \supset q . \supset . \sim q \supset \sim p$

Dem.

$$\begin{array}{lll}
[*2{\cdot}12] & \vdash . q \supset \sim(\sim q) . \supset & \\
[*2{\cdot}05] & \vdash : p \supset q . \supset . p \supset \sim(\sim q) & (1) \\
\left[*2{\cdot}03 \dfrac{\sim q}{q}\right] & \vdash : p \supset \sim(\sim q) . \supset . \sim q \supset \sim p & (2) \\
[\text{Syll}] & \vdash . (1) . (2) . \supset \vdash : p \supset q . \supset . \sim q \supset \sim p &
\end{array}$$

Note. The proposition to be proved will be called "Prop," and when a proof ends, like that of $*2{\cdot}16$, by an implication between asserted propositions, of which the consequent is the proposition to be proved, we shall write "$\vdash$. etc. $\supset \vdash$. Prop". Thus "$\supset \vdash$. Prop" ends a proof, and more or less corresponds to "Q.E.D."

$*2{\cdot}17$. $\vdash : \sim q \supset \sim p . \supset . p \supset q$

Dem.

$$\begin{array}{lll}
\left[*2{\cdot}03 \dfrac{\sim q, p}{p, q}\right] & \vdash : \sim q \supset \sim p . \supset . p \supset \sim(\sim q) & (1) \\
[*2{\cdot}14] & \vdash : \sim(\sim q) \supset q : \supset & \\
[*2{\cdot}05] & \vdash : p \supset \sim(\sim q) . \supset . p \supset q & (2) \\
[\text{Syll}] & \vdash . (1) . (2) . \supset \vdash . \text{Prop} &
\end{array}$$

$*2{\cdot}15$, $*2{\cdot}16$ and $*2{\cdot}17$ are forms of the principle of transposition, and will be all referred to as "Transp."

$*2{\cdot}18$. $\vdash : \sim p \supset p . \supset . p$

Dem.

$$\begin{array}{lll}
[*2{\cdot}12] & \vdash . p \supset \sim(\sim p) . \supset & \\
[*2{\cdot}05] & \vdash . \sim p \supset p . \supset . \sim p \supset \sim(\sim p) & (1) \\
\left[*2{\cdot}01 \dfrac{\sim p}{p}\right] & \vdash : \sim p \supset \sim(\sim p) . \supset . \sim(\sim p) & (2) \\
[\text{Syll}] & \vdash . (1) . (2) . \supset \vdash : \sim p \supset p . \supset . \sim(\sim p) & (3) \\
[*2{\cdot}14] & \vdash . \sim(\sim p) \supset p & (4) \\
[\text{Syll}] & \vdash . (3) . (4) . \supset \vdash . \text{Prop} &
\end{array}$$

This is the complement of the principle of the *reductio ad absurdum*. It

states that a proposition which follows from the hypothesis of its own falsehood is true.

$*2{\cdot}2.\ \vdash : p\,.\supset .\,p \vee q$

Dem.

$$\begin{aligned} &\vdash .\,\text{Add}\,.\supset \vdash : p\,.\supset .\,q \vee p && (1)\\ [\text{Perm}]\ &\vdash : q \vee p\,.\supset .\,p \vee q && (2)\\ [\text{Syll}]\ &\vdash .\,(1)\,.\,(2)\,.\supset \vdash .\,\text{Prop} \end{aligned}$$

$*2{\cdot}21.\ \vdash : \sim p\,.\supset .\,p \supset q \quad \left[*2{\cdot}2\,\dfrac{\sim p}{p}\right]$

The above two propositions are very frequently used.

$*2{\cdot}24.\ \vdash : p\,.\supset .\sim p \supset q \quad [*2{\cdot}21\,.\,\text{Comm}]$

$*2{\cdot}25.\ \vdash :.\, p : \vee : p \vee q\,.\supset .\,q$

Dem.

$$\begin{aligned} &\vdash .\,*2{\cdot}1\,.\supset \vdash : \sim(p \vee q)\,.\vee .\,(p \vee q) :\\ [\text{Assoc}]\ &\supset \vdash : p\,.\vee .\,\{\sim(p \vee q)\,.\vee .\,q\} : \supset \vdash .\,\text{Prop} \end{aligned}$$

$*2{\cdot}26.\ \vdash :.\sim p : \vee : p \supset q\,.\supset .\,q \quad \left[*2{\cdot}25\,\dfrac{\sim p}{p}\right]$

$*2{\cdot}27.\ \vdash :.\, p\,.\supset : p \supset q\,.\supset .\,q \quad [*2{\cdot}26]$

$*2{\cdot}3.\ \vdash : p \vee (q \vee r)\,.\supset .\,p \vee (r \vee q)$

Dem.

$$\begin{aligned} &\left[\text{Perm}\,\frac{q,\ r}{p,\ q}\right] && \vdash : q \vee r\,.\supset .\,r \vee q :\\ &\left[\text{Sum}\,\frac{q \vee r,\ r \vee q}{q,\quad r}\right] && \supset \vdash : p \vee (q \vee r)\,.\supset .\,p \vee (r \vee q) \end{aligned}$$

$*2{\cdot}31.\ \vdash : p \vee (q \vee r)\,.\supset .\,(p \vee q) \vee r$

This proposition and $*2{\cdot}32$ together constitute the associative law for logical addition of propositions. In the proof, the following abbreviation (constantly used hereafter) will be employed*: When we have a series of propositions of the form $a \supset b$, $b \supset c$, $c \supset d$, all asserted, and "$a \supset d$" is the proposition to be proved, the proof in full is as follows:

$$\begin{aligned} [\text{Syll}]\ &\vdash :.\, a \supset b\,.\supset : b \supset c\,.\supset .\,a \supset c && (1)\\ &\vdash : a\,.\supset .\,b && (2)\\ [(1).(2).*1{\cdot}11]\ &\vdash : b \supset c\,.\supset .\,a \supset c && (3)\\ &\vdash : b\,.\supset .\,c && (4)\\ [(3).(4).*1{\cdot}11]\ &\vdash : a\,.\supset .\,c && (5)\\ [\text{Syll}]\ &\vdash :.\, a \supset c\,.\supset : c \supset d\,.\supset .\,a \supset d && (6)\\ [(5).(6).*1{\cdot}11]\ &\vdash : c \supset d\,.\supset .\,a \supset d && (7)\\ &\vdash : c\,.\supset .\,d && (8)\\ [(7).(8).*1{\cdot}11]\ &\vdash : a\,.\supset .\,d \end{aligned}$$

* This abbreviation applies to the same type of cases as those concerned in the note to $*2{\cdot}15$, but is often more convenient than the abbreviation explained in that note.

It is tedious to write out this process in full; we therefore write simply

$$\begin{array}{l} \vdash : a \,.\supset.\, b \,. \\ \text{[etc.]} \supset .\, c \,. \\ \text{[etc.]} \supset .\, d : \supset \vdash .\, \text{Prop}, \end{array}$$

where "$a \supset d$" is the proposition to be proved. We indicate on the left by references in square brackets the propositions in virtue of which the successive implications hold. We put one dot (not two) after "b," to show that it is b, not "$a \supset b$," that implies c. But we put two dots after d, to show that now the whole proposition "$a \supset d$" is concerned. If "$a \supset d$" is not the proposition to be proved, but is to be used subsequently in the proof, we put

$$\begin{array}{l} \vdash : a \,.\supset.\, b \,. \\ \text{[etc.]} \supset .\, c \,. \\ \text{[etc.]} \supset .\, d \end{array} \qquad (1),$$

and then "(1)" means "$a \supset d$." The proof of $*2{\cdot}31$ is as follows:

Dem.

$$\begin{array}{ll} [*2{\cdot}3] \vdash : p \vee (q \vee r) \,.\supset.\, p \vee (r \vee q) \,. & \\ \left[\text{Assoc}\,\dfrac{r,\, q}{q,\, r}\right] & \supset .\, r \vee (p \vee q) \,. \\ \left[\text{Perm}\,\dfrac{r,\, p \vee q}{p,\ \ q}\right] & \supset .\, (p \vee q) \vee r : \supset \vdash .\, \text{Prop} \end{array}$$

$*2{\cdot}32$. $\vdash : (p \vee q) \vee r \,.\supset.\, p \vee (q \vee r)$

Dem.

$$\begin{array}{ll} \left[\text{Perm}\,\dfrac{p \vee q,\, r}{p,\ \ q}\right] \vdash : (p \vee q) \vee r \,.\supset.\, r \vee (p \vee q) & \\ \left[\text{Assoc}\,\dfrac{r,\, p,\, q}{p,\, q,\, r}\right] & \supset .\, p \vee (r \vee q) \\ [*2{\cdot}3] & \supset .\, p \vee (q \vee r) : \supset \vdash .\, \text{Prop} \end{array}$$

$*2{\cdot}33$. $p \vee q \vee r \,.=.\, (p \vee q) \vee r \quad \text{Df}$

This definition serves only for the avoidance of brackets.

$*2{\cdot}36$. $\vdash :.\, q \supset r \,.\supset:\, p \vee q \,.\supset.\, r \vee p$

Dem.

$$\begin{array}{lll} [\text{Perm}] & \vdash : p \vee r \,.\supset.\, r \vee p : & \\ \left[\text{Syll}\,\dfrac{p \vee q,\, p \vee r,\, r \vee p}{p,\quad q,\quad r}\right] & \supset \vdash :.\, p \vee q \,.\supset.\, p \vee r : \supset : p \vee q \,.\supset.\, r \vee p & (1) \\ [\text{Sum}] & \vdash :.\, q \supset r \,.\supset:\, p \vee q \,.\supset.\, p \vee r & (2) \\ \vdash .\, (1) \,.\, (2) \,.\, \text{Syll} \,.\supset \vdash .\, \text{Prop} & & \end{array}$$

$*2{\cdot}37$. $\vdash :.\, q \supset r \,.\supset:\, q \vee p \,.\supset.\, p \vee r$

[Syll . Perm . Sum]

$*2{\cdot}38$. $\vdash :.\, q \supset r \,.\supset:\, q \vee p \,.\supset.\, r \vee p$

[Syll . Perm . Sum]

The proofs of $*2{\cdot}37{\cdot}38$ are exactly analogous to that of $*2{\cdot}36$. (We use "$*2{\cdot}37{\cdot}38$" as an abbreviation for "$*2{\cdot}37$ and $*2{\cdot}38$." Such abbreviations will be used throughout.)

The use of a general principle of deduction, such as either form of "Syll," in a proof, is different from the use of the particular premisses to which the principle of deduction is applied. The principle of deduction gives the general rule according to which the inference is made, but is not itself a premiss in the inference. If we treated it as a premiss, we should need either it or some other general rule to enable us to infer the desired conclusion, and thus we should gradually acquire an increasing accumulation of premisses without ever being able to make any inference. Thus when a general rule is adduced in drawing an inference, as when we write "[Syll] $\vdash . (1) . (2) . \supset \vdash .$ Prop," the mention of "Syll" is only required in order to remind the reader how the inference is drawn.

The rule of inference may, however, also occur as one of the ordinary premisses, that is to say, in the case of "Syll" for example, the proposition "$p \supset q . \supset : q \supset r . \supset . p \supset r$" may be one of those to which our rules of deduction are applied, and it is then an ordinary premiss. The distinction between the two uses of principles of deduction is of some philosophical importance, and in the above proofs we have indicated it by putting the rule of inference in square brackets. It is, however, practically inconvenient to continue to distinguish in the manner of the reference. We shall therefore henceforth both adduce ordinary premisses in square brackets where convenient, and adduce rules of inference, along with other propositions, in asserted premisses, *i.e.* we shall write *e.g.*

"$\vdash . (1) . (2) .$ Syll $. \supset \vdash .$ Prop"

rather than "[Syll] $\vdash . (1) . (2) . \supset \vdash .$ Prop"

$*2{\cdot}4$. $\vdash :. p . \vee . p \vee q : \supset . p \vee q$

Dem.

$\vdash . *2{\cdot}31 . \supset \vdash :. p . \vee . p \vee q : \supset : p \vee p . \vee . q :$

[Taut.$*2{\cdot}38$] $\supset : p \vee q :. \supset \vdash .$ Prop

$*2{\cdot}41$. $\vdash :. q . \vee . p \vee q : \supset . p \vee q$

Dem.

$\left[\text{Assoc}\dfrac{q, p, q}{p, q, r}\right] \vdash :. q . \vee . p \vee q : \supset : p . \vee . q \vee q :$

[Taut.Sum] $\supset : p \vee q :. \supset \vdash .$ Prop

$*2{\cdot}42$. $\vdash :. \sim p . \vee . p \supset q : \supset . p \supset q$ $\left[*2{\cdot}4\dfrac{\sim p}{p}\right]$

$*2{\cdot}43$. $\vdash :. p . \supset . p \supset q : \supset . p \supset q$ [$*2{\cdot}42$]

$*2{\cdot}45$. $\vdash : \sim(p \vee q) . \supset . \sim p$ [$*2{\cdot}2$. Transp]

$*2{\cdot}46$. $\vdash : \sim(p \vee q) . \supset . \sim q$ [$*1{\cdot}3$. Transp]

$*2{\cdot}47.\quad \vdash : \sim(p \vee q) \,.\supset .\, \sim p \vee q \qquad \left[*2{\cdot}45 \,.\, *2{\cdot}2 \dfrac{\sim p}{p} \,.\, \mathrm{Syll}\right]$

$*2{\cdot}48.\quad \vdash : \sim(p \vee q) \,.\supset .\, p \vee \sim q \qquad \left[*2{\cdot}46 \,.\, *1{\cdot}3 \dfrac{\sim q}{q} \,.\, \mathrm{Syll}\right]$

$*2{\cdot}49.\quad \vdash : \sim(p \vee q) \,.\supset .\, \sim p \vee \sim q \qquad \left[*2{\cdot}45 \,.\, *2{\cdot}2 \dfrac{\sim p, \sim q}{p, \quad q} \,.\, \mathrm{Syll}\right]$

$*2{\cdot}5.\quad \vdash : \sim(p \supset q) \,.\supset .\, \sim p \supset q \qquad \left[*2{\cdot}47 \dfrac{\sim p}{p}\right]$

$*2{\cdot}51.\quad \vdash : \sim(p \supset q) \,.\supset .\, p \supset \sim q \qquad \left[*2{\cdot}48 \dfrac{\sim p}{p}\right]$

$*2{\cdot}52.\quad \vdash : \sim(p \supset q) \,.\supset .\, \sim p \supset \sim q \qquad \left[*2{\cdot}49 \dfrac{\sim p}{p}\right]$

$*2{\cdot}521.\quad \vdash : \sim(p \supset q) \,.\supset .\, q \supset p \qquad [*2{\cdot}52{\cdot}17]$

$*2{\cdot}53.\quad \vdash : p \vee q \,.\supset .\, \sim p \supset q$

Dem.

$\vdash . *2{\cdot}12{\cdot}38 \,.\supset \vdash : p \vee q \,.\supset .\, \sim(\sim p) \vee q : \supset \vdash . \mathrm{Prop}$

$*2{\cdot}54.\quad \vdash : \sim p \supset q \,.\supset .\, p \vee q \qquad [*2{\cdot}14{\cdot}38]$

$*2{\cdot}55.\quad \vdash :. \sim p \,.\supset : p \vee q \,.\supset .\, q \qquad [*2{\cdot}53 \,.\, \mathrm{Comm}]$

$*2{\cdot}56.\quad \vdash :. \sim q \,.\supset : p \vee q \,.\supset .\, p \qquad \left[*2{\cdot}55 \dfrac{q, p}{p, q} \,.\, \mathrm{Perm}\right]$

$*2{\cdot}6.\quad \vdash :. \sim p \supset q \,.\supset : p \supset q \,.\supset .\, q$

Dem.

$[*2{\cdot}38] \qquad \vdash :. \sim p \supset q \,.\supset : \sim p \vee q \,.\supset .\, q \vee q \qquad (1)$

$[\mathrm{Taut} \,.\, \mathrm{Syll}] \quad \vdash :. \sim p \vee q \,.\supset .\, q \vee q : \supset : \sim p \vee q \,.\supset .\, q \qquad (2)$

$\vdash . (1) . (2) . \mathrm{Syll} \,.\supset \vdash :. \sim p \supset q \,.\supset : \sim p \vee q \,.\supset .\, q :. \supset \vdash . \mathrm{Prop}$

$*2{\cdot}61.\quad \vdash :. p \supset q \,.\supset : \sim p \supset q \,.\supset .\, q \qquad [*2{\cdot}6 \,.\, \mathrm{Comm}]$

$*2{\cdot}62.\quad \vdash :. p \vee q \,.\supset : p \supset q \,.\supset .\, q \qquad [*2{\cdot}53{\cdot}6 \,.\, \mathrm{Syll}]$

$*2{\cdot}621.\quad \vdash :. p \supset q \,.\supset : p \vee q \,.\supset .\, q \qquad [*2{\cdot}62 \,.\, \mathrm{Comm}]$

$*2{\cdot}63.\quad \vdash :. p \vee q \,.\supset : \sim p \vee q \,.\supset .\, q \qquad [*2{\cdot}62]$

$*2{\cdot}64.\quad \vdash :. p \vee q \,.\supset : p \vee \sim q \,.\supset .\, p \qquad \left[*2{\cdot}63 \dfrac{q, p}{p, q} \,.\, \mathrm{Perm}\right]$

$*2{\cdot}65.\quad \vdash :. p \supset q \,.\supset : p \supset \sim q \,.\supset .\, \sim p \qquad \left[*2{\cdot}64 \dfrac{\sim p}{p}\right]$

$*2{\cdot}67.\quad \vdash :. p \vee q \,.\supset .\, q : \supset .\, p \supset q$

Dem.

$[*2{\cdot}54 . \mathrm{Syll}] \quad \vdash :. p \vee q \,.\supset .\, q : \supset : \sim p \supset q \,.\supset .\, q \qquad (1)$

$[*2{\cdot}24 . \mathrm{Syll}] \quad \vdash :. \sim p \supset q \,.\supset .\, q : \supset .\, p \supset q \qquad (2)$

$\vdash . (1) . (2) . \mathrm{Syll} \,.\supset \vdash . \mathrm{Prop}$

***2·68.** $\vdash :. p \supset q . \supset . q : \supset . p \vee q$

Dem.

$$\left[*2{\cdot}67 \frac{\sim p}{p}\right] \vdash :. p \supset q . \supset . q : \supset . \sim p \supset q \quad (1)$$

$\vdash . (1) . *2{\cdot}54 . \supset \vdash .$ Prop

***2·69.** $\vdash :. p \supset q . \supset . q : \supset : q \supset p . \supset . p$ $\quad \left[*2{\cdot}68 . \text{Perm} . *2{\cdot}62 \frac{q, p}{p, q}\right]$

***2·73.** $\vdash :. p \supset q . \supset : p \vee q \vee r . \supset . q \vee r$ $\quad [*2{\cdot}621{\cdot}38]$

***2·74.** $\vdash :. q \supset p . \supset : p \vee q \vee r . \supset . p \vee r$ $\quad \left[*2{\cdot}73 \frac{q, p}{p, q} . \text{Assoc} . \text{Syll}\right]$

***2·75.** $\vdash :: p \vee q . \supset :. p . \vee . q \supset r : \supset . p \vee r$ $\quad \left[*2{\cdot}74 \frac{\sim q}{q} . *2{\cdot}53{\cdot}31\right]$

***2·76.** $\vdash :. p . \vee . q \supset r : \supset : p \vee q . \supset . p \vee r$ $\quad [*2{\cdot}75 . \text{Comm}]$

***2·77.** $\vdash :. p . \supset . q \supset r : \supset : p \supset q . \supset . p \supset r$ $\quad \left[*2{\cdot}76 \frac{\sim p}{p}\right]$

***2·8.** $\vdash :. q \vee r . \supset : \sim r \vee s . \supset . q \vee s$

Dem.

$\vdash . *2{\cdot}53 . \text{Perm} . \supset \vdash :. q \vee r . \supset : \sim r \supset q :$

$[*2{\cdot}38]$ $\quad \supset : \sim r \vee s . \supset . q \vee s :. \supset \vdash .$ Prop

***2·81.** $\vdash :: q . \supset . r \supset s : \supset :. p \vee q . \supset : p \vee r . \supset . p \vee s$

Dem.

$\vdash . \text{Sum} . \supset \vdash :: q . \supset . r \supset s : \supset :. p \vee q . \supset : p . \vee . r \supset s \quad (1)$

$\vdash . *2{\cdot}76 . \text{Syll} . \supset \vdash :: p \vee q . \supset : p . \vee . r \supset s :. \supset :.$

$\quad p \vee q . \supset : p \vee r . \supset . p \vee s \quad (2)$

$\vdash . (1) . (2) . \supset \vdash .$ Prop

***2·82.** $\vdash :. p \vee q \vee r . \supset : p \vee \sim r \vee s . \supset . p \vee q \vee s$

$$\left[*2{\cdot}8 . *2{\cdot}81 \frac{q \vee r,\ \sim r \vee s,\ q \vee s}{q,\quad r,\quad s}\right]$$

***2·83.** $\vdash :: p . \supset . q \supset r : \supset :. p . \supset . r \supset s : \supset : p . \supset . q \supset s$

$$\left[*2{\cdot}82 \frac{\sim p, \sim q}{p,\ q}\right]$$

***2·85.** $\vdash :. p \vee q . \supset . p \vee r : \supset : p . \vee . q \supset r$

Dem.

$[\text{Add.Syll}] \vdash :. p \vee q . \supset . r : \supset . q \supset r \quad (1)$

$\vdash . *2{\cdot}55 . \supset \vdash :: \sim p . \supset :. p \vee r . \supset . r :.$

$[\text{Syll}]$ $\quad \supset :. p \vee q . \supset . p \vee r : \supset : p \vee q . \supset . r :.$

$[(1).*2{\cdot}83]$ $\quad \supset :. p \vee q . \supset . p \vee r : \supset : q \supset r \quad (2)$

$\vdash . (2) . \text{Comm} . \supset \vdash :. \quad p \vee q . \supset . p \vee r : \supset : \sim p . \supset . q \supset r :$

$[*2{\cdot}54]$ $\quad \supset : p . \vee . q \supset r :. \supset \vdash .$ Prop

***2·86.** $\vdash :. p \supset q . \supset . p \supset r : \supset : p . \supset . q \supset r$ $\quad \left[*2{\cdot}85 \frac{\sim p}{p}\right]$

$*3$. THE LOGICAL PRODUCT OF TWO PROPOSITIONS

Summary of $*3$.

The logical product of two propositions p and q is practically the proposition "p and q are both true." But this as it stands would have to be a new primitive idea. We therefore take as the logical product the proposition $\sim(\sim p \vee \sim q)$, *i.e.* "it is false that either p is false or q is false," which is obviously true when and only when p and q are both true. Thus we put

$*3{\cdot}01$. $p \,.\, q \,.=.\, \sim(\sim p \vee \sim q)$ Df

where "$p \,.\, q$" is the logical product of p and q.

$*3{\cdot}02$. $p \supset q \supset r \,.=.\, p \supset q \,.\, q \supset r$ Df

This definition serves merely to abbreviate proofs.

When we are given two asserted propositional functions "$\vdash . \phi x$" and "$\vdash . \psi x$," we shall have "$\vdash . \phi x \,.\, \psi x$" whenever ϕ and ψ take arguments of the same type. This will be proved for any functions in $*9$; for the present, we are confined to *elementary* propositional functions of elementary propositions. In this case, the result is proved as follows:

By $*1{\cdot}7$, $\sim\phi p$ and $\sim\psi p$ are elementary propositional functions, and therefore, by $*1{\cdot}72$, $\sim\phi p \vee \sim\psi p$ is an elementary propositional function. Hence by $*2{\cdot}11$,

$$\vdash : \sim\phi p \vee \sim\psi p \,.\vee.\, \sim(\sim\phi p \vee \sim\psi p).$$

Hence by $*2{\cdot}32$ and $*1{\cdot}01$,

$$\vdash :. \phi p \,.\supset :\, \psi p \,.\supset.\, \sim(\sim\phi p \vee \sim\psi p),$$

i.e. by $*3{\cdot}01$,

$$\vdash :. \phi p \,.\supset :\, \psi p \,.\supset.\, \phi p \,.\, \psi p.$$

Hence by $*1{\cdot}11$, when we have "$\vdash . \phi p$" and "$\vdash . \psi p$" we have "$\vdash . \phi p \,.\, \psi p$." This proposition is $*3{\cdot}03$. It is to be understood, like $*1{\cdot}72$, as applying also to functions of two or more variables.

The above is the practically most useful form of the axiom of identification of real variables (cf. $*1{\cdot}72$). In practice, when the restriction to *elementary* propositions and propositional functions has been removed, a convenient means by which two functions can often be recognized as taking arguments of the same type is the following:

If ϕx contains, in any way, a constituent $\chi(x, y, z, \ldots)$ and ψx contains, in any way, a constituent $\chi(x, u, v, \ldots)$, then both ϕx and ψx take arguments of the type of the argument x in $\chi(x, y, z, \ldots)$, and therefore both ϕx and ψx take arguments of the same type. Hence, in such a case, if both ϕx and ψx can be asserted, so can $\phi x \,.\, \psi x$.

As an example of the use of this proposition, take the proof of $*3\cdot47$. We there prove

$$\vdash :. p \supset r . q \supset s . \supset : p . q . \supset . q . r \qquad (1)$$

and

$$\vdash :. p \supset r . q \supset s . \supset : q . r . \supset . r . s \qquad (2)$$

and what we wish to prove is

$$p \supset r . q \supset s . \supset : p . q . \supset . r . s,$$

which is $*3\cdot47$. Now in (1) and (2), p, q, r, s are elementary propositions (as everywhere in Section A); hence by $*1\cdot7\cdot71$, applied repeatedly, "$p \supset r . q \supset s . \supset : p . q . \supset . q . r$" and "$p \supset r . q \supset s . \supset : q . r . \supset . r . s$" are elementary propositional functions. Hence by $*3\cdot03$, we have

$$\vdash :: p \supset r . q \supset s . \supset : p . q . \supset . q . r :. p \supset r . q \supset s . \supset : q . r . \supset . r . s,$$

whence the result follows by $*3\cdot43$ and $*3\cdot33$.

The principal propositions of the present number are the following:

$*3\cdot2$. $\vdash :. p . \supset : q . \supset . p . q$

I.e. "p implies that q implies $p . q$," *i.e.* if each of two propositions is true, so is their logical product.

$*3\cdot26$. $\vdash : p . q . \supset . p$

$*3\cdot27$. $\vdash : p . q . \supset . q$

I.e. if the logical product of two propositions is true, then each of the two propositions severally is true.

$*3\cdot3$. $\vdash :. p . q . \supset . r : \supset : p . \supset . q \supset r$

I.e. if p and q jointly imply r, then p implies that q implies r. This principle (following Peano) will be called "exportation," because q is "exported" from the hypothesis. It will be referred to as "Exp."

$*3\cdot31$. $\vdash :. p . \supset . q \supset r : \supset : p . q . \supset . r$

This is the correlative of the above, and will be called (following Peano) "importation" (referred to as "Imp").

$*3\cdot35$. $\vdash : p . p \supset q . \supset . q$

I.e. "if p is true, and q follows from it, then q is true." This will be called the "principle of assertion" (referred to as "Ass"). It differs from $*1\cdot1$ by the fact that it does not apply only when p really is true, but requires merely the *hypothesis* that p is true.

$*3\cdot43$. $\vdash :. p \supset q . p \supset r . \supset : p . \supset . q . r$

I.e. if a proposition implies each of two propositions, then it implies their logical product. This is called by Peano the "principle of composition." It will be referred to as "Comp."

$*3\cdot45$. $\vdash :. p \supset q . \supset : p . r . \supset . q . r$

I.e. both sides of an implication may be multiplied by a common factor. This is called by Peano the "principle of the factor." It will be referred to as "Fact."

∗3·47. $\vdash :. p \supset r . q \supset s . \supset : p . q . \supset . r . s$

I.e. if p implies q and r implies s, then p and q jointly imply r and s jointly. The law of contradiction, "$\vdash . \sim(p . \sim p)$," is proved in this number (∗3·24); but in spite of its fame we have found few occasions for its use.

∗3·01. $p . q . = . \sim(\sim p \vee \sim q)$ Df

∗3·02. $p \supset q \supset r . = . p \supset q . q \supset r$ Df

∗3·03. Given two asserted elementary propositional functions "$\vdash . \phi p$" and "$\vdash . \psi p$" whose arguments are elementary propositions, we have $\vdash . \phi p . \psi p$.

Dem.

$\vdash . *1·7·72 . *2·11 . \supset \vdash : \sim \phi p \vee \sim \psi p . \vee . \sim(\sim \phi p \vee \sim \psi p)$ (1)

$\vdash . (1) . *2·32 . (*1·01) . \supset \vdash :. \phi p . \supset : \psi p . \supset . \sim(\sim \phi p \vee \sim \psi p)$ (2)

$\vdash . (2) . (*3·01) . \supset \vdash :. \phi p . \supset : \psi p . \supset . \phi p . \psi p$ (3)

$\vdash . (3) . *1·11 . \supset \vdash .$ Prop

∗3·1. $\vdash : p . q . \supset . \sim(\sim p \vee \sim q)$ [Id . (∗3·01)]

∗3·11. $\vdash : \sim(\sim p \vee \sim q) . \supset . p . q$ [Id . (∗3·01)]

∗3·12. $\vdash : \sim p . \vee . \sim q . \vee . p . q$ $\left[*2·11 \dfrac{\sim p \vee \sim q}{p}\right]$

∗3·13. $\vdash : \sim(p . q) . \supset . \sim p \vee \sim q$ [∗3·11 . Transp]

∗3·14. $\vdash : \sim p \vee \sim q . \supset . \sim(p . q)$ [∗3·1 . Transp]

∗3·2. $\vdash :. p . \supset : q . \supset . p . q$ [∗3·12]

∗3·21. $\vdash :. q . \supset : p . \supset . p . q$ [∗3·2 . Comm]

∗3·22. $\vdash : p . q . \supset . q . p$

This is one form of the commutative law for logical multiplication. A more complete form is given in ∗4·3.

Dem.

$\left[*3·13 \dfrac{q, p}{p, q}\right]$ $\vdash : \sim(q . p) . \supset . \sim q \vee \sim p .$

[Perm] $\supset . \sim p \vee \sim q .$

[∗3·14] $\supset . \sim(p . q)$ (1)

$\vdash . (1) .$ Transp $. \supset \vdash .$ Prop

Note that, in the above proof, "(1)" stands for the proposition

"$\sim(q . p) . \supset . \sim(p . q)$,"

as was explained in the proof of ∗2·31.

∗3·24. $\vdash . \sim(p . \sim p)$

Dem.

$\left[*2·11 \dfrac{\sim p}{p}\right]$ $\vdash . \sim p \vee \sim(\sim p) . \supset$

$\left[*3·14 \dfrac{\sim p}{q}\right]$ $\vdash . \sim(p . \sim p)$

The above is the law of contradiction.

$*3\cdot26$. $\vdash : p \,.\, q \,.\supset .\, p$

Dem.

$$\begin{array}{llr}
\left[*2\cdot02\dfrac{q,p}{p,q}\right] & \vdash : p \,.\supset .\, q \supset p & (1) \\
[(1).(*1\cdot01)] & \vdash : \sim p \,.\vee .\, \sim q \vee p : & \\
[*2\cdot31] & \supset \vdash : \sim p \vee \sim q \,.\vee .\, p : & \\
\left[*2\cdot53\dfrac{\sim p \vee \sim q,\ p}{p,\quad q}\right] & \supset \vdash : \sim(\sim p \vee \sim q) \,.\supset .\, p & (2) \\
[(2).(*3\cdot01)] & \vdash : p \,.\, q \,.\supset .\, p &
\end{array}$$

$*3\cdot27$. $\vdash : p \,.\, q \,.\supset .\, q$

Dem.

$$\begin{array}{ll}
[*3\cdot22] & \vdash : p \,.\, q \,.\supset .\, q \,.\, p \,. \\
\left[*3\cdot26\dfrac{q,p}{p,q}\right] & \supset .\, q : \supset \vdash .\, \mathrm{Prop}
\end{array}$$

$*3\cdot26\cdot27$ will both be called the "principle of simplification," like $*2\cdot02$, from which they are deduced. They will be referred to as "Simp."

$*3\cdot3$. $\vdash :. p \,.\, q \,.\supset .\, r : \supset : p \,.\supset .\, q \supset r$

Dem.

$$\begin{array}{ll}
[\mathrm{Id}.(*3\cdot01)] & \vdash :. p \,.\, q \,.\supset .\, r : \supset : \sim(\sim p \vee \sim q) \,.\supset .\, r : \\
[\mathrm{Transp}] & \supset : \sim r \,.\supset .\, \sim p \vee \sim q : \\
[\mathrm{Id}.(*1\cdot01)] & \supset : \sim r \,.\supset .\, p \supset \sim q : \\
[\mathrm{Comm}] & \supset : p \,.\supset .\, \sim r \supset \sim q : \\
[\mathrm{Transp}.\mathrm{Syll}] & \supset : p \,.\supset .\, q \supset r :. \supset \vdash .\, \mathrm{Prop}
\end{array}$$

$*3\cdot31$. $\vdash :. p \,.\supset .\, q \supset r : \supset : p \,.\, q \,.\supset .\, r$

Dem.

$$\begin{array}{ll}
[\mathrm{Id}.(*1\cdot01)] & \vdash :. p \,.\supset .\, q \supset r : \supset : \sim p \,.\vee .\, \sim q \vee r : \\
[*2\cdot31] & \supset : \sim p \vee \sim q \,.\vee .\, r : \\
\left[*2\cdot53\dfrac{\sim p \vee \sim q,\ r}{p,\quad q}\right] & \supset : \sim(\sim p \vee \sim q) \,.\supset .\, r : \\
[\mathrm{Id}.(*3\cdot01)] & \supset : p \,.\, q \,.\supset .\, r :. \supset \vdash .\, \mathrm{Prop}
\end{array}$$

$*3\cdot33$. $\vdash : p \supset q \,.\, q \supset r \,.\supset .\, p \supset r$ [Syll . Imp]

$*3\cdot34$. $\vdash : q \supset r \,.\, p \supset q \,.\supset .\, p \supset r$ [Syll . Imp]

These two propositions will hereafter be referred to as "Syll"; they are usually more convenient than either $*2\cdot05$ or $*2\cdot06$.

$*3\cdot35$. $\vdash : p \,.\, p \supset q \,.\supset .\, q$ [$*2\cdot27$. Imp]

$*3\cdot37$. $\vdash :. p \,.\, q \,.\supset .\, r : \supset : p \,.\, \sim r \,.\supset .\, \sim q$

Dem.

$$\begin{array}{llr}
\vdash .\, \mathrm{Transp} \,. & \supset \vdash : q \supset r \,.\supset .\, \sim r \supset \sim q : & \\
[\mathrm{Syll}] & \supset \vdash :. p \,.\supset .\, q \supset r : \supset : p \,.\supset .\, \sim r \supset \sim q & (1) \\
\vdash .\, \mathrm{Exp} \,. & \supset \vdash :. p \,.\, q \,.\supset .\, r : \supset : p \,.\supset .\, q \supset r & (2) \\
\vdash .\, \mathrm{Imp} \,. & \supset \vdash :. p \,.\supset .\, \sim r \supset \sim q : \supset : p \,.\, \sim r \,.\supset .\, \sim q & (3) \\
\vdash .\, (2) \,.\, (1) \,.\, (3) \,.\, \mathrm{Syll} \,.\supset \vdash .\, \mathrm{Prop} & &
\end{array}$$

This is another form of transposition.

∗3·4. $\vdash : p \,.\, q \,.\, \supset \,.\, p \supset q$ [∗2·51 . Transp . (∗1·01 . ∗3·01)]

∗3·41. $\vdash :. \, p \supset r \,.\, \supset : p \,.\, q \,.\, \supset \,.\, r$ [∗3·26 . Syll]

∗3·42. $\vdash :. \, q \supset r \,.\, \supset : p \,.\, q \,.\, \supset \,.\, r$ [∗3·27 . Syll]

∗3·43. $\vdash :. \, p \supset q \,.\, p \supset r \,.\, \supset : p \,.\, \supset \,.\, q \,.\, r$

Dem.

$\vdash \,.\, *3 \cdot 2 \,.\, \supset \vdash :. \, q \,.\, \supset : r \,.\, \supset \,.\, q \,.\, r$ (1)

$\vdash \,.\, (1) \,.\, \text{Syll} \,.\, \supset \vdash :: p \supset q \,.\, \supset :. \, p \,.\, \supset : r \,.\, \supset \,.\, q \,.\, r :.$

[∗2·77] $\supset :. \, p \supset r \,.\, \supset : p \,.\, \supset \,.\, q \,.\, r$ (2)

$\vdash \,.\, (2) \,.\, \text{Imp} \,.\, \supset \vdash \,.\, \text{Prop}$

∗3·44. $\vdash :. \, q \supset p \,.\, r \supset p \,.\, \supset : q \vee r \,.\, \supset \,.\, p$

This principle is analogous to ∗3·43. The analogy between ∗3·43 and ∗3·44 is of a sort which generally subsists between formulae concerning products and formulae concerning sums.

Dem.

$\vdash \,.\, \text{Syll} \,.\, \supset \vdash :. \, {\sim} q \supset r \,.\, r \supset p \,.\, \supset : {\sim} q \supset p :$

[∗2·6] $\supset : q \supset p \,.\, \supset \,.\, p$ (1)

$\vdash \,.\, (1) \,.\, \text{Exp} \,.\, \supset \vdash :: {\sim} q \supset r \,.\, \supset :. \, r \supset p \,.\, \supset : q \supset p \,.\, \supset \,.\, p :.$

[Comm.Imp] $\supset :. \, q \supset p \,.\, r \supset p \,.\, \supset \,.\, p$ (2)

$\vdash \,.\, (2) \,.\, \text{Comm} \,.\, \supset \vdash :. \, q \supset p \,.\, r \supset p \,.\, \supset : {\sim} q \supset r \,.\, \supset \,.\, p :.$

[∗2·53.Syll] $\supset \vdash \,.\, \text{Prop}$

∗3·45. $\vdash :. \, p \supset q \,.\, \supset : p \,.\, r \,.\, \supset \,.\, q \,.\, r$

This principle shows that we may multiply both sides of an implication by a common factor; hence it is called by Peano the "principle of the factor." We shall refer to it as "Fact." It is the analogue, for multiplication, of the primitive proposition ∗1·6.

Dem.

$\vdash \,.\, \text{Syll} \frac{\sim r}{r} \,.\, \supset \vdash :. \, p \supset q \,.\, \supset : q \supset {\sim} r \,.\, \supset \,.\, p \supset {\sim} r :$

[Transp] $\supset : {\sim}(p \supset {\sim} r) \,.\, \supset \,.\, {\sim}(q \supset {\sim} r) :.$

[Id.(∗1·01.∗3·01)] $\supset \vdash \,.\, \text{Prop}$

∗3·47. $\vdash :. \, p \supset r \,.\, q \supset s \,.\, \supset : p \,.\, q \,.\, \supset \,.\, r \,.\, s$

This proposition, or rather its analogue for classes, was proved by Leibniz, and evidently pleased him, since he calls it "præclarum theorema∗."

Dem.

$\vdash \,.\, *3 \cdot 26 \,.\, \supset \vdash :. \, p \supset r \,.\, q \supset s \,.\, \supset : p \supset r :$

[Fact] $\supset : p \,.\, q \,.\, \supset \,.\, r \,.\, q :$

[∗3·22] $\supset : p \,.\, q \,.\, \supset \,.\, q \,.\, r$ (1)

∗ *Philosophical works*, Gerhardt's edition, Vol. VII. p. 223.

$\vdash . *3{\cdot}27 . \supset \vdash :. p \supset r . q \supset s . \supset : q \supset s :$

[Fact] $\quad \supset : q . r . \supset . s . r :$

[$*3{\cdot}22$] $\quad \supset : q . r . \supset . r . s \qquad (2)$

$\vdash . (1) . (2) . *3{\cdot}03 . *2{\cdot}83 . \supset$

$\qquad \vdash :. p \supset r . q \supset s . \supset : p . q . \supset . r . s :. \supset \vdash . \text{Prop}$

$*3{\cdot}48.$ $\vdash :. p \supset r . q \supset s . \supset : p \vee q . \supset . r \vee s$

This theorem is the analogue of $*3{\cdot}47$.

Dem.

$\vdash . *3{\cdot}26 . \supset \vdash :. p \supset r . q \supset s . \supset : p \supset r :$

[Sum] $\quad \supset : p \vee q . \supset . r \vee q :$

[Perm] $\quad \supset : p \vee q . \supset . q \vee r \qquad (1)$

$\vdash . *3{\cdot}27 . \supset \vdash :. p \supset r . q \supset s . \supset : q \supset s :$

[Sum] $\quad \supset : q \vee r . \supset . s \vee r :$

[Perm] $\quad \supset : q \vee r . \supset . r \vee s \qquad (2)$

$\vdash . (1) . (2) . *2{\cdot}83 . \supset$

$\qquad \vdash :. p \supset r . q \supset s . \supset : p \vee q . \supset . r \vee s :. \supset \vdash . \text{Prop}$

$*4$. EQUIVALENCE AND FORMAL RULES

Summary of $*4$.

In this number, we shall be concerned with rules analogous, more or less, to those of ordinary algebra. It is from these rules that the usual "calculus of formal logic" starts. Treated as a "calculus," the rules of deduction are capable of many other interpretations. But all other interpretations depend upon the one here considered, since in all of them we deduce consequences from our rules, and thus presuppose the theory of deduction. One very simple interpretation of the "calculus" is as follows: The entities considered are to be numbers which are all either 0 or 1; "$p \supset q$" is to have the value 0 if p is 1 and q is 0; otherwise it is to have the value 1; $\sim p$ is to be 1 if p is 0, and 0 if p is 1; $p \,.\, q$ is to be 1 if p and q are both 1, and is to be 0 in any other case; $p \vee q$ is to be 0 if p and q are both 0, and is to be 1 in any other case; and the assertion-sign is to mean that what follows has the value 1. Symbolic logic considered as a calculus has undoubtedly much interest on its own account; but in our opinion this aspect has hitherto been too much emphasized, at the expense of the aspect in which symbolic logic is merely the most elementary part of mathematics, and the logical prerequisite of all the rest. For this reason, we shall only deal briefly with what is required for the algebra of symbolic logic.

When each of two propositions implies the other, we say that the two are *equivalent*, which we write "$p \equiv q$." We put

$*4{\cdot}01$. $p \equiv q \,.=.\, p \supset q \,.\, q \supset p$ Df

It is obvious that two propositions are equivalent when, and only when, both are true or both are false. Following Frege, we shall call the *truth-value of a proposition* truth if it is true, and falsehood if it is false. Thus two propositions are equivalent when they have the same truth-value.

It should be observed that, if $p \equiv q$, q may be substituted for p without altering the truth-value of any function of p which involves no primitive ideas except those enumerated in $*1$. This can be proved in each separate case, but not generally, because we have no means of specifying (with our apparatus of primitive ideas) that a function is one which can be built up out of these ideas alone. We shall give the name of a *truth-function* to a function $f(p)$ whose argument is a proposition, and whose truth-value depends only upon the truth-value of its argument. All the functions of propositions with which we shall be specially concerned will be truth-functions, *i.e.* we shall have

$$p \equiv q \,.\supset.\, f(p) \equiv f(q).$$

The reason of this is, that the functions of propositions with which we deal are all built up by means of the primitive ideas of ∗1. But it is not a universal characteristic of functions of propositions to be truth-functions. For example, "A believes p" may be true for one true value of p and false for another.

The principal propositions of this number are the following:

∗4·1. $\vdash : p \supset q \,.\equiv.\, \sim q \supset \sim p$

∗4·11. $\vdash : p \equiv q \,.\equiv.\, \sim p \equiv \sim q$

These are both forms of the "principle of transposition."

∗4·13. $\vdash . \, p \equiv \sim(\sim p)$

This is the principle of double negation, *i.e.* a proposition is equivalent to the falsehood of its negation.

∗4·2. $\vdash . \, p \equiv p$

∗4·21. $\vdash : p \equiv q \,.\equiv.\, q \equiv p$

∗4·22. $\vdash : p \equiv q \,.\, q \equiv r \,.\supset.\, p \equiv r$

These propositions assert that equivalence is *reflexive, symmetrical* and *transitive.*

∗4·24. $\vdash : p \,.\equiv.\, p \,.\, p$

∗4·25. $\vdash : p \,.\equiv.\, p \vee p$

I.e. p is equivalent to "p and p" and to "p or p," which are two forms of the *law of tautology,* and are the source of the principal differences between the algebra of symbolic logic and ordinary algebra.

∗4·3. $\vdash : p \,.\, q \,.\equiv.\, q \,.\, p$

This is the commutative law for the product of propositions.

∗4·31. $\vdash : p \vee q \,.\equiv.\, q \vee p$

This is the commutative law for the sum of propositions.

The associative laws for multiplication and addition of propositions, namely

∗4·32. $\vdash : (p \,.\, q) \,.\, r \,.\equiv.\, p \,.\, (q \,.\, r)$

∗4·33. $\vdash : (p \vee q) \vee r \,.\equiv.\, p \vee (q \vee r)$

The distributive law in the two forms

∗4·4. $\vdash :. \, p \,.\, q \vee r \,.\equiv : p \,.\, q \,.\vee.\, p \,.\, r$

∗4·41. $\vdash :. \, p \,.\vee.\, q \,.\, r : \equiv .\, p \vee q \,.\, p \vee r$

The second of these forms has no analogue in ordinary algebra.

∗4·71. $\vdash :. \, p \supset q \,.\equiv : p \,.\equiv.\, p \,.\, q$

I.e. p implies q when, and only when, p is equivalent to $p \,.\, q$. This proposition is used constantly; it enables us to replace any implication by an equivalence.

∗4·73. $\vdash :. \, q \,.\supset : p \,.\equiv.\, p \,.\, q$

I.e. a true factor may be dropped from or added to a proposition without altering the truth-value of the proposition.

∗4·01. $p \equiv q \,.\, = \,.\, p \supset q \,.\, q \supset p$ Df

∗4·02. $p \equiv q \equiv r \,.\, = \,.\, p \equiv q \,.\, q \equiv r$ Df

This definition serves merely to provide a convenient abbreviation.

∗4·1. $\vdash : p \supset q \,.\, \equiv \,.\, \sim q \supset \sim p$ [∗2·16·17]

∗4·11. $\vdash : p \equiv q \,.\, \equiv \,.\, \sim p \equiv \sim q$ [∗2·16·17 . ∗3·47·22]

∗4·12. $\vdash : p \equiv \sim q \,.\, \equiv \,.\, q \equiv \sim p$ [∗2·03·15]

∗4·13. $\vdash \,.\, p \equiv \sim(\sim p)$ [∗2·12·14]

∗4·14. $\vdash :. \, p \,.\, q \,.\, \supset \,.\, r : \equiv : p \,.\, \sim r \,.\, \supset \,.\, \sim q$ [∗3·37 . ∗4·13]

∗4·15. $\vdash :. \, p \,.\, q \,.\, \supset \,.\, \sim r : \equiv : q \,.\, r \,.\, \supset \,.\, \sim p$ [∗3·22 . ∗4·13·14]

∗4·2. $\vdash \,.\, p \equiv p$ [Id . ∗3·2]

∗4·21. $\vdash : p \equiv q \,.\, \equiv \,.\, q \equiv p$ [∗3·22]

∗4·22. $\vdash : p \equiv q \,.\, q \equiv r \,.\, \supset \,.\, p \equiv r$

Dem.

$\vdash \,.\, ∗3·26 \,.\quad \supset \vdash : p \equiv q \,.\, q \equiv r \,.\, \supset \,.\, p \equiv q \,.$

$[∗3·26] \qquad \supset \,.\, p \supset q$ (1)

$\vdash \,.\, ∗3·27 \,.\quad \supset \vdash : p \equiv q \,.\, q \equiv r \,.\, \supset \,.\, q \equiv r \,.$

$[∗3·26] \qquad \supset \,.\, q \supset r$ (2)

$\vdash \,.\, (1) \,.\, (2) \,.\, ∗2·83 \,.\, \supset \vdash : p \equiv q \,.\, q \equiv r \,.\, \supset \,.\, p \supset r$ (3)

$\vdash \,.\, ∗3·27 \,.\quad \supset \vdash : p \equiv q \,.\, q \equiv r \,.\, \supset \,.\, q \equiv r \,.$

$[∗3·27] \qquad \supset \,.\, r \supset q$ (4)

$\vdash \,.\, ∗3·26 \,.\quad \supset \vdash : p \equiv q \,.\, q \equiv r \,.\, \supset \,.\, p \equiv q \,.$

$[∗3·27] \qquad \supset \,.\, q \supset p$ (5)

$\vdash \,.\, (4) \,.\, (5) \,.\, ∗2·83 \,.\, \supset \vdash : p \equiv q \,.\, q \equiv r \,.\, \supset \,.\, r \supset p$ (6)

$\vdash \,.\, (3) \,.\, (6) \,.\, \mathrm{Comp} \,.\, \supset \vdash \,.\, \mathrm{Prop}$

Note. The above three propositions show that the relation of equivalence is reflexive (∗4·2), symmetrical (∗4·21), and transitive (∗4·22). Implication is reflexive and transitive, but not symmetrical. The properties of being symmetrical, transitive, and (at least within a certain field) reflexive are essential to any relation which is to have the formal characters of equality.

∗4·24. $\vdash : p \,.\, \equiv \,.\, p \,.\, p$

Dem.

$\vdash \,.\, ∗3·26 \,.\, \supset \vdash : p \,.\, p \,.\, \supset \,.\, p$ (1)

$\vdash \,.\, ∗3·2 \,.\quad \supset \vdash :. \, p \,.\, \supset : p \,.\, \supset \,.\, p \,.\, p :.$

$[∗2·43] \quad \supset \vdash : p \,.\, \supset \,.\, p \,.\, p$ (2)

$\vdash \,.\, (1) \,.\, (2) \,.\, ∗3·2 \,.\, \supset \vdash \,.\, \mathrm{Prop}$

∗4·25. $\vdash : p \,.\, \equiv \,.\, p \vee p$ $\left[\mathrm{Taut} \,.\, \mathrm{Add}\, \frac{p}{q}\right]$

Note. ∗4·24·25 are two forms of the *law of tautology*, which is what chiefly distinguishes the algebra of symbolic logic from ordinary algebra.

∗4·3. $\vdash : p \,.\, q \,.\equiv .\, q \,.\, p$ [∗3·22]

Note. Whenever we have, whatever values p and q may have,

$$\phi(p, q) \,.\supset .\, \phi(q, p),$$

we have also

$$\phi(p, q) \,.\equiv .\, \phi(q, p).$$

For $$\{\phi(p, q) \,.\supset .\, \phi(q, p)\}\frac{q, p}{p, q} \,.\supset : \phi(q, p) \,.\supset .\, \phi(p, q).$$

∗4·31. $\vdash : p \vee q \,.\equiv .\, q \vee p$ [Perm]

∗4·32. $\vdash : (p \,.\, q) \,.\, r \,.\equiv .\, p \,.\, (q \,.\, r)$

Dem.

$\vdash .\, ∗4·15 \,. \quad \supset \vdash :. \, p \,.\, q \,.\supset .\, \sim r : \equiv : q \,.\, r \,.\supset .\, \sim p :$

$[∗4·12] \qquad \equiv : p \,.\supset .\, \sim (q \,.\, r)$ (1)

$\vdash .\, (1) \,.\, ∗4·11 \,.\supset \vdash : \sim (p \,.\, q \,.\supset .\, \sim r) \,.\equiv .\, \sim \{p \,.\supset .\, \sim (q \,.\, r)\} :$

$[(∗1·01 . ∗3·01)] \supset \vdash .$ Prop

Note. Here "(1)" stands for "$\vdash :. \, p \,.\, q \,.\supset .\, \sim r : \equiv : p \,.\supset .\, \sim (q \,.\, r)$," which is obtained from the above steps by ∗4·22. The use of ∗4·22 will often be tacit, as above. The principle is the same as that explained in respect of implication in ∗2·31.

∗4·33. $\vdash : (p \vee q) \vee r \,.\equiv .\, p \vee (q \vee r)$ [∗2·31·32]

The above are the associative laws for multiplication and addition. To avoid brackets, we introduce the following definition:

∗4·34. $p \,.\, q \,.\, r \,. = .\, (p \,.\, q) \,.\, r$ Df

∗4·36. $\vdash :. \, p \equiv q \,.\supset : p \,.\, r \,.\equiv .\, q \,.\, r$ [Fact . ∗3·47]

∗4·37. $\vdash :. \, p \equiv q \,.\supset : p \vee r \,.\equiv .\, q \vee r$ [Sum . ∗3·47]

∗4·38. $\vdash :. \, p \equiv r \,.\, q \equiv s \,.\supset : p \,.\, q \,.\equiv .\, r \,.\, s$ [∗3·47 . ∗4·32 . ∗3·22]

∗4·39. $\vdash :. \, p \equiv r \,.\, q \equiv s \,.\supset : p \vee q \,.\equiv .\, r \vee s$ [∗3·48·47 . ∗4·32 . ∗3·22]

∗4·4. $\vdash :. \, p \,.\, q \vee r \,.\equiv : p \,.\, q \,.\vee .\, p \,.\, r$

This is the first form of the distributive law.

Dem.

$\vdash .\, ∗3·2 \,. \qquad \supset \vdash :: p \,.\supset : q \,.\supset .\, p \,.\, q :. \, p \,.\supset : r \,.\supset .\, p \,.\, r ::$

$[\text{Comp}] \qquad \supset \vdash :: p \,.\supset :. \, q \,.\supset .\, p \,.\, q : r \,.\supset .\, p \,.\, r :.$

$[∗3·48] \qquad \supset :. \, q \vee r \,.\supset : p \,.\, q \,.\vee .\, p \,.\, r$ (1)

$\vdash .\, (1) \,.\, \text{Imp} \,. \qquad \supset \vdash :. \, p \,.\, q \vee r \,.\supset : p \,.\, q \,.\vee .\, p \,.\, r$ (2)

$\vdash .\, ∗3·26 \,. \qquad \supset \vdash :. \, p \,.\, q \,.\supset .\, p : p \,.\, r \,.\supset .\, p :.$

$[∗3·44] \qquad \supset \vdash :. \, p \,.\, q \,.\vee .\, p \,.\, r : \supset .\, p$ (3)

$\vdash .\, ∗3·27 \,. \qquad \supset \vdash :. \, p \,.\, q \,.\supset .\, q : p \,.\, r \,.\supset .\, r :.$

$[∗3·48] \qquad \supset \vdash :. \, p \,.\, q \,.\vee .\, p \,.\, r : \supset .\, q \vee r$ (4)

$\vdash .\, (3) \,.\, (4) \,.\, \text{Comp} \,. \supset \vdash :. \, p \,.\, q \,.\vee .\, p \,.\, r : \supset .\, p \,.\, q \vee r$ (5)

$\vdash .\, (2) \,.\, (5) \,. \qquad \supset \vdash .$ Prop

***4·41.** $\vdash :. p . \vee . q . r : \equiv . p \vee q . p \vee r$

This is the second form of the distributive law—a form to which there is nothing analogous in ordinary algebra. By the conventions as to dots, "$p . \vee . q . r$" means "$p \vee (q . r)$."

Dem.

$\vdash . *3{\cdot}26 . \mathrm{Sum} . \quad \supset \vdash :. p . \vee . q . r : \supset . p \vee q$ (1)

$\vdash . *3{\cdot}27 . \mathrm{Sum} . \quad \supset \vdash :. p . \vee . q . r : \supset . p \vee r$ (2)

$\vdash . (1) . (2) . \mathrm{Comp} . \supset \vdash :. p . \vee . q . r : \supset . p \vee q . p \vee r$ (3)

$\vdash . *2{\cdot}53 . *3{\cdot}47 . \quad \supset \vdash :. p \vee q . p \vee r . \supset : \sim p \supset q . \sim p \supset r :$

[Comp] $\quad \supset : \sim p . \supset . q . r :$

[*2·54] $\quad \supset : p . \vee . q . r$ (4)

$\vdash . (3) . (4) . \quad \supset \vdash . \mathrm{Prop}$

***4·42.** $\vdash :. p . \equiv : p . q . \vee . p . \sim q$

Dem.

$\vdash . *3{\cdot}21 . \quad \supset \vdash :. q \vee \sim q . \supset : p . \supset . p . q \vee \sim q :.$

[*2·11] $\quad \supset \vdash : p . \supset . p . q \vee \sim q$ (1)

$\vdash . *3{\cdot}26 . \quad \supset \vdash : p . q \vee \sim q . \supset . p$ (2)

$\vdash . (1) . (2) . \supset \vdash :. p . \equiv : p . q \vee \sim q :$

[*4·4] $\quad \equiv : p . q . \vee . p . \sim q :. \supset \vdash . \mathrm{Prop}$

***4·43.** $\vdash :. p . \equiv : p \vee q . p \vee \sim q$

Dem.

$\vdash . *2{\cdot}2 . \quad \supset \vdash : p . \supset . p \vee q : p . \supset . p \vee \sim q :$

[Comp] $\quad \supset \vdash : p . \supset . p \vee q . p \vee \sim q$ (1)

$\vdash . *2{\cdot}65 \dfrac{\sim p}{p} . \supset \vdash :. \sim p \supset q . \supset : \sim p \supset \sim q . \supset . p :.$

[Imp] $\quad \supset \vdash :. \sim p \supset q . \sim p \supset \sim q . \supset . p :.$

[*2·53.*3·47] $\quad \supset \vdash :. p \vee q . p \vee \sim q . \supset . p$ (2)

$\vdash . (1) . (2) . \quad \supset \vdash . \mathrm{Prop}$

***4·44.** $\vdash :. p . \equiv : p . \vee . p . q$

Dem.

$\vdash . *2{\cdot}2 . \quad \supset \vdash :. p . \supset : p . \vee . p . q$ (1)

$\vdash . \mathrm{Id} . *3{\cdot}26 . \supset \vdash :. p \supset p : p . q . \supset . p :.$

[*3·44] $\quad \supset \vdash :. p . \vee . p . q : \supset . p$ (2)

$\vdash . (1) . (2) . \quad \supset \vdash . \mathrm{Prop}$

***4·45.** $\vdash : p . \equiv . p . p \vee q$ [*3·26 . *2·2]

The following formulae are due to De Morgan, or rather, are the propositional analogues of formulae given by De Morgan for classes. The first of them, it will be observed, merely embodies our definition of the logical product.

$*4\cdot5.\ \vdash :\ p\,.\,q\,.\equiv.\sim(\sim p\vee\sim q)$ $[*4\cdot2\,.\,(*3\cdot01)]$

$*4\cdot51.\ \vdash :\ \sim(p\,.\,q)\,.\equiv.\sim p\vee\sim q$ $[*4\cdot5\cdot12]$

$*4\cdot52.\ \vdash :\ p\,.\sim q\,.\equiv.\sim(\sim p\vee q)$ $\left[*4\cdot5\dfrac{\sim q}{q}\,.\,*4\cdot13\right]$

$*4\cdot53.\ \vdash :\ \sim(p\,.\sim q)\,.\equiv.\sim p\vee q$ $[*4\cdot52\cdot12]$

$*4\cdot54.\ \vdash :\ \sim p\,.\,q\,.\equiv.\sim(p\vee\sim q)$ $\left[*4\cdot5\dfrac{\sim p}{p}\,.\,*4\cdot13\right]$

$*4\cdot55.\ \vdash :\ \sim(\sim p\,.\,q)\,.\equiv.\,p\vee\sim q$ $[*4\cdot54\cdot12]$

$*4\cdot56.\ \vdash :\ \sim p\,.\sim q\,.\equiv.\sim(p\vee q)$ $\left[*4\cdot54\dfrac{\sim q}{q}\,.\,*4\cdot13\right]$

$*4\cdot57.\ \vdash :\sim(\sim p\,.\sim q)\,.\equiv.\,p\vee q$ $[*4\cdot56\cdot12]$

The following formulae are obtained immediately from the above. They are important as showing how to transform implications into sums or into denials of products, and vice versa. It will be observed that the first of them merely embodies the definition $*1\cdot01$.

$*4\cdot6.\ \vdash :\ p\supset q\,.\equiv.\sim p\vee q$ $[*4\cdot2\,.\,(*1\cdot01)]$

$*4\cdot61.\ \vdash :\ \sim(p\supset q)\,.\equiv.\,p\,.\sim q$ $[*4\cdot6\cdot11\cdot52]$

$*4\cdot62.\ \vdash :\ p\supset\sim q\,.\equiv.\sim p\vee\sim q$ $\left[*4\cdot6\ \dfrac{\sim q}{q}\right]$

$*4\cdot63.\ \vdash :\ \sim(p\supset\sim q)\,.\equiv.\,p\,.\,q$ $[*4\cdot62\cdot11\cdot5]$

$*4\cdot64.\ \vdash :\ \sim p\supset q\,.\equiv.\,p\vee q$ $[*2\cdot53\cdot54]$

$*4\cdot65.\ \vdash :\ \sim(\sim p\supset q)\,.\equiv.\sim p\,.\sim q$ $[*4\cdot64\cdot11\cdot56]$

$*4\cdot66.\ \vdash :\ \sim p\supset\sim q\,.\equiv.\,p\vee\sim q$ $\left[*4\cdot64\ \dfrac{\sim q}{q}\right]$

$*4\cdot67.\ \vdash :\sim(\sim p\supset\sim q)\,.\equiv.\sim p\,.\,q$ $[*4\cdot66\cdot11\cdot54]$

$*4\cdot7.\ \vdash :.\,p\supset q\,.\equiv:p\,.\supset.\,p\,.\,q$

Dem.

$\vdash.*3\cdot27\,.\,\text{Syll}\,.\supset\vdash :.\,p\,.\supset.\,p\,.\,q:\supset.\,p\supset q$ (1)

$\vdash.\,\text{Comp}\,.\quad\supset\vdash :.\,p\supset p\,.\,p\supset q\,.\supset:p\,.\supset.\,p\,.\,q:.$

$[\text{Exp}]\quad\supset\vdash ::p\supset p\,.\supset:.\,p\supset q\,.\supset:p\,.\supset.\,p\,.\,q::$

$[\text{Id}]\quad\supset\vdash :.\,p\supset q\,.\supset:p\,.\supset.\,p\,.\,q$ (2)

$\vdash.(1).(2).\quad\supset\vdash.\,\text{Prop}$

$*4\cdot71.\ \vdash :.\,p\supset q\,.\equiv:p\,.\equiv.\,p\,.\,q$

Dem.

$\vdash.*3\cdot21\,.\quad\supset\vdash ::p\,.\,q\,.\supset.\,p:\supset:.\,p\,.\supset.\,p\,.\,q:\supset:p\,.\equiv.\,p\,.\,q::$

$[*3\cdot26]\quad\supset\vdash :.\,p\,.\supset.\,p\,.\,q:\supset:p\,.\equiv.\,p\,.\,q$ (1)

$\vdash.*3\cdot26\,.\quad\supset\vdash :.\,p\,.\equiv.\,p\,.\,q:\supset:p\,.\supset.\,p\,.\,q$ (2)

$\vdash.(1).(2).\quad\supset\vdash :.\,p\,.\supset.\,p\,.\,q:\equiv:p\,.\equiv.\,p\,.\,q$ (3)

$\vdash.(3).*4\cdot7\cdot22\,.\supset\vdash.\,\text{Prop}$

The above proposition is constantly used. It enables us to transform every implication into an equivalence, which is an advantage if we wish to assimilate symbolic logic as far as possible to ordinary algebra. But when symbolic logic is regarded as an instrument of proof, we need implications, and it is usually inconvenient to substitute equivalences. Similar remarks apply to the following proposition.

***4·72.** $\vdash :. p \supset q . \equiv : q . \equiv . p \vee q$

Dem.

$$\begin{array}{ll} \vdash . *4\cdot 1 . \supset \vdash :. p \supset q . \equiv : \sim q \supset \sim p : & \\ \left[*4\cdot 71 \dfrac{\sim q, \sim p}{p, \quad q}\right] & \equiv : \sim q . \equiv . \sim q . \sim p : \\ {[*4\cdot 12]} & \equiv : q . \equiv . \sim(\sim q . \sim p) : \\ {[*4\cdot 57]} & \equiv : q . \equiv . q \vee p : \\ {[*4\cdot 31]} & \equiv : q . \equiv . p \vee q :. \supset \vdash . \text{Prop} \end{array}$$

***4·73.** $\vdash :. q . \supset : p . \equiv . p . q$ [Simp . *4·71]

This proposition is very useful, since it shows that a true factor may be omitted from a product without altering its truth or falsehood, just as a true hypothesis may be omitted from an implication.

***4·74.** $\vdash :. \sim p . \supset : q . \equiv . p \vee q$ [*2·21 . *4·72]

***4·76.** $\vdash :. p \supset q . p \supset r . \equiv : p . \supset . q . r$ $\left[*4\cdot 41 \dfrac{\sim p}{p} . (*1\cdot 01)\right]$

***4·77.** $\vdash :. q \supset p . r \supset p . \equiv : q \vee r . \supset . p$ [*3·44 . Add . *2·2]

***4·78.** $\vdash :. p \supset q . \vee . p \supset r : \equiv : p . \supset . q \vee r$

Dem.

$$\begin{array}{ll} \vdash . *4\cdot 2 . (*1\cdot 01) . \supset \vdash :. p \supset q . \vee . p \supset r : \equiv : \sim p \vee q . \vee . \sim p \vee r : & \\ {[*4\cdot 33]} & \equiv . \sim p . \vee . q \vee \sim p \vee r : \\ {[*4\cdot 31\cdot 37]} & \equiv : \sim p . \vee . \sim p \vee q \vee r : \\ {[*4\cdot 33]} & \equiv : \sim p \vee \sim p . \vee . q \vee r : \\ {[*4\cdot 25\cdot 37]} & \equiv : \sim p . \vee . q \vee r : \\ {[*4\cdot 2 . (*1\cdot 01)]} & \equiv : p . \supset . q \vee r :. \supset \vdash . \text{Prop} \end{array}$$

***4·79.** $\vdash :. q \supset p . \vee . r \supset p : \equiv : q . r . \supset . p$

Dem.

$$\begin{array}{ll} \vdash . *4\cdot 1\cdot 39 . \supset \vdash :. q \supset p . \vee . r \supset p : \equiv : \sim p \supset \sim q . \vee . \sim p \supset \sim r : & \\ {[*4\cdot 78]} & \equiv : \sim p . \supset . \sim q \vee \sim r : \\ {[*2\cdot 15]} & \equiv : \sim(\sim q \vee \sim r) . \supset . p : \\ {[*4\cdot 2 . (*3\cdot 01)]} & \equiv : q . r . \supset . p :. \supset \vdash . \text{Prop} \end{array}$$

Note. The analogues, for classes, of *4·78·79 are false. Take, *e.g.* *4·78, and put $p =$ English people, $q =$ men, $r =$ women. Then p is contained in q or r, but is not contained in q and is not contained in r.

***4·8.** $\vdash : p \supset \sim p . \equiv . \sim p$ [*2·01 . Simp]

***4·81.** $\vdash : \sim p \supset p . \equiv . p$ [*2·18 . Simp]

***4·82.** $\vdash : p \supset q . p \supset \sim q . \equiv . \sim p$ [*2·65 . Imp . *2·21 . Comp]

***4·83.** $\vdash : p \supset q . \sim p \supset q . \equiv . q$ [*2·61 . Imp . Simp . Comp]

Note. *4·82·83 may also be obtained from *4·43, of which they are virtually other forms.

***4·84.** $\vdash :. p \equiv q . \supset : p \supset r . \equiv . q \supset r$ [*2·06 . *3·47]

***4·85.** $\vdash :. p \equiv q . \supset : r \supset p . \equiv . r \supset q$ [*2·05 . *3·47]

***4·86.** $\vdash :. p \equiv q . \supset : p \equiv r . \equiv . q \equiv r$ [*4·21·22]

***4·87.** $\vdash :. p . q . \supset . r : \equiv : p . \supset . q \supset r : \equiv : q . \supset . p \supset r : \equiv : q . p . \supset . r$
[Exp . Comm . Imp]

*4·87 embodies in one proposition the principles of exportation and importation and the commutative principle.

$*5$. MISCELLANEOUS PROPOSITIONS

Summary of $*5$.

The present number consists chiefly of propositions of two sorts: (1) those which will be required as lemmas in one or more subsequent proofs, (2) those which are on their own account illustrative, or would be important in other developments than those that we wish to make. A few of the propositions of this number, however, will be used very frequently. These are:

$*5\cdot1.\quad \vdash : p \,.\, q \,.\supset .\, p \equiv q$

I.e. two propositions are equivalent if they are both true. (The statement that two propositions are equivalent if they are both false is $*5\cdot21$.)

$*5\cdot32.\quad \vdash :.\, p \,.\supset .\, q \equiv r : \equiv : p \,.\, q \,.\equiv .\, p \,.\, r$

I.e. to say that, on the hypothesis p, q and r are equivalent, is equivalent to saying that the joint assertion of p and q is equivalent to the joint assertion of p and r. This is a very useful rule in inference.

$*5\cdot6.\quad \vdash :.\, p \,.\, \sim q \,.\supset .\, r : \equiv : p \,.\supset .\, q \vee r$

I.e. "p and not-q imply r" is equivalent to "p implies q or r."

Among propositions never subsequently referred to, but inserted for their intrinsic interest, are the following: $*5\cdot11\cdot12\cdot13\cdot14$, which state that, given any two propositions p, q, either p or $\sim p$ must imply q, and p must imply either q or not-q, and either p implies q or q implies p; and given any third proposition r, either p implies q or q implies r*.

Other propositions not subsequently referred to are $*5\cdot22\cdot23\cdot24$; in these it is shown that two propositions are not equivalent when, and only when, one is true and the other false, and that two propositions are equivalent when, and only when, both are true or both false. It follows ($*5\cdot24$) that the negation of "$p \,.\, q \,.\vee .\, \sim p \,.\, \sim q$" is equivalent to "$p \,.\, \sim q \,.\vee .\, q \,.\, \sim p$." $*5\cdot54\cdot55$ state that both the product and the sum of p and q are equivalent, respectively, either to p or to q.

The proofs of the following propositions are all easy, and we shall therefore often merely indicate the propositions used in the proofs.

$*5\cdot1.\quad \vdash : p \,.\, q \,.\supset .\, p \equiv q \qquad [*3\cdot4\cdot22]$

$*5\cdot11.\quad \vdash : p \supset q \,.\vee .\, \sim p \supset q \qquad [*2\cdot5\cdot54]$

$*5\cdot12.\quad \vdash : p \supset q \,.\vee .\, p \supset \sim q \qquad [*2\cdot51\cdot54]$

$*5\cdot13.\quad \vdash : p \supset q \,.\vee .\, q \supset p \qquad [*2\cdot521]$

$*5\cdot14.\quad \vdash : p \supset q \,.\vee .\, q \supset r \qquad [\text{Simp} \,.\, \text{Transp} \,.\, *2\cdot21]$

* Cf. Schröder, *Vorlesungen über Algebra der Logik*, Zweiter Band (Leipzig, 1891), pp. 270—271, where the apparent oddity of the above proposition is explained.

***5·15.** $\vdash : p \equiv q \,.\vee.\, p \equiv \sim q$

Dem.

$\vdash . *4\cdot61 . \supset \vdash : \sim(p \supset q) . \supset . p . \sim q .$

$[*5\cdot1] \qquad \supset . p \equiv \sim q :$

$[*2\cdot54] \quad \supset \vdash : p \supset q . \vee . p \equiv \sim q \qquad (1)$

$\vdash . *4\cdot61 . \supset \vdash : \sim(q \supset p) . \supset . q . \sim p .$

$[*5\cdot1] \qquad \supset . q \equiv \sim p .$

$[*4\cdot12] \qquad \supset . p \equiv \sim q :$

$[*2\cdot54] \quad \supset \vdash : q \supset p . \vee . p \equiv \sim q \qquad (2)$

$\vdash . (1) . (2) . *4\cdot41 . \supset \vdash . \mathrm{Prop}$

***5·16.** $\vdash . \sim(p \equiv q . p \equiv \sim q)$

Dem.

$\vdash . *3\cdot26 . \supset \vdash : p \equiv q . p \supset \sim q . \supset . p \supset q . p \supset \sim q .$

$[*4\cdot82] \qquad \supset . \sim p \qquad (1)$

$\vdash . *3\cdot27 . \supset \vdash : p \equiv q . p \supset \sim q . \supset . q \supset p . p \supset \sim q .$

$[\mathrm{Syll}] \qquad \supset . q \supset \sim q .$

$[\mathrm{Abs}] \qquad \supset . \sim q \qquad (2)$

$\vdash . (1) . (2) . \mathrm{Comp} . \supset \vdash : p \equiv q . p \supset \sim q . \supset . \sim p . \sim q .$

$\left[*4\cdot65 \frac{q, p}{p, q}\right] \qquad \supset . \sim(\sim q \supset p) \qquad (3)$

$\vdash . (3) . \mathrm{Exp} . \supset \vdash :. p \equiv q . \supset : p \supset \sim q . \supset . \sim(\sim q \supset p) :$

$[\mathrm{Id.}(*1\cdot01)] \qquad \supset : \sim(p \supset \sim q) . \vee . \sim(\sim q \supset p) :$

$[*4\cdot51.(*4\cdot01)] \qquad \supset : \sim(p \equiv \sim q) :. \supset \vdash . \mathrm{Prop}$

***5·17.** $\vdash : p \vee q . \sim(p . q) . \equiv . p \equiv \sim q$

Dem.

$\vdash . *4\cdot64\cdot21 . \qquad \supset \vdash : p \vee q . \equiv . \sim q \supset p \qquad (1)$

$\vdash . *4\cdot63 . \mathrm{Transp} . \qquad \supset \vdash : \sim(p . q) . \equiv . p \supset \sim q \qquad (2)$

$\vdash . (1) . (2) . *4\cdot38\cdot21 . \supset \vdash . \mathrm{Prop}$

***5·18.** $\vdash : p \equiv q . \equiv . \sim(p \equiv \sim q) \qquad \left[*5\cdot15\cdot16 . *5\cdot17 \frac{p \equiv q,\ p \equiv \sim q}{p,\ \ q}\right]$

***5·19.** $\vdash . \sim(p \equiv \sim p) \qquad \left[*5\cdot18 \frac{p}{q} . *4\cdot2\right]$

***5·21.** $\vdash : \sim p . \sim q . \supset . p \equiv q \qquad [*5\cdot1 . *4\cdot11]$

***5·22.** $\vdash :. \sim(p \equiv q) . \equiv : p . \sim q . \vee . q . \sim p \qquad [*4\cdot61\cdot51\cdot39]$

***5·23.** $\vdash :. p \equiv q . \equiv : p . q . \vee . \sim p . \sim q \qquad \left[*5\cdot18 . *5\cdot22 \frac{\sim q}{q} . *4\cdot13\cdot36\right]$

***5·24.** $\vdash :. \sim(p . q . \vee . \sim p . \sim q) . \equiv : p . \sim q . \vee . q . \sim p \qquad [*5\cdot22\cdot23]$

***5·25.** $\vdash :. p \vee q . \equiv : p \supset q . \supset . q \qquad [*2\cdot62\cdot68]$

From $*5{\cdot}25$ it appears that we might have taken implication, instead of disjunction, as a primitive idea, and have defined "$p \vee q$" as meaning "$p \supset q . \supset . q$." This course, however, requires more primitive propositions than are required by the method we have adopted.

$*5{\cdot}3$. $\vdash :. p . q . \supset . r : \equiv : p . q . \supset . p . r$ [Simp . Comp . Syll]

$*5{\cdot}31$. $\vdash :. r . p \supset q : \supset : p . \supset . q . r$ [Simp . Comp]

$*5{\cdot}32$. $\vdash :. p . \supset . q \equiv r : \equiv : p . q . \equiv . p . r$ [$*4{\cdot}76 . *3{\cdot}3{\cdot}31 . *5{\cdot}3$]

This proposition is constantly required in subsequent proofs.

$*5{\cdot}33$. $\vdash :. p . q \supset r . \equiv : p : p . q . \supset . r$ [$*4{\cdot}73{\cdot}84 . *5{\cdot}32$]

$*5{\cdot}35$. $\vdash :. p \supset q . p \supset r . \supset : p . \supset . q \equiv r$ [Comp . $*5{\cdot}1$]

$*5{\cdot}36$. $\vdash : p . p \equiv q . \equiv . q . p \equiv q$ [Ass . $*4{\cdot}38$]

$*5{\cdot}4$. $\vdash :. p . \supset . p \supset q : \equiv . p \supset q$ [Simp . $*2{\cdot}43$]

$*5{\cdot}41$. $\vdash :. p \supset q . \supset . p \supset r : \equiv : p . \supset . q \supset r$ [$*2{\cdot}77{\cdot}86$]

$*5{\cdot}42$. $\vdash :: p . \supset . q \supset r : \equiv :. p . \supset : q . \supset . p . r$ [$*5{\cdot}3 . *4{\cdot}87$]

$*5{\cdot}44$. $\vdash :: p \supset q . \supset :. p \supset r . \equiv : p . \supset . q . r$ [$*4{\cdot}76 . *5{\cdot}3{\cdot}32$]

$*5{\cdot}5$. $\vdash :. p . \supset : p \supset q . \equiv . q$ [Ass . Exp . Simp]

$*5{\cdot}501$. $\vdash :. p . \supset : q . \equiv . p \equiv q$ [$*5{\cdot}1$. Exp . Ass]

$*5{\cdot}53$. $\vdash :. p \vee q \vee r . \supset . s : \equiv : p \supset s . q \supset s . r \supset s$ [$*4{\cdot}77$]

$*5{\cdot}54$. $\vdash :. p . q . \equiv . p : \vee : p . q . \equiv . q$ [$*4{\cdot}73 . *4{\cdot}44$. Transp . $*5{\cdot}1$]

$*5{\cdot}55$. $\vdash :. p \vee q . \equiv . p : \vee : p \vee q . \equiv . q$ [$*1{\cdot}3 . *5{\cdot}1 . *4{\cdot}74$]

$*5{\cdot}6$. $\vdash :. p . \sim q . \supset . r : \equiv : p . \supset . q \vee r$ $\left[*4{\cdot}87 \dfrac{\sim q}{q} . *4{\cdot}64{\cdot}85\right]$

$*5{\cdot}61$. $\vdash : p \vee q . \sim q . \equiv . p . \sim q$ [$*4{\cdot}74 . *5{\cdot}32$]

$*5{\cdot}62$. $\vdash :. p . q . \vee . \sim q : \equiv . p \vee \sim q$ $\left[*4{\cdot}7 \dfrac{q, p}{p, q}\right]$

$*5{\cdot}63$. $\vdash :. p \vee q . \equiv : p . \vee . \sim p . q$ $\left[*5{\cdot}62 \dfrac{\sim p, q}{q, \ p}\right]$

$*5{\cdot}7$. $\vdash :. p \vee r . \equiv . q \vee r : \equiv : r . \vee . p \equiv q$ [$*4{\cdot}74 . *1{\cdot}3 . *5{\cdot}1 . *4{\cdot}37$]

$*5{\cdot}71$. $\vdash :. q \supset \sim r . \supset : p \vee q . r . \equiv . p . r$

In the following proof, as always henceforth, "Hp" means the hypothesis of the proposition to be proved.

Dem.

$\vdash . *4{\cdot}4 . \quad \supset \vdash :. p \vee q . r . \equiv : p . r . \vee . q . r \qquad (1)$

$\vdash . *4{\cdot}62{\cdot}51 . \supset \vdash :: \text{Hp} . \supset :. \sim (q . r) :.$

$[*4{\cdot}74] \qquad \supset :. p . r . \vee . q . r : \equiv : p . r \qquad (2)$

$\vdash . (1) . (2) . *4{\cdot}22 . \supset \vdash . \text{Prop}$

***5·74.** $\vdash :. p . \supset . q \equiv r : \equiv : p \supset q . \equiv . p \supset r$

Dem.

$\vdash . *5\cdot41 . \supset \vdash :: p \supset q . \supset . p \supset r : \equiv : p . \supset . q \supset r :.$

$p \supset r . \supset . p \supset q : \equiv : p . \supset . r \supset q$ (1)

$\vdash . (1) . *4\cdot38 . \supset \vdash :: p \supset q . \equiv . p \supset r . \equiv :. p . \supset . q \supset r : p . \supset . r \supset q :.$

[*4·76] $\equiv :. p . \supset . q \equiv r :: \supset \vdash .$ Prop

***5·75.** $\vdash :. r \supset \sim q : p . \equiv . q \vee r : \supset : p . \sim q . \equiv . r$

Dem.

$\vdash . *5\cdot6 . \supset \vdash :. \mathrm{Hp} . \supset : p . \sim q . \supset . r$ (1)

$\vdash . *3\cdot27 . \supset \vdash :. \mathrm{Hp} . \supset : q \vee r . \supset . p :$

[*4·77] $\supset : r \supset p$ (2)

$\vdash . *3\cdot26 . \supset \vdash :. \mathrm{Hp} . \supset : r \supset \sim q$ (3)

$\vdash . (2) . (3) . \mathrm{Comp} . \supset \vdash :. \mathrm{Hp} . \supset : r \supset p . r \supset \sim q :$

[Comp] $\supset : r . \supset . p . \sim q$ (4)

$\vdash . (1) . (4) . \mathrm{Comp} . \supset \vdash :. \mathrm{Hp} . \supset : p . \sim q . \equiv . r :. \supset \vdash .$ Prop

SECTION B

THEORY OF APPARENT VARIABLES

*9. EXTENSION OF THE THEORY OF DEDUCTION FROM LOWER TO HIGHER TYPES OF PROPOSITIONS

Summary of *9.

In the present number, we introduce two new primitive ideas, which may be expressed as "ϕx is always* true" and "ϕx is sometimes* true," or, more correctly, as "ϕx always" and "ϕx sometimes." When we assert "ϕx always," we are asserting all values of $\phi \hat{x}$, where "$\phi \hat{x}$" means the function itself, as opposed to an ambiguous value of the function (cf. pp. 15, 40); we are not asserting that ϕx is true for all values of x, because, in accordance with the theory of types, there are values of x for which "ϕx" is meaningless; for example, the function $\phi \hat{x}$ itself must be such a value. We shall denote "ϕx always" by the notation

$$(x) . \phi x,$$

where the "(x)" will be followed by a sufficiently large number of dots to cover the function of which "all values" are concerned. The form in which such propositions most frequently occur is the "formal implication," *i.e.* such a proposition as

$$(x) : \phi x . \supset . \psi x,$$

i.e. "ϕx always implies ψx." This is the form in which we express the universal affirmative "all objects having the property ϕ have the property ψ."

We shall denote "ϕx sometimes" by the notation

$$(\exists x) . \phi x.$$

Here "$\exists$" stands for "there exists," and the whole symbol may be read "there exists an x such that ϕx."

In a proposition of either of the two forms $(x) . \phi x$, $(\exists x) . \phi x$, the x is called an *apparent variable.* A proposition which contains no apparent variables is called "elementary," and a function, all whose values are elementary propositions, is called an elementary function. For reasons explained in Chapter II of the Introduction, it would seem that negation and disjunction and their derivatives must have a different meaning when applied to elementary propositions from that which they have when applied to such propositions as $(x) . \phi x$ or $(\exists x) . \phi x$. If $\phi \hat{x}$ is an elementary function, we will in this number call $(x) . \phi x$ and $(\exists x) . \phi x$ "first-order propositions." Then in virtue of the fact

* We use "always" as meaning "in all cases," not "at all times." A similar remark applies to "sometimes."

that disjunction and negation do not have the same meanings as applied to elementary or to first-order propositions, it follows that, in asserting the primitive propositions of $*1$, we must either confine them, in their application, to propositions of a single type, or we must regard them as the simultaneous assertion of a number of different primitive propositions, corresponding to the different meanings of "disjunction" and "negation." Likewise in regard to the primitive ideas of disjunction and negation, we must either, in the primitive propositions of $*1$, confine them to disjunctions and negations of elementary propositions, or we must regard them as really each multiple, so that in regard to each type of propositions we shall need a new primitive idea of negation and a new primitive idea of disjunction. In the present number, we shall show how, when the primitive ideas of negation and disjunction are restricted to elementary propositions, and the p, q, r of $*1$—$*5$ are therefore necessarily elementary propositions, it is possible to obtain definitions of the negation and disjunction of first-order propositions, and proofs of the analogues, for first-order propositions, of the primitive propositions $*1 \cdot 2$—$\cdot 6$. ($*1 \cdot 1$ and $*1 \cdot 11$ have to be assumed afresh for first-order propositions, and the analogues of $*1 \cdot 7 \cdot 71 \cdot 72$ require a fresh treatment.) It follows that the analogues of the propositions of $*2$—$*5$ follow by merely repeating previous proofs. It follows also that the theory of deduction can be extended from first-order propositions to such as contain two apparent variables, by merely repeating the process which extends the theory of deduction from elementary to first-order propositions. Thus by merely repeating the process set forth in the present number, propositions of any order can be reached. Hence negation and disjunction may be treated in practice as if there were no difference in these ideas as applied to different types; that is to say, when "$\sim p$" or "$p \vee q$" occurs, it is unnecessary in practice to know what is the type of p or q, since the properties of negation and disjunction assumed in $*1$ (which are alone used in proving other properties) can be asserted, without formal change, of propositions of any order or, in the case of $p \vee q$, of any two orders. The limitation, in practice, to the treatment of negation or disjunction as single ideas, the same in all types, would only arise if we ever wished to assume that there is some one function of p whose value is always $\sim p$, whatever may be the order of p, or that there is some one function of p and q whose value is always $p \vee q$, whatever may be the orders of p and q. Such an assumption is not involved so long as p (and q) remain *real* variables, since, in that case, there is no need to give the same meaning to negation and disjunction for different values of p (and q), when these different values are of different types. But if p (or q) is going to be turned into an apparent variable, then since our two primitive ideas $(x) . \phi x$ and $(\exists x) . \phi x$ both demand some definite function ϕ, and restrict the apparent variable to possible arguments for ϕ, it follows that negation and disjunction must, wherever they occur in the expression in which p (or q) is an apparent variable, be restricted to the kind of negation or disjunction

appropriate to a given type or pair of types. Thus, to take an instance, if we assert the law of excluded middle in the form

$$\text{“} \vdash . p \vee \sim p \text{”}$$

there is no need to place any restriction upon p: we may give to p a value of any order, and then give to the negation and disjunction involved those meanings which are appropriate to that order. But if we assert

$$\text{“} \vdash . (p) . p \vee \sim p \text{”}$$

it is necessary, if our symbol is to be significant, that "$p \vee \sim p$" should be the value, for the argument p, of a function ϕp; and this is only possible if the negation and disjunction involved have meanings fixed in advance, and if, therefore, p is limited to one type. Thus the assertion of the law of excluded middle in the form involving a real variable is more general than in the form involving an apparent variable. Similar remarks apply generally where the variable is the argument to a typically ambiguous function.

In what follows the single letters p and q will represent *elementary* propositions, and so will "ϕx," "ψx," etc. We shall show how, assuming the primitive ideas and propositions of ∗1 as applied to elementary propositions, we can define and prove analogous ideas and propositions as applied to propositions of the forms $(x) . \phi x$ and $(\exists x) . \phi x$. By mere repetition of the analogous process, it will then follow that analogous ideas and propositions can be defined and proved for propositions of any order; whence, further, it follows that, in all that concerns disjunction and negation, so long as propositions do not appear as apparent variables, we may wholly ignore the distinction between different types of propositions and between different meanings of negation and disjunction. Since we never have occasion, in practice, to consider propositions as apparent variables, it follows that the hierarchy of propositions (as opposed to the hierarchy of functions) will never be relevant in practice after the present number.

The purpose and interest of the present number are purely philosophical, namely to show how, by means of certain primitive propositions, we can deduce the theory of deduction for propositions containing apparent variables from the theory of deduction for elementary propositions. From the purely technical point of view, the distinction between elementary and other propositions may be ignored, so long as propositions do not appear as apparent variables; we may then regard the primitive propositions of ∗1 as applying to propositions of any type, and proceed as in ∗10, where the purely technical development is resumed.

It should be observed that although, in the present number, we prove that the analogues of the primitive propositions of ∗1, if they hold for propositions containing n apparent variables, also hold for such as contain $n+1$, yet we must not suppose that mathematical induction may be used to infer that the analogues of the primitive propositions of ∗1 hold for propositions

containing any number of apparent variables. Mathematical induction is a method of proof which is not yet applicable, and is (as will appear) incapable of being used freely until the theory of propositions containing apparent variables has been established. What we are enabled to do, by means of the propositions in the present number, is to prove our desired result for any assigned number of apparent variables—say ten—by ten applications of the same proof. Thus we can prove, concerning any assigned proposition, that it obeys the analogues of the primitive propositions of $*1$, but we can only do this by proceeding step by step, not by any such compendious method as mathematical induction would afford. The fact that higher types can only be reached step by step is essential, since to proceed otherwise we should need an apparent variable which would wander from type to type, which would contradict the principle upon which types are built up.

Definition of Negation. We have first to define the negations of $(x) . \phi x$ and $(\exists x) . \phi x$. We define the negation of $(x) . \phi x$ as $(\exists x) . \sim \phi x$, *i.e.* "it is not the case that ϕx is always true" is to mean "it is the case that not-ϕx is sometimes true." Similarly the negation of $(\exists x) . \phi x$ is to be defined as $(x) . \sim \phi x$. Thus we put

$*9{\cdot}01.$ $\sim \{(x) . \phi x\} . = . (\exists x) . \sim \phi x$ Df

$*9{\cdot}02.$ $\sim \{(\exists x) . \phi x\} . = . (x) . \sim \phi x$ Df

To avoid brackets, we shall write $\sim (x) . \phi x$ in place of $\sim \{(x) . \phi x\}$, and $\sim (\exists x) . \phi x$ in place of $\sim \{(\exists x) . \phi x\}$. Thus:

$*9{\cdot}011.$ $\sim (x) . \phi x . = . \sim \{(x) . \phi x\}$ Df

$*9{\cdot}021.$ $\sim (\exists x) . \phi x . = . \sim \{(\exists x) . \phi x\}$ Df

Definition of Disjunction. To define disjunction when one or both of the propositions concerned is of the first order, we have to distinguish six cases, as follows:

$*9{\cdot}03.$ $(x) . \phi x . \vee . p : = . (x) . \phi x \vee p$ Df

$*9{\cdot}04.$ $p . \vee . (x) . \phi x : = . (x) . p \vee \phi x$ Df

$*9{\cdot}05.$ $(\exists x) . \phi x . \vee . p : = . (\exists x) . \phi x \vee p$ Df

$*9{\cdot}06.$ $p . \vee . (\exists x) . \phi x : = . (\exists x) . p \vee \phi x$ Df

$*9{\cdot}07.$ $(x) . \phi x . \vee . (\exists y) . \psi y : = : (x) : (\exists y) . \phi x \vee \psi y$ Df

$*9{\cdot}08.$ $(\exists y) . \psi y . \vee . (x) . \phi x : = : (x) : (\exists y) . \psi y \vee \phi x$ Df

(The definitions $*9{\cdot}07{\cdot}08$ are to apply also when ϕ and ψ are not both elementary functions.)

In virtue of these definitions, the true scope of an apparent variable is always the whole of the asserted proposition in which it occurs, even when, typographically, its scope appears to be only part of the asserted proposition. Thus when $(\exists x) . \phi x$ or $(x) . \phi x$ *appears* as *part* of an asserted proposition, it does not really occur, since the scope of the apparent variable really extends

to the whole asserted proposition. It will be shown, however, that, so far as the theory of deduction is concerned, $(\exists x) . \phi x$ and $(x) . \phi x$ behave like propositions not containing apparent variables.

The definitions of implication, the logical product, and equivalence are to be transferred unchanged to $(x) . \phi x$ and $(\exists x) . \phi x$.

The above definitions can be repeated for successive types, and thus reach propositions of any type.

Primitive Propositions. The primitive propositions required are six in number, and may be divided into three sets of two. We have first two propositions, which effect the passage from elementary to first-order propositions, namely

***9·1.** $\vdash : \phi x . \supset . (\exists z) . \phi z$ Pp

***9·11.** $\vdash : \phi x \vee \phi y . \supset . (\exists z) . \phi z$ Pp

Of these, the first states that, if ϕx is true, then there is a value of $\phi \hat{z}$ which is true; *i.e.* if we can find an instance of a function which is true, then the function is "sometimes true." (When we speak of a function as "sometimes" true, we do not mean to assert that there is *more* than one argument for which it is true, but only that there is *at least* one.) Practically, the above primitive proposition gives the only method of proving "existence-theorems": in order to prove such theorems, it is necessary (and sufficient) to find some instance in which an object possesses the property in question. If we were to assume what may be called "existence-axioms," *i.e.* axioms stating $(\exists z) . \phi z$ for some particular ϕ, these axioms would give other methods of proving existence. Instances of such axioms are the multiplicative axiom (*88) and the axiom of infinity (defined in *120·03). But we have not assumed any such axioms in the present work.

The second of the above primitive propositions is only used once, in proving $(\exists z) . \phi z . \vee . (\exists z) . \phi z : \supset . (\exists z) . \phi z$, which is the analogue of *1·2 (namely $p \vee p . \supset . p$) when p is replaced by $(\exists z) . \phi z$. The effect of this primitive proposition is to emphasize the ambiguity of the z required in order to secure $(\exists z) . \phi z$. We have, of course, in virtue of *9·1,

$$\phi x . \supset . (\exists z) . \phi z \text{ and } \phi y . \supset . (\exists z) . \phi z.$$

But if we try to infer from these that $\phi x \vee \phi y . \supset . (\exists z) . \phi z$, we must use the proposition $q \supset p . r \supset p . \supset . q \vee r \supset p$, where p is $(\exists z) . \phi z$. Now it will be found, on referring to *4·77 and the propositions used in its proof, that this proposition depends upon *1·2, *i.e.* $p \vee p . \supset . p$. Hence it cannot be used by us to prove $(\exists x) . \phi x . \vee . (\exists x) . \phi x : \supset . (\exists x) . \phi x$, and thus we are compelled to assume the primitive proposition *9·11.

We have next two propositions concerned with *inference* to or from propositions containing apparent variables, as opposed to implication. First, we have,

for the new meaning of implication resulting from the above definitions of negation and disjunction, the analogue of $*1\cdot1$, namely

$*9\cdot12$. What is implied by a true premiss is true. Pp.

That is to say, given "$\vdash . p$" and "$\vdash . p \supset q$," we may proceed to "$\vdash . q$," even when the propositions p and q are not elementary. Also, as in $*1\cdot11$, we may proceed from "$\vdash . \phi x$" and "$\vdash . \phi x \supset \psi x$" to "$\vdash . \psi x$," where x is a real variable, and ϕ and ψ are not necessarily elementary functions. It is in this latter form that the axiom is usually needed. It is to be assumed for functions of several variables as well as for functions of one variable.

We have next the primitive proposition which permits the passage from a real to an apparent variable, namely "when ϕy may be asserted, where y may be any possible argument, then $(x) . \phi x$ may be asserted." In other words, when ϕy is true however y may be chosen among possible arguments, then $(x) . \phi x$ is true, *i.e.* all values of ϕ are true. That is to say, if we can assert a wholly ambiguous value ϕy, that must be because all values are true. We may express this primitive proposition by the words: "What is true in *any* case, however the case may be selected, is true in *all* cases." We cannot symbolise this proposition, because if we put

$$\text{"}\vdash : \phi y . \supset . (x) . \phi x\text{"}$$

that means: "However y may be chosen, ϕy implies $(x) . \phi x$," which is in general false. What we mean is: "If ϕy is true however y may be chosen, then $(x) . \phi x$ is true." But we have not supplied a symbol for the mere *hypothesis* of what is *asserted* in "$\vdash . \phi y$," where y is a real variable, and it is not worth while to supply such a symbol, because it would be very rarely required. If, for the moment, we use the symbol $[\phi y]$ to express this hypothesis, then our primitive proposition is

$$\vdash : [\phi y] . \supset . (x) . \phi x \quad \text{Pp.}$$

In practice, this primitive proposition is only used for *inference*, not for implication; that is to say, when we actually have an assertion containing a real variable, it enables us to turn this real variable into an apparent variable by placing it in brackets immediately after the assertion-sign, followed by enough dots to reach to the end of the assertion. This process will be called "turning a real variable into an apparent variable." Thus we may assert our primitive proposition, for technical use, in the form:

$*9\cdot13$. In any assertion containing a real variable, this real variable may be turned into an apparent variable of which all possible values are asserted to satisfy the function in question. Pp.

We have next two primitive propositions concerned with types. These require some preliminary explanations.

Primitive Idea: Individual. We say that x is "individual" if x is neither a proposition nor a function (cf. p. 51).

$*9 \cdot 131$. *Definition of "being of the same type."* The following is a step-by-step definition, the definition for higher types presupposing that for lower types. We say that u and v "are of the same type" if (1) both are individuals, (2) both are elementary functions taking arguments of the same type, (3) u is a function and v is its negation, (4) u is $\phi\hat{x}$ or $\psi\hat{x}$, and v is $\phi\hat{x} \vee \psi\hat{x}$, where $\phi\hat{x}$ and $\psi\hat{x}$ are elementary functions, (5) u is $(y) . \phi(\hat{x}, y)$ and v is $(z) . \psi(\hat{x}, z)$, where $\phi(\hat{x}, \hat{y})$, $\psi(\hat{x}, \hat{y})$ are of the same type, (6) both are elementary propositions, (7) u is a proposition and v is $\sim u$, or (8) u is $(x) . \phi x$ and v is $(y) . \psi y$, where $\phi\hat{x}$ and $\psi\hat{x}$ are of the same type.

Our primitive propositions are:

$*9 \cdot 14$. If "ϕx" is significant, then if x is of the same type as a, "ϕa" is significant, and vice versa. Pp. (Cf. note on $*10 \cdot 121$, p. 140.)

$*9 \cdot 15$. If, for some a, there is a proposition ϕa, then there is a function $\phi\hat{x}$, and vice versa. Pp.

It will be seen that, in virtue of the definitions,

$$(x) . \phi x . \supset . p \text{ means } \sim(x) . \phi x . \vee . p, \text{ i.e. } (\exists x) . \sim\phi x . \vee . p,$$
$$\text{i.e. } (\exists x) . \sim\phi x \vee p, \text{ i.e. } (\exists x) . \phi x \supset p$$
$$(\exists x) . \phi x . \supset . p \text{ means } \sim(\exists x) . \phi x . \vee . p, \text{ i.e. } (x) . \sim\phi x . \vee . p,$$
$$\text{i.e. } (x) . \sim\phi x \vee p, \text{ i.e. } (x) . \phi x \supset p$$

In order to prove that $(x) . \phi x$ and $(\exists x) . \phi x$ obey the same rules of deduction as ϕx, we have to prove that propositions of the forms $(x) . \phi x$ and $(\exists x) . \phi x$ may replace one or more of the propositions p, q, r in $*1 \cdot 2$—$\cdot 6$. When this has been proved, the previous proofs of subsequent propositions in $*2$—$*5$ become applicable. These proofs are given below. Certain other propositions, required in the proofs, are also proved.

$*9 \cdot 2$. $\vdash : (x) . \phi x . \supset . \phi y$

The above proposition states the principle of deduction from the general to the particular, *i.e.* "what holds in all cases, holds in any one case."

Dem.

$$\vdash . *2 \cdot 1 . \supset \vdash . \sim\phi y \vee \phi y \qquad (1)$$
$$\vdash . *9 \cdot 1 . \supset \vdash : \sim\phi y \vee \phi y . \supset . (\exists x) . \sim\phi x \vee \phi y \qquad (2)$$
$$\vdash . (1) . (2) . *1 \cdot 11 . \supset \vdash . (\exists x) . \sim\phi x \vee \phi y \qquad (3)$$
$$[(3).(*9 \cdot 05)] \quad \vdash : (\exists x) . \sim\phi x . \vee . \phi y \qquad (4)$$
$$[(4).(*9 \cdot 01 . *1 \cdot 01)] \quad \vdash : (x) . \phi x . \supset . \phi y$$

In the second line of the above proof, "$\sim\phi y \vee \phi y$" is taken as the value, for the argument y, of the function "$\sim\phi x \vee \phi y$," where x is the argument. A similar method of using $*9 \cdot 1$ is employed in most of the following proofs.

$*1 \cdot 11$ is used, as in the third line of the above proof, in almost all steps except such as are mere applications of definitions. Hence it will not be

further referred to, unless in cases where its employment is obscure or specially important.

✱9·21. $\vdash :. (x) . \phi x \supset \psi x . \supset : (x) . \phi x . \supset . (x) . \psi x$

I.e. if ϕx always implies ψx, then "ϕx always" implies "ψx always." The use of this proposition is constant throughout the remainder of this work.

Dem.

$\vdash . ✱2{\cdot}08 . \quad \supset \vdash : \phi z \supset \psi z . \supset . \phi z \supset \psi z \qquad (1)$

$\vdash . (1) . ✱9{\cdot}1 . \quad \supset \vdash : (\exists y) : \phi z \supset \psi z . \supset . \phi y \supset \psi z \qquad (2)$

$\vdash . (2) . ✱9{\cdot}1 . \quad \supset \vdash :. (\exists x) :. (\exists y) : \phi x \supset \psi x . \supset . \phi y \supset \psi z \qquad (3)$

$\vdash . (3) . ✱9{\cdot}13 . \quad \supset \vdash :: (z) :: (\exists x) :. (\exists y) : \phi x \supset \psi x . \supset . \phi y \supset \psi z \qquad (4)$

$[(4).(✱9{\cdot}06)] \quad \vdash :: (z) :: (\exists x) :. \phi x \supset \psi x . \supset : (\exists y) . \phi y \supset \psi z \qquad (5)$

$[(5).(✱1{\cdot}01.✱9{\cdot}08)] \quad \vdash :. (\exists x) . \sim(\phi x \supset \psi x) : \lor : (z) : (\exists y) . \sim \phi y \lor \psi z \qquad (6)$

$[(6).(✱9{\cdot}08)] \quad \vdash :. (\exists x) . \sim(\phi x \supset \psi x) : \lor : (\exists y) . \sim \phi y . \lor . (z) . \psi z \qquad (7)$

$[(7).(✱1{\cdot}01)] \quad \vdash :. (x) . \phi x \supset \psi x . \supset : (y) . \phi y . \supset . (z) . \psi z$

This is the proposition to be proved, since "$(y) . \phi y$" is the same proposition as "$(x) . \phi x$," and "$(z) . \psi z$" is the same proposition as "$(x) . \psi x$."

✱9·22. $\vdash :. (x) . \phi x \supset \psi x . \supset : (\exists x) . \phi x . \supset . (\exists x) . \psi x$

I.e. if ϕx always implies ψx, then if ϕx is sometimes true, so is ψx. This proposition, like ✱9·21, is constantly used in the sequel.

Dem.

$\vdash . ✱2{\cdot}08 . \quad \supset \vdash : \phi y \supset \psi y . \supset . \phi y \supset \psi y \qquad (1)$

$\vdash . (1) . ✱9{\cdot}1 . \quad \supset \vdash : (\exists z) : \phi y \supset \psi y . \supset . \phi y \supset \psi z \qquad (2)$

$\vdash . (2) . ✱9{\cdot}1 . \quad \supset \vdash :. (\exists x) :. (\exists z) : \phi x \supset \psi x . \supset . \phi y \supset \psi z \qquad (3)$

$\vdash . (3) . ✱9{\cdot}13 . \quad \supset \vdash :: (y) :: (\exists x) :. (\exists z) : \phi x \supset \psi x . \supset . \phi y \supset \psi z \qquad (4)$

$[(4).(✱9{\cdot}06)] \quad \vdash :: (y) :: (\exists x) :. \phi x \supset \psi x . \supset : (\exists z) . \phi y \supset \psi z \qquad (5)$

$[(5).(✱1{\cdot}01.✱9{\cdot}08)] \quad \vdash :: (\exists x) . \sim(\phi x \supset \psi x) : \lor : (y) : (\exists z) . \phi y \supset \psi z \qquad (6)$

$[(6).(✱1{\cdot}01.✱9{\cdot}07)] \quad \vdash :: (\exists x) . \sim(\phi x \supset \psi x) : \lor : (y) . \sim \phi y . \lor . (\exists z) . \psi z \qquad (7)$

$[(7).(✱1{\cdot}01.✱9{\cdot}01{\cdot}02)] \quad \vdash :. (x) . \phi x \supset \psi x . \supset : (\exists y) . \phi y . \supset . (\exists z) . \psi z$

This is the proposition to be proved, because $(\exists y) . \phi y$ is the same proposition as $(\exists x) . \phi x$, and $(\exists z) . \psi z$ is the same proposition as $(\exists x) . \psi x$.

✱9·23. $\vdash : (x) . \phi x . \supset . (x) . \phi x$ [Id . ✱9·13·21]

✱9·24. $\vdash : (\exists x) . \phi x . \supset . (\exists x) . \phi x$ [Id . ✱9·13·22]

✱9·25. $\vdash :. (x) . p \lor \phi x . \supset : p . \lor . (x) . \phi x$ [✱9·23 . (✱9·04)]

We are now in a position to prove the analogues of ✱1·2—·6, replacing one of the letters p, q, r in those propositions by $(x) . \phi x$ or $(\exists x) . \phi x$. The proofs are given below.

***9·3.** $\vdash :. (x) . \phi x . \vee . (x) . \phi x : \supset . (x) . \phi x$

Dem.

$\vdash . *1\cdot2 . \quad \supset \vdash . \phi x \vee \phi x . \supset . \phi x$ (1)

$\vdash . (1) . *9\cdot1 . \quad \supset \vdash : (\exists y) : \phi x \vee \phi y . \supset . \phi x$ (2)

$\vdash . (2) . *9\cdot13 . \quad \supset \vdash :. (x) :. (\exists y) : \phi x \vee \phi y . \supset . \phi x$ (3)

$[(3).(*9\cdot05\cdot01\cdot04)] \quad \vdash :. (x) :. \phi x . \vee . (y) . \phi y : \supset . \phi x$ (4)

$\vdash . (4) . *9\cdot21 . \quad \supset \vdash :. (x) : \phi x . \vee . (y) . \phi y : \supset . (x) . \phi x$ (5)

$[(5).(*9\cdot03)] \quad \vdash :. (x) . \phi x . \vee . (y) . \phi y : \supset . (x) . \phi x :. \supset \vdash . \mathrm{Prop}$

***9·31.** $\vdash :. (\exists x) . \phi x . \vee . (\exists x) . \phi x : \supset . (\exists x) . \phi x$

This is the only proposition which employs *9·11.

Dem.

$\vdash . *9\cdot11\cdot13 . \quad \supset \vdash : (y) : \phi x \vee \phi y . \supset . (\exists z) . \phi z$ (1)

$[(1).(*9\cdot03\cdot02)] \quad \vdash : (\exists y) . \phi x \vee \phi y . \supset . (\exists z) . \phi z$ (2)

$\vdash . (2) . *9\cdot13 . \supset \vdash : (x) : (\exists y) . \phi x \vee \phi y . \supset . (\exists z) . \phi z$ (3)

$[(3).(*9\cdot03\cdot02)] \quad \vdash :. (\exists x) : (\exists y) . \phi x \vee \phi y : \supset . (\exists z) . \phi z$ (4)

$[(4).(*9\cdot05\cdot06)] \quad \vdash :. (\exists x) . \phi x . \vee . (\exists y) . \phi y : \supset . (\exists z) . \phi z$

***9·32.** $\vdash :. q . \supset : (x) . \phi x . \vee . q$

Dem.

$\vdash . *1\cdot3 . \quad \supset \vdash :. q . \supset : \phi x . \vee . q$ (1)

$\vdash . (1) . *9\cdot13 . \supset \vdash :. (x) :. q . \supset : \phi x . \vee . q$

$[*9\cdot25] \quad \supset \vdash :. q . \supset : (x) : \phi x . \vee . q$ (2)

$[(2).(*9\cdot03)] \quad \vdash :. q . \supset : (x) . \phi x . \vee . q$

***9·33.** $\vdash :. q . \supset : (\exists x) . \phi x . \vee . q$ [Proof as above]

***9·34.** $\vdash :. (x) . \phi x . \supset : p . \vee . (x) . \phi x$

Dem.

$\vdash . *1\cdot3 . \quad \supset \vdash : \phi x . \supset . p \vee \phi x$ (1)

$\vdash . (1) . *9\cdot13 . \quad \supset \vdash : (x) : \phi x . \supset . p \vee \phi x$ (2)

$\vdash . (2) . *9\cdot21 . \quad \supset \vdash : (x) . \phi x . \supset . (x) . p \vee \phi x$ (3)

$\vdash . (3) . (*9\cdot04) . \supset \vdash . \mathrm{Prop}$

***9·35.** $\vdash :. (\exists x) . \phi x . \supset : p . \vee . (\exists x) . \phi x$ [Proof as above]

***9·36.** $\vdash :. p . \vee . (x) . \phi x : \supset : (x) . \phi x . \vee . p$

Dem.

$\vdash . *1\cdot4 . \quad \supset \vdash : p \vee \phi x . \supset . \phi x \vee p$ (1)

$\vdash . (1) . *9\cdot13\cdot21 . \quad \supset \vdash : (x) . p \vee \phi x . \supset . (x) . \phi x \vee p$ (2)

$\vdash . (2) . (*9\cdot03\cdot04) . \supset \vdash . \mathrm{Prop}$

***9·361.** $\vdash :. (x) . \phi x . \vee . p : \supset : p . \vee . (x) . \phi x$ [Similar proof]

***9·37.** $\vdash :. p . \vee . (\exists x) . \phi x : \supset : (\exists x) . \phi x . \vee . p$ [Similar proof]

***9·371.** $\vdash :. (\exists x) . \phi x . \vee . p : \supset : p . \vee . (\exists x) . \phi x$ [Similar proof]

***9·4.** $\vdash :: p : \lor : q . \lor . (x) . \phi x :. \supset :. q : \lor : p . \lor . (x) . \phi x$

Dem.

$\vdash . *1{\cdot}5 . *9{\cdot}21 . \supset \vdash :. (x) : p . \lor . q \lor \phi x : \supset : (x) : q . \lor . p \lor \phi x$ (1)

$\vdash . (1) . (*9{\cdot}04) . \supset \vdash .$ Prop

***9·401.** $\vdash :: p : \lor : q . \lor . (\exists x) . \phi x :. \supset :. q : \lor : p . \lor . (\exists x) . \phi x$ [As above]

***9·41.** $\vdash :: p : \lor : (x) . \phi x . \lor . r :. \supset :. (x) . \phi x : \lor : p \lor r$ [As above]

***9·411.** $\vdash :: p : \lor : (\exists x) . \phi x . \lor . r :. \supset :. (\exists x) . \phi x : \lor : p \lor r$ [As above]

***9·42.** $\vdash :: (x) . \phi x : \lor : q \lor r :. \supset :. q : \lor : (x) . \phi x . \lor . r$ [As above]

***9·421.** $\vdash :: (\exists x) . \phi x : \lor : q \lor r :. \supset :. q : \lor : (\exists x) . \phi x . \lor . r$ [As above]

***9·5.** $\vdash :: p \supset q . \supset :. p . \lor . (x) . \phi x : \supset : q . \lor . (x) . \phi x$

Dem.

$\vdash . *1{\cdot}6 . \quad \supset \vdash :. p \supset q . \supset : p \lor \phi y . \supset . q \lor \phi y$ (1)

$\vdash . (1) . *9{\cdot}1 . (*9{\cdot}06) . \quad \supset \vdash :. p \supset q . \supset : (\exists x) : p \lor \phi x . \supset . q \lor \phi y$ (2)

$\vdash . (2) . *9{\cdot}13 . (*9{\cdot}04) . \supset \vdash :: p \supset q . \supset :. (y) :. (\exists x) : p \lor \phi x . \supset . q \lor \phi y$ (3)

$[(3).(*9{\cdot}08)] \quad \vdash :: p \supset q . \supset :. (\exists x) . \sim(p \lor \phi x) . \lor . (y) . q \lor \phi y$ (4)

$[(4).(*9{\cdot}01)] \quad \vdash :: p \supset q . \supset :. (x) . p \lor \phi x . \supset . (y) . q \lor \phi y$ (5)

$[(5).(*9{\cdot}04)] \quad \vdash :: p \supset q . \supset :. p . \lor . (x) . \phi x : \supset : q . \lor . (y) . \phi y$

***9·501.** $\vdash :: p \supset q . \supset :. p . \lor . (\exists x) . \phi x : \supset : q . \lor . (\exists x) . \phi x$ [As above]

***9·51.** $\vdash :: p . \supset . (x) . \phi x : \supset :. p \lor r . \supset : (x) . \phi x . \lor . r$

Dem.

$\vdash . *1{\cdot}6 . \quad \supset \vdash :. p \supset \phi x . \supset : p \lor r . \supset . \phi x \lor r$ (1)

$\vdash . (1) . *9{\cdot}13{\cdot}21 . \quad \supset \vdash :: (x) . p \supset \phi x . \supset :. (x) : p \lor r . \supset . \phi x \lor r$ (2)

$\vdash . (2) . (*9{\cdot}03{\cdot}04) . \supset \vdash .$ Prop

***9·511.** $\vdash :: p . \supset . (\exists x) . \phi x : \supset :. p \lor r . \supset : (\exists x) . \phi x . \lor . r$ [As above]

***9·52.** $\vdash :: (x) . \phi x . \supset . q : \supset :. (x) . \phi x . \lor . r : \supset . q \lor r$

Dem.

$\vdash . *1{\cdot}6 . \quad \supset \vdash :. \phi x \supset q . \supset : \phi x \lor r . \supset . q \lor r$ (1)

$\vdash . (1) . *9{\cdot}13{\cdot}22 . \quad \supset \vdash :: (\exists x) . \phi x \supset q . \supset :. (\exists x) : \phi x \lor r . \supset . q \lor r$ (2)

$\vdash . (2) . (*9{\cdot}05{\cdot}01) . \supset \vdash :: (x) . \phi x . \supset . q : \supset :. (x) . \phi x \lor r . \supset . q \lor r$ (3)

$\vdash . (3) . (*9{\cdot}03) . \quad \supset \vdash .$ Prop

***9·521.** $\vdash :: (\exists x) . \phi x . \supset . q : \supset :. (\exists x) . \phi x . \lor . r : \supset . q \lor r$ [As above]

***9·6.** $(x) . \phi x$, $\sim(x) . \phi x$, $(\exists x) . \phi x$ and $\sim(\exists x) . \phi x$ are of the same type. [*9·131, (7) and (8)]

***9·61.** If $\phi \hat{x}$ and $\psi \hat{x}$ are elementary functions of the same type, there is a function $\phi \hat{x} \lor \psi \hat{x}$.

Dem.

By *9·14·15, there is an a for which "ψa," and therefore "ϕa," are significant, and therefore so is "$\phi a \lor \psi a$," by the primitive idea of disjunction. Hence the result by *9·15.

The same proof holds for functions of any number of variables.

∗9·62. If $\phi(\hat{x}, \hat{y})$ and $\psi\hat{z}$ are elementary functions, and the x-argument to ϕ is of the same type as the argument to ψ, there are functions

$$(y) . \phi(\hat{x}, y) . \mathrm{v} . \psi\hat{x}, \; (\exists y) . \phi(\hat{x}, y) . \mathrm{v} . \psi\hat{x}.$$

Dem.

By ∗9·15, there are propositions $\phi(x, b)$ and ψa, where by hypothesis x and a are of the same type. Hence by ∗9·14 there is a proposition $\phi(a, b)$, and therefore, by the primitive idea of disjunction, there is a proposition $\phi(a, b) \mathrm{v} \psi a$, and therefore, by ∗9·15 and ∗9·03, there is a proposition $(y) . \phi(a, y) . \mathrm{v} . \psi a$. Similarly there is a proposition $(\exists y) . \phi(a, y) . \mathrm{v} . \psi a$. Hence the result, by ∗9·15.

∗9·63. If $\phi(\hat{x}, \hat{y})$, $\psi(\hat{x}, \hat{y})$ are elementary functions of the same type, there are functions $(y) . \phi(\hat{x}, y) . \mathrm{v} . (z) . \psi(\hat{x}, z)$, etc. [Proof as above]

We have now completed the proof that, in the primitive propositions of ∗1, any *one* of the propositions that occur may be replaced by $(x) . \phi x$ or $(\exists x) . \phi x$. It follows that, by merely repeating the proofs, we can show that any other of the propositions that occur in these propositions can be simultaneously replaced by $(x) . \psi x$ or $(\exists x) . \psi x$. Thus all the primitive propositions of ∗1, and therefore all the propositions of ∗2—∗5, hold equally when some or all of the propositions concerned are of one of the forms $(x) . \phi x$, $(\exists x) . \phi x$, which was to be proved.

It follows, by mere repetition of the proofs, that the propositions of ∗1—∗5 hold when p, q, r are replaced by propositions containing any number of apparent variables.

$*10$. THEORY OF PROPOSITIONS CONTAINING ONE APPARENT VARIABLE

Summary of $*10$.

The chief purpose of the propositions of this number is to extend to formal implications (*i.e.* to propositions of the form $(x) . \phi x \supset \psi x$) as many as possible of the propositions proved previously for material implications, *i.e.* for propositions of the form $p \supset q$. Thus *e.g.* we have proved in $*3{\cdot}33$ that

$$p \supset q . q \supset r . \supset . p \supset r.$$

Put $p =$ Socrates is a Greek,
$q =$ Socrates is a man,
$r =$ Socrates is a mortal.

Then we have "if 'Socrates is a Greek' implies 'Socrates is a man,' and 'Socrates is a man' implies 'Socrates is a mortal,' it follows that 'Socrates is a Greek' implies 'Socrates is a mortal.'" But this does not of itself prove that if all Greeks are men, and all men are mortals, then all Greeks are mortals.

Putting $\phi x . = . x$ is a Greek,
$\psi x . = . x$ is a man,
$\chi x . = . x$ is a mortal,

we have to prove

$$(x) . \phi x \supset \psi x : (x) . \psi x \supset \chi x : \supset : (x) . \phi x \supset \chi x.$$

It is such propositions that have to be proved in the present number. It will be seen that formal implication ($(x) . \phi x \supset \psi x$) is a relation of two functions $\phi \hat{x}$ and $\psi \hat{x}$. Many of the formal properties of this relation are analogous to properties of the relation "$p \supset q$" which expresses material implication; it is such analogues that are to be proved in this number.

We shall assume in this number, what has been proved in $*9$, that the propositions of $*1$—$*5$ can be applied to such propositions as $(x) . \phi x$ and $(\exists x) . \phi x$. Instead of the method adopted in $*9$, it is possible to take negation and disjunction as new primitive ideas, as applied to propositions containing apparent variables, and to assume that, with the new meanings of negation and disjunction, the primitive propositions of $*1$ still hold. If this method is adopted, we need not take $(\exists x) . \phi x$ as a primitive idea, but may put

$*10{\cdot}01$. $(\exists x) . \phi x . = . \sim(x) . \sim \phi x$ Df

In order to make it clear how this alternative method can be developed, we shall, in the present number, assume nothing of what has been proved in $*9$ except certain propositions which, in the alternative method, will be primitive propositions, and (what in part characterizes the alternative method)

the applicability to propositions containing apparent variables of analogues of the primitive ideas and propositions of ∗1, and therefore of their consequences as set forth in ∗2—∗5.

The two following definitions merely serve to introduce a notation which is often more convenient than the notation $(x) . \phi x \supset \psi x$ or $(x) . \phi x \equiv \psi x$.

∗10·02. $\phi x \supset_x \psi x . = . (x) . \phi x \supset \psi x$ Df

∗10·03. $\phi x \equiv_x \psi x . = . (x) . \phi x \equiv \psi x$ Df

The first of these notations is due to Peano, who, however, has no notation for $(x) . \phi x$ except in the special case of a formal implication.

The following propositions (∗10·1·11·12·121·122) have already been given in ∗9. ∗10·1 is ∗9·2, ∗10·11 is ∗9·13, ∗10·12 is ∗9·25, ∗10·121 is ∗9·14, and ∗10·122 is ∗9·15. These five propositions must all be taken as primitive propositions in the alternative method; on the other hand, ∗9·1 and ∗9·11 are not required as primitive propositions in the alternative method.

The propositions of the present number are very much used throughout the rest of the work. The propositions most used are the following:

∗10·1. $\vdash : (x) . \phi x . \supset . \phi y$

I.e. what is true in all cases is true in any one case.

∗10·11. If ϕy is true whatever possible argument y may be, then $(x) . \phi x$ is true. In other words, whenever the propositional function ϕy can be asserted, so can the proposition $(x) . \phi x$.

∗10·21. $\vdash :. (x) . p \supset \phi x . \equiv : p . \supset . (x) . \phi x$

∗10·22. $\vdash :. (x) . \phi x . \psi x . \equiv : (x) . \phi x : (x) . \psi x$

The conditions of significance in this proposition demand that ϕ and ψ should take arguments of the same type.

∗10·23. $\vdash :. (x) . \phi x \supset p . \equiv : (\exists x) . \phi x . \supset . p$

I.e. if ϕx always implies p, then if ϕx is ever true, p is true.

∗10·24. $\vdash : \phi y . \supset . (\exists x) . \phi x$

I.e. if ϕy is true, then there is an x for which ϕx is true. This is the sole method of proving existence-theorems.

∗10·27. $\vdash :. (z) . \phi z \supset \psi z . \supset : (z) . \phi z . \supset . (z) . \psi z$

I.e. if ϕz always implies ψz, then "ϕz always" implies "ψz always." The three following propositions, which are equally useful, are analogous to ∗10·27.

∗10·271. $\vdash :. (z) . \phi z \equiv \psi z . \supset : (z) . \phi z . \equiv . (z) . \psi z$

∗10·28. $\vdash :. (x) . \phi x \supset \psi x . \supset : (\exists x) . \phi x . \supset . (\exists x) . \psi x$

∗10·281. $\vdash :. (x) . \phi x \equiv \psi x . \supset : (\exists x) . \phi x . \equiv . (\exists x) . \psi x$

∗10·35. $\vdash :. (\exists x) . p . \phi x . \equiv : p : (\exists x) . \phi x$

∗10·42. $\vdash :. (\exists x) . \phi x . \vee . (\exists x) . \psi x : \equiv . (\exists x) . \phi x \vee \psi x$

∗10·5. $\vdash :. (\exists x) . \phi x . \psi x . \supset : (\exists x) . \phi x : (\exists x) . \psi x$

It should be noticed that whereas $*10\cdot42$ expresses an equivalence, $*10\cdot5$ only expresses an implication. This is the source of many subsequent differences between formulae concerning addition and formulae concerning multiplication.

$*10\cdot51$. $\vdash :. \sim\{(\exists x) . \phi x . \psi x\} . \equiv : \phi x . \supset_x . \sim \psi x$

This proposition is analogous to

$$\vdash : \sim(p . q) . \equiv . p \supset \sim q$$

which results from $*4\cdot63$ by transposition.

Of the remaining propositions of this number, some are employed fairly often, while others are lemmas which are used only once or twice, sometimes at a much later stage.

$*10\cdot01$. $(\exists x) . \phi x . = . \sim(x) . \sim\phi x$ Df

This definition is only to be used when we discard the method of $*9$ in favour of the alternative method already explained. In either case we have

$$\vdash : (\exists x) . \phi x . \equiv . \sim(x) . \sim\phi x.$$

$*10\cdot02$. $\phi x \supset_x \psi x . = . (x) . \phi x \supset \psi x$ Df

$*10\cdot03$. $\phi x \equiv_x \psi x . = . (x) . \phi x \equiv \psi x$ Df

$*10\cdot1$. $\vdash : (x) . \phi x . \supset . \phi y$ [$*9\cdot2$]

$*10\cdot11$. If ϕy is true whatever possible argument y may be, then $(x) . \phi x$ is true. [$*9\cdot13$]

This proposition is, in a sense, the converse of $*10\cdot1$. $*10\cdot1$ may be stated: "What is true of all is true of any," while $*10\cdot11$ may be stated: "What is true of any, however chosen, is true of all."

$*10\cdot12$. $\vdash :. (x) . p \lor \phi x . \supset : p . \lor . (x) . \phi x$ [$*9\cdot25$]

According to the definitions in $*9$, this proposition is a mere example of "$q \supset q$," since by definition the two sides of the implication are different symbols for the same proposition. According to the alternative method, on the contrary, $*10\cdot12$ is a substantial proposition.

$*10\cdot121$. If "ϕx" is significant, then if a is of the same type as x, "ϕa" is significant, and vice versa. [$*9\cdot14$]

It follows from this proposition that two arguments to the same function must be of the same type; for if x and a are arguments to $\phi\hat{x}$, "ϕx" and "ϕa" are significant, and therefore x and a are of the same type. Thus the above primitive proposition embodies the outcome of our discussion of the vicious-circle paradoxes in Chapter II of the Introduction.

$*10\cdot122$. If, for some a, there is a proposition ϕa, then there is a function $\phi\hat{x}$, and vice versa. [$*9\cdot15$]

$*10\cdot13$. If $\phi\hat{x}$ and $\psi\hat{x}$ take arguments of the same type, and we have "$\vdash . \phi x$" and "$\vdash . \psi x$," we shall have "$\vdash . \phi x . \psi x$."

Dem.

By repeated use of ∗9·61·62·63·131 (3), there is a function $\sim\phi\hat{x} \vee \sim\psi\hat{x}$. Hence by ∗2·11 and ∗3·01,

$\vdash : \sim\phi x \vee \sim\psi x \,.\vee.\, \phi x \,.\, \psi x$ (1)

$\vdash . (1) . *2\cdot32 . (*1\cdot01) . \supset \vdash :. \phi x . \supset : \psi x . \supset . \phi x . \psi x$ (2)

$\vdash . (2) . *9\cdot12 . \supset \vdash .$ Prop

∗10·14. $\vdash :. (x) . \phi x : (x) . \psi x : \supset . \phi y . \psi y$

This proposition is true whenever it is significant, but it is not always significant when its hypothesis is significant. For the thesis demands that ϕ and ψ should take arguments of the same type, while the hypothesis does not demand this. Hence, if it is to be applied when ϕ and ψ are given, or when ψ is given as a function of ϕ or vice versa, we must not argue from the hypothesis to the thesis unless, in the supposed case, ϕ and ψ take arguments of the same type.

Dem.

$\vdash . *10\cdot1 . \quad \supset \vdash : (x) . \phi x . \supset . \phi y$ (1)

$\vdash . *10\cdot1 . \quad \supset \vdash : (x) . \psi x . \supset . \psi y$ (2)

$\vdash . (1) . (2) . *10\cdot13 . \supset \vdash : (x) . \phi x . \supset . \phi y : (x) . \psi x . \supset . \psi y :$

$[*3\cdot47] \quad \supset \vdash :. (x) . \phi x : (x) . \psi x : \supset . \phi y . \psi y :. \supset \vdash .$ Prop

∗10·2. $\vdash :. (x) . p \vee \phi x . \equiv : p . \vee . (x) . \phi x$

Dem.

$\vdash . *10\cdot1 . *1\cdot6 . \supset \vdash :. p . \vee . (x) . \phi x : \supset . p \vee \phi y :.$

$[*10\cdot11] \quad \supset \vdash :. (y) :. p . \vee . (x) . \phi x : \supset . p \vee \phi y :.$

$[*10\cdot12] \quad \supset \vdash :. p . \vee . (x) . \phi x : \supset . (y) . p \vee \phi y$ (1)

$\vdash . *10\cdot12 . \quad \supset \vdash :. (y) . p \vee \phi y . \supset : p . \vee . (x) . \phi x$ (2)

$\vdash . (1) . (2) . \quad \supset \vdash .$ Prop

∗10·21. $\vdash :. (x) . p \supset \phi x . \equiv : p . \supset . (x) . \phi x \quad \left[*10\cdot2 \dfrac{\sim p}{p}\right]$

This proposition is much more used than ∗10·2.

∗10·22. $\vdash :. (x) . \phi x . \psi x . \equiv : (x) . \phi x : (x) . \psi x$

Dem.

$\vdash . *10\cdot1 . \quad \supset \vdash : (x) . \phi x . \psi x . \supset . \phi y . \psi y .$ (1)

$[*3\cdot26] \quad \supset . \phi y :$

$[*10\cdot11] \quad \supset \vdash :. (y) : (x) . \phi x . \psi x . \supset . \phi y :.$

$[*10\cdot21] \quad \supset \vdash :. (x) . \phi x . \psi x . \supset . (y) . \phi y$ (2)

$\vdash . (1) . *3\cdot27 . \quad \supset \vdash :. (x) . \phi x . \psi x . \supset . \psi z :.$

$[*10\cdot11] \quad \supset \vdash :. (z) : (x) . \phi x . \psi x . \supset . \psi z :.$

$[*10\cdot21] \quad \supset \vdash :. (x) . \phi x . \psi x . \supset . (z) . \psi z$ (3)

$\vdash . (2) . (3) . \text{Comp} . \supset \vdash :. (x) . \phi x . \psi x . \supset : (y) . \phi y : (z) . \psi z$ (4)

$\vdash . *10\cdot14\cdot11 . \quad \supset \vdash :. (y) :. (x) . \phi x : (x) . \psi x : \supset . \phi y . \psi y :.$

$[*10\cdot21] \quad \supset \vdash :. (x) . \phi x : (x) . \psi x : \supset . (y) . \phi y . \psi y$ (5)

$\vdash . (4) . (5) . \quad \supset \vdash .$ Prop

The above proposition is true whenever it is significant; but, as was pointed out in connexion with $*10{\cdot}14$, it is not always significant when "$(x) . \phi x : (x) . \psi x$" is significant.

$*10{\cdot}221$. If ϕx contains a constituent $\chi(x, y, z, \ldots)$ and ψx contains a constituent $\chi(x, u, v, \ldots)$, where χ is an elementary function and $y, z, \ldots u, v, \ldots$ are either constants or apparent variables, then $\phi\hat{x}$ and $\psi\hat{x}$ take arguments of the same type. This can be proved in each particular case, though not generally, provided that, in obtaining ϕ and ψ from χ, χ is only submitted to negations, disjunctions and generalizations. The process may be illustrated by an example. Suppose ϕx is $(y) . \chi(x, y) . \supset . \theta x$, and ψx is $fx . \supset . (y) . \chi(x, y)$. By the definitions of $*9$, ϕx is $(\exists y) . \sim\chi(x, y) \vee \theta x$, and ψx is $(y) . \sim fx \vee \chi(x, y)$. Hence since the primitive ideas $(x) . Fx$ and $(\exists x) . Fx$ only apply to functions, there are functions $\sim\chi(\hat{x}, \hat{y}) \vee \theta\hat{x}$, $\sim f\hat{x} \vee \chi(\hat{x}, \hat{y})$. Hence there is a proposition $\sim\chi(a, b) \vee \theta a$. Hence, since "$p \vee q$" and "$\sim p$" are only significant when p and q are propositions, there is a proposition $\chi(a, b)$. Similarly, for some u and v, there are propositions $\sim fu \vee \chi(u, v)$ and $\chi(u, v)$. Hence by $*9{\cdot}14$, u and a, v and b are respectively of the same type, and (again by $*9{\cdot}14$) there is a proposition $\sim fa \vee \chi(a, b)$. Hence ($*9{\cdot}15$) there are functions $\sim\chi(a, \hat{y}) \vee \theta a$, $\sim fa \vee \chi(a, \hat{y})$, and therefore there are propositions

$$(\exists y) . \sim\chi(a, y) \vee \theta a, \quad (y) . \sim fa \vee \chi(a, y),$$

i.e. there are propositions ϕa, ψa, which was to be proved. This process can be applied similarly in any other instance.

$*10{\cdot}23$. $\vdash :. (x) . \phi x \supset p . \equiv : (\exists x) . \phi x . \supset . p$

Dem.

$$\begin{array}{ll} \vdash . *4{\cdot}2 . (*9{\cdot}03) . \supset \vdash :. (x) . \sim\phi x \vee p . \equiv : (x) . \sim\phi x . \vee . p : & \\ [(*9{\cdot}02)] \qquad\qquad \equiv . (\exists x) . \phi x . \supset . p & (1) \\ \vdash . (1) . (*1{\cdot}01) . \supset \vdash . \text{Prop} & \end{array}$$

In the above proof, we employ the definitions of $*9$. In the alternative method, in which $(\exists x) . \phi x$ is defined in accordance with $*10{\cdot}01$, the proof proceeds as follows.

$*10{\cdot}23$. $\vdash :. (x) . \phi x \supset p . \equiv : (\exists x) . \phi x . \supset . p$

Dem.

$$\begin{array}{ll} \vdash . \text{Transp} . (*10{\cdot}01) . \supset \vdash :. (\exists x) . \phi x . \supset . p : \equiv : \sim p . \supset . (x) . \sim\phi x : & \\ [*10{\cdot}21] \qquad\qquad \equiv : (x) : \sim p . \supset . \sim\phi x : & (1) \\ [*10{\cdot}1] \qquad\qquad \supset : \sim p . \supset . \sim\phi x : & \\ [\text{Transp}] \qquad\qquad \supset : \phi x . \supset . p :. & \\ [*10{\cdot}11] \qquad \supset \vdash :. (x) :. (\exists x) . \phi x . \supset . p : \supset : \phi x . \supset . p :. & \end{array}$$

[*10·21] $\supset \vdash :. (\exists x) . \phi x . \supset . p : \supset : (x) : \phi x . \supset . p$ (2)

$\vdash$. *10·1 . $\supset \vdash :. (x) : \phi x . \supset . p : \supset : \phi x \supset p :$

[Transp] $\supset : \sim p . \supset . \sim \phi x :.$

[*10·11·21] $\supset \vdash :. (x) : \phi x . \supset . p : \supset : (x) : \sim p . \supset . \sim \phi x :$

[(1)] $\supset : (\exists x) . \phi x . \supset . p$ (3)

$\vdash$. (2) . (3) . $\supset \vdash$. Prop

Whenever we have an asserted proposition of the form $p \supset \phi x$, we can pass by *10·11·21 to an asserted proposition $p . \supset . (x) . \phi x$. This passage is constantly required, as in the last line but one of the above proof. It will be indicated merely by the reference "*10·11·21," and the two steps which it requires will not be separately put down.

***10·24.** $\vdash : \phi y . \supset . (\exists x) . \phi x$

This is *9·1. In the alternative method, the proof is as follows.

Dem.

$\vdash$. *10·1 . $\supset \vdash : (x) . \sim \phi x . \supset . \sim \phi y :$

[Transp] $\supset \vdash : \phi y . \supset . \sim (x) . \sim \phi x :$

[(*10·01)] $\supset \vdash$. Prop

***10·25.** $\vdash : (x) . \phi x . \supset . (\exists x) . \phi x$ [*10·1·24]

***10·251.** $\vdash : (x) . \sim \phi x . \supset . \sim \{(x) . \phi x\}$ [*10·25 . Transp]

***10·252.** $\vdash : \sim \{(\exists x) . \phi x\} . \equiv . (x) . \sim \phi x$ [*4·2 . (*9·02)]

***10·253.** $\vdash : \sim \{(x) . \phi x\} . \equiv . (\exists x) . \sim \phi x$ [*4·2 . (*9·01)]

In the alternative method, in which $(\exists x) . \phi x$ is defined as in *10·01, the proofs of *10·252·253 are as follows.

***10·252.** $\vdash : \sim \{(\exists x) . \phi x\} . \equiv . (x) . \sim \phi x$ [*4·13 . (*10·01)]

***10·253.** $\vdash : \sim \{(x) . \phi x\} . \equiv . (\exists x) . \sim \phi x$

Dem.

$\vdash$. *10·1 . $\supset \vdash : (x) . \phi x . \supset . \phi y .$

[*2·12] $\supset . \sim (\sim \phi y) :$

[*10·11·21] $\supset \vdash : (x) . \phi x . \supset . (y) . \sim (\sim \phi y) :$

[Transp] $\supset \vdash : \sim \{(y) . \sim (\sim \phi y)\} . \supset . \sim \{(x) . \phi x\} :$

[(*10·01)] $\supset \vdash : (\exists y) . \sim \phi y . \quad \supset . \sim \{(x) . \phi x\}$ (1)

$\vdash$. *10·1 . $\supset \vdash : (y) . \sim (\sim \phi y) . \quad \supset . \sim (\sim \phi x) .$

[*2·14] $\supset . \phi x :$

[*10·11·21] $\supset \vdash : (y) . \sim (\sim \phi y) . \quad \supset . (x) . \phi x :$

[Transp] $\supset \vdash : \sim \{(x) . \phi x\} . \quad \supset . \sim \{(y) . \sim (\sim \phi y)\} .$

[(*10·01)] $\supset . (\exists y) . \sim \phi y$ (2)

$\vdash$. (1) . (2) . $\supset \vdash$. Prop

***10·26.** $\vdash :. (z) . \phi z \supset \psi z : \phi x : \supset . \psi x$ [*10·1 . Imp]

This is one form of the syllogism in Barbara. *E.g.* put $\phi z . = . z$ is a man, $\psi z . = . z$ is mortal, $x =$ Socrates. Then the proposition becomes:

"If all men are mortal, and Socrates is a man, then Socrates is mortal."

Another form of the syllogism in Barbara is given in *10·3. The two forms, formerly wrongly identified, were first distinguished by Peano and Frege.

***10·27.** $\vdash :. (z) . \phi z \supset \psi z . \supset : (z) . \phi z . \supset . (z) . \psi z$

This is *9·21. In the alternative method, the proof is as follows.

Dem.

$\vdash . *10{\cdot}14 . \quad \supset \vdash :. (z) . \phi z \supset \psi z : (z) . \phi z : \supset . \phi y \supset \psi y . \phi y .$

[Ass] $\quad \supset . \psi y :.$

[*10·1] $\quad \supset \vdash :. (y) :. (z) . \phi z \supset \psi z : (z) . \phi z : \supset . \psi y :.$

[*10·21] $\quad \supset \vdash :. (z) . \phi z \supset \psi z : (z) . \phi z : \supset . (y) . \psi y \qquad (1)$

$\vdash . (1) . \mathrm{Exp} . \supset \vdash . \mathrm{Prop}$

***10·271.** $\vdash :. (z) . \phi z \equiv \psi z . \supset : (z) . \phi z . \equiv . (z) . \psi z$

Dem.

$\vdash . *10{\cdot}22 . \quad \supset \vdash :. \mathrm{Hp} . \supset : (z) . \phi z \supset \psi z :$

[*10·27] $\quad \supset : (z) . \phi z . \supset . (z) . \psi z \qquad (1)$

$\vdash . *10{\cdot}22 . \quad \supset \vdash :. \mathrm{Hp} . \supset : (z) . \psi z \supset \phi z :$

[*10·27] $\quad \supset : (z) . \psi z . \supset . (z) . \phi z \qquad (2)$

$\vdash . (1) . (2) . \mathrm{Comp} . \supset \vdash . \mathrm{Prop}$

***10·28.** $\vdash :. (x) . \phi x \supset \psi x . \supset : (\exists x) . \phi x . \supset . (\exists x) . \psi x$

This is *9·22. In the alternative method, the proof is as follows.

Dem.

$\vdash . *10{\cdot}1 . \quad \supset \vdash :. (x) . \phi x \supset \psi x . \supset . \phi y \supset \psi y .$

[Transp] $\quad \supset . \sim\psi y \supset \sim\phi y :.$

[*10·11·21] $\supset \vdash :. (x) . \phi x \supset \psi x . \supset : (y) . \sim\psi y \supset \sim\phi y :$

[*10·27] $\quad \supset : (y) . \sim\psi y . \supset . (y) . \sim\phi y :$

[Transp] $\quad \supset : (\exists y) . \phi y . \supset . (\exists y) . \psi y :. \supset \vdash . \mathrm{Prop}$

***10·281.** $\vdash :. (x) . \phi x \equiv \psi x . \supset : (\exists x) . \phi x . \equiv . (\exists x) . \psi x$ [*10·22·28 . Comp]

***10·29.** $\vdash :. (x) . \phi x \supset \psi x : (x) . \phi x \supset \chi x : \equiv : (x) : \phi x . \supset . \psi x . \chi x.$

Dem.

$\vdash . *10{\cdot}22 . \quad \supset \vdash :. (x) . \phi x \supset \psi x : (x) . \phi x \supset \chi x :$

$\quad \equiv : (x) : \phi x \supset \psi x . \phi x \supset \chi x \qquad (1)$

$\vdash . *4{\cdot}76 . \quad \supset \vdash :. \phi x \supset \psi x . \phi x \supset \chi x . \equiv : \phi x . \supset . \psi x . \chi x :.$

[*10·11] $\quad \supset \vdash :. (x) :. \phi x \supset \psi x . \phi x \supset \chi x . \equiv : \phi x . \supset . \psi x . \chi x :.$

[*10·271] $\quad \supset \vdash :. (x) : \phi x \supset \psi x . \phi x \supset \chi x : \equiv : (x) : \phi x . \supset . \psi x . \chi x \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

This is an extension of the principle of composition.

***10·3.** $\vdash :. (x) . \phi x \supset \psi x : (x) . \psi x \supset \chi x : \supset . (x) . \phi x \supset \chi x$

This is the second form of the syllogism in Barbara.

Dem.

$\vdash . *10 \cdot 22 \cdot 221 . \supset \vdash : \mathrm{Hp} . \supset . (x) . \phi x \supset \psi x . \psi x \supset \chi x .$

$[\mathrm{Syll}. *10 \cdot 27] \quad \supset . (x) . \phi x \supset \chi x : \supset \vdash . \mathrm{Prop}$

***10·301.** $\vdash :. (x) . \phi x \equiv \psi x : (x) . \psi x \equiv \chi x : \supset . (x) . \phi x \equiv \chi x$

Dem.

$\vdash . *10 \cdot 22 \cdot 221 . \supset \vdash :. \mathrm{Hp} . \supset : (x) . \phi x \equiv \psi x . \psi x = \chi x :$

$[*4 \cdot 22 . *10 \cdot 27] \quad \supset : (x) . \phi x \equiv \chi x :. \supset \vdash . \mathrm{Prop}$

In the second line of the proofs of *10·3 and *10·301, we abbreviate the process of proof in a way which is often convenient. In *10·3, the full process would be as follows:

$\vdash . \mathrm{Syll} . \supset \vdash : \phi x \supset \psi x . \psi x \supset \chi x . \supset . \phi x \supset \chi x :$

$[*10 \cdot 11] \supset \vdash : (x) : \phi x \supset \psi x . \psi x \supset \chi x . \supset . \phi x \supset \chi x :$

$[*10 \cdot 27] \supset \vdash : (x) . \phi x \supset \psi x . \psi x \supset \chi x . \supset . (x) . \phi x \supset \chi x$

The above two propositions show that formal implication and formal equivalence are transitive relations between functions.

***10·31.** $\vdash :. (x) . \phi x \supset \psi x . \supset : (x) : \phi x . \chi x . \supset . \psi x . \chi x$

Dem.

$\vdash . \mathrm{Fact} . *10 \cdot 11 . \supset \vdash :. (x) :. \phi x \supset \psi x . \supset : \phi x . \chi x . \supset . \psi x . \chi x \quad (1)$

$\vdash . (1) . *10 \cdot 27 . \quad \supset \vdash . \mathrm{Prop}$

***10·311.** $\vdash :. (x) . \phi x \equiv \psi x . \supset : (x) : \phi x . \chi x . \equiv . \psi x . \chi x$

Dem.

$\vdash . *4 \cdot 36 . *10 \cdot 11 . \supset \vdash :. (x) :. \phi x \equiv \psi x . \supset : \phi x . \chi x . \equiv . \psi x . \chi x \quad (1)$

$\vdash . (1) . *10 \cdot 27 . \quad \supset \vdash . \mathrm{Prop}$

The above two propositions are extensions of the principle of the factor.

***10·32.** $\vdash : \phi x \equiv_x \psi x . \equiv . \psi x \equiv_x \phi x$

Dem.

$\vdash . *10 \cdot 22 . \supset \vdash : \phi x \equiv_x \psi x . \equiv . \phi x \supset_x \psi x . \psi x \supset_x \phi x .$

$[*4 \cdot 3] \quad \equiv . \psi x \supset_x \phi x . \phi x \supset_x \psi x .$

$[*10 \cdot 22] \quad \equiv . \psi x \equiv_x \phi x : \supset \vdash . \mathrm{Prop}$

This proposition shows that formal equivalence is symmetrical.

***10·321.** $\vdash : \phi x \equiv_x \psi x . \phi x \equiv_x \chi x . \supset . \psi x \equiv_x \chi x$

Dem.

$\vdash . *10 \cdot 32 . \mathrm{Fact} . \supset \vdash : \mathrm{Hp} . \supset . \psi x \equiv_x \phi x . \phi x \equiv_x \chi x .$

$[*10 \cdot 301] \quad \supset . \psi x \equiv_x \chi x : \supset \vdash . \mathrm{Prop}$

***10·322.** $\vdash : \psi x \equiv_x \phi x . \chi x \equiv_x \phi x . \supset . \psi x \equiv_x \chi x$

Dem.

$\vdash . *10 \cdot 32 . \supset \vdash : \mathrm{Hp} . \supset . \psi x \equiv_x \phi x . \phi x \equiv_x \chi x .$

$[*10 \cdot 301] \quad \supset . \psi x \equiv_x \chi x : \supset \vdash . \mathrm{Prop}$

***10·33.** $\vdash :. (x) : \phi x . p : \equiv : (x) . \phi x : p$

Dem.

$\vdash . *10\cdot1 . \quad \supset \vdash :. (x) : \phi x . p : \supset . \phi y . p .$ (1)

$[*3\cdot27] \quad \supset . p$ (2)

$\vdash . (1) . *3\cdot26 . \supset \vdash :. (x) : \phi x . p : \supset . \phi y :$

$[*10\cdot11\cdot21] \quad \supset \vdash :. (x) : \phi x . p : \supset . (y) . \phi y$ (3)

$\vdash . (2) . (3) . \quad \supset \vdash :. (x) : \phi x . p : \supset : (y) . \phi y : p$ (4)

$\vdash . *10\cdot1 . \quad \supset \vdash :. (y) . \phi y . \quad \supset . \phi x :.$

$[\text{Fact}] \quad \supset \vdash :. (y) . \phi y : p : \supset . \phi x . p :.$

$[*10\cdot11\cdot21] \quad \supset \vdash :. (y) . \phi y : p : \supset : (x) : \phi x . p$ (5)

$\vdash . (4) . (5) . \quad \supset \vdash . \text{Prop}$

***10·34.** $\vdash :. (\exists x) . \phi x \supset p . \equiv : (x) . \phi x . \supset . p$

This follows immediately from *9·05·01 and *1·01. In the alternative method, the proof is as follows.

Dem.

$\vdash . *4\cdot2 . (*10\cdot01) . \supset$

$\vdash :. (\exists x) . \phi x \supset p . \quad \equiv : \sim\{(x) . \sim(\phi x \supset p)\} :$

$[*4\cdot61 . *10\cdot271] \quad \equiv : \sim\{(x) : \phi x . \sim p\} :$

$[*10\cdot33] \quad \equiv : \sim\{(x) . \phi x : \sim p\} :$

$[*4\cdot53] \quad \equiv : \sim\{(x) . \phi x\} . \vee . p :$

$[*4\cdot6] \quad \equiv : (x) . \phi x . \supset . p$

***10·35.** $\vdash :. (\exists x) . p . \phi x . \equiv : p : (\exists x) . \phi x$

Dem.

$\vdash . *3\cdot26 . \quad \supset \vdash : p . \phi x . \supset . p :$

$[*10\cdot11] \quad \supset \vdash : (x) : p . \phi x . \supset . p :$

$[*10\cdot23] \quad \supset \vdash : (\exists x) . p . \phi x . \supset . p$ (1)

$\vdash . *3\cdot27 . \quad \supset \vdash : p . \phi x . \supset . \phi x :$

$[*10\cdot11] \quad \supset \vdash : (x) : p . \phi x . \supset . \phi x :$

$[*10\cdot28] \quad \supset \vdash : (\exists x) . p . \phi x . \supset . (\exists x) . \phi x$ (2)

$\vdash . *3\cdot2 . \quad \supset \vdash :. p . \supset : \phi x . \supset . p . \phi x .$

$[*10\cdot11\cdot21] \supset \vdash :. p . \supset : (x) : \phi x . \supset . p . \phi x :$

$[*10\cdot28] \quad \supset : (\exists x) . \phi x . \supset . (\exists x) . p . \phi x$ (3)

$\vdash . (1) . (2) . (3) . \text{Imp} . \supset \vdash . \text{Prop}$

***10·36.** $\vdash :. (\exists x) . \phi x \vee p . \equiv : (\exists x) . \phi x . \vee . p$

This follows immediately from *9·05. In the alternative method, the proof is as follows.

Dem.

$\vdash . *4\cdot64 . \quad \supset \vdash : \phi x \vee p . \equiv . \sim\phi x \supset p :$

$[*10\cdot11] \quad \supset \vdash : (x) : \phi x \vee p . \equiv . \sim\phi x \supset p :$

$[*10\cdot281] \quad \supset \vdash :. (\exists x) . \phi x \vee p . \equiv : (\exists x) . \sim\phi x \supset p :$

$[*10\cdot34] \quad \equiv : (x) . \sim\phi x . \supset . p :$

$[*4\cdot6 . (*10\cdot01)] \quad \equiv : (\exists x) . \phi x . \vee . p :. \supset \vdash . \text{Prop}$

The above proposition is only required in order to lead to the following:

$*10{\cdot}37$. $\vdash :. (\exists x) . p \supset \phi x . \equiv : p . \supset . (\exists x) . \phi x \quad \left[*10{\cdot}36 \frac{\sim p}{p}\right]$

$*10{\cdot}39$. $\vdash :. \phi x \supset_x \chi x : \psi x \supset_x \theta x : \supset : \phi x . \psi x . \supset_x . \chi x . \theta x$

Dem.

$\vdash . *10{\cdot}22 . \supset \vdash :. \mathrm{Hp} . \supset : (x) : \phi x \supset \chi x . \psi x \supset \theta x :$

$[*3{\cdot}47 . *10{\cdot}27] \quad \supset : (x) : \phi x . \psi x . \supset . \chi x . \theta x :. \supset \vdash . \mathrm{Prop}$

This proposition is only true when the conclusion is significant; the significance of the hypothesis does not insure that of the conclusion. On the conditions of significance, see the remarks on $*10{\cdot}4$, below.

$*10{\cdot}4$. $\vdash :. \phi x \equiv_x \chi x . \psi x \equiv_x \theta x . \supset : \phi x . \psi x . \equiv_x . \chi x . \theta x$

Dem.

$\vdash . *10{\cdot}22 . \quad \supset \vdash :. \mathrm{Hp} . \supset : \phi x \supset_x \chi x . \psi x \supset_x \theta x :$

$[*10{\cdot}39] \quad \supset : \phi x . \psi x . \supset_x . \chi x . \theta x \quad (1)$

Similarly $\quad \vdash :. \mathrm{Hp} . \supset : \chi x . \theta x . \supset_x . \phi x . \psi x \quad (2)$

$\vdash . (1) . (2) . \mathrm{Comp} . \supset \vdash :. \mathrm{Hp} . \supset : \phi x . \psi x . \supset_x . \chi x . \theta x : \chi x . \theta x . \supset_x . \phi x . \psi x :$

$[*10{\cdot}22] \quad \supset : \phi x . \psi x . \equiv_x . \chi x . \theta x :. \supset \vdash . \mathrm{Prop}$

In $*10{\cdot}4$ and many later propositions, as in $*10{\cdot}39$, the conclusion may be not significant when the hypothesis is true. Hence, in order that it may be legitimate to use $*10{\cdot}4$ in *inference, i.e.* to pass from the *assertion* of the hypothesis to the *assertion* of the conclusion, the functions ϕ, ψ, χ, θ must be such as to have overlapping ranges of significance. In virtue of $*10{\cdot}221$, this is secured if they are of the forms $F\{x, \chi(x, \hat{y}, \hat{z}, \ldots)\}$, $f\{x, \chi(x, \hat{y}, \hat{z}, \ldots)\}$, $G\{x, \chi(x, \hat{y}, \hat{z}, \ldots)\}$, $g\{x, \chi(x, \hat{y}, \hat{z}, \ldots)\}$. It is also secured if ϕ and ψ or ϕ and θ or χ and ψ or χ and θ are of such forms, for ϕ and χ must have overlapping ranges of significance if the hypothesis is to be significant, and so must ψ and θ.

$*10{\cdot}41$. $\vdash :. (x) . \phi x . \lor . (x) . \psi x : \supset . (x) . \phi x \lor \psi x$

Dem.

$\vdash . *10{\cdot}1 . \quad \supset \vdash : (x) . \phi x . \supset . \phi y .$

$[*2{\cdot}2] \quad \supset . \phi y \lor \psi y \quad (1)$

$\vdash . *10{\cdot}1 . \quad \supset \vdash : (x) . \psi x . \supset . \psi y .$

$[*1{\cdot}3] \quad \supset . \phi y \lor \psi y \quad (2)$

$\vdash . (1) . (2) . *10{\cdot}13 . \supset \vdash :. (x) . \phi x . \supset . \phi y \lor \psi y : (x) . \psi x . \supset . \phi y \lor \psi y :.$

$[*3{\cdot}44] \quad \supset \vdash :. (x) . \phi x . \lor . (x) . \psi x : \supset . \phi y \lor \psi y$

$[*10{\cdot}11{\cdot}21] \quad \supset \vdash :. (x) . \phi x . \lor . (x) . \psi x : \supset . (y) . \phi y \lor \psi y :. \supset \vdash . \mathrm{Prop}$

Observe that in the above proof the uses of $*2{\cdot}2$ and $*1{\cdot}3$ are only legitimate if ϕy and ψy have overlapping ranges of significance, for otherwise, if y is such that there is a proposition ϕy, it is such that there is no proposition ψy, and conversely.

***10·411.** $\vdash :. \phi x \equiv_x \chi x . \psi x \equiv_x \theta x . \supset : \phi x \vee \psi x . \equiv_x . \chi x \vee \theta x$

Dem.

$\vdash . *10·14 . \supset \vdash :. \mathrm{Hp} . \supset : \phi x \equiv \chi x . \psi x \equiv \theta x :$

[*4·39] $\supset : \phi x \vee \psi x . \equiv . \chi x \vee \theta x$ (1)

$\vdash . (1) . *10·11·21 . \supset \vdash . \mathrm{Prop}$

***10·412.** $\vdash : \phi x \equiv_x \psi x . \equiv . \sim\phi x \equiv_x \sim\psi x$ [*4·11 . *10·11·271]

***10·413.** $\vdash :. \phi x \equiv_x \chi x . \psi x \equiv_x \theta x . \supset : \phi x \supset \psi x . \equiv_x . \chi x \supset \theta x$

Dem.

$\vdash . *10·411·412 . \supset \vdash :. \mathrm{Hp} . \supset : \sim\phi x \vee \psi x . \equiv_x . \sim\chi x \vee \theta x$

[(*1·01)] $\supset : \phi x \supset \psi x . \equiv_x . \chi x \supset \theta x :. \supset \vdash . \mathrm{Prop}$

***10·414.** $\vdash :. \phi x \equiv_x \chi x . \psi x \equiv_x \theta x . \supset : \phi x \equiv \psi x . \equiv_x . \chi x \equiv \theta x$

Dem.

$\vdash . *10·413 \dfrac{\psi, \phi, \theta, \chi}{\phi, \psi, \chi, \theta} . *10·32 . \supset \vdash :. \mathrm{Hp} . \supset : \psi x \supset \phi x . \equiv_x . \theta x \supset \chi x$ (1)

$\vdash . *10·413 . (1) . *10·4 . \supset \vdash . \mathrm{Prop}$

The propositions *10·413·414 are chiefly used in cases where either χ is replaced by ϕ or θ is replaced by ψ, in which case half the hypothesis becomes superfluous, being true by *4·2.

***10·42.** $\vdash :. (\exists x) . \phi x . \vee . (\exists x) . \psi x : \equiv . (\exists x) . \phi x \vee \psi x$

Dem.

$\vdash . *10·22 . \supset \vdash :. (x) . \sim\phi x : (x) . \sim\psi x : \equiv . (x) . \sim\phi x . \sim\psi x :.$

[*4·11] $\supset \vdash :. \sim\{(x) . \sim\phi x : (x) . \sim\psi x\} . \equiv . \sim\{(x) . \sim\phi x . \sim\psi x\} :.$

[*4·51·56.*10·271] $\supset \vdash :. \sim\{(x) . \sim\phi x\} . \vee . \sim\{(x) . \sim\psi x\} :$

$\equiv . \sim\{(x) . \sim(\phi x \vee \psi x)\} :.$

[*10·253] $\supset \vdash :. (\exists x) . \phi x . \vee . (\exists x) . \psi x : \equiv . (\exists x) . \phi x \vee \psi x :.$

$\supset \vdash . \mathrm{Prop}$

This proposition is very frequently used. It should be contrasted with *10·5, in which we have only an implication, not an equivalence.

***10·43.** $\vdash : \phi z \equiv_z \psi z . \phi x . \equiv . \phi z \equiv_z \psi z . \psi x$

Dem.

$\vdash . *10·1 . \supset \vdash : \phi z \equiv_z \psi z . \supset . \phi x \equiv \psi x$ (1)

$\vdash . (1) . *5·32 . \supset \vdash . \mathrm{Prop}$

***10·5.** $\vdash :. (\exists x) . \phi x . \psi x . \supset : (\exists x) . \phi x : (\exists x) . \psi x$

Dem.

$\vdash . *3·26 . *10·11 . \supset \vdash : (x) : \phi x . \psi x . \supset . \phi x :$

[*10·28] $\supset \vdash : (\exists x) . \phi x . \psi x . \supset . (\exists x) . \phi x$ (1)

$\vdash . *3·27 . *10·11 . \supset \vdash :. (x) : \phi x . \psi x . \supset . \psi x :$

[*10·28] $\supset \vdash : (\exists x) . \phi x . \psi x . \supset . (\exists x) . \psi x$ (2)

$\vdash . (1) . (2) . \mathrm{Comp} . \supset \vdash :. \mathrm{Prop}$

The converse of the above proposition is false. The fact that this proposition states an implication, while $*10{\cdot}42$ states an equivalence, is the source of many subsequent differences between formulae concerning logical addition and formulae concerning logical multiplication.

$*10{\cdot}51$. $\vdash :. \sim\{(\exists x) . \phi x . \psi x\} . \equiv : \phi x . \supset_x . \sim\psi x$

Dem.

$\vdash . *10{\cdot}252 . \supset \vdash :. \sim\{(\exists x) . \phi x . \psi x\} . \equiv : (x) . \sim(\phi x . \psi x) :$

$[*4{\cdot}51{\cdot}62 . *10{\cdot}271] \qquad \equiv : (x) : \phi x . \supset . \sim\psi x :. \supset \vdash . \text{Prop}$

$*10{\cdot}52$. $\vdash :. (\exists x) . \phi x . \supset : (x) . \phi x \supset p . \equiv . p$

Dem.

$\vdash . *5{\cdot}5 . \supset \vdash :: \text{Hp} . \supset :. p . \equiv : (\exists x) . \phi x . \supset . p :$

$[*10{\cdot}23] \qquad \equiv : (x) . \phi x \supset p :: \supset \vdash . \text{Prop}$

$*10{\cdot}53$. $\vdash :. \sim(\exists x) . \phi x . \supset : \phi x . \supset_x . \psi x$

Dem.

$\vdash . *2{\cdot}21 . *10{\cdot}11 . \supset$

$\vdash :. (x) :. \sim\phi x . \supset : \phi x . \supset . \psi x :.$

$[*10{\cdot}27] \supset \vdash :. (x) . \sim\phi x . \supset : (x) : \phi x . \supset . \psi x :.$

$[*10{\cdot}252] \supset \vdash :. \sim(\exists x) . \phi x . \supset : (x) : \phi x . \supset . \psi x :. \supset \vdash . \text{Prop}$

$*10{\cdot}541$. $\vdash :. \phi y . \supset_y . p \vee \psi y : \equiv : p . \vee . \phi y \supset_y \psi y$

Dem.

$\vdash . *4{\cdot}2 . (*1{\cdot}01) . \supset \vdash :. \phi y . \supset_y . p \vee \psi y : \equiv : (y) . \sim\phi y \vee p \vee \psi y :$

$[\text{Assoc}. *10{\cdot}271] \qquad \equiv : (y) . p \vee \sim\phi y \vee \psi y :$

$[*10{\cdot}2] \qquad \equiv : p . \vee . (y) . \sim\phi y \vee \psi y :$

$[(*1{\cdot}01)] \qquad \equiv : p . \vee . \phi y \supset_y \psi y :. \supset \vdash . \text{Prop}$

The above proposition is only needed in order to lead to the following:

$10{\cdot}542$. $\vdash :. \phi y . \supset_y . p \supset \psi y : \equiv : p . \supset . \phi y \supset_y \psi y \qquad \left[*10{\cdot}541 \dfrac{\sim p}{p}\right]$

This proposition is a lemma for $*84{\cdot}43$.

$*10{\cdot}55$. $\vdash :. (\exists x) . \phi x . \psi x : \phi x \supset_x \psi x : \equiv : (\exists x) . \phi x : \phi x \supset_x \psi x$

Dem.

$\vdash . *4{\cdot}71 . \supset \vdash :. \phi x \supset \psi x . \supset : \phi x . \psi x . \equiv . \phi x \qquad (1)$

$\vdash . (1) . *10{\cdot}11{\cdot}27 . \supset$

$\vdash :. \phi x \supset_x \psi x . \supset : (x) : \phi x . \psi x . \equiv . \phi x :$

$[*10{\cdot}281] \qquad \supset : (\exists x) . \phi x . \psi x . \equiv . (\exists x) . \phi x \qquad (2)$

$\vdash . (2) . *5{\cdot}32 . \supset \vdash . \text{Prop}$

This proposition is a lemma for $*117{\cdot}12{\cdot}121$.

$*10 \cdot 56$. $\vdash :. \phi x . \supset_x . \psi x : (\exists x) . \phi x . \chi x : \supset . (\exists x) . \psi x . \chi x$

Dem.

$\vdash . *10 \cdot 31 . \quad \supset \vdash :. \phi x . \supset_x . \psi x : \supset : \phi x . \chi x . \supset_x . \psi x . \chi x :$

$[*10 \cdot 28] \qquad \supset : (\exists x) . \phi x . \chi x . \supset . (\exists x) . \psi x . \chi x \quad (1)$

$\vdash . (1) . \mathrm{Imp} . \supset \vdash . \mathrm{Prop}$

This proposition and $*10 \cdot 57$ are used in the theory of series (Part V).

$*10 \cdot 57$. $\vdash :. \phi x . \supset_x . \psi x \vee \chi x : \supset : \phi x \supset_x \psi x . \vee . (\exists x) . \phi x . \chi x$

Dem.

$\vdash . *10 \cdot 51 . \mathrm{Fact} . \supset$

$\vdash :. \phi x . \supset_x . \psi x \vee \chi x : \sim(\exists x) . \phi x . \chi x : \supset : \phi x . \supset_x . \psi x \vee \chi x : \phi x . \supset_x . \sim \chi x :$

$[*10 \cdot 29] \qquad \supset : \phi x . \supset_x . \psi x \vee \chi x . \sim \chi x :$

$[*5 \cdot 61] \qquad \supset : \phi x . \supset_x . \psi x \quad (1)$

$\vdash . (1) . *5 \cdot 6 . \supset \vdash . \mathrm{Prop}$

$*11$. THEORY OF TWO APPARENT VARIABLES

Summary of $*11$.

In this number, the propositions proved for one variable in $*10$ are to be extended to two variables, with the addition of a few propositions having no analogues for one variable, such as $*11\cdot2\cdot21\cdot23\cdot24$ and $*11\cdot53\cdot55\cdot6\cdot7$. "$\phi(x, y)$" stands for a proposition containing x and containing y; when x and y are unassigned, $\phi(x, y)$ is a propositional function of x and y. The definition $*11\cdot01$ shows that "the truth of all values of $\phi(x, y)$" does not need to be taken as a new primitive idea, but is definable in terms of "the truth of all values of ψx." The reason is that, when x is assigned, $\phi(x, y)$ becomes a function of one variable, namely y, whence it follows that, for every possible value of x, "$(y) . \phi(x, y)$" embodies merely the primitive idea introduced in $*9$. But "$(y) . \phi(x, y)$" is again only a function of one variable, namely x, since y has here become an apparent variable. Hence the definition $*11\cdot01$ below is legitimate. We put:

$*11\cdot01$. $(x, y) . \phi(x, y) . = : (x) : (y) . \phi(x, y)$ Df

$*11\cdot02$. $(x, y, z) . \phi(x, y, z) . = : (x) : (y, z) . \phi(x, y, z)$ Df

$*11\cdot03$. $(\exists x, y) . \phi(x, y) . = : (\exists x) : (\exists y) . \phi(x, y)$ Df

$*11\cdot04$. $(\exists x, y, z) . \phi(x, y, z) . = : (\exists x) : (\exists y, z) . \phi(x, y, z)$ Df

$*11\cdot05$. $\phi(x, y) . \supset_{x,y} . \psi(x, y) : = : (x, y) : \phi(x, y) . \supset . \psi(x, y)$ Df

$*11\cdot06$. $\phi(x, y) . \equiv_{x,y} . \psi(x, y) : = : (x, y) : \phi(x, y) . \equiv . \psi(x, y)$ Df

All the above definitions are supposed extended to any number of variables that may occur.

The propositions of this section can all be extended to any finite number of variables; as the analogy is exact, it is not necessary to carry the process beyond two variables in our proofs.

In addition to the definition $*11\cdot01$, we need the primitive proposition that "whatever possible argument x may be, $\phi(x, y)$ is true whatever possible argument y may be" implies the corresponding statement with x and y interchanged except in "$\phi(x, y)$". Either may be taken as the meaning of "$\phi(x, y)$ is true whatever possible arguments x and y may be."

The propositions of the present number are somewhat less used than those of $*10$, but some of them are used frequently. Such are the following:

$*11\cdot1$. $\vdash : (x, y) . \phi(x, y) . \supset . \phi(z, w)$

$*11\cdot11$. If $\phi(z, w)$ is true whatever possible arguments z and w may be, then $(x, y) . \phi(x, y)$ is true

These two propositions are the analogues of $*10\cdot1\cdot11$.

∗11·2. $\vdash : (x, y) . \phi(x, y) . \equiv . (y, x) . \phi(x, y)$

I.e. to say that "for all possible values of x, $\phi(x, y)$ is true for all possible values of y" is equivalent to saying "for all possible values of y, $\phi(x, y)$ is true for all possible values of x."

∗11·3. $\vdash :. p . \supset . (x, y) . \phi(x, y) : \equiv : (x, y) : p . \supset . \phi(x, y)$

This is the analogue of ∗10·21.

∗11·32. $\vdash :. (x, y) : \phi(x, y) . \supset . \psi(x, y) : \supset : (x, y) . \phi(x, y) . \supset . (x, y) . \psi(x, y)$

I.e. "if $\phi(x, y)$ always implies $\psi(x, y)$, then '$\phi(x, y)$ always' implies '$\psi(x, y)$ always.'" This is the analogue of ∗10·27. ∗11·33·34·341 are respectively the analogues of ∗10·271·28·281, and are also much used.

∗11·35. $\vdash :. (x, y) : \phi(x, y) . \supset . p : \equiv : (\exists x, y) . \phi(x, y) . \supset . p$

I.e. if $\phi(x, y)$ always implies p, then if $\phi(x, y)$ is ever true, p is true, and vice versa. This is the analogue of ∗10·23.

∗11·45. $\vdash :. (\exists x, y) : p . \phi(x, y) : \equiv : p : (\exists x, y) . \phi(x, y)$

This is the analogue of ∗10·35.

∗11·54. $\vdash :. (\exists x, y) . \phi x . \psi y . \equiv : (\exists x) . \phi x : (\exists y) . \psi y$

This proposition is useful because it analyses a proposition containing two apparent variables into two propositions which each contain only one. "$\phi x . \psi y$" is a function of two variables, but is compounded of two functions of one variable each. Such a function is like a conic which is two straight lines: it may be called an "analysable" function.

∗11·55. $\vdash :. (\exists x, y) . \phi x . \psi(x, y) . \equiv : (\exists x) : \phi x : (\exists y) . \psi(x, y)$

I.e. to say "there are values of x and y for which $\phi x . \psi(x, y)$ is true" is equivalent to saying "there is a value of x for which ϕx is true and for which there is a value of y such that $\psi(x, y)$ is true."

∗11·6. $\vdash :: (\exists x) :. (\exists y) . \phi(x, y) . \psi y : \chi x :. \equiv :. (\exists y) :. (\exists x) . \phi(x, y) . \chi x : \psi y$

This gives a transformation which is useful in many proofs.

∗11·62. $\vdash :: \phi x . \psi(x, y) . \supset_{x,y} . \chi(x, y) : \equiv :. \phi x . \supset_x : \psi(x, y) . \supset_y . \chi(x, y)$

This transformation also is often useful.

∗11·01. $(x, y) . \phi(x, y) . = : (x) : (y) . \phi(x, y)$ Df

∗11·02. $(x, y, z) . \phi(x, y, z) . = : (x) : (y, z) . \phi(x, y, z)$ Df

∗11·03. $(\exists x, y) . \phi(x, y) . = : (\exists x) : (\exists y) . \phi(x, y)$ Df

∗11·04. $(\exists x, y, z) . \phi(x, y, z) . = : (\exists x) : (\exists y, z) . \phi(x, y, z)$ Df

∗11·05. $\phi(x, y) . \supset_{x,y} . \psi(x, y) : = : (x, y) : \phi(x, y) . \supset . \psi(x, y)$ Df

∗11·06. $\phi(x, y) . \equiv_{x,y} . \psi(x, y) : = : (x, y) : \phi(x, y) . \equiv . \psi(x, y)$ Df

with similar definitions for any number of variables.

∗11·07. "Whatever possible argument x may be, $\phi(x, y)$ is true whatever possible argument y may be" implies the corresponding statement with x and y interchanged except in "$\phi(x, y)$". Pp.

***11·1.** $\vdash : (x, y) . \phi(x, y) . \supset . \phi(z, w)$

Dem.

$$\begin{array}{ll} \vdash . *10{\cdot}1 . \supset \vdash : \mathrm{Hp} . \supset . (y) . \phi(z, y) . & \\ [*10{\cdot}1] \qquad\qquad \supset . \phi(z, w) : \supset \vdash . \mathrm{Prop} & \end{array}$$

***11·11.** If $\phi(z, w)$ is true whatever possible arguments z and w may be, then $(x, y) . \phi(x, y)$ is true.

Dem.

By *10·11, the hypothesis implies that $(y) . \phi(z, y)$ is true whatever possible argument z may be; and this, by *10·11, implies $(x, y) . \phi(x, y)$.

***11·12.** $\vdash :. (x, y) . p \vee \phi(x, y) . \supset : p . \vee . (x, y) . \phi(x, y)$

Dem.

$$\begin{array}{ll} \vdash . *10{\cdot}12 . \supset \vdash :. (y) . p \vee \phi(x, y) . & \supset : p . \vee . (y) . \phi(x, y) :. \\ [*10{\cdot}11{\cdot}27] \supset \vdash :. (x, y) . p \vee \phi(x, y) . \supset : (x) : p . \vee . (y) . \phi(x, y) : & \\ [*10{\cdot}12] & \supset : p . \vee . (x, y) . \phi(x, y) :. \supset \vdash . \mathrm{Prop} \end{array}$$

This proposition is only used for proving *11·2.

***11·13.** If $\phi(\hat{x}, \hat{y})$, $\psi(\hat{x}, \hat{y})$ take their first and second arguments respectively of the same type, and we have "$\vdash . \phi(x, y)$" and "$\vdash . \psi(x, y)$," we shall have "$\vdash . \phi(x, y) . \psi(x, y)$." [Proof as in *10·13]

***11·14.** $\vdash :. (x, y) . \phi(x, y) : (x, y) . \psi(x, y) : \supset : \phi(z, w) . \psi(z, w)$

Dem.

$$\begin{array}{ll} \vdash . *10{\cdot}14 . \supset \vdash :. \mathrm{Hp} . \supset : (y) . \phi(z, y) : (y) . \psi(z, y) & \\ [*10{\cdot}14] \qquad\qquad \supset : \phi(z, w) . \psi(z, w) :. \supset \vdash . \mathrm{Prop} & \end{array}$$

This proposition, like *10·14, is not always significant when its hypothesis is true. *11·13, on the contrary, is always significant when its hypothesis is true. For this reason, *11·13 may always be safely used in *inference*, whereas *11·14 can only be used in *inference* (*i.e.* for the actual assertion of the conclusion when the hypothesis is asserted) if it is known that the conclusion is significant.

***11·2.** $\vdash : (x, y) . \phi(x, y) . \equiv . (y, x) . \phi(x, y)$

Dem.

$$\begin{array}{lll} \vdash . *11{\cdot}1 . & \supset \vdash : (x, y) . \phi(x, y) . \supset . \phi(z, w) & (1) \\ \multicolumn{2}{l}{\vdash . (1) . *11{\cdot}07{\cdot}11 . \supset \vdash :. (w, z) : (x, y) . \phi(x, y) . \supset . \phi(z, w)} & (2) \\ \multicolumn{2}{l}{\vdash . (2) . *11{\cdot}12 \dfrac{\sim\{(x, y) . \phi(x, y)\}}{p} . \supset} & \\ & \vdash :. (x, y) . \phi(x, y) . \supset . (w, z) . \phi(z, w) & (3) \\ \text{Similarly} & \vdash :. (w, z) . \phi(z, w) . \supset . (x, y) . \phi(x, y) & (4) \\ \vdash . (3) . (4) . & \supset \vdash . \mathrm{Prop} & \end{array}$$

Note that "$(w, z) . \phi(z, w)$" is the same proposition as "$(y, x) . \phi(x, y)$"; a proposition is not a function of any apparent variable which occurs in it.

***11·21.** $\vdash : (x, y, z) . \phi(x, y, z) . \equiv . (y, z, x) . \phi(x, y, z)$

Dem.

$$
\begin{aligned}
&[(*11\cdot01\cdot02)] \vdash :: (x, y, z) . \phi(x, y, z) . \equiv :. (x) :. (y) : (z) . \phi(x, y, z) :. \\
&[*11\cdot2] \qquad \equiv :. (y) :. (x) : (z) . \phi(x, y, z) :. \\
&[*11\cdot2 . *10\cdot271] \qquad \equiv :. (y) :. (z) : (x) . \phi(x, y, z) :. \\
&[(*11\cdot01\cdot02)] \qquad \equiv :. (y, z, x) . \phi(x, y, z) :: \supset \vdash . \text{Prop}
\end{aligned}
$$

***11·22.** $\vdash : (\exists x, y) . \phi(x, y) . \equiv . \sim \{(x, y) . \sim \phi(x, y)\}$

Dem.

$$
\begin{aligned}
&\vdash . *10\cdot252 . \text{Transp} . (*11\cdot03) . \supset \\
&\vdash : (\exists x, y) . \phi(x, y) . \quad \equiv . \sim \{(x) : \sim (\exists y) . \phi(x, y)\} . \\
&[*10\cdot252\cdot271] \quad \equiv . \sim \{(x) : (y) . \sim \phi(x, y)\} . \\
&[(*11\cdot01)] \quad \equiv . \sim \{(x, y) . \sim \phi(x, y)\} : \supset \vdash . \text{Prop}
\end{aligned}
$$

***11·23.** $\vdash : (\exists x, y) . \phi(x, y) . \equiv . (\exists y, x) . \phi(x, y)$

Dem.

$$
\begin{aligned}
&\vdash . *11\cdot22 . \supset \vdash : (\exists x, y) . \phi(x, y) . \equiv . \sim \{(x, y) . \sim \phi(x, y)\} . \\
&[*11\cdot2 . \text{Transp}] \qquad \equiv . \sim \{(y, x) . \sim \phi(x, y)\} . \\
&[*11\cdot22] \qquad \equiv . (\exists y, x) . \phi(x, y) : \supset \vdash . \text{Prop}
\end{aligned}
$$

***11·24.** $\vdash : (\exists x, y, z) . \phi(x, y, z) . \equiv . (\exists y, z, x) . \phi(x, y, z)$

Dem.

$$
\begin{aligned}
&[(*11\cdot03\cdot04)] \vdash :: (\exists x, y, z) . \phi(x, y, z) . \equiv :. (\exists x) :. (\exists y) : (\exists z) . \phi(x, y, z) :. \\
&[*11\cdot23] \qquad \equiv :. (\exists y) :. (\exists x) : (\exists z) . \phi(x, y, z) :. \\
&[*11\cdot23 . *10\cdot281] \qquad \equiv :. (\exists y) :. (\exists z) : (\exists x) . \phi(x, y, z) :. \\
&[(*11\cdot03\cdot04)] \qquad \equiv :. (\exists y, z, x) . \phi(x, y, z) :: \supset \vdash . \text{Prop}
\end{aligned}
$$

***11·25.** $\vdash : \sim \{(\exists x, y) . \phi(x, y)\} . \equiv . (x, y) . \sim \phi(x, y)$ [*11·22 . Transp]

***11·26.** $\vdash :. (\exists x) : (y) . \phi(x, y) : \supset : (y) : (\exists x) . \phi(x, y)$

Dem.

$$
\begin{aligned}
&\vdash . *10\cdot1\cdot28 . \supset \vdash :. (\exists x) : (y) . \phi(x, y) : \supset : (\exists x) . \phi(x, y) \qquad (1) \\
&\vdash . (1) . *10\cdot11\cdot21 . \supset \vdash . \text{Prop}
\end{aligned}
$$

Note that the converse of this proposition is false. *E.g.* let $\phi(x, y)$ be the propositional function "if y is a proper fraction, then x is a proper fraction greater than y." Then for all values of y we have $(\exists x) . \phi(x, y)$, so that $(y) : (\exists x) . \phi(x, y)$ is satisfied. In fact "$(y) : (\exists x) . \phi(x, y)$" expresses the proposition: "If y is a proper fraction, then there is always a proper fraction greater than y." But "$(\exists x) : (y) . \phi(x, y)$" expresses the proposition: "There is a proper fraction which is greater than any proper fraction," which is false.

***11·27.** $\vdash :. (\exists x, y) : (\exists z) . \phi(x, y, z) : \equiv : (\exists x) : (\exists y, z) . \phi(x, y, z) :$
$\equiv : (\exists x, y, z) . \phi(x, y, z)$

Dem.

$\vdash . *4{\cdot}2 . (*11{\cdot}03) . \supset$

$\vdash :: (\exists x, y) : (\exists z) . \phi(x, y, z) : \equiv :. (\exists x) :. (\exists y) : (\exists z) . \phi(x, y, z)$ (1)

$\vdash . *4{\cdot}2 . (*11{\cdot}03) . \supset$

$\vdash :. (\exists y) : (\exists z) . \phi(x, y, z) : \equiv : (\exists y, z) . \phi(x, y, z)$ (2)

$\vdash . (2) . *10{\cdot}11{\cdot}281 . \supset$

$\vdash :: (\exists x) :. (\exists y) : (\exists z) . \phi(x, y, z) :. \equiv :. (\exists x) : (\exists y, z) . \phi(x, y, z)$ (3)

$\vdash . (1) . (3) . (*11{\cdot}04) . \supset \vdash .$ Prop

All the propositions of $*10$ have analogues which hold for two or more variables. The more important of these are proved in what follows.

$*11{\cdot}3$. $\vdash :. p . \supset . (x, y) . \phi(x, y) : \equiv : (x, y) : p . \supset . \phi(x, y)$

Dem.

$\vdash . *10{\cdot}21 . \supset \vdash :. p . \supset . (x, y) . \phi(x, y) : \equiv : (x) : p . \supset . (y) . \phi(x, y) :$

[$*10{\cdot}21{\cdot}271$] $\equiv : (x, y) : p . \supset . \phi(x, y) :. \supset \vdash .$ Prop

$*11{\cdot}31$. $\vdash :. (x, y) . \phi(x, y) : (x, y) . \psi(x, y) : \equiv : (x, y) : \phi(x, y) . \psi(x, y)$

Here the conditions of significance on the right-hand side require that ϕ and ψ should take arguments of the same types.

Dem.

$\vdash . *10{\cdot}22 . \supset \vdash :: (x, y) . \phi(x, y) : (x, y) . \psi(x, y) :$

$\equiv :. (x) :. (y) . \phi(x, y) : (y) . \psi(x, y) :.$

[$*10{\cdot}22{\cdot}271$] $\equiv :. (x, y) : \phi(x, y) . \psi(x, y) :: \supset \vdash .$ Prop

The proofs of most of the following propositions are conducted exactly as those of $*11{\cdot}3{\cdot}31$ are conducted: the analogous proposition in $*10$ is used twice, together with $*10{\cdot}27$ or $*10{\cdot}271$ or $*10{\cdot}28$ or $*10{\cdot}281$ as the case may be. When proofs conform to this pattern we shall merely give references to the propositions used.

$*11{\cdot}311$. If $\phi(\hat{x}, \hat{y})$, $\psi(\hat{x}, \hat{y})$ take arguments of the same type, and we have "$\vdash . \phi(x, y)$" and "$\vdash . \psi(x, y)$," we shall have "$\vdash . \phi(x, y) . \psi(x, y)$." [Proof as in $*10{\cdot}13$.]

$*11{\cdot}32$. $\vdash :. (x, y) : \phi(x, y) . \supset . \psi(x, y) : \supset : (x, y) . \phi(x, y) . \supset . (x, y) . \psi(x, y)$ [$*10{\cdot}27$]

$*11{\cdot}33$. $\vdash :. (x, y) : \phi(x, y) . \equiv . \psi(x, y) : \supset : (x, y) . \phi(x, y) . \equiv . (x, y) . \psi(x, y)$ [$*10{\cdot}271$]

$*11{\cdot}34$. $\vdash :. (x, y) : \phi(x, y) . \supset . \psi(x, y) : \supset :$
$(\exists x, y) . \phi(x, y) . \supset . (\exists x, y) . \psi(x, y)$ [$*10{\cdot}27{\cdot}28$]

$*11{\cdot}341$. $\vdash :. (x, y) : \phi(x, y) . \equiv . \psi(x, y) : \supset :$
$(\exists x, y) . \phi(x, y) . \equiv . (\exists x, y) . \psi(x, y)$ [$*10{\cdot}271{\cdot}281$]

$*11{\cdot}35$. $\vdash :. (x, y) : \phi(x, y) . \supset . p : \equiv : (\exists x, y) . \phi(x, y) . \supset . p$ [$*10{\cdot}23{\cdot}271$]

$*11{\cdot}36$. $\vdash : \phi(z, w) . \supset . (\exists x, y) . \phi(x, y)$

Dem.

$\vdash . *11{\cdot}1 . \supset \vdash : (x, y) . \sim\phi(x, y) . \supset . \sim\phi(z, w)$ (1)

$\vdash . (1) .$ Transp $. \supset \vdash .$ Prop

***11·37.** $\vdash :: (x,y) : \phi(x,y) . \supset . \psi(x,y) :. (x,y) : \psi(x,y) . \supset . \chi(x,y) :.$
$\supset : (x,y) : \phi(x,y) . \supset . \chi(x,y)$

Dem.

In the following demonstration, "Hp" means the hypothesis of the proposition to be proved. We shall employ this abbreviation, whenever convenient, in all cases where the proposition to be proved is a hypothetical, *i.e.* is of the form "$p \supset q$." Similarly "Hp (1)" will mean "the hypothesis of (1)," and so on.

$\vdash . *11·31 . \supset \vdash :: \text{Hp} . \supset :. (x,y) :. \phi(x,y) . \supset . \psi(x,y) : \psi(x,y) . \supset . \chi(x,y)$ (1)

$\vdash . \text{Syll} . *11·11 . \supset \vdash :. (x,y) :. \phi(x,y) . \supset . \psi(x,y) : \psi(x,y) . \supset . \chi(x,y) :$
$\supset : \phi(x,y) . \supset . \chi(x,y) :.$

[*11·32] $\supset \vdash :. (x,y) : \phi(x,y) . \supset . \psi(x,y) : \psi(x,y) . \supset . \chi(x,y) :$
$\supset : (x,y) : \phi(x,y) . \supset . \chi(x,y)$ (2)

$\vdash . (1) . (2) . \text{Syll} . \supset \vdash . \text{Prop}$

The above is a type of proof which recurs frequently in what follows. Proofs conforming to this pattern will be indicated only by the numbers of the propositions used.

***11·371.** $\vdash :: (x,y) : \phi(x,y) . \equiv . \psi(x,y) :. (x,y) : \psi(x,y) . \equiv . \chi(x,y) :.$
$\supset :. (x,y) : \phi(x,y) . \equiv . \chi(x,y)$ [*11·31·11·33]

***11·38.** $\vdash :: (x,y) : \phi(x,y) . \supset . \psi(x,y) :. \supset :.$
$(x,y) : \phi(x,y) . \chi(x,y) . \supset . \psi(x,y) . \chi(x,y)$ [Fact . *11·11·32]

***11·39.** $\vdash :: (x,y) : \phi(x,y) . \supset . \psi(x,y) :. (x,y) : \chi(x,y) . \supset . \theta(x,y) :. \supset :.$
$(x,y) : \phi(x,y) . \chi(x,y) . \supset . \psi(x,y) . \theta(x,y)$ [*3·47 . *11·11·32]

***11·391.** $\vdash :: (x,y) : \phi(x,y) . \supset . \psi(x,y) :. (x,y) : \phi(x,y) . \supset . \chi(x,y) :.$
$\equiv : (x,y) : \phi(x,y) . \supset . \psi(x,y) . \chi(x,y)$

Dem.

$\vdash . *4·76 . \quad \supset \vdash :. \phi(x,y) . \supset . \psi(x,y) : \phi(x,y) . \supset . \chi(x,y) :$
$\equiv : \phi(x,y) . \supset . \psi(x,y) . \chi(x,y) :.$

[*11·11·33] $\supset \vdash :. (x,y) : \phi(x,y) . \supset . \psi(x,y) : \phi(x,y) . \supset . \chi(x,y) :$
$\equiv : (x,y) : \phi(x,y) . \supset . \psi(x,y) . \chi(x,y) ::$

[*11·31] $\supset \vdash :: (x,y) : \phi(x,y) . \supset . \psi(x,y) :. (x,y) : \phi(x,y) . \supset . \chi(x,y) :.$
$\equiv : (x,y) : \phi(x,y) . \supset . \psi(x,y) . \chi(x,y) ::$
$\supset \vdash . \text{Prop}$

***11·4.** $\vdash :: (x,y) : \phi(x,y) . \equiv . \psi(x,y) :. (x,y) : \chi(x,y) . \equiv . \theta(x,y) :. \supset :.$
$(x,y) : \phi(x,y) . \chi(x,y) . \equiv . \psi(x,y) . \theta(x,y)$

Dem.

$\vdash . *11·31 . \supset \vdash :: \text{Hp} . \supset :. (x,y) :. \phi(x,y) . \equiv . \psi(x,y) : \chi(x,y) . \equiv . \theta(x,y) :.$

[*4·38.*11·11·32] $\supset :. (x,y) : \phi(x,y) . \chi(x,y) . \equiv . \psi(x,y) . \theta(x,y) ::$
$\supset \vdash . \text{Prop}$

***11·401.** $\vdash :: (x, y) : \phi(x, y) . \equiv . \psi(x, y) : \supset :.$
$(x, y) : \phi(x, y) . \chi(x, y) . \equiv . \psi(x, y) . \chi(x, y)$ $\left[*11\cdot4 \dfrac{\chi}{\theta} . \mathrm{Id}\right]$

***11·41.** $\vdash :. (\exists x, y) . \phi(x, y) : \mathsf{v} : (\exists x, y) . \psi(x, y) :$
$\equiv : (\exists x, y) : \phi(x, y) . \mathsf{v} . \psi(x, y)$ $[*10\cdot42\cdot281]$

***11·42.** $\vdash :. (\exists x, y) . \phi(x, y) . \psi(x, y) . \supset : (\exists x, y) . \phi(x, y) : (\exists x, y) . \psi(x, y)$
$[*10\cdot5]$

***11·421.** $\vdash :. (x, y) . \phi(x, y) . \mathsf{v} . (x, y) . \psi(x, y) : \supset : (x, y) : \phi(x, y) . \mathsf{v} . \psi(x, y)$
$\left[*11\cdot42 \dfrac{\sim\phi, \sim\psi}{\phi, \quad \psi} . \mathrm{Transp} . *4\cdot56\right]$

***11·43.** $\vdash :. (\exists x, y) : \phi(x, y) . \supset . p : \equiv : (x, y) . \phi(x, y) . \supset . p$ $[*10\cdot34\cdot281]$

***11·44.** $\vdash :. (x, y) : \phi(x, y) . \mathsf{v} . p : \equiv : (x, y) . \phi(x, y) . \mathsf{v} . p$ $[*10\cdot2\cdot271]$

***11·45.** $\vdash :. (\exists x, y) : p . \phi(x, y) : \equiv : p : (\exists x, y) . \phi(x, y)$ $[*10\cdot35\cdot281]$

***11·46.** $\vdash :. (\exists x, y) : p . \supset . \phi(x, y) : \equiv : p . \supset . (\exists x, y) . \phi(x, y)$ $[*10\cdot37\cdot281]$

***11·47.** $\vdash :. (x, y) : p . \phi(x, y) : \equiv : p : (x, y) . \phi(x, y)$ $[*10\cdot33\cdot271]$

***11·5.** $\vdash :. (\exists x) : \sim\{(y) . \phi(x, y)\} : \equiv : \sim\{(x, y) . \phi(x, y)\} : \equiv : (\exists x, y) . \sim\phi(x, y)$

Dem.

$\vdash . *10\cdot253 . \supset \vdash :. (\exists x) : \sim\{(y) . \phi(x, y)\} : \equiv : \sim\{(x) : (y) . \phi(x, y)\} :$
$[(*11\cdot01)]$ $\equiv : \sim\{(x, y) . \phi(x, y)\}$ (1)
$\vdash . *10\cdot253 . \supset \vdash : \sim\{(y) . \phi(x, y)\} .$ $\equiv . (\exists y) . \sim\phi(x, y) :$
$[*10\cdot11\cdot281] \supset \vdash :. (\exists x) : \sim\{(y) . \phi(x, y)\} : \equiv : (\exists x) : (\exists y) . \sim\phi(x, y) :$
$[(*11\cdot03)]$ $\equiv : (\exists x, y) . \sim\phi(x, y)$ (2)
$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***11·51.** $\vdash :. (\exists x) : (y) . \phi(x, y) : \equiv : \sim\{(x) : (\exists y) . \sim\phi(x, y)\}$

Dem.

$\vdash . *10\cdot252 . \mathrm{Transp} . \supset \vdash :. (\exists x) : (y) . \phi(x, y) : \equiv : \sim[(x) : \sim(y) . \phi(x, y)]$ (1)
$\vdash . *10\cdot253 . \supset \vdash :. \sim(y) . \phi(x, y) .$ $\equiv : (\exists y) . \sim\phi(x, y) :.$
$[*10\cdot11\cdot271] \supset \vdash :. (x) : \sim(y) . \phi(x, y) :$ $\equiv : (x) : (\exists y) . \sim\phi(x, y) :.$
$[\mathrm{Transp}] \quad \supset \vdash :. \sim[(x) : \sim\{(y) . \phi(x, y)\}] .$ $\equiv : \sim\{(x) : (\exists y) . \sim\phi(x, y)\}$ (2)
$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***11·52.** $\vdash :. (\exists x, y) . \phi(x, y) . \psi(x, y) . \equiv . \sim\{(x, y) : \phi(x, y) . \supset . \sim\psi(x, y)\}$

Dem.

$\vdash . *4\cdot51\cdot62 . \supset$
$\vdash :. \sim\{\phi(x, y) . \psi(x, y)\} .$ $\equiv : \phi(x, y) . \supset . \sim\psi(x, y)$ (1)
$\vdash . (1) . *11\cdot11\cdot33 . \supset$
$\vdash :. (x, y) . \sim\{\phi(x, y) . \psi(x, y)\} : \equiv : (x, y) : \phi(x, y) . \supset . \sim\psi(x, y)$ (2)
$\vdash : (2) . \mathrm{Transp} . *11\cdot22 . \supset \vdash . \mathrm{Prop}$

***11·521.** $\vdash :. \sim(\exists x, y) . \phi(x, y) . \sim\psi(x, y) . \equiv : (x, y) : \phi(x, y) . \supset . \psi(x, y)$
$\left[*11\cdot52 . \mathrm{Transp} . \dfrac{\sim\psi(x, y)}{\psi(x, y)}\right]$

***11·53.** $\vdash :. (x, y) . \phi x \supset \psi y . \equiv : (\exists x) . \phi x . \supset . (y) . \psi y$

Dem.

$\vdash . *10{\cdot}21{\cdot}271 . \supset \vdash :. (x, y) . \phi x \supset \psi y . \equiv : (x) : \phi x . \supset . (y) . \psi y :$

$[*10{\cdot}23] \qquad \equiv : (\exists x) . \phi x . \supset . (y) . \psi y :. \supset \vdash . \mathrm{Prop}$

***11·54.** $\vdash :. (\exists x, y) . \phi x . \psi y . \equiv : (\exists x) . \phi x : (\exists y) . \psi y$

Dem.

$\vdash . *10{\cdot}35 . \supset \vdash :. (\exists y) . \phi x . \psi y . \equiv : \phi x : (\exists y) . \psi y :.$

$[*10{\cdot}11{\cdot}281] \supset \vdash :. (\exists x, y) . \phi x . \psi y . \equiv : (\exists x) : \phi x : (\exists y) . \psi y :$

$[*10{\cdot}35] \qquad \equiv : (\exists x) . \phi x : (\exists y) . \psi y :. \supset \vdash . \mathrm{Prop}$

This proposition is very often used.

***11·55.** $\vdash :. (\exists x, y) . \phi x . \psi (x, y) . \equiv : (\exists x) : \phi x : (\exists y) . \psi (x, y)$

Dem.

$\vdash . *10{\cdot}35 . \supset \vdash :. (\exists y) . \phi x . \psi (x, y) . \equiv : \phi x : (\exists y) . \psi (x, y) :.$

$[*10{\cdot}11] \quad \supset \vdash :. (x) :. (\exists y) . \phi x . \psi (x, y) . \equiv : \phi x : (\exists y) . \psi (x, y) :.$

$[*10{\cdot}281] \quad \supset \vdash :. (\exists x) : (\exists y) . \phi x . \psi (x, y) . \equiv : (\exists x) : \phi x : (\exists y) . \psi (x, y) :. \supset \vdash . \mathrm{Prop}$

This proposition is very often used.

***11·56.** $\vdash :. (x) . \phi x : (y) . \psi y : \equiv : (x, y) . \phi x . \psi y$

Dem.

$\vdash . *10{\cdot}33 . \supset \vdash :: (x) . \phi x : (y) . \psi y : \equiv :. (x) :. \phi x : (y) . \psi y \qquad (1)$

$\vdash . *10{\cdot}33 . \supset \vdash :. \quad \phi x : (y) . \psi y : \equiv : (y) . \phi x . \psi y :.$

$[*10{\cdot}11] \quad \supset \vdash :. (x) :. \phi x : (y) . \psi y : \equiv : (y) . \phi x . \psi y :.$

$[*10{\cdot}271] \quad \supset \vdash :: (x) :. \phi x : (y) . \psi y :. \equiv : (x) : (y) . \phi x . \psi y :$

$[(*11{\cdot}01)] \qquad \equiv : (x, y) . \phi x . \psi y \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***11·57.** $\vdash : (x) . \phi x . \equiv . (x, y) . \phi x . \phi y \quad [*11{\cdot}56 . *4{\cdot}24]$

The use of *4·24 here depends upon the fact that $(x) . \phi x$ and $(y) . \phi y$ are the same proposition.

***11·58.** $\vdash : (\exists x) . \phi x . \equiv . (\exists x, y) . \phi x . \phi y \quad [*11{\cdot}54 . *4{\cdot}24]$

***11·59.** $\vdash :. \phi x . \supset_x . \psi x : \equiv : \phi x . \phi y . \supset_{x,y} . \psi x . \psi y$

Dem.

$\vdash . *11{\cdot}57 . \supset \vdash :. \phi x . \supset_x . \psi x : \equiv : (x, y) : \phi x . \supset . \psi x : \phi y . \supset . \psi y :$

$[*3{\cdot}47 . *11{\cdot}32] \qquad \supset : (x, y) : \phi x . \phi y . \supset . \psi x . \psi y \qquad (1)$

$\vdash . *11{\cdot}1 . \supset \vdash :. (x, y) : \phi x . \phi y . \supset . \psi x . \psi y : \supset : \phi x . \phi y . \supset . \psi x . \psi y \quad (2)$

$\vdash . (2) \frac{x}{y} . *4{\cdot}24 . \supset \vdash :. \mathrm{Hp} (2) . \supset : \phi x . \supset . \psi x \qquad (3)$

$\vdash . (3) . *10{\cdot}11{\cdot}21 . \supset$

$\qquad \vdash :. (x, y) : \phi x . \phi y . \supset . \psi x . \psi y : \supset : \phi x . \supset_x . \psi x \qquad (4)$

$\vdash . (1) . (4) . \supset \vdash . \mathrm{Prop}$

$*11{\cdot}6$. $\vdash :: (\exists x) :. (\exists y) . \phi(x, y) . \psi y : \chi x :. \equiv :. (\exists y) :. (\exists x) . \phi(x, y) . \chi x : \psi y$

This proposition is very frequently employed in subsequent proofs.

Dem.

$\vdash . *10{\cdot}35 . \supset \vdash :. (\exists y) . \phi(x, y) . \psi y : \chi x : \equiv : (\exists y) : \phi(x, y) . \psi y . \chi x :.$

$[*10{\cdot}11{\cdot}281] \supset \vdash :: (\exists x) :. (\exists y) . \phi(x, y) . \psi y : \chi x :$

$\equiv :. (\exists x) :. (\exists y) . \phi(x, y) . \psi y . \chi x :.$

$[*11{\cdot}23] \quad \equiv :. (\exists y) :. (\exists x) . \phi(x, y) . \psi y . \chi x :.$

$[*11{\cdot}341.\mathrm{Perm}] \quad \equiv :. (\exists y) :. (\exists x) . \phi(x, y) . \chi x . \psi y :.$

$[*10{\cdot}35{\cdot}281] \quad \equiv :. (\exists y) :. (\exists x) . \phi(x, y) . \chi x : \psi y :: \supset \vdash . \mathrm{Prop}$

$*11{\cdot}61$. $\vdash :. (\exists y) : \phi x . \supset_x . \psi(x, y) : \supset : \phi x . \supset_x . (\exists y) . \psi(x, y)$

Dem.

$\vdash . *11{\cdot}26 . \supset \vdash :: \mathrm{Hp} . \supset :. (x) :. (\exists y) : \phi x . \supset . \psi(x, y)$ (1)

$\vdash . *10{\cdot}37 . \supset \vdash :. (\exists y) : \phi x . \supset . \psi(x, y) : \supset : \phi x . \supset . (\exists y) . \psi(x, y) :.$

$[*10{\cdot}11{\cdot}27] \supset \vdash :: (x) :. (\exists y) : \phi x . \supset . \psi(x, y) :. \supset :. (x) : \phi x . \supset . (\exists y) . \psi(x, y)$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

$*11{\cdot}62$. $\vdash :: \phi x . \psi(x, y) . \supset_{x, y} . \chi(x, y) : \equiv :. \phi x . \supset_x : \psi(x, y) . \supset_y . \chi(x, y)$

Dem.

$\vdash . *4{\cdot}87 . *11{\cdot}11{\cdot}33 . \supset$

$\vdash :: \phi x . \psi(x, y) . \supset_{x, y} . \chi(x, y) : \equiv :. (x, y) :. \phi x . \supset : \psi(x, y) . \supset . \chi(x, y)$

$[*10{\cdot}21{\cdot}11{\cdot}271] \quad \equiv :. (x) :. \phi x . \supset : (y) : \psi(x, y) . \supset . \chi(x, y) ::$

$\supset \vdash . \mathrm{Prop}$

$*11{\cdot}63$. $\vdash :. \sim(\exists x, y) . \phi(x, y) . \supset : \phi(x, y) . \supset_{x, y} . \psi(x, y)$

Dem.

$\vdash . *2{\cdot}21 . *11{\cdot}11 . \supset \vdash :. (x, y) :. \sim\phi(x, y) . \supset : \phi(x, y) . \supset . \psi(x, y) :.$

$[*11{\cdot}32] \quad \supset \vdash :. (x, y) . \sim\phi(x, y) . \supset : (x, y) : \phi(x, y) . \supset . \psi(x, y) :.$

$[*11{\cdot}25] \quad \supset \vdash :. \sim(\exists x, y) . \phi(x, y) . \supset : (x, y) : \phi(x, y) . \supset . \psi(x, y) :.$

$\supset \vdash . \mathrm{Prop}$

$*11{\cdot}7$. $\vdash :. (\exists x, y) : \phi(x, y) . \vee . \phi(y, x) : \equiv . (\exists x, y) . \phi(x, y)$

Dem.

$\vdash . *11{\cdot}41 . \supset \vdash :. (\exists x, y) : \phi(x, y) . \vee . \phi(y, x) :$

$\equiv : (\exists x, y) . \phi(x, y) . \vee . (\exists x, y) . \phi(y, x) :$

$[*11{\cdot}23] \quad \equiv : (\exists x, y) . \phi(x, y) . \vee . (\exists y, x) . \phi(y, x) :$

$[*4{\cdot}25] \quad \equiv : (\exists x, y) . \phi(x, y) :. \supset \vdash . \mathrm{Prop}$

In the last line of the above proof, use is made of the fact that

$$(\exists x, y) . \phi(x, y) \text{ and } (\exists y, x) . \phi(y, x)$$

are the same proposition.

The first use of the following proposition occurs in the proof of $*234{\cdot}12$. Its utility lies in its enabling us to pass from a hypothesis

$$\phi z . \chi w . \supset_{z, w} . \psi z . \theta w,$$

containing two apparent variables, to the product of two hypotheses each containing only one.

$*11{\cdot}71$. $\vdash :: (\exists z) . \phi z : (\exists w) . \chi w : \supset :.$

$$\phi z . \supset_z . \psi z : \chi w . \supset_w . \theta w : \equiv : \phi z . \chi w . \supset_{z,w} . \psi z . \theta w$$

Dem.

$\vdash . *10{\cdot}1 . *3{\cdot}47 . \supset \vdash :. \phi z . \supset_z . \psi z : \chi w . \supset_w . \theta w :$

$$\supset : \phi z . \chi w . \supset . \psi z . \theta w \qquad (1)$$

$\vdash . (1) . *11{\cdot}11{\cdot}3 . \supset \vdash :. \phi z . \supset_z . \psi z : \chi w . \supset_w . \theta w :$

$$\supset : \phi z . \chi w . \supset_{z,w} . \psi z . \theta w \qquad (2)$$

$\vdash . *10{\cdot}1 . \supset \vdash :: \phi z . \chi w . \supset_{z,w} . \psi z . \theta w : \supset :. \phi z . \chi w . \supset_w . \psi z . \theta w :.$

[$*10{\cdot}28$] $\supset :. (\exists w) . \phi z . \chi w . \supset . (\exists w) . \psi z . \theta w :.$

[$*10{\cdot}35$] $\supset :. \phi z : (\exists w) . \chi w : \supset : \psi z : (\exists w) . \theta w$ (3)

$\vdash . (3) . \text{Comm} . *3{\cdot}26 . \supset \vdash :: (\exists w) . \chi w : \supset :. \phi z . \chi w . \supset_{z,w} . \psi z . \theta w :$

$$\supset : \phi z . \supset . \psi z \qquad (4)$$

$\vdash . (4) . *10{\cdot}11{\cdot}21 . \supset \vdash :: (\exists w) . \chi w . \supset :. \phi z . \chi w . \supset_{z,w} . \psi z . \theta w :$

$$\supset : \phi z . \supset_z . \psi z \qquad (5)$$

Similarly $\vdash :: (\exists z) . \phi z . \supset :. \phi z . \chi w . \supset_{z,w} . \psi z . \theta w :$

$$\supset : \chi w . \supset_w . \theta w \qquad (6)$$

$\vdash . (5) . (6) . *3{\cdot}47 . \text{Comp} . \supset$

$\vdash :: \text{Hp} . \supset :. \phi z . \chi w . \supset_{z,w} . \psi z . \theta w : \supset : \phi z . \supset_z . \psi z : \chi w . \supset_w . \theta w$ (7)

$\vdash . (2) . (7) . \supset \vdash . \text{Prop}$

$*12$. THE HIERARCHY OF TYPES AND THE AXIOM OF REDUCIBILITY

The primitive idea "$(x) . \phi x$" has been explained to mean "ϕx is always true," *i.e.* "all values of ϕx are true." But whatever function ϕ may be, there will be arguments x with which ϕx is meaningless, *i.e.* with which as arguments ϕ does not have any value. The arguments with which ϕx has values form what we will call the "range of significance" of ϕx. A "*type*" is defined as the range of significance of some function. In virtue of $*9 \cdot 14$, if ϕx, ϕy, and ψx are significant, *i.e.* either true or false, so is ψy. From this it follows that two types which have a common member coincide, and that two different types are mutually exclusive. Any proposition of the form $(x) . \phi x$, *i.e.* any proposition containing an apparent variable, determines some type as the range of the apparent variable, the type being fixed by the function ϕ.

The division of objects into types is necessitated by the vicious-circle fallacies which otherwise arise*. These fallacies show that there must be no totalities which, if legitimate, would contain members defined in terms of themselves. Hence any expression containing an apparent variable must not be in the range of that variable, *i.e.* must belong to a different type. Thus the apparent variables contained or presupposed in an expression are what determines its type. This is the guiding principle in what follows.

As explained in $*9$, propositions containing variables are generated from propositional functions which do not contain these apparent variables, by the process of asserting all or some values of such functions. Suppose ϕa is a proposition containing a; we will give the name of *generalization* to the process which turns ϕa into $(x) . \phi x$ or $(\exists x) . \phi x$, and we will give the name of *generalized propositions* to all such as contain apparent variables. It is plain that propositions containing apparent variables presuppose others not containing apparent variables, from which they can be derived by generalization. Propositions which contain no apparent variables we call *elementary propositions*†, and the terms of such propositions, other than functions, we call *individuals*. Then individuals form the first type.

It is unnecessary, in practice, to know what objects belong to the lowest type, or even whether the lowest type of variable occurring in a given context is that of individuals or some other. For in practice only the *relative* types of variables are relevant; thus the lowest type occurring in a given context may be called that of individuals, so far as that context is concerned. Accordingly the above account of individuals is not essential to the truth of what

* Cf. Introduction, Chapter II.

† Cf. pp. 91, 92.

follows; all that is essential is the way in which other types are generated from individuals, however the type of individuals may be constituted.

By applying the process of generalization to individuals occurring in elementary propositions, we obtain new propositions. The legitimacy of this process requires only that no individuals should be propositions. That this is so, is to be secured by the meaning we give to the word *individual*. We may explain an individual as something which exists on its own account; it is then obviously not a proposition, since propositions, as explained in Chapter II of the Introduction (p. 43), are incomplete symbols, having no meaning except in use. Hence in applying the process of generalization to individuals we run no risk of incurring reflexive fallacies. We will give the name of *first-order propositions* to such as contain one or more apparent variables whose possible values are individuals, but contain no other apparent variables. First-order propositions are not all of the same type, since, as was explained in ∗9, two propositions which do not contain the same number of apparent variables cannot be of the same type. But owing to the systematic ambiguity of negation and disjunction, their differences of type may usually be ignored in practice. No reflexive fallacies will result, since no first-order proposition involves any totality except that of individuals.

Let us denote by "$\phi \,!\, \hat{x}$" or "$\phi \,!\, (\hat{x}, \hat{y})$" or etc. an elementary function whose argument or arguments are individual. We will call such a function a *predicative function of an individual*. Such functions, together with those derived from them by generalization, will be called *first-order functions*. In practice we may without risk of reflexive fallacies treat first-order functions as a type, since the only totality they involve is that of individuals, and, by means of the systematic ambiguity of negation and disjunction, any function of a first-order function which will concern us will be significant whatever first-order function is taken as argument, provided the right meanings are given to the negations and disjunctions involved.

For the sake of clearness, we will repeat in somewhat different terms our account of what is meant by a first-order function. Let us give the name of *matrix* to any function, of however many variables, which does not involve any apparent variables. Then any possible function other than a matrix is derived from a matrix by means of generalization, *i.e.* by considering the proposition which asserts that the function in question is true with all possible values or with some value of one of the arguments, the other argument or arguments remaining undetermined. Thus *e.g.* from the function $\phi(x, y)$ we shall be able to derive the four functions

$$(x) \,.\, \phi(x, y), \quad (\exists x) \,.\, \phi(x, y), \quad (y) \,.\, \phi(x, y), \quad (\exists y) \,.\, \phi(x, y),$$

of which the two first are functions of y, while the two last are functions of x. (All *propositions*, with the exception of such as are values of matrices, are also derived from matrices by the above process of generalization. In order to obtain

a proposition from a matrix containing n variables, without assigning values to any of the variables, it is necessary to turn all the variables into apparent variables. Thus if $\phi(x, y)$ is a matrix, $(x, y) . \phi(x, y)$ is a proposition.) We will give the name *first-order matrices* to such as have only individuals for their arguments, and we will give the name of *first-order functions* (of any number of variables) to such as either are first-order matrices or are derived from first-order matrices by generalization applied to some (not all) of the arguments to such matrices. First-order *propositions* will be such as result from applying generalization to *all* the arguments to a first-order matrix.

As we have already stated, the notation "$\phi ! \hat{z}$" is used for any elementary function of one variable. Thus "$\phi ! x$" represents any value of any elementary function of one variable. It will be seen that "$\phi ! x$" is a function of two variables, namely $\phi ! \hat{z}$ and x. Since it contains no apparent variable, it is a matrix, but since it contains a variable (namely $\phi ! \hat{z}$) which is not an individual, it is not a first-order matrix. The same applies to $\phi ! a$, where a is some definite constant. We can build up a number of new matrices, such as

$$\sim \phi ! a, \quad \sim \phi ! x, \quad \phi ! x \vee \phi ! y, \quad \phi ! x \vee \psi ! x, \quad \phi ! x \vee \psi ! y,$$
$$\phi ! x . \supset . \psi ! x, \quad \phi ! x . \psi ! x, \quad \phi ! x \vee \psi ! y \vee \chi ! z, \text{ and so on.}$$

All these are matrices which involve first-order functions among their arguments. Such matrices we will call *second-order matrices.* From these matrices, by applying generalization to their arguments, whether to such as are functions or to such (if any) as are individuals, we obtain new functions and propositions. Such functions (together with second-order matrices) will be called *second-order functions,* and such propositions will be called *second-order propositions.* Thus we are led to the following definitions:

A *second-order matrix* is one which has at least one first-order matrix among its arguments, but has no arguments other than first-order matrices and individuals.

A *second-order function* is one which either is a second-order matrix or results from one by applying generalization to some (not all) of the arguments to a second-order matrix.

A *second-order proposition* is one which results from a second-order matrix by applying generalization to all its arguments.

In addition to the above illustrations of second-order matrices, we may give the following examples of second-order functions:

(1) Functions in which the argument is $\phi ! \hat{z}$: $(x) . \phi ! x$, $(\exists x) . \phi ! x$, $\phi ! a . \supset . \phi ! b$, where a and b are constants, $\phi ! x . \supset_x . g ! x$, where $g ! \hat{z}$ is a constant function, and so on.

(2) Functions in which the arguments are $\phi ! \hat{z}$ and $\psi ! \hat{z}$:

$$\phi ! x . \supset_x . \psi ! x, \quad \phi ! x . \equiv_x . \psi ! x, \quad (\exists x) . \phi x . \psi x, \quad \phi ! a . \supset . \psi ! b,$$

where a and b are constants, and so on

(3) Functions in which the argument is an individual x: $(\phi) . \phi ! x$, $(\exists \phi) . \phi ! x$, $\phi ! x . \supset_\phi . \phi ! a$, where a is constant, and so on.

(4) Functions in which the arguments are $\phi ! \hat{z}$ and x: $\phi ! x$, $\phi ! x . \supset . \phi ! a$, where a is constant, $(\exists \psi) : \phi ! x . \equiv . \psi ! x$, and so on.

Examples of second-order functions might, of course, be multiplied indefinitely, but the above seem sufficient for purposes of illustration.

A second-order matrix of one variable will be called a *predicative second-order function of one variable* or a *predicative function of a first-order matrix*. Thus $\phi ! a$, $\sim \phi ! a$ and $\phi ! a \supset \phi ! b$ are predicative functions of $\phi ! \hat{z}$. Similarly a function of several variables of which at least one is a first-order matrix, while the rest are either individuals or first-order matrices, will be called *predicative* if it is a matrix.

It will be seen, however, that a second-order function may have only individuals for its arguments; instances were given just now under the heading (3). Such functions we shall not call predicative, since predicative functions of individuals have already been defined as being such as are of the first order. Thus the order of a function is not determined by the order of its argument or arguments; indeed, the function may be of any order superior to the order or orders of its arguments.

A variable matrix whose argument is $\phi ! \hat{z}$ will be denoted by $f ! \phi ! \hat{z}$, and generally, a matrix whose arguments are $\phi ! \hat{z}, \psi ! \hat{z}, \ldots x, y, \ldots$ (where there is at least one function among the arguments) will be denoted by

$$f ! (\phi ! \hat{z}, \psi ! \hat{z}, \ldots x, y, \ldots).$$

Such a matrix is not of the first or second order, since it contains the new variable f, whose values are second-order matrices. We proceed to construct new matrices as we did with the matrix $\phi ! \hat{x}$; these constitute *third-order matrices*. These together with the functions derived from them by generalization are called *third-order functions*, and the propositions derived from third-order matrices by generalization are called *third-order propositions*.

In this way we can proceed indefinitely to matrices, functions and propositions of higher and higher orders. We introduce the following definition:

A function is said to be *predicative* when it is a matrix. It will be observed that, in a hierarchy in which all the variables are individuals or matrices, a matrix is the same thing as an elementary function (cf. pp. 127, 128).

"Matrix" or "predicative function" is a primitive idea.

The fact that a function is predicative is indicated, as above, by a note of exclamation after the functional letter.

The variables occurring in the present work, from this point onwards, will all be either individuals or matrices of some order in the above hierarchy. Propositions, which have occurred hitherto as variables, will no longer do so

except in a few isolated cases of which no subsequent use is made. In practice, for the reasons explained on p. 162, a function of a matrix may be regarded as capable of any argument which is a function of the same order and takes arguments of the same type.

In practice, we never need to know the absolute types of our variables, but only their *relative* types. That is to say, if we prove any proposition on the assumption that one of our variables is an individual, and another is a function of order n, the proof will still hold if, in place of an individual, we take a function of order m, and in place of our function of order n we take a function of order $n+m$, with corresponding changes for any other variables that may be involved. This results from the assumption that our primitive propositions are to apply to variables of any order.

We shall use small Latin letters (other than p, q, r, s) for variables of the lowest type concerned in any context. For functions, we shall use the letters ϕ, ψ, χ, θ, f, g, F (except that, at a later stage, F will be defined as a constant relation, and θ will be defined as the order-type of the continuum).

We shall explain later a different hierarchy, that of classes and relations, which is derived from the functional hierarchy explained above, but is more convenient in practice.

When any predicative function, say $\phi ! \hat{z}$, occurs as apparent variable, it would be strictly more correct to indicate the fact by placing "$(\phi ! \hat{z})$" before what follows, as thus: "$(\phi ! \hat{z}) . f(\phi ! \hat{z})$." But for the sake of brevity we write simply "(ϕ)" instead of "$(\phi ! \hat{z})$." Since what follows the ϕ in brackets must always contain ϕ with arguments supplied, no confusion can result from this practice.

It should be observed that, in virtue of the manner in which our hierarchy of functions was generated, non-predicative functions always result from such as are predicative by means of generalization. Hence it is unnecessary to introduce a special notation for non-predicative functions of a given order and taking arguments of a given order. For example, second-order functions of an individual x are always derived by generalization from a matrix

$$f ! (\phi ! \hat{z}, \psi ! \hat{z}, \ldots x, y, z, \ldots),$$

where the functions f, ϕ, ψ, ... are predicative. It is possible, therefore, without loss of generality, to use no apparent variables except such as are predicative.

We require, however, a means of symbolizing a function whose order is not assigned. We shall use "ϕx" or "$f(\chi ! \hat{z})$" or etc. to express a function (ϕ or f) whose order, relatively to its argument, is not given. Such a function cannot be made into an apparent variable, unless we suppose its order previously fixed. As the only purpose of the notation is to avoid the necessity of fixing the order, such a function will not be used as an apparent variable; the only functions which will be so used will be predicative functions, because, as we have just seen, this restriction involves no loss of generality.

We have now to state and explain the *axiom of reducibility.*

It is important to observe that, since there are various types of propositions and functions, and since generalization can only be applied within some one type (or, by means of systematic ambiguity, within some well-defined and completed set of types), all phrases referring to "all propositions" or "all functions," or to "some (undetermined) proposition" or "some (undetermined) function," are *prima facie* meaningless, though in certain cases they are capable of an unobjectionable interpretation. Contradictions arise from the use of such phrases in cases where no innocent meaning can be found.

If mathematics is to be possible, it is absolutely necessary (as explained in the Introduction, Chapter II) that we should have some method of making statements which will usually be equivalent to what we have in mind when we (inaccurately) speak of "all properties of x." (A "property of x" may be defined as a propositional function satisfied by x.) Hence we must find, if possible, some method of reducing the order of a propositional function without affecting the truth or falsehood of its values. This seems to be what common-sense effects by the admission of *classes.* Given any propositional function ψx, of whatever order, this is assumed to be equivalent, for all values of x, to a statement of the form "x belongs to the class α." Now assuming that there is such an entity as the class α, this statement is of the first order, since it involves no allusion to a variable function. Indeed its only practical advantage over the original statement ψx is that it is of the first order. There is no advantage in assuming that there really are such things as classes, and the contradiction about the classes which are not members of themselves shows that, if there are classes, they must be something radically different from individuals. It would seem that the sole purpose which classes serve, and one main reason which makes them linguistically convenient, is that they provide a method of reducing the order of a propositional function. We shall, therefore, not assume anything of what may seem to be involved in the common-sense admission of classes, except this, that every propositional function is equivalent, for all its values, to some predicative function of the same argument or arguments.

This assumption with regard to functions is to be made whatever may be the type of their arguments. Let fu be a function, of any order, of an argument u, which may itself be either an individual or a function of any order. If f is a matrix, we write the function in the form $f \,!\, u$; in such a case we call f a *predicative* function. Thus a predicative function of an individual is a first-order function; and for higher types of arguments, predicative functions take the place that first-order functions take in respect of individuals. We assume, then, that every function of one variable is equivalent, for all its values, to some predicative function of the same argument. This assumption seems to be the essence of the usual assumption of classes; at any rate, it retains as much

of classes as we have any use for, and little enough to avoid the contradictions which a less grudging admission of classes is apt to entail. We will call this assumption the *axiom of classes*, or the *axiom of reducibility*.

We shall assume similarly that every function of two variables is equivalent, for all its values, to a predicative function of those variables, *i.e.* to a matrix. This assumption is what seems to be meant by saying that any statement about two variables defines a relation between them. We will call this assumption the *axiom of relations* or (like the previous axiom) the *axiom of reducibility*.

In dealing with relations between more than two terms, similar assumptions would be needed for three, four, ... variables. But these assumptions are not indispensable for our purpose, and are therefore not made in this work.

Stated in symbols, the two forms of the axiom of reducibility are as follows:

***12·1.** $\vdash : (\exists f) : \phi x \,.\, \equiv_x \,.\, f \,!\, x$ Pp

***12·11.** $\vdash : (\exists f) : \phi (x, y) \,.\, \equiv_{x,y} \,.\, f \,!\, (x, y)$ Pp

We call two functions $\phi \hat{x}$, $\psi \hat{x}$ *formally equivalent* when $\phi x \,.\, \equiv_x \,.\, \psi x$, and similarly we call $\phi (\hat{x}, \hat{y})$ and $\psi (\hat{x}, \hat{y})$ formally equivalent when

$$\phi (x, y) \,.\, \equiv_{x,y} \,.\, \psi (x, y).$$

Thus the above axioms state that any function of one or two variables is formally equivalent to some *predicative* function of one or two variables, as the case may be.

Of the above two axioms, the first is chiefly needed in the theory of classes (*20), and the second in the theory of relations (*21). But the first is also essential to the theory of identity, if identity is to be defined (as we have done, in *13·01); its use in the theory of identity is embodied in the proof of *13·101, below.

We may sum up what has been said in the present number as follows:

(1) A function of the first order is one which involves no variables except individuals, whether as apparent variables or as arguments.

(2) A function of the $(n + 1)$th order is one which has at least one argument or apparent variable of order n, and contains no argument or apparent variable which is not either an individual or a first-order function or a second-order function or ... or a function of order n.

(3) A predicative function is one which contains no apparent variables, *i.e.* is a matrix. It is possible, without loss of generality, to use no variables except matrices and individuals, so long as variable *propositions* are not required.

(4) Any function of one argument or of two is formally equivalent to a predicative function of the same argument or arguments.

$*13$. IDENTITY

Summary of $*13$.

The propositional function "x is identical with y" will be written "$x = y$." We shall find that this use of the sign of equality covers all the common uses of equality that occur in mathematics. The definition is as follows:

$*13{\cdot}01$. $x = y \,.=:\, (\phi) : \phi ! x \,.\supset.\, \phi ! y$ Df

This definition states that x and y are to be called identical when every predicative function satisfied by x is also satisfied by y. We cannot state that *every* function satisfied by x is to be satisfied by y, because x satisfies functions of various orders, and these cannot all be covered by one apparent variable. But in virtue of the axiom of reducibility it follows that, if $x = y$ and x satisfies ψx, where ψ is any function, predicative or non-predicative, then y also satisfies ψy (cf. $*13{\cdot}101$, below). Hence in effect the definition is as powerful as it would be if it could be extended to cover *all* functions of x.

Note that the second sign of equality in the above definition is combined with "Df," and thus is not really the same symbol as the sign of equality which is defined. Thus the definition is not circular, although at first sight it appears so.

The propositions of the present number are constantly referred to. Most of them are self-evident, and the proofs offer no difficulty. The most important of the propositions of this number are the following:

$*13{\cdot}101$. $\vdash : x = y \,.\supset.\, \psi x \supset \psi y$

I.e. if x and y are identical, any property of x is a property of y.

$*13{\cdot}12$. $\vdash : x = y \,.\supset.\, \psi x \equiv \psi y$

This includes $*13{\cdot}101$ together with the fact that if x and y are identical any property of y is a property of x.

$*13{\cdot}15{\cdot}16{\cdot}17$, which state that identity is reflexive, symmetrical and transitive.

$*13{\cdot}191$. $\vdash :. \, y = x \,.\supset_y.\, \phi y : \equiv . \, \phi x$

I.e. to state that everything that is identical with x has a certain property is equivalent to stating that x has that property.

$*13{\cdot}195$. $\vdash : (\exists y) \,.\, y = x \,.\, \phi y \,.\equiv.\, \phi x$

I.e. to state that something identical with x has a certain property is equivalent to saying that x has that property.

$*13{\cdot}22$. $\vdash : (\exists z, w) \,.\, z = x \,.\, w = y \,.\, \phi(z, w) \,.\equiv.\, \phi(x, y)$

This is the analogue of $*13{\cdot}195$ for two variables.

***13·01.** $x=y\,.=:(\phi):\phi!x\,.\supset.\,\phi!y$ Df

The following definitions embody abbreviations which are often convenient.

***13·02.** $x\neq y\,.=.\sim(x=y)$ Df

***13·03.** $x=y=z\,.=.\,x=y\,.\,y=z$ Df

***13·1.** $\vdash:.\,x=y\,.\equiv:\phi!x\,.\supset_\phi.\,\phi!y$ [*4·2 . (*13·01) . (*10·02)]

***13·101.** $\vdash:x=y\,.\supset.\,\psi x\supset\psi y$

Dem.

$\vdash.*12\cdot1.\supset\vdash:.(\exists\phi):.\psi x\,.\equiv.\,\phi!x:\psi y\,.\equiv.\,\phi!y$ (1)

$\vdash.*13\cdot1.\supset\vdash::\mathrm{Hp}.\supset:.\phi!x\,.\supset_\phi.\,\phi!y:.$

[*4·84·85.*10·27] $\supset:.\psi x\,.\equiv.\,\phi!x:\psi y\,.\equiv.\,\phi!y:\supset_\phi:\psi x\,.\supset.\,\psi y:.$

[*10·23] $\supset:.(\exists\phi):\psi x\,.\equiv.\,\phi!x:\psi y\,.\equiv.\,\phi!y:\supset:\psi x\,.\supset.\,\psi y$ (2)

$\vdash.(1).(2).\supset\vdash.\mathrm{Prop}$

In virtue of this proposition, if $x=y$, y satisfies any function, whether predicative or non-predicative, which is satisfied by x. It will be observed that the proof uses the axiom of reducibility (*12·1). But for this axiom, two terms x and y might agree in respect of all *predicative* functions, but not in respect of all non-predicative functions. We should thus be led to identities of different degrees, according to the degree of the functions in respect of which x and y agreed. Strict identity would, in this case, have to be taken as a primitive idea, and *13·101 would have to be a primitive proposition, as would also *13·15·16·17.

***13·11.** $\vdash::x=y\,.\equiv:\phi!x\,.\equiv_\phi.\,\phi!y$

Dem.

$\vdash.*10\cdot22.\quad\supset\vdash:.\phi!x\,.\equiv_\phi.\,\phi!y:\supset:\phi!x\,.\supset_\phi.\,\phi!y:$

[*13·1] $\supset:x=y$ (1)

$\vdash.*13\cdot101.\quad\supset\vdash:.x=y\,.\supset.\,\phi!x\supset\phi!y$ (2)

$\vdash.*13\cdot101.*1\cdot7.\quad\supset\vdash:.x=y\,.\supset.\sim\phi!x\supset\sim\phi!y.$

[Transp] $\supset.\,\phi!y\supset\phi!x$ (3)

$\vdash.(2).(3).\mathrm{Comp}.\supset\vdash:x=y\,.\ \supset.\,\phi!x\equiv\phi!y:$

[*10·11·21] $\supset\vdash:.x=y\,.\supset:\phi!x\,.\equiv_\phi.\,\phi!y$ (4)

$\vdash.(1).(4).\quad\supset\vdash.\mathrm{Prop}$

***13·12.** $\vdash:x=y\,.\supset.\,\psi x\equiv\psi y$

Dem.

$\vdash.*13\cdot101.\mathrm{Comp}.\supset\vdash:x=y\,.\supset.\,\psi x\supset\psi y\,.\sim\psi x\supset\sim\psi y.$

[Transp] $\supset.\,\psi x\equiv\psi y:\supset\vdash.\mathrm{Prop}$

***13·13.** $\vdash:\psi x\,.\,x=y\,.\supset.\,\psi y$ [*13·101 . Comm . Imp]

***13·14.** $\vdash:\psi x\,.\sim\psi y\,.\supset.\,x\neq y$ [*13·13 . *4·14]

***13·15.** $\vdash.\,x=x$ [Id . *10·11 . *13·1]

***13·16.** $\vdash:x=y\,.\equiv.\,y=x$ [*13·11 . *10·32]

***13·17.** $\vdash : x = y \,.\, y = z \,.\, \supset \,.\, x = z$

Dem.

$\vdash \,.\, *13{\cdot}1 \,.\, \supset \vdash :: \text{Hp} \,.\, \supset :. \, \phi\,!\,x \,.\, \supset_\phi \,.\, \phi\,!\,y : \phi\,!\,y \,.\, \supset_\phi \,.\, \phi\,!\,z :.$
$[*10{\cdot}3] \qquad \supset :. \, \phi\,!\,x \,.\, \supset_\phi \,.\, \phi\,!\,z :: \supset \vdash \,.\, \text{Prop}$

In the above use of *10·3, $\phi\,!\,x$, $\phi\,!\,y$, $\phi\,!\,z$ are regarded as three different functions of ϕ, and ϕ replaces the x of *10·3.

The above three propositions show that identity is reflexive (*13·15), symmetrical (*13·16), and transitive (*13·17). These are the three marks of relations having the formal properties which we associate commonly with the sign of equality.

***13·171.** $\vdash : x = y \,.\, x = z \,.\, \supset \,.\, y = z$ [*13·16·17]

***13·172.** $\vdash : y = x \,.\, z = x \,.\, \supset \,.\, y = z$ [*13·16·17]

***13·18.** $\vdash : x = y \,.\, x \neq z \,.\, \supset \,.\, y \neq z$ [*13·17 . *4·14]

***13·181.** $\vdash : x = y \,.\, y \neq z \,.\, \supset \,.\, x \neq z$ [*13·171 . *4·14]

***13·182.** $\vdash :. \, x = y \,.\, \supset : z = x \,.\, \equiv \,.\, z = y$ [*13·17·172 . Exp . Comp]

***13·183.** $\vdash :. \, x = y \,.\, \equiv : z = x \,.\, \equiv_z \,.\, z = y$

Dem.

$\vdash \,.\, *13{\cdot}182 \,.\, *10{\cdot}11{\cdot}21 \,.\, \supset \vdash :. \, x = y \,.\, \supset : z = x \,.\, \equiv_z \,.\, z = y \qquad (1)$
$\vdash \,.\, *10{\cdot}1 \,.\, \supset \vdash :. \, z = x \,.\, \equiv_z \,.\, z = y : \supset : x = x \,.\, \supset \,.\, x = y :$
$[*13{\cdot}15] \qquad \supset : x = y \qquad (2)$
$\vdash \,.\, (1) \,.\, (2) \,.\, \supset \vdash \,.\, \text{Prop}$

***13·19.** $\vdash \,.\, (\exists y) \,.\, y = x$ [*13·15 . *10·24]

***13·191.** $\vdash :. \, y = x \,.\, \supset_y \,.\, \phi y : \equiv \,.\, \phi x$

Dem.

$\vdash \,.\, *10{\cdot}1 \,.\, \supset \vdash :. \, y = x \,.\, \supset_y \,.\, \phi y : \supset : x = x \,.\, \supset \,.\, \phi x :$
$[*13{\cdot}15] \qquad \supset : \phi x \qquad (1)$
$\vdash \,.\, *13{\cdot}12 \,.\, \supset \vdash :. \, y = x \,.\, \supset : \phi x \,.\, \supset \,.\, \phi y :.$
$[\text{Comm}] \qquad \supset \vdash :. \, \phi x \,.\, \supset : y = x \,.\, \supset \,.\, \phi y :.$
$[*10{\cdot}11{\cdot}21] \supset \vdash :. \, \phi x \,.\, \supset : y = x \,.\, \supset_y \,.\, \phi y \qquad (2)$
$\vdash \,.\, (1) \,.\, (2) \,.\, \supset \vdash \,.\, \text{Prop}$

This proposition is constantly used in subsequent proofs.

***13·192.** $\vdash :. \, (\exists c) : x = b \,.\, \equiv_x \,.\, x = c : \psi c : \equiv \,.\, \psi b$

Dem.

$\vdash \,.\, *4{\cdot}2 \,.\, *3{\cdot}2 \,.\, \supset \vdash :: \psi b \,.\, \supset :. \, x = b \,.\, \equiv_x \,.\, x = b : \psi b :.$
$[*10{\cdot}24] \qquad \supset :. \, (\exists c) : x = b \,.\, \equiv_x \,.\, x = c : \psi c \qquad (1)$
$\vdash \,.\, *10{\cdot}1 \,.\, \supset \vdash :. \, x = b \,.\, \equiv_x \,.\, x = c : \psi c : \supset : b = b \,.\, \equiv \,.\, b = c : \psi c :$
$[*5{\cdot}501 . *13{\cdot}15] \qquad \supset : b = c \,.\, \psi c :$
$[*13{\cdot}13] \qquad \supset : \psi b \qquad (2)$
$\vdash \,.\, (2) \,.\, *10{\cdot}11{\cdot}23 \,.\, \supset \vdash :. \, (\exists c) : x = b \,.\, \equiv_x \,.\, x = c : \psi c : \supset \,.\, \psi b \qquad (3)$
$\vdash \,.\, (1) \,.\, (3) \,.\, \supset \vdash \,.\, \text{Prop}$

This proposition is useful in the theory of descriptions (*14).

$*13{\cdot}193.\ \vdash : \phi x \,.\, x = y \,.\, \equiv \,.\, \phi y \,.\, x = y$

Dem.

$$\begin{array}{lll}
\vdash . \text{Simp} . & \supset \vdash : \phi x \,.\, x = y \,.\, \supset \,.\, x = y & (1)\\
\vdash . *13{\cdot}13 . & \supset \vdash : \phi x \,.\, x = y \,.\, \supset \,.\, \phi y & (2)\\
\vdash . (1) . (2) . \text{Comp} . & \supset \vdash : \phi x \,.\, x = y \,.\, \supset \,.\, \phi y \,.\, x = y & (3)\\
\vdash . *13{\cdot}16 . \text{Fact} . & \supset \vdash : \phi y \,.\, x = y \,.\, \supset \,.\, \phi y \,.\, y = x \,. & \\
\left[(3)\dfrac{y, x}{x, y}\right] & \qquad \supset \,.\, \phi x \,.\, y = x \,. & \\
[*13{\cdot}16 . \text{Fact}] & \qquad \supset \,.\, \phi x \,.\, x = y & (4)\\
\vdash . (3) . (4) . \supset \vdash . \text{Prop} & &
\end{array}$$

This proposition is very often used.

$*13{\cdot}194.\ \vdash : \phi x \,.\, x = y \,.\, \equiv \,.\, \phi x \,.\, \phi y \,.\, x = y \quad [*13{\cdot}13 \,.\, *4{\cdot}71]$

This proposition is used in $*37{\cdot}65$ and $*101{\cdot}14$.

$*13{\cdot}195.\ \vdash : (\exists y) \,.\, y = x \,.\, \phi y \,.\, \equiv \,.\, \phi x$

Dem.

$$\begin{array}{lll}
\vdash . *3{\cdot}2 . *13{\cdot}15 . & \supset \vdash : \phi x \,.\, \supset \,.\, x = x \,.\, \phi x \,. & \\
[*10{\cdot}24] & \qquad \supset \,.\, (\exists y) \,.\, y = x \,.\, \phi y & (1)\\
\vdash . *13{\cdot}13 . *10{\cdot}11 . & \supset \vdash :. (y) : y = x \,.\, \phi y \,.\, \supset \,.\, \phi x : & \\
[*10{\cdot}23] & \supset \vdash :. (\exists y) \,.\, y = x \,.\, \phi y \,.\, \supset \,.\, \phi x & (2)\\
\vdash . (1) . (2) . & \supset \vdash . \text{Prop} &
\end{array}$$

The use of this proposition in subsequent proofs is very frequent.

$*13{\cdot}196.\ \vdash :. \sim\phi x \,.\, \equiv : \phi y \,.\, \supset_y \,.\, y \neq x \quad [*13{\cdot}195 \,.\, \text{Transp} \,.\, *10{\cdot}51]$

$*13{\cdot}21.\ \vdash :. z = x \,.\, w = y \,.\, \supset_{z,w} \,.\, \phi(z, w) : \equiv \,.\, \phi(x, y)$

Dem.

$$\begin{array}{ll}
\vdash . *11{\cdot}62 . \supset & \\
\vdash :: z = x \,.\, w = y \,.\, \supset_{z,w} \,.\, \phi(z, w) : \equiv :. z = x \,.\, \supset_z : w = y \,.\, \supset_w \,.\, \phi(z, w) :. & \\
[*13{\cdot}191] & \equiv :. w = y \,.\, \supset_w \,.\, \phi(x, w) :. \\
[*13{\cdot}191] & \equiv :. \phi(x, y) :: \supset \vdash . \text{Prop}
\end{array}$$

This proposition is the analogue, for two variables, of $*13{\cdot}191$.

$*13{\cdot}22.\ \vdash : (\exists z, w) \,.\, z = x \,.\, w = y \,.\, \phi(z, w) \,.\, \equiv \,.\, \phi(x, y)$

Dem.

$$\begin{array}{ll}
\vdash . *11{\cdot}55 . \supset \vdash :. (\exists z, w) \,.\, z = x \,.\, w = y \,.\, \phi(z, w) \,. & \\
 & \equiv : (\exists z) : z = x : (\exists w) \,.\, w = y \,.\, \phi(z, w) : \\
[*13{\cdot}195] & \equiv : (\exists w) \,.\, w = y \,.\, \phi(x, w) : \\
[*13{\cdot}195] & \equiv : \phi(x, y) :. \supset \vdash . \text{Prop}
\end{array}$$

This proposition is the analogue, for two variables, of $*13{\cdot}195$. It is frequently used, especially in the theory of couples ($*54$, $*55$, $*56$).

The following proposition is useful in the theory of types. Its purpose is to show that, if a is any argument for which "ϕa" is significant, *i.e.* for which we have $\phi a \vee \sim\phi a$, then "ϕx" is significant when, and only when, x is either

identical with a or not identical with a. It follows (as will be proved in *20·81) that, if "ϕa" and "ψa" are both significant, the class of values of x for which "ϕx" is significant is the same as the class of those for which "ψx" is significant, *i.e.* two types which have a common member are identical.

In the following proof, the chief point to observe is the use of *10·221. There are two variables, a and x, to be identified. In the first use, we depend upon the fact that ϕa and $x = a$ both occur in both (4) and (5): the occurrence of ϕa in both justifies the identification of the two a's, and when these have been identified, the occurrence of $x = a$ in both justifies the identification of the two x's. (Unless the a's had been already identified, this would not be legitimate, because "$x = a$" is typically ambiguous if neither x nor a is of given type.) The second use of *10·221 is justified by the fact that both ϕa and ϕx occur in both (2) and (6).

***13·3.** $\vdash :: \phi a \lor \sim\phi a . \supset :. \phi x \lor \sim\phi x . \equiv : x = a . \lor . x \neq a$

Dem.

$\vdash . *2{\cdot}11 . \quad \supset \vdash . \phi x \lor \sim\phi x \quad (1)$

$\vdash . (1) . \text{Simp} . \quad \supset \vdash : \phi a \lor \sim\phi a . \supset . \phi x \lor \sim\phi x \quad (2)$

$\vdash . *2{\cdot}11 . \quad \supset \vdash : x = a . \lor . x \neq a \quad (3)$

$\vdash . (3) . \text{Simp} . \quad \supset \vdash :. \phi a \lor \sim\phi a . \supset : x = a . \lor . x \neq a \quad (4)$

$\vdash . *13{\cdot}101 . \text{Comm} . \supset \vdash :. \phi a \lor \sim\phi a . \supset : x = a . \supset . \phi x \lor \sim\phi x \quad (5)$

$\vdash . (4) . (5) . *10{\cdot}13{\cdot}221 . \supset$

$\vdash :: \phi a \lor \sim\phi a . \supset : x = a . \lor . x \neq a :. \phi a \lor \sim\phi a . \supset : x = a . \supset . \phi x \lor \sim\phi x \quad (6)$

$\vdash . (2) . (6) . *10{\cdot}13{\cdot}221 . \supset$

$\vdash :: \phi a \lor \sim\phi a . \supset . \phi x \lor \sim\phi x :. \phi a \lor \sim\phi a . \supset : x = a . \lor . x \neq a :.$

$\phi a \lor \sim\phi a . \supset : x = a . \supset . \phi x \lor \sim\phi x \quad (7)$

$\vdash . (7) . \text{Simp} . \supset$

$\vdash :: \phi a \lor \sim\phi a . \supset . \phi x \lor \sim\phi x :. \phi a \lor \sim\phi a . \supset : x = a . \lor . x \neq a \quad (8)$

$\vdash . (8) . *5{\cdot}35 . \quad \supset \vdash :: \phi a \lor \sim\phi a . \supset :. \phi x \lor \sim\phi x . \equiv : x = a . \lor . x \neq a ::$

$\supset \vdash . \text{Prop}$

$*14$. DESCRIPTIONS

Summary of $*14$.

A *description* is a phrase of the form "the term which etc.," or, more explicitly, "the term x which satisfies $\phi\hat{x}$," where $\phi\hat{x}$ is some function satisfied by one and only one argument. For reasons explained in the Introduction (Chapter III), we do not define "the x which satisfies $\phi\hat{x}$," but we define any proposition in which this phrase occurs. Thus when we say: "The term x which satisfies $\phi\hat{x}$ satisfies $\psi\hat{x}$," we shall mean: "There is a term b such that ϕx is true when, and only when, x is b, and ψb is true." That is, writing "$(\imath x)(\phi x)$" for "the term x which satisfies ϕx," $\psi(\imath x)(\phi x)$ is to mean

$$(\exists b) : \phi x . \equiv_x . x = b : \psi b.$$

This, however, is not yet quite adequate as a definition, for when $(\imath x)(\phi x)$ occurs in a proposition which is part of a larger proposition, there is doubt whether the smaller or the larger proposition is to be taken as the "$\psi(\imath x)(\phi x)$." Take, for example, $\psi(\imath x)(\phi x) . \supset . p$. This may be either

$$(\exists b) : \phi x . \equiv_x . x = b : \psi b : \supset . p$$

or

$$(\exists b) :. \phi x . \equiv_x . x = b : \psi b . \supset . p.$$

If "$(\exists b) : \phi x . \equiv_x . x = b$" is false, the first of these must be true, while the second must be false. Thus it is very necessary to distinguish them.

The proposition which is to be treated as the "$\psi(\imath x)(\phi x)$" will be called the *scope* of $(\imath x)(\phi x)$. Thus in the first of the above two propositions, the scope of $(\imath x)(\phi x)$ is $\psi(\imath x)(\phi x)$, while in the second it is $\psi(\imath x)(\phi x) . \supset . p$. In order to avoid ambiguities as to scope, we shall indicate the scope by writing "$[(\imath x)(\phi x)]$" at the beginning of the scope, followed by enough dots to extend to the end of the scope. Thus of the above two propositions the first is

$$[(\imath x)(\phi x)] . \psi(\imath x)(\phi x) . \supset . p,$$

while the second is

$$[(\imath x)(\phi x)] : \psi(\imath x)(\phi x) . \supset . p.$$

Thus we arrive at the following definition:

$*14{\cdot}01$. $[(\imath x)(\phi x)] . \psi(\imath x)(\phi x) . = : (\exists b) : \phi x . \equiv_x . x = b : \psi b$ Df

It will be found in practice that the scope usually required is the smallest proposition enclosed in dots or brackets in which "$(\imath x)(\phi x)$" occurs. Hence when this scope is to be given to $(\imath x)(\phi x)$, we shall usually omit explicit mention of the scope. Thus *e.g.* we shall have

$$a \neq (\imath x)(\phi x) \ . = : \ (\exists b) : \phi x . \equiv_x . x = b : a \neq b,$$

$$\sim\{a = (\imath x)(\phi x)\} . = . \sim\{(\exists b) : \phi x . \equiv_x . x = b : a = b\}.$$

Of these the first necessarily implies $(\exists b) : \phi x . \equiv_x . x = b$, while the second does not. We put

***14·02.** $E ! (\imath x)(\phi x) . = : (\exists b) : \phi x . \equiv_x . x = b$ Df

This defines: "The x satisfying $\phi \hat{x}$ exists," which holds when, and only when, $\phi \hat{x}$ is satisfied by one value of x and by no other value.

When two or more descriptions occur in the same proposition, there is need of avoiding ambiguity as to which has the larger scope. For this purpose, we put

***14·03.** $[(\imath x)(\phi x), (\imath x)(\psi x)] . f\{(\imath x)(\phi x), (\imath x)(\psi x)\} . = :$

$$[(\imath x)(\phi x)] : [(\imath x)(\psi x)] . f\{(\imath x)(\phi x), (\imath x)(\psi x)\} \quad \text{Df}$$

It will be shown (*14·113) that the truth-value of a proposition containing two descriptions is unaffected by the question which has the larger scope. Hence we shall in general adopt the convention that the description occurring first typographically is to have the larger scope, unless the contrary is expressly indicated. Thus *e.g.*

$$(\imath x)(\phi x) = (\imath x)(\psi x)$$

will mean $(\exists b) : \phi x . \equiv_x . x = b : b = (\imath x)(\psi x)$,

i.e. $(\exists b) :. \phi x . \equiv_x . x = b :. (\exists c) : \psi x . \equiv_x . x = c : b = c.$

By this convention we are able almost always to avoid explicit indication of the order of elimination of two or more descriptions. If, however, we require a larger scope for the later description, we put

***14·04.** $[(\imath x)(\psi x)] . f\{(\imath x)(\phi x), (\imath x)(\psi x)\} . = .$

$$[(\imath x)(\psi x), (\imath x)(\phi x)] . f\{(\imath x)(\phi x), (\imath x)(\psi x)\} \quad \text{Df}$$

Whenever we have $E ! (\imath x)(\phi x)$, $(\imath x)(\phi x)$ behaves, formally, like an ordinary argument to any function in which it may occur. This fact is embodied in the following proposition:

***14·18.** $\vdash :. E ! (\imath x)(\phi x) . \supset : (x) . \psi x . \supset . \psi (\imath x)(\phi x)$

That is to say, when $(\imath x)(\phi x)$ exists, it has any property which belongs to everything. This does not hold when $(\imath x)(\phi x)$ does not exist; for example, the present King of France does not have the property of being either bald or not bald.

If $(\imath x)(\phi x)$ has any property whatever, it must exist. This fact is stated in the proposition:

***14·21.** $\vdash : \psi (\imath x)(\phi x) . \supset . E ! (\imath x)(\phi x)$

This proposition is obvious, since "$E ! (\imath x)(\phi x)$" is, by the definitions, part of "$\psi (\imath x)(\phi x)$." When, in ordinary language or in philosophy, something is said to "exist," it is always something *described*, *i.e.* it is not something immediately presented, like a taste or a patch of colour, but something like "matter" or "mind" or "Homer" (meaning "the author of the Homeric

poems"), which is known by description as "the so-and-so," and is thus of the form $(\imath x)(\phi x)$. Thus in all such cases, the existence of the (grammatical) subject $(\imath x)(\phi x)$ can be analytically inferred from any true proposition having this grammatical subject. It would seem that the word "existence" cannot be significantly applied to subjects immediately given; *i.e.* not only does our definition give no meaning to "$E\,!\,x$," but there is no reason, in philosophy, to suppose that a meaning of existence could be found which would be applicable to immediately given subjects.

Besides the above, the following are among the more useful propositions of the present number.

$*14{\cdot}202.\ \vdash :.\ \phi x \,.\, \equiv_x . \, x = b : \equiv : (\imath x)(\phi x) = b : \equiv : \phi x \,.\, \equiv_x . \, b = x : \equiv : b = (\imath x)(\phi x)$

From the first equivalence in the above, it follows that

$*14{\cdot}204.\ \vdash : E\,!\,(\imath x)(\phi x) \,.\, \equiv . \, (\exists b) \,.\, (\imath x)(\phi x) = b$

I.e. $(\imath x)(\phi x)$ exists when there is something which $(\imath x)(\phi x)$ is.

We have

$*14{\cdot}205.\ \vdash : \psi(\imath x)(\phi x) \,.\, \equiv . \, (\exists b) \,.\, b = (\imath x)(\phi x) \,.\, \psi b$

I.e. $(\imath x)(\phi x)$ has the property ψ when there is something which is $(\imath x)(\phi x)$ and which has the property ψ.

We have to prove that such symbols as "$(\imath x)(\phi x)$" obey the same rules with regard to identity as symbols which directly represent objects. To this, however, there is one partial exception, for instead of having

$$(\imath x)(\phi x) = (\imath x)(\phi x),$$

we only have

$*14{\cdot}28.\ \vdash : E\,!\,(\imath x)(\phi x) \,.\, \equiv . \, (\imath x)(\phi x) = (\imath x)(\phi x)$

I.e. "$(\imath x)(\phi x)$" only satisfies the reflexive property of identity if $(\imath x)(\phi x)$ exists.

The symmetrical property of identity holds for such symbols as $(\imath x)(\phi x)$, without the need of assuming existence, *i.e.* we have

$*14{\cdot}13.\ \vdash : a = (\imath x)(\phi x) \,.\, \equiv . \, (\imath x)(\phi x) = a$

$*14{\cdot}131.\ \vdash : (\imath x)(\phi x) = (\imath x)(\psi x) \,.\, \equiv . \, (\imath x)(\psi x) = (\imath x)(\phi x)$

Similarly the transitive property of identity holds without the need of assuming existence. This is proved in $*14{\cdot}14{\cdot}142{\cdot}144$.

$*14{\cdot}01.\ \ [(\imath x)(\phi x)] \,.\, \psi(\imath x)(\phi x) \,.\, = : (\exists b) : \phi x \,.\, \equiv_x . \, x = b : \psi b \quad \text{Df}$

$*14{\cdot}02.\ \ E\,!\,(\imath x)(\phi x) \,.\, = : (\exists b) : \phi x \,.\, \equiv_x . \, x = b \quad \text{Df}$

$*14{\cdot}03.\ \ [(\imath x)(\phi x), (\imath x)(\psi x)] \,.\, f\{(\imath x)(\phi x), (\imath x)(\psi x)\} \,.\, = :$
$[(\imath x)(\phi x)] : [(\imath x)(\psi x)] \,.\, f\{(\imath x)(\phi x), (\imath x)(\psi x)\} \quad \text{Df}$

$*14{\cdot}04.\ \ [(\imath x)(\psi x)] \,.\, f\{(\imath x)(\phi x), (\imath x)(\psi x)\} \,.\, = .$
$[(\imath x)(\psi x), (\imath x)(\phi x)] \,.\, f\{(\imath x)(\phi x), (\imath x)(\psi x)\} \quad \text{Df}$

$*14{\cdot}1.\ \vdash :. [(\imath x)(\phi x)] . \psi(\imath x)(\phi x) . \equiv : (\exists b) : \phi x . \equiv_x . x = b : \psi b$ [$*4{\cdot}2 . (*14{\cdot}01)$]

In virtue of our conventions as to the scope intended when no scope is explicitly indicated, the above proposition is the same as the following:

$*14{\cdot}101.\ \vdash :. \psi(\imath x)(\phi x) . \equiv : (\exists b) : \phi x . \equiv_x . x = b : \psi b$ [$*14{\cdot}1$]

$*14{\cdot}11.\ \vdash :. E\,!(\imath x)(\phi x) . \equiv : (\exists b) : \phi x . \equiv_x . x = b$ [$*4{\cdot}2 . (*14{\cdot}02)$]

$*14{\cdot}111.\ \vdash :. [(\imath x)(\psi x)] . f\{(\imath x)(\phi x), (\imath x)(\psi x)\} . \equiv :$

$(\exists b, c) : \phi x . \equiv_x . x = b : \psi x . \equiv_x . x = c : f(b, c)$

Dem.

$\vdash . *4{\cdot}2 . (*14{\cdot}04{\cdot}03) . \supset$

$\vdash :: [(\imath x)(\psi x)] . f\{(\imath x)(\phi x), (\imath x)(\psi x)\} . \equiv :.$

$[(\imath x)(\psi x)] : [(\imath x)(\phi x)] . f\{(\imath x)(\phi x), (\imath x)(\psi x)\} :.$

[$*14{\cdot}1$] $\equiv :. [(\imath x)(\psi x)] :. (\exists b) : \phi x . \equiv_x . x = b : f\{b, (\imath x)(\psi x)\} :.$

[$*14{\cdot}1$] $\equiv :. (\exists c) :. \psi x . \equiv_x . x = c :. (\exists b) : \phi x . \equiv_x . x = b : f(b, c) :.$

[$*11{\cdot}55$] $\equiv :. (\exists b, c) : \phi x . \equiv_x . x = c : \psi x . \equiv_x . x = b : f(b, c) :: \supset \vdash .$ Prop

$*14{\cdot}112.\ \vdash :. f\{(\imath x)(\phi x), (\imath x)(\psi x)\} . \equiv :$

$(\exists b, c) : \phi x . \equiv_x . x = b : \psi x . \equiv_x . x = c : f(b, c)$

[Proof as in $*14{\cdot}111$]

In the above proposition, we assume the convention explained on p. 174, after the statement of $*14{\cdot}03$.

$*14{\cdot}113.\ \vdash : [(\imath x)(\psi x)] . f\{(\imath x)(\phi x), (\imath x)(\psi x)\} . \equiv . f\{(\imath x)(\phi x), (\imath x)(\psi x)\}$

[$*14{\cdot}111{\cdot}112$]

This proposition shows that when two descriptions occur in the same proposition, the truth-value of the proposition is unaffected by the question which has the larger scope.

$*14{\cdot}12.\ \vdash :. E\,!(\imath x)(\phi x) . \supset : \phi x . \phi y . \supset_{x,y} . x = y$

Dem.

$\vdash . *14{\cdot}11 \quad \supset \vdash :. Hp . \supset : (\exists b) : \phi x . \equiv_x . x = b$ (1)

$\vdash . *4{\cdot}38 . *10{\cdot}1 . *11{\cdot}11{\cdot}3 . \supset$

$\vdash :. \phi x . \equiv_x . x = b : \supset : \phi x . \phi y . \equiv_{x,y} . x = b . y = b .$

[$*13{\cdot}172$] $\supset_{x,y} . x = y$ (2)

$\vdash . (2) . *10{\cdot}11{\cdot}23 . \quad \supset \vdash :. (\exists b) : \phi x . \equiv_x . x = b : \supset : \phi x . \phi y . \supset_{x,y} . x = y$ (3)

$\vdash . (1) . (3) . \quad \supset \vdash .$ Prop

$*14{\cdot}121.\ \vdash :. \phi x . \equiv_x . x = b : \phi x . \equiv_x . x = c : \supset . b = c$

Dem.

$\vdash . *10{\cdot}1 . \supset \vdash :. Hp . \supset : \phi b . \equiv . b = b : \phi b . \equiv . b = c :$

[$*13{\cdot}15$] $\supset : \phi b : \phi b . \equiv . b = c :$

[Ass] $\supset : b = c :. \supset \vdash .$ Prop

$*14{\cdot}122.\ \vdash :. \phi x . \equiv_x . x = b : \equiv : \phi x . \supset_x . x = b : \phi b :$

$\equiv : \phi x . \supset_x . x = b : (\exists x) . \phi x$

Dem.

$\vdash . *10{\cdot}22 . \quad \supset \vdash :. \phi x . \equiv_x . x = b : \equiv : \phi x . \supset_x . x = b : x = b . \supset_x . \phi x :$

$[*13{\cdot}191] \quad \equiv : \phi x . \supset_x . x = b : \phi b \quad (1)$

$\vdash . *4{\cdot}71 . \quad \supset \vdash :. \phi x . \supset . x = b : \supset : \phi x . \equiv . \phi x . x = b :.$

$[*10{\cdot}11{\cdot}27] \quad \supset \vdash :. \phi x . \supset_x . x = b : \supset : \phi x . \equiv_x . \phi x . x = b :$

$[*10{\cdot}281] \quad \supset : (\exists x) . \phi x . \equiv . (\exists x) . \phi x . x = b .$

$[*13{\cdot}195] \quad \equiv . \phi b \quad (2)$

$\vdash . (2) . *5{\cdot}32 . \supset \vdash :. \phi x . \supset_x . x = b : (\exists x) . \phi x : \equiv : \phi x . \supset_x . x = b : \phi b \quad (3)$

$\vdash . (1) . (3) . \quad \supset \vdash . \mathrm{Prop}$

The two following propositions ($*14{\cdot}123{\cdot}124$) are placed here because of the analogy with $*14{\cdot}122$, but they are not used until we come to the theory of couples ($*55$ and $*56$).

$*14{\cdot}123$. $\vdash :. \phi (z, w) . \equiv_{z,w} . z = x . w = y :$

$\equiv : \phi (z, w) . \supset_{z,w} . z = x . w = y : \phi (x, y) :$

$\equiv : \phi (z, w) . \supset_{z,w} . z = x . w = y : (\exists z, w) . \phi (z, w)$

Dem.

$\vdash . *11{\cdot}31 . \quad \supset \vdash :. \phi (z, w) . \equiv_{z,w} . z = x . w = y :$

$\equiv : \phi (z, w) . \supset_{z,w} . z = x . w = y : z = x . w = y . \supset_{z,w} . \phi (z, w) :$

$[*13{\cdot}21] \quad \equiv : \phi (z, w) . \supset_{z,w} . z = x . w = y : \phi (x, y) \quad (1)$

$\vdash . *4{\cdot}71 . \quad \supset \vdash :. \phi (z, w) . \supset . z = x . w = y :$

$\supset : \phi (z, w) . \equiv . \phi (z, w) . z = x . w = y :.$

$[*11{\cdot}11{\cdot}32] \quad \supset \vdash :. \phi (z, w) . \supset_{z,w} . z = x . w = y :$

$\supset : \phi (z, w) . \equiv_{z,w} . \phi (z, w) . z = x . w = y :$

$[*11{\cdot}341] \quad \supset : (\exists z, w) . \phi (z, w) . \equiv . (\exists z, w) . \phi (z, w) . z = x . w = y .$

$[*13{\cdot}22] \quad \equiv . \phi (x, y) \quad (2)$

$\vdash . (2) . *5{\cdot}32 . \supset \vdash :. \phi (z, w) . \supset_{z,w} . z = x . w = y : (\exists z, w) . \phi (z, w) :$

$\equiv : \phi (z, w) . \supset_{z,w} . z = x . w = y : \phi (x, y) \quad (3)$

$\vdash . (1) . (3) . \quad \supset \vdash . \mathrm{Prop}$

$*14{\cdot}124$. $\vdash :. (\exists x, y) : \phi (z, w) . \equiv_{z,w} . z = x . w = y :$

$\equiv : (\exists x, y) . \phi (x, y) : \phi (z, w) . \phi (u, v) . \supset_{z,w,u,v} . z = u . w = v$

Dem.

$\vdash . *14{\cdot}123 . *3{\cdot}27 . \supset$

$\vdash :. (\exists x, y) : \phi (z, w) . \equiv_{z,w} . z = x . w = y : \supset . (\exists x, y) . \phi (x, y) \quad (1)$

$\vdash . *11{\cdot}1 . *3{\cdot}47 . \supset \vdash :. \phi (z, w) . \equiv_{z,w} . z = x . w = y :$

$\supset : \phi (z, w) . \phi (u, v) . \supset . z = x . w = y . u = x . v = y .$

$[*13{\cdot}172] \quad \supset . z = u . w = v \quad (2)$

$\vdash . (2) . *11{\cdot}11{\cdot}35 . \supset$

$\vdash :. (\exists x, y) : \phi (z, w) . \equiv_{z,w} . z = x . w = y :$

$\supset : \phi (z, w) . \phi (u, v) . \supset . z = u . w = v \quad (3)$

$\vdash . (3) . *11{\cdot}11{\cdot}3 . \supset$

$\vdash :. (\exists x, y) : \phi(z, w) . \equiv_{z,w} . z = x . w = y :$

$\supset : \phi(z, w) . \phi(u, v) . \supset_{z,w,u,v} . z = u . w = v$ (4)

$\vdash . *11{\cdot}1 . \supset \vdash :. \phi(x, y) : \phi(z, w) . \phi(u, v) . \supset_{z,w,u,v} . z = u . w = v :$

$\supset : \phi(x, y) : \phi(z, w) . \phi(x, y) . \supset_{z,w} . z = x . w = y :$

[$*5{\cdot}33$] $\supset : \phi(x, y) : \phi(z, w) . \supset_{z,w} . z = x . w = y :$

[$*14{\cdot}123$] $\supset : \phi(z, w) . \equiv_{z,w} . z = x . w = y$ (5)

$\vdash . (5) . *11{\cdot}11{\cdot}34{\cdot}45 . \supset$

$\vdash :. (\exists x, y) . \phi(x, y) : \phi(z, w) . \phi(u, v) . \supset_{z,w,u,v} . z = u . w = v :$

$\supset : (\exists x, y) : \phi(z, w) . \equiv_{z,w} . z = x . w = y$ (6)

$\vdash . (1) . (4) . (6) . \supset \vdash .$ Prop

$*14{\cdot}13$. $\vdash : a = (\imath x)(\phi x) . \equiv . (\imath x)(\phi x) = a$

Dem.

$\vdash . *14{\cdot}1 . \quad \supset \vdash :. a = (\imath x)(\phi x) . \equiv : (\exists b) : \phi x . \equiv_x . x = b : a = b$ (1)

$\vdash . *13{\cdot}16 . *4{\cdot}36 . \supset \vdash :. \phi x . \equiv_x . x = b : a = b : \equiv : \phi x . \equiv_x . x = b : b = a :$

[$*10{\cdot}11{\cdot}281$] $\supset \vdash :. (\exists b) : \phi x . \equiv_x . x = b : a = b :$

$\equiv : (\exists b) : \phi x . \equiv_x . x = b : b = a :$

[$*14{\cdot}1$] $\equiv : (\imath x)(\phi x) = a$ (2)

$\vdash . (1) . (2) . \quad \supset \vdash .$ Prop

This proposition is not an *immediate* consequence of $*13{\cdot}16$, because "$a = (\imath x)(\phi x)$" is not a value of the function "$x = y$." Similar remarks apply to the following propositions.

$*14{\cdot}131$. $\vdash : (\imath x)(\phi x) = (\imath x)(\psi x) . \equiv . (\imath x)(\psi x) = (\imath x)(\phi x)$

Dem.

$\vdash . *14{\cdot}1 . \supset \vdash :: (\imath x)(\phi x) = (\imath x)(\psi x) . \equiv :. (\exists b) : \phi x . \equiv_x . x = b : b = (\imath x)(\psi x) :.$

[$*14{\cdot}1$] $\equiv :. (\exists b) :. \phi x . \equiv_x . x = b :. (\exists c) : \psi x . \equiv_x . x = c : b = c :.$

[$*11{\cdot}6$] $\equiv :. (\exists c) :. \psi x . \equiv_x . x = c :. (\exists b) : \phi x . \equiv_x . x = b : b = c :.$

[$*14{\cdot}1$] $\equiv :. (\exists c) :. \psi x . \equiv_x . x = c : (\imath x)(\phi x) = c :.$

[$*14{\cdot}13$] $\equiv :. (\exists c) :. \psi x . \equiv_x . x = c : c = (\imath x)(\phi x) :.$

[$*14{\cdot}1$] $\equiv :. (\imath x)(\psi x) = (\imath x)(\phi x) :: \supset \vdash .$ Prop

In the above proposition, in accordance with our convention, the descriptive expression $(\imath x)(\phi x)$ is eliminated before $(\imath x)(\psi x)$, because it occurs first in "$(\imath x)(\phi x) = (\imath x)(\psi x)$"; but in "$(\imath x)(\psi x) = (\imath x)(\phi x)$," $(\imath x)(\psi x)$ is to be first eliminated. The order of elimination makes no difference to the truth-value, as was proved in $*14{\cdot}113$.

The above proposition may also be proved as follows:

$\vdash . *14{\cdot}111 . \supset \vdash :. (\imath x)(\phi x) = (\imath x)(\psi x) .$

$\equiv : (\exists b, c) : \phi x . \equiv_x . x = b : \psi x . \equiv_x . x = c : b = c :$

[$*4{\cdot}3 . *13{\cdot}16 . *11{\cdot}11{\cdot}341$] $\equiv : (\exists b, c) : \psi x . \equiv_x . x = c : \phi x . \equiv_x . x = b : c = b :$

[$*11{\cdot}2 . *14{\cdot}111$] $\equiv : (\imath x)(\psi x) = (\imath x)(\phi x) :. \supset \vdash .$ Prop

***14·14.** $\vdash : a = b \,.\, b = (\iota x)(\phi x) \,.\, \supset \,.\, a = (\iota x)(\phi x)$ [*13·13]

***14·142.** $\vdash : a = (\iota x)(\phi x) \,.\, (\iota x)(\phi x) = (\iota x)(\psi x) \,.\, \supset \,.\, a = (\iota x)(\psi x)$

Dem.

$\vdash \,.\, *14·1 \,.\, \supset \vdash :: \mathrm{Hp} \,.\, \supset :. (\exists b) : \phi x \,.\, \equiv_x \,.\, x = b : a = b :.$

$(\exists c) : \phi x \,.\, \equiv_x \,.\, x = c : c = (\iota x)(\psi x) :.$

[*13·195] $\supset :. \phi x \,.\, \equiv_x \,.\, x = a :. (\exists c) : \phi x \,.\, \equiv_x \,.\, x = c : c = (\iota x)(\psi x) :.$

[*10·35] $\supset :. (\exists c) :. \phi x \,.\, \equiv_x \,.\, x = a : \phi x \,.\, \equiv_x \,.\, x = c : c = (\iota x)(\psi x) :.$

[*14·121] $\supset :. (\exists c) :. \phi x \,.\, \equiv_x \,.\, x = a : a = c : c = (\iota x)(\psi x) :.$

[*3·27.*13·195] $\supset :. a = (\iota x)(\psi x) :: \supset \vdash \,.\, \mathrm{Prop}$

***14·144.** $\vdash : (\iota x)(\phi x) = (\iota x)(\psi x) \,.\, (\iota x)(\psi x) = (\iota x)(\chi x) \,.\, \supset \,.\, (\iota x)(\phi x) = (\iota x)(\chi x)$

Dem.

$\vdash \,.\, *14·111 \,.\, \supset \vdash :: \mathrm{Hp} \,.\, \supset :. (\exists a, b) : \phi x \,.\, \equiv_x \,.\, x = a : \psi x \,.\, \equiv_x \,.\, x = b : a = b :.$

$(\exists c, d) : \psi x \,.\, \equiv_x \,.\, x = c : \chi x \,.\, \equiv_x \,.\, x = d : c = d :.$

[*13·195] $\supset :. (\exists a) : \phi x \,.\, \equiv_x \,.\, x = a : \psi x \,.\, \equiv_x \,.\, x = a :.$

$(\exists c) : \psi x \,.\, \equiv_x \,.\, x = c : \chi x \,.\, \equiv_x \,.\, x = c :.$

[*11·54] $\supset :. (\exists a, c) : \phi x \,.\, \equiv_x \,.\, x = a : \psi x \,.\, \equiv_x \,.\, x = a :$

$\psi x \,.\, \equiv_x \,.\, x = c : \chi x \,.\, \equiv_x \,.\, x = c :.$

[*14·121.*11·42] $\supset :. (\exists a, c) : \phi x \,.\, \equiv_x \,.\, x = a : \chi x \,.\, \equiv_x \,.\, x = c : a = c :.$

[*14·111] $\supset :. (\iota x)(\phi x) = (\iota x)(\chi x) :: \supset \vdash \,.\, \mathrm{Prop}$

***14·145.** $\vdash : a = (\iota x)(\phi x) \,.\, a = (\iota x)(\psi x) \,.\, \supset \,.\, (\iota x)(\phi x) = (\iota x)(\psi x)$

Dem.

$\vdash \,.\, *14·1 \,.\, \supset \vdash :. a = (\iota x)(\phi x) \,.\, \equiv : (\exists b) : \phi x \,.\, \equiv_x \,.\, x = b : a = b :$

[*13·195] $\equiv : \phi x \,.\, \equiv_x \,.\, x = a$ (1)

$\vdash \,.\, (1) \,.\, *14·1 \,.\, \supset \vdash :: \mathrm{Hp} \,.\, \equiv :. \phi x \,.\, \equiv_x \,.\, x = a :. (\exists b) : \psi x \,.\, \equiv_x \,.\, x = b : a = b :.$

[*10·35] $\equiv :. (\exists b) :. \phi x \,.\, \equiv_x \,.\, x = a : \psi x \,.\, \equiv_x \,.\, x = b : a = b :.$

[*14·111] $\supset :. (\iota x)(\phi x) = (\iota x)(\psi x) :: \supset \vdash \,.\, \mathrm{Prop}$

***14·15.** $\vdash :. (\iota x)(\phi x) = b \,.\, \supset : \psi \{(\iota x)(\phi x)\} \,.\, \equiv \,.\, \psi b$

Dem.

$\vdash \,.\, *14·1 \,.\, \supset$

$\vdash :: \mathrm{Hp} \,.\, \supset :. (\exists c) : \phi x \,.\, \equiv_x \,.\, x = c : c = b :.$

[*13·195] $\supset :. \phi x \,.\, \equiv_x \,.\, x = b$ (1)

$\vdash \,.\, (1) \,.\, *14·1 \,.\, \supset$

$\vdash :: \mathrm{Hp} \,.\, \supset :. \psi \{(\iota x)(\phi x)\} \,.\, \equiv : (\exists c) : x = b \,.\, \equiv_x \,.\, x = c : \psi c :$

[*13·192] $\equiv : \psi b :: \supset \vdash \,.\, \mathrm{Prop}$

***14·16.** $\vdash :. (\iota x)(\phi x) = (\iota x)(\psi x) \,.\, \supset : \chi \{(\iota x)(\phi x)\} \,.\, \equiv \,.\, \chi \{(\iota x)(\psi x)\}$

Dem.

$\vdash \,.\, *14·1 \,.\, \supset \vdash :. \mathrm{Hp} \,.\, \supset : (\exists b) : \phi x \,.\, \equiv_x \,.\, x = b : b = (\iota x)(\psi x)$ (1)

$\vdash \,.\, *14·1 \,.\, \supset \vdash :: \phi x \,.\, \equiv_x \,.\, x = b : \supset :.$

$\chi \{(\iota x)(\phi x)\} \,.\, \equiv : (\exists c) : x = b \,.\, \equiv_x \,.\, x = c : \chi c :$

[*13·192] $\equiv : \chi b$ (2)

$\vdash . *14 \cdot 13 \cdot 15 . \supset \vdash :. b = (\imath x)(\psi x) . \supset : \chi b . \equiv . \chi \{(\imath x)(\psi x)\}$ (3)

$\vdash . (2) . (3) . \quad \supset \vdash :. \phi x . \equiv_x . x = b : b = (\imath x)(\psi x) :$

$\supset : \chi \{(\imath x)(\phi x)\} . \equiv . \chi \{(\imath x)(\psi x)\}$ (4)

$\vdash . (1) . (4) . *10 \cdot 1 \cdot 23 . \supset \vdash .$ Prop

$*14 \cdot 17$. $\vdash :. (\imath x)(\phi x) = b . \equiv : \psi ! (\imath x)(\phi x) . \equiv_\psi . \psi ! b$

Dem.

$\vdash . *14 \cdot 15 . *10 \cdot 11 \cdot 21 . \supset$

$\vdash :. (\imath x)(\phi x) = b . \supset : \psi ! (\imath x)(\phi x) . \equiv_\psi . \psi ! b$ (1)

$\vdash . *10 \cdot 1 . *4 \cdot 22 . \supset \vdash :: \chi ! x . \equiv_x . x = b : \psi ! (\imath x)(\phi x) . \equiv_\psi . \psi ! b :$

$\supset : (\imath x)(\phi x) = b . \equiv . b = b :$

[$*13 \cdot 15$] $\supset : (\imath x)(\phi x) = b$ (2)

$\vdash . (2) . \text{Exp} . *10 \cdot 11 \cdot 23 . \supset$

$\vdash :: (\exists \chi) : \chi ! x . \equiv_x . x = b : \supset :. \psi ! (\imath x)(\phi x) . \equiv_\psi . \psi ! b : \supset . (\imath x)(\phi x) = b$ (3)

$\vdash . *12 \cdot 1 . \quad \supset \vdash : (\exists \chi) : \chi ! x . \equiv_x . x = b$ (4)

$\vdash . (3) . (4) . \supset \vdash :. \psi ! (\imath x)(\phi x) . \equiv_\psi . \psi ! b : \supset . (\imath x)(\phi x) = b$ (5)

$\vdash . (1) . (5) . \supset \vdash .$ Prop

It should be observed that we do *not* have

$$(\imath x)(\phi x) = b . \equiv : \psi ! (\imath x)(\phi x) . \supset_\psi . \psi ! b$$

for, if $\sim E ! (\imath x)(\phi x)$, $\psi ! (\imath x)(\phi x)$ is always false, and therefore

$$\psi ! (\imath x)(\phi x) . \supset_\psi . \psi ! b$$

holds for all values of b. But we do have

$*14 \cdot 171$. $\vdash :. (\imath x)(\phi x) = b . \equiv : \psi ! b . \supset_\psi . \psi ! (\imath x)(\phi x)$

Dem.

$\vdash . *14 \cdot 17 . \qquad \supset \vdash :. (\imath x)(\phi x) = b . \supset : \psi ! b . \supset_\psi . \psi ! (\imath x)(\phi x)$ (1)

$\vdash . *10 \cdot 1 . *12 \cdot 1 . \supset \vdash :. \psi ! b . \supset_\psi . \psi ! (\imath x)(\phi x) : \supset : b = b . \supset . (\imath x)(\phi x) = b :$

[$*13 \cdot 15$] $\supset : (\imath x)(\phi x) = b$ (2)

$\vdash . (1) . (2) . \qquad \supset \vdash .$ Prop

$*14 \cdot 18$. $\vdash :. E ! (\imath x)(\phi x) . \supset : (x) . \psi x . \supset . \psi (\imath x)(\phi x)$

Dem.

$\vdash . *10 \cdot 1 . \quad \supset \vdash : (x) . \psi x . \supset . \psi b :$

[Fact] $\supset \vdash :. \phi x . \equiv_x . x = b : (x) . \psi x : \supset : \phi x . \equiv_x . x = b : \psi b :$

[$*10 \cdot 11 \cdot 28$] $\supset \vdash :. (\exists b) : \phi x . \equiv_x . x = b : (x) . \psi x : \supset : (\exists b) : \phi x . \equiv_x . x = b : \psi b :.$

[$*10 \cdot 35$] $\supset \vdash :: (\exists b) : \phi x . \equiv_x . x = b :. (x) . \psi x :. \supset : (\exists b) : \phi x . \equiv_x . x = b : \psi b :.$

[$*14 \cdot 1 \cdot 11$] $\supset \vdash :. E ! (\imath x)(\phi x) : (x) . \psi x : \supset : \psi (\imath x)(\phi x) :. \supset \vdash .$ Prop

The above proposition shows that, provided $(\imath x)(\phi x)$ exists, it has (speaking formally) all the logical properties of symbols which directly represent objects. Hence when $(\imath x)(\phi x)$ exists, the fact that it is an incomplete symbol becomes irrelevant to the truth-values of logical propositions in which it occurs.

$*14\cdot2.\quad \vdash . (\iota x)(x=a)=a$

Dem.

$\vdash . *14\cdot101 . \supset \vdash :. (\iota x)(x=a)=a . \equiv : (\exists b) : x=a . \equiv_x . x=b : b=a :$

$[*13\cdot195] \qquad \equiv : x=a . \equiv_x . x=a \qquad (1)$

$\vdash . (1) . \mathrm{Id} . \supset \vdash . \mathrm{Prop}$

$*14\cdot201.\quad \vdash : E ! (\iota x)(\phi x) . \supset . (\exists x) . \phi x$

Dem.

$\vdash . *14\cdot11 . \supset \vdash :. \mathrm{Hp} . \supset : (\exists b) : \phi x . \equiv_x . x=b :$

$[*10\cdot1] \qquad \supset : (\exists b) : \phi b . \equiv . b=b :$

$[*13\cdot15] \qquad \supset : (\exists b) . \phi b :. \supset \vdash . \mathrm{Prop}$

$*14\cdot202.\quad \vdash :. \phi x . \equiv_x . x=b : \equiv : (\iota x)(\phi x)=b : \equiv : \phi x . \equiv_x . b=x : \equiv : b=(\iota x)(\phi x)$

Dem.

$\vdash . *14\cdot1 . \supset \vdash :. (\iota x)(\phi x)=b . \equiv : (\exists c) : \phi x . \equiv_x . x=c : c=b :$

$[*13\cdot195] \qquad \equiv : \phi x . \equiv_x . x=b :. \supset \vdash . \mathrm{Prop}$

[The second half is proved in the same way as the first half.]

$*14\cdot203.\quad \vdash :. E ! (\iota x)(\phi x) . \equiv : (\exists x) . \phi x : \phi x . \phi y . \supset_{x,y} . x=y$

Dem.

$\vdash . *14\cdot12\cdot201 . \quad \supset \vdash :. E ! (\iota x)(\phi x) . \supset : (\exists x) . \phi x : \phi x . \phi y . \supset_{x,y} . x=y \qquad (1)$

$\vdash . *10\cdot1 . \quad \supset \vdash :. \phi b : \phi x . \phi y . \supset_{x,y} . x=y : \supset : \phi b : \phi x . \phi b . \supset_x . x=b :$

$[*5\cdot33] \qquad \supset : \phi b : \phi x . \supset_x . x=b :$

$[*13\cdot191] \qquad \supset : x=b . \supset_x . \phi x :$

$\qquad \phi x . \supset_x . x=b :$

$[*10\cdot22] \qquad \supset : \phi x . \equiv_x . x=b \qquad (2)$

$\vdash . (2) . *10\cdot1\cdot28 . \supset \vdash :. (\exists b) : \phi b : \phi x . \phi y . \supset_{x,y} . x=y : \supset : (\exists b) : \phi x . \equiv_x . x=b :.$

$[*10\cdot35] \qquad \supset \vdash :. (\exists b) . \phi b : \phi x . \phi y . \supset_{x,y} . x=y : \supset : (\exists b) : \phi x . \equiv_x . x=b :$

$[*14\cdot11] \qquad \supset : E ! (\iota x)(\phi x) \qquad (3)$

$\vdash . (1) . (3) . \quad \supset \vdash . \mathrm{Prop}$

$*14\cdot204.\quad \vdash :. E ! (\iota x)(\phi x) . \equiv : (\exists b) . (\iota x)(\phi x)=b$

Dem.

$\vdash . *14\cdot202 . *10\cdot11 . \supset$

$\vdash :. (b) :. \phi x . \equiv_x . x=b : \equiv : (\iota x)(\phi x)=b :. \supset$

$[*10\cdot281] \vdash :. (\exists b) : \phi x . \equiv_x . x=b : \equiv : (\exists b) . (\iota x)(\phi x)=b \qquad (1)$

$\vdash . (1) . *14\cdot11 . \supset \vdash . \mathrm{Prop}$

$*14\cdot205.\quad \vdash : \psi (\iota x)(\phi x) . \equiv . (\exists b) . b=(\iota x)(\phi x) . \psi b \quad [*14\cdot202\cdot1]$

$*14\cdot21.\quad \vdash : \psi (\iota x)(\phi x) . \supset . E ! (\iota x)(\phi x)$

Dem.

$\vdash . *14\cdot1 . \supset$

$\vdash :. \psi \{(\iota x)(\phi x)\} . \supset : (\exists b) : \phi x . \equiv_x . x=b : \psi b :$

$[*10\cdot5] \qquad \supset : (\exists b) : \phi x . \equiv_x . x=b :$

$[*14\cdot11] \qquad \supset : E ! (\iota x)(\phi x) :. \supset \vdash . \mathrm{Prop}$

This proposition shows that if any true statement can be made about $(\imath x)(\phi x)$, then $(\imath x)(\phi x)$ must exist. Its use throughout the remainder of the work will be very frequent.

When $(\imath x)(\phi x)$ does not exist, there are still true propositions in which "$(\imath x)(\phi x)$" occurs, but it has, in such propositions, a *secondary* occurrence, in the sense explained in Chapter III of the Introduction, *i.e.* the asserted proposition concerned is not of the form $\psi(\imath x)(\phi x)$, but of the form $f\{\psi(\imath x)(\phi x)\}$, in other words, the proposition which is the scope of $(\imath x)(\phi x)$ is only part of the whole asserted proposition.

***14·22.** $\vdash : E!(\imath x)(\phi x) . \equiv . \phi(\imath x)(\phi x)$

Dem.

$\vdash . *14\cdot122 . \quad \supset \vdash :. \phi x . \equiv_x . x = b : \supset . \phi b \qquad (1)$

$\vdash . (1) . *4\cdot71 . \supset \vdash :. \phi x . \equiv_x . x = b : \equiv : \phi x . \equiv_x . x = b : \phi b :.$

$[*10\cdot11\cdot281] \quad \supset \vdash :. (\exists b) : \phi x . \equiv_x . x = b : \equiv : (\exists b) : \phi x . \equiv_x . x = b : \phi b :.$

$[*14\cdot11\cdot101] \quad \supset \vdash : E!(\imath x)(\phi x) . \equiv . \phi(\imath x)(\phi x) : \supset \vdash . \text{Prop}$

As an instance of the above proposition, we may take the following: "The proposition 'the author of Waverley existed' is equivalent to 'the man who wrote Waverley wrote Waverley.'" Thus such a proposition as "the man who wrote Waverley wrote Waverley" does not embody a logically necessary truth, since it would be false if Waverley had not been written, or had been written by two men in collaboration. For example, "the man who squared the circle squared the circle" is a false proposition.

***14·23.** $\vdash : E!(\imath x)(\phi x . \psi x) . \equiv . \phi\{(\imath x)(\phi x . \psi x)\}$

Dem.

$\vdash . *14\cdot22 . \supset \vdash :. E!(\imath x)(\phi x . \psi x) .$

$\equiv : [(\imath x)(\phi x . \psi x)] : \phi\{(\imath x)(\phi x . \psi x)\} . \psi\{(\imath x)(\phi x . \psi x)\}$

$[*10\cdot5 . *3\cdot26] \quad \supset : \phi\{(\imath x)(\phi x . \psi x)\} \qquad (1)$

$\vdash . *14\cdot21 . \supset \vdash : \phi\{(\imath x)(\phi x . \psi x)\} . \supset . E!(\imath x)(\phi x . \psi x) \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

Note that in the second line of the above proof *10·5, not only *3·26, is required. For the scope of the descriptive symbol $(\imath x)(\phi x . \psi x)$ is the whole product $\phi\{(\imath x)(\phi x . \psi x)\} . \psi\{(\imath x)(\phi x . \psi x)\}$, so that, applying *14·1, the proposition on the right in the first line becomes

$$(\exists b) : \phi x . \psi x . \equiv_x . x = b : \phi b . \psi b$$

which, by *10·5 and *3·26, implies

$$(\exists b) : \phi x . \psi x . \equiv_x . x = b : \phi b,$$

i.e. $\qquad \phi\{(\imath x)(\phi x . \psi x)\}.$

***14·24.** $\vdash :. E!(\imath x)(\phi x) . \equiv : [(\imath x)(\phi x)] : \phi y . \equiv_y . y = (\imath x)(\phi x)$

Dem.

$\vdash . *14\cdot1 . \supset \vdash :. [(\imath x)(\phi x)] : \phi y . \equiv_y . y = (\imath x)(\phi x) :$

$\equiv : (\exists b) : \phi y . \equiv_y . y = b : \phi y . \equiv_y . y = b :$

[$*4{\cdot}24 . *10{\cdot}281$] $\quad \equiv : (\exists b) : \phi y . \equiv_y . y = b :$

[$*14{\cdot}11$] $\quad \equiv : E ! (\iota x)(\phi x) :. \supset \vdash . \text{Prop}$

This proposition should be compared with $*14{\cdot}241$, where, in virtue of the smaller scope of $(\iota x)(\phi x)$, we get an implication instead of an equivalence.

$*$**14·241**. $\vdash :. E ! (\iota x)(\phi x) . \supset : \phi y . \equiv_y . y = (\iota x)(\phi x)$

Dem.

$\vdash . *14{\cdot}203 . \supset \vdash :: \text{Hp} . \supset :. \phi y . \phi x . \supset . y = x :.$

[Exp] $\quad \supset :. \phi y . \supset : \phi x . \supset . y = x ::$

[$*10{\cdot}11{\cdot}21$] $\supset \vdash :: \text{Hp} . \supset :. \phi y . \supset : \phi x . \supset_x . y = x :.$

[$*4{\cdot}71$] $\quad \supset :. \phi y . \equiv : \phi y : \phi x . \supset_x . y = x :$

[$*13{\cdot}191$] $\quad \equiv : y = x . \supset_x . \phi x : \phi x . \supset_x . y = x :$

[$*10{\cdot}22$] $\quad \equiv : \phi x . \equiv_x . y = x :$

[$*14{\cdot}202$] $\quad \equiv : y = (\iota x)(\phi x) :: \supset \vdash . \text{Prop}$

$*$**14·242**. $\vdash :. \phi x . \equiv_x . x = b : \supset : \psi b . \equiv . \psi (\iota x)(\phi x)$ [$*14{\cdot}202{\cdot}15$]

$*$**14·25**. $\vdash :. E ! (\iota x)(\phi x) . \supset : \phi x \supset_x \psi x . \equiv . \psi (\iota x)(\phi x)$

Dem.

$\vdash . *4{\cdot}84 . *10{\cdot}27{\cdot}271 . \supset \vdash :: \phi x . \equiv_x . x = b : \supset :. \phi x \supset_x \psi x . \equiv : x = b . \supset_x . \psi x :$

[$*13{\cdot}191$] $\quad \equiv : \psi b :$

[$*14{\cdot}242$] $\quad \equiv . \psi (\iota x)(\phi x) \quad (1)$

$\vdash . (1) . *10{\cdot}11{\cdot}23 . \supset \vdash :. (\exists b) : \phi x . \equiv_x . x = b :$

$\quad \supset : \phi x \supset_x \psi x . \equiv . \psi (\iota x)(\phi x) \quad (2)$

$\vdash . (2) . *14{\cdot}11 . \quad \supset \vdash . \text{Prop}$

$*$**14·26**. $\vdash :. E ! (\iota x)(\phi x) . \supset : (\exists x) . \phi x . \psi x . \equiv . \psi \{(\iota x)(\phi x)\} . \equiv . \phi x \supset_x \psi x$

Dem.

$\vdash . *14{\cdot}11 . \supset$

$\vdash :. \text{Hp} . \supset : (\exists b) : \phi x . \equiv_x . x = b \quad (1)$

$\vdash . *10{\cdot}311 . \supset \vdash :: \phi x . \equiv_x . x = b : \supset :. \phi x . \psi x . \equiv_x . x = b . \psi x :.$

[$*10{\cdot}281$] $\quad \supset :. (\exists x) . \phi x . \psi x . \equiv . (\exists x) . x = b . \psi x .$

[$*13{\cdot}195$] $\quad \equiv . \psi b .$

[$*14{\cdot}242$] $\quad \equiv . \psi \{(\iota x)(\phi x)\} \quad (2)$

$\vdash . (2) . *10{\cdot}11{\cdot}23 . \supset$

$\vdash :. (\exists b) : \phi x . \equiv_x . x = b : \supset : (\exists x) . \phi x . \psi x . \equiv . \psi \{(\iota x)(\phi x)\} \quad (3)$

$\vdash . (1) . (3) . *14{\cdot}25 . \supset \vdash . \text{Prop}$

$*$**14·27**. $\vdash :. E ! (\iota x)(\phi x) . \supset : \phi x \equiv_x \psi x . \equiv . (\iota x)(\phi x) = (\iota x)(\psi x)$

Dem.

$\vdash . *4{\cdot}86{\cdot}21 . \quad \supset \vdash :: \phi x . \equiv \ . x = b : \supset :. \phi x . \equiv . \psi x : \equiv : \psi x . \equiv . x = b \quad (1)$

$\vdash . (1) . *10{\cdot}11{\cdot}27 . \supset \vdash :: \phi x . \equiv_x . x = b : \supset :. (x) :. \phi x . \equiv . \psi x : \equiv : \psi x . \equiv . x = b :.$

[$*10{\cdot}271$] $\quad \supset :. \phi x . \equiv_x . \psi x : \equiv : \psi x . \equiv_x . x = b :$

[$*14{\cdot}202$] $\quad \equiv : b = (\iota x)(\psi x) :$

[$*14{\cdot}242$] $\quad \equiv : (\iota x)(\phi x) = (\iota x)(\psi x) \quad (2)$

$\vdash . (2) . *10{\cdot}11{\cdot}23 . *14{\cdot}11 . \supset \vdash . \text{Prop}$

***14·271.** $\vdash :. \phi x . \equiv_x . \psi x : \supset : E ! (\imath x)(\phi x) . \equiv . E ! (\imath x)(\psi x)$

Dem.

$\vdash . *4·86 . \supset \vdash :: \phi x \equiv \psi x . \supset :. \phi x . \equiv . x = b : \equiv : \psi x . \equiv . x = b ::$

$[*10·11·27] \supset \vdash :: Hp . \supset :. (x) :. \phi x . \equiv . x = b : \equiv : \psi x . \equiv . x = b :.$

$[*10·271] \supset :. (x) : \phi x . \equiv . x = b : \equiv : (x) : \psi x . \equiv . x = b ::$

$[*10·11·21] \supset \vdash :: Hp . \supset :. (b) :. \phi x . \equiv_x . x = b : \equiv : \psi x . \equiv_x . x = b :.$

$[*10·281] \supset :. (\exists b) : \phi x . \equiv_x . x = b : \equiv : (\exists b) : \psi x . \equiv_x . x = b ::$

$\supset \vdash . Prop$

***14·272.** $\vdash :. \phi x . \equiv_x . \psi x : \supset : \chi (\imath x)(\phi x) . \equiv . \chi (\imath x)(\psi x)$

Dem.

$\vdash . *4·86 . \supset \vdash :: \phi x \equiv \psi x . \supset :. \phi x . \equiv . x = b : \equiv : \psi x . \equiv . x = b :.$

$[*10·11·414] \supset \vdash :: Hp . \supset :. \phi x . \equiv_x . x = b : \equiv : \psi x . \equiv_x . x = b :.$

$[Fact] \supset :. \phi x . \equiv_x . x = b : \chi b : \equiv : \psi x . \equiv_x . x = b : \chi b :.$

$[*10·11·21] \supset \vdash :: Hp . \supset :. (b) :. \phi x . \equiv_x . x = b : \chi b : \equiv : \psi x . \equiv_x . x = b : \chi b :.$

$[*10·281] \supset :. (\exists b) : \phi x . \equiv_x . x = b : \chi b : \equiv$

$: (\exists b) : \psi x . \equiv_x . x = b : \chi b :.$

$[*14·101] \supset :. \chi (\imath x)(\phi x) . \equiv . \chi (\imath x)(\psi x) :: \supset \vdash . Prop$

The above two propositions show that $E ! (\imath x)(\phi x)$ and $\chi (\imath x)(\phi x)$ are "extensional" properties of $\phi \hat{x}$, *i.e.* their truth-value is unchanged by the substitution, for $\phi \hat{x}$, of any formally equivalent function $\psi \hat{x}$.

***14·28.** $\vdash : E ! (\imath x)(\phi x) . \equiv . (\imath x)(\phi x) = (\imath x)(\phi x)$

Dem.

$\vdash . *13·15 . *4·73 . \supset \vdash :. \phi x . \equiv_x . x = b : \equiv : \phi x . \equiv_x . x = b : b = b \quad (1)$

$\vdash . (1) . *10·11·281 . \supset$

$\vdash :. (\exists b) : \phi x . \equiv_x . x = b : \equiv : (\exists b) : \phi x . \equiv_x . x = b : b = b \quad (2)$

$\vdash . (2) . *14·1·11 . \supset \vdash . Prop$

This proposition states that $(\imath x)(\phi x)$ is identical with itself whenever it exists, but not otherwise. Thus for example the proposition "the present King of France is the present King of France" is false.

The purpose of the following propositions is to show that, when $E ! (\imath x)(\phi x)$, the scope of $(\imath x)(\phi x)$ does not matter to the truth-value of any proposition in which $(\imath x)(\phi x)$ occurs. This proposition cannot be proved generally, but it can be proved in each particular case. The following propositions show the method, which proceeds always by means of *14·242, *10·23 and *14·11. The proposition can be proved generally when $(\imath x)(\phi x)$ occurs in the form $\chi (\imath x)(\phi x)$, and $\chi (\imath x)(\phi x)$ occurs in what we may call a "truth-function," *i.e.* a function whose truth or falsehood depends only upon the truth or falsehood of its argument or arguments. This covers all the cases with which we are ever concerned. That is to say, if $\chi (\imath x)(\phi x)$ occurs in any of the ways which can be generated by the processes of *1—*11, then, provided $E ! (\imath x)(\phi x)$, the truth-value of $f \{[(\imath x)(\phi x)] . \chi (\imath x)(\phi x)\}$ is the same as that of

$$[(\imath x)(\phi x)] . f \{\chi (\imath x)(\phi x)\}.$$

This is proved in the following proposition. In this proposition, however, the use of propositions as apparent variables involves an apparatus not required elsewhere, and we have therefore not used this proposition in subsequent proofs.

***14·3.** $\vdash :. p \equiv q . \supset_{p,q} . f(p) \equiv f(q) : \mathrm{E}\,!\,(\imath x)(\phi x) : \supset :$
$f\{[(\imath x)(\phi x)] . \chi(\imath x)(\phi x)\} . \equiv . [(\imath x)(\phi x)] . f\{\chi(\imath x)(\phi x)\}$

Dem.

$\vdash . *14{\cdot}242 . \supset$

$\vdash :. \phi x . \equiv_x . x = b : \supset : [(\imath x)(\phi x)] . \chi(\imath x)(\phi x) . \equiv . \chi b$ (1)

$\vdash . (1) . \supset \vdash :. p \equiv q . \supset_{p,q} . f(p) \equiv f(q) : \phi x . \equiv_x . x = b : \supset :$
$f\{[(\imath x)(\phi x)] . \chi(\imath x)(\phi x)\} . \equiv . f(\chi b)$ (2)

$\vdash . *14{\cdot}242 . \supset$

$\vdash :. \phi x . \equiv_x . x = b : \supset : [(\imath x)(\phi x)] . f\{\chi(\imath x)(\phi x)\} . \equiv . f(\chi b)$ (3)

$\vdash . (2) . (3) . \supset$

$\vdash :. p \equiv q . \supset_{p,q} . f(p) \equiv f(q) : \phi x . \equiv_x . x = b : \supset :$
$f\{[(\imath x)(\phi x)] . \chi(\imath x)(\phi x)\} . \equiv . [(\imath x)(\phi x)] . f\{\chi(\imath x)(\phi x)\}$ (4)

$\vdash . (4) . *10{\cdot}23 . *14{\cdot}11 . \supset \vdash . \mathrm{Prop}$

The following propositions are immediate applications of the above. They are, however, independently proved, because *14·3 introduces propositions (p, q namely) as apparent variables, which we have not done elsewhere, and cannot do legitimately without the explicit introduction of the hierarchy of propositions with a reducibility-axiom such as *12·1.

***14·31.** $\vdash :: \mathrm{E}\,!\,(\imath x)(\phi x) . \supset :. [(\imath x)(\phi x)] . p \lor \chi(\imath x)(\phi x) .$
$\equiv : p . \lor . [(\imath x)(\phi x)] . \chi(\imath x)(\phi x)$

Dem.

$\vdash . *14{\cdot}242 . \supset \vdash :. \phi x . \equiv_x . x = b : \supset : [(\imath x)(\phi x)] . p \lor \chi(\imath x)(\phi x) . \equiv . p \lor \chi b$ (1)

$\vdash . *14{\cdot}242 . \supset \vdash :. \phi x . \equiv_x . x = b : \supset : [(\imath x)(\phi x)] . \chi(\imath x)(\phi x) . \equiv . \chi b :$

[*4·37] $\supset : p \lor [(\imath x)(\phi x)] \chi(\imath x)(\phi x) . \equiv . p \lor \chi b$ (2)

$\vdash . (1) . (2) . \supset \vdash :. \phi x . \equiv_x . x = b : \supset : [(\imath x)(\phi x)] . p \lor \chi(\imath x)(\phi x) .$
$\equiv . p \lor [(\imath x)(\phi x)] \chi(\imath x)(\phi x)$ (3)

$\vdash . (3) . *10{\cdot}23 . *14{\cdot}11 . \supset \vdash . \mathrm{Prop}$

The following propositions are proved in precisely the same way as *14·31; hence we shall merely give references to the propositions used in the proofs.

***14·32.** $\vdash :. \mathrm{E}\,!\,(\imath x)(\phi x) . \equiv : [(\imath x)(\phi x)] . \sim\chi(\imath x)(\phi x) .$
$\equiv . \sim\{[(\imath x)(\phi x)] . \chi(\imath x)(\phi x)\}$

[*14·242 . *4·11 . *10·23 . *14·11]

The equivalence asserted here fails when $\sim\mathrm{E}\,!\,(\imath x)(\phi x)$. Thus, for example, let ϕy be "y is King of France." Then $(\imath x)(\phi x) =$ the King of France. Let χy be "y is bald." Then $[(\imath x)(\phi x)] . \sim\chi(\imath x)(\phi x) . = .$ the King of France exists and is not bald; but $\sim\{[(\imath x)(\phi x)] . \chi(\imath x)(\phi x)\} . = .$ it is false that the King of France exists and is bald. Of these the first is false, the second true.

Either might be meant by "the King of France is not bald," which is ambiguous; but it would be more natural to take the first (false) interpretation as the meaning of the words. If the King of France existed, the two would be equivalent; thus as applied to the King of England, both are true or both false.

$*14{\cdot}33.$ $\vdash :: E\,!\,(\imath x)(\phi x) \,.\supset :. [(\imath x)(\phi x)] \,.\, p \supset \chi(\imath x)(\phi x) \,.$
$$\equiv : p \,.\supset.\, [(\imath x)(\phi x)] \,.\, \chi(\imath x)(\phi x)$$
[$*14{\cdot}242 \,.\, *4{\cdot}85 \,.\, *10{\cdot}23 \,.\, *14{\cdot}11$]

$*14{\cdot}331.$ $\vdash :: E\,!\,(\imath x)(\phi x) \,.\supset :. [(\imath x)(\phi x)] \,.\, \chi(\imath x)(\phi x) \supset p \,.$
$$\equiv : [(\imath x)(\phi x)] \,.\, \chi(\imath x)(\phi x) \,.\supset.\, p$$
[$*4{\cdot}84 \,.\, *14{\cdot}242 \,.\, *10{\cdot}23 \,.\, *14{\cdot}11$]

$*14{\cdot}332.$ $\vdash :: E\,!\,(\imath x)(\phi x) \,.\supset :. [(\imath x)(\phi x)] \,.\, p \equiv \chi(\imath x)(\phi x) \,.\equiv$
$$: p \,.\equiv.\, [(\imath x)(\phi x)] \,.\, \chi(\imath x)(\phi x)$$
[$*4{\cdot}86 \,.\, *14{\cdot}242 \,.\, *10{\cdot}23 \,.\, *14{\cdot}11$]

$*14{\cdot}34.$ $\vdash :. p : [(\imath x)(\phi x)] \,.\, \chi(\imath x)(\phi x) : \equiv : [(\imath x)(\phi x)] : p \,.\, \chi(\imath x)(\phi x)$

This proposition does not require the hypothesis $E\,!\,(\imath x)(\phi x)$.

Dem.

$\vdash \,.\, *14{\cdot}1 \,.\supset$

$\vdash :. p : [(\imath x)(\phi x)] \,.\, \chi(\imath x)(\phi x) : \equiv : p : (\exists b) : \phi x \,.\equiv_x.\, x = b : \chi b :$

[$*10{\cdot}35$] $\equiv : (\exists b) : p : \phi x \,.\equiv_x.\, x = b : \chi b :$

[$*14{\cdot}1$] $\equiv : [(\imath x)(\phi x)] : p \,.\, \chi(\imath x)(\phi x) :. \supset \vdash .$ Prop

Propositions of the above type might be continued indefinitely, but as they are proved on a uniform plan, it is unnecessary to go beyond the fundamental cases of $p \vee q$, $\sim p$, $p \supset q$ and $p \,.\, q$.

It should be observed that the proposition in which $(\imath x)(\phi x)$ has the larger scope always implies the corresponding one in which it has the smaller scope, but the converse implication only holds if either (a) we have $E\,!\,(\imath x)(\phi x)$ or (b) the proposition in which $(\imath x)(\phi x)$ has the smaller scope implies $E\,!\,(\imath x)(\phi x)$. The second case occurs in $*14{\cdot}34$, and is the reason why we get an equivalence without the hypothesis $E\,!\,(\imath x)(\phi x)$. The proposition in which $(\imath x)(\phi x)$ has the larger scope always implies $E\,!\,(\imath x)(\phi x)$, in virtue of $*14{\cdot}21$.

SECTION C

CLASSES AND RELATIONS

$*20$. GENERAL THEORY OF CLASSES

Summary of $*20$.

The following theory of classes, although it provides a notation to represent them, avoids the assumption that there are such things as classes. This it does by merely defining propositions in whose expression the symbols representing classes occur, just as, in $*14$, we defined propositions containing descriptions.

The characteristics of a class are that it consists of all the terms satisfying some propositional function, so that every propositional function determines a class, and two functions which are formally equivalent (*i.e.* such that whenever either is true, the other is true also) determine the same class, while conversely two functions which determine the same class are formally equivalent. When two functions are formally equivalent, we shall say that they have the same *extension*. The incomplete symbols which take the place of classes serve the purpose of technically providing something identical in the case of two functions having the same extension; without something to represent classes, we cannot, for example, count the combinations that can be formed out of a given set of objects.

Propositions in which a function ϕ occurs may depend, for their truth-value, upon the particular function ϕ, or they may depend only upon the *extension* of ϕ. In the former case, we will call the proposition concerned an *intensional* function of ϕ; in the latter case, an *extensional* function of ϕ. Thus, for example, $(x) . \phi x$ or $(\exists x) . \phi x$ is an extensional function of ϕ, because, if ϕ is formally equivalent to ψ, *i.e.* if $\phi x . \equiv_x . \psi x$, we have $(x) . \phi x . \equiv . (x) . \psi x$ and $(\exists x) . \phi x . \equiv . (\exists x) . \psi x$. But on the other hand "I believe $(x) . \phi x$" is an *intensional* function, because, even if $\phi x . \equiv_x . \psi x$, it by no means follows that I believe $(x) . \psi x$ provided I believe $(x) . \phi x$. The mark of an extensional function f of a function $\phi ! \hat{z}$ is

$$\phi ! x . \equiv_x . \psi ! x : \supset_{\phi, \psi} : f(\phi ! \hat{z}) . \equiv . f(\psi ! \hat{z}).$$

(We write "$\phi ! \hat{z}$" when we wish to speak of the function itself as opposed to its argument.) The functions of functions with which mathematics is specially concerned are all extensional.

When a function of $\phi ! \hat{z}$ is extensional, it may be regarded as being about the class determined by $\phi ! \hat{z}$, since its truth-value remains unchanged so long as the class is unchanged. Hence we require, for the theory of classes, a method of obtaining an extensional function from any given function of a function. This is effected by the following definition:

***20·01.** $f\{\hat{z}(\psi z)\} \,.=:\, (\exists\phi) : \phi\,!\,x \,.\equiv_x.\, \psi x : f\{\phi\,!\,\hat{z}\}$ Df

Here $f\{\hat{z}(\psi z)\}$ is in reality a function of $\psi\hat{z}$, which is defined whenever $f\{\phi\,!\,\hat{z}\}$ is significant for predicative functions $\phi\,!\,\hat{z}$. But it is convenient to regard $f\{\hat{z}(\psi z)\}$ as though it had an argument $\hat{z}(\psi z)$, which we will call "the class determined by the function $\psi\hat{z}$." It will be proved shortly that $f\{\hat{z}(\psi z)\}$ is always an *extensional* function of $\psi\hat{z}$, and that, applying the definition of identity (*13·01) to the fictitious objects $\hat{z}(\phi z)$ and $\hat{z}(\psi z)$, we have

$$\hat{z}(\phi z) = \hat{z}(\psi z) \,.\equiv: (x) : \phi x \,.\equiv.\, \psi x.$$

This last is the distinguishing characteristic of classes, and justifies us in treating $\hat{z}(\psi z)$ as the class determined by $\psi\hat{z}$.

With regard to the scope of $\hat{z}(\psi z)$, and to the order of elimination of two such expressions, we shall adopt the same conventions as were explained in *14 for $(\imath x)(\phi x)$. The condition corresponding to

$$\text{E}\,!\,(\imath x)(\psi x) \text{ is } (\exists\phi) : \phi\,!\,x \,.\equiv_x.\, \psi x,$$

which is always satisfied because of *12·1.

Following Peano, we shall use the notation

$$x \,\epsilon\, \hat{z}(\psi z)$$

to express "x is a member of the class determined by $\psi\hat{z}$." We therefore introduce the following definition:

***20·02.** $x \,\epsilon\, (\phi\,!\,\hat{z}) \,.=.\, \phi\,!\,x$ Df

In this form, the definition is never used; it is introduced for the sake of the proposition

$$\vdash :.\, x \,\epsilon\, \hat{z}(\psi z) \,.\equiv: (\exists\phi) : \psi y \,.\equiv_y.\, \phi\,!\,y : \phi\,!\,x$$

which results from *20·02 and *20·01, and leads to

$$\vdash : x \,\epsilon\, \hat{z}(\psi z) \,.\equiv.\, \psi x$$

by the help of *12·1.

We shall use small Greek letters (other than $\epsilon, \iota, \pi, \phi, \psi, \chi, \theta$) to represent classes, *i.e.* to stand for symbols of the form $\hat{z}(\phi z)$ or $\hat{z}(\phi\,!\,z)$. When a small Greek letter occurs as apparent variable, it is to be understood to stand for a symbol of the form $\hat{z}(\phi\,!\,z)$, where ϕ is properly the apparent variable concerned. The use of single letters in place of such symbols as $\hat{z}(\phi z)$ or $\hat{z}(\phi\,!\,z)$ is practically almost indispensable, since otherwise the notation rapidly becomes intolerably cumbrous. Thus "$x \,\epsilon\, \alpha$" will mean "x is a member of the class α," and may be used wherever no special defining function of the class α is in question.

The following definition defines what is meant by a *class.*

***20·03.** $\text{Cls} = \hat{\alpha}\{(\exists\phi) \,.\, \alpha = \hat{z}(\phi\,!\,z)\}$ Df

Note that the expression "$\hat{\alpha}\{(\exists\phi) \,.\, \alpha = \hat{z}(\phi\,!\,z)\}$" has no meaning in isolation: we have merely defined (in *20·01) certain *uses* of such expressions. What the above definition decides is that the symbol "Cls" may replace the symbol "$\hat{\alpha}\{(\exists\phi) \,.\, \alpha = \hat{z}(\phi\,!\,z)\}$," wherever the latter occurs, and that the

meaning of the combination of symbols concerned is to be unchanged thereby. Thus "Cls," also, has no meaning in isolation, but merely in certain uses.

The above definition, like many future definitions, is ambiguous as to type. The Latin letter z, according to our conventions, is to represent the lowest type concerned; thus ϕ is of the type next above this. It is convenient to speak of a class as being of the same type as its defining function; thus α is of the type next above that of z, and "Cls" is of the type next above that of α. Thus the type of "Cls" is fixed relatively to the lowest type concerned; but if, in two different contexts, different types are the lowest concerned, the meaning of "Cls" will be different in these two contexts. The meaning of "Cls" only becomes definite when the lowest type concerned is specified.

Equality between classes is defined by applying $*13{\cdot}01$, symbolically unchanged, to their defining functions, and then using $*20{\cdot}01$.

The propositions of the present number may be divided into three sets. First, we have those that deal with the fundamental properties of classes; these end with $*20{\cdot}43$. Then we have a set of propositions dealing with both classes and descriptions; these extend from $*20{\cdot}5$ to $*20{\cdot}59$ (with the exception of $*20{\cdot}53{\cdot}54$). Lastly, we have a set of propositions designed to prove that classes of classes have all the same formal properties as classes of individuals.

In the first set, the principal propositions are the following.

$*20{\cdot}15$. $\vdash :. \psi x . \equiv_x . \chi x : \equiv . \hat{z}(\psi z) = \hat{z}(\chi z)$

I.e. two classes are identical when, and only when, their defining functions are formally equivalent. This is the principal property of classes.

$*20{\cdot}31$. $\vdash :. \hat{z}(\psi z) = \hat{z}(\chi z) . \equiv : x \epsilon \hat{z}(\psi z) . \equiv_x . x \epsilon \hat{z}(\chi z)$

I.e. two classes are identical when, and only when, they have the same members.

$*20{\cdot}43$. $\vdash :. \alpha = \beta . \equiv : x \epsilon \alpha . \equiv_x . x \epsilon \beta$

This is the same proposition as $*20{\cdot}31$, merely employing Greek letters in place of $\hat{z}(\psi z)$ and $\hat{z}(\chi z)$.

$*20{\cdot}18$. $\vdash :. \hat{z}(\phi z) = \hat{z}(\psi z) . \supset : f\{\hat{z}(\phi z)\} . \equiv . f\{\hat{z}(\psi z)\}$

I.e. if two classes are identical, any property of either belongs also to the other. This is the analogue of $*13{\cdot}12$.

$*20{\cdot}2{\cdot}21{\cdot}22$, which prove that identity between classes is reflexive, symmetrical and transitive.

$*20{\cdot}3$. $\vdash : x \epsilon \hat{z}(\psi z) . \equiv . \psi x$

I.e. a term belongs to a class when, and only when, it satisfies the defining function of the class.

In the second set of propositions ($*20{\cdot}5$—$\cdot59$), we show that, under suitable circumstances, expressions such as $(\imath x)(\phi x)$ may be substituted for x in $*20{\cdot}3$

and various other propositions of the first set, and we prove a few properties of such expressions as "$(\imath\alpha)(f\alpha)$," *i.e.* "the class which satisfies the function f." Here it is to be remembered that "α" stands for "$\hat{z}(\phi z)$," and that "$f\alpha$" therefore stands for "$f\{\hat{z}(\phi z)\}$." This is, in reality, a function of $\phi\hat{z}$, namely the extensional function associated with $f(\psi ! \hat{z})$ by means of $*20{\cdot}01$. Thus an expression containing a variable class is always an abbreviation for an expression containing a variable function.

In the third set of propositions, we prove that variable classes satisfy all the primitive propositions assumed for variable individuals or functions, whence it follows, by merely repeating the proofs of the first set of propositions ($*20{\cdot}1$—$\cdot43$), that classes of classes have all the formal properties of classes of individuals or functions. We shall never have occasion explicitly to consider classes of functions, but classes of classes will occur constantly—for example, every cardinal number will be defined as a class of classes. Classes of relations, which will also frequently occur, will be considered in $*21$.

$*20{\cdot}01$. $f\{\hat{z}(\psi z)\} . = : (\exists\phi) : \phi ! x . \equiv_x . \psi x : f\{\phi ! \hat{z}\}$ Df

$*20{\cdot}02$. $x \in (\phi ! \hat{z}) . = . \phi ! x$ Df

$*20{\cdot}03$. $\mathrm{Cls} = \hat{\alpha}\{(\exists\phi) . \alpha = \hat{z}(\phi ! z)\}$ Df

The three following definitions serve merely for purposes of abbreviation.

$*20{\cdot}04$. $x, y \in \alpha . = . x \in \alpha . y \in \alpha$ Df

$*20{\cdot}05$. $x, y, z \in \alpha . = . x, y \in \alpha . z \in \alpha$ Df

$*20{\cdot}06$. $x \sim\in \alpha . = . \sim(x \in \alpha)$ Df

The following definitions merely extend to symbols representing classes the definitions which have already been given for other symbols, with the smallest possible modifications.

$*20{\cdot}07$. $(\alpha) . f\alpha . = . (\phi) . f\{\hat{z}(\phi ! z)\}$ Df

$*20{\cdot}071$. $(\exists\alpha) . f\alpha . = . (\exists\phi) . f\{\hat{z}(\phi ! z)\}$ Df

$*20{\cdot}072$. $[(\imath\alpha)(\phi\alpha)] . f(\imath\alpha)(\phi\alpha) . = : (\exists\gamma) : \phi\alpha . \equiv_\alpha . \alpha = \gamma : f\gamma$ Df

$*20{\cdot}08$. $f\{\hat{\alpha}(\psi\alpha)\} . = : (\exists\phi) : \psi\alpha . \equiv_\alpha . \phi ! \alpha : f(\phi ! \hat{\alpha})$ Df

$*20{\cdot}081$. $\alpha \in \psi ! \hat{\alpha} . = . \psi ! \alpha$ Df

The propositions which follow give the most general properties of classes.

$*20{\cdot}1$. $\vdash :. f\{\hat{z}(\psi z)\} . \equiv : (\exists\phi) : \phi ! x . \equiv_x . \psi x : f\{\phi ! \hat{z}\}$ [$*4{\cdot}2 . (*20{\cdot}01)$]

$*20{\cdot}11$. $\vdash :. \psi x . \equiv_x . \chi x : \supset : f\{\hat{z}(\psi z)\} . \equiv . f\{\hat{z}(\chi z)\}$

Dem.

$\vdash . *4{\cdot}86 . \supset \vdash :: \mathrm{Hp} . \supset :. \phi ! x . \equiv_x . \psi x : \equiv_\phi : \phi ! x . \equiv_x . \chi x :.$

[$*4{\cdot}36$] $\supset :. \phi ! x . \equiv_x . \psi x : f\{\phi ! \hat{z}\} : \equiv_\phi : \phi ! x . \equiv_x . \chi x : f\{\phi ! \hat{z}\} :.$

[$*10{\cdot}281$] $\supset :. (\exists\phi) : \phi ! x . \equiv_x . \psi x : f\{\phi ! \hat{z}\} :$

$\equiv : (\exists\phi) : \phi ! x . \equiv_x . \chi x : f\{\phi ! \hat{z}\} :.$

[$*20{\cdot}1$] $\supset :. f\{\hat{z}(\psi z)\} . \equiv . f\{\hat{z}(\chi z)\} :: \supset \vdash . \mathrm{Prop}$

This proves that every proposition about a class expresses an extensional property of the determining function of the class, and therefore does not depend for its truth or falsehood upon the particular function selected for determining the class, but only upon the extension of the determining function.

$*20\cdot111$. $\vdash :. f(\phi ! \hat{z}) . \equiv_\phi . g(\phi ! \hat{z}) : \supset : f\{\hat{z}(\phi ! z)\} . \equiv_\phi . g\{\hat{z}(\phi ! z)\}$

Dem.

$\vdash .\text{Fact}. \quad \supset \vdash :: \text{Hp} . \supset :. \phi ! x . \equiv_x . \psi ! x : f(\psi ! \hat{z}) : \equiv : \phi ! x . \equiv_x . \psi ! x : g(\psi ! \hat{z}) ::$

$[*10\cdot11\cdot21] \supset \vdash :: \text{Hp} . \supset :. \phi ! x . \equiv_x . \psi ! x : f(\psi ! \hat{z}) : \equiv_\psi : \phi ! x . \equiv_x . \psi ! x : g(\psi ! \hat{z}) :.$

$[*10\cdot281] \quad \supset :. (\exists \psi) : \phi ! x . \equiv_x . \psi ! x : f(\psi ! \hat{z}) : \equiv : (\exists \psi) : \phi ! x . \equiv_x . \psi ! x : g(\psi ! \hat{z}) :.$

$[*20\cdot1] \quad \supset :. f\{\hat{z}(\phi ! x)\} . \equiv . g\{\hat{z}(\phi ! x)\} \quad (1)$

$\vdash . (1) . *10\cdot11\cdot21 . \supset \vdash . \text{Prop}$

$*20\cdot112$. $\vdash :. (\exists g) :. f\{\hat{z}(\phi ! z)\} . \equiv_\phi . g ! \{\hat{z}(\phi ! z)\}$

Dem.

$\vdash . *12\cdot1 . \supset \vdash :. (\exists g) : f(\phi ! \hat{z}) . \equiv_\phi . g ! (\phi ! \hat{z}) \quad (1)$

$\vdash . (1) . *20\cdot111 . \supset \vdash . \text{Prop}$

Thus the axiom of reducibility still holds for classes as arguments.

$*20\cdot12$. $\vdash : (\exists \phi) : \phi ! x . \equiv_x . \psi x : f\{\hat{z}(\psi z)\} . \equiv . f\{\hat{z}(\phi ! z)\} \quad [*20\cdot11 . *12\cdot1]$

$*20\cdot13$. $\vdash :. \psi x . \equiv_x . \chi x : \supset . \hat{z}(\psi z) = \hat{z}(\chi z)$

The meaning of "$\hat{z}(\psi z) = \hat{z}(\chi z)$" is obtained by a double application of $*20\cdot01$ to $*13\cdot01$, remembering the convention that $\hat{z}(\psi z)$ is to have a larger scope than $\hat{z}(\chi z)$ because it occurs first.

Dem.

$\vdash . *20\cdot1 . \supset \vdash :: \hat{z}(\psi z) = \hat{z}(\chi z) . \equiv :. (\exists \phi) : \psi x . \equiv_x . \phi ! x : \phi ! \hat{z} = \hat{z}(\chi z) :.$

$[*20\cdot1] \quad \equiv :. (\exists \phi, \theta) :. \psi x . \equiv_x . \phi ! x : \chi x . \equiv_x . \theta ! x : \phi ! \hat{z} = \theta ! \hat{z} \quad (1)$

$\vdash . *12\cdot1 . *10\cdot321 . \supset$

$\vdash :: \text{Hp} . \supset :. (\exists \phi) : \psi x . \equiv_x . \phi ! x : \chi x . \equiv_x . \phi ! x :.$

$[*13\cdot195] \supset :. (\exists \phi, \theta) :. \psi x . \equiv_x . \phi ! x : \chi x . \equiv_x . \theta ! x : \phi ! \hat{z} = \theta ! \hat{z} \quad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

$*20\cdot14$. $\vdash :. \hat{z}(\psi z) = \hat{z}(\chi z) . \supset : \psi x . \equiv_x . \chi x$

Dem.

$\vdash . *20\cdot1 . \supset \vdash :: \hat{z}(\psi z) = \hat{z}(\chi z) . \equiv :. (\exists \phi) : \psi x . \equiv_x . \phi ! x : \phi ! \hat{z} = \hat{z}(\chi z) :.$

$[*20\cdot1] \quad \equiv :. (\exists \phi, \theta) :. \psi x . \equiv_x . \phi ! x : \chi x . \equiv_x . \theta ! x : \phi ! \hat{z} = \theta ! \hat{z} :.$

$[*13\cdot195] \quad \equiv :. (\exists \phi) :. \psi x . \equiv_x . \phi ! x : \chi x . \equiv_x . \phi ! x :.$

$[*10\cdot322] \quad \supset :. \psi x . \equiv_x . \chi x :: \supset \vdash . \text{Prop}$

This proposition is the converse of $*20\cdot13$.

$*20\cdot15$. $\vdash :. \psi x . \equiv_x . \chi x : \equiv . \hat{z}(\psi z) = \hat{z}(\chi z) \quad [*20\cdot13\cdot14]$

This proposition states that two functions determine the same class when, and only when, they are formally equivalent, *i.e.* are satisfied by the same set of values. This is the essential property of classes, and gives the justification of the definition $*20\cdot01$.

$*20{\cdot}151$. $\vdash . (\exists\phi) . \hat{z}(\psi z) = \hat{z}(\phi ! z)$

Dem.

$\vdash . *20{\cdot}15 . \quad \supset \vdash :. \psi x . \equiv_x . \phi ! x : \supset . \hat{z}(\psi z) = \hat{z}(\phi ! z) :.$

$[*10{\cdot}11{\cdot}28] \quad \supset \vdash :. (\exists\phi) : \psi x . \equiv_x . \phi ! x : \supset . (\exists\phi) . \hat{z}(\psi z) = \hat{z}(\phi ! z) \quad (1)$

$\vdash . (1) . *12{\cdot}1 . \supset \vdash .$ Prop

In virtue of this proposition, all classes can be obtained from predicative functions. This fact is especially important when classes are used as apparent variables. For in that case, according to the definitions $*20{\cdot}07{\cdot}071$, the apparent variable really involved is a predicative function. In virtue of $*20{\cdot}151$, this places no limitation upon the classes concerned, except the limitation which inevitably results from the nature of their membership. A class, therefore, unlike a function, has its order completely determined by the order of its possible members, *i.e.* of the arguments which render its defining function significant.

$*20{\cdot}16$. $\vdash : (\exists\phi) : f\{\hat{z}(\psi z)\} . \equiv . f\{\hat{z}(\phi ! z)\}$ $[*20{\cdot}12]$

$*20{\cdot}17$. $\vdash : (\phi) . f\{\hat{z}(\phi ! z)\} . \supset . f\{\hat{z}(\psi z)\}$ $[*20{\cdot}16 . *10{\cdot}1]$

$*20{\cdot}18$. $\vdash :. \hat{z}(\phi z) = \hat{z}(\psi z) . \supset : f\{\hat{z}(\phi z)\} . \equiv . f\{\hat{z}(\psi z)\}$ $[*20{\cdot}11{\cdot}15]$

$*20{\cdot}19$. $\vdash :. \hat{z}(\psi z) = \hat{z}(\chi z) . \equiv : (f) : f ! \hat{z}(\psi z) . \supset . f ! \hat{z}(\chi z)$

Dem.

$\vdash . *20{\cdot}18 . *10{\cdot}11{\cdot}21 . \supset \vdash :. \hat{z}(\psi z) = \hat{z}(\chi z) . \supset :$
$(f) : f ! \hat{z}(\psi z) . \supset . f ! \hat{z}(\chi z) \quad (1)$

$\vdash . *20{\cdot}18{\cdot}15 . \supset \vdash :: \phi ! x . \equiv_x . \psi x : \theta ! x . \equiv_x . \chi x : f ! \hat{z}(\psi z) . \supset . f ! \hat{z}(\chi z) : \supset :$
$f ! \hat{z}(\phi ! z) . \supset . f ! \hat{z}(\theta ! z) \quad (2)$

$\vdash . (2) . *10{\cdot}11{\cdot}27{\cdot}33 . \supset$

$\vdash :: \phi ! x . \equiv_x . \psi x : \theta ! x . \equiv_x . \chi x :. (f) : f ! \hat{z}(\psi z) . \supset . f ! \hat{z}(\chi z) :. \supset :.$
$(f) : f ! \hat{z}(\phi ! z) . \supset . f ! \hat{z}(\theta ! z) :.$

$[*20{\cdot}112 . *10{\cdot}1] \quad \supset :. \phi ! x . \equiv_x . \phi ! x : \supset : \phi ! x . \equiv_x . \theta ! x :.$

$[*4{\cdot}2] \quad \supset :. \phi ! x . \equiv_x . \theta ! x :.$

$[*10{\cdot}301{\cdot}32 . \text{Hp}] \quad \supset :. \psi x . \equiv_x . \chi x :.$

$[*20{\cdot}15] \quad \supset :. \hat{z}(\psi z) = \hat{z}(\chi z) \quad (3)$

$\vdash . (3) . *10{\cdot}11{\cdot}23{\cdot}35 . \supset$

$\vdash :: (\exists\phi, \theta) : \phi ! x . \equiv_x . \psi x : \theta ! x . \equiv_x . \chi x :. (f) : f ! \hat{z}(\psi z) . \supset . f ! \hat{z}(\chi z) :.$
$\supset . \hat{z}(\psi z) = \hat{z}(\chi z) \quad (4)$

$\vdash . (4) . *12{\cdot}1 . \supset \vdash :. (f) : f ! \hat{z}(\psi z) . \supset . f ! \hat{z}(\chi z) : \supset . \hat{z}(\psi z) = \hat{z}(\chi z) \quad (5)$

$\vdash . (1) . (5) . \quad \supset \vdash .$ Prop

$*20{\cdot}191$. $\vdash :. \hat{z}(\psi z) = \hat{z}(\chi z) . \equiv : (f) : f ! \hat{z}(\psi z) . \equiv . f ! \hat{z}(\chi z)$
$[*20{\cdot}18{\cdot}19 . *10{\cdot}22]$

$*20{\cdot}2$. $\vdash . \hat{z}(\phi z) = \hat{z}(\phi z)$

Dem.

$\vdash . *20{\cdot}15 . \supset \vdash :. \hat{z}(\phi z) = \hat{z}(\phi z) . \equiv : \phi x . \equiv_x . \phi x \quad (1)$

$\vdash . (1) . *4{\cdot}2 . *10{\cdot}11 . \supset \vdash .$ Prop

***20·21.** $\vdash : \hat{z}(\phi z) = \hat{z}(\psi z) . \equiv . \hat{z}(\psi z) = \hat{z}(\phi z)$ [*20·15 . *10·32]

***20·22.** $\vdash : \hat{z}(\phi z) = \hat{z}(\psi z) . \hat{z}(\psi z) = \hat{z}(\chi z) .\supset . \hat{z}(\phi z) = \hat{z}(\chi z)$ [*20·15 . *10·301]

The above propositions are not *immediate* consequences of *13·15·16·17, for a reason analogous to that explained in the note to *14·13, namely because $f\{\hat{z}(\phi z)\}$ is not a value of fx, and therefore in particular "$\hat{z}(\phi z) = \hat{z}(\psi z)$" is not a value of "$x = y$."

***20·23.** $\vdash : \hat{z}(\phi z) = \hat{z}(\psi z) . \hat{z}(\phi z) = \hat{z}(\chi z) . \supset . \hat{z}(\psi z) = \hat{z}(\chi z)$ [*20·21·22]

***20·24.** $\vdash : \hat{z}(\psi z) = \hat{z}(\phi z) . \hat{z}(\chi z) = \hat{z}(\phi z) . \supset . \hat{z}(\psi z) = \hat{z}(\chi z)$ [*20·21·22]

***20·25.** $\vdash :. \alpha = \hat{z}(\phi z) . \equiv_\alpha . \alpha = \hat{z}(\psi z) : \equiv . \hat{z}(\phi z) = \hat{z}(\psi z)$

Dem.

$\vdash . *10·1 . \quad \supset \vdash :. \alpha = \hat{z}(\phi z) . \equiv_\alpha . \alpha = \hat{z}(\psi z) : \supset :$

$\hat{z}(\phi z) = \hat{z}(\phi z) . \equiv . \hat{z}(\phi z) = \hat{z}(\psi z) :$

[*20·2] $\supset : \hat{z}(\phi z) = \hat{z}(\psi z)$ (1)

$\vdash . *20·22 . \quad \supset \vdash : \alpha = \hat{z}(\phi z) . \hat{z}(\phi z) = \hat{z}(\psi z) . \supset . \alpha = \hat{z}(\psi z) :$

[Exp.Comm] $\supset \vdash :. \hat{z}(\phi z) = \hat{z}(\psi z) . \supset : \alpha = \hat{z}(\phi z) . \supset . \alpha = \hat{z}(\psi z)$ (2)

$\vdash . *20·24 . \quad \supset \vdash :. \hat{z}(\phi z) = \hat{z}(\psi z) . \alpha = \hat{z}(\psi z) . \supset . \alpha = \hat{z}(\phi z) :.$

[Exp] $\supset \vdash :. \hat{z}(\phi z) = \hat{z}(\psi z) . \supset : \alpha = \hat{z}(\psi z) . \supset . \alpha = \hat{z}(\phi z)$ (3)

$\vdash . (2) . (3) . \quad \supset \vdash :. \hat{z}(\phi z) = \hat{z}(\psi z) . \supset : \alpha = \hat{z}(\phi z) . \equiv . \alpha = \hat{z}(\psi z) :.$

[*10·11·21] $\supset \vdash :. \hat{z}(\phi z) = \hat{z}(\psi z) . \supset : \alpha = \hat{z}(\phi z) . \equiv_\alpha . \alpha = \hat{z}(\psi z)$ (4)

$\vdash . (1) . (4) . \quad \supset \vdash .$ Prop

***20·3.** $\vdash : x \in \hat{z}(\psi z) . \equiv . \psi x$

Dem.

$\vdash . *20·1 . \supset$

$\vdash :: x \in \hat{z}(\psi z) . \equiv :. (\exists \phi) :. \psi y . \equiv_y . \phi ! y : x \in (\phi ! \hat{z}) :.$

[(*20·02)] $\equiv :. (\exists \phi) :. \psi y . \equiv_y . \phi ! y : \phi ! x :.$

[*10·43] $\equiv :. (\exists \phi) :. \psi y . \equiv_y . \phi ! y : \psi x :.$

[*10·35] $\equiv :. (\exists \phi) : \psi y . \equiv_y . \phi ! y :. \psi x :.$

[*12·1] $\equiv :. \psi x :: \supset \vdash .$ Prop

This proposition shows that x is a member of the class determined by ψ when, and only when, x satisfies ψ.

***20·31.** $\vdash :. \hat{z}(\psi z) = \hat{z}(\chi z) . \equiv : x \in \hat{z}(\psi z) . \equiv_x . x \in \hat{z}(\chi z)$ [*20·15·3]

***20·32.** $\vdash . \hat{x}\{x \in \hat{z}(\phi z)\} = \hat{z}(\phi z)$ [*20·3·15]

***20·33.** $\vdash :. \alpha = \hat{z}(\phi z) . \equiv : x \in \alpha . \equiv_x . \phi x$

Dem.

$\vdash . *20·31 . \quad \supset \vdash :. \alpha = \hat{z}(\phi z) . \equiv : x \in \alpha . \equiv_x . x \in \hat{z}(\phi z)$ (1)

$\vdash . (1) . *20·3 . \supset \vdash .$ Prop

Here α is written in place of some expression of the form $\hat{z}(\psi z)$. The use of the single Greek letter is more convenient whenever the determining function is irrelevant.

***20·34.** $\vdash :. x = y . \equiv : x \epsilon \alpha . \supset_\alpha . y \epsilon \alpha$

Dem.

$\vdash . *4\cdot2 . (*20\cdot07) . \supset \vdash :. x \epsilon \alpha . \supset_\alpha . y \epsilon \alpha : \equiv : x \epsilon \hat{z}(\phi ! z) . \supset_\phi . y \epsilon \hat{z}(\phi ! z) :$

$[*20\cdot3] \qquad \equiv : \phi ! x . \supset_\phi . \phi ! y :$

$[*13\cdot1] \qquad \equiv : x = y :. \supset \vdash . \text{Prop}$

The above proposition and *20·25 illustrate the use of Greek letters as apparent variables.

***20·35.** $\vdash :. x = y . \equiv : x \epsilon \alpha . \equiv_\alpha . y \epsilon \alpha$ [*20·3 . *13·11]

***20·4.** $\vdash : \alpha \epsilon \text{Cls} . \equiv . (\exists \phi) . \alpha = \hat{z}(\phi ! z)$ [*20·3 . (*20·03)]

***20·41.** $\vdash . \hat{z}(\psi z) \epsilon \text{Cls}$ [*20·4·151]

***20·42.** $\vdash . \hat{z}(z \epsilon \alpha) = \alpha$

A Greek letter, such as α, is merely an abbreviation for an expression of the form $\hat{z}(\phi z)$; thus this proposition is *20·32 repeated.

Dem.

$\vdash . *20\cdot3 . *10\cdot11 . \supset \vdash : x \epsilon \hat{z}(\psi z) . \equiv_x . \psi x :$

$[*20\cdot15] \qquad \supset \vdash . \hat{x}\{x \epsilon \hat{z}(\psi z)\} = \hat{x}(\psi x) . \supset \vdash . \text{Prop}$

***20·43.** $\vdash :. \alpha = \beta . \equiv : x \epsilon \alpha . \equiv_x . x \epsilon \beta$ [*20·31]

The following propositions deal with cases in which both classes and descriptions occur. In such cases, we shall, in the absence of any indication to the contrary, adopt the convention that the descriptions are to have a larger scope than the classes, in applying the definitions *14·01 and *20·01.

***20·5.** $\vdash : (\iota x)(\phi x) \epsilon \hat{z}(\psi z) . \equiv . \psi\{(\iota x)(\phi x)\}$

Dem.

$\vdash . *14\cdot1 . \supset \vdash :: (\iota x)(\phi x) \epsilon \hat{z}(\psi z) . \equiv :. (\exists c) : \phi x . \equiv_x . x = c : c \epsilon \hat{z}(\psi z) :.$

$[*20\cdot3] \qquad \equiv :. (\exists c) : \phi x . \equiv_x . x = c : \psi c :.$

$[*14\cdot1] \qquad \equiv :. \psi\{(\iota x)(\phi x)\} :: \supset \vdash . \text{Prop}$

***20·51.** $\vdash :. (\iota x)(\phi x) = b . \equiv : (\iota x)(\phi x) \epsilon \alpha . \equiv_\alpha . b \epsilon \alpha$

Dem.

$\vdash . *20\cdot5\cdot3 . \supset$

$\vdash :. (\iota x)(\phi x) \epsilon \hat{z}(\psi ! z) . \equiv . b \epsilon \hat{z}(\psi ! z) : \equiv : \psi ! (\iota x)(\phi x) . \equiv . \psi ! b :. \supset$

$[*10\cdot11] \vdash :. (\iota x)(\phi x) \epsilon \alpha . \equiv_\alpha . b \epsilon \alpha : \equiv : \psi ! (\iota x)(\phi x) . \equiv_\psi . \psi ! b :$

$[*14\cdot17] \qquad \equiv : (\iota x)(\phi x) = b :. \supset \vdash . \text{Prop}$

***20·52.** $\vdash :. E ! (\iota x)(\phi x) . \equiv : (\exists b) : (\iota x)(\phi x) \epsilon \alpha . \equiv_\alpha . b \epsilon \alpha$

Dem.

$\vdash . *20\cdot51 . *10\cdot11\cdot281 . \supset$

$\vdash :. (\exists b) . (\iota x)(\phi x) = b . \equiv : (\exists b) : (\iota x)(\phi x) \epsilon \alpha . \equiv_\alpha . b \epsilon \alpha \qquad (1)$

$\vdash . (1) . *14\cdot204 . \supset \vdash . \text{Prop}$

***20·53.** $\vdash :. \beta = \alpha . \supset_\beta . \phi\beta : \equiv . \phi\alpha$

This is the analogue of *13·191.

Dem.

$\vdash . *10 \cdot 1 . \quad \supset \vdash :. \beta = \alpha . \supset_\beta . \phi\beta : \supset : \alpha = \alpha . \supset . \phi\alpha :$

$[*20 \cdot 2] \quad \supset : \phi\alpha \quad (1)$

$\vdash . *20 \cdot 18 \cdot 21 . \supset \vdash :. \beta = \alpha . \supset : \phi\alpha . \supset . \phi\beta :.$

$[\text{Comm}] \quad \supset \vdash :. \phi\alpha . \supset : \beta = \alpha . \supset . \phi\beta :.$

$[*10 \cdot 11 \cdot 21] \quad \supset \vdash :. \phi\alpha . \supset : \beta = \alpha . \supset_\beta . \phi\beta \quad (2)$

$\vdash . (1) . (2) . \quad \supset \vdash . \text{Prop}$

***20·54.** $\vdash : (\exists\beta) . \beta = \alpha . \phi\beta . \equiv . \phi\alpha$

This proposition is the analogue of *13·195.

Dem.

$\vdash . *20 \cdot 18 . *10 \cdot 11 . \supset \vdash : \beta = \alpha . \phi\beta . \supset_\beta . \phi\alpha :$

$[*10 \cdot 23] \quad \supset \vdash : (\exists\beta) . \beta = \alpha . \phi\beta . \supset . \phi\alpha \quad (1)$

$\vdash . *20 \cdot 2 . *3 \cdot 2 . \quad \supset \vdash : \phi\alpha . \supset . \alpha = \alpha . \phi\alpha .$

$[*10 \cdot 24] \quad \supset . (\exists\beta) . \beta = \alpha . \phi\beta \quad (2)$

$\vdash . (1) . (2) . \quad \supset \vdash . \text{Prop}$

***20·55.** $\vdash . \hat{z}(\phi z) = (\imath\alpha)(x \in \alpha . \equiv_x . \phi x)$

Dem.

$\vdash . *20 \cdot 33 . \supset \vdash :: x \in \alpha . \equiv_x . \phi x : \equiv_\alpha . \alpha = \hat{z}(\phi z) :.$

$[*20 \cdot 54] \quad \supset \vdash :. (\exists\beta) :. x \in \alpha . \equiv_x . \phi x : \equiv_\alpha . \alpha = \beta :. \hat{z}(\phi z) = \beta :.$

$[*14 \cdot 1] \quad \supset \vdash . \hat{z}(\phi z) = (\imath\alpha)(x \in \alpha . \equiv_x . \phi x) . \supset \vdash . \text{Prop}$

***20·56.** $\vdash . E ! (\imath\alpha)(x \in \alpha . \equiv_x . \phi x) \quad [*20 \cdot 55 . *14 \cdot 21]$

***20·57.** $\vdash :. \hat{z}(\phi z) = (\imath\alpha)(f\alpha) . \supset : g\{\hat{z}(\phi z)\} . \equiv . g\{(\imath\alpha)(f\alpha)\}$

Dem.

$\vdash . *14 \cdot 1 . \quad \supset \vdash :: \text{Hp} . \equiv :. (\exists\beta) : f\alpha . \equiv_\alpha . \alpha = \beta : \hat{z}(\phi z) = \beta :.$

$[*20 \cdot 54] \quad \equiv :. f\alpha . \equiv_\alpha . \alpha = \hat{z}(\phi z) \quad (1)$

$\vdash . *14 \cdot 1 . \quad \supset \vdash :. g\{(\imath\alpha)(f\alpha)\} . \equiv : (\exists\beta) : f\alpha . \equiv_\alpha . \alpha = \beta : g\beta \quad (2)$

$\vdash . (1) . (2) . \supset \vdash :: \text{Hp} . \supset :. g\{(\imath\alpha)(f\alpha)\} . \equiv : (\exists\beta) : \alpha = \hat{z}(\phi z) . \equiv_\alpha . \alpha = \beta : g\beta :$

$[*13 \cdot 183] \quad \equiv : (\exists\beta) . \hat{z}(\phi z) = \beta . g\beta :$

$[*20 \cdot 54] \quad \equiv : g\{\hat{z}(\phi z)\} :: \supset \vdash . \text{Prop}$

***20·58.** $\vdash . \hat{z}(\phi z) = (\imath\alpha)\{\alpha = \hat{z}(\phi z)\}$

Dem.

$\vdash . *4 \cdot 2 . *10 \cdot 11 . \supset \vdash : \alpha = \hat{z}(\phi z) . \equiv_\alpha . \alpha = \hat{z}(\phi z) :$

$[*20 \cdot 54] \quad \supset \vdash :. (\exists\beta) :. \alpha = \hat{z}(\phi z) . \equiv_\alpha . \alpha = \beta : \hat{z}(\phi z) = \beta :.$

$[*14 \cdot 1] \quad \supset \vdash . \hat{z}(\phi z) = (\imath\alpha)\{\alpha = \hat{z}(\phi z)\} . \supset \vdash . \text{Prop}$

***20·59.** $\vdash : \hat{z}(\phi z) = (\imath\alpha)(f\alpha) . \equiv . (\imath\alpha)(f\alpha) = \hat{z}(\phi z)$

Dem.

$\vdash . *20 \cdot 1 . \supset \vdash :. \hat{z}(\phi z) = (\imath\alpha)(f\alpha) . \equiv : (\exists\psi) : \phi x . \equiv_x . \psi ! x : \psi ! \hat{z} = (\imath\alpha)(f\alpha) :$

$[*14 \cdot 13] \quad \equiv : (\exists\psi) : \phi x . \equiv_x . \psi ! x : (\imath\alpha)(f\alpha) = \psi ! \hat{z} :$

$[*20 \cdot 1] \quad \equiv : (\imath\alpha)(f\alpha) = \hat{z}(\phi z) :. \supset \vdash . \text{Prop}$

In the following propositions, we shall prove that classes have all the formal properties of individuals, and have the same relations to classes of classes as individuals have to classes of individuals. It is only necessary to prove the analogues of our primitive propositions, and of our definitions in cases where their analogues are not themselves definitions. We shall take the propositions ∗10·1·11·12·121·122, rather than those of ∗9, and we shall prove the analogue of ∗10·01. As was pointed out in ∗10, we shall thus have proved everything upon which subsequent proofs depend. The analogues of ∗20·01·02 and of ∗14·01 remain definitions, but those of ∗10·01 and ∗13·01 become propositions to be proved. ∗9·131 must be extended by the definition: Two classes are "of the same type" when they have predicative defining functions of the same type. In addition to these, we have to prove the analogues of ∗10·1·11·12·121·122, ∗11·07 and ∗12·1·11. When these have been proved, the analogues of other propositions follow by merely repeating previous proofs. These analogues will, therefore, be quoted by the numbers of the original propositions whose analogues they are.

∗20·6. $\vdash : (\exists \alpha) . f\alpha . \equiv . \sim \{(\alpha) . \sim f\alpha\}$

Dem.

$\vdash . *4 \cdot 2 . (*20 \cdot 071) . \supset$

$\vdash : (\exists \alpha) . f\alpha . \equiv . (\exists \phi) . f\{\hat{z}(\phi ! z)\} .$

$[(*10 \cdot 01)] \quad \equiv . \sim [(\phi) . \sim f\{\hat{z}(\phi ! z)\}] .$

$[(*20 \cdot 07)] \quad \equiv . \sim \{(\alpha) . \sim f\alpha\} : \supset \vdash . \text{Prop}$

This is the analogue of ∗10·01.

∗20·61. $\vdash : (\alpha) . f\alpha . \supset . f\beta$

Dem.

$\vdash . *10 \cdot 1 . (*20 \cdot 07) . \supset \vdash : (\alpha) . f\alpha . \supset . f\{\hat{z}(\phi ! z)\} : \supset \vdash . \text{Prop}$

This is the analogue of ∗10·1.

In practice we also need

$$\vdash : (\alpha) . f\alpha . \supset . f\{\hat{z}(\psi z)\}.$$

This is ∗20·17.

We need further $\vdash . (\exists \alpha) . \hat{z}(\psi z) = \alpha.$

This is ∗20·41.

∗20·62. When $f\beta$ is true, whatever possible argument of the form $\hat{z}(\phi ! z)$ β may be, then $(\alpha) . f\alpha$ is true.

This is the analogue of ∗10·11.

Dem.

$\vdash . *10 \cdot 11 . \supset .$ when $f\{\hat{z}(\phi ! z)\}$ is true, whatever possible argument ϕ may be, then $(\phi) . f\{\hat{z}(\phi ! z)\}$ is true, *i.e.* (by ∗20·07), $(\alpha) . f\alpha$ is true.

∗20·63. $\vdash :. (\alpha) . p \lor f\alpha . \supset : p . \lor . (\alpha) . f\alpha$

This is the analogue of ∗10·12.

Dem.

$\vdash . *4\cdot2 . (*20\cdot07) . \supset$

$\vdash :. (\alpha) . p \lor f\alpha . \equiv : (\phi) . p \lor f\{\hat{z}(\phi ! z)\} :$

$[*10\cdot12] \quad \equiv : p . \lor . (\phi) . f\{\hat{z}(\phi ! z)\} :$

$[(*20\cdot07)] \quad \equiv : p . \lor . (\alpha) . f\alpha :. \supset \vdash . \text{Prop}$

***20·631**. If "$f\alpha$" is significant, then if β is of the same type as α, "$f\beta$" is significant, and vice versa.

This is the analogue of *10·121.

Dem.

By *20·151, α is of the form $\hat{z}(\phi ! z)$, and therefore, by *20·01, $f\alpha$ is a function of $\phi ! \hat{z}$. Similarly β is of the form $\hat{z}(\psi ! z)$, and $f\beta$ is a function of $\psi ! \hat{z}$. Hence by applying *10·121 to $\phi ! \hat{z}$ and $\psi ! \hat{z}$ the result follows.

***20·632**. If, for some α, there is a proposition $f\alpha$, then there is a function $f\hat{\alpha}$, and vice versa.

Dem.

By the definition in *20·01, $f\{\hat{z}(\psi ! z)\}$ is a function of $\psi ! \hat{z}$. Hence the proposition follows from *10·122.

***20·633**. "Whatever possible class α may be, $f(\alpha, \beta)$ is true whatever possible class β may be" implies the corresponding statement with α and β interchanged except in "$f(\alpha, \beta)$."

This is the analogue of *11·07, and follows at once from *11·07 because $f(\alpha, \beta)$ is a function of the defining functions of α and β.

***20·64**. $\vdash :. (\alpha) . f\alpha : (\alpha) . g\alpha : \supset . f\beta . g\beta$

Dem.

$\vdash . *4\cdot2 . (*20\cdot07) . \supset$

$\vdash :. (\alpha) . f\alpha : (\alpha) . g\alpha : \equiv : (\phi) . f\{\hat{z}(\phi ! z)\} : (\phi) . g\{\hat{z}(\phi ! z)\} :$

$[*10\cdot14] \quad \supset : f\{\hat{z}(\psi ! z)\} . g\{\hat{z}(\psi ! z)\} :. \supset \vdash . \text{Prop}$

Observe that "β" is merely an abbreviation for any symbol of the form $\hat{z}(\psi ! z)$. This is why nothing further is required in the above proof.

The above proposition is the analogue of *10·14. Like that proposition, it requires, for the significance of the conclusion, that f and g should be functions which take arguments of the same type. This is not required for the significance of the hypothesis. Hence, though the above proposition is true whenever it is significant, it is not true whenever its hypothesis is significant.

***20·7**. $\vdash : (\exists g) : f\alpha . \equiv_\alpha . g ! \alpha \quad [*20\cdot112]$

This is the analogue of *12·1.

***20·701**. $\vdash : (\exists g) : f\{\hat{z}(\phi ! z), x\} . \equiv_{\phi, x} . g ! \{\hat{z}(\phi ! z), x\}$

[The proof proceeds as in *20·112, using *12·11 instead of *12·1.]

***20·702.** $\vdash : (\exists g) : f\{x, \hat{z}(\phi ! z)\} . \equiv_{\phi, x} . g ! \{x, \hat{z}(\phi ! z)\}$

[Proof as in *20·701.]

***20·703.** $\vdash : (\exists g) : f\{\hat{z}(\phi ! z), \hat{z}(\psi ! z)\} . \equiv_{\phi, \psi} . g ! \{\hat{z}(\phi ! z), \hat{z}(\psi ! z)\}$

Dem.

$\vdash . *10\cdot311 . \supset \vdash :. f\{\chi ! \hat{z}, \theta ! \hat{z}\} . \equiv_{\chi, \theta} . g ! \{\chi ! \hat{z}, \theta ! \hat{z}\} : \supset :$

$\phi ! x \equiv_x \chi ! x . \psi ! x \equiv_x \theta ! x . f\{\chi ! \hat{z}, \theta ! \hat{z}\} . \equiv_{\chi, \theta} .$

$\phi ! x \equiv_x \chi ! x . \psi ! x \equiv_x \theta ! x . g ! \{\chi ! \hat{z}, \theta ! \hat{z}\}$ (1)

$\vdash . (1) . *11\cdot11\cdot3\cdot341 . \supset$

$\vdash :. \mathrm{Hp}(1) . \quad \supset : (\exists \chi, \theta) . \phi ! x \equiv_x \chi ! x . \psi ! x \equiv_x \theta ! x . f\{\chi ! \hat{z}, \theta ! \hat{z}\} . \equiv_{\phi, \psi} .$

$(\exists \chi, \theta) . \phi ! x \equiv_x \chi ! x . \psi ! x \equiv_x \theta ! x . g ! \{\chi ! \hat{z}, \theta ! \hat{z}\} :$

[*20·1.*10·35] $\supset : f\{\hat{z}(\phi ! z), \hat{z}(\psi ! z)\} . \equiv_{\phi, \psi} . g ! \{\phi ! \hat{z}, \psi ! \hat{z}\}$ (2)

$\vdash . (2) . *10\cdot11\cdot281 . \supset$

$\vdash :. (\exists g) : f\{\chi ! \hat{z}, \theta ! \hat{z}\} . \equiv_{\chi, \theta} . g ! \{\chi ! \hat{z}, \theta ! \hat{z}\} : \supset :$

$(\exists g) : f\{\hat{z}(\phi ! z), \hat{z}(\psi ! z)\} . \equiv_{\phi, \psi} . g ! \{\hat{z}(\phi ! z), \hat{z}(\psi ! z)\}$ (3)

$\vdash . (3) . *12\cdot11 . \supset \vdash . \mathrm{Prop}$

*20·701·702·703 give the analogues, for classes, of *12·11.

***20·71.** $\vdash :. \alpha = \beta . \equiv : g ! \alpha . \supset_g . g ! \beta$ [*20·19]

This is the analogue of *13·01.

This completes the proof that all propositions hitherto given apply to classes as well as to individuals. Precisely similar reasoning extends this result to classes of classes, classes of classes of classes, etc.

From the above propositions it appears that, although expressions such as $\hat{z}(\phi z)$ have no meaning in isolation, yet those of their formal properties with which we have been hitherto concerned are the same as the corresponding properties of symbols which have a meaning in isolation. Hence nothing in the apparatus hitherto introduced requires us to determine whether a given symbol stands for a class or not, unless the symbol occurs in a way in which only a class can occur significantly. This is an important result, which enables us to give much greater generality to our propositions than would otherwise be possible.

The two following propositions (*20·8·81) are consequences of *13·3. The "type" of any object x will be defined in *63 as the class of terms either identical with x or not identical with x. We may define the "type of the arguments to $\phi\hat{z}$" as the class of arguments x for which "ϕx" is significant, *i.e.* the class $\hat{x}(\phi x \lor \sim \phi x)$. Then the first of the following propositions shows that if "ϕa" is significant, the type of the arguments to $\phi\hat{z}$ is the type of a; the second proposition shows that, if "ϕa" and "ψa" are both significant, the type of the arguments to $\phi\hat{z}$ is the same as the type of the arguments to $\psi\hat{z}$, because each is the type of a. *20·8 will be used in *63·11, which is a fundamental proposition in the theory of relative types.

***20·8.** $\vdash : \phi a \lor \sim \phi a \,.\supset .\, \hat{x}(\phi x \lor \sim \phi x) = \hat{x}(x = a \,.\lor .\, x \neq a)$

Dem.

$\vdash . *13·3 . *10·11·21 . \supset$

$\vdash :: \mathrm{Hp} . \supset :. \phi x \lor \sim \phi x . \equiv_x : x = a . \lor . x \neq a :.$

$[*20·15] \supset :. \hat{x}(\phi x \lor \sim \phi x) = \hat{x}(x = a . \lor . x \neq a) :: \supset \vdash . \mathrm{Prop}$

***20·81.** $\vdash : \phi a \lor \sim \phi a \,.\, \psi a \lor \sim \psi a \,.\supset .\, \hat{x}(\phi x \lor \sim \phi x) = \hat{x}(\psi x \lor \sim \psi x)$

Dem.

$\vdash . *20·8 . \supset \vdash : \mathrm{Hp} . \supset . \hat{x}(\phi x \lor \sim \phi x) = \hat{x}(x = a . \lor . x \neq a)$ (1)

$\vdash . *20·8 . \supset \vdash : \mathrm{Hp} . \supset . \hat{x}(\psi x \lor \sim \psi x) = \hat{x}(x = a . \lor . x \neq a)$ (2)

$\vdash . (1) . (2) . *10·121·13 . \mathrm{Comp} . \supset$

$\vdash : \mathrm{Hp} . \supset . \hat{x}(\phi x \lor \sim \phi x) = \hat{x}(x = a . \lor . x \neq a) . \hat{x}(\psi x \lor \sim \psi x) = \hat{x}(x = a . \lor . x \neq a) .$

$[*20·24] \supset . \hat{x}(\phi x \lor \sim \phi x) = \hat{x}(\psi x \lor \sim \psi x) : \supset \vdash . \mathrm{Prop}$

In the third line of the above proof, the use of *10·121 depends upon the fact that the "a" in both (1) and (2) must be such as to render the hypothesis significant, *i.e.* such as to render

$$\text{“}\phi a \lor \sim \phi a \,.\, \psi a \lor \sim \psi a\text{”}$$

significant. Hence the "a" in (1) and the "a" in (2) must be of the same type, by *10·121, and hence by *10·13 we can assert the product of (1) and (2), identifying the two "a's."

Since a type is the range of significance of a function, if ϕx is a function which is always true, $\hat{z}(\phi z)$ must be a type. For if a function is always true, the arguments for which it is true are the same as the arguments for which it is significant; hence $\hat{z}(\phi z)$ is the range of significance of ϕx, if $(x) . \phi x$ holds. Thus any class α is a type if $(x) . x \in \alpha$. It follows that, whatever function ϕ may be, $\hat{x}(\phi x \lor \sim \phi x)$ is a type; and in particular, $\hat{x}(x = a . \lor . x \neq a)$ is a type. Since a is a member of this class, this class is the type to which a belongs. In virtue of *20·8, if ϕa is significant, the type to which a belongs is the class of arguments for which ϕx is significant, *i.e.* $\hat{x}(\phi x \lor \sim \phi x)$. And if there is any argument a for which ϕa and ψa are both significant, then $\phi \hat{x}$ and $\psi \hat{x}$ have the same range of significance, in virtue of *20·81.

$*21$. GENERAL THEORY OF RELATIONS

Summary of $*21$.

The definitions and propositions of this number are exactly analogous to those of $*20$, from which they differ by being concerned with functions of two variables instead of one. A *relation*, as we shall use the word, will be understood in extension: it may be regarded as the class of couples (x, y) for which some given function $\psi(x, y)$ is true. Its relation to the function $\psi(\hat{x}, \hat{y})$ is just like that of the class to its determining function. We put

$*21\cdot01$. $f\{\hat{x}\hat{y}\psi(x, y)\} . = : (\exists\phi) : \phi!(x, y) . \equiv_{x,y} . \psi(x, y) : f\{\phi!(\hat{u}, \hat{v})\}$ Df

Here "$\hat{x}\hat{y}\psi(x, y)$" has no meaning in isolation, but only in certain of its uses. In $*21\cdot01$ the *alphabetical* order of u and v corresponds to the *typographical* order of $\hat{x}$ and $\hat{y}$ in $f\{\hat{x}\hat{y}\psi(x, y)\}$, so that

$$f\{\hat{y}\hat{x}\psi(x, y)\} . = : (\exists\phi) : \phi!(x, y) . \equiv_{x,y} . \psi(x, y) : f\{\phi!(\hat{v}, \hat{u})\} \quad \text{Df}$$

This is important in relation to the substitution-convention below.

It will be shown that

$$\hat{x}\hat{y}\psi(x, y) = \hat{x}\hat{y}\chi(x, y) . \equiv : \psi(x, y) . \equiv_{x,y} . \chi(x, y),$$

i.e. that two relations, as above defined, are identical when, and only when, they are satisfied by the same pair of arguments.

For substitution in $\phi!(\hat{x}, \hat{y})$ and $\phi!(\hat{y}, \hat{x})$, we adopt the convention that when a function (as opposed to its values) is represented in a form involving $\hat{x}$ and $\hat{y}$, or any other two letters of the alphabet, the value of this function for the arguments a and b is to be found by substituting a for $\hat{x}$ and b for $\hat{y}$, while the value for the arguments b and a is to be found by substituting b for $\hat{x}$ and a for $\hat{y}$. That is, the argument mentioned first is to be substituted for the letter which comes first in the alphabet, and the argument mentioned second for the later letter; thus the mode of substitution depends upon the *alphabetical* order of the letters which have circumflexes and the *typographical* order of the other letters.

The above convention as to order is presupposed in the following definition, where a is the first argument mentioned and b the second:

$*21\cdot02$. $a\{\phi!(\hat{x}, \hat{y})\}b . = . \phi!(a, b)$ Df

Hence, following the convention,

$$b\{\phi!(\hat{x}, \hat{y})\}a . = . \phi!(b, a) \quad \text{Df}$$
$$a\{\phi!(\hat{y}, \hat{x})\}b . = . \phi!(b, a) \quad \text{Df}$$
$$b\{\phi!(\hat{y}, \hat{x})\}a . = . \phi!(a, b) \quad \text{Df}$$

This definition is not used as it stands, but is introduced for the sake of

$$a\{\hat{x}\hat{y}\psi(x, y)\}b . \equiv : (\exists\phi) : \phi!(x, y) . \equiv_{x,y} . \psi(x, y) : \phi!(a, b)$$

which results from $*21 \cdot 01 \cdot 02$. We shall use capital Latin letters to represent variable expressions of the form $\hat{x}\hat{y}\phi\,!\,(x, y)$, just as we used Greek letters for variable expressions of the form $\hat{z}\,(\phi\,!\,z)$. If a capital Latin letter, say R, is used as an apparent variable, it is supposed that the R which occurs in the form "(R)" or "$(\exists R)$" is to be replaced by "(ϕ)" or "$(\exists\phi)$," while the R which occurs later is to be replaced by "$\hat{x}\hat{y}\phi\,!\,(x, y)$." In fact we put

$$(R) \,.\, fR \,. = .\, (\phi) \,.\, f\{\hat{x}\hat{y}\phi\,!\,(x, y)\} \quad \text{Df.}$$

The use of single letters for such expressions as $\hat{x}\hat{y}\phi\,(x, y)$ is a practically indispensable convenience.

The following is the definition of the class of relations:

$*21 \cdot 03$. $\mathrm{Rel} = \hat{R}\,\{(\exists\phi) \,.\, R = \hat{x}\hat{y}\phi\,!\,(x, y)\}$ Df

Similar remarks apply to it as to the definition of "Cls" ($*20 \cdot 03$).

In virtue of the definitions $*21 \cdot 01 \cdot 02$ and the convention as to capital Latin letters, the notation "xRy" will mean "x has the relation R to y." This notation is practically convenient, and will, after the preliminaries, wholly replace the cumbrous notation $x\,\{\hat{x}\hat{y}\phi\,(x, y)\}\,y$.

The proofs of the propositions of this number are usually omitted, since they are exactly analogous to those of $*20$, merely substituting $*12 \cdot 11$ for $*12 \cdot 1$, and propositions in $*11$ for propositions in $*10$.

The propositions of this number, like those of $*20$, fall into three sections. Those of the second section are seldom referred to. Those of the third section, extending to relations the formal properties hitherto assumed or proved for individuals and functions, are not explicitly referred to in the sequel, but are constantly relevant, namely whenever a proposition which has been assumed or proved for individuals and functions is applied to relations. The principal propositions of the first section are the following.

$*21 \cdot 15$. $\vdash :.\, \psi\,(x, y) \,.\, \equiv_{x,y} .\, \chi\,(x, y) : \equiv .\, \hat{x}\hat{y}\psi\,(x, y) = \hat{x}\hat{y}\chi\,(x, y)$

I.e. two relations are identical when, and only when, their defining functions are formally equivalent.

$*21 \cdot 31$. $\vdash :.\, \hat{x}\hat{y}\psi\,(x, y) = \hat{x}\hat{y}\chi\,(x, y) \,.\, \equiv : x\,\{\hat{x}\hat{y}\psi\,(x, y)\}\,y \,.\, \equiv_{x,y} .\, x\,\{\hat{x}\hat{y}\chi\,(x, y)\}\,y$

I.e. two relations are identical when, and only when, they hold between the same pairs of terms. The same fact is expressed by the following proposition:

$*21 \cdot 43$. $\vdash :.\, R = S \,.\, \equiv : xRy \,.\, \equiv_{x,y} .\, xSy$

$*21 \cdot 2 \cdot 21 \cdot 22$ show that identity of relations is reflexive, symmetrical and transitive.

$*21 \cdot 3$. $\vdash : x\,\{\hat{x}\hat{y}\psi\,(x, y)\}\,y \,.\, \equiv .\, \psi\,(x, y)$

I.e. two terms have a given relation when, and only when, they satisfy its defining function.

***21·151.** $\vdash . (\exists\phi) . \hat{x}\hat{y}\psi(x, y) = \hat{x}\hat{y}\phi!(x, y)$

I.e. every relation can be defined by a predicative function. Hence when, using *21·07 or *21·071, we have a relation as apparent variable, and are therefore confined to predicative defining functions, there is no loss of generality.

***21·01.** $f\{\hat{x}\hat{y}\psi(x, y)\} . = : (\exists\phi) : \phi!(x, y) . \equiv_{x,y} . \psi(x, y) : f\{\phi!(\hat{u}, \hat{v})\}$ Df

On the convention as to order in *21·01·02, cf. p. 200, and thus relate $\hat{u}, \hat{v}$ to $\hat{x}, \hat{y}$ so that

$$f\{\hat{y}\hat{x}\psi(x, y)\} . = : (\exists\phi) : \phi!(x, y) . \equiv_{x,y} . \psi(x, y) : f\{\phi!(\hat{v}, \hat{u})\} \quad \text{Df}$$

***21·02.** $a\{\phi!(\hat{x}, \hat{y})\}b . = . \phi!(a, b)$ Df

***21·03.** $\text{Rel} = \hat{R}\{(\exists\phi) . R = \hat{x}\hat{y}\phi!(x, y)\}$ Df

The following definitions merely extend to relations, with as little modification as possible, the definitions already given for other symbols.

***21·07.** $(R) . fR . = . (\phi) . f\{\hat{x}\hat{y}\phi!(x, y)\}$ Df

***21·071.** $(\exists R) . fR . = . (\exists\phi) . f\{\hat{x}\hat{y}\phi!(x, y)\}$ Df

***21·072.** $[(\iota R)(\phi R)] . f(\iota R)(\phi R) . = : (\exists S) : \phi R . \equiv_R . R = S : fS$ Df

***21·08.** $f\{\hat{R}\hat{S}\psi(R, S)\} . = : (\exists\phi) : \psi(R, S) . \equiv_{R,S} . \phi!(R, S) : f\{\phi!(\hat{R}, \hat{S})\}$ Df

***21·081.** $P\{\phi!(\hat{R}, \hat{S})\}Q . = . \phi!(P, Q)$ Df

The convention as to typographic and alphabetic order is here retained.

***21·082.** $f\{\hat{R}(\psi R)\} . = : (\exists\phi) : \psi R . \equiv_R . \phi!R : f(\phi!\hat{R})$ Df

***21·083.** $R \epsilon \phi!\hat{R} . = . \phi!R$ Df

***21·1.** $\vdash :. f\{\hat{x}\hat{y}\psi(x, y)\} . \equiv : (\exists\phi) : \phi!(x, y) . \equiv_{x,y} . \psi(x, y) : f\{\phi!(\hat{u}, \hat{v})\}$
[*4·2 . (*21·01)]

***21·11.** $\vdash :. \psi(x, y) . \equiv_{x,y} . \chi(x, y) : \supset : f\{\hat{x}\hat{y}\psi(x, y)\} . \equiv . f\{\hat{x}\hat{y}\chi(x, y)\}$
[*4·86·36 . *10·281 . *21·1]

This proposition proves that every proposition about a relation expresses an extensional property of the determining function.

***21·111.** $\vdash :. f\{\phi!(\hat{x}, \hat{y})\} . \equiv_\phi . g\{\phi!(x, y)\} : \supset : f\{\hat{x}\hat{y}\phi!(x, y)\} . \equiv_\phi . g\{\hat{x}\hat{y}\phi!(x, y)\}$
[Fact . *11·11·3 . *10·281 . *21·1]

***21·112.** $\vdash :. (\exists g) :. f\{\hat{x}\hat{y}\phi!(x, y)\} . \equiv_\phi . g!\{\hat{x}\hat{y}\phi!(x, y)\}$ [*12·1 . *21·111]

It is *12·1, not *12·11, which is required in this proposition, because we are concerned with a function (f) of *one* variable, namely ϕ, although that one variable is itself a function of two variables.

***21·12.** $\vdash :. (\exists\phi) :. \phi!(x, y) . \equiv_{x,y} . \psi(x, y) : f\{\hat{x}\hat{y}\psi(x, y)\} . \equiv . f\{\hat{x}\hat{y}\phi!(x, y)\}$
[*21·11 . *12·11]

This is the first use of the primitive proposition *12·11, except in *20·701·702·703.

***21·13.** $\vdash :. \psi(x, y) . \equiv_{x,y} . \chi(x, y) : \supset . \hat{x}\hat{y}\psi(x, y) = \hat{x}\hat{y}\chi(x, y)$
[*21·1 . *12·11 . *13·195]

***21·14.** $\vdash :. \hat{x}\hat{y}\,\psi(x,y) = \hat{x}\hat{y}\,\chi(x,y) . \supset : \psi(x,y) . \equiv_{x,y} . \chi(x,y)$
[Proof as in *20·14]

***21·15.** $\vdash :. \psi(x,y) . \equiv_{x,y} . \chi(x,y) : \equiv . \hat{x}\hat{y}\,\psi(x,y) = \hat{x}\hat{y}\,\chi(x,y)$ [*21·13·14]

This proposition states that two double functions determine the same relation when, and only when, they are formally equivalent, *i.e.* are satisfied by the same pairs of arguments. This is a fundamental property of relations as defined above (*21·01).

***21·151.** $\vdash . (\exists\phi) . \hat{x}\hat{y}\,\psi(x,y) = \hat{x}\hat{y}\,\phi!(x,y)$ [*21·15 . *12·11]

***21·16.** $\vdash : (\exists\phi) : f\{\hat{x}\hat{y}\,\psi(x,y)\} . \equiv . f\{\hat{x}\hat{y}\,\phi!(x,y)\}$ [*21·12]

***21·17.** $\vdash : (\phi) . f\{\hat{x}\hat{y}\,\phi!(x,y)\} . \supset . f\{\hat{x}\hat{y}\,\psi(x,y)\}$ [*21·16 . *10·1]

***21·18.** $\vdash :. \hat{x}\hat{y}\,\phi(x,y) = \hat{x}\hat{y}\,\psi(x,y) . \supset : f\{\hat{x}\hat{y}\,\phi(x,y)\} . \equiv . f\{\hat{x}\hat{y}\,\psi(x,y)\}$
[*21·11·15]

***21·19.** $\vdash :. \hat{x}\hat{y}\,\psi(x,y) = \hat{x}\hat{y}\,\chi(x,y) . \equiv : (f) : f!\,\hat{x}\hat{y}\,\psi(x,y) . \supset . f!\,\hat{x}\hat{y}\,\chi(x,y)$
[*21·18 . *10·11·21 . *21·1 . *10·35 . (*13·01) . *21·112 . *10·301]

***21·191.** $\vdash :. \hat{x}\hat{y}\,\psi(x,y) = \hat{x}\hat{y}\,\chi(x,y) . \equiv : (f) : f!\,\hat{x}\hat{y}\,\psi(x,y) . \equiv . f!\,\hat{x}\hat{y}\,\chi(x,y)$
[*21·18·19]

***21·2.** $\vdash . \hat{x}\hat{y}\,\phi(x,y) = \hat{x}\hat{y}\,\phi(x,y)$ [*21·15 . *4·2]

***21·21.** $\vdash : \hat{x}\hat{y}\,\phi(x,y) = \hat{x}\hat{y}\,\psi(x,y) . \equiv . \hat{x}\hat{y}\,\psi(x,y) = \hat{x}\hat{y}\,\phi(x,y)$ [*21·15 . *10·32]

***21·22.** $\vdash : \hat{x}\hat{y}\,\phi(x,y) = \hat{x}\hat{y}\,\psi(x,y) . \hat{x}\hat{y}\,\psi(x,y) = \hat{x}\hat{y}\,\chi(x,y) . \supset .$
$\hat{x}\hat{y}\,\phi(x,y) = \hat{x}\hat{y}\,\chi(x,y)$ [*21·15 . *10·301]

***21·23.** $\vdash : \hat{x}\hat{y}\,\phi(x,y) = \hat{x}\hat{y}\,\psi(x,y) . \hat{x}\hat{y}\,\phi(x,y) = \hat{x}\hat{y}\,\chi(x,y) . \supset .$
$\hat{x}\hat{y}\,\psi(x,y) = \hat{x}\hat{y}\,\chi(x,y)$ [*21·21·22]

***21·24.** $\vdash : \hat{x}\hat{y}\,\psi(x,y) = \hat{x}\hat{y}\,\phi(x,y) . \hat{x}\hat{y}\,\chi(x,y) = \hat{x}\hat{y}\,\phi(x,y) . \supset .$
$\hat{x}\hat{y}\,\psi(x,y) = \hat{x}\hat{y}\,\chi(x,y)$ [*21·21·22]

***21·3.** $\vdash : x\,\{\hat{x}\hat{y}\,\psi(x,y)\}\,y . \equiv . \psi(x,y)$ [*21·1·02 . *10·43·35 . *12·11]

This shows that x has to y the relation determined by ψ when, and only when, x and y satisfy $\psi(x,y)$.

Note that the primitive proposition *12·11 is again required here.

***21·31.** $\vdash :. \hat{x}\hat{y}\,\psi(x,y) = \hat{x}\hat{y}\,\chi(x,y) . \equiv : x\,\{\hat{x}\hat{y}\,\psi(x,y)\}\,y . \equiv_{x,y} . x\,\{\hat{x}\hat{y}\,\chi(x,y)\}\,y$
[*21·15·3]

***21·32.** $\vdash . \hat{x}\hat{y}\,[x\,\{\hat{x}\hat{y}\,\phi(x,y)\}\,y] = \hat{x}\hat{y}\,\phi(x,y)$ [*21·3·15]

***21·33.** $\vdash :. R = \hat{x}\hat{y}\,\phi(x,y) . \equiv : xRy . \equiv_{x,y} . \phi(x,y)$ [*21·31·3]

Here R is written for some expression of the form $\hat{x}\hat{y}\,\psi(x,y)$. The use of a single capital letter for a relation is convenient whenever the determining function is irrelevant.

***21·4.** $\vdash : R \in \mathrm{Rel} . \equiv . (\exists\phi) . R = \hat{x}\hat{y}\,\phi!(x,y)$ [*20·3 . (*21·03)]

***21·41.** $\vdash . \hat{x}\hat{y}\,\phi(x,y) \in \mathrm{Rel}$ [*21·4·151]

***21·42.** $\vdash . \hat{x}\hat{y}\,(xRy) = R$ [*21·3·15]

***21·43.** $\vdash :. R = S . \equiv : xRy . \equiv_{x,y} . xSy$ [*21·15·3]

***20·5·51·52** have no analogues in the theory of relations.

***21·53.** $\vdash :. S = R \,.\, \supset_S . \, \phi S : \equiv . \, \phi R$ [*10·1 . *21·2·18·21 . Comm . *10·11·21]

***21·54.** $\vdash :. (\exists S) . \, S = R \,.\, \phi S \,.\, \equiv . \, \phi R$ [*21·18 . *10·11·23 . *21·2 . *10·24]

***21·55.** $\vdash . \, \hat{x}\hat{y} \phi (x, y) = (\imath R) \{ xRy \,.\, \equiv_{x,y} . \, \phi (x, y) \}$ [*21·33·54 . *14·1]

***21·56.** $\vdash . \, \mathrm{E} \,!\, (\imath R) \{ xRy \,.\, \equiv_{x,y} . \, \phi (x, y) \}$ [*21·55 . *14·21]

***21·57.** $\vdash :. \hat{x}\hat{y} \phi (x, y) = (\imath R) (fR) \,.\, \supset : g \{ \hat{x}\hat{y} \phi (x, y) \} \,.\, \equiv . \, g \{ (\imath R) (fR) \}$ [*14·1 . *21·54 . *13·183]

***21·58.** $\vdash : \hat{x}\hat{y} \phi (x, y) = (\imath R) \{ R = \hat{x}\hat{y} \phi (x, y) \}$ [*4·2 . *10·11 . *21·54 . *14·1]

The following propositions are the analogues of *20·6 ff., and have a similar purpose.

***21·6.** $\vdash : (\exists R) . \, fR \,.\, \equiv . \sim \{ (R) . \sim fR \}$ [Proof as in *20·6]

***21·61.** $\vdash : (R) . \, fR \,.\, \supset . \, fS$ [Proof as in *20·61]

***21·62.** When fR is true, whatever possible argument of the form $\hat{x}\hat{y} \phi \,!\, (x, y)$ R may be, $(R) . \, fR$ is true. [Proof as in *20·62]

***21·63.** $\vdash :. (R) . \, p \vee fR \,.\, \supset : p \,.\, \vee . \, (R) . \, fR$ [Proof as in *20·63]

***21·631.** If "fR" is significant, then if S is of the same type as R, "fS" is significant, and vice versa. [Proof as in *20·631]

***21·632.** If, for some R, there is a proposition fR, then there is a function $f\hat{R}$, and vice versa. [Proof as in *20·632]

***21·633.** "Whatever possible relation R may be, $f(R, S)$ is true whatever possible relation S may be" implies "whatever possible relation S may be, $f(R, S)$ is true whatever possible relation R may be."
[Proof as in *20·633]

***21·64.** $\vdash :. (R) . \, fR : (R) . \, gR : \supset . \, fS \,.\, gS$ [Proof as in *20·64]

***21·7.** $\vdash : (\exists g) : fR \,.\, \equiv_R . \, g \,!\, R$ [Proof as in *20·7]

***21·701.** $\vdash : (\exists g) : f(R, x) \,.\, \equiv_{R,x} . \, g \,!\, (R, x)$ [Proof as in *20·701]

***21·702.** $\vdash : (\exists g) : f(x, R) \,.\, \equiv_{R,x} . \, g \,!\, (R, x)$ [Proof as in *20·702]

***21·703.** $\vdash : (\exists g) : f(R, S) \,.\, \equiv_{R,S} . \, g \,!\, (R, S)$ [Proof as in *20·703]

***21·704.** $\vdash : (\exists g) : f(R, \alpha) \,.\, \equiv_{R,\alpha} . \, g \,!\, (R, \alpha)$ [Proof as in *20·703]

***21·705.** $\vdash : (\exists g) : f(\alpha, R) \,.\, \equiv_{\alpha, R} . \, g \,!\, (\alpha, R)$ [Proof as in *20·703]

***21·71.** $\vdash :. R = S \,.\, \equiv : g \,!\, R \,.\, \supset_g . \, g \,!\, S$ [Proof as in *20·71]

From the above propositions it appears that relations, like classes, have all the formal properties which they would have if they were symbols having a meaning in isolation. Hence unless a symbol occurs in a way in which only a relation can occur significantly, we do not need to decide whether it stands for a relation or not. This result, like the corresponding result for classes mentioned at the end of *20, is important as giving greater generality to our propositions than they would otherwise possess. The results obtained in *20 and *21 for classes and relations whose members or terms are neither classes nor relations can be extended, by mere repetition of the proofs, to classes of classes, classes of relations, relations of classes, relations of relations, and so on.

$*22$. CALCULUS OF CLASSES

Summary of $*22$.

In this number we reach what was historically the starting-point of symbolic logic. The Greek letters used (except ϕ, ψ, χ, θ) are always to stand for expressions of the form $\hat{x}(\phi ! x)$, or, where the Greek letters are not apparent variables, $\hat{x}(\phi x)$. The small Latin letters may either be such as have a meaning in isolation, or may represent classes or relations; this is possible in virtue of the notes at the ends of $*20$ and $*21$. We put:

$*22{\cdot}01$. $\alpha \subset \beta \,.=:\, x \in \alpha \,.\supset_x .\, x \in \beta$ Df

This defines "the class α is contained in the class β," or "all α's are β's."

$*22{\cdot}02$. $\alpha \cap \beta = \hat{x}(x \in \alpha \,.\, x \in \beta)$ Df

This defines the logical product or common part of two classes α and β.

$*22{\cdot}03$. $\alpha \cup \beta = \hat{x}(x \in \alpha \,.\vee.\, x \in \beta)$ Df

This defines the logical sum of two classes; it is the class consisting of all the members of one together with all the members of the other.

$*22{\cdot}04$. $-\alpha = \hat{x}(x \sim\in \alpha)$ Df

This defines the negation of a class. It is read "not-α." It does not contain every object x concerning which "$x \in \alpha$" is *not true*, but only those objects concerning which "$x \in \alpha$" is *false*; *i.e.* it excludes those objects for which "$x \in \alpha$" is meaningless. Thus it consists of all objects, of the type next below α, which are not members of α; but it does not contain objects of any other type but this.

$*22{\cdot}05$. $\alpha - \beta = \alpha \cap -\beta$ Df

This definition gives an abbreviation which is often convenient.

The postulates required for the algebra of logic have been enumerated by Huntington*. In our notation, they are as follows.

We assume a class K, with two rules of combination, namely $\cup$ and $\cap$; and we then require the following ten postulates:

I *a*. $a \cup b$ is in the class whenever a and b are in the class.
I *b*. $a \cap b$ is in the class whenever a and b are in the class.
II *a*. There is an element Λ such that $a \cup \Lambda = a$ for every element a.
II *b*. There is an element V such that $a \cap \mathrm{V} = a$ for every element a.
III *a*. $a \cup b = b \cup a$ whenever a, b, $a \cup b$ and $b \cup a$ are in the class.
III *b*. $a \cap b = b \cap a$ whenever a, b, $a \cap b$ and $b \cap a$ are in the class.

* *Trans. Amer. Math. Soc.* Vol. 5, July 1904, p. 292.

IV a. $a \cup (b \cap c) = (a \cup b) \cap (a \cup c)$ whenever $a, b, c, a \cup b, a \cup c, b \cap c, a \cup (b \cap c)$, and $(a \cup b) \cap (a \cup c)$ are in the class.

IV b. $a \cap (b \cup c) = (a \cap b) \cup (a \cap c)$ whenever $a, b, c, a \cap b, a \cap c, b \cup c, a \cap (b \cup c)$, and $(a \cap b) \cup (a \cap c)$ are in the class.

V. If the elements Λ and V in postulates II a and II b exist and are unique, then for every element a there is an element $-a$ such that $a \cup -a = V$ and $a \cap -a = \Lambda$.

VI. There are at least two elements, x and y, in the class, such that $x \neq y$.

The form of the above postulates is such that they are mutually independent, *i.e.* any nine of them are satisfied by interpretations of the symbols which do not satisfy the remaining one.

For our purposes, "K" must be replaced by "Cls." Λ and V will be the null-class and the universal class, which are defined in ∗24. Then the above ten postulates are proved below, as follows:

I a, in ∗22·37, namely "$\vdash . \alpha \cup \beta \in \text{Cls}$"
I b, in ∗22·36, namely "$\vdash . \alpha \cap \beta \in \text{Cls}$"
II a, in ∗24·24, namely "$\vdash . \alpha \cup \Lambda = \alpha$"
II b, in ∗24·26, namely "$\vdash . \alpha \cap V = \alpha$"
III a, in ∗22·57, namely "$\vdash . \alpha \cup \beta = \beta \cup \alpha$"
III b, in ∗22·51, namely "$\vdash . \alpha \cap \beta = \beta \cap \alpha$"
IV a, in ∗22·69, namely "$\vdash . (\alpha \cup \beta) \cap (\alpha \cup \gamma) = \alpha \cup (\beta \cap \gamma)$"
IV b, in ∗22·68, namely "$\vdash . (\alpha \cap \beta) \cup (\alpha \cap \gamma) = \alpha \cap (\beta \cup \gamma)$"
V, in ∗24·21·22, namely "$\vdash . \alpha \cap -\alpha = \Lambda$" and "$\vdash . \alpha \cup -\alpha = V$"
VI, in ∗24·1, namely "$\vdash . \Lambda \neq V$"

Hence, assuming Huntington's analysis of the postulates for the formal algebra of logic, the propositions proved in what follows suffice to establish that this algebra holds for classes. The corresponding propositions of ∗23 and ∗25 prove that it holds for relations, substituting Rel, $\dot{\cup}$, $\dot{\cap}$, $\dot{\Lambda}$, $\dot{V}$ for Cls, $\cup$, $\cap$, Λ, V.

The principal propositions of the present number are the following:

(1) Those embodying the formal rules:

∗22·51. $\vdash . \alpha \cap \beta = \beta \cap \alpha$

∗22·57. $\vdash . \alpha \cup \beta = \beta \cup \alpha$

These embody the commutative law.

∗22·52. $\vdash . (\alpha \cap \beta) \cap \gamma = \alpha \cap (\beta \cap \gamma)$

∗22·7. $\vdash . (\alpha \cup \beta) \cup \gamma = \alpha \cup (\beta \cup \gamma)$

These embody the associative law.

∗22·5. $\vdash . \alpha \cap \alpha = \alpha$

∗22·56. $\vdash . \alpha \cup \alpha = \alpha$

These embody the law of tautology.

∗22·68. $\vdash . (\alpha \cap \beta) \cup (\alpha \cap \gamma) = \alpha \cap (\beta \cup \gamma)$

∗22·69. $\vdash . (\alpha \cup \beta) \cap (\alpha \cup \gamma) = \alpha \cup (\beta \cap \gamma)$

These embody the distributive law. It will be seen that the second results from the first by everywhere interchanging the signs of addition and multiplication.

∗22·8. $\vdash . -(-\alpha) = \alpha$

This is the principle of double negation.

∗22·81. $\vdash : \alpha \subset \beta . \equiv . -\beta \subset -\alpha$

This is the principle of transposition.

(2) Other useful propositions:

∗22·44. $\vdash : \alpha \subset \beta . \beta \subset \gamma . \supset . \alpha \subset \gamma$

∗22·441. $\vdash : \alpha \subset \beta . x \in \alpha . \supset . x \in \beta$

These embody the two forms of the syllogism in Barbara.

∗22·62. $\vdash : \alpha \subset \beta . \equiv . \alpha \cup \beta = \beta$

∗22·621. $\vdash : \alpha \subset \beta . \equiv . \alpha \cap \beta = \alpha$

These two propositions enable us to transform any inclusion ($\alpha \subset \beta$) into an equation.

∗22·91. $\vdash . \alpha \cup \beta = \alpha \cup (\beta - \alpha)$

I.e. "α or β" is identical with "α or the part of β which is excluded from α."

∗22·01. $\alpha \subset \beta . = : x \in \alpha . \supset_x . x \in \beta$ Df

∗22·02. $\alpha \cap \beta = \hat{x}(x \in \alpha . x \in \beta)$ Df

∗22·03. $\alpha \cup \beta = \hat{x}(x \in \alpha . \vee . x \in \beta)$ Df

∗22·04. $-\alpha = \hat{x}(x \sim\in \alpha)$ Df

∗22·05. $\alpha - \beta = \alpha \cap -\beta$ Df

∗22·1. $\vdash :. \alpha \subset \beta . \equiv : x \in \alpha . \supset_x . x \in \beta$ [∗4·2 . (∗22·01)]

∗22·2. $\vdash . \alpha \cap \beta = \hat{x}(x \in \alpha . x \in \beta)$ [∗20·2 . (∗22·02)]

∗22·3. $\vdash . \alpha \cup \beta = \hat{x}(x \in \alpha . \vee . x \in \beta)$ [∗20·2 . (∗22·03)]

∗22·31. $\vdash . -\alpha = \hat{x}(x \sim\in \alpha)$ [∗20·2 . (∗22·04)]

∗22·32. $\vdash . \alpha - \beta = \hat{x}(x \in \alpha . x \sim\in \beta)$ [∗20·2 . (∗22·05) . ∗22·2 . ∗20·32]

∗22·33. $\vdash : x \in \alpha \cap \beta . \equiv . x \in \alpha . x \in \beta$ [∗20·3 . ∗22·2]

∗22·34. $\vdash :. x \in \alpha \cup \beta . \equiv : x \in \alpha . \vee . x \in \beta$ [∗20·3 . ∗22·3]

∗22·35. $\vdash : x \in -\alpha . \equiv . x \sim\in \alpha$ [∗20·3 . ∗22·31]

∗22·351. $\vdash . -\alpha \neq \alpha$

Dem.

$\vdash . ∗22·35 . ∗5·19 . \supset \vdash : \sim\{x \in -\alpha . \equiv . x \in \alpha\} :$

[∗10·11] $\supset \vdash : (x) : \sim\{x \in -\alpha . \equiv . x \in \alpha\} :$

[∗10·251] $\supset \vdash : \sim\{(x) : x \in -\alpha . \equiv . x \in \alpha\} :$

[∗20·43.Transp] $\supset \vdash : \sim(-\alpha = \alpha) : \supset \vdash .$ Prop

This proposition is used in proving that the null-class is not identical with the class containing everything (*24·1), which is used to show that at least two classes exist. Our axioms do not suffice to prove that more than one *individual* exists, but they prove the existence of at least two *classes* and at least two *relations*.

***22·36.** $\vdash . \alpha \cap \beta \in \text{Cls}$ [*20·41]

***22·37.** $\vdash . \alpha \cup \beta \in \text{Cls}$ [*20·41]

***22·38.** $\vdash . -\alpha \in \text{Cls}$ [*20·41]

***22·39.** $\vdash . \hat{z}(\phi z) \cap \hat{z}(\psi z) = \hat{z}(\phi z . \psi z)$

Dem.

$\vdash . *22·33 . \supset \vdash : x \in \hat{z}(\phi z) \cap \hat{z}(\psi z) . \equiv . x \in \hat{z}(\phi z) . x \in \hat{z}(\psi z) .$
[*20·3] $\equiv . \phi x . \psi x$ (1)
$\vdash . (1) . *20·33 . \supset \vdash . \text{Prop}$

***22·391.** $\vdash . \hat{z}(\phi z) \cup \hat{z}(\psi z) = \hat{z}(\phi z \vee \psi z)$ [Similar proof]

***22·392.** $\vdash . -\hat{z}(\phi z) = \hat{z}(\sim \phi z)$ [Similar proof]

***22·4.** $\vdash :. \alpha \subset \beta . \beta \subset \alpha . \equiv : x \in \alpha . \equiv_x . x \in \beta$

Dem.

$\vdash . *22·1 . \supset \vdash :: \alpha \subset \beta . \equiv : x \in \alpha . \supset_x . x \in \beta :. \beta \subset \alpha . \equiv : x \in \beta . \supset_x . x \in \alpha :.$
[*4·38] $\supset \vdash :: \alpha \subset \beta . \beta \subset \alpha . \equiv :. x \in \alpha . \supset_x . x \in \beta : x \in \beta . \supset_x . x \in \alpha :.$
[*10·22] $\equiv :. x \in \alpha . \equiv_x . x \in \beta :: \supset \vdash . \text{Prop}$

***22·41.** $\vdash : \alpha \subset \beta . \beta \subset \alpha . \equiv . \alpha = \beta$ [*22·4 . *20·43]

***22·42.** $\vdash . \alpha \subset \alpha$ [Id . *10·11]

***22·43.** $\vdash : \alpha \cap \beta \subset \alpha$ [*3·26 . *10·11]

***22·44.** $\vdash : \alpha \subset \beta . \beta \subset \gamma . \supset . \alpha \subset \gamma$ [*10·3]

This is one form of the syllogism in Barbara. Another form is the following:

***22·441.** $\vdash : \alpha \subset \beta . x \in \alpha . \supset . x \in \beta$ [*10·1 . Imp]

***22·45.** $\vdash : \alpha \subset \beta . \alpha \subset \gamma . \equiv . \alpha \subset \beta \cap \gamma$

Dem.

$\vdash . *22·1 . \supset \vdash :. \alpha \subset \beta . \alpha \subset \gamma . \equiv : x \in \alpha . \supset_x . x \in \beta : x \in \alpha . \supset_x . x \in \gamma :$
[*10·29] $\equiv : x \in \alpha . \supset_x . x \in \beta . x \in \gamma :$
[*22·33.*10·413] $\equiv : x \in \alpha . \supset_x . x \in \beta \cap \gamma :. \supset \vdash . \text{Prop}$

***22·46.** $\vdash : x \in \alpha . \alpha \subset \beta . \supset . x \in \beta$ [*22·441 . Perm]

***22·47.** $\vdash : \alpha \subset \gamma . \supset . \alpha \cap \beta \subset \gamma$ [22·43·44]

***22·48.** $\vdash : \alpha \subset \beta . \supset . \alpha \cap \gamma \subset \beta \cap \gamma$ [*10·31]

***22·481.** $\vdash : \alpha = \beta . \supset . \alpha \cap \gamma = \beta \cap \gamma$

Dem.

$\vdash . *22·41 . \supset :. \text{Hp} . \supset : \alpha \subset \beta . \beta \subset \alpha :$
[*22·48] $\supset : \alpha \cap \gamma \subset \beta \cap \gamma . \beta \cap \gamma \subset \alpha \cap \gamma :$
[*22·41] $\supset : \alpha \cap \gamma = \beta \cap \gamma :. \supset \vdash . \text{Prop}$

*22·49. $\vdash : \alpha \subset \beta \,.\, \gamma \subset \delta \,.\, \supset \,.\, \alpha \cap \gamma \subset \beta \cap \delta$ [*10·39]

*22·5. $\vdash \,.\, \alpha \cap \alpha = \alpha$

Dem.

$\vdash \,.\, *22{\cdot}33 \,.\, \supset \vdash :. \, x \in \alpha \cap \alpha \,.\, \equiv : x \in \alpha \,.\, x \in \alpha :$

[*4·24] $\equiv : x \in \alpha$ (1)

$\vdash \,.\, (1) \,.\, *10{\cdot}11 \,.\, *20{\cdot}43 \,.\, \supset \vdash \,.\,$ Prop

The above is the law of tautology for the logical multiplication of classes.

*22·51. $\vdash \,.\, \alpha \cap \beta = \beta \cap \alpha$ [*22·33 . *4·3 . *10·11 . *20·43]

*22·52. $\vdash \,.\, (\alpha \cap \beta) \cap \gamma = \alpha \cap (\beta \cap \gamma)$ [*22·33 . *4·32 . *10·11 . *20·43]

Thus logical multiplication of classes obeys the commutative and associative laws. References to *22·33·34·35 and to *20·43 will in future often be omitted.

*22·53. $\alpha \cap \beta \cap \gamma = (\alpha \cap \beta) \cap \gamma$ Df

This definition serves merely for the avoidance of brackets.

*22·54. $\vdash :. \, \alpha = \beta \,.\, \supset : \alpha \subset \gamma \,.\, \equiv \,.\, \beta \subset \gamma$ [*20·18]

*22·55. $\vdash :. \, \alpha = \beta \,.\, \supset : \gamma \subset \alpha \,.\, \equiv \,.\, \gamma \subset \beta$ [*20·18]

*22·551. $\vdash : \alpha = \beta \,.\, \supset \,.\, \alpha \cup \gamma = \beta \cup \gamma$ [*10·411]

*22·56. $\vdash \,.\, \alpha \cup \alpha = \alpha$ [*4·25 . *10·11]

The above is the law of tautology for the logical addition of classes.

*22·57. $\vdash \,.\, \alpha \cup \beta = \beta \cup \alpha$ [*4·31 . *10·11]

*22·58. $\vdash \,.\, \alpha \subset \alpha \cup \beta \,.\, \beta \subset \alpha \cup \beta$ [*1·3 . *2·2]

*22·59. $\vdash : \alpha \subset \gamma \,.\, \beta \subset \gamma \,.\, \equiv \,.\, \alpha \cup \beta \subset \gamma$

Dem.

$\vdash \,.\, *22{\cdot}1 \,.\, \supset \vdash :: \mathrm{Hp} \,.\, \equiv :. \, x \in \alpha \,.\, \supset_x \,.\, x \in \gamma : x \in \beta \,.\, \supset_x \,.\, x \in \gamma :.$

[*10·22] $\equiv :. \, (x) :. \, x \in \alpha \,.\, \supset \,.\, x \in \gamma : x \in \beta \,.\, \supset \,.\, x \in \gamma :.$

[*4·77.*10·271] $\equiv :. \, (x) :. \, x \in \alpha \,.\, \mathrm{v} \,.\, x \in \beta : \supset \,.\, x \in \gamma :.$

[*22·34.*10·413] $\equiv :. \, (x) : x \in \alpha \cup \beta \,.\, \supset \,.\, x \in \gamma :: \supset \vdash \,.\,$ Prop

The analogue of *4·78, *i.e.*

$$\alpha \subset \beta \,.\, \mathrm{v} \,.\, \alpha \subset \gamma : \equiv \,.\, \alpha \subset \beta \cup \gamma$$

is false. We have only

$$\alpha \subset \beta \,.\, \mathrm{v} \,.\, \alpha \subset \gamma : \supset \,.\, \alpha \subset \beta \cup \gamma.$$

A similar remark applies to the analogue of *4·79. Cf. *22·64·65.

*22·6. $\vdash :. \, x \in \alpha \cup \beta \,.\, \equiv : \alpha \subset \gamma \,.\, \beta \subset \gamma \,.\, \supset_\gamma \,.\, x \in \gamma$

Dem.

$\vdash \,.\, *22{\cdot}59 \,.\, \supset \vdash :. \, \alpha \subset \gamma \,.\, \beta \subset \gamma \,.\, \supset : x \in \alpha \cup \beta \,.\, \supset \,.\, x \in \gamma :.$

[Comm] $\supset \vdash :. \, x \in \alpha \cup \beta \,.\, \supset : \alpha \subset \gamma \,.\, \beta \subset \gamma \,.\, \supset \,.\, x \in \gamma :.$

[*10·11·21] $\supset \vdash :. \, x \in \alpha \cup \beta \,.\, \supset : \alpha \subset \gamma \,.\, \beta \subset \gamma \,.\, \supset_\gamma \,.\, x \in \gamma$ (1)

$\vdash \,.\, *10{\cdot}1 \,.\, \supset \vdash :. \, \alpha \subset \gamma \,.\, \beta \subset \gamma \,.\, \supset_\gamma \,.\, x \in \gamma : \supset : \alpha \subset \alpha \cup \beta \,.\, \beta \subset \alpha \cup \beta \,.\, \supset \,.\, x \in \alpha \cup \beta :$

[*22·58] $\supset : x \in \alpha \cup \beta$ (2)

$\vdash \,.\, (1) \,.\, (2) \,.\, \supset \vdash \,.\,$ Prop

***22·61.** $\vdash : \alpha \subset \beta . \supset . \alpha \subset \beta \cup \gamma$ [*22·44·58]

***22·62.** $\vdash : \alpha \subset \beta . \equiv . \alpha \cup \beta = \beta$

Dem.

$\vdash . *4{\cdot}72 . \supset \vdash :: x \in \alpha . \supset . x \in \beta : \equiv :. x \in \alpha . \vee . x \in \beta : \equiv . x \in \beta :.$

[*22·34] $\equiv :. x \in \alpha \cup \beta . \equiv . x \in \beta$ (1)

$\vdash . (1) . *10{\cdot}271 . \supset \vdash :: \alpha \subset \beta . \equiv :. x \in \alpha \cup \beta . \equiv_x . x \in \beta :.$

[*20·43] $\equiv :. \alpha \cup \beta = \beta :: \supset \vdash .$ Prop

***22·621.** $\vdash : \alpha \subset \beta . \equiv . \alpha \cap \beta = \alpha$ [*4·71]

The proof proceeds as in *22·62. The proposition *22·621 is one of the most useful propositions in the present number.

***22·63.** $\vdash : \alpha \cup (\alpha \cap \beta) = \alpha$ [*4·44]

The process of obtaining *22·63 from *4·44 is of the same kind as the process employed in the proofs that have been written out in this number. Hence only *4·44 is referred to. We shall similarly restrict references for later propositions in this number. The process is always roughly as follows: p, q, r are replaced by $x \in \alpha$, $x \in \beta$, $x \in \gamma$; then *10·11 is applied, and such further propositions of *10 as may be required, together with *22·33·34·35.

***22·631.** $\vdash . \alpha \cap (\alpha \cup \beta) = \alpha$ [*22·58·621]

***22·632.** $\vdash : \alpha = \beta . \supset . \alpha = \alpha \cap \beta$ [*22·42·621]

***22·633.** $\vdash : \alpha \subset \beta . \supset . \alpha \cup \gamma = (\alpha \cap \beta) \cup \gamma$ [*22·551·621]

***22·64.** $\vdash :. \alpha \subset \gamma . \vee . \beta \subset \gamma : \supset . \alpha \cap \beta \subset \gamma$

Dem.

$\vdash . *22{\cdot}47{\cdot}51 . \supset \vdash : \alpha \subset \gamma . \supset . \alpha \cap \beta \subset \gamma : \beta \subset \gamma . \supset . \alpha \cap \beta \subset \gamma$ (1)

$\vdash . (1) . *4{\cdot}77 . \supset \vdash .$ Prop

The converse of this proposition does not hold, because the converse of *10·41 does not hold.

***22·65.** $\vdash :. \alpha \subset \beta . \vee . \alpha \subset \gamma : \supset . \alpha \subset \beta \cup \gamma$ [*22·61·57 . *4·77]

Here again the converse is untrue.

***22·66.** $\vdash : \alpha \subset \beta . \supset . \alpha \cup \gamma \subset \beta \cup \gamma$ [*2·38]

***22·68.** $\vdash . (\alpha \cap \beta) \cup (\alpha \cap \gamma) = \alpha \cap (\beta \cup \gamma)$

Dem.

$\vdash . *22{\cdot}34 . \supset \vdash :: x \in \{(\alpha \cap \beta) \cup (\alpha \cap \gamma)\} . \equiv :. x \in \alpha \cap \beta . \vee . x \in \alpha \cap \gamma :.$

[*22·33] $\equiv :. x \in \alpha . x \in \beta . \vee . x \in \alpha . x \in \gamma :.$

[*4·4] $\equiv :. x \in \alpha : x \in \beta . \vee . x \in \gamma :.$

[*22·34] $\equiv :. x \in \alpha . x \in \beta \cup \gamma :.$

[*22·33] $\equiv :. x \in \alpha \cap (\beta \cup \gamma)$ (1)

$\vdash . (1) . *10{\cdot}11 . *20{\cdot}43 . \supset \vdash .$ Prop

***22·69.** $\vdash . (\alpha \cup \beta) \cap (\alpha \cup \gamma) = \alpha \cup (\beta \cap \gamma)$ [Similar proof, by *4·41]

The above propositions *22·68·69 are the two forms of the distributive law. Note that either results from the other by interchanging the signs of addition and multiplication.

***22·7.** $\vdash . (\alpha \cup \beta) \cup \gamma = \alpha \cup (\beta \cup \gamma)$ [*4·33]

***22·71.** $\alpha \cup \beta \cup \gamma = (\alpha \cup \beta) \cup \gamma$ Df

***22·72.** $\vdash : \alpha \subset \gamma . \beta \subset \delta . \supset . \alpha \cup \beta \subset \gamma \cup \delta$ [*3·48]

***22·73.** $\vdash : \alpha = \gamma . \beta = \delta . \supset . \alpha \cup \beta = \gamma \cup \delta$ [*10·411]

***22·74.** $\vdash : \alpha \cap \beta \subset \gamma . \alpha \cap \gamma \subset \beta . \equiv . \alpha \cap \beta = \alpha \cap \gamma$

Dem.

$\vdash . *22{\cdot}43 . *4{\cdot}73 . \supset \vdash : \alpha \cap \beta \subset \gamma . \equiv . \alpha \cap \beta \subset \alpha . \alpha \cap \beta \subset \gamma .$

[*22·45] $\equiv . \alpha \cap \beta \subset \alpha \cap \gamma$ (1)

$\vdash . (1) \frac{\gamma, \beta}{\beta, \gamma} . \quad \supset \vdash : \alpha \cap \gamma \subset \beta . \equiv . \alpha \cap \gamma \subset \alpha \cap \beta$ (2)

$\vdash . (1) . (2) . *4{\cdot}38 . \supset \vdash : \alpha \cap \beta \subset \gamma . \alpha \cap \gamma \subset \beta . \equiv . \alpha \cap \beta \subset \alpha \cap \gamma . \alpha \cap \gamma \subset \alpha \cap \beta .$

[*22·41] $\equiv . \alpha \cap \beta = \alpha \cap \gamma : \supset \vdash .$ Prop

***22·8.** $\vdash . -(-\alpha) = \alpha$ [*4·13]

***22·81.** $\vdash : \alpha \subset \beta . \equiv . -\beta \subset -\alpha$ [*4·1]

***22·811.** $\vdash : \alpha \subset -\beta . \equiv . \beta \subset -\alpha$ [*4·1 . *22·8]

***22·82.** $\vdash : \alpha \cap \beta \subset \gamma . \equiv . \alpha - \gamma \subset -\beta$ [*4·14]

***22·83.** $\vdash : \alpha = \beta . \equiv . -\alpha = -\beta$ [*4·11]

***22·831.** $\vdash : \alpha = -\beta . \equiv . \beta = -\alpha$ [*4·12]

***22·84.** $\vdash . -(\alpha \cap \beta) = -\alpha \vee -\beta$ [*4·51]

***22·85.** $\vdash . \alpha \cap \beta = -(-\alpha \vee -\beta)$ [*22·84·831]

***22·86.** $\vdash . -(-\alpha \cap -\beta) = \alpha \cup \beta$ [*4·57]

***22·87.** $\vdash . -\alpha \cap -\beta = -(\alpha \cup \beta)$ [*22·86·831]

*22·84·85·86·87 are De Morgan's formulae.

***22·88.** $\vdash . (x) . x \in (\alpha \cup -\alpha)$ [*2·11]

This is a form of the law of excluded middle.

***22·89.** $\vdash . (x) . x \sim\in (\alpha - \alpha)$ [*3·24]

This is a form of the law of contradiction.

***22·9.** $\vdash . (\alpha \cup \beta) - \beta = \alpha - \beta$ [*5·61]

***22·91.** $\vdash . \alpha \cup \beta = \alpha \cup (\beta - \alpha)$

Dem.

$\vdash . *5{\cdot}63 . \quad \supset \vdash :. x \in \alpha . \vee . x \in \beta : \equiv : x \in \alpha . \vee . x \in \beta . x \sim\in \alpha :.$

[*22·33·34·35] $\supset \vdash :. x \in \alpha \cup \beta . \equiv : x \in \alpha . \vee . x \in (\beta - \alpha) :$

[*22·34] $\equiv : x \in \alpha \cup (\beta - \alpha)$ (1)

$\vdash . (1) . *10{\cdot}11 . *20{\cdot}43 . \supset \vdash .$ Prop

***22·92.** $\vdash : \alpha \subset \beta . \supset . \beta = \alpha \cup (\beta - \alpha)$ [*22·91·62]

***22·93.** $\vdash . \alpha - \beta = \alpha - (\alpha \cap \beta)$

Dem.

$\vdash . *4{\cdot}73 . \text{Transp} . \supset \vdash :. x \epsilon \alpha . \supset : x \sim\epsilon \beta . \equiv . \sim(x \epsilon \alpha . x \epsilon \beta) .$

[*22·33] $\equiv . x \sim\epsilon (\alpha \cap \beta) :.$

[*5·32] $\supset \vdash :. x \epsilon \alpha . x \sim\epsilon \beta . \equiv . x \epsilon \alpha . x \sim\epsilon (\alpha \cap \beta) :.$

[*22·35·33] $\supset \vdash : x \epsilon \alpha - \beta . \equiv . x \epsilon \alpha - (\alpha \cap \beta) :$

[*10·11.*20·43] $\supset \vdash . \alpha - \beta = \alpha - (\alpha \cap \beta) . \supset \vdash . \text{Prop}$

***22·94.** $\vdash : (\alpha) . f\alpha . \equiv . (\alpha) . f(-\alpha)$

Dem.

$\vdash . *10{\cdot}1 .$ $\supset \vdash : (\alpha) . f\alpha . \supset . f(-\alpha) :$

[*10·11·21] $\supset \vdash : (\alpha) . f\alpha . \supset . (\alpha) . f(-\alpha)$ (1)

$\vdash . *10{\cdot}1 .$ $\supset \vdash : (\alpha) . f(-\alpha) . \supset . f\{-(-\alpha)\} .$

[*22·8.*20·18] $\supset . f\alpha :$

[*10·11·21] $\supset \vdash : (\alpha) . f(-\alpha) . \supset . (\alpha) . f\alpha$ (2)

$\vdash . (1) . (2) .$ $\supset \vdash . \text{Prop}$

This proposition is used in connection with mathematical induction, in *90·102, which is required for the proof of *90·132, which is one of the fundamental propositions in the theory of mathematical induction.

***22·95.** $\vdash : (\exists \alpha) . f\alpha . \equiv . (\exists \alpha) . f(-\alpha)$

Dem.

$\vdash . *22{\cdot}94 . \supset \vdash : (\alpha) . \sim f\alpha . \equiv . (\alpha) . \sim f(-\alpha)$ (1)

$\vdash . (1) . \text{Transp} . *20{\cdot}6 . \supset \vdash . \text{Prop}$

$*23$. CALCULUS OF RELATIONS

Summary of $*23$.

The definitions and propositions of this number are to be exact analogues of those of $*22$. Properties of relations which have no analogues for classes will not be dealt with till Section D. Proofs will be omitted in the present number, as they are precisely analogous to those of analogous propositions in $*22$. In this number, as always in future, capital Latin letters stand for expressions of the form $\hat{x}\hat{y}\phi\,!\,(x, y)$, or, where they are not being used as apparent variables, for $\hat{x}\hat{y}\phi\,(x, y)$. The principal propositions of this number are the analogues of those of $*22$.

$*23{\cdot}01$. $R \mathrel{\dot{\subset}} S \,.=:\, xRy \,.\supset_{x,y}.\, xSy$ Df

$*23{\cdot}02$. $R \mathrel{\dot{\cap}} S = \hat{x}\hat{y}\,(xRy \,.\, xSy)$ Df

$*23{\cdot}03$. $R \mathrel{\dot{\cup}} S = \hat{x}\hat{y}\,(xRy \,.\vee.\, xSy)$ Df

$*23{\cdot}04$. $\dot{-} R = \hat{x}\hat{y}\,\{\sim(xRy)\}$ Df

$*23{\cdot}05$. $R \mathrel{\dot{-}} S = R \mathrel{\dot{\cap}} \dot{-} S$ Df

Similar remarks apply to these definitions as to those of $*22$.

$*23{\cdot}1$. $\vdash :.\, R \mathrel{\dot{\subset}} S \,.\equiv:\, xRy \,.\supset_{x,y}.\, xSy$

$*23{\cdot}2$. $\vdash .\, R \mathrel{\dot{\cap}} S = \hat{x}\hat{y}\,(xRy \,.\, xSy)$

$*23{\cdot}3$. $\vdash .\, R \mathrel{\dot{\cup}} S = \hat{x}\hat{y}\,(xRy \,.\vee.\, xSy)$

$*23{\cdot}31$. $\vdash .\, \dot{-} R = \hat{x}\hat{y}\,\{\sim(xRy)\}$

$*23{\cdot}32$. $\vdash .\, R \mathrel{\dot{-}} S = \hat{x}\hat{y}\,\{xRy \,.\sim(xSy)\}$

$*23{\cdot}33$. $\vdash : x\,(R \mathrel{\dot{\cap}} S)\,y \,.\equiv.\, xRy \,.\, xSy$

$*23{\cdot}34$. $\vdash :.\, x\,(R \mathrel{\dot{\cup}} S)\,y \,.\equiv:\, xRy \,.\vee.\, xSy$

$*23{\cdot}35$. $\vdash : x \mathrel{\dot{-}} Ry \,.\equiv.\sim(xRy)$

$*23{\cdot}351$. $\vdash .\, \dot{-} R \neq R$

$*23{\cdot}36$. $\vdash .\, R \mathrel{\dot{\cap}} S \in \text{Rel}$

$*23{\cdot}37$. $\vdash .\, R \mathrel{\dot{\cup}} S \in \text{Rel}$

$*23{\cdot}38$. $\vdash .\, \dot{-} R \in \text{Rel}$

$*23{\cdot}39$. $\vdash .\, \hat{x}\hat{y}\phi\,(x, y) \mathrel{\dot{\cap}} \hat{x}\hat{y}\psi\,(x, y) = \hat{x}\hat{y}\,\{\phi\,(x, y) \,.\, \psi\,(x, y)\}$

$*23{\cdot}391$. $\vdash .\, \hat{x}\hat{y}\phi\,(x, y) \mathrel{\dot{\cup}} \hat{x}\hat{y}\psi\,(x, y) = \hat{x}\hat{y}\,\{\phi\,(x, y) \,.\vee.\, \psi\,(x, y)\}$

$*23{\cdot}392$. $\vdash .\, \dot{-} \hat{x}\hat{y}\phi\,(x, y) = \hat{x}\hat{y}\,\{\sim\phi\,(x, y)\}$

$*23{\cdot}4$. $\vdash :.\, R \mathrel{\dot{\subset}} S \,.\, S \mathrel{\dot{\subset}} R \,.\equiv:\, xRy \,.\equiv_{x,y}.\, xSy$

$*23{\cdot}41$. $\vdash : R \mathrel{\dot{\subset}} S \,.\, S \mathrel{\dot{\subset}} R \,.\equiv.\, R = S$

$*23{\cdot}42$. $\vdash .\, R \mathrel{\dot{\subset}} R$

$*23{\cdot}43$. $\vdash .\, R \mathrel{\dot{\cap}} S \mathrel{\dot{\subset}} R$

$*23{\cdot}44$. $\vdash : R \mathrel{\dot{\subset}} S \,.\, S \mathrel{\dot{\subset}} T \,.\supset.\, R \mathrel{\dot{\subset}} T$

$*23{\cdot}441$. $\vdash : R \mathrel{\dot{\subset}} S \,.\, xRy \,.\supset.\, xSy$

*23·45. $\vdash : R \mathrel{\dot\subset} S . R \mathrel{\dot\subset} T . \supset . R \mathrel{\dot\subset} S \mathbin{\dot\cap} T$

*23·46. $\vdash : xRy . R \mathrel{\dot\subset} S . \supset . xSy$

*23·47. $\vdash : R \mathrel{\dot\subset} T . \supset . R \mathbin{\dot\cap} S \mathrel{\dot\subset} T$

*23·48. $\vdash : R \mathrel{\dot\subset} S . \supset . R \mathbin{\dot\cap} T \mathrel{\dot\subset} S \mathbin{\dot\cap} T$

*23·481. $\vdash : R = S . \supset . R \mathbin{\dot\cap} T = S \mathbin{\dot\cap} T$

*23·49. $\vdash : P \mathrel{\dot\subset} Q . R \mathrel{\dot\subset} S . \supset . P \mathbin{\dot\cap} R \mathrel{\dot\subset} Q \mathbin{\dot\cap} S$

*23·5. $\vdash . R \mathbin{\dot\cap} R = R$

*23·51. $\vdash . R \mathbin{\dot\cap} S = S \mathbin{\dot\cap} R$

*23·52. $\vdash . (R \mathbin{\dot\cap} S) \mathbin{\dot\cap} T = R \mathbin{\dot\cap} (S \mathbin{\dot\cap} T)$

*23·53. $R \mathbin{\dot\cap} S \mathbin{\dot\cap} T = (R \mathbin{\dot\cap} S) \mathbin{\dot\cap} T$ Df

*23·54. $\vdash :. R = S . \supset : R \mathrel{\dot\subset} T . \equiv . S \mathrel{\dot\subset} T$

*23·55. $\vdash :. R = S . \supset : T \mathrel{\dot\subset} R . \equiv . T \mathrel{\dot\subset} S$

*23·551. $\vdash : R = S . \supset . R \mathbin{\dot\cup} T = S \mathbin{\dot\cup} T$

*23·56. $\vdash . R \mathbin{\dot\cup} R = R$

*23·57. $\vdash . R \mathbin{\dot\cup} S = S \mathbin{\dot\cup} R$

*23·58. $\vdash . R \mathrel{\dot\subset} R \mathbin{\dot\cup} S . S \mathrel{\dot\subset} R \mathbin{\dot\cup} S$

*23·59. $\vdash : R \mathrel{\dot\subset} T . S \mathrel{\dot\subset} T . \equiv . R \mathbin{\dot\cup} S \mathrel{\dot\subset} T$

*23·6. $\vdash :. x(R \mathbin{\dot\cup} S)y . \equiv : R \mathrel{\dot\subset} T . S \mathrel{\dot\subset} T . \supset_T . xTy$

*23·61. $\vdash : R \mathrel{\dot\subset} S . \supset . R \mathrel{\dot\subset} S \mathbin{\dot\cup} T$

*23·62. $\vdash : R \mathrel{\dot\subset} S . \equiv . R \mathbin{\dot\cup} S = S$

*23·621. $\vdash : R \mathrel{\dot\subset} S . \equiv . R \mathbin{\dot\cap} S = R$

*23·63. $\vdash . R \mathbin{\dot\cup} (R \mathbin{\dot\cap} S) = R$

*23·631. $\vdash . R \mathbin{\dot\cap} (R \mathbin{\dot\cup} S) = R$

*23·632. $\vdash : R = S . \supset . R = R \mathbin{\dot\cap} S$

*23·633. $\vdash : R \mathrel{\dot\subset} S . \supset . R \mathbin{\dot\cup} T = (R \mathbin{\dot\cap} S) \mathbin{\dot\cup} T$

*23·64. $\vdash :. R \mathrel{\dot\subset} T . \vee . S \mathrel{\dot\subset} T : \supset . R \mathbin{\dot\cap} S \mathrel{\dot\subset} T$

*23·65. $\vdash :. R \mathrel{\dot\subset} S . \vee . R \mathrel{\dot\subset} T : \supset . R \mathrel{\dot\subset} S \mathbin{\dot\cup} T$

*23·66. $\vdash : R \mathrel{\dot\subset} S . \supset . R \mathbin{\dot\cup} T \mathrel{\dot\subset} S \mathbin{\dot\cup} T$

*23·68. $\vdash . (R \mathbin{\dot\cap} S) \mathbin{\dot\cup} (R \mathbin{\dot\cap} T) = R \mathbin{\dot\cap} (S \mathbin{\dot\cup} T)$

*23·69. $\vdash . (R \mathbin{\dot\cup} S) \mathbin{\dot\cap} (R \mathbin{\dot\cup} T) = R \mathbin{\dot\cup} (S \mathbin{\dot\cap} T)$

*23·7. $\vdash . (R \mathbin{\dot\cup} S) \mathbin{\dot\cup} T = R \mathbin{\dot\cup} (S \mathbin{\dot\cup} T)$

*23·71. $R \mathbin{\dot\cup} S \mathbin{\dot\cup} T = (R \mathbin{\dot\cup} S) \mathbin{\dot\cup} T$ Df

*23·72. $\vdash : P \mathrel{\dot\subset} R . Q \mathrel{\dot\subset} S . \supset . P \mathbin{\dot\cup} Q \mathrel{\dot\subset} R \mathbin{\dot\cup} S$

*23·73. $\vdash : P = R . Q = S . \supset . P \mathbin{\dot\cup} Q = R \mathbin{\dot\cup} S$

*23·74. $\vdash : P \mathbin{\dot\cap} Q \mathrel{\dot\subset} R . P \mathbin{\dot\cap} R \mathrel{\dot\subset} Q . \equiv . P \mathbin{\dot\cap} Q = P \mathbin{\dot\cap} R$

*23·8. $\vdash . \dot{-}(\dot{-} R) = R$

*23·81. $\vdash : R \mathrel{\dot\subset} S . \equiv . \dot{-} S \mathrel{\dot\subset} \dot{-} R$

*23·811. $\vdash : R \mathrel{\dot\subset} \dot{-} S . \equiv . S \mathrel{\dot\subset} \dot{-} R$

*23·82. $\vdash : R \mathbin{\dot\cap} S \mathrel{\dot\subset} T . \equiv . R \mathbin{\dot{-}} T \mathrel{\dot\subset} \dot{-} S$

*23·83. $\vdash : R = S . \equiv . \dot{-} R = \dot{-} S$

$*23{\cdot}831.\ \vdash : R = \dot{-} S . \equiv . S = \dot{-} R$

$*23{\cdot}84.\ \vdash . \dot{-}(R \dot{\cap} S) = \dot{-} R \dot{\cup} \dot{-} S$

$*23{\cdot}85.\ \vdash . R \dot{\cap} S = \dot{-}(\dot{-} R \dot{\cup} \dot{-} S)$

$*23{\cdot}86.\ \vdash . \dot{-}(\dot{-} R \dot{\cap} \dot{-} S) = R \dot{\cup} S$

$*23{\cdot}87.\ \vdash . \dot{-} R \dot{\cap} \dot{-} S = \dot{-}(R \dot{\cup} S)$

$*23{\cdot}88.\ \vdash . (x, y) . x (R \dot{\cup} \dot{-} R) y$

$*23{\cdot}89.\ \vdash . (x, y) . \sim \{x (R \dot{-} R) y\}$

$*23{\cdot}9.\ \vdash . (R \dot{\cup} S) \dot{-} S = R \dot{-} S$

$*23{\cdot}91.\ \vdash . R \dot{\cup} S = R \dot{\cup} (S \dot{-} R)$

$*23{\cdot}92.\ \vdash : R \mathrel{\dot{\subset}} S . \supset . S = R \dot{\cup} (S \dot{-} R)$

$*23{\cdot}93.\ \vdash . R \dot{-} S = R \dot{-} (R \dot{\cap} S)$

$*23{\cdot}94.\ \vdash : (R) . fR . \equiv . (R) . f(\dot{-} R)$

$*23{\cdot}95.\ \vdash : (\exists R) . fR . \equiv . (\exists R) . f(\dot{-} R)$

$*24$. THE UNIVERSAL CLASS, THE NULL-CLASS, AND THE EXISTENCE OF CLASSES

Summary of $*24$.

The universal class, denoted by V, is the class of all objects of the type which, in the given context, is being denoted by small Latin letters, *i.e.* of the lowest type concerned. Thus V, like "Cls," is ambiguous as to type. Its definition is as follows:

$*24{\cdot}01$. $V = \hat{x}(x = x)$ Df

Any other property possessed by everything would do as well as "$x = x$," but this is the only such property which we have hitherto studied.

The null-class, denoted by Λ, is the class which has no members. Like V, it is ambiguous as to type. We use the same symbol, Λ, for null-classes of various types; but these null-classes differ. The type of Λ is determined by that of the terms x concerning which "$x \epsilon \Lambda$" is false: whatever x may be, "$x \epsilon \Lambda$" will not represent a *true* proposition, but unless x is of the appropriate type, "$x \epsilon \Lambda$" will be meaningless, not false. Thus Λ is of the type next above that of an x concerning which "$x \epsilon \Lambda$" is significant and false. The definition of Λ is

$*24{\cdot}02$. $\Lambda = -V$ Df

When a class α is not null, so that it has one or more members, it is said to *exist*. (This sense of "existence" must not be confused with that defined in $*14{\cdot}02$.) We write "$\exists ! \alpha$" for "α exists." The definition is

$*24{\cdot}03$. $\exists ! \alpha \,.=.\, (\exists x) \,.\, x \epsilon \alpha$ Df

In the present number, we shall deal first with the properties of Λ and V, then with those of existence. In comparing the algebra of symbolic logic with ordinary algebra, Λ takes the place of 0, while V combines the properties of 1 and of ∞.

Among the more important properties of Λ and V which are proved in this number are the following:

$*24{\cdot}1$. $\vdash . \Lambda \neq V$

I.e. "nothing is not everything." This is useful as giving us the existence of at least two classes. If the monistic philosophers were right in maintaining that only one individual exists, there would be only two classes, Λ and V, V being (in that case) the class whose only member is the one individual. Our primitive propositions do not require the existence of more than one individual.

∗24·102·103 show that any function which is always true determines the universal class, and any function which is always false determines the null-class.

∗24·21·22 give forms of the laws of contradiction and excluded middle, namely "nothing is both α and not-α" ($\alpha \cap -\alpha = \Lambda$) and "everything is either α or not-α" ($\alpha \cup -\alpha = \mathrm{V}$).

∗24·23·24·26·27 give the properties of Λ and V with respect to addition and multiplication, namely: multiplication by V and addition of Λ make no change in a class (∗24·26·24); addition of V gives V, and multiplication by Λ gives Λ (∗24·27·23). It will be observed that the properties of Λ and V result from each other by interchanging addition and multiplication.

∗24·3. $\vdash : \alpha \subset \beta \,.\equiv .\, \alpha - \beta = \Lambda$

I.e. "α is contained in β" is equivalent to "nothing is α but not β."

∗24·311. $\vdash : \alpha \subset -\beta \,.\equiv .\, \alpha \cap \beta = \Lambda$

I.e. "no α is a β" is equivalent to "nothing is both α and β."

∗24·411. $\vdash : \beta \subset \alpha \,.\supset .\, \alpha = \beta \cup (\alpha - \beta)$

∗24·43. $\vdash : \alpha - \beta \subset \gamma \,.\equiv .\, \alpha \subset \beta \cup \gamma$

As a rule, propositions concerning V are much less used than the correlative propositions concerning Λ.

The properties of the existence of classes result from those of Λ, owing to the fact that $\exists ! \alpha$ is the contradictory of $\alpha = \Lambda$, as is proved in ∗24·54. Thus we have, in virtue of ∗24·3,

∗24·55. $\vdash : \sim(\alpha \subset \beta) \,.\equiv .\, \exists ! \alpha - \beta$

I.e. "not all α's are β's" is equivalent to "there are α's which are not β's." This is the familiar proposition of formal logic, that the contradictory of the universal affirmative is the particular negative.

We have

∗24·56. $\vdash :.\, \exists ! (\alpha \cup \beta) \,.\equiv :\, \exists ! \alpha \,.\vee .\, \exists ! \beta$

∗24·561. $\vdash : \exists ! (\alpha \cap \beta) \,.\supset .\, \exists ! \alpha \,.\, \exists ! \beta$

I.e. if a sum exists, then one of the summands exists, and vice versa; and if a product exists, both the factors exist (but not vice versa).

The proofs of propositions in the present number offer no difficulty.

∗24·01. $\mathrm{V} = \hat{x}(x = x)$ Df

∗24·02. $\Lambda = -\mathrm{V}$ Df

∗24·03. $\exists ! \alpha \,.=.\, (\exists x) \,.\, x \in \alpha$ Df

∗24·1. $\vdash .\, \Lambda \neq \mathrm{V}$ [∗22·351 . (∗24·02)]

∗24·101. $\vdash .\, \mathrm{V} = -\Lambda$ [∗22·831 . (∗24·02)]

***24·102.** $\vdash : (x) . \phi x . \equiv . \hat{z}(\phi z) = \mathrm{V}$

Dem.

$\vdash . *13\cdot15 . *5\cdot501 . \supset \vdash :. \phi x . \equiv : \phi x . \equiv . x = x :.$

$[*10\cdot11\cdot271] \quad \supset \vdash :. (x) . \phi x . \equiv : (x) : \phi x . \equiv . x = x :$

$[*20\cdot15] \quad \equiv : \hat{z}(\phi z) = \hat{z}(x = x) :$

$[(*24\cdot01)] \quad \equiv : \hat{z}(\phi z) = \mathrm{V} :. \supset \vdash . \mathrm{Prop}$

Thus any function which is always true determines the universal class, and vice versa.

***24·103.** $\vdash : (x) . \sim\phi x . \equiv . \hat{z}(\phi z) = \Lambda$

Dem.

$\vdash . *24\cdot102 . \supset \vdash :. (x) . \sim\phi x . \equiv : \hat{z}(\sim\phi z) = \mathrm{V} :$

$[*22\cdot392] \quad \equiv : -\hat{z}(\phi z) = \mathrm{V} :$

$[*22\cdot831] \quad \equiv : \hat{z}(\phi z) = -\mathrm{V} :$

$[(*24\cdot02)] \quad \equiv : \hat{z}(\phi z) = \Lambda :. \supset \vdash . \mathrm{Prop}$

***24·104.** $\vdash . (x) . x \in \mathrm{V}$

Dem.

$\vdash . *20\cdot3 . \supset \vdash : x \in \mathrm{V} . \equiv . x = x \quad (1)$

$\vdash . (1) . *13\cdot15 . *10\cdot11\cdot271 . \supset \vdash . \mathrm{Prop}$

***24·105.** $\vdash . (x) . x \sim\in \Lambda$

Dem.

$\vdash . *22\cdot35 . \supset \vdash : x \in \Lambda . \equiv . x \sim\in \mathrm{V} :$

$[*4\cdot12] \quad \supset \vdash : x \sim\in \Lambda . \equiv . x \in \mathrm{V} \quad (1)$

$\vdash . (1) . *10\cdot11\cdot271 . *24\cdot104 . \supset \vdash . \mathrm{Prop}$

***24·11.** $\vdash . (\alpha) . \alpha \subset \mathrm{V}$

Dem.

$\vdash . *24\cdot104 . *10\cdot1 . \supset \vdash . x \in \mathrm{V} .$

$[\mathrm{Simp}] \quad \supset \vdash : x \in \alpha . \supset . x \in \mathrm{V} :$

$[*10\cdot11 . *22\cdot1] \quad \supset \vdash : \alpha \subset \mathrm{V} :$

$[*10\cdot11] \quad \supset \vdash : (\alpha) . \alpha \subset \mathrm{V} : \supset \vdash . \mathrm{Prop}$

***24·12.** $\vdash . (\alpha) . \Lambda \subset \alpha$

Dem.

$\vdash . *24\cdot105 . *10\cdot1 . \quad \supset \vdash . x \sim\in \Lambda .$

$[*2\cdot21] \quad \supset \vdash : x \in \Lambda . \supset . x \in \alpha \quad (1)$

$\vdash . (1) . *10\cdot11 . *22\cdot1 . \supset \vdash . \mathrm{Prop}$

***24·13.** $\vdash : \alpha = \Lambda . \equiv . \alpha \subset \Lambda$

Dem.

$\vdash . *24\cdot12 . *4\cdot73 . \supset \vdash : \alpha \subset \Lambda . \equiv . \alpha \subset \Lambda . \Lambda \subset \alpha .$

$[*22\cdot41] \quad \equiv . \alpha = \Lambda : \supset \vdash . \mathrm{Prop}$

***24·14.** $\vdash : (x) . x \in \alpha . \equiv . \alpha = \mathrm{V}$

Dem.

$\vdash . *24\cdot102 . \supset \vdash : (x) . x \in \alpha . \equiv . \hat{x}(x \in \alpha) = \mathrm{V} .$

$[*20\cdot32] \quad \equiv . \alpha = \mathrm{V} : \supset \vdash . \mathrm{Prop}$

***24·141.** $\vdash : \mathrm{V} \subset \alpha . \equiv . \mathrm{V} = \alpha$

Dem.

$$\vdash . *24\cdot11 . *4\cdot73 . \supset \vdash : \mathrm{V} \subset \alpha . \equiv . \alpha \subset \mathrm{V} . \mathrm{V} \subset \alpha .$$
$$[*22\cdot41] \qquad \equiv . \alpha = \mathrm{V} : \supset \vdash . \mathrm{Prop}$$

***24·15.** $\vdash : (x) . x \sim\epsilon\, \alpha . \equiv . \alpha = \Lambda$

Dem.

$$\vdash . *24\cdot103 . \supset \vdash : (x) . x \sim\epsilon\, \alpha . \equiv . \hat{x}(x \,\epsilon\, \alpha) = \Lambda .$$
$$[*20\cdot32] \qquad \equiv . \alpha = \Lambda : \supset \vdash . \mathrm{Prop}$$

***24·17.** $\vdash : \alpha = \mathrm{V} . \equiv . -\alpha = \Lambda$ [$*22\cdot83 . (*24\cdot02)$]

***24·21.** $\vdash . \alpha \cap -\alpha = \Lambda$ [$*24\cdot103 . *22\cdot89$]

***24·22.** $\vdash . \alpha \cup -\alpha = \mathrm{V}$ [$*22\cdot88 . *24\cdot102$]

***24·23.** $\vdash . \alpha \cap \Lambda = \Lambda$ [$*24\cdot12 . *22\cdot621$]

***24·24.** $\vdash . \alpha \cup \Lambda = \alpha$ [$*24\cdot12 . *22\cdot62$]

The above two propositions ($*24\cdot23\cdot24$) exhibit the algebraical analogy of Λ to zero.

***24·26.** $\vdash . \alpha \cap \mathrm{V} = \alpha$ [$*22\cdot621 . *24\cdot11$]

This exhibits the analogy of V to 1.

***24·27.** $\vdash . \alpha \cup \mathrm{V} = \mathrm{V}$ [$*22\cdot62 . *24\cdot11$]

This exhibits the analogy of V to ∞.

***24·3.** $\vdash : \alpha \subset \beta . \equiv . \alpha - \beta = \Lambda$

Dem.

$$\vdash . *4\cdot53\cdot6 . \supset$$
$$\vdash :. x \,\epsilon\, \alpha . \supset . x \,\epsilon\, \beta : \equiv : \sim(x \,\epsilon\, \alpha . x \sim\epsilon\, \beta) :$$
$$[*22\cdot35] \qquad \equiv : \sim(x \,\epsilon\, \alpha . x \,\epsilon\, -\beta) :$$
$$[*22\cdot33] \qquad \equiv : \sim(x \,\epsilon\, \alpha - \beta) \qquad (1)$$
$$\vdash . (1) . *10\cdot11\cdot271 . \supset$$
$$\vdash : \alpha \subset \beta . \equiv . (x) . \sim(x \,\epsilon\, \alpha - \beta) .$$
$$[*24\cdot15] \quad \equiv . \alpha - \beta = \Lambda : \supset \vdash . \mathrm{Prop}$$

The above proposition is very frequently used.

***24·31.** $\vdash : \alpha \subset \beta . \equiv . -\alpha \cup \beta = \mathrm{V}$

Dem.

$$\vdash . *4\cdot6 . \quad \supset \vdash :. x \,\epsilon\, \alpha . \supset . x \,\epsilon\, \beta : \equiv : x \sim\epsilon\, \alpha . \vee . x \,\epsilon\, \beta :.$$
$$[*10\cdot11\cdot271] \supset \vdash :. \alpha \subset \beta . \equiv : (x) : x \sim\epsilon\, \alpha . \vee . x \,\epsilon\, \beta :$$
$$[*22\cdot35] \qquad \equiv : (x) : x \,\epsilon\, -\alpha . \vee . x \,\epsilon\, \beta :$$
$$[*22\cdot34] \qquad \equiv : (x) . x \,\epsilon\, (-\alpha \cup \beta) :$$
$$[*24\cdot14] \qquad \equiv : -\alpha \cup \beta = \mathrm{V} :. \supset \vdash . \mathrm{Prop}$$

This proposition is the correlative of $*24\cdot3$, but, unlike that proposition, it is not useful in the sequel. Every proposition concerning Λ has a correlative concerning V, but we shall often not give these correlatives, since they are seldom required for subsequent proofs.

***24·311.** $\vdash : \alpha \subset -\beta . \equiv . \alpha \cap \beta = \Lambda$

Dem.

$\vdash . *22\cdot35 . \supset \vdash :. x \epsilon \alpha . \supset . x \epsilon -\beta : \equiv : x \epsilon \alpha . \supset . x \sim\epsilon \beta :$

[*4·51·62] $\equiv : \sim(x \epsilon \alpha . x \epsilon \beta) :$

[*22·33] $\equiv : \sim(x \epsilon \alpha \cap \beta)$ (1)

$\vdash . (1) . *10\cdot11\cdot271 . \supset \vdash : \alpha \subset -\beta . \equiv . (x) . x \sim\epsilon \alpha \cap \beta .$

[*24·15] $\equiv . \alpha \cap \beta = \Lambda : \supset \vdash .$ Prop

***24·312.** $\vdash : -\alpha \subset \beta . \equiv . \alpha \cup \beta = V$

Dem.

$\vdash . *22\cdot35 . \supset \vdash :. -\alpha \subset \beta . \equiv : x \sim\epsilon \alpha . \supset_x . x \epsilon \beta :$

[*4·64] $\equiv : (x) : x \epsilon \alpha . \vee . x \epsilon \beta :$

[*22·34] $\equiv : (x) . x \epsilon \alpha \cup \beta :$

[*24·14] $\equiv : \alpha \cup \beta = V :. \supset \vdash .$ Prop

***24·313.** $\vdash : \alpha \cap \beta = \Lambda . \equiv . \alpha = \alpha - \beta$ [*24·311 . *22·621]

***24·32.** $\vdash :. \alpha \cup \beta = \Lambda . \equiv . \alpha = \Lambda . \beta = \Lambda$

Dem.

$\vdash . *24\cdot13 . \supset \vdash :. \alpha \cup \beta = \Lambda . \equiv : \alpha \cup \beta \subset \Lambda :$

[*22·59] $\equiv : \alpha \subset \Lambda . \beta \subset \Lambda :$

[*24·13] $\equiv : \alpha = \Lambda . \beta = \Lambda :. \supset \vdash .$ Prop

***24·33.** $\vdash : \alpha = V . \supset . \alpha \cup \beta = V$

Dem.

$\vdash . *22\cdot551 . \supset \vdash : \text{Hp} . \supset . \alpha \cup \beta = V \cup \beta$

[*24·27.*22·57] $= V : \supset \vdash .$ Prop

***24·34.** $\vdash : \alpha = \Lambda . \supset . \alpha \cap \beta = \Lambda$ [*22·481 . *24·23]

***24·35.** $\vdash : \alpha = V . \supset . \alpha \cap \beta = \beta$ [*22·481 . *24·26]

***24·36.** $\vdash : \alpha = \Lambda . \supset . \alpha \cup \beta = \beta$ [*22·551 . *24·24]

***24·37.** $\vdash :. \alpha \cap \beta = \Lambda . \equiv : x \epsilon \alpha . y \epsilon \beta . \supset_{x,y} . x \neq y$

Dem.

$\vdash . *24\cdot15 . \supset \vdash :. \alpha \cap \beta = \Lambda . \equiv : (x) . x \sim\epsilon (\alpha \cap \beta) :$

[*22·33] $\equiv : (x) . \sim(x \epsilon \alpha . x \epsilon \beta) :$

[*13·191] $\equiv : (x, y) : x = y . \supset . \sim(x \epsilon \alpha . y \epsilon \beta) :$

[Transp] $\equiv : (x, y) : x \epsilon \alpha . y \epsilon \beta . \supset . x \neq y :. \supset \vdash .$ Prop

***24·38.** $\vdash :. \alpha \cap \beta = \Lambda . \supset : \alpha \neq \beta . \vee . \alpha = \Lambda . \beta = \Lambda$

Dem.

$\vdash . *22\cdot481 . \supset \vdash : \alpha \cap \beta = \Lambda . \alpha = \beta . \supset . \alpha \cap \alpha = \Lambda .$

[*22·5] $\supset . \alpha = \Lambda .$

[*20·23] $\supset . \alpha = \Lambda . \beta = \Lambda$ (1)

$\vdash . (1) . \text{Exp} . \supset \vdash :. \alpha \cap \beta = \Lambda . \supset : \alpha = \beta . \supset . \alpha = \Lambda . \beta = \Lambda :$

[*4·6] $\supset : \alpha \neq \beta . \vee . \alpha = \Lambda . \beta = \Lambda :. \supset \vdash .$ Prop

***24·39.** $\vdash :. \alpha \cap \beta = \Lambda . \equiv : x \epsilon \alpha . \supset_x . x \sim\epsilon \beta$ [*24·311 . *22·35]

***24·4.** $\vdash : \alpha \cap \beta = \Lambda . \equiv . (\alpha \cup \beta) - \alpha = \beta . \equiv . (\alpha \cup \beta) - \beta = \alpha$

Dem.

$\vdash . *24{\cdot}311 . \supset \vdash : \alpha \cap \beta = \Lambda . \equiv . \beta \subset -\alpha .$
[*22·621] $\equiv . \beta - \alpha = \beta .$
[*22·9] $\equiv . (\alpha \cup \beta) - \alpha = \beta$ (1)
$\vdash . (1) \dfrac{\beta, \alpha}{\alpha, \beta} . \supset \vdash : \beta \cap \alpha = \Lambda . \equiv . (\beta \cup \alpha) - \beta = \alpha :$
[*22·51·57] $\supset \vdash : \alpha \cap \beta = \Lambda . \equiv . (\alpha \cup \beta) - \beta = \alpha$ (2)
$\vdash . (1) . (2) . \supset \vdash .$ Prop

***24·401.** $\vdash : \beta \subset \alpha . \supset . (\beta \cup \gamma) - \alpha = \gamma - \alpha$

Dem.

$\vdash . *22{\cdot}68 . \supset \vdash . (\beta \cup \gamma) - \alpha = (\beta - \alpha) \cup (\gamma - \alpha)$ (1)
$\vdash . *24{\cdot}3 . \supset \vdash : \mathrm{Hp} . \supset . \beta - \alpha = \Lambda$ (2)
$\vdash . (1) . (2) . \supset \vdash : \mathrm{Hp} . \supset . (\beta \cup \gamma) - \alpha = \Lambda \cup (\gamma - \alpha)$
[*24·24] $= \gamma - \alpha : \supset \vdash .$ Prop

***24·402.** $\vdash : \alpha \cap \beta = \Lambda . \xi \subset \alpha . \eta \subset \beta . \supset . \xi \cap \eta = \Lambda$

Dem.

$\vdash . *22{\cdot}49 . \supset \vdash : \mathrm{Hp} . \supset . \xi \cap \eta \subset \alpha \cap \beta .$
[*22·55] $\supset . \xi \cap \eta \subset \Lambda .$
[*24·13] $\supset . \xi \cap \eta = \Lambda : \supset \vdash .$ Prop

***24·41.** $\vdash . \alpha = (\alpha \cap \beta) \cup (\alpha - \beta)$

Dem.

$\vdash . *22{\cdot}68 . \supset \vdash . (\alpha \cap \beta) \cup (\alpha - \beta) = \alpha \cap (\beta \cup -\beta)$
[*24·22] $= \alpha \cap \mathrm{V}$
[*24·26] $= \alpha . \supset \vdash .$ Prop

***24·411.** $\vdash : \beta \subset \alpha . \supset . \alpha = \beta \cup (\alpha - \beta)$

Dem.

$\vdash . *22{\cdot}633 \dfrac{\beta, \alpha, \alpha - \beta}{\alpha, \beta, \gamma} . \supset \vdash : \beta \subset \alpha . \supset . \beta \cup (\alpha - \beta) = (\alpha \cap \beta) \cup (\alpha - \beta)$
[*24·41] $= \alpha : \supset \vdash .$ Prop

***24·412.** $\vdash : \beta \subset \alpha . \gamma \subset \beta . \supset . (\alpha - \beta) \cup (\beta - \gamma) = \alpha - \gamma$

Dem.

$\vdash . *24{\cdot}41 . \supset \vdash : \mathrm{Hp} . \supset . (\alpha - \beta) \cup (\beta - \gamma) = (\alpha - \beta \cap \gamma) \cup (\alpha - \beta - \gamma) \cup (\beta - \gamma)$
[*24·3·23] $= (\alpha - \beta - \gamma) \cup (\beta - \gamma)$
[*22·68] $= \{(\alpha - \beta) \cup \beta\} - \gamma$
[*24·411] $= \alpha - \gamma : \supset \vdash .$ Prop

This proposition is used in *234·181, in the theory of continuous functions.

***24·42.** $\vdash : \alpha \cap \beta \subset \gamma . \alpha - \beta \subset \gamma . \equiv . \alpha \subset \gamma$

Dem.

$\vdash . *22{\cdot}59 . \supset \vdash : \alpha \cap \beta \subset \gamma . \alpha - \beta \subset \gamma . \equiv . (\alpha \cap \beta) \cup (\alpha - \beta) \subset \gamma .$
[*24·41] $\equiv . \alpha \subset \gamma : \supset \vdash .$ Prop

***24·43.** $\vdash : \alpha - \beta \subset \gamma . \equiv . \alpha \subset \beta \cup \gamma$

Dem.

$\vdash . *5{\cdot}6 . \quad \supset \vdash :: x \in \alpha . x \sim\in \beta . \supset . x \in \gamma : \equiv :. x \in \alpha . \supset : x \in \beta . \vee . x \in \gamma :.$
$[*22{\cdot}35{\cdot}33] \supset \vdash :: x \in \alpha - \beta . \supset . x \in \gamma : \equiv :. x \in \alpha . \supset : x \in \beta . \vee . x \in \gamma :.$
$[*22{\cdot}34] \qquad \equiv :. x \in \alpha . \supset . x \in (\beta \cup \gamma) \qquad (1)$
$\vdash . (1) . *10{\cdot}11{\cdot}271 . \supset \vdash . \text{Prop}$

***24·431.** $\vdash . (\alpha \cup \gamma) \cap (\beta \cup -\gamma) = (\alpha \cap \beta) \cup (\alpha - \gamma) \cup (\beta \cap \gamma)$

This and the following proposition are lemmas for *24·44.

Dem.

$\vdash . *22{\cdot}68 . \supset \vdash . (\alpha \cup \gamma) \cap (\beta \cup -\gamma) = \{(\alpha \cup \gamma) \cap \beta\} \cup \{(\alpha \cup \gamma) \cap -\gamma\}$
$[*22{\cdot}68] \qquad = (\alpha \cap \beta) \cup (\gamma \cap \beta) \cup (\alpha - \gamma) \cup (\gamma - \gamma)$
$[*24{\cdot}21] \qquad = (\alpha \cap \beta) \cup (\gamma \cap \beta) \cup (\alpha - \gamma) \cup \Lambda$
$[*24{\cdot}24] \qquad = (\alpha \cap \beta) \cup (\gamma \cap \beta) \cup (\alpha - \gamma)$
$[*22{\cdot}51{\cdot}57] \qquad = (\alpha \cap \beta) \cup (\alpha - \gamma) \cup (\beta \cap \gamma) . \supset \vdash . \text{Prop}$

***24·432.** $\vdash . (\alpha - \gamma) \cup (\beta \cap \gamma) = (\alpha \cap \beta) \cup (\alpha - \gamma) \cup (\beta \cap \gamma)$

Dem.

$\vdash . *24{\cdot}22{\cdot}35 . \supset \vdash . \alpha \cap \beta = (\alpha \cap \beta) \cap (\gamma \cup -\gamma)$
$[*22{\cdot}68] \qquad = (\alpha \cap \beta \cap \gamma) \cup (\alpha \cap \beta - \gamma)$
$[*22{\cdot}51] \qquad = (\alpha \cap \beta \cap \gamma) \cup (\alpha \cap -\gamma \cap \beta) .$
$[*22{\cdot}551] \quad \supset \vdash . (\alpha \cap \beta) \cup (\alpha - \gamma) = (\alpha \cap \beta \cap \gamma) \cup (\alpha \cap -\gamma \cap \beta) \cup (\alpha - \gamma)$
$[*22{\cdot}63] \qquad = (\alpha \cap \beta \cap \gamma) \cup (\alpha - \gamma)$
$[*22{\cdot}57] \qquad = (\alpha - \gamma) \cup (\alpha \cap \beta \cap \gamma) .$
$[*22{\cdot}551] \quad \supset \vdash . (\alpha \cap \beta) \cup (\alpha - \gamma) \cup (\beta \cap \gamma) = (\alpha - \gamma) \cup (\alpha \cap \beta \cap \gamma) \cup (\beta \cap \gamma)$
$[*22{\cdot}63] \qquad = (\alpha - \gamma) \cup (\beta \cap \gamma) . \supset \vdash . \text{Prop}$

***24·44.** $\vdash . (\alpha \cup \gamma) \cap (\beta \cup -\gamma) = (\alpha \cap -\gamma) \cup (\beta \cap \gamma) \quad [*24{\cdot}431{\cdot}432]$

***24·45.** $\vdash : (\alpha \cap \gamma) \cup (\beta - \gamma) = \Lambda . \equiv . \beta \subset \gamma . \gamma \subset -\alpha$

Dem.

$\vdash . *24{\cdot}32 . \supset \vdash : (\alpha \cap \gamma) \cup (\beta - \gamma) = \Lambda . \equiv . \alpha \cap \gamma = \Lambda . \beta - \gamma = \Lambda .$
$[*24{\cdot}3{\cdot}311] \qquad \equiv . \gamma \subset -\alpha . \beta \subset \gamma : \supset \vdash . \text{Prop}$

***24·46.** $\vdash : (\alpha \cap \gamma) \cup (\beta - \gamma) = \Lambda . \supset . \alpha \cap \beta = \Lambda$

Dem.

$\vdash . *24{\cdot}45 . *22{\cdot}44 . \supset \vdash : \text{Hp} . \supset . \beta \subset -\alpha .$
$[*22{\cdot}811] \qquad \supset . \alpha \subset -\beta .$
$[*24{\cdot}311] \qquad \supset . \alpha \cap \beta = \Lambda : \supset \vdash . \text{Prop}$

The following propositions, down to *24·495 inclusive, are lemmas inserted for use in much later propositions, most of them being only used a few times.

***24·47.** $\vdash : \alpha \cap \beta = \Lambda \,.\, \alpha \cup \beta = \gamma \,.\equiv.\, \alpha \subset \gamma \,.\, \beta = \gamma - \alpha$

Dem.

$\vdash . *24{\cdot}311 . \supset \vdash : \alpha \cap \beta = \Lambda . \equiv . \beta \subset -\alpha$ (1)

$\vdash . *22{\cdot}41 . \supset \vdash : \alpha \cup \beta = \gamma . \equiv . \alpha \cup \beta \subset \gamma . \gamma \subset \alpha \cup \beta .$

[*22·59.*24·43] $\equiv . \alpha \subset \gamma . \beta \subset \gamma . \gamma - \alpha \subset \beta$ (2)

$\vdash . (1) . (2) . \supset \vdash : \alpha \cap \beta = \Lambda . \alpha \cup \beta = \gamma . \equiv . \beta \subset -\alpha . \alpha \subset \gamma . \beta \subset \gamma . \gamma - \alpha \subset \beta .$

[*4·3] $\equiv . \alpha \subset \gamma . \beta \subset \gamma . \beta \subset -\alpha . \gamma - \alpha \subset \beta .$

[*22·45] $\equiv . \alpha \subset \gamma . \beta \subset \gamma - \alpha . \gamma - \alpha \subset \beta .$

[*22·41] $\equiv . \alpha \subset \gamma . \beta = \gamma - \alpha : \supset \vdash .$ Prop

***24·48.** $\vdash :. \xi \subset \alpha . \xi' \subset \alpha . \eta \subset \beta . \eta' \subset \beta . \alpha \cap \beta = \Lambda . \supset :$
$\xi \cup \eta = \xi' \cup \eta' . \equiv . \xi = \xi' . \eta = \eta'$

Dem.

$\vdash . *22{\cdot}73 . \quad \supset \vdash : \xi = \xi' . \eta = \eta' . \supset . \xi \cup \eta = \xi' \cup \eta'$ (1)

$\vdash . *22{\cdot}481 . \quad \supset \vdash :. \xi \cup \eta = \xi' \cup \eta' . \supset : (\xi \cup \eta) \cap \alpha = (\xi' \cup \eta') \cap \alpha :$

[*22·68] $\supset : (\xi \cap \alpha) \cup (\eta \cap \alpha) = (\xi' \cap \alpha) \cup (\eta' \cap \alpha)$ (2)

$\vdash . *22{\cdot}621 . \quad \supset \vdash : \xi \subset \alpha . \supset . \xi \cap \alpha = \xi : \xi' \subset \alpha . \supset . \xi' \cap \alpha = \xi' :$

[*3·47] $\supset \vdash : \xi \subset \alpha . \xi' \subset \alpha . \supset . \xi \cap \alpha = \xi . \xi' \cap \alpha = \xi'$ (3)

$\vdash . *22{\cdot}48 . \quad \supset \vdash : \eta \subset \beta . \supset . \eta \cap \alpha \subset \alpha \cap \beta :$

[*22·55] $\supset \vdash : \eta \subset \beta . \alpha \cap \beta = \Lambda . \supset . \eta \cap \alpha \subset \Lambda .$

[*24·13] $\supset . \eta \cap \alpha = \Lambda$ (4)

Similarly $\vdash : \eta' \subset \beta . \alpha \cap \beta = \Lambda . \supset . \eta' \cap \alpha = \Lambda$ (5)

$\vdash . (3) . (4) . \quad \supset \vdash :. \text{Hp} . \supset : (\xi \cap \alpha) \cup (\eta \cap \alpha) = \xi \cup \Lambda$

[*24·24] $= \xi$ (6)

$\vdash . (3) . (5) . \quad \supset \vdash :. \text{Hp} . \supset : (\xi' \cap \alpha) \cup (\eta' \cap \alpha) = \xi' \cup \Lambda$

[*24·24] $= \xi'$ (7)

$\vdash . (2) . (6) . (7) . \supset \vdash :. \text{Hp} . \supset : \xi \cup \eta = \xi' \cup \eta' . \supset . \xi = \xi'$ (8)

Similarly $\vdash :. \text{Hp} . \supset : \xi \cup \eta = \xi' \cup \eta' . \supset . \eta = \eta'$ (9)

$\vdash . (1) . (8) . (9) . \supset \vdash .$ Prop

The above proposition, besides being used in the next two, is used in the theory of couples (*54·6), in the theory of greater and less (*117·632), and in the chapter on the ordering of classes by the principle of first differences (*170·68).

***24·481.** $\vdash :. \alpha \cap \beta = \Lambda . \alpha \cap \gamma = \Lambda . \supset : \alpha \cup \beta = \alpha \cup \gamma . \equiv . \beta = \gamma$

Dem.

$\vdash . *24{\cdot}48 \dfrac{\alpha, -\alpha, \alpha. \alpha, \beta, \gamma}{\alpha, \beta, \xi, \xi', \eta, \eta'} . \supset$

$\vdash :. \alpha \subset \alpha . \alpha \subset \alpha . \beta \subset -\alpha . \gamma \subset -\alpha . \alpha - \alpha = \Lambda . \supset :$
$\alpha \cup \beta = \alpha \cup \gamma . \equiv . \alpha = \alpha . \beta = \gamma$ (1)

$\vdash . *22{\cdot}42 . *24{\cdot}21 . \supset$

$\vdash :. \alpha \subset \alpha . \alpha \subset \alpha . \beta \subset -\alpha . \gamma \subset -\alpha . \alpha - \alpha = \Lambda . \equiv . \beta \subset -\alpha . \gamma \subset -\alpha .$

$[*24\cdot311] \qquad \equiv . \alpha \cap \beta = \Lambda . \alpha \cap \gamma = \Lambda \quad (2)$

$\vdash . *20\cdot2 . *4\cdot73 . \supset \vdash : \alpha = \alpha . \beta = \gamma . \equiv . \beta = \gamma \quad (3)$

$\vdash . (1) . (2) . (3) . \supset \vdash . \text{Prop}$

The above proposition is used in the theory of selections (*83·74), in the theory of greater and less (*117·582), and in the theory of transfinite induction (*257).

***24·482.** $\vdash :. \xi \subset \alpha . \eta \subset \beta . \alpha \cap \beta = \Lambda . \supset : \xi \cup \eta = \alpha \cup \beta . \equiv . \xi = \alpha . \eta = \beta$

$$\left[*24\cdot48 \frac{\alpha, \beta}{\xi', \eta'} . *22\cdot42 \right]$$

The above proposition is used in the theory of convergence (*232·34).

***24·49.** $\vdash :. \alpha \cap \beta = \Lambda . \supset : \alpha \subset \beta \cup \gamma . \equiv . \alpha \subset \gamma$

Dem.

$\vdash . *22\cdot621 . \supset \vdash : \alpha \subset \beta \cup \gamma . \equiv . \alpha = \alpha \cap (\beta \cup \gamma)$

$[*22\cdot68] \qquad = (\alpha \cap \beta) \cup (\alpha \cap \gamma) \quad (1)$

$\vdash . *24\cdot24 . \supset \vdash : \alpha \cap \beta = \Lambda . \supset . (\alpha \cap \beta) \cup (\alpha \cap \gamma) = \alpha \cap \gamma \quad (2)$

$\vdash . (1) . (2) . \supset \vdash :. \text{Hp} . \supset : \alpha \subset \beta \cup \gamma . \equiv . \alpha = \alpha \cap \gamma .$

$[*22\cdot621] \qquad \equiv . \alpha \subset \gamma : \supset \vdash . \text{Prop}$

***24·491.** $\vdash : \beta \cap \gamma = \Lambda . \alpha \subset \beta \cup \gamma .$
$\supset . \alpha - \beta = \alpha \cap \gamma . \alpha - \gamma = \alpha \cap \beta . \alpha = (\alpha - \beta) \cup (\alpha - \gamma)$

Dem.

$\vdash . *22\cdot621 . \quad \supset \vdash : \text{Hp} . \supset . \alpha = \alpha \cap (\beta \cup \gamma) .$

$[*22\cdot481] \qquad \supset . \alpha - \gamma = \alpha \cap (\beta \cup \gamma) - \gamma$

$[*24\cdot4] \qquad = \alpha \cap \beta \quad (1)$

Similarly $\quad \vdash : \text{Hp} . \supset . \alpha - \beta = \alpha \cap \gamma \quad (2)$

$\vdash . (1) . (2) . \quad \supset \vdash : \text{Hp} . \supset . (\alpha - \beta) \cup (\alpha - \gamma) = (\alpha \cap \gamma) \cup (\alpha \cap \beta)$

$[*22\cdot68] \qquad = \alpha \cap (\gamma \cup \beta)$

$[*22\cdot621] \qquad = \alpha \quad (3)$

$\vdash . (1) . (2) . (3) . \supset \vdash . \text{Prop}$

The above proposition is used in the theory of selections (*83·63·65) and in the theory of segments of a series (*211·84).

***24·492.** $\vdash : \beta \subset \alpha . \alpha - \beta = \gamma . \supset . \alpha - \gamma = \beta$

Dem.

$\vdash . *22\cdot481 . \supset \vdash : \text{Hp} . \supset . \alpha - \gamma = \alpha - (\alpha - \beta)$

$[*22\cdot8\cdot86] \qquad = \alpha \cap (-\alpha \cup \beta)$

$[*22\cdot8\cdot9] \qquad = \alpha \cap \beta$

$[*22\cdot621] \qquad = \beta : \supset \vdash . \text{Prop}$

The above proposition is used fairly frequently, especially in the theory of series. It is first used in *93·273, in the theory of "generations."

***24·493.** $\vdash : \beta \cap \gamma = \Lambda . \supset . \alpha = (\alpha - \beta) \cup (\alpha - \gamma)$

Dem.

$\vdash . *22·84 . *24·17 . \supset \vdash : \mathrm{Hp} . \supset . -\beta \cup -\gamma = \mathrm{V} .$

$[*24·26] \quad \supset . \alpha = \alpha \cap (-\beta \cup -\gamma)$

$[*22·68] \quad = (\alpha - \beta) \cup (\alpha - \gamma) : \supset \vdash . \mathrm{Prop}$

***24·494.** $\vdash : \xi \subset \alpha . \eta \subset \beta . \alpha \cap \beta = \Lambda . \supset . (\xi \cup \eta) - \alpha = \eta . (\xi \cup \eta) - \beta = \xi$

Dem.

$\vdash . *24·3 . \quad \supset \vdash : \mathrm{Hp} . \supset . \xi - \alpha = \Lambda \quad (1)$

$\vdash . *24·311 . \quad \supset \vdash : \mathrm{Hp} . \supset . \beta \subset -\alpha .$

$[*22·44] \quad \supset . \eta \subset -\alpha .$

$[*22·621] \quad \supset . \eta - \alpha = \eta \quad (2)$

$\vdash . *22·68 . \quad \supset \vdash . (\xi \cup \eta) - \alpha = (\xi - \alpha) \cup (\eta - \alpha) \quad (3)$

$\vdash . (1) . (2) . (3) . *24·24 . \supset \vdash : \mathrm{Hp} . \supset . (\xi \cup \eta) - \alpha = \eta \quad (4)$

Similarly $\quad \vdash : \mathrm{Hp} . \supset . (\xi \cup \eta) - \beta = \xi \quad (5)$

$\vdash . (4) . (5) . \quad \supset \vdash . \mathrm{Prop}$

This proposition is used in the theory of selections (*83·63 and *88·45).

***24·495.** $\vdash : \alpha \cap \gamma = \Lambda . \supset . (\alpha \cup \gamma) - (\beta \cup \gamma) = \alpha - \beta$

Dem.

$\vdash . *22·87·68 . \supset$

$\vdash . (\alpha \cup \gamma) - (\beta \cup \gamma) = (\alpha - \beta - \gamma) \cup (\gamma - \beta - \gamma)$

$[*24·21] \quad = \alpha - \beta - \gamma \quad (1)$

$\vdash . *24·311 . *22·621 . \supset \vdash : \mathrm{Hp} . \supset . \alpha - \gamma = \alpha \quad (2)$

$\vdash . (1) . (2) . \quad \supset \vdash . \mathrm{Prop}$

The above proposition is used in the theory of minimum points (*205·83·832·84).

In the remainder of this number we shall be concerned with the existence of classes. Many of the properties of the existence of classes follow from the fact that to say a class exists is equivalent to saying that the class is not equal to the null-class. This is proved in *24·54.

***24·5.** $\vdash : \exists ! \alpha . \equiv . (\exists x) . x \in \alpha \quad [*4·2 . (*24·03)]$

***24·51.** $\vdash : \sim \exists ! \alpha . \equiv . \alpha = \Lambda$

Dem.

$\vdash . *24·5 . \supset \vdash : \sim \exists ! \alpha . \equiv . \sim \{(\exists x) . x \in \alpha\} .$

$[*10·252] \quad \equiv . (x) . x \sim\in \alpha .$

$[*24·15] \quad \equiv . \alpha = \Lambda : \supset \vdash . \mathrm{Prop}$

***24·52.** $\vdash . \exists ! \mathrm{V} \quad [*24·51·1 . \mathrm{Transp}]$

This proposition states that the class of all objects of the type in question is not null, but has at least one member. The assumption that there is some-

thing, which is equivalent to this proposition, is implicit in the proposition $*10\cdot1$, that what is true always is true in any instance. This would not hold if there were no instances of anything; hence it implies the existence of something. It will be observed that the above proposition ($*24\cdot52$) depends on $*24\cdot1$, which depends on $*22\cdot351$, which depends on $*10\cdot251$, which depends on $*10\cdot24$, which depends on $*10\cdot1$ or on $*9\cdot1$. The assumption that there is something is involved in the use of the real variable, which would otherwise be meaningless. This is made explicit in $*9\cdot1$, and in the proof of $*9\cdot2$, which is the same proposition as $*10\cdot1$.

$*24\cdot53$. $\vdash . \sim\exists ! \Lambda$ [$*24\cdot51 . *20\cdot2$]

$*24\cdot54$. $\vdash : \exists ! \alpha . \equiv . \alpha \neq \Lambda$ [$*24\cdot51$. Transp]

$*24\cdot55$. $\vdash : \sim(\alpha \subset \beta) . \equiv . \exists ! \alpha - \beta$ [$*24\cdot3$. Transp . $*24\cdot54$]

$*24\cdot56$. $\vdash :. \exists ! (\alpha \cup \beta) . \equiv : \exists ! \alpha . \vee . \exists ! \beta$ [$*10\cdot42 . *22\cdot34$]

$*24\cdot561$. $\vdash : \exists ! (\alpha \cap \beta) . \supset . \exists ! \alpha . \exists ! \beta$ [$*10\cdot5 . *22\cdot33$]

$*24\cdot57$. $\vdash :. \alpha \cap \beta = \Lambda . \supset : \exists ! \alpha . \supset . \alpha \neq \beta$

Dem.

$\vdash . *22\cdot481 . \supset \vdash : \alpha \cap \beta = \Lambda . \alpha = \beta . \supset . \alpha \cap \alpha = \Lambda .$
[$*22\cdot5$] $\supset . \alpha = \Lambda .$
[$*24\cdot51$] $\supset . \sim\exists ! \alpha$ (1)
$\vdash . (1) . \text{Exp} . \text{Transp} . \supset \vdash . \text{Prop}$

$*24\cdot571$. $\vdash : \exists ! \alpha . \alpha = \beta . \supset . \exists ! (\alpha \cap \beta)$

Dem.

$\vdash . *24\cdot57 . \text{Comm} . \supset \vdash :. \exists ! \alpha . \supset : \alpha \cap \beta = \Lambda . \supset . \alpha \neq \beta :$
[Transp] $\supset : \alpha = \beta . \supset . \alpha \cap \beta \neq \Lambda .$
[$*24\cdot54$] $\supset . \exists ! (\alpha \cap \beta)$ (1)
$\vdash . (1) . \text{Imp} . \supset \vdash . \text{Prop}$

$*24\cdot58$. $\vdash :. \alpha \subset \beta . \supset : \exists ! \alpha . \supset . \exists ! \beta$ [$*10\cdot28$]

$*24\cdot6$. $\vdash :. \alpha \subset \beta \supset : \alpha \neq \beta . \equiv . \exists ! \beta - \alpha$

Dem.

$\vdash . *22\cdot41 . \text{Transp} . \supset \vdash :. \text{Hp} . \supset : \alpha \neq \beta . \supset . \sim(\beta \subset \alpha) .$
[$*24\cdot55$] $\supset . \exists ! \beta - \alpha$ (1)
$\vdash . *24\cdot21 . \supset \vdash : \alpha = \beta . \supset . \beta - \alpha = \Lambda$ (2)
$\vdash . (2) . \text{Transp} . *24\cdot54 . \supset \vdash : \exists ! \beta - \alpha . \supset . \alpha \neq \beta$ (3)
$\vdash . (1) . (3) . \supset \vdash . \text{Prop}$

$*24\cdot61$. $\vdash : \sim\exists ! \beta . \supset . \alpha \cup \beta = \alpha$ [$*24\cdot51\cdot24$]

$*24\cdot62$. $\vdash : \sim\exists ! \beta . \supset . \alpha \cap \beta = \Lambda$ [$*24\cdot51\cdot23$]

***24·63.** $\vdash :. \Lambda \sim\epsilon \kappa . \equiv : \alpha \epsilon \kappa . \supset_\alpha . \exists ! \alpha$

In this proposition, the conditions of significance require that κ should be a class of classes. The condition "$\alpha \epsilon \kappa . \supset_\alpha . \exists ! \alpha$" is one required as hypothesis in many propositions. In virtue of the above proposition, this hypothesis may be replaced by "$\Lambda \sim\epsilon \kappa$."

Dem.

$\vdash . *13\cdot191 . \supset \vdash :. \Lambda \sim\epsilon \kappa . \equiv : \alpha = \Lambda . \supset_\alpha . \alpha \sim\epsilon \kappa :$

[Transp] $\equiv : \alpha \epsilon \kappa . \supset_\alpha . \alpha \neq \Lambda :$

[*24·54] $\equiv : \alpha \epsilon \kappa . \supset_\alpha . \exists ! \alpha :. \supset \vdash . \text{Prop}$

This proposition is frequently used in later parts of the work. We often have to deal with classes of existent classes, and the most convenient form in which to state that all the members of a class of classes exist is "$\Lambda \sim\epsilon \kappa$."

$*25$. THE UNIVERSAL RELATION, THE NULL RELATION, AND THE EXISTENCE OF RELATIONS

Summary of $*25$.

This number contains the analogues, for relations, of the definitions and propositions of $*24$. Proofs will not be given, as they proceed precisely as in $*24$.

The universal relation, denoted by $\dot{V}$, is the relation which holds between any two terms whatever of the appropriate types, whatever these may be in the given context. The null relation, $\dot{\Lambda}$, is the relation which does not hold between any pair of terms whatever, its type being fixed by the types of the terms concerning which the denial that it holds is significant. A relation R is said to *exist* when there is at least one pair of terms between which it holds; "R exists" is written "$\dot{\exists} ! R$."

The propositions of this number are much less often referred to than those of $*24$, but for the sake of uniformity we have given the analogues of all propositions in $*24$, with the same numeration (except for the integral part).

All the remarks made in $*24$ apply, *mutatis mutandis*, in the present number.

$*25 \cdot 01$. $\dot{V} = \hat{x}\hat{y}\,(x = x \,.\, y = y)$ Df

$*25 \cdot 02$. $\dot{\Lambda} = \dot{-}\,\dot{V}$ Df

$*25 \cdot 03$. $\dot{\exists} ! R \,.=.\, (\exists x, y) \,.\, xRy$ Df

$*25 \cdot 1$. $\vdash . \dot{\Lambda} \neq \dot{V}$

$*25 \cdot 101$. $\vdash . \dot{V} = \dot{-}\,\dot{\Lambda}$

$*25 \cdot 102$. $\vdash : (x, y) \,.\, \phi(x, y) \,.\equiv.\, \hat{x}\hat{y}\,\phi(x, y) = \dot{V}$

$*25 \cdot 103$. $\vdash : (x, y) \,.\, {\sim}\phi(x, y) \,.\equiv.\, \hat{x}\hat{y}\,\phi(x, y) = \dot{\Lambda}$

$*25 \cdot 104$. $\vdash . (x, y) \,.\, x\dot{V}y$

$*25 \cdot 105$. $\vdash . (x, y) \,.\, {\sim}(x\dot{\Lambda}y)$

$*25 \cdot 11$. $\vdash . (R) \,.\, R \mathrel{\dot{\subset}} \dot{V}$

$*25 \cdot 12$. $\vdash . (R) \,.\, \dot{\Lambda} \mathrel{\dot{\subset}} R$

$*25 \cdot 13$. $\vdash : R = \dot{\Lambda} \,.\equiv.\, R \mathrel{\dot{\subset}} \dot{\Lambda}$

$*25 \cdot 14$. $\vdash : (x, y) \,.\, xRy \,.\equiv.\, R = V$

$*25 \cdot 141$. $\vdash : \dot{V} \mathrel{\dot{\subset}} R \,.\equiv.\, V = R$

$*25 \cdot 15$. $\vdash : (x, y) \,.\, {\sim}(xRy) \,.\equiv.\, R = \dot{\Lambda}$

$*25 \cdot 17$. $\vdash : R = \dot{V} \,.\equiv.\, \dot{-}R = \dot{\Lambda}$

$*25 \cdot 21$. $\vdash . R \mathbin{\dot{\cap}} \dot{-}R = \dot{\Lambda}$

∗25·22. $\vdash . R \mathbin{\dot\cup} \dot{-} R = \dot{\mathrm{V}}$

∗25·23. $\vdash . R \mathbin{\dot\cap} \dot\Lambda = \dot\Lambda$

∗25·24. $\vdash . R \mathbin{\dot\cup} \dot\Lambda = R$

∗25·26. $\vdash . R \mathbin{\dot\cap} \dot{\mathrm{V}} = R$

∗25·27. $\vdash . R \mathbin{\dot\cup} \dot{\mathrm{V}} = \dot{\mathrm{V}}$

∗25·3. $\vdash : R \mathrel{\dot\subset} S . \equiv . R \mathbin{\dot-} S = \dot\Lambda$

∗25·31. $\vdash : R \mathrel{\dot\subset} S . \equiv . \dot{-} R \mathbin{\dot\cup} S = \mathrm{V}$

∗25·311. $\vdash : R \mathrel{\dot\subset} \dot{-} S . \equiv . R \mathbin{\dot\cap} S = \dot\Lambda$

∗25·312. $\vdash : \dot{-} R \mathrel{\dot\subset} S . \equiv . R \mathbin{\dot\cup} S = \dot{\mathrm{V}}$

∗25·313. $\vdash : R \mathbin{\dot\cap} S = \dot\Lambda . \equiv . R \mathbin{\dot-} S = R$

∗25·32. $\vdash : R \mathbin{\dot\cup} S = \dot\Lambda . \equiv . R = \dot\Lambda . S = \dot\Lambda$

∗25·33. $\vdash : R = \dot{\mathrm{V}} . \supset . R \mathbin{\dot\cup} S = \dot{\mathrm{V}}$

∗25·34. $\vdash : R = \dot\Lambda . \supset . R \mathbin{\dot\cap} S = \dot\Lambda$

∗25·35. $\vdash : R = \dot{\mathrm{V}} . \supset . R \mathbin{\dot\cap} S = S$

∗25·36. $\vdash : R = \dot\Lambda . \supset . R \mathbin{\dot\cup} S = S$

∗25·37. $\vdash :: R \mathbin{\dot\cap} S = \dot\Lambda . \equiv :. xRy . zSw . \supset_{x,y,z,w} : x \neq z . \vee . y \neq w$

∗25·38. $\vdash :. R \mathbin{\dot\cap} S = \dot\Lambda . \supset : R \neq S . \vee . R = \dot\Lambda . S = \dot\Lambda$

∗25·39. $\vdash :. R \mathbin{\dot\cap} S = \dot\Lambda . \equiv : xRy . \supset_{x,y} . \sim(xSy)$

∗25·4. $\vdash : P \mathbin{\dot\cap} Q = \dot\Lambda . \equiv . (P \mathbin{\dot\cup} Q) \mathbin{\dot-} P = Q . \equiv . (P \mathbin{\dot\cup} Q) \mathbin{\dot-} Q = P$

∗25·401. $\vdash : Q \mathrel{\dot\subset} P . \supset . (Q \mathbin{\dot\cup} R) \mathbin{\dot-} P = R \mathbin{\dot-} P$

∗25·402. $\vdash : P \mathbin{\dot\cap} Q = \dot\Lambda . R \mathrel{\dot\subset} P . S \mathrel{\dot\subset} Q . \supset . R \mathbin{\dot\cap} S = \dot\Lambda$

∗25·41. $\vdash . R = (R \mathbin{\dot\cap} S) \mathbin{\dot\cup} (R \mathbin{\dot-} S)$

∗25·411. $\vdash : S \mathrel{\dot\subset} R . \supset . R = S \mathbin{\dot\cup} (R \mathbin{\dot-} S)$

∗25·412. $\vdash : Q \mathrel{\dot\subset} P . S \mathrel{\dot\subset} Q . \supset . (P \mathbin{\dot-} Q) \mathbin{\dot\cup} (Q \mathbin{\dot-} S) = P \mathbin{\dot-} S$

∗25·42. $\vdash : P \mathbin{\dot\cap} Q \mathrel{\dot\subset} R . P \mathbin{\dot-} Q \mathrel{\dot\subset} R . \equiv . P \mathrel{\dot\subset} R$

∗25·43. $\vdash : P \mathbin{\dot-} Q \mathrel{\dot\subset} R . \equiv . P \mathrel{\dot\subset} Q \mathbin{\dot\cup} R$

∗25·431. $\vdash . (P \mathbin{\dot\cup} R) \mathbin{\dot\cap} (Q \mathbin{\dot\cup} \dot{-} R) = (P \mathbin{\dot\cap} Q) \mathbin{\dot\cup} (P \mathbin{\dot-} R) \mathbin{\dot\cup} (Q \mathbin{\dot\cap} R)$

∗25·432. $\vdash . (P \mathbin{\dot-} R) \mathbin{\dot\cup} (Q \mathbin{\dot\cap} R) = (P \mathbin{\dot\cap} Q) \mathbin{\dot\cup} (P \mathbin{\dot-} R) \mathbin{\dot\cup} (Q \mathbin{\dot\cap} R)$

∗25·44. $\vdash . (P \mathbin{\dot\cup} R) \mathbin{\dot\cap} (Q \mathbin{\dot\cup} \dot{-} R) = (P \mathbin{\dot\cap} \dot{-} R) \mathbin{\dot\cup} (Q \mathbin{\dot\cap} R)$

∗25·45. $\vdash : (P \mathbin{\dot\cap} R) \mathbin{\dot\cup} (Q \mathbin{\dot-} R) = \dot\Lambda . \equiv . Q \mathrel{\dot\subset} R . R \mathrel{\dot\subset} \dot{-} P$

∗25·46. $\vdash : (P \mathbin{\dot\cap} R) \mathbin{\dot\cup} (Q \mathbin{\dot-} R) = \dot\Lambda . \supset . P \mathbin{\dot\cap} Q = \dot\Lambda$

∗25·47. $\vdash : P \mathbin{\dot\cap} Q = \dot\Lambda . P \mathbin{\dot\cup} Q = R . \equiv . P \mathrel{\dot\subset} R . Q = R \mathbin{\dot-} P$

∗25·48. $\vdash :. R \mathrel{\dot\subset} P . R' \mathrel{\dot\subset} P . S \mathrel{\dot\subset} Q . S' \mathrel{\dot\subset} Q . P \mathbin{\dot\cap} Q = \dot\Lambda . \supset :$
$R \mathbin{\dot\cup} S = R' \mathbin{\dot\cup} S' . \equiv . R = R' . S = S'$

∗25·481. $\vdash :. P \mathbin{\dot\cap} Q = \dot\Lambda . P \mathbin{\dot\cap} R = \dot\Lambda . \supset : P \mathbin{\dot\cup} Q = P \mathbin{\dot\cup} R . \equiv . Q = R$

∗25·482. $\vdash :. R \mathrel{\dot\subset} P . S \mathrel{\dot\subset} Q . P \mathbin{\dot\cap} Q = \dot\Lambda . \supset : R \mathbin{\dot\cup} S = P \mathbin{\dot\cup} Q . \equiv . R = P . S = Q$

∗25·49. $\vdash :. P \mathbin{\dot\cap} Q = \dot\Lambda . \supset : P \mathrel{\dot\subset} Q \mathbin{\dot\cup} R . \equiv . P \mathrel{\dot\subset} R$

$*25\cdot491.\ \vdash : Q \mathbin{\dot{\cap}} R = \dot{\Lambda} \,.\, P \mathrel{\dot{\subset}} Q \mathbin{\dot{\cup}} R \,.\supset.$
$P \mathbin{\dot{-}} Q = P \mathbin{\dot{\cap}} R \,.\, P \mathbin{\dot{-}} R = P \mathbin{\dot{\cap}} Q \,.\, P = (P \mathbin{\dot{-}} Q) \mathbin{\dot{\cup}} (P \mathbin{\dot{-}} R)$

$*25\cdot492.\ \vdash : Q \mathrel{\dot{\subset}} P \,.\, P \mathbin{\dot{-}} Q = R \,.\supset.\, P \mathbin{\dot{-}} R = Q$

$*25\cdot493.\ \vdash : Q \mathbin{\dot{\cap}} R = \dot{\Lambda} \,.\supset.\, P = (P \mathbin{\dot{-}} Q) \mathbin{\dot{\cup}} (P \mathbin{\dot{-}} R)$

$*25\cdot494.\ \vdash : R \mathrel{\dot{\subset}} P \,.\, S \mathrel{\dot{\subset}} Q \,.\, P \mathbin{\dot{\cap}} Q = \dot{\Lambda} \,.\supset.\, (R \mathbin{\dot{\cup}} S) \mathbin{\dot{-}} P = S \,.\, (R \mathbin{\dot{\cup}} S) \mathbin{\dot{-}} Q = R$

$*25\cdot495.\ \vdash : P \mathbin{\dot{\cap}} R = \dot{\Lambda} \,.\supset.\, (P \mathbin{\dot{\cup}} R) \mathbin{\dot{-}} (Q \mathbin{\dot{\cup}} R) = P \mathbin{\dot{-}} Q$

$*25\cdot5.\ \vdash : \dot{\exists}\,!\, R \,.\equiv.\, (\exists x, y) \,.\, xRy$

$*25\cdot51.\ \vdash : \sim \dot{\exists}\,!\, R \,.\equiv.\, R = \dot{\Lambda}$

$*25\cdot52.\ \vdash \,.\, \dot{\exists}\,!\, \dot{V}$

$*25\cdot53.\ \vdash \,.\, \sim \dot{\exists}\,!\, \dot{\Lambda}$

$*25\cdot54.\ \vdash : \dot{\exists}\,!\, R \,.\equiv.\, R \neq \dot{\Lambda}$

$*25\cdot55.\ \vdash : \sim (R \mathrel{\dot{\subset}} S) \,.\equiv.\, \dot{\exists}\,!\, R \mathbin{\dot{-}} S$

$*25\cdot56.\ \vdash :.\, \dot{\exists}\,!\, (R \mathbin{\dot{\cup}} S) \,.\equiv:\, \dot{\exists}\,!\, R \,.\vee.\, \dot{\exists}\,!\, S$

$*25\cdot561.\ \vdash : \dot{\exists}\,!\, (R \mathbin{\dot{\cap}} S) \,.\supset.\, \dot{\exists}\,!\, R \,.\, \dot{\exists}\,!\, S$

$*25\cdot57.\ \vdash :.\, R \mathbin{\dot{\cap}} S = \dot{\Lambda} \,.\supset:\, \dot{\exists}\,!\, R \,.\supset.\, R \neq S$

$*25\cdot571.\ \vdash : \dot{\exists}\,!\, R \,.\, R = S \,.\supset.\, \dot{\exists}\,!\, (R \mathbin{\dot{\cap}} S)$

$*25\cdot58.\ \vdash :.\, R \mathrel{\dot{\subset}} S \,.\supset:\, \dot{\exists}\,!\, R \,.\supset.\, \dot{\exists}\,!\, S$

$*25\cdot6.\ \vdash :.\, R \mathrel{\dot{\subset}} S \,.\supset:\, R \neq S \,.\equiv.\, \dot{\exists}\,!\, S \mathbin{\dot{-}} R$

$*25\cdot61.\ \vdash : \sim \dot{\exists}\,!\, S \,.\supset.\, R \mathbin{\dot{\cup}} S = R$

$*25\cdot62.\ \vdash : \sim \dot{\exists}\,!\, S \,.\supset.\, R \mathbin{\dot{\cap}} S = \dot{\Lambda}$

$*25\cdot63.\ \vdash :.\, \dot{\Lambda} \sim \epsilon\, \kappa \,.\equiv:\, R \,\epsilon\, \kappa \,.\supset_R.\, \dot{\exists}\,!\, R$

SECTION D

LOGIC OF RELATIONS

In the present section we shall be concerned with such of the general properties of relations as have no analogues in the theory of classes. The notations introduced in this section will be used constantly throughout the rest of the work, and the ideas expressed in the definitions will be found to be of fundamental importance.

$*30$. DESCRIPTIVE FUNCTIONS

Summary of $*30$.

The functions hitherto considered, with the exception of a few particular functions such as $\alpha \cap \beta$, have been propositional, *i.e.* have had propositions for their values. But the ordinary functions of mathematics, such as x^2, $\sin x$, $\log x$, are not propositional. Functions of this kind always mean "the term having such and such a relation to x." For this reason they may be called *descriptive* functions, because they *describe* a certain term by means of its relation to their argument. Thus "$\sin \pi/2$" describes the number 1; yet propositions in which $\sin \pi/2$ occurs are not the same as they would be if 1 were substituted for $\sin \pi/2$. This appears *e.g.* from the proposition "$\sin \pi/2 = 1$," which conveys valuable information, whereas "$1 = 1$" is trivial. Descriptive functions, like descriptions in general, have no meaning by themselves, but only as constituents of propositions*.

The general definition of a descriptive function is:

$*30 \cdot 01$. $R`y = (\imath x)(xRy)$ Df

That is, "$R`y$" is to mean "the term $x$ which has the relation $R$ to $y$." If there are several terms or none having the relation $R$ to $y$, all propositions about $R`y$, *i.e.* all propositions of the form "$\phi(R`y)$," will be false. The apostrophe in "$R`y$" may be read "of." Thus if R is the relation of father to son, "$R`y$" means "the father of $y$." If $R$ is the relation of son to father, "$R`y$" means "the son of y"; in this case, all propositions of the form "$\phi(R`y)$" will be false unless y has one son and no more.

All the functions that occur in ordinary mathematics are instances of the above definition; all are obtained in the above manner from some relation. Thus in our notation "$R`y$" takes the place of what would commonly be "$fy$," this latter notation being reserved for *propositional* functions. We should write "$\sin `y$" in place of "$\sin y$," using "$\sin$" to express the relation of x to y when $x = \sin y$.

A definition such as $R`y = (\imath x)(xRy)$, where the meaning given to the term defined is a description, must be understood to mean that the term defined (in this case $R`y$) and the description assigned as its meaning (in this case $(\imath x)(xRy)$) are to be interchangeable in use: the definition is, in a sense, more purely symbolic than other definitions, since the description assigned as the meaning has itself no meaning except in use. It would perhaps be more formally correct to write

$$f(R`y) . = . f\{(\imath x)(xRy)\} \quad \text{Df.}$$

* Cf. $*14$, above.

But even this definition would not be quite complete, because it omits mention of the *scope* of the two descriptions $R`y$ and $(\imath x)(xRy)$. Thus the complete form would be

$$[R`y] . f(R`y) . = . [(\imath x)(xRy)] . f\{(\imath x)(xRy)\} \quad \text{Df.}$$

But it is unnecessary to adopt this form of definition, provided it is understood that the definition $*30{\cdot}01$ means that "$R`y$" may be written for "$(\imath x)(xRy)$" *everywhere, i.e.* in indications of scope as well as elsewhere. The *use* of the definition occurs always in accordance with the proposition:

$$\vdash : [R`y] . f(R`y) . \equiv . [(\imath x)(xRy)] . f(\imath x)(xRy),$$

which is $*30{\cdot}1$, below.

It is to be observed that $*30{\cdot}01$ does not necessarily involve

$$R`y = (\imath x)(xRy).$$

For this, by the definition, is equivalent to

$$(\imath x)(xRy) = (\imath x)(xRy),$$

which, by $*14{\cdot}28$, only holds when $E ! (\imath x)(xRy)$, *i.e.* when there is one term, and no more, which has the relation R to y.

All the conventions as to scope explained in $*14$ are to be transferred to $R`x$, *i.e.*, in the absence of any contrary indication, the scope of $R`x$ is to be the smallest proposition, enclosed in dots or other brackets, in which the $R`x$ in question occurs.

We put

$*30{\cdot}02$. $R`S`y = R`(S`y)$ Df

This definition serves merely for the avoidance of brackets. It is to be interpreted as meaning

$$[R`S`y] . f(R`S`y) . = . [R`(S`y)] . f\{R`(S`y)\} \quad \text{Df.}$$

In future, we shall often define a new expression as having a descriptive phrase for its meaning; in such a case, the definition is always to be interpreted as above. That is, any proposition in which the new expression occurs is to be the proposition which is obtained by substituting the old expression for the new one wherever the latter occurs.

$R`(S`y)$, in the above, is to be interpreted by first treating $S`y$ as if it were not a descriptive symbol, and applying $*30{\cdot}01$ and $*14{\cdot}01$ or $*14{\cdot}02$ to $R`(S`y)$, and by then applying $*30{\cdot}01$ and $*14{\cdot}01$ or $*14{\cdot}02$ to $S`y$.

The majority of the propositions of the present number are immediate consequences of the corresponding propositions in $*14$. Thus $*14{\cdot}31$—$\cdot 34$ and $*14{\cdot}113$ lead immediately to $*30{\cdot}12$—$\cdot 16$, which show that, either always or when $R`y$ exists, the "scope" of $R`y$ or of $R`y$ and $S`y$ makes no difference to the truth-values of such propositions as we are concerned with. We have

$*30{\cdot}18$. $\vdash :. E ! R`y : (z) . \phi z : \supset . \phi(R`y)$

so that what holds of everything holds of $R\text{‘}y$, provided $R\text{‘}y$ exists. This results immediately from $*14\cdot18$, and shows that, provided $R\text{‘}y$ exists, the fact that "$R\text{‘}y$" is an incomplete symbol does not prevent its being substituted as a value of z whenever we have $(z) \,.\, \phi z$, or an assertion of the propositional function ϕz.

One of the most used propositions of this number is:

$*30\cdot3$. $\vdash :. x = R\text{‘}y \,.\equiv : z\breve{R}y \,.\equiv_z .\, z = x$

which results immediately from $*14\cdot202$. The following analogous proposition results from the above by means of $*14\cdot122$:

$*30\cdot31$. $\vdash :. x = R\text{‘}y \,.\equiv : xRy : zRy \,.\supset_z .\, z = x$

I.e. "$x = R\text{‘}y$" involves, in addition to "xRy," the statement that whatever has the relation R to y is identical with x.

A proposition constantly referred to is:

$*30\cdot37$. $\vdash : \mathrm{E}\,!\, R\text{‘}y \,.\, y = z \,.\supset .\, R\text{‘}y = R\text{‘}z$

In the hypothesis, $\mathrm{E}\,!\, R\text{‘}y$ might be replaced by $\mathrm{E}\,!\, R\text{‘}z$, but one or other of them is essential. For, by $*14\cdot21$, "$R\text{‘}y = R\text{‘}z$" implies $\mathrm{E}\,!\, R\text{‘}y$ and $\mathrm{E}\,!\, R\text{‘}z$ (these are equivalent when $y = z$), and therefore cannot be true when $R\text{‘}y$ and $R\text{‘}z$ do not exist.

The use of $*30\cdot37$ is chiefly in cases where y or z or both are replaced by descriptive functions. Suppose, for example, that z is replaced by $S\text{‘}w$. By $*30\cdot18$, we may substitute $S\text{‘}w$ for z if $S\text{‘}w$ exists. By $*14\cdot21$, both sides of the implication in $*30\cdot37$ will become false if $S\text{‘}w$ does not exist, and therefore the implication will still hold. Hence whether $S\text{‘}w$ exists or not, we may substitute it for z and obtain

$$\vdash : \mathrm{E}\,!\, R\text{‘}y \,.\, y = S\text{‘}w \,.\supset .\, R\text{‘}y = R\text{‘}S\text{‘}w.$$

In like manner, if we replace y by $T\text{‘}v$, we obtain

$$\vdash : \mathrm{E}\,!\, R\text{‘}T\text{‘}v \,.\, T\text{‘}v = S\text{‘}w \,.\supset .\, R\text{‘}T\text{‘}v = R\text{‘}S\text{‘}w.$$

A very important proposition is:

$*30\cdot4$. $\vdash :. \mathrm{E}\,!\, R\text{‘}y \,.\supset : a = R\text{‘}y \,.\equiv .\, aRy$

This proposition states that, provided $R\text{‘}y$ exists, to say that a is *the* term which has the relation R to y is equivalent to saying that a has the relation R to y. Thus for example "a is the occupier of the house y" is equivalent to "a occupies the house y," "a is the writer of Waverley" is equivalent to "a wrote Waverley," "a is the father of y" is equivalent to "a begot y." But we cannot argue from "John Smith inhabits London" to "John Smith is *the* inhabitant of London."

We shall introduce in this and subsequent sections many constant relations for which $\mathrm{E}\,!\, R\text{‘}y$ is always true. When R is such that $\mathrm{E}\,!\, R\text{‘}y$ is always true, we have, in virtue of $*30\cdot4$,

$$a = R\text{‘}y \,.\equiv .\, aRy$$

for every possible value of y. The following proposition is useful in cases where both R and S are such that R‘y and S‘y always exist:

$*30·41.\ \vdash :. (y) . R‘y = S‘y . \equiv : (y) . E ! R‘y : R = S$

Thus if we know that R‘y and S‘y are always identical, we know not only that R and S are identical, but also that R‘y (and therefore S‘y) always exists.

$*30·01.\ R‘y = (\imath x)(xRy)$ Df

$*30·02.\ R‘S‘y = R‘(S‘y)$ Df

In interpreting R‘(S‘y), S‘y is to be treated as an ordinary symbol until R‘(S‘y) has been eliminated by $*30·01$ and $*14·01$ or $*14·02$, and then the above definitions are to be applied to S‘y.

$*30·1.\ \vdash : [R‘y] . f(R‘y) . \equiv . [(\imath x)(xRy)] . f(\imath x)(xRy)$ [$*4·2 . (*30·01)$]

$*30·11.\ \vdash :. [R‘y] . f(R‘y) . \equiv : (\exists b) : xRy . \equiv_x . x = b : fb$ [$*30·1 . *14·1$]

The following propositions are immediate applications of $*14·31$ ff., made in accordance with $*30·1$.

$*30·12.\ \vdash :: E ! R‘y . \supset :. [R‘y] . p \vee \chi(R‘y) . \equiv : p . \vee . [R‘y] . \chi(R‘y)$ [$*14·31$]

$*30·13.\ \vdash :: E ! R‘y . \supset :. [R‘y] . \sim\chi(R‘y) . \equiv . \sim\{[R‘y] . \chi(R‘y)\}$ [$*14·32$]

$*30·14.\ \vdash :: E ! R‘y . \supset :. [R‘y] . p \supset \chi(R‘y) . \equiv : p . \supset . [R‘y] . \chi(R‘y)$ [$*14·33$]

$*30·141.\ \vdash :: E ! R‘y . \supset :. [R‘y] . \chi(R‘y) \supset p . \equiv : [R‘y] . \chi(R‘y) . \supset . p$ [$*14·331$]

$*30·142.\ \vdash :: E ! R‘y . \supset :. [R‘y] . p \equiv \chi(R‘y) . \equiv : p . \equiv . [R‘y] . \chi(R‘y)$ [$*14·332$]

$*30·15.\ \vdash :. p : [R‘y] . \chi(R‘y) : \equiv : [R‘y] . p . \chi(R‘y)$ [$*14·34$]

The following two propositions are immediate consequences of $*14·113·112$.

$*30·16.\ \vdash : [R‘y] . f(R‘y, S‘z) . \equiv . [S‘z] . f(R‘y, S‘z)$ [$*14·113$]

$*30·17.\ \vdash :. [R‘y] . f(R‘y, S‘z) . \equiv :$
$(\exists b, c) : xRy . \equiv_x . x = b : xSz . \equiv_x . x = c : f(b, c)$ [$*14·112$]

$*30·18.\ \vdash :. E ! R‘y : (z) . \phi z : \supset . \phi(R‘y)$ [$*14·18$]

$*30·19.\ \vdash :. R‘y = b . \supset : \psi(R‘y) . \equiv . \psi b$ [$*14·15$]

$*30·2.\ \vdash :. E ! R‘y . \equiv : (\exists b) : xRy . \equiv_x . x = b$ [$*4·2 . *14·11 . (*30·01)$]

In proving $*30·2$, we have to use the definition $*30·01$, not $*30·1$, because E ! $(\imath x)(\phi x)$ is not of the form $f(\imath x)(\phi x)$. This appears if we attempt to apply the definition $*14·01$ to E ! $(\imath x)(\phi x)$, which leads to an expression containing the meaningless constituent E ! b. But by the definition $*30·01$, every typographical occurrence of the symbol "R‘y" means what results when this symbol is replaced by "$(\imath x)(xRy)$," hence "E ! R‘y" means "E ! $(\imath x)(xRy)$."

$*30{\cdot}21.\ \vdash :: E\,!\,R\text{‘}y\,.\equiv :.\,(\exists x)\,.\,xRy : xRy\,.\,zRy\,.\supset_{x,z}.\,x=z$
$[*14{\cdot}203\,.\,(*30{\cdot}01)]$

$*30{\cdot}22.\ \vdash : E\,!\,R\text{‘}y\,.\equiv .\,R\text{‘}y=(\iota x)(xRy)\quad [*14{\cdot}28\,.\,(*30{\cdot}01)]$

Note that we do not necessarily have

$$R\text{‘}y=(\iota x)(xRy),$$

which is only true when $E\,!\,R\text{‘}y$.

$*30{\cdot}3.\ \vdash :.\,x=R\text{‘}y\,.\equiv : zRy\,.\equiv_z .\,z=x\qquad [*14{\cdot}202]$

$*30{\cdot}31.\ \vdash :.\,x=R\text{‘}y\,.\equiv : xRy : zRy\,.\supset_z .\,z=x\quad [*14{\cdot}122\,.\,*30{\cdot}3]$

$*30{\cdot}32.\ \vdash : E\,!\,R\text{‘}y\,.\equiv .\,(R\text{‘}y)\,Ry\qquad [*14{\cdot}22]$

$*30{\cdot}33.\ \vdash :: E\,!\,R\text{‘}y\,.\supset :.\,\psi(R\text{‘}y) : \equiv : (\exists x)\,.\,xRy\,.\,\psi x : \equiv : xRy\,.\supset_x .\,\psi x$
$[*14{\cdot}26]$

$*30{\cdot}34.\ \vdash :.\,xRy\,.\equiv_x .\,xSy : \supset : E\,!\,R\text{‘}y\,.\equiv .\,E\,!\,S\text{‘}y\quad [*14{\cdot}271]$

$*30{\cdot}341.\ \vdash :.\,xRy\,.\equiv_x .\,xSy : \supset : E\,!\,R\text{‘}y\,.\equiv .\,R\text{‘}y=S\text{‘}y$

Dem.

$\vdash .\,*14{\cdot}21\,.\qquad \supset \vdash : R\text{‘}y=S\text{‘}y\,.\supset .\,E\,!\,R\text{‘}y\qquad (1)$
$\vdash .\,*14{\cdot}27\,.\,\mathrm{Comm}\,.\supset \vdash :.\,\mathrm{Hp}\,.\supset : E\,!\,R\text{‘}y\,.\supset .\,R\text{‘}y=S\text{‘}y\qquad (2)$
$\vdash .\,(1)\,.\,(2)\,.\qquad \supset \vdash .\,\mathrm{Prop}$

$*30{\cdot}35.\ \vdash :.\,R=S\,.\supset : E\,!\,R\text{‘}y\,.\equiv .\,E\,!\,S\text{‘}y\quad [*30{\cdot}34\,.\,*21{\cdot}43]$

$*30{\cdot}36.\ \vdash : E\,!\,R\text{‘}y\,.\,R=S\,.\supset .\,R\text{‘}y=S\text{‘}y\quad [*14{\cdot}27\,.\,\mathrm{Imp}\,.\,*21{\cdot}43]$

$*30{\cdot}37.\ \vdash : E\,!\,R\text{‘}y\,.\,y=z\,.\supset .\,R\text{‘}y=R\text{‘}z$

Dem.

$\vdash .\,*14{\cdot}28\,.\qquad \supset \vdash : E\,!\,R\text{‘}y\,.\supset .\,R\text{‘}y=R\text{‘}y\qquad (1)$
$\vdash .\,*13{\cdot}12\,.\qquad \supset \vdash :.\,y=z\,.\supset : R\text{‘}y=R\text{‘}y\,.\equiv .\,R\text{‘}y=R\text{‘}z\qquad (2)$
$\vdash .\,(1)\,.\,(2)\,.\,\mathrm{Ass}\,.\supset \vdash .\,\mathrm{Prop}$

This proposition is very frequently used.

$*30{\cdot}4.\ \vdash :.\,E\,!\,R\text{‘}y\,.\supset : a=R\text{‘}y\,.\equiv .\,aRy\quad [*14{\cdot}241]$

This is a very important proposition, of which the use is constant.

$*30{\cdot}41.\ \vdash :.\,(y)\,.\,R\text{‘}y=S\text{‘}y\,.\equiv : (y)\,.\,E\,!\,R\text{‘}y : R=S$

Dem.

$\vdash .\,*14{\cdot}21\,.\,*10{\cdot}11{\cdot}27\,.\supset \vdash : (y)\,.\,R\text{‘}y=S\text{‘}y\,.\supset .\,(y)\,.\,E\,!\,R\text{‘}y\qquad (1)$
$\vdash .\,*14{\cdot}13{\cdot}142\,.\qquad \supset \vdash :.\,(y)\,.\,R\text{‘}y=S\text{‘}y\,.\supset : (x,y) : x=R\text{‘}y\,.\equiv .\,x=S\text{‘}y :$
$[(1).*30{\cdot}4]\qquad \supset : (x,y) : xRy\,.\equiv .\,xSy :$
$[*21{\cdot}43]\qquad \supset : R=S\qquad (2)$
$\vdash .\,*30{\cdot}36\,.\qquad \supset \vdash : E\,!\,R\text{‘}y\,.\,R=S\,.\supset .\,R\text{‘}y=S\text{‘}y :$
$[*10{\cdot}11{\cdot}27{\cdot}35]\qquad \supset \vdash :.\,(y)\,.\,E\,!\,R\text{‘}y : R=S : \supset .\,(y)\,.\,R\text{‘}y=S\text{‘}y\qquad (3)$
$\vdash .\,(1)\,.\,(2)\,.\,(3)\,.\qquad \supset \vdash .\,\mathrm{Prop}$

***30·42.** $\vdash :. (y) . E ! R`y . \supset : (y) . R`y = S`y . \equiv . R = S$ [*30·41]

The hypothesis $(y) . E ! R`y$ is fulfilled by a number of important special relations, of which examples will occur in the subsequent numbers of the present section.

***30·5.** $\vdash : E ! P`Q`z . \supset . E ! Q`z$

Dem.

$$\begin{array}{ll} \vdash . *30\cdot2 . \supset \vdash :. E ! P`Q`z . \equiv : (\exists b) : xP(Q`z) . \equiv_x . x = b : & \\ [*10\cdot1] & \supset : (\exists b) : bP(Q`z) . \equiv . b = b : \\ [*13\cdot15] & \supset : (\exists b) . bP(Q`z) : \\ [*14\cdot21] & \supset : E ! Q`z :. \supset \vdash . \text{Prop} \end{array}$$

***30·501.** $\vdash : \phi(P`Q`z) . \equiv . (\exists b, c) . c = Q`z . b = P`c . \phi b$

On the meaning of "$\phi(P`Q`z)$," see note to the definition *30·02.

Dem.

$$\begin{array}{ll} \vdash . *14\cdot1\cdot122 . \supset \vdash :: \phi(P`Q`z) . \equiv :. (\exists b) : bP(Q`z) : xP(Q`z) . \supset_x . x = b : \phi b :. & \\ [*14\cdot205] & \equiv :. (\exists b) :. (\exists c) : c = Q`z : bPc : xPc . \supset_x . x = b : \phi b :. \\ [*14\cdot122\cdot202] & \equiv :. (\exists b, c) . c = Q`z . b = P`c . \phi b :: \supset \vdash . \text{Prop} \end{array}$$

***30·51.** $\vdash : b = P`Q`z . \equiv . (\exists c) . b = P`c . c = Q`z$ [*30·501 . *13·195]

***30·52.** $\vdash : E ! P`Q`z . \equiv . (\exists b, c) . b = P`c . c = Q`z$ [*30·51 . *14·204]

$*31$. CONVERSES OF RELATIONS

Summary of $*31$.

If R is a relation, the relation which y has to x when xRy is called the *converse* of R. Thus *greater* is the converse of *less*, *before* of *after*, *husband* of *wife*. The converse of identity is identity, and the converse of diversity is diversity. The converse of R is written $\breve{R}$ (read "R-converse"). When $R = \breve{R}$, R is called a *symmetrical* relation, otherwise it is called *not-symmetrical*. When R is incompatible with $\breve{R}$, R is called *asymmetrical*. Thus "cousin" is symmetrical, "brother" is not-symmetrical (because when x is the brother of y, y may be either the brother or the sister of x), and "husband" is asymmetrical.

The relation of $\breve{R}$ to R is called "Cnv." It will be shown that every relation has one, and only one, converse; hence, applying the notation of $*30$, that one is $\mathrm{Cnv}`R$. Thus $\breve{R} = \mathrm{Cnv}`R$. We have thus two notations for the converse of R; the second is more convenient for the converse of a relation not denoted by a single letter.

The more important propositions of the present number are the following:

$*31{\cdot}13$. $\vdash . \mathrm{E} ! \, \mathrm{Cnv}`P$

I.e. any relation P has a converse. Hence the relation "Cnv" verifies the hypothesis $(y) . \mathrm{E} ! \, R`y$, *i.e.* we have $(P) . \mathrm{E} ! \, \mathrm{Cnv}`P$.

$*31{\cdot}32$. $\vdash : P = Q . \equiv . \breve{P} = \breve{Q}$

I.e. two relations are identical when, and only when, their converses are identical.

$*31{\cdot}33$. $\vdash . \mathrm{Cnv}`\mathrm{Cnv}`P = P$

I.e. any relation is the converse of its converse.

Very many of the subsequent uses of the notion of the converse of a relation require only the propositions which embody the definitions of $\breve{P}$ and Cnv, namely

$*31{\cdot}11$. $\vdash : x\breve{P}y . \equiv . yPx$

and

$*31{\cdot}131$. $\vdash : x (\mathrm{Cnv}`P) \, y . \equiv . yPx$

***31·01.** $\text{Cnv} = \hat{Q}\hat{P}\{xQy . \equiv_{x,y} . yPx\}$ Df

***31·02.** $\breve{P} = \hat{x}\hat{y}(yPx)$ Df

***31·1.** $\vdash :. Q \,\text{Cnv}\, P . \equiv : xQy . \equiv_{x,y} . yPx$ [*21·3 . (*31·01)]

***31·101.** $\vdash : Q \,\text{Cnv}\, P . R \,\text{Cnv}\, P . \supset . Q = R$

Dem.

$\vdash . *31·1 . \supset \vdash :. \text{Hp} . \supset : xQy . \equiv_{x,y} . yPx : xRy . \equiv_{x,y} . yPx :$

[*11·371] $\supset : xQy . \equiv_{x,y} . xRy :$

[*21·43] $\supset : Q = R :. \supset \vdash . \text{Prop}$

***31·11.** $\vdash : x\breve{P}y . \equiv . yPx$ [*21·3 . (*31·02)]

***31·111.** $\vdash . \breve{P} \,\text{Cnv}\, P$ [*31·1·11]

***31·12.** $\vdash . \breve{P} = \text{Cnv}`P$

Dem.

$\vdash . *31·101 . \supset \vdash : Q \,\text{Cnv}\, P . \breve{P} \,\text{Cnv}\, P . \supset . Q = \breve{P} :$

[*31·111] $\supset \vdash : Q \,\text{Cnv}\, P . \supset . Q = \breve{P}$ (1)

$\vdash . (1) . *10·11 . *31·111 . \supset$

$\vdash : \breve{P} \,\text{Cnv}\, P : Q \,\text{Cnv}\, P . \supset_Q . Q = \breve{P} :$

[*30·31] $\supset \vdash . \breve{P} = \text{Cnv}`P$

***31·13.** $\vdash . \text{E}!\, \text{Cnv}`P$ [*14·21 . *31·12]

***31·131.** $\vdash : x(\text{Cnv}`P)y . \equiv . yPx$ [*31·11·12 . *21·43]

***31·132.** $\vdash : Q \,\text{Cnv}\, P . \equiv . Q = \text{Cnv}`P . \equiv . Q = \breve{P}$ [*30·4 . *31·13·12]

***31·14.** $\vdash . \text{Cnv}`(P \dot{\cap} Q) = \text{Cnv}`P \dot{\cap} \text{Cnv}`Q$

Dem.

$\vdash . *31·131 . \supset \vdash : x\{\text{Cnv}`(P \dot{\cap} Q)\}y . \equiv . y(P \dot{\cap} Q)x .$

[*21·33] $\equiv . yPx . yQx .$

[*31·131] $\equiv . x(\text{Cnv}`P)y . x(\text{Cnv}`Q)y .$

[*21·33] $\equiv . x\{\text{Cnv}`P \dot{\cap} \text{Cnv}`Q\}y$ (1)

$\vdash . (1) . *11·11 . *21·43 . \supset \vdash . \text{Prop}$

***31·15.** $\vdash . \text{Cnv}`(P \dot{\cup} Q) = \text{Cnv}`P \dot{\cup} \text{Cnv}`Q$ [Similar proof]

***31·16.** $\vdash . \text{Cnv}`\dot{-}P = \dot{-}(\text{Cnv}`P)$

Dem.

$\vdash . *31·131 . \supset \vdash : x(\text{Cnv}`\dot{-}P)y . \equiv . y\dot{-}Px .$

[*23·35] $\equiv . \sim(yPx) .$

[*31·131] $\equiv . \sim\{x(\text{Cnv}`P)y\} .$

[*23·35] $\equiv . x\{\dot{-}(\text{Cnv}`P)\}y$ (1)

$\vdash . (1) . *11·11 . *21·43 . \supset \vdash . \text{Prop}$

***31·17.** $\vdash :. y = \breve{P}`x . \equiv : xPz . \equiv_z . z = y$ [*30·3 . *31·11]

***31·18.** $\vdash :. E ! \breve{P}`x . \equiv : (\exists y) : xPz . \equiv_z . z = y$ [*30·2 . *31·11]

***31·21.** $\vdash . \text{Cnv}`\dot{\Lambda} = \dot{\Lambda}$

Dem.

$\vdash . *31·131 . \supset \vdash : x (\text{Cnv}`\dot{\Lambda}) y . \equiv . y \dot{\Lambda} x :$

[*25·105] $\supset \vdash . \sim x (\text{Cnv}`\dot{\Lambda}) y$ (1)

$\vdash . (1) . *11·11 . *25·15 . \supset \vdash . \text{Prop}$

***31·22.** $\vdash . \text{Cnv}`\dot{V} = \dot{V}$ [Similar proof]

***31·23.** $\vdash : \breve{P} = \dot{V} . \equiv . P = \dot{V}$

Dem.

$\vdash . *25·14 . \supset \vdash : \breve{P} = \dot{V} . \equiv . (x, y) . x\breve{P}y .$

[*31·11.*11·33] $\equiv . (x, y) . yPx .$

[*11·2] $\equiv . (y, x) . yPx .$

[*25·14] $\equiv . P = \dot{V} : \supset \vdash . \text{Prop}$

***31·24.** $\vdash : \breve{P} = \dot{\Lambda} . \equiv . P = \dot{\Lambda}$ [Similar proof]

***31·32.** $\vdash : P = Q . \equiv . \breve{P} = \breve{Q}$

Dem.

$\vdash . *21·43 . \supset \vdash :. P = Q . \equiv : xPy . \equiv_{x,y} . xQy :$

[*4·86·21.*31·11] $\equiv : y\breve{P}x . \equiv_{x,y} . y\breve{Q}x :$

[*11·2] $\equiv : y\breve{P}x . \equiv_{y,x} . y\breve{Q}x :$

[*21·43] $\equiv : \breve{P} = \breve{Q} :. \supset \vdash . \text{Prop}$

***31·33.** $\vdash . \text{Cnv}`\text{Cnv}`P = P$

Dem.

$\vdash . *31·131 . \supset \vdash : x (\text{Cnv}`\text{Cnv}`P) y . \equiv . y (\text{Cnv}`P) x .$

[*31·131] $\equiv . xPy$ (1)

$\vdash . (1) . *11·11 . *21·43 . \supset \vdash . \text{Prop}$

***31·34.** $\vdash : P = \breve{Q} . \equiv . Q = \breve{P}$

Dem.

$\vdash . *31·32 . \supset \vdash : P = \breve{Q} . \equiv . \breve{P} = \text{Cnv}`\breve{Q}$

[*31·12·32] $= \text{Cnv}`\text{Cnv}`Q$

[*31·33] $= Q : \supset \vdash . \text{Prop}$

***31·4.** $\vdash : P \mathrel{\dot{\subset}} Q . \equiv . \breve{P} \mathrel{\dot{\subset}} \breve{Q}$ [*31·11 . *11·33]

***31·41.** $\vdash : P \mathrel{\dot{\subset}} \breve{Q} . \equiv . \breve{P} \mathrel{\dot{\subset}} Q$ [*31·4·33·12]

***31·5.** $\vdash : \dot{\exists} ! P . \equiv . \dot{\exists} ! \breve{P}$ [*31·24 . Transp . *25·54]

$*31\cdot51$. $\vdash :(P) . f\breve{P} . \equiv .(P) . fP$

Dem.

$$\vdash . *10\cdot1 . \quad \supset \vdash :(P) . fP . \supset . f\breve{P} :$$

$$[*10\cdot11\cdot21] \supset \vdash :(P) . fP . \supset .(P) . f\breve{P} \qquad (1)$$

$$\vdash . *10\cdot1 . *31\cdot12 . \supset$$

$$\vdash :(P) . f\breve{P} . \supset . f(\mathrm{Cnv}`\breve{P}) .$$

$$[*31\cdot33\cdot12] \qquad \supset . fP :$$

$$[*10\cdot11\cdot21] \supset \vdash :(P) . f\breve{P} . \supset .(P) . fP \qquad (2)$$

$$\vdash .(1) .(2) . \supset \vdash . \mathrm{Prop}$$

$*31\cdot52$. $\vdash :(\exists P) . f\breve{P} . \equiv .(\exists P) . fP \quad [*31\cdot51 . \mathrm{Transp}]$

*32. REFERENTS AND RELATA OF A GIVEN TERM WITH RESPECT TO A GIVEN RELATION

Summary of *32.

Given any relation R, the class of terms which have the relation R to a given term y are called the *referents* of y, and the class of terms to which a given term x has the relation R are called the *relata* of x. We shall denote by $\overrightarrow{R}$ the relation of the class of referents of y to y, and by $\overleftarrow{R}$ the relation of the class of relata of x to x. It is convenient also to have a notation for the relations of $\overrightarrow{R}$ and $\overleftarrow{R}$ to R. We shall denote the relation of $\overrightarrow{R}$ to R by "sg," where "sg" stands for "sagitta." Similarly we shall denote by "gs" the relation of $\overleftarrow{R}$ to R, to suggest an arrow running from right to left instead of from left to right. $\overrightarrow{R}$ and $\overleftarrow{R}$ are chiefly useful for the sake of the descriptive functions to which they give rise; thus $\overrightarrow{R}\text{‘}y = \hat{x}(xRy)$ and $\overleftarrow{R}\text{‘}x = \hat{y}(xRy)$. Thus *e.g.* if R is the relation of parent to son, $\overrightarrow{R}\text{‘}y =$ the parents of y, $\overleftarrow{R}\text{‘}x =$ the sons of x. If R is the relation of less to greater among numbers of any kind, $\overrightarrow{R}\text{‘}y =$ numbers less than y, and $\overleftarrow{R}\text{‘}x =$ numbers greater than x. When $R\text{‘}y$ exists, $\overrightarrow{R}\text{‘}y$ is the class whose only member is $R\text{‘}y$. But when there are many terms having the relation R to y, $\overrightarrow{R}\text{‘}y$, which is the class of those terms, supplies a notation which cannot be supplied by $R\text{‘}y$. And similarly if there are many terms to which x has the relation R, $\overleftarrow{R}\text{‘}x$ supplies the notation for these terms. Thus for example let R be the relation "sin," *i.e.* the relation which x has to y when $x = \sin y$. Then "$\overleftarrow{\sin}\text{‘}x$" represents all values of y such that $x = \sin y$, *i.e.* all values of $\sin^{-1} x$ or $\arcsin x$. Unlike the usual symbol, it is not ambiguous, since instead of representing some one of these values, it represents the class of them.

The definitions of $\overrightarrow{R}$, $\overleftarrow{R}$, sg, gs are as follows:

***32·01.** $\overrightarrow{R} = \hat{\alpha}\hat{y}\{\alpha = \hat{x}(xRy)\}$ Df

***32·02.** $\overleftarrow{R} = \hat{\beta}\hat{x}\{\beta = \hat{y}(xRy)\}$ Df

***32·03.** $\mathrm{sg} = \hat{A}\hat{R}(A = \overrightarrow{R})$ Df

***32·04.** $\mathrm{gs} = \hat{A}\hat{R}(A = \overleftarrow{R})$ Df

In virtue of the above definitions, we shall have $\mathrm{sg}\text{‘}R = \overrightarrow{R}$, $\mathrm{gs}\text{‘}R = \overleftarrow{R}$. This gives an alternative notation which is convenient in dealing with a relation not represented by a single letter.

It should be observed that if R is a homogeneous relation (*i.e.* one in which referents and relata are of the same type), then $\overrightarrow{R}$ and $\overleftarrow{R}$ are not homogeneous, but relate a class to objects of the type of its members.

In virtue of the definitions of $\overrightarrow{R}$ and $\overleftarrow{R}$, we shall have

***32·13.** $\vdash . \overrightarrow{R}`y = \hat{x}(xRy)$

***32·131.** $\vdash . \overleftarrow{R}`x = \hat{y}(xRy)$

Thus by *14·21, we always have $E ! \overrightarrow{R}`y$ and $E ! \overleftarrow{R}`x$. Thus whatever relation R may be, we have $(y) . E ! \overrightarrow{R}`y$ and $(x) . E ! \overleftarrow{R}`x$. We do not in general have $(y) . \exists ! \overrightarrow{R}`y$ or $(x) . \exists ! \overleftarrow{R}`x$. Thus taking R to be the relation of parent and child, $\overrightarrow{R}`y =$ the parents of $y$ and $\overleftarrow{R}`x =$ the children of x. Thus $\overleftarrow{R}`x = \Lambda$, *i.e* $\sim \exists ! \overleftarrow{R}`x$, when x is childless, and $\overrightarrow{R}`y = \Lambda$, *i.e.* $\sim \exists ! \overrightarrow{R}`y$, when y is Adam or Eve. The two sorts of existence, $E ! \overrightarrow{R}`y$ and $\exists ! \overrightarrow{R}`y$, can both be *significantly* predicated of $\overrightarrow{R}`y$, because "$\overrightarrow{R}`y$" is a descriptive function whose value is a class; and the same applies to $\overleftarrow{R}`x$. It will be seen that (by *14·21) $\exists ! \overrightarrow{R}`y . \supset . E ! \overrightarrow{R}`y$, but the converse implication does not hold in general.

We have

***32·16.** $\vdash : \overrightarrow{R} = \overrightarrow{S} . \equiv . \overleftarrow{R} = \overleftarrow{S} . \equiv . R = S$

Aso by *32·18·181,

$$\vdash : x \in \overrightarrow{R}`y . \equiv . xRy . \equiv . y \in \overleftarrow{R}`x.$$

Thus by the use of $\overrightarrow{R}`y$ or $\overleftarrow{R}`x$, every statement of the form "xRy" can be reduced to a statement asserting membership of a class. Since, however, the class in question is given by a descriptive function, and descriptive functions are defined by means of relations, we do not thus obtain a method of reducing the theory of relations to the theory of classes.

***32·01.** $\overrightarrow{R} = \hat{\alpha}\hat{y}\{\alpha = \hat{x}(xRy)\}$ Df

***32·02.** $\overleftarrow{R} = \hat{\beta}\hat{x}\{\beta = \hat{y}(xRy)\}$ Df

***32·03.** $\text{sg} = \hat{A}\hat{R}(A = \overrightarrow{R})$ Df

***32·04.** $\text{gs} = \hat{A}\hat{R}(A = \overleftarrow{R})$ Df

***32·1.** $\vdash : \alpha \overrightarrow{R} y . \equiv . \alpha = \hat{x}(xRy)$ [*21·3 . (*32·01)]

***32·101.** $\vdash : \beta \overleftarrow{R} x . \equiv . \beta = \hat{y}(xRy)$ [*21·3 . (*32·02)]

***32·11.** $\vdash . \hat{x}(xRy) = \overrightarrow{R}`y$ [*32·1 . *30·3]

$*32\cdot111.\ \vdash . \hat{y}(xRy) = \overleftarrow{R}`x$ [$*32\cdot101 . *30\cdot3$]

$*32\cdot12.\ \vdash . E ! \overrightarrow{R}`y$ [$*32\cdot11 . *14\cdot21$]

$*32\cdot121.\ \vdash . E ! \overleftarrow{R}`x$ [$*32\cdot111 . *14\cdot21$]

"$E ! \overrightarrow{R}`y$" must not be confounded with "$\exists ! \overrightarrow{R}`y$." The former means that there is such a class as $\overrightarrow{R}`y$, which, as we have just seen, is always true; the latter means that $\overrightarrow{R}`y$ is not null, which is only true if y is a term to which some other term has the relation R. Note that, by $*14\cdot21$, both $\exists ! \overrightarrow{R}`y$ and $\sim\exists ! \overrightarrow{R}`y$ imply $E ! \overrightarrow{R}`y$. The contradictory of $\exists ! \overrightarrow{R}`y$ is not $\sim\exists ! \overrightarrow{R}`y$, but $\sim\{[\overrightarrow{R}`y] . \exists ! \overrightarrow{R}`y\}$. This last would not imply $E ! \overrightarrow{R}`y$, but for the fact that $E ! \overrightarrow{R}`y$ is always true.

$*32\cdot13.\ \vdash . \overrightarrow{R}`y = \hat{x}(xRy)$ [$*32\cdot11 . *20\cdot59$]

$*32\cdot131.\ \vdash . \overleftarrow{R}`x = \hat{y}(xRy)$ [$*32\cdot111 . *20\cdot59$]

$*32\cdot132.\ \vdash : \alpha \overrightarrow{R} y . \equiv . \alpha = \overrightarrow{R}`y . \equiv . \alpha = \hat{x}(xRy)$ [$*32\cdot1\cdot13 . *20\cdot57$]

$*32\cdot133.\ \vdash : \beta \overleftarrow{R} x . \equiv . \beta = \overleftarrow{R}`x . \equiv . \beta = \hat{y}(xRy)$ [$*32\cdot101\cdot131 . *20\cdot57$]

The use of $*20\cdot57$ will in general be tacit. It happens constantly that we have propositions such as $*32\cdot13$, in which a descriptive expression is shown to be identical with a class. In such cases, whenever the properties of the class are asserted of the descriptive expression, $*20\cdot57$ is relevant.

$*32\cdot14.\ \vdash : \overrightarrow{R} = \overrightarrow{S} . \equiv . R = S$

Dem.

$$
\begin{aligned}
&\vdash . *21\cdot43 . \supset \vdash :: \overrightarrow{R} = \overrightarrow{S} . \equiv :. \alpha \overrightarrow{R} y . \equiv_{\alpha, y} . \alpha \overrightarrow{S} y :. \\
&[*32\cdot1] \qquad \equiv :. \alpha = \hat{x}(xRy) . \equiv_{\alpha, y} . \alpha = \hat{x}(xSy) :. \\
&[*11\cdot2] \qquad \equiv :. (y) :. \alpha = \hat{x}(xRy) . \equiv_{\alpha} . \alpha = \hat{x}(xSy) :. \\
&[*20\cdot25] \qquad \equiv :. (y) : \hat{x}(xRy) = \hat{x}(xSy) :. \\
&[*20\cdot15] \qquad \equiv :. (y) :. (x) : xRy . \equiv . xSy :. \\
&[*11\cdot2] \qquad \equiv :. (x, y) : xRy . \equiv . xSy :. \\
&[*21\cdot43] \qquad \equiv :. R = S :: \supset \vdash . \text{Prop}
\end{aligned}
$$

$*32\cdot15.\ \vdash : \overleftarrow{R} = \overleftarrow{S} . \equiv . R = S$ [Similar proof]

$*32\cdot16.\ \vdash : \overrightarrow{R} = \overrightarrow{S} . \equiv . \overleftarrow{R} = \overleftarrow{S} . \equiv . R = S$ [$*32\cdot14\cdot15$]

$*32\cdot18.\ \vdash : x \in \overrightarrow{R}`y . \equiv . xRy$ [$*32\cdot13 . *20\cdot33$]

$*32\cdot181.\ \vdash : y \in \overleftarrow{R}`x . \equiv . xRy$ [$*32\cdot131 . *20\cdot33$]

$*32\cdot182.\ \vdash : x \in \overrightarrow{R}`y . \equiv . y \in \overleftarrow{R}`x$ [$*32\cdot18\cdot181$]

The transformation from "xRy" to "$x \in \overrightarrow{R}`y$" is one commonly effected in language. *E.g.* suppose "$xRy$" is "$x$ loves $y$," then "$x \in \overrightarrow{R}`y$" is "$x$ is a lover of y."

***32·19.** $\vdash : R \subseteq S . \supset . \overrightarrow{R}`y \subset \overrightarrow{S}`y . \overleftarrow{R}`x \subset \overleftarrow{S}`x$

Dem.

$$\begin{array}{lll} \vdash . *32\cdot18 . \supset \vdash :. \mathrm{Hp} . \supset : x \in \overrightarrow{R}`y . \supset_x . x \in \overrightarrow{S}`y : & & \\ [*22\cdot1] \qquad \supset : \overrightarrow{R}`y \subset \overrightarrow{S}`y & & (1) \\ \vdash . *32\cdot181 . \supset \vdash :. \mathrm{Hp} . \supset : y \in \overleftarrow{R}`x . \supset_y . y \in \overleftarrow{S}`x : & & \\ [*22\cdot1] \qquad \supset : \overleftarrow{R}`x \subset \overleftarrow{S}`x & & (2) \\ \vdash . (1) . (2) . \supset \vdash . \mathrm{Prop} & & \end{array}$$

***32·2.** $\vdash : A \,\mathrm{sg}\, R . \equiv . A = \overrightarrow{R}$ [*21·3 . (*32·03)]

***32·201.** $\vdash : A \,\mathrm{gs}\, R . \equiv . A = \overleftarrow{R}$ [*21·3 . (*32·04)]

***32·21.** $\vdash . \overrightarrow{R} = \mathrm{sg}`R$ [*32·2 . *30·3]

***32·211.** $\vdash . \overleftarrow{R} = \mathrm{gs}`R$ [*32·201 . *30·3]

***32·22.** $\vdash . \mathrm{E!}\, \mathrm{sg}`R$ [*32·21 . *14·21]

***32·221.** $\vdash . \mathrm{E!}\, \mathrm{gs}`R$ [*32·211 . *14·21]

***32·23.** $\vdash . \mathrm{sg}`R = \overrightarrow{R}$ [*32·21 . *21·2·57]

***32·231.** $\vdash . \mathrm{gs}`R = \overleftarrow{R}$ [*32·211 . *21·2·57]

***32·24.** $\vdash . \mathrm{sg}`\breve{R} = \mathrm{gs}`R$

Dem.

$$\begin{array}{lll} \vdash . *32\cdot23 . (*32\cdot01) . & \supset \vdash . \mathrm{sg}`\breve{R} = \hat{\alpha}\hat{y}\, \{\alpha = \hat{x}\,(x\breve{R}y)\} . & \\ [*21\cdot33] & \supset \vdash : \alpha\,(\mathrm{sg}`\breve{R})\,y . \equiv . \alpha = \hat{x}\,(x\breve{R}y) . & \\ [*31\cdot11 . *20\cdot15] & \equiv . \alpha = \hat{x}\,(yRx) . & \\ [*32\cdot101] & \equiv . \alpha \overleftarrow{R} x . & \\ [*32\cdot211] & \equiv . \alpha\,(\mathrm{gs}`R)\,x & (1) \\ \vdash . (1) . *11\cdot11 . *21\cdot43 . \supset \vdash . \mathrm{Prop} & & \end{array}$$

***32·241.** $\vdash . \mathrm{gs}`\breve{R} = \mathrm{sg}`R$ [Similar proof]

***32·25.** $\vdash : A \,\mathrm{sg}\, R . \equiv . A = \mathrm{sg}`R$ [*30·4 . *32·22]

***32·251.** $\vdash : A \,\mathrm{gs}\, R . \equiv . A = \mathrm{gs}`R$ [*30·4 . *32·221]

***32·3.** $\vdash . \{\mathrm{sg}`(R \dot{\cap} S)\}`y = \overrightarrow{R}`y \cap \overrightarrow{S}`y$

Note that we do *not* have

$$\mathrm{sg}`(R \dot{\cap} S) = \mathrm{sg}`R \dot{\cap} \mathrm{sg}`S.$$

Dem.

$\vdash . *32 \cdot 23 \cdot 13 . \supset \vdash . \{\mathrm{sg}'(R \dot{\cap} S)\}'y = \hat{x}\{x(R \dot{\cap} S)y\}$

[*23·33] $= \hat{x}(xRy . xSy)$

[*22·39] $= \hat{x}(xRy) \cap \hat{x}(xSy)$

[*32·13] $= \overrightarrow{R}'y \cap \overrightarrow{S}'y . \supset \vdash . \mathrm{Prop}$

***32·31.** $\vdash . \{\mathrm{gs}'(R \dot{\cap} S)\}'x = \overleftarrow{R}'x \cap \overleftarrow{S}'x$

***32·32.** $\vdash . \{\mathrm{sg}'(R \dot{\cup} S)\}'y = \overrightarrow{R}'y \cup \overrightarrow{S}'y$

***32·33.** $\vdash . \{\mathrm{gs}'(R \dot{\cup} S)\}'x = \overleftarrow{R}'x \cup \overleftarrow{S}'x$

***32·34.** $\vdash . \{\mathrm{sg}'(\dot{-} R)\}'y = - \overrightarrow{R}'y$

***32·35.** $\vdash . \{\mathrm{gs}'(\dot{-} R)\}'x = - \overleftarrow{R}'x$

The proofs of the above propositions are similar to that of *32·3.

***32·4.** $\vdash :. E ! R'z . \equiv : \exists ! \overrightarrow{R}'z : x, y \in \overrightarrow{R}'z . \supset_{x,y} . x = y$ [*30·21 . *32·18]

***32·41.** $\vdash :. E ! S'y . \supset : \overrightarrow{R}'y = \overrightarrow{S}'y . \equiv . R'y = S'y$

Dem.

$\vdash . *4 \cdot 86 . \quad \supset \vdash :: xSy . \equiv_x . x = b : \supset :.$

$xRy . \equiv_x . xSy : \equiv : xRy . \equiv_x . x = b$ (1)

$\vdash . (1) . *5 \cdot 32 . \supset \vdash :. xSy . \equiv_x . x = b : xRy . \equiv_x . xSy : \equiv :$

$xSy . \equiv_x . x = b : xRy . \equiv_x . x = b$ (2)

$\vdash . (2) . *10 \cdot 11 \cdot 281 . *32 \cdot 18 \cdot 181 . \supset$

$\vdash :. (\exists b) : xSy . \equiv_x . x = b : \overrightarrow{R}'y = \overrightarrow{S}'y : \equiv : (\exists b) : xSy . \equiv_x . x = b : xRy . \equiv_x . x = b :$

[*30·3.*14·13] $\equiv : (\exists b) : xSy . \equiv_x . x = b : R'y = b :$

[*14·101] $\equiv : R'y = S'y$ (3)

$\vdash . (3) . *30 \cdot 2 . \supset \vdash :. E ! S'y . \overrightarrow{R}'y = \overrightarrow{S}'y . \equiv . R'y = S'y :. \supset \vdash . \mathrm{Prop}$

***32·42.** $\vdash :. \overrightarrow{R}'y = \overrightarrow{S}'y . \supset : E ! R'y . \equiv . E ! S'y$ [*30·34 . *32·18]

*33. DOMAINS, CONVERSE DOMAINS, AND FIELDS OF RELATIONS

Summary of *33.

If R is any relation, the *domain* of R, which we denote by $\mathrm{D}\text{‘}R$, is the class of terms which have the relation R to something or other; the *converse domain*, $\text{Ɑ}\text{‘}R$, is the class of terms to which something or other has the relation R; and the *field*, $C\text{‘}R$, is the sum of the domain and the converse domain. (Note that the field is only significant when R is a *homogeneous* relation.)

The above notations $\mathrm{D}\text{‘}R$, $\text{Ɑ}\text{‘}R$, $C\text{‘}R$ are derivative from the notations D, Ɑ, C for the relations, to a relation, of its domain, converse domain, and field respectively. We are to have

$$\mathrm{D}\text{‘}R = \hat{x}\{(\exists y) . xRy\}$$
$$\text{Ɑ}\text{‘}R = \hat{y}\{(\exists x) . xRy\}$$
$$C\text{‘}R = \hat{x}\{(\exists y) : xRy . \vee . yRx\};$$

hence we define D, Ɑ, C as follows:

***33·01.** $\mathrm{D} = \hat{\alpha}\hat{R}\,[\alpha = \hat{x}\{(\exists y) . xRy\}]$ Df

***33·02.** $\text{Ɑ} = \hat{\beta}\hat{R}\,[\beta = \hat{y}\{(\exists x) . xRy\}]$ Df

***33·03.** $C = \hat{\gamma}\hat{R}\,[\gamma = \hat{x}\{(\exists y) : xRy . \vee . yRx\}]$ Df

The letter C is chosen as the initial of the word "campus." We require one other definition, namely of the relation of x to R when x is a member of the field of R. This relation, which we will call F, is defined as follows:

***33·04.** $F = \hat{x}\hat{R}\{(\exists y) : xRy . \vee . yRx\}$ Df

We shall find that $C = \overrightarrow{F}$. $\breve{\mathrm{D}}$ will be the relation of a relation to its domain, $\overleftarrow{\mathrm{D}}\text{‘}\alpha$ will be the class of relations having α for their domain. Similar remarks apply to Ɑ and C. The *field* of a relation is specially important in connection with series.

The propositions of this number are constantly used throughout the remainder of the work. The ideas of the domain, converse domain, and field are very general, and have somewhat different uses for relations of different kinds. Consider first the sort of relation that gives rise to a descriptive function $R\text{‘}y$. For this we require that $R\text{‘}y$ should exist whenever there is anything having the relation R to y, *i.e.* that there should never be more than one term having the relation R to a given term y. In this case, the values of y for which $R\text{‘}y$ exists will constitute the "converse domain" of R, *i.e.* $\text{Ɑ}\text{‘}R$, and the values which $R\text{‘}y$ assumes for various values of y will

constitute the "domain" of R, *i.e.* $D'R$. Thus the converse domain is the class of possible arguments for the descriptive function $R'y$, and the domain is the class of all values of the function. Thus, for example, if R is the relation of the square of an integer y to y, then $R'y =$ the square of y, provided y is an integer. In this case, $\text{Ꮖ}'R$ is the class of integers, and $D'R$ is the class of perfect squares. Or again, suppose R is the relation of wife to husband; then $R'y =$ the wife of y, $\text{Ꮖ}'R =$ married men, $D'R =$ married women. In such cases, the *field* usually has little importance; and if the values of the function $R'y$ are not of the same type as its arguments, *i.e.* if the relation R is not *homogeneous*, the field is meaningless. Thus, for example, if R is a homogeneous relation, $\overrightarrow{R}$ and $\overleftarrow{R}$ are not homogeneous, and therefore "$C'\overrightarrow{R}$" and "$C'\overleftarrow{R}$" are meaningless.

Let us next suppose that R is the sort of relation that generates a series, say the relation of less to greater among integers. Then $D'R =$ all integers that are less than some other integer = all integers, $\text{Ꮖ}'R =$ all integers that are greater than some other integer = all integers except 0. In this case, $C'R =$ all integers that are either greater or less than some other integer = all integers. Generally, if R generates a series, $D'R =$ all members of the series except the last (if any), $\text{Ꮖ}'R =$ all members of the series except the first (if any), and $C'R =$ all members of the series. In this case, "xFR" expresses the fact that x is a member of the series. Thus when R generates a series, $C'R$ becomes important, and the relation F is likely to be useful.

We shall have occasion to deal with many relations having some of the properties of series, and with many propositions which, though only important in connection with serial relations, hold much more generally. In such cases, the field of a relation is likely to be important. Thus in the section on Induction (Part II, Section E), where we are preparing the way for the construction of serial relations by means of a certain kind of non-serial relation, and throughout relation-arithmetic (Part IV), the fields of relations will occur constantly. But in the earlier parts of the work, it is chiefly domains and converse domains that occur.

Among the more important properties of domains, converse domains and fields, which are proved in the present number, are the following.

We have always $E\,!\,D'R$, $E\,!\,\text{Ꮖ}'R$, $E\,!\,C'R$ (*33·12·121·122). (The last of these, however, is only significant when R is homogeneous.)

*33·13. $\vdash : x \in D'R \,.\equiv .\, (\exists y) \,.\, xRy$

*33·131. $\vdash : y \in \text{Ꮖ}'R \,.\equiv .\, (\exists x) \,.\, xRy$

*33·132. $\vdash :. x \in C'R \,.\equiv : (\exists y) : xRy \,.\vee .\, yRx$

*33·14. $\vdash : xRy \,.\supset .\, x \in D'R \,.\, y \in \text{Ꮖ}'R$

*33·16. $\vdash .\, C'R = D'R \cup \text{Ꮖ}'R$

***33·2·21·22.** The converse domain of a relation is the domain of its converse, the domain of a relation is the converse domain of its converse, and the field of a relation is the field of its converse.

***33·24.** $\vdash : \exists ! \mathrm{D}\text{‘}R . \equiv . \exists ! \text{Ɑ}\text{‘}R . \equiv . \exists ! C\text{‘}R . \equiv . \dot{\exists} ! R$

***33·4.** $\vdash . \mathrm{D}\text{‘}R = \hat{x}\{\exists ! \overleftarrow{R}\text{‘}x\}$

with corresponding propositions (*33·41·42) for $\text{Ɑ}\text{‘}R$ and $C\text{‘}R$.

***33·43.** $\vdash : \mathrm{E} ! R\text{‘}y . \supset . y \in \text{Ɑ}\text{‘}R . R\text{‘}y \in \mathrm{D}\text{‘}R$

***33·431.** $\vdash : (y) . \mathrm{E} ! R\text{‘}y . \supset . (\beta) . \beta \subset \text{Ɑ}\text{‘}R$

***33·5.** $\vdash . C = \overrightarrow{F}$

***33·51.** $\vdash : x \in C\text{‘}R . \equiv . xFR$

The proofs of propositions concerning Ɑ and C are usually similar to those for D, and are therefore often omitted.

***33·01.** $\mathrm{D} = \hat{\alpha}\hat{R}\,[\alpha = \hat{x}\{(\exists y) . xRy\}]$ Df

***33·02.** $\text{Ɑ} = \hat{\beta}\hat{R}\,[\beta = \hat{y}\{(\exists x) . xRy\}]$ Df

***33·03.** $C = \hat{\gamma}\hat{R}\,[\gamma = \hat{x}\{(\exists y) : xRy . \vee . yRx\}]$ Df

***33·04.** $F = \hat{x}\hat{R}\,\{(\exists y) : xRy . \vee . yRx\}$ Df

***33·1.** $\vdash : \alpha \mathrm{D} R . \equiv . \alpha = \hat{x}\{(\exists y) . xRy\}$ [*21·3 . (*33·01)]

***33·101.** $\vdash : \beta \text{Ɑ} R . \equiv . \beta = \hat{y}\{(\exists x) . xRy\}$

***33·102.** $\vdash : \gamma C R . \equiv . \gamma = \hat{x}\{(\exists y) : xRy . \vee . yRx\}$

***33·103.** $\vdash :. xFR . \equiv : (\exists y) : xRy . \vee . yRx$

***33·11.** $\vdash . \mathrm{D}\text{‘}R = \hat{x}\{(\exists y) . xRy\}$ [*33·1 . *30·3 . *20·59]

***33·111.** $\vdash . \text{Ɑ}\text{‘}R = \hat{y}\{(\exists x) . xRy\}$

***33·112.** $\vdash . C\text{‘}R = \hat{x}\{(\exists y) : xRy . \vee . yRx\}$

***33·12.** $\vdash . \mathrm{E} ! \mathrm{D}\text{‘}R$ [*33·11 . *14·21]

***33·121.** $\vdash . \mathrm{E} ! \text{Ɑ}\text{‘}R$

***33·122.** $\vdash . \mathrm{E} ! C\text{‘}R$

***33·123.** $\vdash : \alpha \mathrm{D} R . \equiv . \alpha = \mathrm{D}\text{‘}R$ [*30·4 . *33·12]

***33·124.** $\vdash : \beta \text{Ɑ} R . \equiv . \beta = \text{Ɑ}\text{‘}R$ [*30·4 . *33·121]

***33·125.** $\vdash : \gamma C R . \equiv . \gamma = C\text{‘}R$ [*30·4 . *32·123]

***33·13.** $\vdash : x \in \mathrm{D}\text{‘}R . \equiv . (\exists y) . xRy$ [*33·11 . *20·3·57]

***33·131.** $\vdash : y \in \text{Ɑ}\text{‘}R . \equiv . (\exists x) . xRy$

***33·132.** $\vdash :. x \in C\text{‘}R . \equiv : (\exists y) : xRy . \vee . yRx$

***33·14.** $\vdash : xRy . \supset . x \in \mathrm{D}\text{‘}R . y \in \text{Ɑ}\text{‘}R$

Dem.

$\vdash . *10\cdot24 . \supset \vdash :. \mathrm{Hp} . \supset : (\exists y) . xRy : (\exists x) . xRy :$

[*33·13·131] $\supset : x \in \mathrm{D}\text{‘}R . y \in \text{Ɑ}\text{‘}R :. \supset \vdash . \mathrm{Prop}$

***33·15.** $\vdash . \overrightarrow{R}`y \subset \mathrm{D}`R$

Dem.

$\vdash . *32\cdot18 . \supset \vdash : x \in \overrightarrow{R}`y . \supset_x . xRy .$

$[*10\cdot24] \qquad \supset_x . (\exists y) . xRy .$

$[*33\cdot13] \qquad \supset_x . x \in \mathrm{D}`R : \supset \vdash . \mathrm{Prop}$

***33·151.** $\vdash . \overleftarrow{R}`x \subset \text{Ꮖ}`R$

***33·152.** $\vdash . \overrightarrow{R}`x \cup \overleftarrow{R}`x \subset C`R$

***33·16.** $\vdash . C`R = \mathrm{D}`R \cup \text{Ꮖ}`R$

Dem.

$\vdash . *33\cdot132 . *10\cdot42 . \supset$

$\vdash :. x \in C`R . \equiv : (\exists y) . xRy . \vee . (\exists y) . yRx :$

$[*33\cdot13\cdot131] \equiv : x \in \mathrm{D}`R . \vee . x \in \text{Ꮖ}`R :$

$[*22\cdot34] \qquad \equiv : x \in \mathrm{D}`R \cup \text{Ꮖ}`R \qquad (1)$

$\vdash . (1) . *10\cdot11 . *20\cdot43 . \supset \vdash . \mathrm{Prop}$

***33·161.** $\vdash . \mathrm{D}`R \subset C`R . \text{Ꮖ}`R \subset C`R \quad [*33\cdot16 . *22\cdot58]$

***33·17.** $\vdash : xRy . \supset . x, y \in C`R \quad [*33\cdot14\cdot161]$

***33·18.** $\vdash : \mathrm{D}`R = \text{Ꮖ}`R . \supset . \mathrm{D}`R = C`R$

Dem.

$\vdash . *22\cdot56 . \supset \vdash : \mathrm{D}`R = \text{Ꮖ}`R . \supset . \mathrm{D}`R = \mathrm{D}`R \cup \text{Ꮖ}`R$

$[*33\cdot16] \qquad = C`R : \supset \vdash . \mathrm{Prop}$

***33·181.** $\vdash : \text{Ꮖ}`R \subset \mathrm{D}`R . \equiv . \mathrm{D}`R = C`R$

Dem.

$\vdash . *22\cdot62 . \supset \vdash : \text{Ꮖ}`R \subset \mathrm{D}`R . \equiv . \mathrm{D}`R = \mathrm{D}`R \cup \text{Ꮖ}`R$

$[*33\cdot16] \qquad = C`R : \supset \vdash . \mathrm{Prop}$

***33·182.** $\vdash : \mathrm{D}`R \subset \text{Ꮖ}`R . \equiv . \text{Ꮖ}`R = C`R$ [Similar proof]

If R is the sort of relation which generates a series, so that "xRy" may be read "x precedes y," then $\text{Ꮖ}`R \subset \mathrm{D}`R$ is the condition that the series may have no last term, since it states that every term which follows some term precedes some other term, and is therefore not the last of the series.

***33·2.** $\vdash . \text{Ꮖ}`R = \mathrm{D}`\breve{R}$

Dem.

$\vdash . *31\cdot11 . *10\cdot11 . \supset \vdash : xRy . \equiv_x . y\breve{R}x :$

$[*10\cdot281] \qquad \supset \vdash : (\exists x) . xRy . \equiv . (\exists x) . y\breve{R}x :$

$[*33\cdot13\cdot131] \qquad \supset \vdash : y \in \text{Ꮖ}`R . \equiv . y \in \mathrm{D}`\breve{R} \qquad (1)$

$\vdash . (1) . *10\cdot11 . *20\cdot43 . \supset \vdash . \mathrm{Prop}$

***33·21.** $\vdash . \mathrm{D}`R = \text{Ꮖ}`\breve{R}$ [Similar proof]

***33·22.** $\vdash . C'R = C'\breve{R}$

Dem.

$\vdash . *33\cdot16\cdot2\cdot21 . \supset \vdash . C'R = \text{Ɑ}'\breve{R} \cup \text{D}'\breve{R}$

$[*33\cdot16] \qquad\qquad = C'\breve{R} . \supset \vdash . \text{Prop}$

***33·24.** $\vdash : \exists ! \text{D}'R . \equiv . \exists ! \text{Ɑ}'R . \equiv . \exists ! C'R . \equiv . \dot{\exists} ! R$

Dem.

$\vdash . *33\cdot13 . \supset \vdash :. \exists ! \text{D}'R . \equiv : (\exists x) : (\exists y) . xRy :$

$[*25\cdot5.(*11\cdot03)] \qquad \equiv : \dot{\exists} ! R \qquad (1)$

$\vdash . *33\cdot131 . \supset \vdash :. \exists ! \text{Ɑ}'R . \equiv : (\exists y) : (\exists x) . xRy :$

$[*11\cdot2] \qquad \equiv : (\exists x, y) . xRy :$

$[*25\cdot5] \qquad \equiv : \dot{\exists} ! R \qquad (2)$

$\vdash . *33\cdot132 . \supset \vdash :: \exists ! C'R . \equiv :. (\exists x) :. (\exists y) : xRy . \vee . yRx :.$

$[*11\cdot7] \qquad \equiv :. (\exists x, y) . xRy :.$

$[*25\cdot5] \qquad \equiv :. \dot{\exists} ! R \qquad (3)$

$\vdash . (1) . (2) . (3) . \supset \vdash . \text{Prop}$

***33·241.** $\vdash : \text{D}'R = \Lambda . \equiv . \text{Ɑ}'R = \Lambda . \equiv . C'R = \Lambda . \equiv . R = \dot{\Lambda}$

$[*33\cdot24 . \text{Transp} . *24\cdot51 . *25\cdot51]$

***33·25.** $\vdash . \text{D}'(R \dot{\cap} S) \subset \text{D}'R \cap \text{D}'S$

Dem.

$\vdash . *33\cdot13 . \supset \vdash :. x \in \text{D}'(R \dot{\cap} S) . \equiv : (\exists y) . x (R \dot{\cap} S) y :$

$[*21\cdot33.*10\cdot281] \qquad \equiv : (\exists y) . xRy . xSy :$

$[*10\cdot5] \qquad \supset : (\exists y) . xRy : (\exists y) . xSy :$

$[*33\cdot13] \qquad \supset : x \in \text{D}'R . x \in \text{D}'S :$

$[*21\cdot33] \qquad \supset : x \in \text{D}'R \cap \text{D}'S \qquad (1)$

$\vdash . (1) . *10\cdot11 . \supset \vdash . \text{Prop}$

***33·251.** $\vdash . \text{Ɑ}'(R \dot{\cap} S) \subset \text{Ɑ}'R \cap \text{Ɑ}'S$ [Similar proof]

***33·252.** $\vdash . C'(R \dot{\cap} S) \subset C'R \cap C'S$ [Similar proof]

***33·26.** $\vdash . \text{D}'(R \dot{\cup} S) = \text{D}'R \cup \text{D}'S$

Dem.

$\vdash . *33\cdot13 . \supset \vdash :. x \in \text{D}'(R \dot{\cup} S) . \equiv : (\exists y) . x (R \dot{\cup} S) y :$

$[*23\cdot34.*10\cdot281] \qquad \equiv : (\exists y) : xRy . \vee . xSy :$

$[*10\cdot42] \qquad \equiv : (\exists y) . xRy : \vee : (\exists y) . xSy :$

$[*33\cdot13] \qquad \equiv : x \in \text{D}'R . \vee . x \in \text{D}'S :$

$[*22\cdot34] \qquad \equiv : x \in \text{D}'R \cup \text{D}'S \qquad (1)$

$\vdash . (1) . *10\cdot11 . *20\cdot43 . \supset \vdash . \text{Prop}$

***33·261.** $\vdash . \text{Ɑ}'(R \dot{\cup} S) = \text{Ɑ}'R \cup \text{Ɑ}'S$ [Similar proof]

***33·262.** $\vdash . C'(R \dot{\cup} S) = C'R \cup C'S$ $[*33\cdot26\cdot261\cdot16]$

***33·263.** $\vdash : R \mathrel{\dot\subset} S . \supset . \mathrm{D}`R \subset \mathrm{D}`S$

Dem.

$\vdash . *23{\cdot}1 . \supset \vdash :. \mathrm{Hp} . \supset : xRy . \supset_{x,y} . xSy :$

[*10·28·27] $\supset : (x) : (\exists y) . xRy . \supset . (\exists y) . xSy :$

[*33·13] $\supset : (x) : x \in \mathrm{D}`R . \supset . x \in \mathrm{D}`S :$

[*23·1] $\supset : \mathrm{D}`R \subset \mathrm{D}`S :. \supset \vdash . \mathrm{Prop}$

***33·264.** $\vdash : R \mathrel{\dot\subset} S . \supset . \text{Ɑ}`R \subset \text{Ɑ}`S$ [Similar proof]

***33·265.** $\vdash : R \mathrel{\dot\subset} S . \supset . C`R \subset C`S$ [*33·263·264·16 . *22·72]

***33·27.** $\vdash . C`R = \mathrm{D}`(R \mathbin{\dot\cup} \breve{R})$

Dem.

$\vdash . *33{\cdot}16{\cdot}2 . \supset \vdash . C`R = \mathrm{D}`R \cup \mathrm{D}`\breve{R}$

[*33·26] $= \mathrm{D}`(R \mathbin{\dot\cup} \breve{R}) . \supset \vdash . \mathrm{Prop}$

***33·271.** $\vdash . C`R = \text{Ɑ}`(R \mathbin{\dot\cup} \breve{R})$ [Similar proof]

***33·272.** $\vdash . \mathrm{D}`(R \mathbin{\dot\cup} \breve{R}) = \text{Ɑ}`(R \mathbin{\dot\cup} \breve{R}) = C`(R \mathbin{\dot\cup} \breve{R}) = C`R$ [*33·27·271·16]

***33·28.** $\vdash . \mathrm{D}`\dot{\mathrm{V}} = \text{Ɑ}`\dot{\mathrm{V}} = C`\dot{\mathrm{V}} = \mathrm{V}$

Dem.

$\vdash . *10{\cdot}25 . *25{\cdot}104 . \supset \vdash :. (x) : (\exists y) . x \dot{\mathrm{V}} y :. (x) : (\exists y) . y \dot{\mathrm{V}} x :.$

[*33·13·131] $\supset \vdash :. (x) . x \in \mathrm{D}`\dot{\mathrm{V}} : (x) . x \in \text{Ɑ}`\dot{\mathrm{V}} :.$

[*24·14] $\supset \vdash : \mathrm{D}`\dot{\mathrm{V}} = \mathrm{V} . \text{Ɑ}`\dot{\mathrm{V}} = \mathrm{V}$ (1)

[*33·16] $\supset \vdash . C`\dot{\mathrm{V}} = \mathrm{V} \cup \mathrm{V}$

[*22·56] $= \mathrm{V}$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***33·29.** $\vdash . \mathrm{D}`\dot\Lambda = \text{Ɑ}`\dot\Lambda = C`\dot\Lambda = \Lambda$ [*33·241 . *21·2]

***33·3.** $\vdash :. \alpha \subset \mathrm{D}`R . \equiv : x \in \alpha . \supset_x . \exists ! \overleftarrow{R}`x$

Dem.

$\vdash . *32{\cdot}181 . \supset \vdash :. x \in \alpha . \supset_x . \exists ! \overleftarrow{R}`x : \equiv : x \in \alpha . \supset_x . (\exists y) . xRy :$

[*33·13] $\equiv : x \in \alpha . \supset_x . x \in \mathrm{D}`R :. \supset \vdash . \mathrm{Prop}$

***33·31.** $\vdash :. \beta \subset \text{Ɑ}`R . \equiv : y \in \beta . \supset_y . \exists ! \overrightarrow{R}`y$ [Proof as in *33·3]

The three following propositions are used in the theory of selections (*80, *83 and *85). The second of them is also used in the theory of greater and less (*117) and in the theory of transitive relations (*201).

***33·32.** $\vdash : \mathrm{D}`R \cap \mathrm{D}`S = \Lambda . \supset . R \mathbin{\dot\cap} S = \dot\Lambda$

The converse of this proposition is not true.

Dem.

$\vdash . *23{\cdot}33 . \quad \supset \vdash : x (R \mathbin{\dot\cap} S) y . \supset . xRy . xSy .$

[*33·14 . *22·33] $\supset . x \in \mathrm{D}`R \cap \mathrm{D}`S .$

$[*10\cdot24] \qquad \supset . \exists ! \text{D}'R \cap \text{D}'S$ (1)

$\vdash . (1) . \text{Transp} . \quad \supset \vdash : \text{D}'R \cap \text{D}'S = \Lambda . \supset . \sim \{x (R \dot{\cap} S) y\}$ (2)

$\vdash . (2) . *11\cdot11\cdot3 . \supset \vdash : \text{D}'R \cap \text{D}'S = \Lambda . \supset . (x, y) . \sim \{x (R \dot{\cap} S) y\} .$

$[*25\cdot15] \qquad \supset . R \dot{\cap} S = \dot{\Lambda} : \supset \vdash . \text{Prop}$

$*33\cdot33$. $\vdash : \text{Ɑ}'R \cap \text{Ɑ}'S = \Lambda . \supset . R \dot{\cap} S = \dot{\Lambda}$ [Proof as in $*33\cdot32$]

$*33\cdot34$. $\vdash : C'R \cap C'S = \Lambda . \supset . R \dot{\cap} S = \dot{\Lambda}$

Dem.

$\vdash . *33\cdot161 . *22\cdot49 . \supset \vdash . \text{D}'R \cap \text{D}'S \subset C'R \cap C'S .$

$[*24\cdot13] \qquad \supset \vdash : C'R \cap C'S = \Lambda . \supset . \text{D}'R \cap \text{D}'S = \Lambda .$

$[*33\cdot32] \qquad \supset . R \dot{\cap} S = \dot{\Lambda} : \supset \vdash . \text{Prop}$

$*33\cdot35$. $\vdash :. \text{D}'R \subset \alpha . \equiv : xRy . \supset_{x,y} . x \in \alpha$

Dem.

$\vdash . *33\cdot13 . \supset \vdash :. \text{D}'R \subset \alpha . \equiv : (\exists y) . xRy . \supset_x . x \in \alpha :$

$[*10\cdot23] \qquad \equiv : xRy . \supset_{x,y} . x \in \alpha :. \supset \vdash . \text{Prop}$

$*33\cdot351$. $\vdash :. \text{Ɑ}'R \subset \alpha . \equiv : xRy . \supset_{x,y} . y \in \alpha$ [Proof as in $*33\cdot35$]

$*33\cdot352$. $\vdash :. C'R \subset \alpha . \equiv : xRy . \supset_{x,y} . x, y \in \alpha$

Dem.

$\vdash . *33\cdot16 . *22\cdot59 . \supset$

$\vdash :. C'R \subset \alpha . \equiv : \text{D}'R \subset \alpha . \text{Ɑ}'R \subset \alpha :$

$[*33\cdot35\cdot351] \equiv : xRy . \supset_{x,y} . x \in \alpha : xRy . \supset_{x,y} . y \in \alpha :$

$[*11\cdot391] \qquad \equiv : xRy . \supset_{x,y} . x, y \in \alpha :. \supset \vdash . \text{Prop}$

The two following propositions ($*33\cdot4\cdot41$) are very frequently used.

$*33\cdot4$. $\vdash . \text{D}'R = \hat{x} \{\exists ! \overleftarrow{R}'x\}$

Dem.

$\vdash . *33\cdot13 . \supset \vdash : x \in \text{D}'R . \equiv . (\exists y) . xRy .$

$[*32\cdot181] \qquad \equiv . (\exists y) . y \in \overleftarrow{R}'x .$

$[*24\cdot5] \qquad \equiv . \exists ! \overleftarrow{R}'x$ (1)

$\vdash . (1) . *10\cdot11 . *20\cdot33 . \supset \vdash . \text{Prop}$

$*33\cdot41$. $\vdash . \text{Ɑ}'R = \hat{y} \{\exists ! \overrightarrow{R}'y\}$ [Similar proof]

$*33\cdot42$. $\vdash . C'R = \hat{x} \{\exists ! (\overrightarrow{R}'x \cup \overleftarrow{R}'x)\}$

Dem.

$\vdash . *33\cdot4\cdot41\cdot16 . \supset \vdash . C'R = \hat{x} \{\exists ! \overrightarrow{R}'x\} \cup \hat{x} \{\exists ! \overleftarrow{R}'x\}$

$[*22\cdot391] \qquad = \hat{x} \{\exists ! \overrightarrow{R}'x . \vee . \exists ! \overleftarrow{R}'x\}$

$[*24\cdot56 . *20\cdot15] \qquad = \hat{x} \{\exists ! (\overrightarrow{R}'x \cup \overleftarrow{R}'x)\} . \supset \vdash . \text{Prop}$

***33·43.** $\vdash : \mathrm{E}\,!\, R\text{‘}y \,.\supset .\, y \in \text{Ɑ}\text{‘}R \,.\, R\text{‘}y \in \mathrm{D}\text{‘}R$

Dem.

$\vdash . *30{\cdot}32 \,.\supset \vdash : \mathrm{E}\,!\, R\text{‘}y \,.\supset .\, (R\text{‘}y)\, Ry \,.$

$[*33{\cdot}14] \qquad \supset .\, y \in \text{Ɑ}\text{‘}R \,.\, R\text{‘}y \in \mathrm{D}\text{‘}R : \supset \vdash .\, \mathrm{Prop}$

***33·431.** $\vdash : (y) \,.\, \mathrm{E}\,!\, R\text{‘}y \,.\supset .\, (\beta) \,.\, \beta \subset \text{Ɑ}\text{‘}R$

Dem.

$\vdash . *33{\cdot}43 \,. \qquad \supset \vdash :. \mathrm{Hp} \,.\supset : y \in \text{Ɑ}\text{‘}R \,.$

$[\mathrm{Simp}] \qquad \supset : y \in \beta \,.\supset .\, y \in \text{Ɑ}\text{‘}R \qquad (1)$

$\vdash . (1) . *10{\cdot}11{\cdot}21 \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, \beta \subset \text{Ɑ}\text{‘}R \qquad (2)$

$\vdash . (2) . *10{\cdot}11{\cdot}21 \,.\supset \vdash .\, \mathrm{Prop}$

***33·432.** $\vdash : (y) \,.\, \mathrm{E}\,!\, R\text{‘}y \,.\supset .\, \text{Ɑ}\text{‘}R = \mathrm{V}$

Dem.

$\vdash . *33{\cdot}43 . *10{\cdot}11{\cdot}27 \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, (y) \,.\, y \in \text{Ɑ}\text{‘}R \,.$

$[*24{\cdot}14] \qquad \supset .\, \text{Ɑ}\text{‘}R = \mathrm{V} : \supset \vdash .\, \mathrm{Prop}$

***33·44.** $\vdash : \mathrm{E}\,!\, \breve{R}\text{‘}x \,.\supset .\, x \in \mathrm{D}\text{‘}R \,.\, \breve{R}\text{‘}x \in \text{Ɑ}\text{‘}R$

Dem.

$\vdash . *33{\cdot}43 \dfrac{\breve{R}}{R} \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, x \in \text{Ɑ}\text{‘}\breve{R} \,.\, \breve{R}\text{‘}x \in \mathrm{D}\text{‘}\breve{R} \,.$

$[*33{\cdot}2{\cdot}21] \qquad \supset .\, x \in \mathrm{D}\text{‘}R \,.\, \breve{R}\text{‘}x \in \text{Ɑ}\text{‘}R : \supset \vdash .\, \mathrm{Prop}$

***33·45.** $\vdash :. y \in \text{Ɑ}\text{‘}R \cup \text{Ɑ}\text{‘}S \,.\supset_y .\, R\text{‘}y = S\text{‘}y : \supset .\, R = S$

Note that by our conventions as to denoting expressions, the scope of both $R\text{‘}y$ and $S\text{‘}y$ in the above is "$R\text{‘}y = S\text{‘}y$," and $R\text{‘}y$ is to be first eliminated.

Dem.

$\vdash . *30{\cdot}11 \,.\supset \vdash :: R\text{‘}y = S\text{‘}y \,.\equiv :. (\exists b) : xRy \,.\equiv_x .\, x = b : b = S\text{‘}y :.$

$[*30{\cdot}11] \qquad \equiv :. (\exists b) :. xRy \,.\equiv_x .\, x = b :. (\exists c) : xSy \,.\equiv_x .\, x = c : b = c :.$

$[*13{\cdot}195] \qquad \equiv :. (\exists b) : xRy \,.\equiv_x .\, x = b : xSy \,.\equiv_x .\, x = b :.$

$[*10{\cdot}322] \qquad \supset :. xRy \,.\equiv_x .\, xSy \qquad (1)$

$\vdash . (1) \,.\supset \vdash :: \mathrm{Hp} \,. \qquad \supset :. y \in \text{Ɑ}\text{‘}R \cup \text{Ɑ}\text{‘}S \,.\supset : xRy \,.\equiv .\, xSy :.$

$[*5{\cdot}32] \qquad \supset :. y \in \text{Ɑ}\text{‘}R \cup \text{Ɑ}\text{‘}S \,.\, xRy \,.\equiv .\, y \in \text{Ɑ}\text{‘}R \cup \text{Ɑ}\text{‘}S \,.\, xSy :.$

$[*33{\cdot}14 . *4{\cdot}71] \qquad \supset :. xRy \,.\equiv .\, xSy \qquad (2)$

$\vdash . (2) . *11{\cdot}11{\cdot}3 \,.\supset \vdash :. \mathrm{Hp} \,.\supset : (x, y) : xRy \,.\equiv .\, xSy :$

$[*21{\cdot}43] \qquad \supset : R = S :. \supset \vdash .\, \mathrm{Prop}$

***33·46.** $\vdash :. x \in \mathrm{D}\text{‘}R \cup \mathrm{D}\text{‘}S \,.\supset_x .\, \breve{R}\text{‘}x = \breve{S}\text{‘}x : \supset .\, R = S$ [Proof as in *33·45]

***33·47.** $\vdash :. y \in \text{Ɑ}\text{‘}R \cup \text{Ɑ}\text{‘}S \,.\supset_y .\, \overrightarrow{R}\text{‘}y = \overrightarrow{S}\text{‘}y : \supset .\, R = S$

Dem.

$\vdash . *33{\cdot}41 \,.\, \mathrm{Transp} \,.\supset \vdash : y \sim\in \text{Ɑ}\text{‘}R \cup \text{Ɑ}\text{‘}S \,.\supset .\, \overrightarrow{R}\text{‘}y = \Lambda \,.\, \overrightarrow{S}\text{‘}y = \Lambda \qquad (1)$

$\vdash . (1) . *13\cdot172 . *4\cdot83 . \supset \vdash : \text{Hp} . \supset . (y) . \overrightarrow{R}\text{‘}y = \overrightarrow{S}\text{‘}y .$

$[*30\cdot41] \qquad \supset . \overrightarrow{R} = \overrightarrow{S} .$

$[*32\cdot14] \qquad \supset . R = S : \supset \vdash . \text{Prop}$

***33·48.** $\vdash :. x \in \text{D‘}R \cup \text{D‘}S . \supset_x . \overleftarrow{R}\text{‘}x = \overleftarrow{S}\text{‘}x : \supset . R = S$ [Proof as in *33·47]

***33·5.** $\vdash . C = \overrightarrow{F}$

Dem.

$\vdash . *32\cdot1 . \supset \vdash :. \alpha \overrightarrow{F} R . \equiv . \alpha = \hat{x}(xFR)$

$[*33\cdot103] \qquad = \hat{x}\{(\exists y) : xRy . \vee . yRx\} .$

$[*33\cdot102] \qquad \equiv . \alpha CR \qquad (1)$

$\vdash . (1) . *11\cdot11 . *21\cdot43 . \supset \vdash . \text{Prop}$

***33·51.** $\vdash : x \in C\text{‘}R . \equiv . xFR$ [*33·132·103]

F is useful in ordinal arithmetic, where we are concerned with a series generated by a relation P, and "xFP" expresses the fact that x is a member of this series. The above two propositions (*33·5·51) will be much used in Part IV, where we deal with the foundations of ordinal arithmetic, but will not often be referred to elsewhere.

***33·6.** $\vdash : R \in \overleftarrow{\text{D}}\text{‘}\alpha . \equiv . \alpha = \text{D‘}R$

Dem.

$\vdash . *32\cdot181 . \supset \vdash : R \in \overleftarrow{\text{D}}\text{‘}\alpha . \equiv . \alpha \text{D} R .$

$[*33\cdot123] \qquad \equiv . \alpha = \text{D‘}R : \supset \vdash . \text{Prop}$

***33·61.** $\vdash : R \in \overleftarrow{\text{Ɑ}}\text{‘}\alpha . \equiv . \alpha = \text{Ɑ‘}R$

***33·62.** $\vdash : R \in \overleftarrow{C}\text{‘}\alpha . \equiv . \alpha = C\text{‘}R$

$*$34. THE RELATIVE PRODUCT OF TWO RELATIONS

Summary of $*$34.

The relative product of two relations R and S is the relation which holds between x and z when there is an intermediate term y such that x has the relation R to y and y has the relation S to z. Thus *e.g.* the relative product of *brother* and *father* is *paternal uncle*; the relative product of *father* and *father* is *paternal grandfather*; and so on. The relative product of R and S is denoted by "$R \mid S$"; the definition is:

$*$34·01. $R \mid S = \hat{x}\hat{z}\{(\exists y) \,.\, xRy \,.\, ySz\}$ Df

This definition is only significant when $\text{ᗡ}`R$ and $\mathrm{D}`S$ belong to the same type.

The relative product of R and R is called the square of R; we put

$*$34·02. $R^2 = R \mid R$ Df

$*$34·03. $R^3 = R^2 \mid R$ Df

The most useful propositions in the present number are the following:

$*$34·2. $\vdash \,.\, \mathrm{Cnv}`(R \mid S) = \breve{S} \mid \breve{R}$

I.e. the converse of a relative product is obtained by turning each factor into its converse and reversing the order of the factors.

$*$34·21. $\vdash \,.\, (P \mid Q) \mid R = P \mid (Q \mid R)$

I.e. the relative product obeys the associative law.

$*$34·25. $\vdash \,.\, P \mid (Q \mathbin{\dot\cup} R) = (P \mid Q) \mathbin{\dot\cup} (P \mid R)$

$*$34·26. $\vdash \,.\, (P \mathbin{\dot\cup} Q) \mid R = (P \mid R) \mathbin{\dot\cup} (Q \mid R)$

I.e. the relative product obeys the distributive law with respect to the logical addition of relations. (For logical multiplication instead of logical addition, we only get inclusion instead of identity; cf. $*$34·23·24.)

$*$34·34. $\vdash : R \mathrel{\dot\subset} P \,.\, S \mathrel{\dot\subset} Q \,.\, \supset \,.\, R \mid S \mathrel{\dot\subset} P \mid Q$

$*$34·36. $\vdash \,.\, \mathrm{D}`(P \mid Q) \subset \mathrm{D}`P \,.\, \text{ᗡ}`(P \mid Q) \subset \text{ᗡ}`Q$

$*$34·41. $\vdash : \mathrm{E}\,!\, P`Q`z \,.\, \supset \,.\, P`Q`z = (P \mid Q)`z$

$*$34·01. $R \mid S = \hat{x}\hat{z}\{(\exists y) \,.\, xRy \,.\, ySz\}$ Df

$*$34·02. $R^2 = R \mid R$ Df

$*$34·03. $R^3 = R^2 \mid R$ Df

$*$34·1. $\vdash : x(R \mid S)z \,.\, \equiv \,.\, (\exists y) \,.\, xRy \,.\, ySz$ [$*$21·3 . ($*$34·01)]

***34·11.** $\vdash : x(R|S)z . \equiv . \exists ! (\overleftarrow{R}`x \cap \overrightarrow{S}`z)$

Dem.

$\vdash . *34·1 . *32·18·181 . \supset$

$\vdash : x(R|S)z . \equiv . (\exists y) . y \in \overleftarrow{R}`x . y \in \overrightarrow{S}`z .$

$[*22·33] \quad \equiv . (\exists y) . y \in \overleftarrow{R}`x \cap \overrightarrow{S}`z .$

$[*24·5] \quad \equiv . \exists ! (\overleftarrow{R}`x \cap \overrightarrow{S}`z) : \supset \vdash . \text{Prop}$

***34·12.** $\vdash . R|S = \hat{x}\hat{z} \{\exists ! (\overleftarrow{R}`x \cap \overrightarrow{S}`x)\}$ [*21·33 . *34·11]

***34·2.** $\vdash . \text{Cnv}`(R|S) = \breve{S} | \breve{R}$

Dem.

$\vdash . *31·131 . \supset \vdash : x \{\text{Cnv}`(R|S)\} z . \equiv . z(R|S)x .$

$[*34·1] \quad \equiv . (\exists y) . zRy . ySx .$

$[*31·11] \quad \equiv . (\exists y) . y\breve{R}z . x\breve{S}y .$

$[*34·1] \quad \equiv . x(\breve{S}|\breve{R})z \quad (1)$

$\vdash . (1) . *11·11 . *21·43 . \supset \vdash . \text{Prop}$

***34·202.** $\vdash . R|S = (\text{Cnv}`\breve{R})|S$

Dem.

$\vdash . *31·131 . \supset \vdash : x(\text{Cnv}`\breve{R})y . ySz . \equiv . y\breve{R}x . ySz .$

$[*31·11] \quad \equiv . xRy . ySz \quad (1)$

$\vdash . (1) . *10·11·281 . *34·1 . \supset \vdash : x \{(\text{Cnv}`\breve{R})|S\} z . \equiv . x(R|S)z \quad (2)$

$\vdash . (2) . *11·11 . *21·43 . \supset \vdash . \text{Prop}$

***34·203.** $\vdash . R|S = R|(\text{Cnv}`\breve{S})$ [Similar proof]

***34·21.** $\vdash . (P|Q)|R = P|(Q|R)$

Dem.

$\vdash . *34·1 . *10·281 . \supset \vdash :: (\exists z) . x(P|Q)z . zRw . \equiv :. (\exists z) : (\exists y) . xPy . yQz : zRw :.$

$[*11·6] \quad \equiv :. (\exists y) :. xPy : (\exists z) . yQz . zRw :.$

$[*34·1 . *10·281] \quad \equiv :. (\exists y) . xPy . y(Q|R)w \quad (1)$

$\vdash . (1) . *11·11 . *34·1 . *21·43 . \supset \vdash . \text{Prop}$

***34·22.** $P|Q|R = (P|Q)|R$ Df

This definition serves merely for the avoidance of brackets.

***34·23.** $\vdash . P|(Q \dot{\cap} R) \subseteq (P|Q) \dot{\cap} (P|R)$

Dem.

$\vdash . *34·1 . \supset$

$\vdash :. x\{P|(Q \dot{\cap} R)\}y . \equiv : (\exists z) . xPz . z(Q \dot{\cap} R)y :$

$[*23·33] \quad \equiv : (\exists z) . xPz . zQy . zRy :$

$[*10·5] \quad \supset : (\exists z) . xPz . zQy : (\exists z) . xPz . zRy :$

$[*34\cdot1] \qquad \supset : x(P \mid Q)y \,.\, x(P \mid R)y :$

$[*23\cdot33] \qquad \supset : x\{(P \mid Q) \dot\cap (P \mid R)\}y \qquad (1)$

$\vdash . (1) . *11\cdot11 . \supset \vdash . \text{Prop}$

The converse of the above is not true.

***34·24.** $\vdash . (P \dot\cap Q) \mid R \mathrel{\dot\subset} (P \mid R) \dot\cap (Q \mid R)$ [Similar proof]

***34·25.** $\vdash . P \mid (Q \dot\cup R) = (P \mid Q) \dot\cup (P \mid R)$

Dem.

$\vdash . *23\cdot34 . *10\cdot281 . \supset$

$\vdash :. (\exists z) . xPz . z(Q \dot\cup R)y . \equiv : (\exists z) : xPz : zQy . \vee . zRy :$

$[*4\cdot4 . *10\cdot281] \qquad \equiv : (\exists z) : xPz . zQy . \vee . xPz . zRy :$

$[*10\cdot42] \qquad \equiv : (\exists z) . xPz . zQy : \vee : (\exists z) . xPz . zRy :$

$[*34\cdot1] \qquad \equiv : x(P \mid Q)y . \vee . x(P \mid R)y :$

$[*23\cdot34] \qquad \equiv : x(P \mid Q \dot\cup P \mid R)y \qquad (1)$

$\vdash . (1) . *11\cdot11 . *34\cdot1 . \supset \vdash . \text{Prop}$

***34·26.** $\vdash . (P \dot\cup Q) \mid R = (P \mid R) \dot\cup (Q \mid R)$ [Similar proof]

The above two forms of the distributive law, and the associative law (*34·21), are the only ones of the usual formal laws that hold for the relative product. The commutative law, in particular, does not hold in general.

***34·27.** $\vdash : R = R' . \supset . R \mid P = R' \mid P$

Dem.

$\vdash . *21\cdot43 . \supset \vdash :. \text{Hp} . \supset : (x, y) : xRy . \equiv . xR'y :$

$[*11\cdot401] \qquad \supset : (x, y) : xRy . yPz . \equiv_z . xR'y . yPz :$

$[*10\cdot281] \qquad \supset : (x) : (\exists y) . xRy . yPz . \equiv_z . (\exists y) . xR'y . yPz :$

$[*21\cdot15] \qquad \supset : R \mid P = R' \mid P :. \supset \vdash . \text{Prop}$

***34·28.** $\vdash : R = R' . \supset . P \mid R = P \mid R'$ [Similar proof]

***34·29.** $\vdash : R = R' . \supset . P \mid R \mid Q = P \mid R' \mid Q$

Dem.

$\vdash . *34\cdot27 . \supset \vdash : \text{Hp} . \supset . R \mid Q = R' \mid Q .$

$[*34\cdot28] \qquad \supset . P \mid R \mid Q = P \mid R' \mid Q : \supset \vdash . \text{Prop}$

In proving the equality of two relations, say R and S, we usually establish first an asserted proposition of the form

$$xRy . \equiv . xSy$$

or
$$\text{Hp} . \supset : xRy . \equiv . xSy.$$

We then proceed by *11·11 (together with *11·3 in the second case) to

$$(x, y) : xRy . \equiv . xSy \text{ or } \text{Hp} . \supset : (x, y) : xRy . \equiv . xSy,$$

whence the result follows by *21·43. We shall in future omit these steps, and write "$\supset \vdash . \text{Prop}$" after we have established

$$xRy . \equiv . xSy \text{ or } \text{Hp} . \supset : xRy . \equiv . xSy.$$

A similar ellipsis will be made in proving the equality of classes.

***34·3.** $\vdash : \dot{\exists} ! (P \mid Q) . \equiv . \exists ! (\text{Д}`P \cap \text{D}`Q)$

Dem.

$\vdash . *25\cdot5 . \supset$

$\vdash :: \dot{\exists} ! (P \mid Q) . \equiv :. (\exists x, y) . x (P \mid Q) y :.$

[*34·1] $\equiv :. (\exists x, y) : (\exists z) . xPz . zQy :.$

[*11·27] $\equiv :. (\exists x, y, z) . xPz . zQy :.$

[*11·24] $\equiv :. (\exists z, x, y) . xPz . zQy :.$

[*11·27] $\equiv :. (\exists z) :. (\exists x, y) . xPz . zQy :.$

[*11·54] $\equiv :. (\exists z) :. (\exists x) . xPz : (\exists y) . zQy :.$

[*33·13·131] $\equiv :. (\exists z) :. z \in \text{Д}`P . z \in \text{D}`Q :.$

[*22·33] $\equiv :. (\exists z) :. z \in \text{Д}`P \cap \text{D}`Q :.$

[*24·5] $\equiv :. \exists ! (\text{Д}`P \cap \text{D}`Q) :: \supset \vdash . \text{Prop}$

***34·301.** $\vdash : \text{Д}`P \cap \text{D}`Q = \Lambda . \equiv . P \mid Q = \dot{\Lambda}$ [*34·3 . Transp]

***34·302.** $\vdash : C`P \cap C`Q = \Lambda . \supset . P \mid Q = \dot{\Lambda} . Q \mid P = \dot{\Lambda}$

Dem.

$\vdash . *33\cdot16 . \supset \vdash : \text{Hp} . \supset . \text{Д}`P \cap \text{D}`Q = \Lambda . \text{Д}`Q \cap \text{D}`P = \Lambda .$

[*34·301] $\supset . P \mid Q = \dot{\Lambda} . Q \mid P = \dot{\Lambda} : \supset \vdash . \text{Prop}$

***34·31.** $\vdash : \dot{\exists} ! (P \mid Q) . \supset . \dot{\exists} ! P . \dot{\exists} ! Q$

Dem.

$\vdash . *34\cdot3 . \supset \vdash : \text{Hp} . \supset . \exists ! (\text{Д}`P \cap \text{D}`Q) .$

[*24·561] $\supset . \exists ! \text{Д}`P . \exists ! \text{D}`Q .$

[*33·24] $\supset . \dot{\exists} ! P . \dot{\exists} ! Q : \supset \vdash . \text{Prop}$

***34·32.** $\vdash :. P = \dot{\Lambda} . \vee . Q = \dot{\Lambda} : \supset . P \mid Q = \dot{\Lambda}$ [*34·31 . Transp . *25·51]

***34·33.** $\vdash : x \in \text{D}`R . \equiv . x (R \mid \breve{R}) x$

Dem.

$\vdash . *33\cdot13 . \supset \vdash : x \in \text{D}`R . \equiv . (\exists y) . xRy .$

[*4·24] $\equiv . (\exists y) . xRy . xRy .$

[*31·11] $\equiv . (\exists y) . xRy . y\breve{R}x .$

[*34·1] $\equiv . x (R \mid \breve{R}) x : \supset \vdash . \text{Prop}$

***34·34.** $\vdash : R \subset\!\!\!\cdot\; P . S \subset\!\!\!\cdot\; Q . \supset . R \mid S \subset\!\!\!\cdot\; P \mid Q$

Dem.

$\vdash . *23\cdot1 . \supset \vdash :. \text{Hp} . \supset : xRy . \supset_{x,y} . xPy : ySz . \supset_{y,z} . yQz :$

[*11·2.*10·1·41] $\supset : xRy . \supset . xPy : ySz . \supset . yQz :$

[*3·47] $\supset : xRy . ySz . \supset . xPy . yQz$ (1)

$\vdash . (1) . *10\cdot11\cdot21\cdot28 . \supset$

$\vdash :. \text{Hp} . \supset : (\exists y) . xRy . ySz . \supset . (\exists y) . xPy . yQz :$

[*34·1] $\supset : x (R \mid S) z . \supset . x (P \mid Q) z$ (2)

$\vdash . (2) . *11\cdot11\cdot3 . \supset \vdash . \text{Prop}$

***34·35.** $\vdash : \dot{\exists} ! R \,.\, \text{Ɑ}`R \subset \mathrm{D}`P \,.\supset .\, \dot{\exists} ! R \mid P$

Dem.

$\vdash . *33·24 . \supset \vdash : \mathrm{Hp} . \supset . \exists ! \text{Ɑ}`R$ (1)

$\vdash . *22·621 . \supset \vdash : \mathrm{Hp} . \supset . \text{Ɑ}`R = \text{Ɑ}`R \cap \mathrm{D}`P$ (2)

$\vdash . (1) . (2) . \supset \vdash : \mathrm{Hp} . \supset . \exists ! \text{Ɑ}`R \cap \mathrm{D}`P .$

[*34·3] $\supset . \dot{\exists} ! R \mid P : \supset \vdash . \mathrm{Prop}$

***34·351.** $\vdash : \dot{\exists} ! R \,.\, \mathrm{D}`R \subset \text{Ɑ}`P \,.\supset .\, \dot{\exists} ! P \mid R$ [Proof as in *34·35]

***34·36.** $\vdash . \mathrm{D}`(P \mid Q) \subset \mathrm{D}`P \,.\, \text{Ɑ}`(P \mid Q) \subset \text{Ɑ}`Q$

Dem.

$\vdash . *33·13 . \supset \vdash :. x \in \mathrm{D}`(P \mid Q) . \supset : (\exists z) . x (P \mid Q) z :$

[*34·1] $\supset : (\exists z, y) . xPy . yQz :$

[*11·23] $\supset : (\exists y, z) . xPy . yQz :$

[*11·55.*10·5] $\supset : (\exists y) . xPy :$

[*33·13] $\supset : x \in \mathrm{D}`P$ (1)

Similarly $\vdash :. z \in \text{Ɑ}`(P \mid Q) . \supset : z \in \text{Ɑ}`P$ (2)

$\vdash . (1) . (2) . *10·11 . \supset \vdash . \mathrm{Prop}$

The following proposition is a lemma for *95·31.

***34·361.** $\vdash : \dot{\exists} ! R \,.\, \mathrm{D}`R \subset \text{Ɑ}`P \,.\, \text{Ɑ}`R \subset \mathrm{D}`Q \,.\supset .\, \dot{\exists} ! P \mid R \mid Q$

Dem.

$\vdash . *34·35 . \supset \vdash : \mathrm{Hp} . \supset . \dot{\exists} ! R \mid Q$ (1)

$\vdash . *34·36 . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{D}`(R \mid Q) \subset \text{Ɑ}`P$ (2)

$\vdash . (1) . (2) . *34·351 . \supset \vdash . \mathrm{Prop}$

***34·37.** $\vdash . C`(P \mid Q) \subset \mathrm{D}`P \cup \text{Ɑ}`Q$ [*34·36 . *33·161 . *22·72]

***34·38.** $\vdash . C`(P \mid Q) \subset C`P \cup C`Q$ [*34·37 . *33·161 . *22·72]

***34·4.** $\vdash : b = P`c \,.\, c = Q`z \,.\supset .\, b = (P \mid Q)`z$

Dem.

$\vdash . *30·31 . \supset \vdash : \mathrm{Hp} . \supset . bPc . cQz .$

[*34·1] $\supset . b (P \mid Q) z$ (1)

$\vdash . *30·31 . \supset \vdash :. \mathrm{Hp} . \supset : yQz . \supset_y . y = c :$

[Fact] $\supset : xPy . yQz . \supset_{x,y} . xPy . y = c .$

[*13·13] $\supset_{x,y} . xPc$ (2)

$\vdash . *30·31 . \supset \vdash :. \mathrm{Hp} . \supset : xPc . \supset_x . x = b$ (3)

$\vdash . (2) . (3) . \supset \vdash :. \mathrm{Hp} . \supset : xPy . yQz . \supset_{x,y} . x = b :$

[*10·23] $\supset : (\exists y) . xPy . yQz . \supset_x . x = b :$

[*34·1] $\supset : x (P \mid Q) z . \supset_x . x = b$ (4)

$\vdash . (1) . (4) . *30·31 . \supset \vdash . \mathrm{Prop}$

***34·41.** $\vdash : \mathrm{E} ! P`Q`z \,.\supset .\, P`Q`z = (P \mid Q)`z$

Dem.

$\vdash . *30·52 . \supset \vdash : \mathrm{Hp} . \supset . (\exists b, c) . b = P`c . c = Q`z .$

[*30·51.*34·4] $\supset . (\exists b) . b = P`Q`z . b = (P \mid Q)`z .$

[*14·145] $\supset . P`Q`z = (P \mid Q)`z : \supset \vdash . \mathrm{Prop}$

The above proposition is no longer true if we change the hypothesis into $E\,!\,(P \mid Q)\text{‘}z$, since $(P \mid Q)\text{‘}z$ may exist when $P\text{‘}Q\text{‘}z$ does not. Suppose, *e.g.*, that Q is the relation of child to father, and P the relation of daughter to father. Then $(P \mid Q)\text{‘}z =$ the granddaughter of z, but $P\text{‘}Q\text{‘}z =$ the daughter of the child of z. The first exists whenever z has only one granddaughter, while the second requires further that z should have only one child.

For the same reason we do not have

$$b = (P \mid Q)\text{‘}z \,.\supset.\, (\exists c) \,.\, b = P\text{‘}c \,.\, c = Q\text{‘}z.$$

This will hold if P, Q are one-many relations (cf. *71), but not in general otherwise.

***34·42.** $\vdash : (z) \,.\, R\text{‘}z = P\text{‘}Q\text{‘}z \,.\supset.\, R = P \mid Q$

Dem.

$\vdash .\, *14{\cdot}21 \,.\quad \supset \vdash :.\, \text{Hp} \,.\supset: (z) \,.\, E\,!\,R\text{‘}z : (z) \,.\, E\,!\,P\text{‘}Q\text{‘}z \qquad (1)$

$\vdash .\, (1) \,.\, *34{\cdot}41 \,.\supset \vdash :.\, \text{Hp} \,.\supset: (z) \,.\, R\text{‘}z = (P \mid Q)\text{‘}z :$

$[*30{\cdot}42.(1)] \qquad \supset : R = P \mid Q :.\supset \vdash .\, \text{Prop}$

***34·5.** $\vdash : xR^2y \,.\equiv.\, (\exists z) \,.\, xRz \,.\, zRy \quad [*34{\cdot}1 \,.\, (*34{\cdot}02)]$

***34·51.** $\vdash : xR^3y \,.\equiv.\, (\exists z, w) \,.\, xRz \,.\, zRw \,.\, wRy$

Dem.

$\vdash .\, *34{\cdot}1 \,.\, (*34{\cdot}03) \,.\supset$

$\vdash :.\, xR^3y \,.\equiv: (\exists w) \,.\, xR^2w \,.\, wRy :$

$[*34{\cdot}5] \qquad \equiv : (\exists w) : (\exists z) \,.\, xRz \,.\, zRw : wRy :$

$[*11{\cdot}55] \qquad \equiv : (\exists w, z) \,.\, xRz \,.\, zRw \,.\, wRy :$

$[*11{\cdot}2] \qquad \equiv : (\exists z, w) \,.\, xRz \,.\, zRw \,.\, wRy :.\supset \vdash .\, \text{Prop}$

***34·52.** $\vdash .\, R^3 = R \mid R^2 \qquad [*34{\cdot}21]$

***34·53.** $\vdash : \dot{\exists}\,!\,R^2 \,.\equiv.\, \exists\,!\,D\text{‘}R \cap \text{Ɑ}\text{‘}R \qquad [*34{\cdot}3]$

***34·531.** $\vdash : D\text{‘}R \cap \text{Ɑ}\text{‘}R = \Lambda \,.\equiv.\, R^2 = \dot{\Lambda} \qquad [*34{\cdot}53 \,.\, \text{Transp}]$

***34·54.** $\vdash : xRx \,.\supset.\, xR^2x$

Dem.

$\vdash .\, *4{\cdot}24 \,.\supset \vdash : xRx \,.\supset.\, xRx \,.\, xRx \,.$

$[*10{\cdot}24] \qquad \supset .\, (\exists y) \,.\, xRy \,.\, yRx \,.$

$[*34{\cdot}5] \qquad \supset .\, xR^2x : \supset \vdash .\, \text{Prop}$

***34·55.** $\vdash :.\, R^2 \subseteq S \,.\equiv: xRy \,.\, yRz \,.\supset_{x,y,z} .\, xSz \qquad [*34{\cdot}5 \,.\, *10{\cdot}23]$

***34·56.** $\vdash .\, D\text{‘}R^2 \subset D\text{‘}R \,.\, \text{Ɑ}\text{‘}R^2 \subset \text{Ɑ}\text{‘}R \,.\, C\text{‘}R^2 \subset C\text{‘}R \qquad [*34{\cdot}36{\cdot}38]$

***34·6.** $\vdash .\, (R \dot{\cap} S)^2 \subseteq R^2 \dot{\cap} S^2$

Dem.

$\vdash .\, *34{\cdot}5 \,.\supset \vdash :.\, x\,(R \dot{\cap} S)^2\, y \,.\equiv: (\exists z) \,.\, x\,(R \dot{\cap} S)\, z \,.\, z\,(R \dot{\cap} S)\, y :$

$[*23{\cdot}33 . *10{\cdot}281] \qquad \equiv : (\exists z) \,.\, xRz \,.\, xSz \,.\, zRy \,.\, zSy :$

$[*4{\cdot}3 . *10{\cdot}281] \qquad \equiv : (\exists z) \,.\, xRz \,.\, zRy \,.\, xSz \,.\, zSy :$

[$*10 \cdot 5$] $\supset : (\exists z) . xRz . zRy : (\exists z) . xSz . zSy :$

[$*34 \cdot 5$] $\supset : xR^2y . xS^2y :$

[$*23 \cdot 33$] $\supset : x(R^2 \dot{\cap} S^2) y$ (1)

$\vdash . (1) . *11 \cdot 11 . \supset \vdash .$ Prop

$*34 \cdot 62$. $\vdash . (R \cup S)^2 = R^2 \cup R \mid S \cup S \mid R \cup S^2$

Dem.

$\vdash . *34 \cdot 26 . \supset \vdash . (R \cup S)^2 = R \mid (R \cup S) \cup S \mid (R \cup S)$

[$*34 \cdot 25$] $= R^2 \cup R \mid S \cup S \mid R \cup S^2 . \supset \vdash .$ Prop

The above proposition is a lemma for $*160 \cdot 51$, as is also $*34 \cdot 73$, which employs the above proposition.

$*34 \cdot 63$. $\vdash . \text{Cnv}\text{‘}(R^2) = (\text{Cnv}\text{‘}R)^2$

Dem.

$\vdash . *31 \cdot 131 . \supset$

$\vdash :. x \{\text{Cnv}\text{‘}(R^2)\} y . \equiv : yR^2x :$

[$*34 \cdot 5$] $\equiv : (\exists z) . yRz . zRx :$

[$*31 \cdot 131 . *10 \cdot 281$] $\equiv : (\exists z) . x\breve{R}z . z\breve{R}y :$

[$*31 \cdot 131 . *34 \cdot 5$] $\equiv : x (\text{Cnv}\text{‘}R)^2 y : \supset \vdash .$ Prop

$*34 \cdot 7$. $\vdash . \text{Cnv}\text{‘}(S \mid \breve{S}) = S \mid \breve{S}$

Dem.

$\vdash . *34 \cdot 2 . \supset \vdash . \text{Cnv}\text{‘}(S \mid \breve{S}) = (\text{Cnv}\text{‘}\breve{S}) \mid \breve{S}$

[$*34 \cdot 202$] $= S \mid \breve{S} . \supset \vdash .$ Prop

Thus $S \mid \breve{S}$ is always a symmetrical relation, *i.e.* one which is equal to its converse.

$*34 \cdot 701$. $\vdash . \text{Cnv}\text{‘}(\breve{S} \mid S) = \breve{S} \mid S$ [$*34 \cdot 2 \cdot 203$]

$*34 \cdot 702$. $\vdash . C\text{‘}(S \mid \breve{S}) = \text{D}\text{‘}S$

Dem.

$\vdash . *34 \cdot 37 . \supset \vdash . C\text{‘}(S \mid \breve{S}) \subset \text{D}\text{‘}S \cup \text{Ꮷ}\text{‘}\breve{S}$

[$*33 \cdot 21$] $\subset \text{D}\text{‘}S$ (1)

$\vdash . *33 \cdot 13 . \supset \vdash : x \in \text{D}\text{‘}S . \supset . (\exists y) . xSy .$

[$*31 \cdot 11$] $\supset . (\exists y) . xSy . y\breve{S}x .$

[$*34 \cdot 1$] $\supset . x (S \mid \breve{S}) x .$

[$*33 \cdot 17$] $\supset . x \in C\text{‘}(S \mid \breve{S})$ (2)

$\vdash . (1) . (2) . *10 \cdot 11 . \supset \vdash .$ Prop

$*34 \cdot 703$. $\vdash . C\text{‘}(\breve{S} \mid S) = \text{Ꮷ}\text{‘}S$ [Similar proof]

***34·73.** $\vdash : C'P \cap C'Q = \Lambda . \supset . (P \cup Q)^2 = P^2 \cup Q^2$

Dem.

$\vdash . *34·302 . \supset \vdash : \text{Hp} . \supset . P \mid Q = \dot{\Lambda} . Q \mid P = \dot{\Lambda} .$

[*25·24] $\supset . P^2 \cup Q^2 = P^2 \cup P \mid Q \cup Q \mid P \cup Q^2$

[*34·62] $= (P \cup Q)^2 : \supset \vdash . \text{Prop}$

***34·8.** $\vdash : R = \breve{R} . R^2 \subset R . \supset . R = R^2 = R \mid \breve{R}$

Dem.

$\vdash . *34·28 . \quad \supset \vdash : R = \breve{R} . \supset . R^2 = R \mid \breve{R}$ (1)

$\vdash . *34·33 . *33·14 . \quad \supset \vdash : xRy . \supset . x (R \mid \breve{R}) x$ (2)

$\vdash . (1) . (2) . \quad \supset \vdash :. R = \breve{R} . \supset : xRy . \supset . xR^2x$ (3)

$\vdash . (3) . *23·1 . \quad \supset \vdash :. R = \breve{R} . R^2 \subset R . \supset : xRy . \supset . xRx :$

[*4·7] $\supset : xRy . \supset . xRx . xRy .$

[*10·24.*34·5] $\supset . xR^2y$ (4)

$\vdash . (4) . *11·11·3 . \quad \supset \vdash : \text{Hp} . \supset . R \subset R^2$ (5)

$\vdash . *3·27 . \quad \supset \vdash : \text{Hp} . \supset . R^2 \subset R$ (6)

$\vdash . (5) . (6) . *23·41 . \supset \vdash : \text{Hp} . \supset . R = R^2$ (7)

$\vdash . (1) . (7) . \quad \supset \vdash . \text{Prop}$

The hypothesis of the above proposition is the hypothesis that R is symmetrical ($R = \breve{R}$) and transitive ($R^2 \subset R$). These are the formal properties of those relations which can suitably be regarded as expressing equality in some respect.

***34·81.** $\vdash : R = \breve{R} . R^2 \subset R . \equiv . R = \breve{R} . R^2 = R$ [*34·8 . *4·71]

The following propositions are lemmas for *34·85, which is used in *72·64.

***34·82.** $\vdash :. R = \breve{R} . R^2 \subset R . \supset : x \in D'R . \equiv . xRx$

Dem.

$\vdash . *34·33 . \supset \vdash : x \in D'R . \equiv . x (R \mid \breve{R}) x$ (1)

$\vdash . *34·8 . \quad \supset \vdash :. \text{Hp} . \supset : x (R \mid \breve{R}) x . \equiv . xRx$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***34·83.** $\vdash : R = \breve{R} . R^2 \subset R . xRy . \supset . \overleftarrow{R}'x = \overleftarrow{R}'y$

Dem.

$\vdash . *31·11 . \supset \vdash :. \text{Hp} . \supset : yRx :$

[*3·2] $\supset : xRz . \supset . yRx . xRz .$

[*34·55.Hp] $\supset . yRz$ (1)

$\vdash . *3·2 . \quad \supset \vdash :. \text{Hp} . \supset : yRz . \supset . xRy . yRz .$

[*34·55.Hp] $\supset . xRz$ (2)

$\vdash . (1) . (2) . \supset \vdash :. \text{Hp} . \supset : xRz . \equiv . yRz :$

[*10·11·21.*20·15.*32·111] $\supset : \overleftarrow{R}'x = \overleftarrow{R}'y :. \supset \vdash . \text{Prop}$

***34·84.** $\vdash : R = \breve{R} \,.\, R^2 \subset R \,.\, y \in \mathrm{D}\text{‘}R \,.\, \overleftarrow{R}\text{‘}x = \overleftarrow{R}\text{‘}y \,.\, \supset .\, xRy$

Dem.

$\vdash .\, *34\cdot82 \,. \quad \supset \vdash : \mathrm{Hp} \,.\, \supset .\, yRy$ (1)

$\vdash .\, *32\cdot181 \,.\, *20\cdot31 \,.\, \supset \vdash :. \mathrm{Hp} \,.\, \supset : xRz \,.\, \equiv_z .\, yRz :$

$[*10\cdot1] . \quad \supset : xRy \,.\, \equiv .\, yRy$ (2)

$\vdash .\, (1) \,.\, (2) \,. \quad \supset \vdash .\, \mathrm{Prop}$

***34·841.** $\vdash : R = \breve{R} \,.\, R^2 \subset R \,.\, x \in \mathrm{D}\text{‘}R \,.\, \overleftarrow{R}\text{‘}x = \overleftarrow{R}\text{‘}y \,.\, \supset .\, xRy$

Dem.

$\vdash .\, *34\cdot84 \dfrac{y, x}{x, y} \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset .\, yRx \,.$

$[*31\cdot11.\mathrm{Hp}] \quad \supset .\, xRy : \supset \vdash .\, \mathrm{Prop}$

***34·85.** $\vdash :. R = \breve{R} \,.\, R^2 \subset R \,.\, \supset : xRy \,.\, \equiv .\, x \in \mathrm{D}\text{‘}R \,.\, \overleftarrow{R}\text{‘}x = \overleftarrow{R}\text{‘}y$

$[*34\cdot83\cdot841 \,.\, *33\cdot14]$

$*35$. RELATIONS WITH LIMITED DOMAINS AND CONVERSE DOMAINS

Summary of $*35$.

In this section, we have to consider the relation derived from a given relation R by limiting either its domain or its converse domain to members of some assigned class. A relation R with its domain limited to members of α is written "$\alpha \rceil R$"; with its converse domain limited to members of β, it is written "$R \upharpoonright \beta$"; with both limitations, it is written "$\alpha \rceil R \upharpoonright \beta$." Thus *e.g.* "brother" and "sister" express the same relation (that of a common parentage), with the domain limited in the first case to males, in the second to females. "The relation of white employers to coloured employees" is a relation limited both as to its domain and as to its converse domain. We put

$*35\cdot01$. $\alpha \rceil R = \hat{x}\hat{y}\,(x \in \alpha \,.\, xRy)$ Df

with similar definitions for $R \upharpoonright \alpha$ and $\alpha \rceil R \upharpoonright \beta$.

A particularly important case is the case in which the same limitation is imposed on the domain and on the converse domain, *i.e.* where we have a relation of the form "$\alpha \rceil R \upharpoonright \alpha$." In this case, the limitation to members of α may be more briefly stated as being imposed on the *field*. For this case, it is convenient to adopt "$R \,\underline{\upharpoonright}\, \alpha$" as an alternative notation. This case will be considered in $*36$.

It is convenient to consider in the present connection the relation between x and y which is constituted by x being a member of α and y being a member of β. This relation will be denoted by "$\alpha \uparrow \beta$." Thus we put

$*35\cdot04$. $\alpha \uparrow \beta = \hat{x}\hat{y}\,(x \in \alpha \,.\, y \in \beta)$ Df

The chief importance of relations with limited *fields* arises in the theory of series. Given a series generated by a relation R, let α be a class consisting of part of this series. Then α is the field of the relation $\alpha \rceil R \upharpoonright \alpha$ or $R \,\underline{\upharpoonright}\, \alpha$, and it is this relation which is the generating relation of the series of members of α in the same order which they have as parts of the original series. Thus parts of a series, considered not merely as classes but as series, are dealt with by means of serial relations with limited fields.

Relations with limited *domains* are not nearly so much used as relations with limited *converse domains*. Relations with limited converse domains play a great part in arithmetic, especially in establishing the formal laws. What is wanted in such cases is a one-one relation correlating two classes or two series. That is, we want a relation such that not only does $R \text{‘} y$ exist whenever $y \in \text{Ɑ}\text{‘}R$, but also $\breve{R}\text{‘}x$ exists whenever $x \in \text{D}\text{‘}R$. The kind of relation which is most frequently found to effect such a correlation is some such relation as D

or Ɑ or C, or some other constant relation for which we always have $E!\, R‘y$, with its converse domain so limited that, subject to the limitation, only one value of y gives any given value of $R‘y$. Thus for example let λ be a class of relations no two of which have the same domain; then $D \upharpoonright \lambda$ will give a one-one correlation of these relations with their domains: if $R, S \in \lambda$, we shall have

$$D‘R = D‘S \,.\supset.\, R = S.$$

We shall also have $D‘R = (D \upharpoonright \lambda)‘R$ and $D‘S = (D \upharpoonright \lambda)‘S$. Moreover the converse domain of $D \upharpoonright \lambda$ is λ, and the domain of $D \upharpoonright \lambda$ is the class of domains of members of λ. Thus $D \upharpoonright \lambda$ gives a one-one correlation of λ with the domains of members of λ. It is chiefly in such ways that relations with limited converse domains are useful.

For purposes of reference, a great many propositions are given in the present number, but the propositions that will be used frequently are comparatively few. Among these are the following:

***35·21.** $\vdash . \alpha \upharpoonleft R \upharpoonright \beta = (\alpha \upharpoonleft R) \upharpoonright \beta = \alpha \upharpoonleft (R \upharpoonright \beta)$

***35·31.** $\vdash . (R \upharpoonright \alpha) \upharpoonright \beta = R \upharpoonright (\alpha \cap \beta)$

***35·354.** $\vdash . (R \upharpoonright \alpha) \,|\, S = R \,|\, \alpha \upharpoonleft S$

I.e. in a relative product it makes no difference whether we limit the converse domain of the first factor, or the domain of the second.

***35·412.** $\vdash . R \upharpoonright (\beta \cup \beta') = R \upharpoonright \beta \,\dot{\cup}\, R \upharpoonright \beta'$

***35·452.** $\vdash : \text{Ɑ}‘R \subset \beta \,.\supset.\, R \upharpoonright \beta = R$

***35·48.** $\vdash : \text{Ɑ}‘P \subset \alpha \,.\supset.\, P \,|\, (\alpha \upharpoonleft R) = P \,|\, R$

***35·52.** $\vdash . \mathrm{Cnv}‘(R \upharpoonright \beta) = \beta \upharpoonleft \breve{R}$

***35·61.** $\vdash . D‘(\alpha \upharpoonleft R) = \alpha \cap D‘R$

***35·64.** $\vdash . \text{Ɑ}‘(R \upharpoonright \beta) = \beta \cap \text{Ɑ}‘R$

***35·65.** $\vdash : \beta \subset \text{Ɑ}‘R \,.\supset.\, \text{Ɑ}‘(R \upharpoonright \beta) = \beta$

The hypothesis $\beta \subset \text{Ɑ}‘R$ is fulfilled in the great majority of cases in which we have occasion to use $R \upharpoonright \beta$.

***35·66.** $\vdash : \text{Ɑ}‘R \subset \beta \,.\equiv.\, R \upharpoonright \beta = R$

***35·7.** $\vdash : \phi\{(R \upharpoonright \beta)‘y\} \,.\equiv.\, y \in \beta \,.\, \phi(R‘y)$

This proposition is used very frequently, owing to the fact that limitation of the converse domain is chiefly applied to such relations as give rise to descriptive functions (*e.g.* D, Ɑ, C).

***35·71.** $\vdash :. y \in \beta \,.\supset_y.\, R‘y = S‘y : \supset .\, R \upharpoonright \beta = S \upharpoonright \beta$

This proposition is useful for a reason similar to that which makes *35·7 useful.

***35·82.** $\vdash . \alpha \uparrow \beta = \alpha \upharpoonleft \dot{V} \upharpoonright \beta$

Owing to this proposition, the properties of $\alpha \uparrow \beta$ can be deduced from the already proved properties of $\alpha \upharpoonleft R \upharpoonright \beta$, by putting $R = \dot{V}$.

The relation "$\alpha \uparrow \beta$" is what may be called an "analysable" relation, *i.e.* it holds between x and y when $x \in \alpha$ and $y \in \beta$, *i.e.* when x has a property independent of y, and y has a property independent of x.

***35·85.** $\vdash : \exists ! \beta \,.\, \supset \,.\, \text{D‘}(\alpha \uparrow \beta) = \alpha$

***35·86.** $\vdash : \exists ! \alpha \,.\, \supset \,.\, \text{Ɑ‘}(\alpha \uparrow \beta) = \beta$

If either α or β is null, so is $\alpha \uparrow \beta$ (*35·88).

***35·01.** $\alpha \rceil R = \hat{x}\hat{y}\,(x \in \alpha \,.\, xRy)$ Df

***35·02.** $R \upharpoonright \beta = \hat{x}\hat{y}\,(xRy \,.\, y \in \beta)$ Df

***35·03.** $\alpha \rceil R \upharpoonright \beta = \hat{x}\hat{y}\,(x \in \alpha \,.\, xRy \,.\, y \in \beta)$ Df

***35·04.** $\alpha \uparrow \beta = \hat{x}\hat{y}\,(x \in \alpha \,.\, y \in \beta)$ Df

***35·05.** $R‘x \uparrow \beta = (R‘x) \uparrow \beta$ Df

The last definition serves merely for the avoidance of brackets.

***35·1.** $\vdash : x\,(\alpha \rceil R)\,y \,.\equiv .\, x \in \alpha \,.\, xRy$ [*21·3 . (*35·01)]

***35·101.** $\vdash : x\,(R \upharpoonright \beta)\,y \,.\equiv .\, xRy \,.\, y \in \beta$

***35·102.** $\vdash : x\,(\alpha \rceil R \upharpoonright \beta)\,y \,.\equiv .\, x \in \alpha \,.\, xRy \,.\, y \in \beta$

***35·103.** $\vdash : x\,(\alpha \uparrow \beta)\,y \,.\equiv .\, x \in \alpha \,.\, y \in \beta$

***35·11.** $\vdash \,.\, \alpha \rceil R \upharpoonright \beta = (\alpha \rceil R) \dot{\cap} (R \upharpoonright \beta)$

Dem.

$\vdash \,.\, *35·102 \,.\supset \vdash : x\,(\alpha \rceil R \upharpoonright \beta)\,y \,.\equiv .\, x \in \alpha \,.\, xRy \,.\, y \in \beta \,.$

[*4·24] $\equiv .\, x \in \alpha \,.\, xRy \,.\, xRy \,.\, y \in \beta \,.$

[*35·1·101] $\equiv .\, x\,(\alpha \rceil R)\,y \,.\, x\,(R \upharpoonright \beta)\,y \,.$

[*23·33] $\equiv .\, x\,\{(\alpha \rceil R) \dot{\cap} (R \upharpoonright \beta)\}\,y : \supset \vdash .$ Prop

***35·12.** $\vdash \,.\, (\alpha \rceil R) \dot{\cap} (S \upharpoonright \beta) = \alpha \rceil (R \dot{\cap} S) \upharpoonright \beta$

Dem.

$\vdash \,.\, *23·33 \,.\supset \vdash : x\,\{(\alpha \rceil R) \dot{\cap} (S \upharpoonright \beta)\}\,y \,.\equiv .\, x\,(\alpha \rceil R)\,y \,.\, x\,(S \upharpoonright \beta)\,y \,.$

[*35·1·101] $\equiv .\, x \in \alpha \,.\, xRy \,.\, xSy \,.\, y \in \beta \,.$

[*23·33] $\equiv .\, x \in \alpha \,.\, x\,(R \dot{\cap} S)\,y \,.\, y \in \beta \,.$

[*35·102] $\equiv .\, x\,\{\alpha \rceil (R \dot{\cap} S) \upharpoonright \beta\}\,y : \supset \vdash .$ Prop

***35·13.** $\vdash \,.\, (\alpha \rceil R) \dot{\cap} (\beta \rceil S) = (\alpha \cap \beta) \rceil (R \dot{\cap} S)$

Dem.

$\vdash \,.\, *23·33 \,.\supset \vdash : x\,\{(\alpha \rceil R) \dot{\cap} (\beta \rceil S)\}\,y \,.\equiv .\, x\,(\alpha \rceil R)\,y \,.\, x\,(\beta \rceil S)\,y \,.$

[*35·1] $\equiv .\, x \in \alpha \,.\, xRy \,.\, x \in \beta \,.\, xSy \,.$

[*22·33.*23·33] $\equiv .\, x \in (\alpha \cap \beta) \,.\, x\,(R \dot{\cap} S)\,y \,.$

[*35·1] $\equiv .\, x\,\{(\alpha \cap \beta) \rceil (R \dot{\cap} S)\}\,y : \supset \vdash .$ Prop

***35·14.** $\vdash \,.\, (R \upharpoonright \alpha) \dot{\cap} (S \upharpoonright \beta) = (R \dot{\cap} S) \upharpoonright (\alpha \cap \beta)$ [Similar proof to *35·13]

∗35·15. $\vdash . (\alpha \upharpoonleft R \upharpoonright \beta) \dot{\cap} (\alpha' \upharpoonleft S \upharpoonright \beta') = (\alpha \cap \alpha') \upharpoonleft (R \dot{\cap} S) \upharpoonright (\beta \cap \beta')$

Dem.

$\vdash . *35{\cdot}11 . \supset$

$\vdash . (\alpha \upharpoonleft R \upharpoonright \beta) \dot{\cap} (\alpha' \upharpoonleft S \upharpoonright \beta') = (\alpha \upharpoonleft R) \dot{\cap} (R \upharpoonright \beta) \dot{\cap} (\alpha' \upharpoonleft S) \dot{\cap} (S \upharpoonright \beta')$

[∗35·13·14] $= \{(\alpha \cap \alpha') \upharpoonleft (R \dot{\cap} S)\} \dot{\cap} \{(R \dot{\cap} S) \upharpoonright (\beta \cap \beta')\}$

[∗35·11] $= \{(\alpha \cap \alpha') \upharpoonleft (R \dot{\cap} S) \upharpoonright (\beta \cap \beta')\} . \supset \vdash .$ Prop

∗35·16. $\vdash . (\alpha \upharpoonleft R) \dot{\cap} S = \alpha \upharpoonleft (R \dot{\cap} S) = R \dot{\cap} \alpha \upharpoonleft S$ [Similar proof to ∗35·13]

∗35·17. $\vdash . (R \upharpoonright \beta) \dot{\cap} S = (R \dot{\cap} S) \upharpoonright \beta = R \dot{\cap} S \upharpoonright \beta$ [Similar proof to ∗35·13]

∗35·18. $\vdash . (\alpha \upharpoonleft R \upharpoonright \beta) \dot{\cap} S = \alpha \upharpoonleft (R \dot{\cap} S) \upharpoonright \beta = R \dot{\cap} \alpha \upharpoonleft S \upharpoonright \beta$

[Similar proof to ∗35·15]

∗35·21. $\vdash . \alpha \upharpoonleft R \upharpoonright \beta = (\alpha \upharpoonleft R) \upharpoonright \beta = \alpha \upharpoonleft (R \upharpoonright \beta)$

Dem.

$\vdash . *35{\cdot}102 . \supset \vdash : x (\alpha \upharpoonleft R \upharpoonright \beta) y . \equiv . x \epsilon \alpha . xRy . y \epsilon \beta .$

[∗35·1] $\equiv . x (\alpha \upharpoonleft R) y . y \epsilon \beta .$

[∗35·101] $\equiv . x \{(\alpha \upharpoonleft R) \upharpoonright \beta\} y$ (1)

$\vdash . *35{\cdot}102 . \supset \vdash : x (\alpha \upharpoonleft R \upharpoonright \beta) y . \equiv . x \epsilon \alpha . xRy . y \epsilon \beta .$

[∗35·101] $\equiv . x \epsilon \alpha . x (R \upharpoonright \beta) y .$

[∗35·1] $\equiv . x \{\alpha \upharpoonleft (R \upharpoonright \beta)\} y$ (2)

$\vdash . (1) . (2) . \supset \vdash .$ Prop

∗35·22. $\vdash . (\alpha \upharpoonleft R) | S = \alpha \upharpoonleft (R | S)$

Dem.

$\vdash . *34{\cdot}1 . \supset \vdash :. x \{(\alpha \upharpoonleft R) | S\} y . \equiv : (\exists z) . x (\alpha \upharpoonleft R) z . zSy :$

[∗35·1] $\equiv : (\exists z) . x \epsilon \alpha . xRz . zSy :$

[∗10·35] $\equiv : x \epsilon \alpha : (\exists z) . xRz . zSy .$

[∗34·1] $\equiv : x \epsilon \alpha . x (R | S) y :$

[∗35·1] $\equiv : x \{\alpha \upharpoonleft (R | S)\} y :. \supset \vdash .$ Prop

∗35·23. $\vdash . S | (R \upharpoonright \beta) = (S | R) \upharpoonright \beta$ [Similar proof to ∗35·22]

∗35·24. $\alpha \upharpoonleft R | S = (\alpha \upharpoonleft R) | S$ Df

∗35·25. $S | R \upharpoonright \beta = (S | R) \upharpoonright \beta$ Df

∗35·26. $\vdash . (\alpha \upharpoonleft R) | (S \upharpoonright \beta) = \alpha \upharpoonleft (R | S) \upharpoonright \beta = \{\alpha \upharpoonleft (R | S)\} \upharpoonright \beta = \alpha \upharpoonleft \{(R | S) \upharpoonright \beta\}$

$= \{(\alpha \upharpoonleft R) | S\} \upharpoonright \beta = \alpha \upharpoonleft \{R | (S \upharpoonright \beta)\}$

$= (\alpha \upharpoonleft R | S) \upharpoonright \beta = \alpha \upharpoonleft (R | S \upharpoonright \beta)$

Dem.

$\vdash . *34{\cdot}1 . \supset \vdash :. x \{(\alpha \upharpoonleft R) | (S \upharpoonright \beta)\} y . \equiv : (\exists z) . x (\alpha \upharpoonleft R) z . z (S \upharpoonright \beta) y :$

[∗35·1·101] $\equiv : (\exists z) . x \epsilon \alpha . xRz . zSy . y \epsilon \beta :$

[∗10·35] $\equiv : x \epsilon \alpha . y \epsilon \beta : (\exists z) . xRz . zSy :$

[∗34·1] $\equiv : x \epsilon \alpha . x (R | S) y . y \epsilon \beta :$

[∗35·102] $\equiv : x \{\alpha \upharpoonleft (R | S) \upharpoonright \beta\} y$ (1)

$\vdash . (1) . *35{\cdot}21{\cdot}22{\cdot}23 . (*35{\cdot}24{\cdot}25) . \supset \vdash .$ Prop

✳35·27. $\alpha \upharpoonleft R \mid S \upharpoonright \beta = (\alpha \upharpoonleft R \mid S) \upharpoonright \beta$ Df

✳35·31. $\vdash . (R \upharpoonright \alpha) \upharpoonright \beta = R \upharpoonright (\alpha \cap \beta)$

Dem.

$\vdash . *35\cdot101 . \supset \vdash : x \{(R \upharpoonright \alpha) \upharpoonright \beta\} y . \equiv . x (R \upharpoonright \alpha) y . y \in \beta .$

$[*35\cdot101] \qquad \equiv . xRy . y \in \alpha . y \in \beta .$

$[*22\cdot33] \qquad \equiv . xRy . y \in \alpha \cap \beta .$

$[*35\cdot101] \qquad \equiv . x \{R \upharpoonright (\alpha \cap \beta)\} y : \supset \vdash . \text{Prop}$

✳35·32. $\vdash . \alpha \upharpoonleft (\beta \upharpoonleft R) = (\alpha \cap \beta) \upharpoonleft R$ [Proof similar to that of ✳35·31]

✳35·33. $\vdash . (\alpha \upharpoonleft R \upharpoonright \beta) \upharpoonright \gamma = \{\alpha \upharpoonleft R \upharpoonright (\beta \cap \gamma)\}$ [Proof similar to that of ✳35·31]

✳35·34. $\vdash . \alpha \upharpoonleft (\beta \upharpoonleft R \upharpoonright \gamma) = \{(\alpha \cap \beta) \upharpoonleft R \upharpoonright \gamma\}$ [Proof similar to that of ✳35·31]

✳35·35. $\vdash . \alpha \upharpoonleft R = (\alpha \cap \mathrm{D}'R) \upharpoonleft R$

Dem.

$\vdash . *35\cdot1 . \supset \vdash : x (\alpha \upharpoonleft R) y . \equiv . x \in \alpha . xRy .$

$[*33\cdot14] \qquad \equiv . x \in \alpha . x \in \mathrm{D}'R . xRy .$

$[*22\cdot33 . *35\cdot1] \qquad \equiv . x \{(\alpha \cap \mathrm{D}'R) \upharpoonleft R\} y : \supset \vdash . \text{Prop}$

✳35·351. $\vdash . R \upharpoonright \beta = R \upharpoonright (\beta \cap \text{Ɑ}'R)$ [Proof as in ✳35·35]

✳35·352. $\vdash . \alpha \upharpoonleft R \upharpoonright \beta = (\alpha \cap \mathrm{D}'R) \upharpoonleft R \upharpoonright (\beta \cap \text{Ɑ}'R)$ [Proof as in ✳35·35]

✳35·354. $\vdash . (R \upharpoonright \alpha) \mid S = R \mid \alpha \upharpoonleft S$

Dem.

$\vdash . *34\cdot1 . *35\cdot101 . \supset$

$\vdash : x \{(R \upharpoonright \alpha) \mid S\} z . \equiv . (\exists y) . xRy . y \in \alpha . ySz .$

$[*35\cdot1] \qquad \equiv . (\exists y) . xRy . y (\alpha \upharpoonleft S) z .$

$[*34\cdot1] \qquad \equiv . x \{R \mid (\alpha \upharpoonleft S)\} z : \supset \vdash . \text{Prop}$

✳35·41. $\vdash . (\alpha \cup \alpha') \upharpoonleft R = \alpha \upharpoonleft R \mathbin{\dot\cup} \alpha' \upharpoonleft R$ [✳35·1 . ✳22·34]

✳35·412. $\vdash . R \upharpoonright (\beta \cup \beta') = R \upharpoonright \beta \mathbin{\dot\cup} R \upharpoonright \beta'$ [✳35·101 . ✳22·34]

✳35·413. $\vdash . (\alpha \cup \alpha') \upharpoonleft R \upharpoonright (\beta \cup \beta') = (\alpha \upharpoonleft R \upharpoonright \beta) \mathbin{\dot\cup} (\alpha \upharpoonleft R \upharpoonright \beta')$

$\mathbin{\dot\cup} (\alpha' \upharpoonleft R \upharpoonright \beta) \mathbin{\dot\cup} (\alpha' \upharpoonleft R \upharpoonright \beta')$ [✳35·102 . ✳22·34]

✳35·42. $\vdash . \alpha \upharpoonleft (R \mathbin{\dot\cup} S) = (\alpha \upharpoonleft R) \mathbin{\dot\cup} (\alpha \upharpoonleft S)$ [✳35·1 . ✳23·34]

✳35·421. $\vdash . (R \mathbin{\dot\cup} S) \upharpoonright \beta = (R \upharpoonright \beta) \mathbin{\dot\cup} (S \upharpoonright \beta)$ [✳35·101 . ✳23·34]

✳35·422. $\vdash . \alpha \upharpoonleft (R \mathbin{\dot\cup} S) \upharpoonright \beta = (\alpha \upharpoonleft R \upharpoonright \beta) \mathbin{\dot\cup} (\alpha \upharpoonleft S \upharpoonright \beta)$ [✳35·102 . ✳23·34]

✳35·43. $\vdash : \alpha \subset \beta . \supset . \alpha \upharpoonleft R \Subset \beta \upharpoonleft R$

Dem.

$\vdash . *35\cdot1 . \supset \vdash :. \alpha \subset \beta . \supset : x (\alpha \upharpoonleft R) y . \equiv . x \in \alpha . xRy .$

$[*22\cdot1] \qquad \supset . x \in \beta . xRy .$

$[*35\cdot1] \qquad \supset . x (\beta \upharpoonleft R) y :. \supset \vdash . \text{Prop}$

✳35·431. $\vdash : \beta \subset \gamma . \supset . R \upharpoonright \beta \Subset R \upharpoonright \gamma$ [Proof similar to that of ✳35·43]

✳35·432. $\vdash : \alpha \subset \gamma . \beta \subset \delta . \supset . \alpha \upharpoonleft R \upharpoonright \beta \Subset \gamma \upharpoonleft R \upharpoonright \delta$

[Proof similar to that of ✳35·43]

$*35{\cdot}44$. $\vdash . \alpha \upharpoonleft R \subseteq R$

Dem.

$\vdash . *35{\cdot}1 . \supset \vdash : x(\alpha \upharpoonleft R) y . \supset . x \in \alpha . xRy .$

$[*3{\cdot}27] \quad \supset . xRy : \supset \vdash . \text{Prop}$

$*35{\cdot}441$. $\vdash . R \upharpoonright \beta \subseteq R$ [Proof similar to that of $*35{\cdot}44$]

$*35{\cdot}442$. $\vdash . \alpha \upharpoonleft R \upharpoonright \beta \subseteq R$ [Proof similar to that of $*35{\cdot}44$]

$*35{\cdot}451$. $\vdash : \text{D}`R \subset \alpha . \supset . \alpha \upharpoonleft R = R$

Dem.

$\vdash . *4{\cdot}71 . \supset \vdash :. \text{Hp} . \supset : x \in \text{D}`R . \equiv . x \in \text{D}`R . x \in \alpha :$

$[*4{\cdot}36] \quad \supset : x \in \text{D}`R . xRy . \equiv . x \in \text{D}`R . xRy . x \in \alpha \quad (1)$

$\vdash . *33{\cdot}14 . *4{\cdot}71 . \quad \supset \vdash : xRy . \equiv . x \in \text{D}`R . xRy \quad (2)$

$\vdash . (1) . (2) . \supset \vdash :. \text{Hp} . \supset : xRy . \equiv . xRy . x \in \alpha .$

$[*35{\cdot}1] \quad \equiv . x(\alpha \upharpoonleft R) y :. \supset \vdash . \text{Prop}$

$*35{\cdot}452$. $\vdash : \text{Q}`R \subset \beta . \supset . R \upharpoonright \beta = R$ [Similar proof]

$*35{\cdot}453$. $\vdash : \text{D}`R \subset \alpha . \supset . \alpha \upharpoonleft R \upharpoonright \beta = R \upharpoonright \beta$ [Similar proof]

$*35{\cdot}454$. $\vdash : \text{Q}`R \subset \beta . \supset . \alpha \upharpoonleft R \upharpoonright \beta = \alpha \upharpoonleft R$ [Similar proof]

$*35{\cdot}46$. $\vdash : R \subseteq S . \supset . \alpha \upharpoonleft R \subseteq \alpha \upharpoonleft S$

Dem.

$\vdash . *23{\cdot}1 . \supset \vdash :. \text{Hp} . \supset : xRy . \supset . xSy :$

$[\text{Fact}] \quad \supset : x \in \alpha . xRy . \supset . x \in \alpha . xSy :$

$[*35{\cdot}1] \quad \supset : x(\alpha \upharpoonleft R) y . \supset . x(\alpha \upharpoonleft S) y :. \supset \vdash . \text{Prop}$

$*35{\cdot}461$. $\vdash : R \subseteq S . \supset . R \upharpoonright \beta \subseteq S \upharpoonright \beta$ [Similar proof]

$*35{\cdot}462$. $\vdash : R \subseteq S . \supset . \alpha \upharpoonleft R \upharpoonright \beta \subseteq \alpha \upharpoonleft S \upharpoonright \beta$ [Similar proof]

$*35{\cdot}471$. $\vdash : \text{Q}`P \cap \alpha = \Lambda . \supset . P \mid (\alpha \upharpoonleft R) = \dot{\Lambda}$

Dem.

$\vdash . *34{\cdot}1 . \supset \vdash : x \{P \mid (\alpha \upharpoonleft R)\} z . \supset . (\exists y) . xPy . y(\alpha \upharpoonleft R) z .$

$[*35{\cdot}1] \quad \supset . (\exists y) . xPy . y \in \alpha . yRz .$

$[*33{\cdot}14 . *10{\cdot}5] \quad \supset . (\exists y) . y \in \text{Q}`P . y \in \alpha .$

$[*22{\cdot}33 . *24{\cdot}5] \quad \supset . \exists ! \text{Q}`P \cap \alpha \quad (1)$

$\vdash . (1) . \text{Transp} . *24{\cdot}51 . \supset$

$\vdash : \text{Q}`P \cap \alpha = \Lambda . \supset . \sim x \{P \mid (\alpha \upharpoonleft R)\} z :$

$[*11{\cdot}11{\cdot}3] \quad \supset \vdash : \text{Q}`P \cap \alpha = \Lambda . \supset . (x, z) . \sim x \{P \mid (\alpha \upharpoonleft R)\} z .$

$[*25{\cdot}15] \quad \supset . P \mid (\alpha \upharpoonleft R) = \dot{\Lambda} : \supset \vdash . \text{Prop}$

$*35{\cdot}472$. $\vdash : \text{D}`P \cap \alpha = \Lambda . \supset . (R \upharpoonright \alpha) \mid P = \dot{\Lambda}$

$*35{\cdot}473$. $\vdash : \text{Q}`P \cap \alpha = \Lambda . \supset . P \mid (\alpha \upharpoonleft R \upharpoonright \beta) = \dot{\Lambda}$

$*35{\cdot}474$. $\vdash : \text{D}`P \cap \beta = \Lambda . \supset . (\alpha \upharpoonleft R \upharpoonright \beta) \mid P = \dot{\Lambda}$

∗35·48. $\vdash : \text{Ɑ}\text{‘}P \subset \alpha \,.\, \supset \,.\, P \,|\, (\alpha \upharpoonleft R) = P \,|\, R$

Dem.

$$\vdash \,.\, {*}22{\cdot}1 \,.\quad \supset \vdash :.\, \mathrm{Hp} \,.\, \supset : y \,\epsilon\, \text{Ɑ}\text{‘}P \,.\, \supset_y \,.\, y \,\epsilon\, \alpha :$$
$$[{*}4{\cdot}71] \qquad \supset : y \,\epsilon\, \text{Ɑ}\text{‘}P \,.\, y \,\epsilon\, \alpha \,.\, \equiv_y \,.\, y \,\epsilon\, \text{Ɑ}\text{‘}P :$$
$$[{*}10{\cdot}311] \qquad \supset : xPy \,.\, y \,\epsilon\, \text{Ɑ}\text{‘}P \,.\, y \,\epsilon\, \alpha \,.\, \equiv_y \,.\, xPy \,.\, y \,\epsilon\, \text{Ɑ}\text{‘}P \quad (1)$$
$$\vdash \,.\, {*}33{\cdot}14 \,.\, {*}4{\cdot}71 \,.\, \supset \vdash : xPy \,.\, y \,\epsilon\, \text{Ɑ}\text{‘}P \,.\, \equiv \,.\, xPy \quad (2)$$
$$\vdash \,.\, (1) \,.\, (2) \,.\, \supset \vdash :.\, \mathrm{Hp} \,.\, \supset : xPy \,.\, y \,\epsilon\, \alpha \,.\, \equiv_y \,.\, xPy :$$
$$[{*}10{\cdot}311] \qquad \supset : xPy \,.\, y \,\epsilon\, \alpha \,.\, yRz \,.\, \equiv_y \,.\, xPy \,.\, yRz :$$
$$[{*}35{\cdot}1] \qquad \supset : xPy \,.\, y \,(\alpha \upharpoonleft R)\, z \,.\, \equiv_y \,.\, xPy \,.\, yRz :$$
$$[{*}10{\cdot}281] \qquad \supset : (\exists y) \,.\, xPy \,.\, y \,(\alpha \upharpoonleft R)\, z \,.\, \equiv \,.\, (\exists y) \,.\, xPy \,.\, yRz :$$
$$[{*}34{\cdot}1] \qquad \supset : x \,(P \,|\, \alpha \upharpoonleft R)\, z \,.\, \equiv \,.\, x \,(P \,|\, R)\, z :.\, \supset \vdash \,.\, \mathrm{Prop}$$

∗35·481. $\vdash : \mathrm{D}\text{‘}R \subset \beta \,.\, \supset \,.\, (P \upharpoonright \beta) \,|\, R = P \,|\, R$ [Similar proof]

∗35·51. $\vdash \,.\, \mathrm{Cnv}\text{‘}(\alpha \upharpoonleft R) = \breve{R} \upharpoonright \alpha$

Dem.

$$\vdash \,.\, {*}31{\cdot}131 \,.\, \supset \vdash : x \,\{\mathrm{Cnv}\text{‘}(\alpha \upharpoonleft R)\}\, y \,.\, \equiv \,.\, y \,(\alpha \upharpoonleft R)\, x \,.$$
$$[{*}35{\cdot}1] \qquad \equiv \,.\, y \,\epsilon\, \alpha \,.\, yRx \,.$$
$$[{*}31{\cdot}11] \qquad \equiv \,.\, x\breve{R}y \,.\, y \,\epsilon\, \alpha \,.$$
$$[{*}35{\cdot}101] \qquad \equiv \,.\, x \,(\breve{R} \upharpoonright \alpha)\, y : \supset \vdash \,.\, \mathrm{Prop}$$

∗35·52. $\vdash \,.\, \mathrm{Cnv}\text{‘}(R \upharpoonright \beta) = \beta \upharpoonleft \breve{R}$ [Proof similar to that of ∗35·51]

∗35·53. $\vdash \,.\, \mathrm{Cnv}\text{‘}(\alpha \upharpoonleft R \upharpoonright \beta) = \beta \upharpoonleft \breve{R} \upharpoonright \alpha$ [Proof similar to that of ∗35·51]

∗35·61. $\vdash \,.\, \mathrm{D}\text{‘}(\alpha \upharpoonleft R) = \alpha \cap \mathrm{D}\text{‘}R$

Dem.

$$\vdash \,.\, {*}33{\cdot}13 \,.\, \supset \vdash :.\, x \,\epsilon\, \mathrm{D}\text{‘}(\alpha \upharpoonleft R) \,.\, \equiv : (\exists y) \,.\, x \,(\alpha \upharpoonleft R)\, y :$$
$$[{*}35{\cdot}1] \qquad \equiv : (\exists y) \,.\, x \,\epsilon\, \alpha \,.\, xRy :$$
$$[{*}10{\cdot}35] \qquad \equiv : x \,\epsilon\, \alpha : (\exists y) \,.\, xRy :$$
$$[{*}33{\cdot}13] \qquad \equiv : x \,\epsilon\, \alpha \,.\, x \,\epsilon\, \mathrm{D}\text{‘}R :$$
$$[{*}22{\cdot}33] \qquad \equiv : x \,\epsilon\, (\alpha \cap \mathrm{D}\text{‘}R) :.\, \supset \vdash \,.\, \mathrm{Prop}$$

∗35·62. $\vdash : \alpha \subset \mathrm{D}\text{‘}R \,.\, \supset \,.\, \mathrm{D}\text{‘}(\alpha \upharpoonleft R) = \alpha$ [∗35·61 . ∗22·621]

∗35·63. $\vdash : \mathrm{D}\text{‘}R \subset \alpha \,.\, \equiv \,.\, \alpha \upharpoonleft R = R$

Dem.

$$\vdash \,.\, {*}35{\cdot}61 \,.\, \supset \vdash : \alpha \upharpoonleft R = R \,.\, \supset \,.\, \alpha \cap \mathrm{D}\text{‘}R = \mathrm{D}\text{‘}R \,.$$
$$[{*}22{\cdot}621] \qquad \supset \,.\, \mathrm{D}\text{‘}R \subset \alpha \quad (1)$$
$$\vdash \,.\, (1) \,.\, {*}35{\cdot}451 \,.\, \supset \vdash \,.\, \mathrm{Prop}$$

∗35·64. $\vdash \,.\, \text{Ɑ}\text{‘}(R \upharpoonright \beta) = \beta \cap \text{Ɑ}\text{‘}R$ [Proof as in ∗35·61]

∗35·641. $\vdash : \alpha \cap \mathrm{D}\text{‘}R = \Lambda \,.\, \supset \,.\, \alpha \upharpoonleft R = \dot{\Lambda}$ [∗35·61 . ∗33·241]

∗35·642. $\vdash : \alpha \cap \text{Ɑ}\text{‘}R = \Lambda \,.\, \supset \,.\, R \upharpoonright \alpha = \dot{\Lambda}$ [∗35·64 . ∗33·241]

∗35·643. $\vdash : \alpha \cap \mathrm{D}\text{‘}R = \Lambda \,.\, \supset \,.\, \alpha \upharpoonleft (R \,\dot{\cup}\, S) = \alpha \upharpoonleft S$ [∗35·641·42]

***35·644**. $\vdash : \alpha \cap \text{G}'R = \Lambda . \supset . (R \cup S) \upharpoonright \alpha = S \upharpoonright \alpha$ [*35·642·421]

***35·65**. $\vdash : \beta \subset \text{G}'R . \supset . \text{G}'(R \upharpoonright \beta) = \beta$ [*35·64 . *22·621]

***35·66**. $\vdash : \text{G}'R \subset \beta . \equiv . R \upharpoonright \beta = R$ [Proof as in *35·63]

***35·671**. $\vdash . \text{D}'(R \mid S) = \text{D}'(R \upharpoonright \text{D}'S)$

Dem.

$\vdash . *33{\cdot}13 . \supset \vdash :. x \in \text{D}'(R \mid S) . \equiv : (\exists y) . x (R \mid S) y :$

[*34·1] $\equiv : (\exists y, z) . xRz . zSy :$

[*11·23] $\equiv : (\exists z, y) . xRz . zSy :$

[*10·35] $\equiv : (\exists z) : xRz : (\exists y) . zSy :$

[*33·13] $\equiv : (\exists z) . xRz . z \in \text{D}'S :$

[*35·101] $\equiv : (\exists z) . x (R \upharpoonright \text{D}'S) z :$

[*33·13] $\equiv : x \in \text{D}'(R \upharpoonright \text{D}'S) :. \supset \vdash . \text{Prop}$

***35·672**. $\vdash . \text{G}'(R \mid S) = \text{G}'(\text{G}'R \upharpoonleft S)$ [Similar proof]

***35·68**. $\vdash : \alpha \cap \beta = \Lambda . \supset . (\alpha \upharpoonleft R \upharpoonright \beta)^2 = \dot{\Lambda}$

Dem.

$\vdash . *35{\cdot}61{\cdot}64{\cdot}21 . \supset \vdash . \text{D}'(\alpha \upharpoonleft R \upharpoonright \beta) \subset \alpha . \text{G}'(\alpha \upharpoonleft R \upharpoonright \beta) \subset \beta .$

[*22·49.*24·13] $\supset \vdash : \alpha \cap \beta = \Lambda . \supset . \text{D}'(\alpha \upharpoonleft R \upharpoonright \beta) \cap \text{G}'(\alpha \upharpoonleft R \upharpoonright \beta) = \Lambda .$

[*34·531] $\supset . (\alpha \upharpoonleft R \upharpoonright \beta)^2 = \dot{\Lambda} : \supset \vdash . \text{Prop}$

***35·7**. $\vdash : \phi \{(R \upharpoonright \beta)'y\} . \equiv . y \in \beta . \phi (R'y)$

This proposition is very often used in the later parts of the work.

Dem.

$\vdash . *14{\cdot}21 . \supset \vdash : \phi \{(R \upharpoonright \beta)'y\} . \supset . \text{E} ! (R \upharpoonright \beta)'y .$

[*33·43] $\supset . y \in \text{G}'(R \upharpoonright \beta) .$

[*35·64] $\supset . y \in \beta$ (1)

$\vdash . (1) . *4{\cdot}71 . \supset \vdash : \phi \{(R \upharpoonright \beta)'y\} . \equiv . y \in \beta . \phi \{(R \upharpoonright \beta)'y\}$ (2)

$\vdash . *4{\cdot}73 . *35{\cdot}101 . \supset \vdash :. y \in \beta . \supset : x (R \upharpoonright \beta) y . \equiv_x . xRy :$

[*14·272] $\supset : \phi \{(R \upharpoonright \beta)'y\} . \equiv . \phi (R'y)$ (3)

$\vdash . (3) . *5{\cdot}32 . \supset \vdash : y \in \beta . \phi \{(R \upharpoonright \beta)'y\} . \equiv . y \in \beta . \phi (R'y)$ (4)

$\vdash . (2) . (4) . \supset \vdash . \text{Prop}$

***35·71**. $\vdash :. y \in \beta . \supset_y . R'y = S'y : \supset . R \upharpoonright \beta = S \upharpoonright \beta$

Dem.

$\vdash . *4{\cdot}7 . \supset \vdash :. \text{Hp} . \supset : y \in \beta . \supset_y . y \in \beta . R'y = S'y :$

[*35·7] $\supset : y \in \beta . \supset_y . (R \upharpoonright \beta)'y = (S \upharpoonright \beta)'y :$

[*35·64] $\supset : y \in \text{G}'(R \upharpoonright \beta) \cup \text{G}'(S \upharpoonright \beta) . \supset_y . (R \upharpoonright \beta)'y = (S \upharpoonright \beta)'y :$

[*33·45] $\supset : R \upharpoonright \beta = S \upharpoonright \beta :. \supset \vdash . \text{Prop}$

***35·75**. $\vdash . \Lambda \upharpoonleft R = R \upharpoonright \Lambda = \Lambda \upharpoonleft R \upharpoonright \beta = \alpha \upharpoonleft R \upharpoonright \Lambda = \dot{\Lambda}$

Dem.

$\vdash . *35{\cdot}61 . \supset \vdash . \text{D}'(\Lambda \upharpoonleft R) = \Lambda .$

[*33·241] $\supset \vdash . \Lambda \upharpoonleft R = \dot{\Lambda}$ (1)

$\vdash . *35{\cdot}64 . \quad \supset \vdash . \text{Ɑ}{}^{\backprime}(R \restriction \Lambda) = \Lambda .$

$[*33{\cdot}241] \quad \supset \vdash . R \restriction \Lambda = \dot{\Lambda}$ (2)

$\vdash . *35{\cdot}441{\cdot}21 . \supset \vdash . \Lambda \upharpoonleft R \restriction \beta \mathrel{\dot{\subset}} \Lambda \upharpoonleft R .$

$[(1) . *25{\cdot}13] \quad \supset \vdash . \Lambda \upharpoonleft R \restriction \beta = \dot{\Lambda}$ (3)

$\vdash . *35{\cdot}44{\cdot}21 . \supset \vdash . \alpha \upharpoonleft R \restriction \Lambda \mathrel{\dot{\subset}} R \restriction \Lambda .$

$[(2) . *25{\cdot}13] \quad \supset \vdash . \alpha \upharpoonleft R \restriction \Lambda = \dot{\Lambda}$ (4)

$\vdash . (1) . (2) . (3) . (4) . \supset \vdash . \text{Prop}$

***35·76.** $\vdash . \text{V} \upharpoonleft R = R \restriction \text{V} = \text{V} \upharpoonleft R \restriction \text{V} = R$

Dem.

$\vdash . *35{\cdot}1 . \supset \vdash : x(\text{V} \upharpoonleft R) y . \equiv . x \in \text{V} . xRy .$

$[*24{\cdot}104 . *4{\cdot}73] \quad \equiv . xRy$ (1)

$\vdash . *35{\cdot}101 . \supset \vdash : x(R \restriction \text{V}) y . \equiv . xRy . y \in \text{V} .$

$[*24{\cdot}104 . *4{\cdot}73] \quad \equiv . xRy$ (2)

$\vdash . *35{\cdot}102 . \supset \vdash : x(\text{V} \upharpoonleft R \restriction \text{V}) y . \equiv . x \in \text{V} . xRy . y \in \text{V} .$

$[*24{\cdot}104 . *4{\cdot}73] \quad \equiv . xRy$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash . \text{Prop}$

The rest of this number, down to *35·93 exclusive, is concerned with $\alpha \uparrow \beta$, except *35·81·812.

***35·81.** $\vdash : x(\alpha \upharpoonleft \dot{\text{V}}) y . \equiv . x \in \alpha \quad [*35{\cdot}1 . *25{\cdot}104]$

***35·812.** $\vdash : x(\dot{\text{V}} \restriction \beta) y . \equiv . y \in \beta \quad [*35{\cdot}101 . *25{\cdot}104]$

***35·82.** $\vdash . \alpha \uparrow \beta = \alpha \upharpoonleft \dot{\text{V}} \restriction \beta$

Dem.

$\vdash . *35{\cdot}103 . \supset \vdash : x(\alpha \uparrow \beta) y . \equiv . x \in \alpha . y \in \beta .$

$[*25{\cdot}104] \quad \equiv . x \in \alpha . x \dot{\text{V}} y . y \in \beta .$

$[*35{\cdot}102] \quad \equiv . x(\alpha \upharpoonleft \dot{\text{V}} \restriction \beta) y : \supset \vdash . \text{Prop}$

***35·822.** $\vdash . \alpha \upharpoonleft R \restriction \beta = R \mathbin{\dot{\cap}} (\alpha \uparrow \beta)$

Dem.

$\vdash . *35{\cdot}102 . \supset \vdash : x(\alpha \upharpoonleft R \restriction \beta) y . \equiv . x \in \alpha . xRy . y \in \beta .$

$[*4{\cdot}3] \quad \equiv . xRy . x \in \alpha . y \in \beta .$

$[*35{\cdot}103] \quad \equiv . xRy . x(\alpha \uparrow \beta) y .$

$[*23{\cdot}33] \quad \equiv . x \{R \mathbin{\dot{\cap}} (\alpha \uparrow \beta)\} y : \supset \vdash . \text{Prop}$

***35·83.** $\vdash : \text{D}{}^{\backprime}R \subset \alpha . \text{Ɑ}{}^{\backprime}R \subset \beta . \equiv . R \mathrel{\dot{\subset}} \alpha \uparrow \beta$

Dem.

$\vdash . *33{\cdot}14 . \quad \supset \vdash :. xRy . \supset : x \in \text{D}{}^{\backprime}R . y \in \text{Ɑ}{}^{\backprime}R :$

$[*22{\cdot}46] \quad \supset : \text{D}{}^{\backprime}R \subset \alpha . \text{Ɑ}{}^{\backprime}R \subset \beta . \supset . x \in \alpha . y \in \beta$ (1)

$\vdash . (1) . \text{Comm} . \supset \vdash :. \text{D}{}^{\backprime}R \subset \alpha . \text{Ɑ}{}^{\backprime}R \subset \beta . \supset : xRy . \supset . x \in \alpha . y \in \beta .$

$[*35{\cdot}103] \quad \supset . x(\alpha \uparrow \beta) y$ (2)

$\vdash . *35{\cdot}103 . \quad \supset \vdash :. R \mathrel{\dot{\subset}} \alpha \uparrow \beta . \supset : xRy . \supset_{x,y} . x \in \alpha . y \in \beta :$

$[*33{\cdot}35{\cdot}351] \quad \supset : \text{D}{}^{\backprime}R \subset \alpha . \text{Ɑ}{}^{\backprime}R \subset \beta$ (3)

$\vdash . (2) . (3) . \quad \supset \vdash . \text{Prop}$

***35·831.** $\vdash . \dot{-} (\alpha \uparrow \beta) = (-\alpha \uparrow \beta) \mathbin{\dot{\cup}} (\alpha \uparrow -\beta) \mathbin{\dot{\cup}} (-\alpha \uparrow -\beta)$

Dem.

$\vdash . *23{\cdot}35 . \supset \vdash :: x \{\dot{-}(\alpha \uparrow \beta)\} y . \equiv :. \sim \{x (\alpha \uparrow \beta) y\} :.$

[*35·103] $\equiv :. \sim (x \epsilon \alpha . y \epsilon \beta) :.$

[*4·51] $\equiv :. x \sim \epsilon \alpha . \mathsf{v} . y \sim \epsilon \beta :.$

[*4·42] $\equiv :. x \sim \epsilon \alpha : y \epsilon \beta . \mathsf{v} . y \sim \epsilon \beta :. \mathsf{v} :. x \epsilon \alpha . \mathsf{v} . x \sim \epsilon \alpha : y \sim \epsilon \beta :.$

[*4·4] $\equiv :. x \sim \epsilon \alpha . y \epsilon \beta . \mathsf{v} . x \sim \epsilon \alpha . y \sim \epsilon \beta . \mathsf{v} . x \epsilon \alpha . y \sim \epsilon \beta . \mathsf{v} . x \sim \epsilon \alpha . y \sim \epsilon \beta :.$

[*4·25·31·37]

$\equiv :. x \sim \epsilon \alpha . y \epsilon \beta . \mathsf{v} . x \epsilon \alpha . y \sim \epsilon \beta . \mathsf{v} . x \sim \epsilon \alpha . y \sim \epsilon \beta :.$

[*22·35] $\equiv :. x \epsilon -\alpha . y \epsilon \beta . \mathsf{v} . x \epsilon \alpha . y \epsilon -\beta . \mathsf{v} . x \epsilon -\alpha . y \epsilon -\beta :.$

[*35·103] $\equiv :. x (-\alpha \uparrow \beta) y . \mathsf{v} . x (\alpha \uparrow -\beta) y . \mathsf{v} . x (-\alpha \uparrow -\beta) y :.$

[*23·34] $\equiv :. x \{(-\alpha \uparrow \beta) \mathbin{\dot{\cup}} (\alpha \uparrow -\beta) \mathbin{\dot{\cup}} (-\alpha \uparrow -\beta)\} y :: \supset \vdash .$ Prop

***35·832.** $\vdash . \dot{-} (\alpha \upharpoonleft R \upharpoonright \beta) = (-\alpha \uparrow \beta) \mathbin{\dot{\cup}} (\alpha \uparrow -\beta) \mathbin{\dot{\cup}} (-\alpha \uparrow -\beta) \mathbin{\dot{\cup}} \dot{-} R$

[*35·822·831 . Transp . *23·84]

***35·834.** $\vdash . (\alpha \uparrow \beta) \mathbin{\dot{\cap}} (\gamma \uparrow \delta) = (\alpha \cap \gamma) \uparrow (\beta \cap \delta)$

Dem.

$\vdash . *35{\cdot}103 . \supset$

$\vdash : x \{(\alpha \uparrow \beta) \mathbin{\dot{\cap}} (\gamma \uparrow \delta)\} y . \equiv . x \epsilon \alpha . y \epsilon \beta . x \epsilon \gamma . y \epsilon \delta .$

[*22·33.*35·103] $\equiv . x \{(\alpha \cap \gamma) \uparrow (\beta \cap \delta)\} y : \supset \vdash .$ Prop

***35·84.** $\vdash . \mathrm{Cnv}{}^{\backprime}(\alpha \uparrow \beta) = \beta \uparrow \alpha$ [*35·103 . *31·131]

***35·85.** $\vdash : \exists ! \beta . \supset . \mathrm{D}{}^{\backprime}(\alpha \uparrow \beta) = \alpha$

Dem.

$\vdash . *35{\cdot}103 . *10{\cdot}281 . \supset$

$\vdash :. (\exists y) . x (\alpha \uparrow \beta) y . \equiv : (\exists y) . x \epsilon \alpha . y \epsilon \beta :$

[*10·35] $\equiv : x \epsilon \alpha : (\exists y) . y \epsilon \beta :$

[*24·5] $\equiv : x \epsilon \alpha . \exists ! \beta$ (1)

$\vdash . (1) . *33{\cdot}13 . *10{\cdot}35 . \supset \vdash .$ Prop

***35·86.** $\vdash : \exists ! \alpha . \supset . \mathrm{Q}{}^{\backprime}(\alpha \uparrow \beta) = \beta$ [Similar proof]

***35·87.** $\vdash : \dot{\exists} ! (\alpha \uparrow \beta) . \equiv . \exists ! \alpha . \exists ! \beta$

Dem.

$\vdash . *35{\cdot}103 . \supset \vdash :. \dot{\exists} ! (\alpha \uparrow \beta) . \equiv : (\exists x, y) . x \epsilon \alpha . y \epsilon \beta :$

[*11·54] $\equiv : (\exists x) . x \epsilon \alpha : (\exists y) . y \epsilon \beta :$

[*24·5] $\equiv : \exists ! \alpha . \exists ! \beta :. \supset \vdash .$ Prop

***35·88.** $\vdash :. \alpha \uparrow \beta = \dot{\Lambda} . \equiv : \alpha = \Lambda . \mathsf{v} . \beta = \Lambda$

[*35·87 . Transp . *24·51 . *25·51]

***35·881.** $\vdash : \mathrm{Q}{}^{\backprime}R \subset \alpha . \supset . R \mid (\alpha \uparrow \beta) = \mathrm{D}{}^{\backprime}R \uparrow \beta$

Dem.

$\vdash . *34{\cdot}1 . *35{\cdot}103 . \supset$

$\vdash : x \{R \mid (\alpha \uparrow \beta)\} y . \equiv . (\exists z) . x R z . z \epsilon \alpha . y \epsilon \beta$ (1)

$\vdash . *33{\cdot}14 . \supset \vdash :. \text{C}'R \subset \alpha . \supset : xRz . \supset . z \epsilon \alpha :$

[$*4{\cdot}73$] $\supset : xRz . \equiv . xRz \,.\, z \epsilon \alpha$ (2)

$\vdash . (1) . (2) . \supset \vdash :: \text{Hp} . \supset :. x \{R \mid (\alpha \uparrow \beta)\} y . \equiv : (\exists z) . xRz . y \epsilon \beta :$

[$*10{\cdot}35$] $\equiv : (\exists z) . xRz : y \epsilon \beta :$

[$*33{\cdot}13$] $\equiv : x \epsilon \text{D}'R . y \epsilon \beta :$

[$*35{\cdot}103$] $\equiv : x (\text{D}'R \uparrow \beta) y :: \supset \vdash . \text{Prop}$

$*35{\cdot}882$. $\vdash : \text{D}'R \subset \beta . \supset . (\alpha \uparrow \beta) \mid R = \alpha \uparrow \text{C}'R$ [Similar proof]

$*35{\cdot}89$. $\vdash : \exists ! \beta . \supset . (\alpha \uparrow \beta) \mid (\beta \uparrow \gamma) = (\alpha \uparrow \gamma) : \sim \exists ! \beta . \supset . (\alpha \uparrow \beta) \mid (\beta \uparrow \gamma) = \dot{\Lambda}$

Dem.

$\vdash . *34{\cdot}1 . \supset \vdash :. x \{(\alpha \uparrow \beta) \mid (\beta \uparrow \gamma)\} z .$

$\equiv : (\exists y) . x (\alpha \uparrow \beta) y . y (\beta \uparrow \gamma) z :$

[$*35{\cdot}103$] $\equiv : (\exists y) . x \epsilon \alpha . y \epsilon \beta . y \epsilon \beta . z \epsilon \gamma :$

[$*4{\cdot}24$] $\equiv : (\exists y) . x \epsilon \alpha . y \epsilon \beta . z \epsilon \gamma :$

[$*10{\cdot}35$] $\equiv : \exists ! \beta : x \epsilon \alpha . z \epsilon \gamma :$

[$*35{\cdot}103$] $\equiv : \exists ! \beta : x (\alpha \uparrow \gamma) z$ (1)

$\vdash . (1) . \supset \vdash :: \exists ! \beta . \supset : x \{(\alpha \uparrow \beta) \mid (\beta \uparrow \gamma)\} z . \equiv . x (\alpha \uparrow \gamma) z :.$

$\sim (\exists ! \beta) . \supset : \sim [x \{(\alpha \uparrow \beta) \mid (\beta \uparrow \gamma)\} z] :: \supset \vdash . \text{Prop}$

$*35{\cdot}891$. $\vdash :. \exists ! \beta . \vee . \sim \exists ! \alpha : \supset . (\alpha \uparrow \beta) \mid (\beta \uparrow \alpha) = (\alpha \uparrow \alpha)$

Dem.

$\vdash . *35{\cdot}88 . \supset \vdash : \sim \exists ! \alpha . \supset . \alpha \uparrow \alpha = \dot{\Lambda} . \alpha \uparrow \beta = \dot{\Lambda} .$

[$*34{\cdot}32$] $\supset . \alpha \uparrow \alpha = \dot{\Lambda} . (\alpha \uparrow \beta) \mid (\beta \uparrow \alpha) = \dot{\Lambda} .$

[$*21{\cdot}24$] $\supset . (\alpha \uparrow \alpha) = (\alpha \uparrow \beta) \mid (\beta \uparrow \alpha)$ (1)

$\vdash . (1) . *35{\cdot}89 . \supset \vdash . \text{Prop}$

$*35{\cdot}892$. $\vdash : (\alpha \uparrow \alpha)^2 = (\alpha \uparrow \alpha)$ $\left[*35{\cdot}891 \dfrac{\alpha}{\beta}\right]$

$*35{\cdot}895$. $\vdash : \alpha \cap \beta = \Lambda . \supset . (\alpha \uparrow \beta)^2 = \dot{\Lambda}$ [$*35{\cdot}68{\cdot}82$]

$*35{\cdot}9$. $\vdash . \text{D}'(\alpha \uparrow \alpha) = \text{C}'(\alpha \uparrow \alpha) = C'(\alpha \uparrow \alpha) = \alpha$

Dem.

$\vdash . *35{\cdot}85{\cdot}86 . \supset \vdash : \exists ! \alpha . \supset . \text{D}'(\alpha \uparrow \alpha) = \alpha . \text{C}'(\alpha \uparrow \alpha) = \alpha$ (1)

$\vdash . *35{\cdot}88 . \supset \vdash : \sim \exists ! \alpha . \supset . \sim \dot{\exists} ! (\alpha \uparrow \alpha) .$

[$*33{\cdot}29$] $\supset . \text{D}'(\alpha \uparrow \alpha) = \Lambda . \text{C}'(\alpha \uparrow \alpha) = \Lambda .$

[$*24{\cdot}51$] $\supset . \text{D}'(\alpha \uparrow \alpha) = \alpha . \text{C}'(\alpha \uparrow \alpha) = \alpha$ (2)

$\vdash . (1) . (2) . *4{\cdot}83 . \supset \vdash . \text{D}'(\alpha \uparrow \alpha) = \text{C}'(\alpha \uparrow \alpha) = \alpha . \supset \vdash . \text{Prop}$

$*35{\cdot}91$. $\vdash : R \mathrel{\dot\subset} \alpha \uparrow \alpha . \equiv . C'R \subset \alpha$

Dem.

$\vdash . *35{\cdot}103 . \supset \vdash :. R \mathrel{\dot\subset} \alpha \uparrow \alpha . \equiv : xRy . \supset_{x,y} . x, y \epsilon \alpha :$

[$*33{\cdot}352$] $\equiv : C'R \subset \alpha :. \supset \vdash . \text{Prop}$

$*35{\cdot}92$. $\vdash :. (\exists \alpha) . P = \alpha \uparrow \alpha . \supset : R \mathrel{\dot\subset} P . \equiv . C'R \subset C'P$ [$*35{\cdot}9{\cdot}91$]

***35·93.** $\vdash : (R) . \phi(\text{D}`R) . \equiv . (\alpha) . \phi\alpha$

Dem.

$\vdash . *33\cdot12 . *14\cdot18 . \supset \vdash : (\alpha) . \phi\alpha . \supset . \phi(\text{D}`R) :$

$[*10\cdot11\cdot21] \qquad \supset \vdash : (\alpha) . \phi\alpha . \supset . (R) . \phi(\text{D}`R) \qquad (1)$

$\vdash . *10\cdot1 . \qquad \supset \vdash : (R) . \phi(\text{D}`R) . \supset . \phi\{\text{D}`(\alpha \uparrow \alpha)\} .$

$[*35\cdot9] \qquad \supset . \phi\alpha :$

$[*10\cdot11\cdot21] \qquad \supset \vdash : (R) . \phi(\text{D}`R) . \supset . (\alpha) . \phi\alpha \qquad (2)$

$\vdash . (1) . (2) . \qquad \supset \vdash .$ Prop

***35·931.** $\vdash : (R) . \phi(\text{Ɑ}`R) . \equiv . (\alpha) . \phi\alpha$ [Proof as in *35·93]

***35·932.** $\vdash : (R) . \phi(C`R) . \equiv . (\alpha) . \phi\alpha$ [Proof as in *35·93]

***35·94.** $\vdash : (\exists R) . \phi(\text{D}`R) . \equiv . (\exists\alpha) . \phi\alpha$ [*35·93 . Transp]

***35·941.** $\vdash : (\exists R) . \phi(\text{Ɑ}`R) . \equiv . (\exists\alpha) . \phi\alpha$ [*35·931 . Transp]

***35·942.** $\vdash : (\exists R) . \phi(C`R) . \equiv . (\exists\alpha) . \phi\alpha$ [*35·932 . Transp]

$*36$. RELATIONS WITH LIMITED FIELDS

Summary of $*36$.

In this number we are concerned with the special case in which the same limitation is imposed upon the domain and the converse domain of a relation. In this case, the same result is achieved by imposing the limitation on the field. It is convenient to be able to regard $\alpha \upharpoonleft P \upharpoonright \alpha$ as a descriptive function of α or of P, which we secure by the notation $P \sqsubset \alpha$, whence, as will be explained in $*38$, $P \sqsubset {}^{\backprime}\alpha$ and $\sqsubset \alpha {}^{\backprime} P$ will both mean $P \sqsubset \alpha$. If P is a serial relation, and $\alpha \subset C{}^{\backprime}P$, "$P \sqsubset \alpha$" will stand for "the terms of α arranged in the order determined by P," or, as we may call it briefly, "α in the P-order." $P \sqsubset \alpha$ is defined as follows:

$*36{\cdot}01$. $P \sqsubset \alpha = \alpha \upharpoonleft P \upharpoonright \alpha$ Df

We thus have

$*36{\cdot}13$. $\vdash : x (P \sqsubset \alpha) y . \equiv . x, y \in \alpha . xPy$

Most of the propositions concerning $P \sqsubset \alpha$ demand that P should have some at least of the characteristics of a *serial* relation. Hence the propositions concerning $P \sqsubset \alpha$ which can be given in the present number are, for the most part, not the most useful propositions concerning $P \sqsubset \alpha$. The most useful propositions in the present number are the following:

$*36{\cdot}25$. $\vdash : C{}^{\backprime}P \subset \alpha . \equiv . P \sqsubset \alpha = P$

$*36{\cdot}29$. $\vdash . P \sqsubset \alpha = P \mathbin{\dot{\cap}} \alpha \uparrow \alpha$

$*36{\cdot}3$. $\vdash . P \sqsubset \alpha = P \sqsubset (\alpha \cap C{}^{\backprime}P)$

$*36{\cdot}33$. $\vdash . P \sqsubset C{}^{\backprime}P = P$

$*36{\cdot}01$. $P \sqsubset \alpha = \alpha \upharpoonleft P \upharpoonright \alpha$ Df

$*36{\cdot}11$. $\vdash . P \sqsubset \alpha = \alpha \upharpoonleft P \upharpoonright \alpha$ [($*36{\cdot}01$)]

$*36{\cdot}13$. $\vdash : x (P \sqsubset \alpha) y . \equiv . x, y \in \alpha . xPy$ [$*36{\cdot}11 . *35{\cdot}102$]

The following propositions are obtained from those of $*35$ by means of $*36{\cdot}11$, which, as it is used in each case, is not referred to again.

$*36{\cdot}2$. $\vdash . P \sqsubset \alpha \mathbin{\dot{\cap}} Q \sqsubset \beta = (P \mathbin{\dot{\cap}} Q) \sqsubset (\alpha \cap \beta)$ [$*35{\cdot}15$]

$*36{\cdot}201$. $\vdash . P \sqsubset \alpha \mathbin{\dot{\cap}} P \sqsubset \beta = P \sqsubset (\alpha \cap \beta)$ [$*36{\cdot}2$]

$*36{\cdot}202$. $\vdash . P \sqsubset \alpha \mathbin{\dot{\cap}} Q \sqsubset \alpha = (P \mathbin{\dot{\cap}} Q) \sqsubset \alpha$ [$*36{\cdot}2$]

$*36{\cdot}203$. $\vdash . P \sqsubset \alpha \mathbin{\dot{\cap}} Q = (P \mathbin{\dot{\cap}} Q) \sqsubset \alpha$ [$*35{\cdot}18$]

$*36{\cdot}21$. $\vdash . (P \sqsubset \alpha) \sqsubset \beta = P \sqsubset (\alpha \cap \beta)$ [$*35{\cdot}33{\cdot}34$]

***36·22.** $\vdash . (P \upharpoonright \alpha) | (Q \upharpoonright \alpha) \subseteq (P | Q) \upharpoonright \alpha$

Dem.

$\vdash . *36\cdot13 . *34\cdot1 . \supset \vdash : x \{(P \upharpoonright \alpha) | (Q \upharpoonright \alpha)\} z . \equiv . (\exists y) . x, y, z \in \alpha . xPy . yQz .$

[*10·5] $\supset . (\exists y) . x, z \in \alpha . xPy . yQz$ (1)

$\vdash . (1) . *10\cdot35 . *34\cdot1 . \supset \vdash .$ Prop

***36·23.** $\vdash . (P \cup Q) \upharpoonright \alpha = P \upharpoonright \alpha \cup Q \upharpoonright \alpha$ [*35·422]

***36·24.** $\vdash : \alpha \subset \beta . \supset . P \upharpoonright \alpha \subseteq P \upharpoonright \beta$ [*35·432]

***36·241.** $\vdash : P \subseteq Q . \supset . P \upharpoonright \alpha \subseteq Q \upharpoonright \alpha$ [*35·462]

***36·25.** $\vdash : C`P \subset \alpha . \equiv . P \upharpoonright \alpha = P$

Dem.

$\vdash . *36\cdot13 . *4\cdot7 . \supset \vdash :. P \upharpoonright \alpha = P . \equiv : xPy . \supset_{x,y} . x, y \in \alpha :$

[*33·352] $\equiv : C`P \subset \alpha :. \supset \vdash .$ Prop

***36·26.** $\vdash : C`P \cap \alpha = \Lambda . \supset . P | (Q \upharpoonright \alpha) = \dot{\Lambda} . (Q \upharpoonright \alpha) | P = \dot{\Lambda}$ [*35·473·474]

***36·27.** $\vdash : P \upharpoonright \Lambda = \dot{\Lambda}$ [*35·75]

***36·28.** $\vdash . P \upharpoonright V = P$ [*35·76]

***36·29.** $\vdash . P \upharpoonright \alpha = P \dot{\cap} \alpha \uparrow \alpha$ [*35·822]

***36·3.** $\vdash . P \upharpoonright \alpha = P \upharpoonright (\alpha \cap C`P)$

Dem.

$\vdash . *33\cdot17 . *4\cdot71 . \supset \vdash : xPy . \equiv . x, y \in C`P . xPy :$

[Fact] $\supset \vdash : x, y \in \alpha . xPy . \equiv . x, y \in \alpha . x, y \in C`P . xPy .$

[*22·33] $\equiv . x, y \in \alpha \cap C`P . xPy .$

[*36·13] $\equiv . x \{P \upharpoonright (\alpha \cap C`P)\} y$ (1)

$\vdash . (1) . *36\cdot13 . \supset \vdash .$ Prop

***36·31.** $\vdash : \alpha \cap C`P = \Lambda . \supset . P \upharpoonright \alpha = \dot{\Lambda}$ [*36·3·27]

***36·32.** $\vdash : \alpha \cap C`P = \beta \cap C`P . \supset . P \upharpoonright \alpha = P \upharpoonright \beta$ [*36·3]

***36·33.** $\vdash . P \upharpoonright C`P = P$ [*36·25]

***36·34.** $\vdash . \mathrm{Cnv}`P \upharpoonright \alpha = (\breve{P}) \upharpoonright \alpha$ [*35·53]

***36·35.** $\vdash . (P \upharpoonright \alpha)^2 \subseteq (P^2) \upharpoonright \alpha$ [*36·22]

***36·4.** $\vdash :. \alpha \cap D`R = \Lambda . \vee . \alpha \cap Ɑ`R = \Lambda : \supset . (R \cup S) \upharpoonright \alpha = S \upharpoonright \alpha$

Dem.

$\vdash . *35\cdot643 . \supset \vdash : \alpha \cap D`R = \Lambda . \supset . \alpha \upharpoonleft (R \cup S) = \alpha \upharpoonleft S .$

[*35·21] $\supset . (R \cup S) \upharpoonright \alpha = S \upharpoonright \alpha$ (1)

Similarly $\vdash : \alpha \cap Ɑ`R = \Lambda . \supset . (R \cup S) \upharpoonright \alpha = S \upharpoonright \alpha$ (2)

$\vdash . (1) . (2) . \supset \vdash .$ Prop

$*$37. PLURAL DESCRIPTIVE FUNCTIONS

Summary of $*37$.

In this number, we introduce what may be regarded as the plural of $R`y$. "$R`y$" was defined to mean "the term which has the relation R to y." We now introduce the notation "$R``\beta$" to mean "the terms which have the relation $R$ to members of $\beta$." Thus if $\beta$ is the class of great men, and $R$ is the relation of wife to husband, $R``\beta$ will mean "wives of great men." If β is the class of fractions of the form $1 - 1/2^n$ for integral values of n, and R is the relation "less than," $R``\beta$ will be the class of fractions each of which is less than some member of this class of fractions, *i.e.* $R``\beta$ will be the class of proper fractions. Generally, $R``\beta$ is the class of those referents which have relata that are members of β.

We require also a notation for the relation of $R``\beta$ to β. This relation we will call R_ϵ. Thus R_ϵ is the relation which holds between two classes α and β when α consists of all terms which have the relation R to some member of β.

A specially important case arises when $R`y$ always exists if $y \epsilon \beta$. In this case, $R``\beta$ is the class of all terms of the form $R`y$ when $y \epsilon \beta$. We will denote the hypothesis that $R`y$ always exists if $y \epsilon \beta$ by the notation $E!! R``\beta$, meaning "the R's of β's exist."

The definitions are as follows:

$*37{\cdot}01$. $R``\beta = \hat{x}\{(\exists y) . y \epsilon \beta . xRy\}$ Df

$*37{\cdot}02$. $R_\epsilon = \hat{\alpha}\hat{\beta}(\alpha = R``\beta)$ Df

$*37{\cdot}03$. $\breve{R}_\epsilon = \text{Cnv}`(R_\epsilon)$ Df

This definition serves merely for the avoidance of brackets. Without it, "$\breve{R}_\epsilon$" would be ambiguous as between $(\breve{R})_\epsilon$ and $\text{Cnv}`(R_\epsilon)$, which are not equal. In all cases in which a suffix occurs, we shall adopt the same convention, *i.e.* we shall always put

$$\breve{R}_{\text{suffix}} = \text{Cnv}`(R_{\text{suffix}}).$$

$*37{\cdot}04$. $R```\kappa = R_\epsilon``\kappa$ Df

Thus $R```\kappa$ consists of all classes which have the relation R_ϵ to some member of κ. $R```\kappa$ is only significant when κ is a class of classes relatively to members of the converse domain of R; in this case, $R```\kappa$ is a class of classes relatively to members of the domain of R.

$*37{\cdot}05$. $E!! R``\beta . = : y \epsilon \beta . \supset_y . E! R`y$ Df

Here the symbol "$E!! R``\beta$" must be treated as a whole, *i.e.* we must not regard it as making an assertion about $R``\beta$. If $R``\beta = \alpha$, we must not suppose

that we shall be able to put "E !! α," which would be nonsense, just as "E ! x" is nonsense even when $x = R\text{‘}y$ and E ! $R\text{‘}y$.

The notation $R\text{‘‘}\alpha$, introduced in the present number, is extremely useful, and embodies a very important idea. Its use is somewhat different according to the kind of relation concerned. Consider first the kind of relation which leads to a descriptive function, say D. If λ is a class of relations, D‘‘λ is the class of the domains of these relations. In this case, D‘‘λ is a class each of whose members is of the form D‘R, where $R \epsilon \lambda$. Again, let us denote by "$\times n$" the relation of m to $m \times n$; then if we denote by "NC" the class of cardinal numbers, $\times n\text{‘‘}NC$ will denote all numbers that result from multiplying a cardinal number by n, *i.e.* all multiples of n. Thus *e.g.* $\times 2\text{‘‘}NC$ will be the class of even numbers. If R is a correlation between two classes α and β, *i.e.* a relation such that, if $y \epsilon \beta$, $R\text{‘}y$ exists and is a member of α, while conversely, if $x \epsilon \alpha$, $\breve{R}\text{‘}x$ exists and is a member of β, then $\alpha = R\text{‘‘}\beta$, and we may regard R as a transformation applied to each member of β and giving rise to a member of α. It is by means of such transformations that two classes are shown to be *similar*, *i.e.* to have the same (cardinal) number of terms.

In the case of serial relations, the utility of the notation $R\text{‘‘}\beta$ is somewhat different. Suppose, for example, that R is the relation of less to greater among real numbers. Then if β is any class of real numbers, $R\text{‘‘}\beta$ will be the segment of real numbers determined by β, *i.e.* the class of real numbers which are less than the limit or maximum of β. In any series, if β is a class contained in the series and R is the generating relation of the series, $R\text{‘‘}\beta$ is the segment determined by β. If β has either a limit or a maximum, say x, $R\text{‘‘}\beta$ will be $\overrightarrow{R}\text{‘}x$. But if β has neither a limit nor a maximum, $R\text{‘‘}\beta$ will be what we may call an "irrational" segment of the series. We shall see at a later stage that the real numbers may be identified with the segments of the series of rationals, *i.e.* if R is the relation of less to greater among rationals, the real numbers will be all classes such as $R\text{‘‘}\beta$, for different values of β. The real numbers which correspond to rationals will be those resulting from a β which has a limit or maximum; the irrationals will be those resulting from a β which has no limit or maximum.

The present number may be divided into various sections, as follows: (1) First, we have various elementary properties of the terms defined at the beginning of the number; this section ends with $*37{\cdot}29$. (2) We have next a set of propositions dealing with relative products, and with such symbols as $P\text{‘‘}Q\text{‘‘}\gamma$, $P\text{‘‘}Q\text{‘‘‘}\kappa$, and so on. The central proposition here is

$*37{\cdot}33$. $\vdash . (P \mid Q)\text{‘‘}\gamma = P\text{‘‘}Q\text{‘‘}\gamma$

By the definition, $Q\text{‘‘‘}\kappa = Q_\epsilon\text{‘‘}\kappa$. Thus $P\text{‘‘}Q\text{‘‘‘}\kappa = (P \mid Q_\epsilon)\text{‘‘}\kappa$. This connects propositions concerning such symbols as $P\text{‘‘}Q\text{‘‘‘}\kappa$ with propositions concerning

relative products. This second section consists of the propositions from $*37\cdot3$ to $*37\cdot39$. (3) We have next a set of propositions on relations with limited domains and converse domains. The chief of these are

$*37\cdot401$. $\vdash . \mathrm{D}`(R \upharpoonright \beta) = R``\beta$

$*37\cdot412$. $\vdash . (R \upharpoonright \alpha)``\beta = R``(\alpha \cap \beta)$

$*37\cdot41$. $\vdash . \mathrm{D}`(R \llcorner \alpha) = \alpha \cap R``\alpha . \mathrm{G}`(R \llcorner \alpha) = \alpha \cap \breve{R}``\alpha$

These propositions on relations with limited domains and converse domains, together with certain others naturally connected with them, extend from $*37\cdot4$ to $*37\cdot52$. (4) We next have a number of very important propositions on the consequences of the hypothesis $\mathrm{E}!! R``\beta$, *i.e.* the hypothesis that, for any argument which is a member of β, R gives rise to a descriptive function $R`y$. The chief proposition in this section is

$*37\cdot6$. $\vdash : \mathrm{E}!! R``\beta . \supset . R``\beta = \hat{x}\{(\exists y) . y \in \beta . x = R`y\}$

Propositions with the hypothesis $\mathrm{E}!! R``\beta$ are applied to the cases of $\overrightarrow{R}$ and $\overleftarrow{R}$, in which the hypothesis is verified. This section extends from $*37\cdot6$ to $*37\cdot791$. (5) Finally, we have three propositions on the relative product of $\alpha \uparrow \beta$ with other relations. These propositions are useful in relation-arithmetic (Part IV).

The propositions of the present number which are most used in the sequel, apart from those already mentioned, are the following (omitting such as merely embody definitions):

$*37\cdot15$. $\vdash . R``\alpha \subset \mathrm{D}`R$

$*37\cdot16$. $\vdash . \breve{R}``\alpha \subset \mathrm{G}`R$

$*37\cdot2$. $\vdash : \alpha \subset \beta . \supset . P``\alpha \subset P``\beta$

$*37\cdot22$. $\vdash . P``(\alpha \cup \beta) = P``\alpha \cup P``\beta$

$*37\cdot25$. $\vdash . \mathrm{D}`R = R``\mathrm{G}`R . \mathrm{G}`R = \breve{R}``\mathrm{D}`R$

$*37\cdot26$. $\vdash . R``\beta = R``(\beta \cap \mathrm{G}`R)$

$*37\cdot265$. $\vdash . R``\alpha = R``(\alpha \cap C`R) . \breve{R}``\alpha = \breve{R}``(\alpha \cap C`R)$

$*37\cdot29$. $\vdash . R``\Lambda = \Lambda . \breve{R}``\Lambda = \Lambda$

$*37\cdot32$. $\vdash . \mathrm{D}`(P \mid Q) = P``\mathrm{D}`Q . \mathrm{G}`(P \mid Q) = \breve{Q}``\mathrm{G}`P$

$*37\cdot45$. $\vdash :. (y) . \mathrm{E}! R`y . \supset : \exists ! R``\beta . \equiv . \exists ! \beta$

$*37\cdot46$. $\vdash : x \in R``\alpha . \equiv . \exists ! \alpha \cap \overleftarrow{R}`x$

$*37\cdot61$. $\vdash :: \mathrm{E}!! R``\beta . \supset :. R``\beta \subset \alpha . \equiv : y \in \beta . \supset_y . R`y \in \alpha$

For example, let R be the relation of father to son, β the class of Etonians, α the class of rich men; then "$R``\beta \subset \alpha$" states "all fathers of Etonians are rich," while "$y \in \beta . \supset_y . R`y \in \alpha$" states "if a boy is an Etonian, his father

must be rich." In virtue of the above proposition, these two statements are equivalent.

***37·62.** $\vdash : \text{E} ! R`y \,.\, y \in \alpha \,.\, \supset \,.\, R`y \in R``\alpha$

***37·63.** $\vdash :: \text{E} !! R``\alpha \,.\, \supset :. \, x \in R``\alpha \,.\, \supset_x \,.\, \psi x : \equiv : y \in \alpha \,.\, \supset_y \,.\, \psi(R`y)$

***37·01.** $R``\beta = \hat{x}\{(\exists y) \,.\, y \in \beta \,.\, xRy\}$ Df

***37·02.** $R_\epsilon = \hat{\alpha}\hat{\beta}\,(\alpha = R``\beta)$ Df

***37·03.** $\breve{R}_\epsilon = \text{Cnv}`(R_\epsilon)$ Df

***37·04.** $R```\kappa = R_\epsilon``\kappa$ Df

***37·05.** $\text{E} !! R``\beta \,.=: y \in \beta \,.\, \supset_y \,.\, \text{E} ! R`y$ Df

***37·1.** $\vdash : x \in R``\beta \,.\equiv.\, (\exists y) \,.\, y \in \beta \,.\, xRy$ [*20·3 . (*37·01)]

***37·101.** $\vdash : \alpha R_\epsilon \beta \,.\equiv.\, \alpha = R``\beta$ [*21·3 . (*37·02)]

***37·102.** $\vdash : \alpha(\breve{R})_\epsilon \beta \,.\equiv.\, \alpha = \breve{R}``\beta$ [*37·101]

***37·103.** $\vdash : \alpha \in R```\kappa \,.\equiv.\, (\exists \beta) \,.\, \beta \in \kappa \,.\, \alpha = R``\beta \,.\equiv.\, \alpha \in R_\epsilon``\kappa$
[*37·1·101 . (*37·04)]

***37·104.** $\vdash :. \text{E} !! R``\beta \,.\equiv: y \in \beta \,.\, \supset_y \,.\, \text{E} ! R`y$ [*4·2 . (*37·05)]

***37·105.** $\vdash : x \in \breve{R}``\beta \,.\equiv.\, (\exists y) \,.\, y \in \beta \,.\, yRx$ [*37·1 . *31·11]

***37·106.** $\vdash :. \text{E} ! R`x \,.\, \supset : x \in \breve{R}``\beta \,.\equiv.\, R`x \in \beta$

Dem.

$\vdash \,.\, *37{\cdot}105 \,.\, *30{\cdot}4 \,.\, \supset \vdash :. \text{Hp} \,.\, \supset : x \in \breve{R}``\beta \,.\equiv.\, (\exists y) \,.\, y \in \beta \,.\, y = R`x \,.$
[*14·205] $\equiv .\, R`x \in \beta :. \supset \vdash \,.\, \text{Prop}$

***37·11.** $\vdash \,.\, R_\epsilon`\beta = R``\beta$ [*37·101 . *30·3]

***37·111.** $\vdash \,.\, \text{E} ! R_\epsilon`\beta$ [*37·11 . *14·21]

***37·12.** $\vdash : (\beta) \,.\, R``\beta = Q`\beta \,.\equiv.\, R_\epsilon = Q$ [*30·42 . *37·111·11]

***37·13.** $\vdash : P = Q \,.\, \supset \,.\, P``\beta = Q``\beta$

Dem.

$\vdash \,.\, *21{\cdot}43 \,.\, \supset \vdash :. \text{Hp} \,.\, \supset : xPy \,.\equiv_{x,y}.\, xQy :$
[Fact] $\supset : y \in \beta \,.\, xPy \,.\equiv_{x,y}.\, y \in \beta \,.\, xQy :$
[*10·281] $\supset : (\exists y) \,.\, y \in \beta \,.\, xPy \,.\equiv_x.\, (\exists y) \,.\, y \in \beta \,.\, xQy :$
[*37·1] $\supset : x \in P``\beta \,.\equiv_x.\, x \in Q``\beta :. \supset \vdash \,.\, \text{Prop}$

***37·131.** $\vdash : P = Q \,.\, \supset \,.\, P_\epsilon = Q_\epsilon$

Dem.

$\vdash \,.\, *37{\cdot}13 \,.\, \supset \vdash :. \text{Hp} \,.\, \supset : \alpha = P``\beta \,.\equiv_{\alpha,\beta}.\, \alpha = Q``\beta :$
[*37·101] $\supset : \alpha P_\epsilon \beta \,.\equiv_{\alpha,\beta}.\, \alpha Q_\epsilon \beta :. \supset \vdash \,.\, \text{Prop}$

***37·14.** $\vdash : P = Q . \equiv . P_\epsilon = Q_\epsilon$

Dem.

$\vdash . *37·101 . *21·15 . \supset$

$\vdash :. P_\epsilon = Q_\epsilon . \quad \equiv : \alpha = P``\beta . \equiv_{\alpha,\beta} . \alpha = Q``\beta :$

$[*13·183] \quad \equiv : (\beta) . P``\beta = Q``\beta :$

$[*37·1.*20·15] \equiv : (\beta, x) : (\exists y) . y \epsilon \beta . xPy . \equiv . (\exists y) . y \epsilon \beta . xQy :$

$[*10·1] \quad \supset : (x) : (\exists y) . y \epsilon \hat{z}(z = w) . xPy . \equiv . (\exists y) . y \epsilon \hat{z}(z = w) . xQy :$

$[*20·3] \quad \supset : (x) : (\exists y) . y = w . xPy . \equiv . (\exists y) . y = w . xQy :$

$[*13·195] \quad \supset : (x) : xPw . \equiv . xQw$ (1)

$\vdash . (1) . *10·11·21 . *11·2 . \supset$

$\vdash :. P_\epsilon = Q_\epsilon . \quad \supset : (x, w) : xPw . \equiv . xQw :$

$[*21·43] \quad \supset : P = Q$ (2)

$\vdash . (2) . *37·131 . \supset \vdash . \text{Prop}$

***37·15.** $\vdash . R``\alpha \subset \text{D}`R$

Dem.

$\vdash . *37·1 . \supset \vdash : x \epsilon R``\alpha . \supset . (\exists y) . y \epsilon \alpha . xRy .$

$[*10·5] \quad \supset . (\exists y) . xRy .$

$[*33·13] \quad \supset . x \epsilon \text{D}`R : \supset \vdash . \text{Prop}$

***37·16.** $\vdash . \breve{R}``\alpha \subset \text{G}`R \quad [*37·15 \frac{\breve{R}}{R} . *33·2]$

***37·17.** $\vdash :. R``\beta \subset \alpha . \equiv : y \epsilon \beta . xRy . \supset_{x,y} . x \epsilon \alpha$

Dem.

$\vdash . *37·1 . \supset \vdash :. R``\beta \subset \alpha . \equiv : (\exists y) . y \epsilon \beta . xRy . \supset_x . x \epsilon \alpha :$

$[*10·23] \quad \equiv : y \epsilon \beta . xRy . \supset_{x,y} . x \epsilon \alpha :. \supset \vdash . \text{Prop}$

***37·171.** $\vdash :. \breve{R}``\alpha \subset \beta . \equiv : x \epsilon \alpha . xRy . \supset_{x,y} . y \epsilon \beta$

Dem.

$\vdash . *37·105 . \supset \vdash :. \breve{R}``\alpha \subset \beta . \equiv : (\exists x) . x \epsilon \alpha . xRy . \supset_y . y \epsilon \beta :$

$[*10·23] \quad \equiv : x \epsilon \alpha . xRy . \supset_{x,y} . y \epsilon \beta :. \supset \vdash . \text{Prop}$

***37·18.** $\vdash : y \epsilon \beta . \supset . \overrightarrow{R}`y \subset R``\beta$

Dem.

$\vdash . *32·18 . \supset \vdash :. \text{Hp} . \supset : x \epsilon \overrightarrow{R}`y . \supset . xRy . y \epsilon \beta .$

$[*37·1] \quad \supset . x \epsilon R``\beta :. \supset \vdash . \text{Prop}$

***37·181.** $\vdash : x \epsilon \alpha . \supset . \overleftarrow{R}`x \subset \breve{R}``\alpha$ [Proof as in *37·18]

***37·2.** $\vdash : \alpha \subset \beta . \supset . P``\alpha \subset P``\beta$

Dem.

$\vdash . *22·1 . \supset \vdash :. \text{Hp} . \supset : y \epsilon \alpha . \supset_y . y \epsilon \beta :$

$[*10·31] \quad \supset : y \epsilon \alpha . xPy . \supset_y . y \epsilon \beta . xPy :$

$[*10·28] \quad \supset : (\exists y) . y \epsilon \alpha . xPy . \supset . (\exists y) . y \epsilon \beta . xPy :$

$[*37·1] \quad \supset : x \epsilon P``\alpha . \supset . x \epsilon P``\beta :. \supset \vdash . \text{Prop}$

The above proposition (∗37·2) is one of the forms of asyllogistic inference due to Leibniz's teacher Jungius. The instance given by Jungius is: "Circulus est figura; ergo qui circulum describit, is figuram describit*." Here the class of circles is our α, the class of figures is our β, and the relation of describing is our P.

∗37·201. $\vdash : P \Subset Q . \supset . P``\alpha \subset Q``\alpha$ [Similar proof]

∗37·202. $\vdash : \alpha \subset \beta . P \Subset Q . \supset . P``\alpha \subset Q``\beta$ [∗37·2·201]

∗37·21. $\vdash . P``(\alpha \cap \beta) \subset P``\alpha \cap P``\beta$

Dem.

$\vdash . ∗37·1 . \supset \vdash :. x \in P``(\alpha \cap \beta) . \equiv : (\exists y) . y \in \alpha \cap \beta . xPy :$
[∗22·33] $\equiv : (\exists y) . y \in \alpha . y \in \beta . xPy :$
[∗10·5] $\supset : (\exists y) . y \in \alpha . xPy : (\exists y) . y \in \beta . xPy :$
[∗37·1] $\supset : x \in P``\alpha . x \in P``\beta :$
[∗22·33] $\supset : x \in P``\alpha \cap P``\beta :. \supset \vdash .$ Prop

∗37·211. $\vdash . (P \dot{\cap} Q)``\alpha \subset P``\alpha \cap Q``\alpha$ [Similar proof]

∗37·212. $\vdash . (P \dot{\cap} Q)``(\alpha \cap \beta) \subset P``\alpha \cap P``\beta \cap Q``\alpha \cap Q``\beta$ [∗37·21·211]

∗37·22. $\vdash . P``(\alpha \cup \beta) = P``\alpha \cup P``\beta$

This proposition is very frequently used. The fact that here we have identity, while in ∗37·21 we only have inclusion, is due to the fact that ∗10·42 states an equivalence, while ∗10·5 only states an implication.

Dem.

$\vdash . ∗37·1 . \supset \vdash :. x \in P``(\alpha \cup \beta) . \equiv : (\exists y) . y \in \alpha \cup \beta . xPy :$
[∗22·34] $\equiv : (\exists y) : y \in \alpha . \vee . y \in \beta : xPy :$
[∗4·4] $\equiv : (\exists y) : y \in \alpha . xPy . \vee . y \in \beta . xPy :$
[∗10·42] $\equiv : (\exists y) . y \in \alpha . xPy : \vee : (\exists y) . y \in \beta . xPy :$
[∗37·1] $\equiv : x \in P``\alpha . \vee . x \in P``\beta :$
[∗22·34] $\equiv : x \in P``\alpha \cup P``\beta :. \supset \vdash .$ Prop

∗37·221. $\vdash . (P \dot{\cup} Q)``\alpha = P``\alpha \cup Q``\alpha$ [Similar proof]

∗37·222. $\vdash . (P \dot{\cup} Q)``(\alpha \cup \beta) = P``\alpha \cup P``\beta \cup Q``\alpha \cup Q``\beta$ [∗37·22·221]

∗37·23. $\vdash . \mathrm{D}`R_\epsilon = \hat{\alpha} \{(\exists \beta) . \alpha = R``\beta\}$ [∗37·101 . ∗33·11]

∗37·231. $\vdash . \mathrm{C\!\!I}`R_\epsilon = \mathrm{Cls}$

The type of "Cls" here is that type whose members are of the same type as $\mathrm{C\!\!I}`R$. In the proof, use is made of the convention that a Greek letter always stands for an expression of the form $\hat{z}(\phi ! z)$.

Dem.

$\vdash . ∗37·101 . \quad \supset \vdash : \alpha R_\epsilon \hat{z}(\phi ! z) . \equiv . \alpha = R``\hat{z}(\phi ! z) :$
[∗10·11·281] $\supset \vdash : (\exists \alpha) . \alpha R_\epsilon \hat{z}(\phi ! z) . \equiv . (\exists \alpha) . \alpha = R``\hat{z}(\phi ! z) :$
[∗33·131] $\supset \vdash : \hat{z}(\phi ! z) \in \mathrm{C\!\!I}`R_\epsilon . \equiv . (\exists \alpha) . \alpha = R``\hat{z}(\phi ! z)$ (1)

* We quote from Couturat, *La Logique de Leibniz*, Chapter III, § 15 (p. 75 *n.*).

$\vdash . *20\cdot2 . (*37\cdot01) . \supset \vdash : \hat{x}\{(\exists y) . y \epsilon \hat{z}(\phi ! z) . xRy\} = R``\hat{z}(\phi ! z) :$

$[*10\cdot11\cdot24] \quad \supset \vdash : (\phi) : (\exists \alpha) . \alpha = R``\hat{z}(\phi ! z)$ (2)

$\vdash . (1) . (2) . *2\cdot02 . \supset \vdash : \hat{z}(\phi ! z) \epsilon \mathrm{Cls} . \supset . \hat{z}(\phi ! z) \epsilon \text{Œ}`R_\epsilon$ (3)

$\vdash . *20\cdot41 . *2\cdot02 . \supset \vdash : \hat{z}(\phi ! z) \epsilon \text{Œ}`R_\epsilon . \supset . \hat{z}(\phi ! z) \epsilon \mathrm{Cls}$ (4)

$\vdash . (3) . (4) . \quad \supset \vdash . \mathrm{Prop}$

As appears in the above proof, it is necessary, when a proposition containing "Cls" is to be proved, to abandon the notation with Greek letters, and revert to the explicit functional notation.

***37·24.** $\vdash : \alpha \epsilon \mathrm{D}`R_\epsilon . \supset . \alpha \subset \mathrm{D}`R$

Dem.

$\vdash . *33\cdot13 . *37\cdot101 . \supset \vdash :: \alpha \epsilon \mathrm{D}`R_\epsilon . \equiv :. (\exists \beta) . \alpha = R``\beta :.$

$[*20\cdot33 . *37\cdot1] \quad \equiv :. (\exists \beta) : x \epsilon \alpha . \equiv_x . (\exists y) . y \epsilon \beta . xRy :.$

$[*11\cdot61] \quad \supset :. x \epsilon \alpha . \supset_x : (\exists \beta, y) . y \epsilon \beta . xRy :$

$[*11\cdot23] \quad \supset_x : (\exists y, \beta) . y \epsilon \beta . xRy :$

$[*11\cdot55] \quad \supset_x : (\exists y) : xRy : (\exists \beta) . y \epsilon \beta :$

$[*10\cdot5] \quad \supset_x : (\exists y) . xRy :$

$[*33\cdot13] \quad \supset_x : x \epsilon \mathrm{D}`R :: \supset \vdash . \mathrm{Prop}$

***37·25.** $\vdash . \mathrm{D}`R = R``\text{Œ}`R . \text{Œ}`R = \breve{R}``\mathrm{D}`R$

Dem.

$\vdash . *33\cdot13 . \supset \vdash : x \epsilon \mathrm{D}`R . \equiv . (\exists y) . xRy .$

$[*33\cdot14 . *4\cdot71] \quad \equiv . (\exists y) . y \epsilon \text{Œ}`R . xRy .$

$[*37\cdot1] \quad \equiv . x \epsilon R``\text{Œ}`R$ (1)

$\vdash . *33\cdot131 . \supset \vdash : y \epsilon \text{Œ}`R . \equiv . (\exists x) . xRy .$

$[*33\cdot14 . *4\cdot71] \quad \equiv . (\exists x) . x \epsilon \mathrm{D}`R . xRy .$

$[*37\cdot105] \quad \equiv . y \epsilon \breve{R}``\mathrm{D}`R$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***37·26.** $\vdash . R``\beta = R``(\beta \cap \text{Œ}`R)$

Dem.

$\vdash . *37\cdot1 . \supset \vdash :. x \epsilon R``\beta . \equiv : (\exists y) . y \epsilon \beta . xRy :$

$[*33\cdot14 . *4\cdot71] \quad \equiv : (\exists y) . y \epsilon \beta . y \epsilon \text{Œ}`R . xRy :$

$[*22\cdot33] \quad \equiv : (\exists y) . y \epsilon \beta \cap \text{Œ}`R . xRy :$

$[*37\cdot1] \quad \equiv : x \epsilon R``(\beta \cap \text{Œ}`R) :. \supset \vdash . \mathrm{Prop}$

***37·261.** $\vdash . \breve{R}``\beta = \breve{R}``(\beta \cap \mathrm{D}`R)$ [*37·26 . *33·21]

***37·262.** $\vdash : \alpha \cap \text{Œ}`R = \beta \cap \text{Œ}`R . \supset . R``\alpha = R``\beta$ [*37·26]

***37·263.** $\vdash : \alpha \cap \mathrm{D}`R = \beta \cap \mathrm{D}`R . \supset . \breve{R}``\alpha = \breve{R}``\beta$ [*37·261]

***37·264.** $\vdash : \exists ! \alpha \cap R``\beta . \equiv . (\exists x, y) . x \epsilon \alpha . y \epsilon \beta . xRy . \equiv . \mathrm{E} ! \beta \cap \breve{R}``\alpha$

Dem.

$\vdash . *22\cdot33 . *37\cdot1 . \supset \vdash :. \exists ! \alpha \cap R``\beta . \equiv : (\exists x) : x \epsilon \alpha : (\exists y) . y \epsilon \beta . xRy :$ (1)

[*11·55] $\equiv : (\exists x, y) . x \in \alpha . y \in \beta . xRy$ (2)

$\vdash . (1) . *11\cdot6 . \supset \vdash :. \exists ! \alpha \cap R``\beta . \equiv : (\exists y) : y \in \beta : (\exists x) . x \in \alpha . xRy :$

[*37·105] $\equiv : (\exists y) . y \in \beta . y \in \breve{R}``\alpha :$

[*22·33] $\equiv : \exists ! \beta \cap \breve{R}``\alpha$ (3)

$\vdash . (2) . (3) . \supset \vdash .$ Prop

***37·265.** $\vdash . R``\alpha = R``(\alpha \cap C`R) . \breve{R}``\alpha = \breve{R}``(\alpha \cap C`R)$

Dem.

$\vdash . *33\cdot161 . *22\cdot621 . \supset \vdash . \text{Ɑ}`R = C`R \cap \text{Ɑ}`R .$

[*22·481] $\supset \vdash . \alpha \cap \text{Ɑ}`R = \alpha \cap C`R \cap \text{Ɑ}`R .$

[*37·262] $\supset \vdash . R``\alpha = R``(\alpha \cap C`R)$ (1)

$\vdash . (1) . *33\cdot22 . \supset \vdash .$ Prop

***37·27.** $\vdash : \text{Ɑ}`R \subset \beta . \supset . \text{D}`R = R``\beta$ [*22·621 . *37·25·26]

***37·271.** $\vdash : \text{D}`R \subset \alpha . \supset . \text{Ɑ}`R = \breve{R}``\alpha$ [*22·621 . *37·25·261]

***37·28.** $\vdash . R``\text{V} = \text{D}`R . \breve{R}``\text{V} = \text{Ɑ}`R$ [*37·27·271 . *24·11]

***37·29.** $\vdash . R``\Lambda = \Lambda . \breve{R}``\Lambda = \Lambda$

Dem.

$\vdash . *10\cdot5 . \supset \vdash : (\exists y) . y \in \Lambda . xRy . \supset . (\exists y) . y \in \Lambda$ (1)

$\vdash . (1) . \text{Transp} . *24\cdot53 . \supset \vdash . \sim(\exists y) . y \in \Lambda . xRy .$

[*37·1] $\supset \vdash . \sim \exists ! R``\Lambda .$

[*24·51] $\supset \vdash . R``\Lambda = \Lambda$ (2)

$\vdash . (2) \frac{\breve{R}}{R} .$ $\supset \vdash . \breve{R}``\Lambda = \Lambda$ (3)

$\vdash . (2) . (3) . \supset \vdash .$ Prop

***37·3.** $\vdash . \{\text{sg}`(P \mid Q)\}`z = P``\overrightarrow{Q}`z$

Dem.

$\vdash . *32\cdot23\cdot13 . \supset$

$\vdash . \{\text{sg}`(P \mid Q)\}`z = \hat{x}\,\{x\,(P \mid Q)\,z\}$

[*34·1] $= \hat{x}\,\{(\exists y) . xPy . yQz\}$

[*32·18] $= \hat{x}\,\{(\exists y) . xPy . y \in \overrightarrow{Q}`z\}$

[(*37·01)] $= P``\overrightarrow{Q}`z . \supset \vdash .$ Prop

***37·301.** $\vdash . \{\text{gs}`(P \mid Q)\}`x = \breve{Q}``\overleftarrow{P}`x$ [Similar proof]

***37·302.** $\vdash : R = P \mid Q . \supset . \overrightarrow{R}`z = P``\overrightarrow{Q}`z . \overleftarrow{R}`x = \breve{Q}``\overleftarrow{P}`x$

[*37·3·301 . *32·23·231·16]

***37·31.** $\vdash . \text{sg}`(P \mid Q) = P_\epsilon \mid \overrightarrow{Q}$

Dem.

$\vdash . *37\cdot11\cdot3 . \supset \vdash . (z) . \{\text{sg}`(P \mid Q)\}`z = P_\epsilon`\overrightarrow{Q}`z$ (1)

$\vdash . (1) . *34\cdot42 . \supset \vdash .$ Prop

***37·311.** $\vdash . \mathrm{gs}'(P \mid Q) = (\breve{Q})_\epsilon \mid \overleftarrow{P}$ [Similar proof]

***37·32.** $\vdash . \mathrm{D}'(P \mid Q) = P``\mathrm{D}'Q . \text{Ɑ}'(P \mid Q) = \breve{Q}``\text{Ɑ}'P$

Dem.

$\vdash . *33·13 . *34·1 . \supset$

$\vdash :. x \in \mathrm{D}'(P \mid Q) . \equiv : (\exists z) : (\exists y) . xPy . yQz :$

$[*11·23] \quad \equiv : (\exists y) : (\exists z) . xPy . yQz :$

$[*11·55] \quad \equiv : (\exists y) : xPy : (\exists z) . yQz :$

$[*33·13] \quad \equiv : (\exists y) . xPy . y \in \mathrm{D}'Q :$

$[*37·1] \quad \equiv : x \in P``\mathrm{D}'Q$ (1)

$\vdash . (1) . *10·11 . *20·43 . \supset$

$\vdash . \mathrm{D}'(P \mid Q) = P``\mathrm{D}'Q$ (2)

$\vdash . *33·2 . \quad \supset \vdash . \text{Ɑ}'(P \mid Q) = \mathrm{D}'\mathrm{Cnv}'(P \mid Q)$

$[*34·2] \quad = \mathrm{D}'(\breve{Q} \mid \breve{P})$

$[(2)] \quad = \breve{Q}``\mathrm{D}'\breve{P}$

$[*33·2] \quad = \breve{Q}``\text{Ɑ}'P$ (3)

$\vdash . (2) . (3) . \supset \vdash .$ Prop

***37·321.** $\vdash : \text{Ɑ}'P \subset \mathrm{D}'Q . \supset . \mathrm{D}'(P \mid Q) = \mathrm{D}'P$ [*37·32·27]

***37·322.** $\vdash : \mathrm{D}'Q \subset \text{Ɑ}'P . \supset . \text{Ɑ}'(P \mid Q) = \text{Ɑ}'Q$ [*37·32·271]

***37·323.** $\vdash : \text{Ɑ}'P = \mathrm{D}'Q . \supset . \mathrm{D}'(P \mid Q) = \mathrm{D}'P . \text{Ɑ}'(P \mid Q) = \text{Ɑ}'Q$ [*37·321·322]

***37·33.** $\vdash . (P \mid Q)``\gamma = P``Q``\gamma$

Dem.

$\vdash . *37·1 . \supset \vdash :. x \in (P \mid Q)``\gamma . \equiv : (\exists z) . z \in \gamma . x(P \mid Q)z :$

$[*34·1 . *11·55] \quad \equiv : (\exists z, y) . z \in \gamma . xPy . yQz :$

$[*11·23] \quad \equiv : (\exists y, z) . xPy . yQz . z \in \gamma :$

$[*11·55] \quad \equiv : (\exists y) : xPy : (\exists z) . yQz . z \in \gamma :$

$[*37·1] \quad \equiv : (\exists y) . xPy . y \in Q``\gamma :$

$[*37·1] \quad \equiv : x \in P``Q``\gamma :. \supset \vdash .$ Prop

***37·34.** $\vdash . (P \mid Q)_\epsilon = P_\epsilon \mid Q_\epsilon$

Dem.

$\vdash . *37·11 . \supset \vdash . (P \mid Q)_\epsilon{}'\gamma = (P \mid Q)``\gamma$

$[*37·33] \quad = P``Q``\gamma$

$[*37·11] \quad = P_\epsilon{}'Q_\epsilon{}'\gamma$ (1)

$\vdash . (1) . *10·11 . *34·42 . \supset \vdash .$ Prop

***37·341.** $\vdash . \{\mathrm{Cnv}'(P \mid Q)\}_\epsilon = (\breve{Q})_\epsilon \mid (\breve{P})_\epsilon$ [*34·2 . *37·34]

***37·35.** $\vdash : (z) . R'z = P'Q'z . \supset . (\gamma) . R``\gamma = P``Q``\gamma$

Dem.

$\vdash . *34·42 . \supset \vdash : \mathrm{Hp} . \supset . R = P \mid Q .$

$[*37·13] \quad \supset . R``\gamma = (P \mid Q)``\gamma$

$[*37·33] \quad = P``Q``\gamma : \supset \vdash .$ Prop

∗37·351. $\vdash : (\alpha) . R'\alpha = P'Q''\alpha . \supset . (\kappa) . R''\kappa = P''Q'''\kappa$

$\left[*37{\cdot}35 \dfrac{Q_\epsilon}{Q} . *37{\cdot}11 . (*37{\cdot}04) \right]$

∗37·352. $\vdash : (\alpha) . R''\alpha = P'Q''\alpha . \supset . (\kappa) . R'''\kappa = P''Q'''\kappa$

$\left[*37{\cdot}351 \dfrac{R_\epsilon}{R} . *37{\cdot}11 . (*37{\cdot}04) \right]$

∗37·353. $\vdash : (z) . R'S'z = P'Q'z . \supset . (\gamma) . R''S''\gamma = P''Q''\gamma$

Dem.

$\vdash . *14{\cdot}21 . \supset \vdash : \text{Hp} . \supset . (z) . \text{E} ! R'S'z .$
[∗34·41] $\supset . (z) . R'S'z = (R \mid S)'z .$
[∗14·131·144] $\supset . (z) . (R \mid S)'z = P'Q'z .$
[∗37·35] $\supset . (\gamma) . (R \mid S)''\gamma = P''Q''\gamma .$
[∗37·33] $\supset . (\gamma) . R''S''\gamma = P''Q''\gamma : \supset \vdash .$ Prop

∗37·354. $\vdash : (\alpha) . R'S'\alpha = P'Q''\alpha . \supset . (\kappa) . R''S''\kappa = P''Q'''\kappa$ $\left[*37{\cdot}353 \dfrac{Q_\epsilon}{Q} \right]$

∗37·355. $\vdash : (z) . R'S'z = P''Q'z . \supset . (\gamma) . R''S''\gamma = P'''Q''\gamma$ $\left[*37{\cdot}353 \dfrac{P_\epsilon}{P} \right]$

∗37·36. $\vdash . \text{D}'R^2 = R''\text{D}'R . \text{Cl}'R^2 = \breve{R}''\text{Cl}'R$ [∗37·32]

∗37·37. $\vdash . (R^2)_\epsilon = (R_\epsilon)^2$ [∗37·34]

∗37·371. $R_\epsilon{}^2 = (R_\epsilon)^2$ Df

This definition serves merely for the avoidance of brackets. Like ∗37·03, this definition will be extended to all suffixes.

∗37·38. $\vdash . \overrightarrow{R^2}'x = R''\overrightarrow{R}'x$ [∗37·3]

∗37·39. $\vdash . R^{2}{}''\alpha = R''R''\alpha$ [∗37·33]

∗37·4. $\vdash . \text{Cl}'(\alpha \upharpoonleft R) = \breve{R}''\alpha$

Dem.

$\vdash . *33{\cdot}131 . *35{\cdot}1 . \supset \vdash : y \,\epsilon\, \text{Cl}'(\alpha \upharpoonleft R) . \equiv . (\exists x) . x \,\epsilon\, \alpha . xRy .$
[∗37·105] $\equiv . y \,\epsilon\, \breve{R}''\alpha : \supset \vdash .$ Prop

∗37·401. $\vdash . \text{D}'(R \upharpoonright \beta) = R''\beta$ [Similar proof]

∗37·402. $\vdash . \text{D}'(\alpha \upharpoonleft R \upharpoonright \beta) = \alpha \cap R''\beta . \text{Cl}'(\alpha \upharpoonleft R \upharpoonright \beta) = \beta \cap \breve{R}''\alpha$

Dem.

$\vdash . *33{\cdot}13 . *35{\cdot}102 . \supset$
$\vdash :. x \,\epsilon\, \text{D}'(\alpha \upharpoonleft R \upharpoonright \beta) . \equiv : (\exists y) . x \,\epsilon\, \alpha . xRy . y \,\epsilon\, \beta :$
[∗10·35] $\equiv : x \,\epsilon\, \alpha : (\exists y) . xRy . y \,\epsilon\, \beta :$
[∗37·1] $\equiv : x \,\epsilon\, \alpha . x \,\epsilon\, R''\beta :$
[∗22·33] $\equiv : x \,\epsilon\, \alpha \cap R''\beta$ (1)

Similarly

$\vdash : y \,\epsilon\, \text{Cl}'(\alpha \upharpoonleft R \upharpoonright \beta) . \equiv . y \,\epsilon\, \beta \cap \breve{R}''\alpha$ (2)

$\vdash . (1) . (2) . \supset \vdash .$ Prop

***37·41.** $\vdash : \mathrm{D}\text{‘}(R \mathbin{\lceil\!\!\upharpoonright} \alpha) = \alpha \cap R\text{“}\alpha \,.\, \text{Ɑ}\text{‘}(R \mathbin{\lceil\!\!\upharpoonright} \alpha) = \alpha \cap \breve{R}\text{“}\alpha$ [*37·402 . *36·11]

***37·411.** $\vdash . (\alpha \upharpoonleft R)\text{“}\beta = \mathrm{D}\text{‘}(\alpha \upharpoonleft R \upharpoonright \beta) = \alpha \cap R\text{“}\beta$

Dem.

$\vdash . *37{\cdot}401 . \supset \vdash . (\alpha \upharpoonleft R)\text{“}\beta = \mathrm{D}\text{‘}(\alpha \upharpoonleft R) \upharpoonright \beta$

[*35·21] $= \mathrm{D}\text{‘}(\alpha \upharpoonleft R \upharpoonright \beta)$ (1)

$\vdash . (1) . *37{\cdot}402 . \supset \vdash .$ Prop

***37·412.** $\vdash . (R \upharpoonright \alpha)\text{“}\beta = R\text{“}(\alpha \cap \beta)$

Dem.

$\vdash . *37{\cdot}401 . \supset \vdash . (R \upharpoonright \alpha)\text{“}\beta = \mathrm{D}\text{‘}(R \upharpoonright \alpha) \upharpoonright \beta$

[*35·31] $= \mathrm{D}\text{‘}R \upharpoonright (\alpha \cap \beta)$

[*37·401] $= R\text{“}(\alpha \cap \beta) . \supset \vdash .$ Prop

***37·413.** $\vdash . (R \mathbin{\lceil\!\!\upharpoonright} \alpha)\text{“}\beta = \alpha \cap R\text{“}(\alpha \cap \beta)$

Dem.

$\vdash . *37{\cdot}411 . *35{\cdot}21 . \supset \vdash . (R \mathbin{\lceil\!\!\upharpoonright} \alpha)\text{“}\beta = \alpha \cap (R \upharpoonright \alpha)\text{“}\beta$

[*37·412] $= \alpha \cap R\text{“}(\alpha \cap \beta) . \supset \vdash .$ Prop

***37·42.** $\vdash : R\text{“}\beta \subset \alpha . \supset . (\alpha \upharpoonleft R)\text{“}\beta = R\text{“}\beta$ [*37·411 . *22·621]

***37·421.** $\vdash : \beta \subset \alpha . \supset . (R \upharpoonright \alpha)\text{“}\beta = R\text{“}\beta$ [*37·412 . *22·621]

***37·43.** $\vdash :. \beta \subset \text{Ɑ}\text{‘}R . \supset : \exists ! R\text{“}\beta . \equiv . \exists ! \beta$

Dem.

$\vdash . *37{\cdot}401 . *35{\cdot}65 . \supset \vdash :. \mathrm{Hp} . \supset : R\text{“}\beta = \mathrm{D}\text{‘}(R \upharpoonright \beta) . \beta = \text{Ɑ}\text{‘}(R \upharpoonright \beta)$ (1)

$\vdash . (1) . *33{\cdot}24 . \quad \supset \vdash .$ Prop

***37·431.** $\vdash :. \alpha \subset \mathrm{D}\text{‘}R . \supset : \exists ! \breve{R}\text{“}\alpha . \equiv . \exists ! \alpha$ [Proof as in *37·43]

***37·44.** $\vdash :. \text{Ɑ}\text{‘}R = \mathrm{V} . \supset : \exists ! R\text{“}\beta . \equiv . \exists ! \beta$ [*37·43 . *24·11]

***37·441.** $\vdash :. \mathrm{D}\text{‘}R = \mathrm{V} . \supset : \exists ! \breve{R}\text{“}\alpha . \equiv . \exists ! \alpha$ [Proof as in *37·44]

***37·45.** $\vdash :. (y) . \mathrm{E} ! R\text{‘}y . \supset : \exists ! R\text{“}\beta . \equiv . \exists ! \beta$ [*33·431 . *37·43]

***37·451.** $\vdash :. (x) . \mathrm{E} ! \breve{R}\text{‘}x . \supset : \exists ! \breve{R}\text{“}\alpha . \equiv . \exists ! \alpha$ [Proof as in *37·45]

***37·46.** $\vdash : x \in R\text{“}\alpha . \equiv . \exists ! \alpha \cap \overleftarrow{R}\text{‘}x$ [*37·1 . *32·181]

***37·461.** $\vdash : x \sim\in R\text{“}\alpha . \equiv . \alpha \cap \overleftarrow{R}\text{‘}x = \Lambda . \equiv . \overleftarrow{R}\text{‘}x \subset -\alpha$ [*37·46 . *24·311]

***37·462.** $\vdash : x \sim\in \breve{R}\text{“}\alpha . \equiv . \alpha \cap \overrightarrow{R}\text{‘}x = \Lambda . \equiv . \overrightarrow{R}\text{‘}x \subset -\alpha$ [*37·461 . *32·241]

***37·47.** $\vdash : \exists ! \alpha . \equiv . \exists ! R\text{“‘}\alpha . \equiv . \exists ! \breve{R}\text{“‘}\alpha$

Dem.

$\vdash . *37{\cdot}45{\cdot}111 . \supset \vdash : \exists ! \alpha . \equiv . \exists ! R_\epsilon\text{“}\alpha .$

[(*37·04)] $\equiv . \exists ! R\text{“‘}\alpha$ (1)

$\vdash . (1) \dfrac{\breve{R}}{R} . \quad \supset \vdash : \exists ! \alpha . \equiv . \exists ! \breve{R}\text{“‘}\alpha$ (2)

$\vdash . (1) . (2) . \quad \supset \vdash .$ Prop

$*37{\cdot}5$. $\vdash : (\beta) . P``\beta = Q`\beta . \supset . (\kappa) . P```\kappa = Q``\kappa$

Dem.

$\vdash . *37{\cdot}12 . \supset \vdash : \text{Hp} . \supset . P_\epsilon = Q .$

$[*37{\cdot}13] \quad \supset . P_\epsilon``\kappa = Q``\kappa .$

$[(*37{\cdot}04)] \quad \supset . P```\kappa = Q``\kappa : \supset \vdash . \text{Prop}$

$*37{\cdot}501$. $\vdash . \beta \cap \text{Œ}`R \subset \breve{R}``R``\beta$

Dem.

$\vdash . *37{\cdot}1 . *10{\cdot}24 . \supset \vdash : y \in \beta . xRy . \supset . x \in R``\beta :$

$[\text{Exp}.*10{\cdot}11{\cdot}21] \quad \supset \vdash :. y \in \beta . \supset : xRy . \supset_x : x \in R``\beta :$

$[*4{\cdot}7] \quad \supset : xRy . \supset_x . xRy . x \in R``\beta :$

$[*10{\cdot}28] \quad \supset : (\exists x) . xRy . \supset . (\exists x) . xRy . x \in R``\beta :$

$[*33{\cdot}131 . *37{\cdot}105] \quad \supset : y \in \text{Œ}`R . \supset . y \in \breve{R}``R``\beta \quad (1)$

$\vdash . (1) . \text{Imp} . *22{\cdot}33 . \supset$

$\vdash : y \in \beta \cap \text{Œ}`R . \supset . y \in \breve{R}``R``\beta : \supset \vdash . \text{Prop}$

$*37{\cdot}502$. $\vdash . \alpha \cap \text{D}`R \subset R``\breve{R}``\alpha$ [Similar proof]

$*37{\cdot}51$. $\vdash : \beta \subset \text{Œ}`R . \equiv . \beta \subset \breve{R}``R``\beta$

Dem.

$\vdash . *37{\cdot}501 . *22{\cdot}621 . \supset \vdash : \beta \subset \text{Œ}`R . \supset . \beta \subset \breve{R}``R``\beta \quad (1)$

$\vdash . *37{\cdot}16 . \quad \supset \vdash : \beta \subset \breve{R}``R``\beta . \supset . \beta \subset \text{Œ}`R \quad (2)$

$\vdash . (1) . (2) . \quad \supset \vdash . \text{Prop}$

$*37{\cdot}52$. $\vdash : \alpha \subset \text{D}`R . \equiv . \alpha \subset R``\breve{R}``\alpha$ [Similar proof]

The following propositions, down to $*37{\cdot}7$ exclusive, are concerned with the special properties of $R``\beta$ which result from the hypothesis $\text{E}\,!!\, R``\beta$, defined in $*37{\cdot}05$. The hypothesis $\text{E}\,!!\, R``\beta$ is important, because it has many consequences and is satisfied in many cases with which we wish to deal.

$*37{\cdot}6$. $\vdash : \text{E}\,!!\, R``\beta . \supset . R``\beta = \hat{x}\{(\exists y) . y \in \beta . x = R`y\}$

This proposition is very important, and is used constantly.

Dem.

$\vdash . *37{\cdot}104 . \supset \vdash :: \text{Hp} . \supset :. y \in \beta . \supset_y : \text{E}\,!\, R`y :$

$[*30{\cdot}4] \quad \supset_y : x = R`y . \equiv . xRy :.$

$[*5{\cdot}32] \quad \supset :. y \in \beta . x = R`y . \equiv_y . y \in \beta . xRy :.$

$[*10{\cdot}281] \quad \supset :. (\exists y) . y \in \beta . x = R`y . \equiv . (\exists y) . y \in \beta . xRy .$

$[*37{\cdot}1] \quad \equiv . x \in R``\beta \quad (1)$

$\vdash . (1) . *10{\cdot}11{\cdot}21 . *20{\cdot}33 . \supset \vdash . \text{Prop}$

$*37{\cdot}601$. $\vdash : (x) . \text{E}\,!\, R`x . \supset . R``\text{V} = \hat{x}\{(\exists y) . x = R`y\}$

Dem.

$\vdash . *2{\cdot}02 . *10{\cdot}11{\cdot}27 . \supset \vdash :. \text{Hp} . \supset : x \in \text{V} . \supset_x . \text{E}\,!\, R`x :$

$[*37{\cdot}104] \supset : \mathrm{E}\,!!\, R``\mathrm{V} :$

$[*37{\cdot}6] \quad \supset : R``\mathrm{V} = \hat{x}\{(\exists y) . y \in \mathrm{V} . x = R`y\}$ (1)

$\vdash . *24{\cdot}104 . *4{\cdot}73 . \supset \vdash : y \in \mathrm{V} . x = R`y . \equiv . x = R`y :$

$[*10{\cdot}11{\cdot}281] \quad \supset \vdash : (\exists y) . y \in \mathrm{V} . x = R`y . \equiv . (\exists y) . x = R`y :$

$[*20{\cdot}15] \quad \supset \vdash : \hat{x}\{(\exists y) . y \in \mathrm{V} . x = R`y\} = \hat{x}\{(\exists y) . x = R`y\}$ (2)

$\vdash . (1) . (2) . \quad \supset \vdash . \mathrm{Prop}$

$*37{\cdot}61$. $\vdash :: \mathrm{E}\,!!\, R``\beta . \supset :. R``\beta \subset \alpha . \equiv : y \in \beta . \supset_y . R`y \in \alpha$

Dem.

$\vdash . *37{\cdot}17 . \supset \vdash :: R``\beta \subset \alpha . \equiv :. y \in \beta . xRy . \supset_{x,y} . x \in \alpha :.$

$[*11{\cdot}2{\cdot}62] \quad \equiv :. y \in \beta . \supset_y : xRy . \supset_x . x \in \alpha$ (1)

$\vdash . *37{\cdot}104 . \supset \vdash ::. \mathrm{Hp} . \supset :: y \in \beta . \supset_y :. \mathrm{E}\,!\, R`y :.$

$[*30{\cdot}33] \quad \supset_y :. R`y \in \alpha . \equiv : xRy . \supset_x . x \in \alpha$ (2)

$\vdash . (1) . (2) . \supset \vdash :: \mathrm{Hp} . \supset :. R``\beta \subset \alpha . \equiv : y \in \beta . \supset_y . R`y \in \alpha :: \supset \vdash . \mathrm{Prop}$

$*37{\cdot}62$. $\vdash : \mathrm{E}\,!\, R`y . y \in \alpha . \supset . R`y \in R``\alpha$

Dem.

$\vdash . *30{\cdot}33 . \supset$

$\vdash :: \mathrm{E}\,!\, R`y . \supset :. R`y \in R``\alpha . \equiv : xRy . \supset_x . x \in R``\alpha$ (1)

$\vdash . *3{\cdot}2 . \quad \supset \vdash :. y \in \alpha . \supset : xRy . \supset . y \in \alpha . xRy .$

$[*10{\cdot}24 . *37{\cdot}1] \quad \supset . x \in R``\alpha$ (2)

$\vdash . (2) . *10{\cdot}11{\cdot}21 . \supset \vdash :. y \in \alpha . \supset : xRy . \supset_x . x \in R``\alpha$ (3)

$\vdash . (1) . (3) . \quad \supset \vdash . \mathrm{Prop}$

The above is the type of inference concerning which Jevons says*: "I remember the late Prof. De Morgan remarking that all Aristotle's logic could not prove that 'Because a horse is an animal, the head of a horse is the head of an animal.'" It must be confessed that this was a merit in Aristotle's logic, since the proposed inference is fallacious without the added premiss "E ! the head of the horse in question." *E.g.* it does not hold for an oyster or a hydra. But with the addition $\mathrm{E}\,!\, R`y$, the above proposition gives an important and common type of asyllogistic inference.

$*37{\cdot}63$. $\vdash :: \mathrm{E}\,!!\, R``\alpha . \supset :. x \in R``\alpha . \supset_x . \psi x : \equiv : y \in \alpha . \supset_y . \psi(R`y)$

Dem.

$\vdash . *37{\cdot}1 . \quad \supset \vdash :: x \in R``\alpha . \supset_x . \psi x : \equiv :. (\exists y) . y \in \alpha . xRy . \supset_x . \psi x :.$

$[*10{\cdot}23] \quad \equiv :. y \in \alpha . xRy . \supset_{x,y} . \psi x :.$

$[*11{\cdot}2{\cdot}62] \quad \equiv :. y \in \alpha . \supset_y : xRy . \supset_x . \psi x$ (1)

$\vdash . *37{\cdot}104 . \supset \vdash ::. \mathrm{Hp} . \supset :: y \in \alpha . \supset_y :. \mathrm{E}\,!\, R`y :.$

$[*30{\cdot}33] \quad \supset_y :. \psi(R`y) . \equiv : xRy . \supset_x . \psi x$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

This proposition is very frequently used.

* *Principles of Science*, chap. I. (p. 18 of edition of 1887).

***37·64.** $\vdash :. \mathrm{E}!! R``\alpha . \supset : (\exists y) . y \in \alpha . \psi(R`y) . \equiv . (\exists x) . x \in R``\alpha . \psi x$

Dem.

$\vdash . *30·33 . \supset \vdash :: \mathrm{Hp} . \supset :. y \in \alpha . \supset : \psi(R`y) . \equiv . (\exists x) . xRy . \psi x :.$

[*5·32] $\supset :. y \in \alpha . \psi(R`y) . \equiv : y \in \alpha : (\exists x) . xRy . \psi x$ (1)

$\vdash . (1) . *10·11·21·281 . \supset$

$\vdash :: \mathrm{Hp} . \supset :. (\exists y) . y \in \alpha . \psi(R`y) . \equiv : (\exists y) : y \in \alpha : (\exists x) . xRy . \psi x :$

[*11·6] $\equiv : (\exists x) : (\exists y) . y \in \alpha . xRy : \psi x :$

[*37·1] $\equiv : (\exists x) . x \in R``\alpha . \psi x :: \supset \vdash . \mathrm{Prop}$

***37·65.** $\vdash : \mathrm{E}!! R``\beta . \alpha \subset R``\beta . \supset . \alpha = R``(\breve{R}``\alpha \cap \beta)$

Dem.

$\vdash . *30·21 . *3·27 . \supset \vdash :: \mathrm{Hp} . \supset :. y \in \beta . \supset_y : zRy . xRy . \supset . z = x$ (1)

$\vdash . *37·1 . \supset \vdash :. \mathrm{Hp} . \supset :$

$x \in R``(\breve{R}``\alpha \cap \beta) . \equiv . (\exists y) . y \in \breve{R}``\alpha \cap \beta . xRy .$

[*37·105.*11·55] $\equiv . (\exists y, z) . z \in \alpha . zRy . y \in \beta . xRy .$

[(1).*4·71] $\equiv . (\exists y, z) . z \in \alpha . zRy . y \in \beta . xRy . z = x .$

[*13·194] $\equiv . (\exists y, z) . z \in \alpha . y \in \beta . xRy . z = x .$

[*13·195] $\equiv . (\exists y) . x \in \alpha . y \in \beta . xRy .$

[*10·35.*37·1] $\equiv . x \in \alpha . x \in R``\beta .$

[*4·71.Hp] $\equiv . x \in \alpha :. \supset \vdash . \mathrm{Prop}$

***37·66.** $\vdash :. \mathrm{E}!! R``\beta . \supset : \alpha \subset R``\beta . \equiv . (\exists \gamma) . \gamma \subset \beta . \alpha = R``\gamma$

Dem.

$\vdash . *37·65 . \mathrm{Exp} . *13·195 . *22·43 . \supset$

$\vdash :. \mathrm{Hp} . \supset : \alpha \subset R``\beta . \supset . (\exists \gamma) . \gamma \subset \beta . \alpha = R``\gamma$ (1)

$\vdash . *37·2 . *13·13 . \supset \vdash : \gamma \subset \beta . \alpha = R``\gamma . \supset . \alpha \subset R``\beta :$

[*10·11·23] $\supset \vdash : (\exists \gamma) . \gamma \subset \beta . \alpha = R``\gamma . \supset . \alpha \subset R``\beta$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***37·67.** $\vdash :. z \in \gamma . \supset_z . \mathrm{E} ! R`S`z : \supset . R``S``\gamma = \hat{x}\{(\exists z) . z \in \gamma . x = R`S`z\}$

Dem.

$\vdash . *34·41 . \supset \vdash : \mathrm{Hp} . z \in \gamma . \supset_z . R`S`z = (R \mid S)`z$ (1)

$\vdash . (1) . *14·21 . \supset \vdash : \mathrm{Hp} . z \in \gamma . \supset_z . \mathrm{E} ! (R \mid S)`z$ (2)

$\vdash . (2) . *37·6 . \supset \vdash : \mathrm{Hp} . \supset . (R \mid S)``\gamma = \hat{x}\{(\exists z) . z \in \gamma . x = (R \mid S)`\gamma\}$

[(1)] $= \hat{x}\{(\exists z) . z \in \gamma . x = R`S`\gamma\}$ (3)

$\vdash . *37·33 . \supset \vdash . R``S``\gamma = (R \mid S)``\gamma$ (4)

$\vdash . (3) . (4) . \supset \vdash . \mathrm{Prop}$

***37·68.** $\vdash :. z \in \gamma . \supset_z . P`Q`z = R`z : \supset . P``Q``\gamma = R``\gamma$

Dem.

$\vdash . *14·21 . \supset \vdash : \mathrm{Hp} . z \in \gamma . \supset . \mathrm{E} ! P`Q`z . \mathrm{E} ! R`z .$

[*34·41] $\supset . P`Q`z = (P \mid Q)`z . \mathrm{E} ! R`z .$ (1)

[*14·21·131·144.Hp] $\supset . \mathrm{E} ! (P \mid Q)`z . (P \mid Q)`z = R`z$ (2)

$\vdash . *37{\cdot}33 . \supset \vdash . P``Q``\gamma = (P \mid Q)``\gamma$ (3)

$\vdash . (2) . (3) . *37{\cdot}6 . \supset$

$\vdash : \mathrm{Hp} . \supset . P``Q``\gamma = \hat{x}\{(\exists z) . z \in \gamma . x = (P \mid Q)`z\}$

$[(2)] \qquad = \hat{x}\{(\exists z) . z \in \gamma . x = R`z\}$

$[*37{\cdot}6 . (1)] \qquad = R``z : \supset \vdash . \mathrm{Prop}$

***37·69.** $\vdash :. y \in \beta . \supset_y . R`y = S`y : \supset . R``\beta = S``\beta$

Dem.

$\vdash . *14{\cdot}21 . \supset \vdash :: \mathrm{Hp} . \supset :. y \in \beta . \supset . E ! R`y . E ! S`y :.$ (1)

$[*30{\cdot}4] \qquad \supset :. y \in \beta . \supset : xRy . \equiv . x = R`y .$

$[*14{\cdot}142] \qquad \equiv . x = S`y .$

$[*30{\cdot}4 . (1)] \qquad \equiv . xSy :.$

$[*5{\cdot}32] \qquad \supset :. y \in \beta . xRy . \equiv . y \in \beta . xSy$ (2)

$\vdash . (2) . *10{\cdot}11{\cdot}21{\cdot}281 . \supset$

$\vdash :. \mathrm{Hp} . \supset : (\exists y) . y \in \beta . xRy . \equiv . (\exists y) . y \in \beta . xSy :$

$[*37{\cdot}1] \quad \supset : x \in R``\beta . \equiv . x \in S``\beta :. \supset \vdash . \mathrm{Prop}$

A specially important case of $R``\beta$ is $\overrightarrow{R}``\beta$ or $\overleftarrow{R}``\beta$. This case will be further studied later (in *70); for the present, we shall only give a few preliminary propositions about it. It will be observed that the hypothesis $E !! \overrightarrow{R}``\beta$ or $E !! \overleftarrow{R}``\beta$ is always verified, in virtue of *32·12·121. Hence the following applications of *37·6 ff.:

***37·7.** $\vdash . \overrightarrow{R}``\beta = \hat{\alpha}\{(\exists y) . y \in \beta . \alpha = \overrightarrow{R}`y\}$ [*37·6 . *32·12]

***37·701.** $\vdash . \overleftarrow{R}``\alpha = \hat{\beta}\{(\exists x) . x \in \alpha . \beta = \overleftarrow{R}`x\}$ [*37·6 . *32·121]

***37·702.** $\vdash :. \overrightarrow{R}``\beta \subset \kappa . \equiv : y \in \beta . \supset_y . \overrightarrow{R}`y \in \kappa$ [*37·61]

***37·703.** $\vdash :. \overleftarrow{R}``\beta \subset \kappa . \equiv : x \in \beta . \supset_x . \overleftarrow{R}`x \in \kappa$ [*37·61]

***37·704.** $\vdash : y \in \alpha . \supset . \overrightarrow{R}`y \in \overrightarrow{R}``\alpha$ [*37·62 . *32·12]

***37·705.** $\vdash : x \in \alpha . \supset . \overleftarrow{R}`x \in \overleftarrow{R}``\alpha$ [*37·62 . *32·121]

***37·706.** $\vdash :. \alpha \in \overrightarrow{R}``\beta . \supset_\alpha . \psi\alpha : \equiv : y \in \beta . \supset_y . \psi(\overrightarrow{R}`y)$ [*37·63]

***37·707.** $\vdash :. \beta \in \overleftarrow{R}``\alpha . \supset_\beta . \psi\beta : \equiv : x \in \alpha . \supset_x . \psi(\overleftarrow{R}`x)$ [*37·63]

***37·708.** $\vdash :. (\exists \alpha) . \alpha \in \overrightarrow{R}``\beta . \psi\alpha . \equiv . (\exists y) . y \in \beta . \psi(\overrightarrow{R}`y)$ [*37·64]

***37·709.** $\vdash :. (\exists \alpha) . \alpha \in \overleftarrow{R}``\beta . \psi\alpha . \equiv . (\exists x) . x \in \beta . \psi(\overleftarrow{R}`x)$ [*37·64]

***37·71.** $\vdash : \kappa \subset \overrightarrow{R}``\beta . \supset . \kappa = \overrightarrow{R}``\{(\mathrm{Cnv}`\overrightarrow{R})``\kappa \cap \beta\}$ [*37·65]

***37·711.** $\vdash : \kappa \subset \overleftarrow{R}``\beta . \supset . \kappa = \overleftarrow{R}``\{(\mathrm{Cnv}`\overleftarrow{R})``\kappa \cap \beta\}$ [*37·65]

***37·712.** $\vdash : \kappa \subset \overrightarrow{R}``\beta . \equiv . (\exists \gamma) . \gamma \subset \beta . \kappa = \overrightarrow{R}``\gamma$ [*37·66]

***37·713.** $\vdash : \kappa \subset \overleftarrow{R}``\beta . \equiv . (\exists \gamma) . \gamma \subset \beta . \kappa = \overleftarrow{R}``\gamma$ [*37·66]

***37·72.** $\vdash : R = P \mid Q . \supset . \overrightarrow{R}``\gamma = P```\overrightarrow{Q}``\gamma$

Dem.

$\vdash . *37\cdot11\cdot302 . \supset \vdash : \mathrm{Hp} . \supset . (z) . P_\epsilon`\overrightarrow{Q}`z = \overrightarrow{R}`z .$

$[*37\cdot68] \qquad \supset . P_\epsilon``\overrightarrow{Q}``\gamma = \overrightarrow{R}``\gamma .$

$[(*37\cdot04)] \qquad \supset . P```\overrightarrow{Q}``\gamma = \overrightarrow{R}``\gamma : \supset \vdash . \mathrm{Prop}$

***37·721.** $\vdash : R = P \mid Q . \supset . \overleftarrow{R}``\gamma = \breve{Q}```\overleftarrow{P}``\gamma$ [Proof as in *37·72]

***37·73.** $\vdash : \exists ! \beta . \equiv . \exists ! \overrightarrow{R}``\beta . \equiv . \exists ! \overleftarrow{R}``\beta$ [*37·45 . *32·12·121]

***37·731.** $\vdash : \beta = \Lambda . \equiv . \overrightarrow{R}``\beta = \Lambda . \equiv . \overleftarrow{R}``\beta = \Lambda$ [*37·73 . Transp]

Observe that the Λ's which occur in this proposition will not be all of the same type. *E.g.* if R relates individuals to individuals, the first Λ will be the class of no individuals, while the second and third will be the class of no classes. Thus the ambiguity which attaches to the type of Λ must be differently determined for different occurrences of Λ in this proposition. In general, when this is the case with our ambiguous symbols, we shall adopt a notation which indicates the fact. But when the ambiguous symbol is Λ, it seems hardly worth while.

***37·74.** $\vdash :. \beta \subset \mathrm{G}`R . \equiv : \alpha \epsilon \overrightarrow{R}``\beta . \supset_\alpha . \exists ! \alpha$

Dem.

$\vdash . *37\cdot706 . \supset \vdash :. \alpha \epsilon \overrightarrow{R}``\beta . \supset_\alpha . \exists ! \alpha : \equiv : y \epsilon \beta . \supset_y . \exists ! \overrightarrow{R}`y :$

$[*33\cdot31] \qquad \equiv : \beta \subset \mathrm{G}`R :. \supset \vdash . \mathrm{Prop}$

***37·75.** $\vdash :. \alpha \subset \mathrm{D}`R . \equiv : \beta \epsilon \overleftarrow{R}``\alpha . \supset_\beta . \exists ! \beta$ [Proof as in *37·74]

***37·76.** $\vdash . \overrightarrow{R}``\beta \subset \mathrm{Cls}$

Dem.

$\vdash . *37\cdot7 . \supset \vdash :. \alpha \epsilon \overrightarrow{R}``\beta . \supset : (\exists y) . y \epsilon \beta . \alpha = \overrightarrow{R}`y :$

$[*10\cdot5] \qquad \supset : (\exists y) . \alpha = \overrightarrow{R}`y :$

$[*32\cdot13] \qquad \supset : (\exists y) . \alpha = \hat{x}(xRy) :$

$[*20\cdot16] \qquad \supset : (\exists \phi) . \alpha = \hat{x}(\phi ! x) :$

$[*20\cdot4] \qquad \supset : \alpha \epsilon \mathrm{Cls} :. \supset \vdash . \mathrm{Prop}$

***37·761.** $\vdash . \overleftarrow{R}``\alpha \subset \mathrm{Cls}$ [Proof as in *37·76]

***37·77.** $\vdash : \alpha \epsilon \overrightarrow{R}``\mathrm{G}`R . \supset_\alpha . \exists ! \alpha$ [*37·74 . *22·42]

***37·771.** $\vdash : \beta \epsilon \overleftarrow{R}``\mathrm{D}`R . \supset_\beta . \exists ! \beta$ [Proof as in *37·77]

***37·772.** $\vdash . \Lambda \sim\epsilon \overrightarrow{R}``\mathrm{G}`R$ [*37·77 . *24·63]

***37·773.** $\vdash . \Lambda \sim\epsilon \overleftarrow{R}``\mathrm{D}`R$ [*37·771 . *24·63]

***37·78.** $\vdash . \mathrm{D}`\overrightarrow{R} = \overrightarrow{R}``\mathrm{V}$ [*37·28]

***37·781.** $\vdash . \mathrm{D}`\overleftarrow{R} = \overleftarrow{R}``\mathrm{V}$ [*37·28]

***37·79.** $\vdash . \overrightarrow{R}``\mathrm{V} = \hat{\alpha}\{(\exists y) . \alpha = \overrightarrow{R}`y\}$ [*37·601 . *32·12]

***37·791.** $\vdash . \overleftarrow{R}``\mathrm{V} = \hat{\beta}\{(\exists x) . \beta = \overleftarrow{R}`x\}$ [*37·601 . *32·121]

***37·8.** $\vdash . (\alpha \uparrow \beta) \mid S = \alpha \uparrow \breve{S}``\beta$

Dem.

$\vdash . *35·103 . *34·1 . \supset \vdash : x\{(\alpha \uparrow \beta) \mid S\} z . \equiv . (\exists y) . x \in \alpha . y \in \beta . ySz .$

[*10·35.*37·105] $\equiv . x \in \alpha . z \in \breve{S}``\beta .$

[*35·103] $\equiv . x (\alpha \uparrow \breve{S}``\beta) z : \supset \vdash .$ Prop

***37·81.** $\vdash . R \mid (\alpha \uparrow \beta) = (R``\alpha) \uparrow \beta$ [Proof as in *37·8]

***37·82.** $\vdash . R \mid (\alpha \uparrow \beta) \mid S = (R``\alpha) \uparrow (\breve{S}``\beta)$ [*37·8·81]

$*38$. RELATIONS AND CLASSES DERIVED FROM A DOUBLE DESCRIPTIVE FUNCTION

Summary of $*38$.

A double descriptive function is a non-propositional function of two arguments, such as $\alpha \cap \beta$, $\alpha \cup \beta$, $R \dot{\cap} S$, $R \dot{\cup} S$, $R \mid S$, $\alpha \rceil R$, $R \upharpoonright \alpha$, $R \,\underline{\upharpoonright}\, \alpha$. The propositions of the present number apply to all such functions, assuming the notation to be (as in the above instances) a functional sign placed between the two arguments. In order to deal with all analogous cases at once, we shall in this number adopt the notation

$$x \female y,$$

where "$\female$" stands for any such sign as $\cap$, $\cup$, $\dot{\cap}$, $\dot{\cup}$, $\mid$, $\rceil$, $\upharpoonright$, $\underline{\upharpoonright}$, or any functional sign to be hereafter defined and satisfying the condition

$$(x, y) \,.\, \mathrm{E}\,!\,(x \female y).$$

The derived relations and classes with which we shall be concerned may be illustrated by taking the case of $\alpha \cap \beta$. The relation of $\alpha \cap \beta$ to β will be written $\alpha \cap$, and the relation of $\alpha \cap \beta$ to α will be written $\cap \beta$. Thus we shall have

$$\vdash \,.\, \alpha \cap \beta = \alpha \cap {}^{\backprime}\beta = \cap \beta {}^{\backprime}\alpha.$$

The utility of this notation is chiefly due to the possibility of such notations as $\alpha \cap {}^{\backprime\backprime}\kappa$ and $\cap \beta {}^{\backprime\backprime}\kappa$. For example, take such a phrase as "the foreign members of English Clubs." Then if we put $\alpha =$ foreigners, $\kappa =$ English Clubs, we have

$\alpha \cap {}^{\backprime\backprime}\kappa =$ the classes of foreign members of the various English Clubs.

Or again, let α be a conic, and κ a pencil of lines; then

$\alpha \cap {}^{\backprime\backprime}\kappa =$ the various pairs of points in which members of κ meet α.

In this case, since $\alpha \cap \beta = \beta \cap \alpha$, we have $\alpha \cap = \cap \alpha$. But when the function concerned is not commutative, this does not hold. Thus for example we do not have $R \mid = \mid R$.

The notations of this number will be frequently applied hereafter to $R \mid S$. In accordance with what was said above, we write $R \mid$ for the relation of $R \mid S$ to S, and $\mid S$ for the relation of $R \mid S$ to R. Hence we have

$$R \mid {}^{\backprime}S = \mid S {}^{\backprime}R = R \mid S.$$

Hence $\mid S {}^{\backprime\backprime}\lambda$ will be the class of relations obtained by taking members of λ and relatively multiplying them by S. Thus if λ were the class of relations first cousin, second cousin, etc., and S were the relation of parent to child, $\mid S {}^{\backprime\backprime}\lambda$ would be the class of relations first cousin once removed, second cousin once removed, etc., taken in the sense which goes from the older to the younger generation.

It is often convenient to be able to exhibit $|S``\lambda$ and kindred expressions as descriptive functions of the first argument instead of the second. For this purpose we put

$$\lambda \underset{,,}{|} S = |S``\lambda$$

with similar notations for other descriptive double functions. We then have, just as in the case of $R|S$,

$$\lambda \underset{,,}{|} `S = \underset{,,}{|} S`\lambda = \lambda \underset{,,}{|} S.$$

This enables us to form the class $\lambda \underset{,,}{|} ``\mu$. This class is chiefly useful because the members of its members (*i.e.* $s`\lambda \underset{,,}{|} ``\mu$, as we shall define it in $*40$) constitute the class of all products $R|S$ that can be formed of a member of λ and a member of μ.

Thus we are led to three general definitions for descriptive double functions, namely (if $x \female y$ be any such function)

$x \female$ is the relation of $x \female y$ to y for any y,
$\female y$ " " " " " x " x,
$\alpha \underset{,,}{\female} y$ is the class of values of $x \female y$ when x is an α.

Since $\alpha \underset{,,}{\female} y$ is again a descriptive double function, the first two of the above definitions can be applied to it. The third definition, for typographical reasons, cannot be applied conveniently, though theoretically it is of course applicable. The relations $x \female$ and $\female y$ represent the general idea contained in some of the uses in mathematics of the term "operation," *e.g.* $+1$ is the operation of adding 1.

The uses of the notations introduced in the present number occur chiefly in arithmetic (Parts III and IV). Few propositions can be given at this stage, since most of the important uses of the notation here introduced depend upon the substitution of some special function for the general function "$\female$" here used. In the present number, the propositions given are all immediate consequences of the definitions.

$*38{\cdot}01$. $x \female = \hat{u}\hat{y}\,(u = x \female y)$ Df

$*38{\cdot}02$. $\female y = \hat{u}\hat{x}\,(u = x \female y)$ Df

$*38{\cdot}03$. $\alpha \underset{,,}{\female} y = \female y``\alpha$ Df

$*38{\cdot}1$. $\vdash : u\,(x \female)\,y \,.\equiv.\, u = x \female y$ [($*38{\cdot}01$)]

$*38{\cdot}101$. $\vdash : u\,(\female y)\,x \,.\equiv.\, u = x \female y$ [($*38{\cdot}02$)]

$*38{\cdot}11$. $\vdash .\, x \female `y = \female y`x = x \female y$ [$*38{\cdot}1{\cdot}101 \,.\, *30{\cdot}3$]

$*38{\cdot}12$. $\vdash .\, E!\, x \female `y \,.\, E!\, \female y`x$ [$*38{\cdot}11 \,.\, *14{\cdot}21$]

$*38{\cdot}13$. $\vdash : u \in x \female ``\alpha \,.\equiv.\, (\exists y) \,.\, y \in \alpha \,.\, u = x \female y$ [$*38{\cdot}1 \,.\, *37{\cdot}1$]

$*38{\cdot}131$. $\vdash : u \in \female y``\alpha \,.\equiv.\, (\exists x) \,.\, x \in \alpha \,.\, u = x \female y$ [$*38{\cdot}101 \,.\, *37{\cdot}1$]

***38·2.** $\vdash . \alpha \underset{,,}{♀} y = ♀ y``\alpha$ [(*38·03)]

***38·21.** $\vdash . \alpha \underset{,,}{♀} y = \hat{u}\{(\exists x) . x \in \alpha . u = x ♀ y\}$ [*38·2·131]

***38·22.** $\vdash . \alpha \underset{,,}{♀} `y = \underset{,,}{♀} y`\alpha = \alpha \underset{,,}{♀} y$ [*38·11]

***38·23.** $\vdash . \mathrm{E} ! \alpha \underset{,,}{♀} `y . \mathrm{E} ! \underset{,,}{♀} y`\alpha$ [*38·22 . *14·21]

***38·24.** $\vdash : \exists ! \alpha \underset{,,}{♀} y . \equiv . \exists ! \alpha$

Dem.

$\vdash . *38·2 . *37·29 . \text{Transp} . \supset \vdash : \exists ! \alpha \underset{,,}{♀} y . \supset . \exists ! \alpha$ (1)

$\vdash . *38·21 . \supset \vdash : x \in \alpha . \supset . (x ♀ y) \in \alpha \underset{,,}{♀} y .$

[*10·24] $\supset . \exists ! \alpha \underset{,,}{♀} y$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***38·3.** $\vdash . \alpha \underset{,,}{♀} ``\beta = \hat{\gamma}\{(\exists y) . y \in \beta . \gamma = \alpha \underset{,,}{♀} y\} = \hat{\gamma}\{(\exists y) . y \in \beta . \gamma = ♀ y``\alpha\}$
[*38·13·2]

***38·31.** $\vdash . \underset{,,}{♀} y``\kappa = \hat{\gamma}\{(\exists \alpha) . \alpha \in \kappa . \gamma = \alpha \underset{,,}{♀} y\} = \hat{\gamma}\{(\exists \alpha) . \alpha \in \kappa . \gamma = ♀ y``\alpha\} = ♀ y```\kappa$
[*38·131·2 . *37·103]

NOTE TO SECTION D

General Observations on Relations. The notion of "relation" is so general that it is important to realize the different sorts of relations to which the notations defined in the preceding section may be applied. It often happens that a proposition which holds for any relation is only important for relations of certain kinds; hence it is desirable that the reader should have in mind some of the principal kinds of relations. Of the various uses to which different sorts of relations may be put, there are three which are specially important, namely (1) to give rise to descriptive functions, (2) to establish correlations between different classes, (3) to generate series. Let us consider these in succession.

(1) In order that a relation R may give rise to a descriptive function, it must be such that the referent is unique when the relatum is given. Thus, for example, the relations Cnv, $\overrightarrow{R}$, $\overleftarrow{R}$, D, Ɑ, C, R_ϵ, defined above, all give rise to descriptive functions. In general, if R gives rise to a descriptive function, there will be a certain class, namely Ɑ‘R, to which the argument of the function must belong in order that the function may have a value for that argument. For example, taking the sine as an illustration, and writing "sin‘y" instead of "sin y," y must be a number in order that sin‘y may exist. Then $\breve{\text{sin}}$ is the relation of y to x when $x = \text{sin}‘y$. If we put α = numbers between $-\pi/2$ and $\pi/2$, both included, $\text{sin} \restriction \alpha$ will be the relation of x to y when $x = \text{sin}‘y$ and $-\pi/2 \leqslant y \leqslant \pi/2$. The converse of this relation, which is $\alpha \upharpoonleft \breve{\text{sin}}$, will also give rise to a descriptive function; thus $(\alpha \upharpoonleft \breve{\text{sin}})‘x$ = that value of $\text{sin}^{-1} x$ which lies between $-\pi/2$ and $\pi/2$. This illustrates a case which arises very frequently, namely, that a relation R does not, as it stands, give rise to a descriptive function, but does do so when its domain or converse domain is suitably limited. Thus for example the relation "parent" does not give rise to a descriptive function, but does do so when its domain is limited to males or limited to females. The relation "square root," similarly, gives rise to a descriptive function when its domain is limited to positive numbers, or limited to negative numbers. The relation "wife" gives rise to a descriptive function when its converse domain is limited to Christian men, but not when Mohammedans are included. The domain of a relation which gives rise to a descriptive function without limiting its domain or converse domain consists of all possible values of the function; the converse domain consists of all possible arguments to the function. Again, if R gives rise to a descriptive function, $\overleftarrow{R}‘x$ will be the class of those arguments for which the value of the function is x. Thus $\overleftarrow{\text{sin}}‘x$ consists of all numbers

whose sine is x, *i.e.* all values of $\sin^{-1} x$. Again, $\sin``\alpha$ will be the sines of the various members of $\alpha$. If $\alpha$ is a class of numbers, then, by the notation of $*38$, $2 \times ``\alpha$ will be the doubles of those numbers, $3 \times ``\alpha$ the trebles of them, and so on. To take another illustration, let $\alpha$ be a pencil of lines, and let $R`x$ be the intersection of a line $x$ with a given transversal. Then $R``\alpha$ will be the intersections of lines belonging to the pencil with the transversal.

(2) Relations which establish a correlation between two classes are really a particular case of relations giving rise to descriptive functions, namely the case in which the converse relation also gives rise to a descriptive function. In this case, the relation is "one-one," *i.e.* given the referent, the relatum is determinate, and vice versa. A relation which is to be conceived as a correlation will generally be denoted by S or T. In such cases, we are as a rule less interested in the particular terms x and y for which xRy, than in classes of such terms. We generally, in such cases, have some class β contained in the converse domain of our relation S, and we have a class α such that $\alpha = S``\beta$. In this case, the relation $S$ correlates the members of $\alpha$ and the members of $\beta$. We shall have also $\beta = \breve{S}``\alpha$, so that, for such a relation, the correlation is reciprocal. Such relations are fundamental in arithmetic, since they are used in defining what is meant by saying that two classes (or series) have the same cardinal (or ordinal) number of terms.

(3) Relations which give rise to series will in general be denoted by P or Q, and in propositions whose chief importance lies in their application to series we shall also, as a rule, denote a variable relation by P or Q. When P is used, it may be read as "precedes." Then $\breve{P}$ may be read "follows," $\overrightarrow{P}`x$ may be read "predecessors of $x$," $\overleftarrow{P}`x$ may be read "followers of x." $\text{D}`P$ will be all members of the series generated by $P$ except the last (if any), $\text{Ɑ}`P$ will be all members of the series except the first (if any), $C`P$ will be all the members of the series. $P``\alpha$ will consist of all terms preceding some member of $\alpha$. Suppose, for example, that our series is the series of real numbers, and that $\alpha$ is the class of members of an ascending series $x_1, x_2, x_3, \ldots x_\nu, \ldots$. Then $P``\alpha$ will be the segment of the real numbers defined by this series, *i.e.* it will be all the predecessors of the limit of the series. (In the event of the series $x_1, x_2, x_3, \ldots x_\nu, \ldots$ growing without limit, $P``\alpha$ will be the whole series of real numbers.)

It very often happens that a relation has more or less of a serial character, without having all the characteristics necessary for generating series. Take, for example, the relation of son to father. It is obvious that by means of this relation series can be generated which start from any man and end with Adam. But these series are not the field of the relation in question; moreover this relation is not *transitive*, *i.e.* a son of a son of x is not a son of x. If, however, we substitute for "son" the relation "descendant in the direct

male line" (which can be defined in terms of "son" by the method explained in $*90$ and $*91$), and if we limit the converse domain of this relation to ancestors of x in the direct male line, we obtain a new relation which *is* serial, and has for its field x and all his ancestors in the direct male line. Again, one relation may generate a number of series, as for example the relation "x is east of y." If x and y are points on the earth's surface, and in the eastern hemisphere, this relation generates one series for every parallel of latitude. By confining the field of the relation further to one parallel of latitude, we obtain a relation which generates a series. (The reason for confining x and y to one hemisphere is to insure that the relation shall be transitive, since otherwise we might have x east of y and y east of z, but x west of z.)

A relation may have the characteristics of all the three kinds of relations, provided we include in the third kind all those which lead to series by some such limitations as those just described. For example, the relation $+1$, *i.e.* (in virtue of the notation of $*38$) the relation of $x+1$ to x, where x is supposed to be a finite cardinal integer, has the characteristics of all three kinds of relations. In the first place, it leads to the descriptive function $(+1)`x$, *i.e.* $x+1$. In the second place, it correlates with any class α of numbers the class obtained by adding 1 to each member of α, *i.e.* $(+1)``\alpha$. This correlation may be used to prove that the number of finite integers is infinite (in one of the two senses of the word "infinite"); for if we take as our class $\alpha$ all the natural numbers including 0, the class $(+1)``\alpha$ consists of all the natural numbers except 0, so that the natural numbers can be correlated with a proper part* of themselves. Again, the relation $+1$ may be used, like that of father to son, to generate a series, namely the usual series of the natural numbers in order of magnitude, in which each has to its immediate predecessor the relation $+1$. Thus this relation partakes of the characteristics of all three kinds of relations.

* *I.e.* a part not the whole. On this definition of infinity, see $*124$.

SECTION E

PRODUCTS AND SUMS OF CLASSES

Summary of Section E.

In the present section, we make an extension of $\alpha \cap \beta$, $\alpha \cup \beta$, $R \dot{\cap} S$, $R \dot{\cup} S$. Given a class of classes, say κ, the *product* of κ (which is denoted by $p`\kappa$) is the common part of all the members of κ, *i.e.* the class consisting of those terms which belong to every member of κ. The definition is

$$p`\kappa = \hat{x}(\alpha \in \kappa \,.\supset_\alpha .\, x \in \alpha) \quad \text{Df.}$$

If κ has only two members, α and β say, $p`\kappa = \alpha \cap \beta$. If $\kappa$ has three members, $\alpha, \beta, \gamma$, then $p`\kappa = \alpha \cap \beta \cap \gamma$; and so on. But this process can only be continued to a finite number of terms, whereas the definition of $p`\kappa$ does not require that κ should be finite. This notion is chiefly important in connection with the lower limits of series. For example, let λ be the class of rational numbers whose square is greater than 2, and let "xMy" mean "$x < y$, where x and y are rationals." Then if $x \in \lambda$, $\overrightarrow{M}`x$ will be the class of rationals less than x. Thus $\overrightarrow{M}``\lambda$ will be the class of such classes as $\overrightarrow{M}`x$, where $x \in \lambda$. Thus the product of $\overrightarrow{M}``\lambda$, which we call $p`\overrightarrow{M}``\lambda$, will be the class of rationals which are less than every member of $\lambda$, *i.e.* the class of rationals whose squares are less than 2. Each member of $\overrightarrow{M}``\lambda$ is a segment of the series of rationals, and $p`\overrightarrow{M}``\lambda$ is the lower limit of these segments. It is thus that we prove the existence of lower limits of series of segments.

Similarly the *sum* of a class of classes κ is defined as the class consisting of all terms belonging to *some* member of κ; *i.e.*

$$s`\kappa = \hat{x}\{(\exists\alpha) \,.\, \alpha \in \kappa \,.\, x \in \alpha\} \quad \text{Df,}$$

i.e. x belongs to the sum of κ if x belongs to some κ. This notion plays the same part for upper limits of series of segments as $p`\kappa$ plays for lower limits. It has, however, many more other uses than $p`\kappa$, and is altogether a more important conception. Thus in cardinal arithmetic, if no two members of κ have any term in common, the arithmetical sum of the numbers of members possessed by the various members of κ is the number of members possessed by $s`\kappa$.

The product of a class of relations (λ say) is the relation which holds between x and y when x and y have every relation of the class λ. The definition is

$$\dot{p}`\lambda = \hat{x}\hat{y}(R \in \lambda \,.\supset_R .\, xRy) \quad \text{Df.}$$

The properties of $\dot{p}`\lambda$ are analogous to those of $p`\kappa$, but its uses are fewer.

The sum of a class of relations (λ say) is the relation which holds between x and y whenever there is a relation of the class λ which holds between x and y. The definition is

$$\dot{s}'\lambda = \hat{x}\hat{y}\{(\exists R) . R \epsilon \lambda . xRy\} \quad \text{Df.}$$

This conception, though less important than $s'\kappa$, is more important than $\dot{p}'\lambda$. The summation of series and ordinal numbers depends upon it, though the connection is less immediate than that of the summation of cardinal numbers with $s'\kappa$.

Instead of defining $p'\kappa$, $s'\kappa$, $\dot{p}'\lambda$, $\dot{s}'\lambda$, it would be formally more correct to define p, s, $\dot{p}$ and $\dot{s}$, which are the relations giving rise to the above descriptive functions. Thus we should have

$$p = \hat{\beta}\hat{\kappa}\{\beta = \hat{x}(\alpha \epsilon \kappa . \supset_\alpha . x \epsilon \alpha)\} \quad \text{Df,}$$

whence we should proceed to

$$\vdash : \beta p \kappa . \equiv . \beta = \hat{x}(\alpha \epsilon \kappa . \supset_\alpha . x \epsilon \alpha),$$

$$\vdash . p'\kappa = \hat{x}(\alpha \epsilon \kappa . \supset_\alpha . x \epsilon \alpha),$$

and $\qquad \vdash . \text{E}\,!\, p'\kappa.$

But in cases where the relation, as opposed to the descriptive function, is very seldom required, it is simpler and easier to give the definition of the descriptive function in the first instance. In such cases, the relation is always tacitly assumed to be also defined; *i.e.* when we give a definition of the form

$$R'x = S'x \quad \text{Df,}$$

where S is some previously defined relation, we always assume that this definition is to be regarded as derived from

$$R = \hat{u}\hat{x}(u = S'x) \quad \text{Df.}$$

In addition to products and sums, we deal, in the present section, with certain properties of the relations $R|$ and $|S$, the meanings of which result from the notation introduced in $*38$. Such relations are very useful in arithmetic. The reason for dealing with them in the present section is that a large proportion of the propositions to be proved involve sums of classes of classes or relations.

$*40$. PRODUCTS AND SUMS OF CLASSES OF CLASSES

Summary of $*40$.

In this number, we introduce the two notations (explained above)

$$p`\kappa = \hat{x}(\alpha \in \kappa \,.\supset_\alpha .\, x \in \alpha) \quad \text{Df}$$

$$s`\kappa = \hat{x}\{(\exists\alpha) \,.\, \alpha \in \kappa \,.\, x \in \alpha\} \quad \text{Df}$$

Both these notations will be found increasingly useful as we proceed, but $s`\kappa$ remains more useful than $p`\kappa$ throughout. It is required for the significance of $p`\kappa$ and $s`\kappa$ that κ should be a class of classes.

In the present number, the most useful propositions are the following:

$*40\cdot12$. $\vdash : \alpha \in \kappa \,.\supset .\, p`\kappa \subset \alpha$

I.e. the product of κ is contained in every member of κ.

$*40\cdot13$. $\vdash : \alpha \in \kappa \,.\supset .\, \alpha \subset s`\kappa$

I.e. every member of κ is contained in the sum of κ.

$*40\cdot15$. $\vdash :. \beta \subset p`\kappa \,.\equiv : \gamma \in \kappa \,.\supset_\gamma .\, \beta \subset \gamma$

I.e. β is contained in the product of κ if β is contained in every member of κ, and vice versa.

$*40\cdot151$. $\vdash :. s`\kappa \subset \beta \,.\equiv : \gamma \in \kappa \,.\supset_\gamma .\, \gamma \subset \beta$

I.e. the sum of κ is contained in β if every member of κ is contained in β, and vice versa.

$*40\cdot2$. $\vdash : \kappa = \Lambda \,.\supset .\, p`\kappa = \mathrm{V}$

I.e. the product of the null-class of classes is the universal class. This may seem paradoxical at first sight, but it is really not so. The fewer members κ has, the larger, speaking generally, $p`\kappa$ becomes. If $\kappa$ has no members, then $\kappa$ has no members to which a given term $x$ does not belong, and therefore $x$ belongs to $p`\kappa$.

$*40\cdot23$. $\vdash : \exists ! \kappa \,.\supset .\, p`\kappa \subset s`\kappa$

I.e. unless κ is null, its product is contained in its sum.

$*40\cdot38$. $\vdash .\, R``s`\kappa = s`R```\kappa$

This proposition is very often used in arithmetic. What it states is as follows: Given a class of classes κ, take its sum, $s`\kappa$, and then consider all the terms that have the relation $R$ to some member of $s`\kappa$; this gives the class $R``s`\kappa$; next, take each separate member of $\kappa$, say $\alpha$, and form the class $R``\alpha$, consisting of all terms having the relation R to some member of α. The class of all such classes as $R``\alpha$, for various α's which are members of κ, is $R```\kappa$; the sum of this class, by the above proposition, is the same as $R``s`\kappa$.

$*40\cdot4$. $\vdash :. \mathrm{E} !! R``\beta \,.\supset .\, s`R``\beta = \hat{x}\{(\exists y) \,.\, y \in \beta \,.\, x \in R`y\}$

This proposition requires, for significance, that $R`y$ should always be a

class. The proposition states that, if $R`y$ always exists when $y \in \beta$, then the sum of all classes which have the relation $R$ to some member of $\beta$ consists of all members of such classes as $R`y$, where $y \in \beta$.

***40·5.** $\vdash . s`\overrightarrow{R}``\beta = R``\beta$

This proposition results from *40·4 by substituting $\overrightarrow{R}$ for R in that proposition.

***40·51.** $\vdash . p`\overrightarrow{R}``\beta = \hat{x}\{y \in \beta . \supset_y . xRy\}$

In virtue of *40·5, $p`\overrightarrow{R}``\beta$ is correlative to $R``\beta$. Thus if R is a serial relation, $p`\overrightarrow{R}``\beta$ consists of terms preceding the whole of $\beta$, and $R``\beta$ consists of terms preceding part of β. If β has a lower limit, it will be the upper limit or maximum of $p`\overrightarrow{R}``\beta$; if $\beta$ has an upper limit, it will be the upper limit of $R``\beta$.

***40·61.** $\vdash : \exists ! \beta . \supset . p`\overrightarrow{R}``\beta \subset R``\beta . p`\overleftarrow{R}``\beta \subset \breve{R}``\beta$

In this proposition the hypothesis is essential, since, if $\beta = \Lambda$, $p`\overrightarrow{R}``\beta = V$ and $R``\beta = \Lambda$.

***40·01.** $p`\kappa = \hat{x}(\alpha \in \kappa . \supset_\alpha . x \in \alpha)$ Df

***40·02.** $s`\kappa = \hat{x}\{(\exists \alpha) . \alpha \in \kappa . x \in \alpha\}$ Df

***40·1.** $\vdash :. x \in p`\kappa . \equiv : \alpha \in \kappa . \supset_\alpha . x \in \alpha$ [*20·3 . (*40·01)]

***40·11.** $\vdash : x \in s`\kappa . \equiv . (\exists \alpha) . \alpha \in \kappa . x \in \alpha$ [*20·3 . (*40·02)]

***40·12.** $\vdash : \alpha \in \kappa . \supset . p`\kappa \subset \alpha$

Dem.

$\vdash$. *40·1 . *10·1 . $\supset \vdash :. x \in p`\kappa . \supset : \alpha \in \kappa . \supset . x \in \alpha :.$

[Comm] $\supset \vdash :. \alpha \in \kappa . \supset : x \in p`\kappa . \supset . x \in \alpha$ (1)

$\vdash$. (1) . *10·11·21 . *22·1 . $\supset \vdash$. Prop

***40·13.** $\vdash : \alpha \in \kappa . \supset . \alpha \subset s`\kappa$

Dem.

$\vdash$. *40·11 . *10·24 . $\supset \vdash : \alpha \in \kappa . x \in \alpha . \supset . x \in s`\kappa :$

[Exp] $\supset \vdash :. \alpha \in \kappa . \supset : x \in \alpha . \supset . x \in s`\kappa$ (1)

$\vdash$. (1) . *10·11·21 . *22·1 . $\supset \vdash$. Prop

***40·14.** $\vdash : \alpha \in \kappa . x \in p`\kappa . \supset . x \in \alpha$ [*40·12 . Imp]

***40·141.** $\vdash : \alpha \in \kappa . x \in \alpha . \supset . x \in s`\kappa$ [*40·11 . *10·24]

***40·15.** $\vdash :. \beta \subset p`\kappa . \equiv : \gamma \in \kappa . \supset_\gamma . \beta \subset \gamma$

Dem.

$\vdash$. *40·1 . $\supset \vdash :: \beta \subset p`\kappa : \equiv :. x \in \beta . \supset_x : \gamma \in \kappa . \supset_\gamma . x \in \gamma :.$

[*11·62] $\equiv :. (x, \gamma) : x \in \beta . \gamma \in \kappa . \supset . x \in \gamma :.$

[*4·3·84.*11·33] $\equiv :. (x, \gamma) : \gamma \in \kappa . x \in \beta . \supset . x \in \gamma :.$

[*11·2·62] $\equiv :. \gamma \in \kappa . \supset_\gamma : x \in \beta . \supset_x . x \in \gamma :.$

[*22·1] $\equiv :. \gamma \in \kappa . \supset_\gamma . \beta \subset \gamma :: \supset \vdash$. Prop

***40·151.** $\vdash :. s'\kappa \subset \beta . \equiv : \gamma \in \kappa . \supset_\gamma . \gamma \subset \beta$

Dem.

$\vdash . *40\cdot11 . \supset \vdash :: s'\kappa \subset \beta . \equiv :. (\exists\gamma) . \gamma \in \kappa . x \in \gamma . \supset_x . x \in \beta :.$
[*10·23] $\equiv :. (\gamma, x) :. \gamma \in \kappa . x \in \gamma . \supset . x \in \beta :.$
[*11·62] $\equiv :. (\gamma) :. \gamma \in \kappa . \supset : (x) : x \in \gamma . \supset . x \in \beta :.$
[*22·1] $\equiv :. \gamma \in \kappa . \supset_\gamma . \gamma \subset \beta :: \supset \vdash$. Prop

This proposition is frequently used.

***40·16.** $\vdash : \kappa \subset \lambda . \supset . p'\lambda \subset p'\kappa$

Dem.

$\vdash . *10\cdot1 . \supset \vdash ::$ Hp $. \supset :. \gamma \in \kappa . \supset . \gamma \in \lambda :.$
[Syll] $\supset :. \gamma \in \lambda . \supset . x \in \gamma : \supset : \gamma \in \kappa . \supset . x \in \gamma$ (1)
$\vdash . (1) . *10\cdot11\cdot21 . \supset$
$\vdash ::$ Hp . $\supset :. (\gamma) :. \gamma \in \lambda . \supset . x \in \gamma : \supset : \gamma \in \kappa . \supset . x \in \gamma :.$
[*10·27] $\supset :. (\gamma) : \gamma \in \lambda . \supset . x \in \gamma : \supset : (\gamma) : \gamma \in \kappa . \supset . x \in \gamma :.$
[*40·1] $\supset :. x \in p'\lambda . \supset . x \in p'\kappa$ (2)
$\vdash . (2) . *10\cdot11\cdot21 . \supset \vdash$. Prop

***40·161.** $\vdash : \kappa \subset \lambda . \supset . s'\kappa \subset s'\lambda$

Dem.

$\vdash . *10\cdot1 . \supset \vdash :.$ Hp $. \supset : \gamma \in \kappa . \supset . \gamma \in \lambda :$
[Fact] $\supset : \gamma \in \kappa . x \in \gamma . \supset . \gamma \in \lambda . x \in \gamma :$
[*10·11·28] $\supset : (\exists\gamma) . \gamma \in \kappa . x \in \gamma . \supset . (\exists\gamma) . \gamma \in \lambda . x \in \gamma :$
[*40·11] $\supset : x \in s'\kappa . \supset . x \in s'\lambda$ (1)
$\vdash . (1) . *10\cdot11\cdot21 . \supset \vdash$. Prop

***40·17.** $\vdash . p'\kappa \cup p'\lambda \subset p'(\kappa \cap \lambda)$

Dem.

$\vdash . *22\cdot34 . \supset \vdash :: x \in p'\kappa \cup p'\lambda . \equiv :. x \in p'\kappa . \vee . x \in p'\lambda :.$
[*40·1] $\equiv :. \gamma \in \kappa . \supset_\gamma . x \in \gamma : \vee : \gamma \in \lambda . \supset_\gamma . x \in \gamma :.$
[*10·41] $\supset :. (\gamma) :. \gamma \in \kappa . \supset . x \in \gamma : \vee : \gamma \in \lambda . \supset . x \in \gamma :.$
[*4·79] $\supset :. (\gamma) : \gamma \in \kappa . \gamma \in \lambda . \supset . x \in \gamma :.$
[*22·33] $\supset :. (\gamma) : \gamma \in \kappa \cap \lambda . \supset . x \in \gamma :.$
[*40·1] $\supset :. x \in p'(\kappa \cap \lambda)$ (1)
$\vdash . (1) . *10\cdot11 . \supset \vdash$. Prop

***40·171.** $\vdash . s'\kappa \cup s'\lambda = s'(\kappa \cup \lambda)$

Dem.

$\vdash . *22\cdot34 . \supset \vdash :: x \in s'\kappa \cup s'\lambda . \equiv :. x \in s'\kappa . \vee . x \in s'\lambda :.$
[*40·11] $\equiv :. (\exists\gamma) . \gamma \in \kappa . x \in \gamma : \vee : (\exists\gamma) . \gamma \in \lambda . x \in \gamma :.$
[*10·42] $\equiv :. (\exists\gamma) : \gamma \in \kappa . x \in \gamma . \vee . \gamma \in \lambda . x \in \gamma :.$
[*4·4] $\equiv :. (\exists\gamma) :. \gamma \in \kappa . \vee . \gamma \in \lambda : x \in \gamma :.$

[$*22\cdot34$] $\equiv :. (\exists\gamma) . \gamma \in \kappa \cup \lambda . x \in \gamma :.$
[$*40\cdot11$] $\equiv :. x \in s`(\kappa \cup \lambda) :: \supset \vdash .$ Prop

$*40\cdot18$. $\vdash . p`(\kappa \cup \lambda) = p`\kappa \cap p`\lambda$

Dem.

$\vdash . *40\cdot1 . \supset \vdash :: x \in p`(\kappa \cup \lambda) . \equiv :. \gamma \in \kappa \cup \lambda . \supset_\gamma . x \in \gamma :.$
[$*22\cdot34$] $\equiv :. (\gamma) :. \gamma \in \kappa . \vee . \gamma \in \lambda : \supset . x \in \gamma :.$
[$*4\cdot77$] $\equiv :. (\gamma) :. \gamma \in \kappa . \supset . x \in \gamma : \gamma \in \lambda . \supset . x \in \gamma :.$
[$*10\cdot22\cdot221$] $\equiv :. (\gamma) : \gamma \in \kappa . \supset . x \in \gamma :. (\gamma) : \gamma \in \lambda . \supset . x \in \gamma :.$
[$*40\cdot1$] $\equiv :. x \in p`\kappa . x \in p`\lambda :.$
[$*22\cdot33$] $\equiv :. x \in p`\kappa \cap p`\lambda :: \supset \vdash .$ Prop

$*40\cdot181$. $\vdash . s`(\kappa \cap \lambda) \subset s`\kappa \cap s`\lambda$

Dem.

$\vdash . *40\cdot11 . \supset \vdash :: x \in s`(\kappa \cap \lambda) . \equiv :. (\exists\gamma) . \gamma \in \kappa \cap \lambda . x \in \gamma :.$
[$*22\cdot33$] $\equiv :. (\exists\gamma) . \gamma \in \kappa . \gamma \in \lambda . x \in \gamma :.$
[$*10\cdot5$] $\supset :. (\exists\gamma) . \gamma \in \kappa . x \in \gamma : (\exists\gamma) . \gamma \in \lambda . x \in \gamma :.$
[$*40\cdot11 . *22\cdot33$] $\supset :. x \in s`\kappa \cap s`\lambda :: \supset \vdash .$ Prop

$*40\cdot19$. $\vdash :: x \in s`\kappa . \equiv :. \gamma \in \kappa . \supset_\gamma . \gamma \subset \beta : \supset_\beta . x \in \beta$

This proposition is the extension of $*22\cdot6$.

Dem.

$\vdash . *40\cdot151 . \supset$
$\vdash :: \gamma \in \kappa . \supset_\gamma . \gamma \subset \beta : \supset_\beta . x \in \beta :. \equiv :. s`\kappa \subset \beta . \supset_\beta . x \in \beta$ (1)
$\vdash . *10\cdot1 . \supset \vdash :. s`\kappa \subset \beta . \supset_\beta . x \in \beta : \supset : s`\kappa \subset s`\kappa . \supset . x \in s`\kappa :$
[$*22\cdot42$] $\supset : x \in s`\kappa$ (2)
$\vdash . *22\cdot46 . \supset \vdash :. x \in s`\kappa . s`\kappa \subset \beta . \supset . x \in \beta :.$
[Exp] $\supset \vdash :. x \in s`\kappa . \supset : s`\kappa \subset \beta . \supset . x \in \beta :.$
[$*10\cdot11\cdot21$] $\supset \vdash :. x \in s`\kappa . \supset : s`\kappa \subset \beta . \supset_\beta . x \in \beta$ (3)
$\vdash . (2) . (3) . \supset \vdash :. s`\kappa \subset \beta . \supset_\beta . x \in \beta : \equiv . x \in s`\kappa$ (4)
$\vdash . (1) . (4) . \supset \vdash .$ Prop

$*40\cdot2$. $\vdash : \kappa = \Lambda . \supset . p`\kappa = V$

Dem.

$\vdash . *24\cdot5\cdot51 . \supset \vdash :. \text{Hp} . \supset : \sim(\exists\alpha) . \alpha \in \kappa :$
[$*10\cdot53$] $\supset : (\alpha) : \alpha \in \kappa . \supset . x \in \alpha :$
[$*40\cdot1$] $\supset : x \in p`\kappa$ (1)
$\vdash . (1) . *10\cdot11\cdot21 . \supset \vdash : \text{Hp} . \supset . (x) . x \in p`\kappa .$
[$*24\cdot14$] $\supset . p`\kappa = V : \supset \vdash .$ Prop

$*40\cdot21$. $\vdash : \kappa = \Lambda . \supset . s`\kappa = \Lambda$

Dem.

$\vdash . *24\cdot51 . \supset \vdash : \text{Hp} . \supset . \sim(\exists\alpha) . \alpha \in \kappa .$
[$*10\cdot5 .$Transp] $\supset . \sim(\exists\alpha) . \alpha \in \kappa . x \in \alpha .$

$$[*40\cdot11 . \text{Transp}] \qquad \supset . x \sim\epsilon s`\kappa \qquad (1)$$

$$\vdash . (1) . *10\cdot11\cdot21 . \supset \vdash : \text{Hp} . \supset . (x) . x \sim\epsilon s`\kappa .$$

$$[*24\cdot15] \qquad \supset . s`\kappa = \Lambda : \supset \vdash . \text{Prop}$$

In the above proposition, the two Λ's are of different types, since κ is of the type next above that of $s`\kappa$. Thus it would be more correct to write

$$\vdash : \kappa = \Lambda \cap \text{Cls} . \supset . s`\kappa = \Lambda \cap \text{V}.$$

But in the case of Λ it is not very important to keep the types distinct.

***40·22.** $\vdash : \Lambda \epsilon \kappa . \supset . p`\kappa = \Lambda$

Dem.

$$\vdash . *40\cdot12 . \supset \vdash : \text{Hp} . \supset . p`\kappa \subset \Lambda .$$

$$[*24\cdot13] \qquad \supset . p`\kappa = \Lambda : \supset \vdash . \text{Prop}$$

In this proposition, the two Λ's are of the same type.

***40·221.** $\vdash : \text{V} \epsilon \kappa . \supset . s`\kappa = \text{V}$

Dem.

$$\vdash . *40\cdot13 . \supset \vdash : \text{Hp} . \supset . \text{V} \subset s`\kappa .$$

$$[*24\cdot141] \qquad \supset . s`\kappa = \text{V} : \supset \vdash . \text{Prop}$$

***40·23.** $\vdash : \exists ! \kappa . \supset . p`\kappa \subset s`\kappa$

Dem.

$$\vdash . *40\cdot12\cdot13 . \supset \vdash : \alpha \epsilon \kappa . \supset . p`\kappa \subset \alpha . \alpha \subset s`\kappa .$$

$$[*22\cdot44] \qquad \supset . p`\kappa \subset s`\kappa :$$

$$[*10\cdot11\cdot23] \qquad \supset \vdash : (\exists \alpha) . \alpha \epsilon \kappa . \supset . p`\kappa \subset s`\kappa : \supset \vdash . \text{Prop}$$

Observe that the hypothesis $\exists ! \kappa$ is essential to this proposition, since when $\kappa = \Lambda$, $p`\kappa = \text{V}$ and $s`\kappa = \Lambda$. Thus

$$\vdash : \exists ! \kappa . \equiv . p`\kappa \subset s`\kappa.$$

***40·24.** $\vdash :. \exists ! \kappa : \gamma \epsilon \kappa . \supset_\gamma . \beta \subset \gamma : \supset . \beta \subset s`\kappa$

Dem.

$$\vdash . *40\cdot15 . \supset \vdash :. \gamma \epsilon \kappa . \supset_\gamma . \beta \subset \gamma : \supset . \beta \subset p`\kappa \qquad (1)$$

$$\vdash . *40\cdot23 . \supset \vdash : \exists ! \kappa . \supset . p`\kappa \subset s`\kappa \qquad (2)$$

$$\vdash . (1) . (2) . \supset \vdash : \text{Hp} . \supset . \beta \subset p`\kappa . p`\kappa \subset s`\kappa .$$

$$[*22\cdot44] \qquad \supset . \beta \subset s`\kappa : \supset \vdash . \text{Prop}$$

The above proposition is used in the proof of *215·25.

***40·25.** $\vdash : x \epsilon s`\kappa . \equiv . \exists ! \kappa \cap \hat{\alpha}(x \epsilon \alpha)$

Dem.

$$\vdash . *22\cdot33 . \supset \vdash : \exists ! \kappa \cap \hat{\alpha}(x \epsilon \alpha) . \equiv . (\exists \gamma) . \gamma \epsilon \kappa . \gamma \epsilon \hat{\alpha}(x \epsilon \alpha) .$$

$$[*20\cdot3] \qquad \equiv . (\exists \gamma) . \gamma \epsilon \kappa . x \epsilon \gamma .$$

$$[*40\cdot11] \qquad \equiv . x \epsilon s`\kappa : \supset \vdash . \text{Prop}$$

***40·26.** $\vdash : \exists ! s`\kappa . \equiv . (\exists \alpha) . \alpha \epsilon \kappa . \exists ! \alpha$

Dem.

$$\vdash . *40\cdot11 . \supset \vdash :. \exists ! s`\kappa . \equiv : (\exists x) : (\exists \alpha) . \alpha \epsilon \kappa . x \epsilon \alpha :$$

$$[*11\cdot23\cdot55] \qquad \equiv : (\exists \alpha) : \alpha \epsilon \kappa : (\exists x) . x \epsilon \alpha :$$

$$[*24\cdot5] \qquad \equiv : (\exists \alpha) . \alpha \epsilon \kappa . \exists ! \alpha :. \supset \vdash . \text{Prop}$$

The following proposition is used in the proof of $*216{\cdot}51$.

$*40{\cdot}27$. $\vdash :. \alpha \cap s{}^{\backprime}\kappa = \Lambda . \equiv : \gamma \in \kappa . \supset_\gamma . \alpha \cap \gamma = \Lambda$

Dem.

$\vdash . *24{\cdot}311 . \supset$

$\vdash :: \alpha \cap s{}^{\backprime}\kappa = \Lambda . \equiv :. s{}^{\backprime}\kappa \subset - \alpha :.$

$[*22{\cdot}1{\cdot}35] \quad \equiv :. x \in s{}^{\backprime}\kappa . \supset_x . x \sim\in \alpha :.$

$[*40{\cdot}1] \quad \equiv :. (\exists\gamma) . \gamma \in \kappa . x \in \gamma . \supset_x . x \sim\in \alpha :.$

$[*10{\cdot}23] \quad \equiv :. \gamma \in \kappa . x \in \gamma . \supset_{x,\gamma} . x \sim\in \alpha :.$

$[*11{\cdot}2{\cdot}62] \quad \equiv :. \gamma \in \kappa . \supset_\gamma : x \in \gamma . \supset_x . x \sim\in \alpha :.$

$[*24{\cdot}39] \quad \equiv :. \gamma \in \kappa . \supset_\gamma . \alpha \cap \gamma = \Lambda :: \supset \vdash . \text{Prop}$

The following propositions are only significant when R is a relation whose domain consists of classes, for they concern $p{}^{\backprime}R{}^{\backprime\backprime}\alpha$ or $s{}^{\backprime}R{}^{\backprime\backprime}\alpha$, and therefore require that $R{}^{\backprime\backprime}\alpha$ should be a class of classes.

$*40{\cdot}3$. $\vdash . p{}^{\backprime}R{}^{\backprime\backprime}(\alpha \cup \beta) = p{}^{\backprime}R{}^{\backprime\backprime}\alpha \cap p{}^{\backprime}R{}^{\backprime\backprime}\beta \quad [*37{\cdot}22 . *40{\cdot}18]$

$*40{\cdot}31$. $\vdash . s{}^{\backprime}R{}^{\backprime\backprime}(\alpha \cup \beta) = s{}^{\backprime}R{}^{\backprime\backprime}\alpha \cup s{}^{\backprime}R{}^{\backprime\backprime}\beta \quad [*37{\cdot}22 . *40{\cdot}171]$

$*40{\cdot}32$. $\vdash . p{}^{\backprime}R{}^{\backprime\backprime}\alpha \cup p{}^{\backprime}R{}^{\backprime\backprime}\beta \subset p{}^{\backprime}R{}^{\backprime\backprime}(\alpha \cap \beta)$

Dem.

$\vdash . *37{\cdot}21 . \supset \vdash . R{}^{\backprime\backprime}(\alpha \cap \beta) \subset R{}^{\backprime\backprime}\alpha \cap R{}^{\backprime\backprime}\beta .$

$[*40{\cdot}16] \quad \supset \vdash . p{}^{\backprime}(R{}^{\backprime\backprime}\alpha \cap R{}^{\backprime\backprime}\beta) \subset p{}^{\backprime}R{}^{\backprime\backprime}(\alpha \cap \beta) \qquad (1)$

$\vdash . *40{\cdot}17 . \supset \vdash . p{}^{\backprime}R{}^{\backprime\backprime}\alpha \cup p{}^{\backprime}R{}^{\backprime\backprime}\beta \subset p{}^{\backprime}(R{}^{\backprime\backprime}\alpha \cap R{}^{\backprime\backprime}\beta) \qquad (2)$

$\vdash . (1) . (2) . *22{\cdot}44 . \supset \vdash . \text{Prop}$

$*40{\cdot}33$. $\vdash . s{}^{\backprime}R{}^{\backprime\backprime}(\alpha \cap \beta) \subset s{}^{\backprime}R{}^{\backprime\backprime}\alpha \cap s{}^{\backprime}R{}^{\backprime\backprime}\beta \quad [*37{\cdot}21 . *40{\cdot}161 . *40{\cdot}181]$

The following propositions no longer require that the domain of R should be composed of classes.

$*40{\cdot}35$. $\vdash . p{}^{\backprime}R{}^{\backprime\backprime\backprime}\kappa = \hat{x}\{\beta \in \kappa . \supset_\beta . x \in R{}^{\backprime\backprime}\beta\}$

Dem.

$\vdash . *40{\cdot}1 . \supset \vdash :. x \in p{}^{\backprime}R{}^{\backprime\backprime\backprime}\kappa . \equiv : \gamma \in R{}^{\backprime\backprime\backprime}\kappa . \supset_\gamma . x \in \gamma :$

$[*37{\cdot}103] \quad \equiv : (\exists\beta) . \beta \in \kappa . \gamma = R{}^{\backprime\backprime}\beta . \supset_\gamma . x \in \gamma :$

$[*10{\cdot}23] \quad \equiv : \beta \in \kappa . \gamma = R{}^{\backprime\backprime}\beta . \supset_{\beta,\gamma} . x \in \gamma :$

$[*13{\cdot}191] \quad \equiv : \beta \in \kappa . \supset_\beta . x \in R{}^{\backprime\backprime}\beta \qquad (1)$

$\vdash . (1) . *10{\cdot}11 . *20{\cdot}3 . \supset \vdash . \text{Prop}$

$*40{\cdot}36$. $\vdash . s{}^{\backprime}R{}^{\backprime\backprime\backprime}\kappa = \hat{x}\{(\exists\beta) . \beta \in \kappa . x \in R{}^{\backprime\backprime}\beta\}$ [Similar proof]

$*40{\cdot}37$. $\vdash . R{}^{\backprime\backprime}p{}^{\backprime}\kappa \subset p{}^{\backprime}R{}^{\backprime\backprime\backprime}\kappa$

Dem.

$\vdash . *37{\cdot}1 . \supset \vdash :: x \in R{}^{\backprime\backprime}p{}^{\backprime}\kappa . \equiv :. (\exists y) . y \in p{}^{\backprime}\kappa . xRy :.$

$[*40{\cdot}1] \quad \equiv :. (\exists y) : \beta \in \kappa . \supset_\beta . y \in \beta : xRy :.$

$[*10{\cdot}33] \quad \equiv :. (\exists y) :. (\beta) : \beta \in \kappa . \supset . y \in \beta : xRy :.$

$[*11{\cdot}26] \quad \supset :. (\beta) :. (\exists y) : \beta \in \kappa . \supset . y \in \beta : xRy :.$

$[*5{\cdot}31] \quad \supset :. (\beta) :. (\exists y) : \beta \in \kappa . \supset . y \in \beta . xRy :.$

[∗10·37] $\supset :. (\beta) :. \beta \in \kappa . \supset . (\exists y) . y \in \beta . xRy :.$

[∗37·1] $\supset :. (\beta) : \beta \in \kappa . \supset . x \in R``\beta :.$

[∗40·35] $\supset :. x \in p`R```\kappa :: \supset \vdash .$ Prop

∗40·38. $\vdash . R``s`\kappa = s`R```\kappa$

Dem.

$\vdash . ∗37·1 . \supset \vdash :: x \in R``s`\kappa . \equiv :. (\exists y) . y \in s`\kappa . xRy :.$

[∗40·11] $\equiv :. (\exists y) :. (\exists \alpha) . \alpha \in \kappa . y \in \alpha : xRy :.$

[∗11·6] $\equiv :. (\exists \alpha) :. \alpha \in \kappa : (\exists y) . y \in \alpha . xRy :.$

[∗37·1] $\equiv :. (\exists \alpha) . \alpha \in \kappa . x \in R``\alpha :.$

[∗40·36] $\equiv :. x \in s`R```\kappa :: \supset \vdash .$ Prop

This proposition is frequently used in the proofs of arithmetical propositions.

∗40·4. $\vdash : E!! R``\beta . \supset . s`R``\beta = \hat{x} \{(\exists y) . y \in \beta . x \in R`y\}$

This proposition is only significant when $D`R \subset$ Cls.

Dem.

$\vdash . ∗37·6 . \supset \vdash : Hp . \supset . R``\beta = \hat{\alpha} \{(\exists y) . y \in \beta . \alpha = R`y\}$ (1)

$\vdash . (1) . ∗40·11 . \supset$

$\vdash :: Hp . \supset :. x \in s`R``\beta . \equiv : (\exists \alpha) : (\exists y) . y \in \beta . \alpha = R`y : x \in \alpha :$

[∗11·6] $\equiv : (\exists y) : y \in \beta : (\exists \alpha) . \alpha = R`y . x \in \alpha :$

[∗14·205] $\equiv : (\exists y) . y \in \beta . x \in R`y :: \supset \vdash .$ Prop

∗40·41. $\vdash : E!! R``\beta . \supset . p`R``\beta = \hat{x} \{y \in \beta . \supset_y . x \in R`y\}$ [Similar proof]

∗40·42. $\vdash : (x) . R`x = P`x \cup Q`x . \supset . s`R``\alpha = s`(P``\alpha \cup Q``\alpha) = s`P``\alpha \cup s`Q``\alpha$

Dem.

$\vdash . ∗14·21 . \supset \vdash : Hp . \supset . (x) . E!R`x . E!P`x . E!Q`x$ (1)

$\vdash . (1) . ∗40·4 . \supset \vdash : Hp . \supset . s`R``\alpha = \hat{x} \{(\exists y) . y \in \alpha . x \in R`y\}$

[Hp] $= \hat{x} \{(\exists y) . y \in \alpha . x \in P`y \cup Q`y\}$

[∗22·34] $= \hat{x} \{(\exists y) : y \in \alpha : x \in P`y . \vee . x \in Q`y\}$

[∗4·4.∗10·42] $= \hat{x} \{(\exists y) . y \in \alpha . x \in P`y . \vee . (\exists y) . y \in \alpha . x \in Q`y\}$

[(1).∗40·4] $= \hat{x} \{x \in s`P``\alpha . \vee . x \in s`Q``\alpha\}$

[∗20·42.∗22·34] $= s`P``\alpha \cup s`Q``\alpha$

[∗40·171] $= s`(P``\alpha \cup Q``\alpha) : \supset \vdash .$ Prop

This proposition is used in ∗40·57, where we take $R = C$, $P = D$, $Q = \text{Ꮯ}$.

∗40·43. $\vdash :: E!! R``\beta . \supset :. s`R``\beta \subset \alpha . \equiv : y \in \beta . \supset_y . R`y \subset \alpha$

Dem.

$\vdash . ∗37·63 . \supset \vdash :: Hp . \supset :. y \in \beta . \supset_y . R`y \subset \alpha : \equiv : \gamma \in R``\beta . \supset_\gamma . \gamma \subset \alpha :$

[∗40·151] $\equiv : s`R``\beta \subset \alpha :: \supset \vdash .$ Prop

∗40·44. $\vdash :: E!! R``\beta . \supset :. \alpha \subset p`R``\beta . \equiv : y \in \beta . \supset_y . \alpha \subset R`y$

Dem.

$\vdash . ∗37·63 . \supset \vdash :: Hp . \supset :. y \in \beta . \supset_y . \alpha \subset R`y : \equiv : \gamma \in R``\beta . \supset_\gamma . \alpha \subset \gamma :$

[∗40·15] $\equiv : \alpha \subset p`R``\beta :: \supset \vdash .$ Prop

The following proposition is used in the proof of $*84{\cdot}44$.

$*40{\cdot}45$. $\vdash :. y \epsilon \beta . \supset_y . R{}^{\backprime}y \subset S{}^{\backprime}y : \supset . s{}^{\backprime}R{}^{\backprime\backprime}\beta \subset s{}^{\backprime}S{}^{\backprime\backprime}\beta$

Dem.

$\vdash . *14{\cdot}21 . \supset \vdash :. \mathrm{Hp} . \supset : \mathrm{E}!!\, S{}^{\backprime\backprime}\beta . \mathrm{E}!!\, R{}^{\backprime\backprime}\beta :$ (1)

$[*37{\cdot}62 . *40{\cdot}13] \quad \supset : y \epsilon \beta . \supset_y . S{}^{\backprime}y \subset s{}^{\backprime}S{}^{\backprime\backprime}\beta :$

$[\mathrm{Hp}] \quad \supset : y \epsilon \beta . \supset_y . R{}^{\backprime}y \subset s{}^{\backprime}S{}^{\backprime\backprime}\beta :$

$[*40{\cdot}43 . (1)] \quad \supset : s{}^{\backprime}R{}^{\backprime\backprime}\beta \subset s{}^{\backprime}S{}^{\backprime\backprime}\beta :. \supset \vdash . \mathrm{Prop}$

The following proposition is used in the proof of $*94{\cdot}402$.

$*40{\cdot}451$. $\vdash :. y \epsilon \beta . \supset_y . R{}^{\backprime}y \subset S{}^{\backprime}y : \supset . p{}^{\backprime}R{}^{\backprime\backprime}\beta \subset p{}^{\backprime}S{}^{\backprime\backprime}\beta$

Dem.

$\vdash . *14{\cdot}21 . *37{\cdot}62 . *40{\cdot}12 . \supset \vdash :. \mathrm{Hp} . \supset : y \epsilon \beta . \supset . p{}^{\backprime}R{}^{\backprime\backprime}\beta \subset R{}^{\backprime}y .$

$[\mathrm{Hp}] \quad \supset . p{}^{\backprime}R{}^{\backprime\backprime}\beta \subset S{}^{\backprime}y .$

$[*40{\cdot}44] \quad \supset : p{}^{\backprime}R{}^{\backprime\backprime}\beta \subset p{}^{\backprime}S{}^{\backprime\backprime}\beta :. \supset \vdash . \mathrm{Prop}$

$*40{\cdot}5$. $\vdash . s{}^{\backprime}\overrightarrow{R}{}^{\backprime\backprime}\beta = R{}^{\backprime\backprime}\beta$

Dem.

$\vdash . *32{\cdot}12 . *40{\cdot}4 . \supset \vdash . s{}^{\backprime}\overrightarrow{R}{}^{\backprime\backprime}\beta = \hat{x}\{(\exists y) . y \epsilon \beta . x \epsilon \overrightarrow{R}{}^{\backprime}y\}$

$[*32{\cdot}18] \quad = \hat{x}\{(\exists y) . y \epsilon \beta . xRy\}$

$[(*37{\cdot}01)] \quad = R{}^{\backprime\backprime}\beta . \supset \vdash . \mathrm{Prop}$

$*40{\cdot}51$. $\vdash . p{}^{\backprime}\overrightarrow{R}{}^{\backprime\backprime}\beta = \hat{x}\{y \epsilon \beta . \supset_y . xRy\}$ $[*32{\cdot}12 . *40{\cdot}41 . *32{\cdot}18]$

$p{}^{\backprime}\overrightarrow{R}{}^{\backprime\backprime}\beta$ is the class of terms each of which has the relation R to *every* member of β, just as $R{}^{\backprime\backprime}\beta$ is the class of terms each of which has the relation R to *some* member of β. In the theory of series, $p{}^{\backprime}\overrightarrow{R}{}^{\backprime\backprime}\beta$ plays an important part, correlative to that played by $R{}^{\backprime\backprime}\beta$ (which is $s{}^{\backprime}\overrightarrow{R}{}^{\backprime\backprime}\beta$, by $*40{\cdot}5$). If β is a class contained in a series whose generating relation is R, $p{}^{\backprime}\overrightarrow{R}{}^{\backprime\backprime}\beta$ will be the predecessors of all members of β, while $R{}^{\backprime\backprime}\beta$ will be the predecessors of some β.

$*40{\cdot}52$. $\vdash . s{}^{\backprime}\overleftarrow{R}{}^{\backprime\backprime}\beta = \breve{R}{}^{\backprime\backprime}\beta$ [Proof as in $*40{\cdot}5$]

$*40{\cdot}53$. $\vdash . p{}^{\backprime}\overleftarrow{R}{}^{\backprime\backprime}\beta = \hat{y}\{x \epsilon \beta . \supset_x . xRy\}$ [Proof as in $*40{\cdot}51$]

$*40{\cdot}54$. $\vdash . p{}^{\backprime}\overrightarrow{R}{}^{\backprime\backprime}\beta = \hat{x}(\beta \subset \overleftarrow{R}{}^{\backprime}x)$ $[*40{\cdot}51 . *32{\cdot}181]$

$*40{\cdot}55$. $\vdash . p{}^{\backprime}\overleftarrow{R}{}^{\backprime\backprime}\alpha = \hat{y}(\alpha \subset \overrightarrow{R}{}^{\backprime}y)$ $[*40{\cdot}53 . *32{\cdot}18]$

From this point onwards to $*40{\cdot}69$, the propositions are inserted on account of their use in the theory of series.

$*40{\cdot}56$. $\vdash . s{}^{\backprime}C{}^{\backprime\backprime}\lambda = F{}^{\backprime\backprime}\lambda$ $[*33{\cdot}5 . *40{\cdot}5]$

In the above proposition, the conditions of significance require that λ should be a class of relations.

$*40{\cdot}57$. $\vdash . s{}^{\backprime}C{}^{\backprime\backprime}\lambda = s{}^{\backprime}(\mathrm{D}{}^{\backprime\backprime}\lambda \cup \mathrm{G}{}^{\backprime\backprime}\lambda) = s{}^{\backprime}\mathrm{D}{}^{\backprime\backprime}\lambda \cup s{}^{\backprime}\mathrm{G}{}^{\backprime\backprime}\lambda$ $[*40{\cdot}42 . *33{\cdot}16]$

✱40·6. $\vdash . p`\overrightarrow{R}``\Lambda = V . p`\overleftarrow{R}``\Lambda = V$ [✱37·29 . ✱40·2]

✱40·61. $\vdash : \exists ! \beta . \supset . p`\overrightarrow{R}``\beta \subset R``\beta . p`\overleftarrow{R}``\beta \subset \breve{R}``\beta$

Dem.

$\vdash . ✱37·73 . \supset \vdash : \mathrm{Hp} . \supset . \exists ! \overrightarrow{R}``\beta .$

[✱40·23] $\supset . p`\overrightarrow{R}``\beta \subset s`\overrightarrow{R}``\beta .$

[✱40·5] $\supset . p`\overrightarrow{R}``\beta \subset R``\beta$ (1)

Similarly $\vdash : \mathrm{Hp} . \supset . p`\overleftarrow{R}``\beta \subset \breve{R}``\beta$ (2)

$\vdash . (1) . (2) . \supset \vdash .$ Prop

✱40·62. $\vdash : \exists ! \beta . \supset . p`\overrightarrow{R}``\beta \subset C`R . p`\overleftarrow{R}``\beta \subset C`R$
[✱40·61 . ✱37·15·16 . ✱33·161]

The two following propositions (✱40·63·64) are used in proving ✱40·65, which is used in ✱204·63.

✱40·63. $\vdash : \exists ! \beta - \text{Ɑ}`R . \supset . p`\overrightarrow{R}``\beta = \Lambda$

Dem.

$\vdash . ✱33·41 . \text{Transp} . \supset \vdash : x \sim\in \text{Ɑ}`R . \supset . \overrightarrow{R}`x = \Lambda$ (1)

$\vdash . ✱37·704 . \supset \vdash : x \in \beta . \supset . \overrightarrow{R}`x \in \overrightarrow{R}``\beta$ (2)

$\vdash . (1) . (2) . ✱22·32 . \supset \vdash : x \in \beta - \text{Ɑ}`R . \supset . \overrightarrow{R}`x \in \overrightarrow{R}``\beta . \overrightarrow{R}`x = \Lambda .$

[✱20·57] $\supset . \Lambda \in \overrightarrow{R}``\beta .$

[✱40·22] $\supset . p`\overrightarrow{R}``\beta = \Lambda$ (3)

$\vdash . (3) . ✱10·11·23 . \supset \vdash .$ Prop

✱40·64. $\vdash : \exists ! \beta - \mathrm{D}`R . \supset . p`\overleftarrow{R}``\beta = \Lambda$ [Proof as in ✱40·63]

✱40·65. $\vdash : \exists ! \beta - C`R . \supset . p`\overrightarrow{R}``\beta = \Lambda . p`\overleftarrow{R}``\beta = \Lambda$ [✱40·63·64 . ✱33·16]

✱40·66. $\vdash :. \alpha \subset p`\overrightarrow{R}``\beta . \equiv : x \in \alpha . y \in \beta . \supset_{x,y} . xRy$

Dem.

$\vdash . ✱40·51 . \supset \vdash :: \alpha \subset p`\overrightarrow{R}``\beta . \equiv :. \alpha \subset \hat{x}(y \in \beta . \supset_y . xRy) :.$

[✱20·3] $\equiv :. x \in \alpha . \supset_x : y \in \beta . \supset_y . xRy :.$

[✱11·62] $\equiv :. (x, y) :. x \in \alpha . y \in \beta . \supset . xRy :: \supset \vdash .$ Prop

✱40·67. $\vdash :. \beta \subset p`\overleftarrow{R}``\alpha . \equiv : x \in \alpha . y \in \beta . \supset_{x,y} . xRy : \equiv . \alpha \subset p`\overrightarrow{R}``\beta$
[Proof as in ✱40·66]

✱40·68. $\vdash . \alpha \cap p`\overleftarrow{P}``\alpha \subset \breve{P}``p`\overleftarrow{P}``\alpha$

Dem.

$\vdash . ✱40·53 . \supset \vdash :. x \in \alpha \cap p`\overleftarrow{P}``\alpha . \supset : x \in \alpha : y \in \alpha . \supset_y . yPx :$

[✱10·26] $\supset : xPx : y \in \alpha . \supset_y . yPx :$

$[*10\cdot24] \qquad \supset : (\exists z) : zPx : y \epsilon \alpha . \supset_y . yPz :$

$[*40\cdot53 . *37\cdot105] \qquad \supset : x \epsilon \breve{P}``p`\overleftarrow{P}``\alpha :. \supset \vdash . \text{Prop}$

This proposition is used in the theory of series ($*206\cdot2$).

$*40\cdot681$. $\vdash . \alpha \cap p`\overrightarrow{P}``\alpha \subset P``p`\overrightarrow{P}``\alpha$ [Proof as in $*40\cdot68$]

The following proposition is used in $*211\cdot56$.

$*40\cdot682$. $\vdash : \exists ! \alpha \cap p`\overleftarrow{P}``\beta . \supset . \beta \subset P``\alpha$

Dem.

$\vdash . *40\cdot53 . \supset \vdash :. \text{Hp} . \supset : (\exists x) : x \epsilon \alpha : y \epsilon \beta . \supset_y . yPx :$

$[*5\cdot31] \qquad \supset : (\exists x) : y \epsilon \beta . \supset_y . x \epsilon \alpha . yPx :$

$[*11\cdot61] \qquad \supset : y \epsilon \beta . \supset_y . (\exists x) . x \epsilon \alpha . yPx .$

$[*37\cdot1] \qquad \supset_y . y \epsilon P``\alpha :. \supset \vdash . \text{Prop}$

$*40\cdot69$. $\vdash : \exists ! C`P \cap p`\overleftarrow{P}``\alpha . \equiv . \dot{\exists} ! P . \exists ! p`\overleftarrow{P}``\alpha$

Dem.

$\vdash . *33\cdot24 . *24\cdot561 . \supset \vdash : \exists ! C`P \cap p`\overleftarrow{P}``\alpha . \supset . \dot{\exists} ! P . \exists ! p`\overleftarrow{P}``\alpha \quad (1)$

$\vdash . *40\cdot62 . \qquad \supset \vdash : \exists ! \alpha . \exists ! p`\overleftarrow{P}``\alpha . \supset . \exists ! C`P \cap p`\overleftarrow{P}``\alpha \quad (2)$

$\vdash . *40\cdot6 . \qquad \supset \vdash :. \alpha = \Lambda . \supset : C`P \cap p`\overleftarrow{P}``\alpha = C`P :$

$[*33\cdot24] \qquad \supset : \dot{\exists} ! P . \supset . \exists ! C`P \cap p`\overleftarrow{P}``\alpha \quad (3)$

$\vdash . (2) . (3) . *4\cdot83 . \quad \supset \vdash : \dot{\exists} ! P . \exists ! p`\overleftarrow{P}``\alpha . \supset . \exists ! C`P \cap p`\overleftarrow{P}``\alpha \quad (4)$

$\vdash . (1) . (4) . \qquad \supset \vdash . \text{Prop}$

The above propositions concerning $p`\overrightarrow{R}``\beta$ and $p`\overleftarrow{R}``\beta$ of course have analogues for $s`\overrightarrow{R}``\beta$ and $s`\overleftarrow{R}``\beta$. But owing to $*40\cdot5$, these analogues are more simply stated as properties of $R``\beta$ and $\breve{R}``\beta$. Thus, for example, $*37\cdot264$ is the analogue of $*40\cdot67$. The above propositions concerning $p`\overrightarrow{R}``\beta$ and $p`\overleftarrow{R}``\beta$ will be used in the theory of series, but until we reach that stage they will seldom be referred to.

$*40\cdot7$. $\vdash . s`\alpha \underset{,,}{\female}``\beta = \hat{z} \{(\exists x, y) . x \epsilon \alpha . y \epsilon \beta . z = x \female y\}$

Dem.

$\vdash . *40\cdot11 . *38\cdot3 . \supset$

$\vdash . s`\alpha \underset{,,}{\female}``\beta = \hat{z} \{(\exists \gamma, y) . y \epsilon \beta . \gamma = \female y``\alpha . z \epsilon \gamma\}$

$[*38\cdot131] \quad = \hat{z} \{(\exists \gamma, x, y) . y \epsilon \beta . \gamma = \female y``\alpha . x \epsilon \alpha . z = x \female y\}$

$[*13\cdot19] \quad = \hat{z} \{(\exists x, y) . x \epsilon \alpha . y \epsilon \beta . z = x \female y\} . \supset \vdash . \text{Prop}$

This proposition is of considerable importance, since it gives a compact form for the class of all values of the function $x \female y$ obtained by taking x in the class α and y in the class β. Thus, for example, suppose α is the class of numbers which are multiples of 3, and β is the class of numbers which are multiples of 5, and $x \times y$ represents the arithmetical product of x and y,

then $s'\alpha \underset{\prime\prime}{\times}{}''\beta$ will be the class of products of multiples of 3 and multiples of 5, *i.e.* the class of multiples of 15. Again suppose α and β are both classes of relations; then $s'\alpha \underset{\prime\prime}{|}{}''\beta$ will be all relative products $R|S$ obtained by choosing R in the class α and S in the class β.

***40·71.** $\vdash . s'\underset{\prime\prime}{♀}y''\kappa = (s'\kappa)\underset{\prime\prime}{♀}y = ♀y''s'\kappa$

Dem.

$$\vdash . *40{\cdot}38 . *38{\cdot}31 . \supset \vdash . s'\underset{\prime\prime}{♀}y''\kappa = ♀y''s'\kappa$$
$$[*38{\cdot}2] \qquad = (s'\kappa)\underset{\prime\prime}{♀}y . \supset \vdash . \text{Prop}$$

The hypothesis $\breve{R}''\alpha \subset \alpha$, which appears in *40·8·81, is one which plays an important part at a later stage. In the theory of induction (Part II, Section E) it characterizes a *hereditary* class, and in the theory of series it characterizes an *upper section* (when combined with $\alpha \subset C'R$).

***40·8.** $\vdash :: \alpha \epsilon \kappa . \supset_\alpha . \breve{R}''\alpha \subset \alpha : \supset . \breve{R}''s'\kappa \subset s'\kappa$

Dem.

$$\vdash . *37{\cdot}171 . \supset \vdash :: \text{Hp} . \supset :. \alpha \epsilon \kappa . \supset_\alpha : x \epsilon \alpha . xRy . \supset_{x,y} . y \epsilon \alpha :.$$
$$[*11{\cdot}62] \qquad \supset :. \alpha \epsilon \kappa . x \epsilon \alpha . xRy . \supset_{\alpha,x,y} . y \epsilon \alpha :.$$
$$[*40{\cdot}13] \qquad \supset_{\alpha,x,y} . y \epsilon s'\kappa :.$$
$$[*40{\cdot}11 . *10{\cdot}23] \qquad \supset :. x \epsilon s'\kappa . xRy . \supset_{x,y} . y \epsilon s'\kappa :.$$
$$[*37{\cdot}171] \qquad \supset :. \breve{R}''s'\kappa \subset s'\kappa :: \supset \vdash . \text{Prop}$$

***40·81.** $\vdash :. \alpha \epsilon \kappa . \supset_\alpha . \breve{R}''\alpha \subset \alpha : \supset . \breve{R}''p'\kappa \subset p'\kappa$

Dem.

$$\vdash . *37{\cdot}171 . \supset \vdash ::. \text{Hp} . \supset :: \alpha \epsilon \kappa . \supset : x \epsilon \alpha . xRy . \supset . y \epsilon \alpha ::$$
$$[\text{Exp.Comm}] \qquad \supset :: xRy . \supset :. \alpha \epsilon \kappa . \supset : x \epsilon \alpha . \supset . y \epsilon \alpha :.$$
$$[*2{\cdot}77] \qquad \supset :. \alpha \epsilon \kappa . \supset . x \epsilon \alpha : \supset : \alpha \epsilon \kappa . \supset . y \epsilon \alpha \quad (1)$$
$$\vdash . (1) . *10{\cdot}11{\cdot}21{\cdot}27 . \supset$$
$$\vdash ::. \text{Hp} . \supset :: xRy . \supset :. \alpha \epsilon \kappa . \supset_\alpha . x \epsilon \alpha : \supset : \alpha \epsilon \kappa . \supset_\alpha . y \epsilon \alpha :.$$
$$\supset :. x \epsilon p'\kappa . \supset . y \epsilon p'\kappa ::$$
$$[\text{Imp}] \qquad \supset :: x \epsilon p'\kappa . xRy . \supset . y \epsilon p'\kappa \quad (2)$$
$$\vdash . (2) . *37{\cdot}171 . \supset \vdash . \text{Prop}$$

$*41$. THE PRODUCT AND SUM OF A CLASS OF RELATIONS

Summary of $*41$.

The propositions to be given in this number, down to $*41\cdot3$ exclusive, are the analogues of those of $*40$, excluding those from $*40\cdot3$ onwards, which have no analogues. Proofs will not be given, in this number, when they are exactly analogous to those of propositions with the same decimal part in $*40$. The smaller importance of $\dot{p}\text{‘}\lambda$ and $\dot{s}\text{‘}\lambda$, as compared with $p\text{‘}\lambda$ and $s\text{‘}\lambda$, is illustrated by the smaller number of propositions in $*41$ as compared with $*40$.

Our definitions are

$*41\cdot01$. $\dot{p}\text{‘}\lambda = \hat{x}\hat{y}\,(R \epsilon \lambda \,.\supset_R .\, xRy)$ Df

$*41\cdot02$. $\dot{s}\text{‘}\lambda = \hat{x}\hat{y}\,\{(\exists R) \,.\, R \epsilon \lambda \,.\, xRy\}$ Df

Of the propositions preceding $*41\cdot3$, which are analogues of propositions in $*40$, the only two that are frequently used are

$*41\cdot13$. $\vdash : R \epsilon \lambda \,.\supset.\, R \mathrel{\dot{\subset}} \dot{s}\text{‘}\lambda$

$*41\cdot151$. $\vdash :.\, \dot{s}\text{‘}\lambda \mathrel{\dot{\subset}} S \,.\equiv :\, R \epsilon \lambda \,.\supset_R .\, R \mathrel{\dot{\subset}} S$

Of the remaining propositions of this number, which have no analogues in $*40$, the most important are $*41\cdot43\cdot44\cdot45$, namely

$$\mathrm{D}\text{‘}\dot{s}\text{‘}\lambda = s\text{‘}\mathrm{D}\text{‘‘}\lambda, \quad \text{ꓷ}\text{‘}\dot{s}\text{‘}\lambda = s\text{‘}\text{ꓷ}\text{‘‘}\lambda, \quad C\text{‘}\dot{s}\text{‘}\lambda = s\text{‘}C\text{‘‘}\lambda.$$

These propositions are constantly required in the theory of selections (Part II, Section D) and in relation-arithmetic. Most of the other propositions of this number are used only once or not at all.

$*41\cdot01$. $\dot{p}\text{‘}\lambda = \hat{x}\hat{y}\,(R \epsilon \lambda \,.\supset_R .\, xRy)$ Df

$*41\cdot02$. $\dot{s}\text{‘}\lambda = \hat{x}\hat{y}\,\{(\exists R) \,.\, R \epsilon \lambda \,.\, xRy\}$ Df

$*41\cdot1$. $\vdash :.\, x\,(\dot{p}\text{‘}\lambda)\,y \,.\equiv :\, R \epsilon \lambda \,.\supset_R .\, xRy$

$*41\cdot11$. $\vdash : x\,(\dot{s}\text{‘}\lambda)\,y \,.\equiv.\, (\exists R) \,.\, R \epsilon \lambda \,.\, xRy$

$*41\cdot12$. $\vdash : R \epsilon \lambda \,.\supset.\, \dot{p}\text{‘}\lambda \mathrel{\dot{\subset}} R$

$*41\cdot13$. $\vdash : R \epsilon \lambda \,.\supset.\, R \mathrel{\dot{\subset}} \dot{s}\text{‘}\lambda$

$*41\cdot14$. $\vdash : R \epsilon \lambda \,.\, x\,(\dot{p}\text{‘}\lambda)\,y \,.\supset.\, xRy$

$*41\cdot141$. $\vdash : R \epsilon \lambda \,.\, xRy \,.\supset.\, x\,(\dot{s}\text{‘}\lambda)\,y$

$*41\cdot15$. $\vdash :.\, S \mathrel{\dot{\subset}} \dot{p}\text{‘}\lambda \,.\equiv :\, R \epsilon \lambda \,.\supset_R .\, S \mathrel{\dot{\subset}} R$

$*41\cdot151$. $\vdash :.\, \dot{s}\text{‘}\lambda \mathrel{\dot{\subset}} S \,.\equiv :\, R \epsilon \lambda \,.\supset_R .\, R \mathrel{\dot{\subset}} S$

$*41\cdot16$. $\vdash : \lambda \subset \mu \,.\supset.\, \dot{p}\text{‘}\mu \mathrel{\dot{\subset}} \dot{p}\text{‘}\lambda$

$*41\cdot161$. $\vdash : \lambda \subset \mu \,.\supset.\, \dot{s}\text{‘}\lambda \mathrel{\dot{\subset}} \dot{s}\text{‘}\mu$

$*41\cdot17$. $\vdash .\, \dot{p}\text{‘}\lambda \mathbin{\dot{\cup}} \dot{p}\text{‘}\mu \mathrel{\dot{\subset}} \dot{p}\text{‘}(\lambda \cap \mu)$

***41·171.** $\vdash . \dot{s}‘\lambda \mathbin{\dot{\cup}} \dot{s}‘\mu = \dot{s}‘(\lambda \cup \mu)$

***41·18.** $\vdash . \dot{p}‘(\lambda \cup \mu) = \dot{p}‘\lambda \mathbin{\dot{\cap}} \dot{p}‘\mu$

***41·181.** $\vdash . \dot{s}‘(\lambda \cap \mu) \mathrel{\dot{\subset}} \dot{s}‘\lambda \mathbin{\dot{\cap}} \dot{s}‘\mu$

***41·19.** $\vdash :: x\,(\dot{s}‘\lambda)\,y . \equiv :. R \epsilon \lambda . \supset_R . R \mathrel{\dot{\subset}} S : \supset_S . xSy$

***41·2.** $\vdash : \lambda = \Lambda . \supset . \dot{p}‘\lambda = \dot{V}$

***41·21.** $\vdash : \lambda = \Lambda . \supset . \dot{s}‘\lambda = \dot{\Lambda}$

***41·22.** $\vdash : \dot{\Lambda} \epsilon \lambda . \supset . \dot{p}‘\lambda = \dot{\Lambda}$

***41·221.** $\vdash : \dot{V} \epsilon \lambda . \supset . \dot{s}‘\lambda = \dot{V}$

***41·23.** $\vdash : \exists ! \lambda . \supset . \dot{p}‘\lambda \mathrel{\dot{\subset}} \dot{s}‘\lambda$

***41·24.** $\vdash :. \exists ! \lambda : R \epsilon \lambda . \supset_R . S \mathrel{\dot{\subset}} R : \supset . S \mathrel{\dot{\subset}} \dot{s}‘\lambda$

***41·25.** $\vdash : x\,(\dot{s}‘\lambda)\,y . \equiv . \exists ! \lambda \cap \hat{R}\,(xRy)$

***41·26.** $\vdash : \dot{\exists} ! \dot{s}‘\lambda . \equiv . (\exists R) . R \epsilon \lambda . \dot{\exists} ! R$

***41·27.** $\vdash :. P \mathbin{\dot{\cap}} \dot{s}‘\lambda = \dot{\Lambda} . \equiv : R \epsilon \lambda . \supset_R . P \mathbin{\dot{\cap}} R = \dot{\Lambda}$

***41·3.** $\vdash . \mathrm{Cnv}‘\dot{p}‘\lambda = \dot{p}‘\mathrm{Cnv}“\lambda$

Dem.

$\vdash . *31·131 . \supset$

$\vdash :. y\,(\mathrm{Cnv}‘\dot{p}‘\lambda)\,x . \equiv : x\,(\dot{p}‘\lambda)\,y :$

[*41·1] $\equiv : R \epsilon \lambda . \supset_R . xRy :$

[*31·131] $\equiv : R \epsilon \lambda . \supset_R . y\,(\mathrm{Cnv}‘R)\,x :$

[*37·63.*31·13] $\equiv : P \epsilon \mathrm{Cnv}“\lambda . \supset_P . yPx :$

[*41·1] $\equiv : y\,(\dot{p}‘\mathrm{Cnv}“\lambda)\,x :. \supset \vdash .$ Prop

***41·31.** $\vdash . \mathrm{Cnv}‘\dot{s}‘\lambda = \dot{s}‘\mathrm{Cnv}“\lambda$ [Proof as in *41·3]

***41·32.** $\vdash . \mathrm{Cnv}“\dot{p}“\kappa = \dot{p}“\mathrm{Cnv}“‘\kappa$ [*41·3 . *37·354]

***41·33.** $\vdash . \mathrm{Cnv}“\dot{s}“\kappa = \dot{s}“\mathrm{Cnv}“‘\kappa$ [*41·31 . *37·354]

***41·34.** $\vdash . \dot{s}‘\alpha \upharpoonleft “\lambda = \alpha \upharpoonleft \dot{s}‘\lambda$

Dem.

$\vdash . *41·11 . *38·13 . *13·195 . \supset \vdash :. x\,(\dot{s}‘\alpha \upharpoonleft “\lambda)\,y . \equiv : (\exists P) . P \epsilon \lambda . x\,(\alpha \upharpoonleft P)\,y :$

[*35·1] $\equiv : (\exists P) . P \epsilon \lambda . x \epsilon \alpha . xPy :$

[*10·35] $\equiv : x \epsilon \alpha : (\exists P) . P \epsilon \lambda . xPy :$

[*41·11.*35·1] $\equiv : x\,(\alpha \upharpoonleft \dot{s}‘\lambda)\,y :. \supset \vdash .$ Prop

***41·341.** $\vdash . \dot{s}‘\upharpoonright \alpha“\lambda = (\dot{s}‘\lambda) \upharpoonright \alpha$ [Proof as in *41·34]

***41·342.** $\vdash . \dot{s}‘\upharpoonleft\!\upharpoonright \alpha“\lambda = (\dot{s}‘\lambda) \upharpoonleft\!\upharpoonright \alpha$

Dem.

$\vdash . *36·11 . *35·21 . \supset \vdash . \dot{s}‘\upharpoonleft\!\upharpoonright \alpha“\lambda = \dot{s}‘\alpha \upharpoonleft “\upharpoonright \alpha“\lambda$

[*41·34] $= \alpha \upharpoonleft (\dot{s}‘\upharpoonright \alpha“\lambda)$

[*41·341] $= \alpha \upharpoonleft (\dot{s}‘\lambda) \upharpoonright \alpha$

[*36·11] $= (\dot{s}‘\lambda) \upharpoonleft\!\upharpoonright \alpha . \supset \vdash .$ Prop

The following proposition is used in $*85{\cdot}22$.

$*41{\cdot}35$. $\vdash . \dot{s}\text{‘}M \upharpoonright \text{“}\kappa = M \upharpoonright s\text{‘}\kappa$

Dem.

$\vdash . *41{\cdot}11 . *38{\cdot}13 . \supset \vdash : x (\dot{s}\text{‘}M \upharpoonright \text{“}\kappa) y . \equiv . (\exists \alpha) . \alpha \in \kappa . x (M \upharpoonright \alpha) y .$

$[*35{\cdot}101] \quad \equiv . (\exists \alpha) . \alpha \in \kappa . y \in \alpha . xMy .$

$[*40{\cdot}11 . *35{\cdot}101] \quad \equiv . x (M \upharpoonright s\text{‘}\kappa) y : \supset \vdash . \text{Prop}$

$*41{\cdot}351$. $\vdash . \dot{s}\text{‘} \urcorner M \text{“}\kappa = (s\text{‘}\kappa) \urcorner M$ [Proof as in $*41{\cdot}35$]

$*41{\cdot}4$. $\vdash . \text{D}\text{‘}\dot{p}\text{‘}\lambda \subset p\text{‘}\text{D}\text{“}\lambda$

Dem.

$\vdash . *33{\cdot}13 . \supset$

$\vdash :: x \in \text{D}\text{‘}\dot{p}\text{‘}\lambda . \equiv :. (\exists y) . x (\dot{p}\text{‘}\lambda) y :.$

$[*41{\cdot}1] \quad \equiv :. (\exists y) : R \in \lambda . \supset_R . xRy :.$

$[*11{\cdot}61] \quad \supset :. R \in \lambda . \supset_R . (\exists y) . xRy :.$

$[*33{\cdot}13] \quad \supset :. R \in \lambda . \supset_R . x \in \text{D}\text{‘}R :.$

$[*40{\cdot}41 . *33{\cdot}12] \supset :. x \in p\text{‘}\text{D}\text{“}\lambda :: \supset \vdash . \text{Prop}$

$*41{\cdot}41$. $\vdash . \text{Ⅽ}\text{‘}\dot{p}\text{‘}\lambda \subset p\text{‘}\text{Ⅽ}\text{“}\lambda$ [Proof as in $*41{\cdot}4$]

$*41{\cdot}42$. $\vdash . C\text{‘}\dot{p}\text{‘}\lambda \subset p\text{‘}C\text{“}\lambda$

Dem.

$\vdash . *33{\cdot}132 . \supset \vdash ::. x \in C\text{‘}\dot{p}\text{‘}\lambda . \equiv :: (\exists y) : x (\dot{p}\text{‘}\lambda) y . \lor . y (\dot{p}\text{‘}\lambda) x ::$

$[*41{\cdot}1] \quad \equiv :: (\exists y) :: R \in \lambda . \supset_R . xRy : \lor : R \in \lambda . \supset_R . yRx ::$

$[*10{\cdot}41{\cdot}221] \quad \supset :: (\exists y) :: (R) :. R \in \lambda . \supset . xRy : \lor : R \in \lambda . \supset . yRx ::$

$[*4{\cdot}78] \quad \supset :: (\exists y) :: (R) :. R \in \lambda . \supset : xRy . \lor . yRx ::$

$[*11{\cdot}61] \quad \supset :: (R) :: R \in \lambda . \supset : (\exists y) : xRy . \lor . yRx :$

$[*33{\cdot}132] \quad \supset : x \in C\text{‘}R ::$

$[*40{\cdot}41 . *33{\cdot}122] \quad \supset :: x \in p\text{‘}C\text{“}\lambda ::. \supset \vdash . \text{Prop}$

$*41{\cdot}43$. $\vdash . \text{D}\text{‘}\dot{s}\text{‘}\lambda = s\text{‘}\text{D}\text{“}\lambda$

Dem.

$\vdash . *33{\cdot}13 . \supset \vdash :. x \in \text{D}\text{‘}\dot{s}\text{‘}\lambda . \equiv : (\exists y) . x (\dot{s}\text{‘}\lambda) y :$

$[*41{\cdot}11] \quad \equiv : (\exists y) : (\exists R) . R \in \lambda . xRy :$

$[*11{\cdot}23{\cdot}55] \quad \equiv : (\exists R) : R \in \lambda : (\exists y) . xRy :$

$[*33{\cdot}13] \quad \equiv : (\exists R) . R \in \lambda . x \in \text{D}\text{‘}R :$

$[*40{\cdot}4 . *33{\cdot}12] \quad \equiv : x \in s\text{‘}\text{D}\text{“}\lambda :. \supset \vdash . \text{Prop}$

$*41{\cdot}44$. $\vdash . \text{Ⅽ}\text{‘}\dot{s}\text{‘}\lambda = s\text{‘}\text{Ⅽ}\text{“}\lambda$ [Proof as in $*41{\cdot}43$]

$*41{\cdot}45$. $\vdash . C\text{‘}\dot{s}\text{‘}\lambda = s\text{‘}C\text{“}\lambda$

Dem.

$\vdash . *33{\cdot}16 . \supset \vdash . C\text{‘}\dot{s}\text{‘}\lambda = \text{D}\text{‘}\dot{s}\text{‘}\lambda \cup \text{Ⅽ}\text{‘}\dot{s}\text{‘}\lambda$

$[*41{\cdot}43{\cdot}44] \quad = s\text{‘}\text{D}\text{“}\lambda \cup s\text{‘}\text{Ⅽ}\text{“}\lambda$

$[*40{\cdot}57] \quad = s\text{‘}C\text{“}\lambda . \supset \vdash . \text{Prop}$

***41·5.** $\vdash . \dot{p}`\lambda \,|\, \dot{p}`\mu \subset \dot{p}`s`\lambda \underset{,,}{|} ``\mu$

Dem.

$\vdash . *34·1 . \supset$

$\vdash :: x (\dot{p}`\lambda \,|\, \dot{p}`\mu) z . \equiv :. (\exists y) . x (\dot{p}`\lambda) y . y (\dot{p}`\mu) z :.$

[*41·1] $\equiv :. (\exists y) :. P \epsilon \lambda . \supset_P . xPy : Q \epsilon \mu . \supset_Q . yQz :.$

[*11·56] $\equiv :. (\exists y) :. (P, Q) : P \epsilon \lambda . \supset . xPy : Q \epsilon \mu . \supset . yQz :.$

[*11·37·39] $\supset :. (\exists y) :. (P, Q) : P \epsilon \lambda . Q \epsilon \mu . \supset . xPy . yQz :.$

[*11·61] $\supset :. (P, Q) :. P \epsilon \lambda . Q \epsilon \mu . \supset . (\exists y) . xPy . yQz .$

[*34·1] $\supset . x (P \,|\, Q) z :.$

[*13·191] $\supset :. (P, Q, R) :. P \epsilon \lambda . Q \epsilon \mu . R = P \,|\, Q . \supset . xRz :.$

[*11·21·35] $\supset :. (R) : (\exists P, Q) . P \epsilon \lambda . Q \epsilon \mu . R = P \,|\, Q . \supset . xRz :$

[*40·7] $\supset :. (R) : R \epsilon s`\lambda \underset{,,}{|} ``\mu . \supset . xRz :.$

[*41·1] $\supset :. x (\dot{p}`s`\lambda \underset{,,}{|} ``\mu) z :: \supset \vdash . \text{Prop}$

***41·51.** $\vdash . \dot{s}`\lambda \,|\, \dot{s}`\mu = \dot{s}`s`\lambda \underset{,,}{|} ``\mu$

Dem.

$\vdash . *34·1 . \supset$

$\vdash :: x (\dot{s}`\lambda \,|\, \dot{s}`\mu) z . \equiv :. (\exists y) . x (\dot{s}`\lambda) y . y (\dot{s}`\mu) z :.$

[*41·11] $\equiv :. (\exists y) :. (\exists P) . P \epsilon \lambda . xPy : (\exists Q) . Q \epsilon \mu . yQz :.$

[*11·54] $\equiv :. (\exists y) :. (\exists P, Q) : P \epsilon \lambda . xPy . Q \epsilon \mu . yQz :.$

[*11·24·27] $\equiv :. (\exists P, Q) :. (\exists y) . P \epsilon \lambda . xPy . Q \epsilon \mu . yQz :.$

[*10·35] $\equiv :. (\exists P, Q) :. P \epsilon \lambda . Q \epsilon \mu : (\exists y) . xPy . yQz :.$

[*34·1] $\equiv :. (\exists P, Q) : P \epsilon \lambda . Q \epsilon \mu . x (P \,|\, Q) z :.$

[*13·195] $\equiv :. (\exists P, Q, R) . P \epsilon \lambda . Q \epsilon \mu . R = P \,|\, Q . xRz :.$

[*11·24.*40·7] $\equiv :. (\exists R) . R \epsilon s`\lambda \underset{,,}{|} ``\mu . xRz :.$

[*41·11] $\equiv :. x (\dot{s}`s`\lambda \underset{,,}{|} ``\mu) z :: \supset \vdash . \text{Prop}$

The above proposition, which is used in *92·31, states that, if λ and μ are classes of relations, the relative product of the relational sum of λ and the relational sum of μ is the relational sum of all the relative products formed of a member of λ and a member of μ.

The following proposition is used in *96·111.

***41·52.** $\vdash :. \alpha \upharpoonright \dot{s}`\lambda \subset Q . \equiv : P \epsilon \lambda . \supset_P . \alpha \upharpoonright P \subset Q$

Dem.

$\vdash . *35·1 . *41·11 . \supset$

$\vdash :: \alpha \upharpoonright \dot{s}`\lambda \subset Q . \equiv :. x \epsilon \alpha : (\exists P) . P \epsilon \lambda . xPy : \supset_{x,y} . xQy :.$

[*10·35·23] $\equiv :. x \epsilon \alpha . P \epsilon \lambda . xPy . \supset_{P,x,y} . xQy :.$

[*35·1] $\equiv :. P \epsilon \lambda . x (\alpha \upharpoonright P) y . \supset_{P,x,y} . xQy :.$

[*11·62] $\equiv :. P \epsilon \lambda . \supset_P . \alpha \upharpoonright P \subset Q :: \supset \vdash . \text{Prop}$

The following proposition is used in *162·32 and in *166·461.

***41·6.** $\vdash :. y \in \beta . \supset_y . P\text{‘}y = Q\text{‘}y \cup R\text{‘}y : \supset . \dot{s}\text{‘}P\text{“}\beta = \dot{s}\text{‘}Q\text{“}\beta \cup \dot{s}\text{‘}R\text{“}\beta$

Dem.

$\vdash . *37\cdot6 . *14\cdot21 . *41\cdot11 . *13\cdot195 . \supset$

$\vdash :: \mathrm{Hp} . \supset :. u (\dot{s}\text{‘}P\text{“}\beta) v . \equiv : (\exists y) . y \in \beta . u (P\text{‘}y) v :$

[Hp] $\equiv : (\exists y) . y \in \beta . u (Q\text{‘}y \cup R\text{‘}y) v :$

[*23·34.*10·42] $\equiv : (\exists y) . y \in \beta . u (Q\text{‘}y) v . \vee . (\exists y) . y \in \beta . u (R\text{‘}y) v :$

[*37·6.*41·11] $\equiv : u (\dot{s}\text{‘}Q\text{“}\beta) v . \vee . u (\dot{s}\text{‘}R\text{“}\beta) v :: \supset \vdash . \mathrm{Prop}$

$*42$. MISCELLANEOUS PROPOSITIONS

Summary of $*42$.

The present number contains various propositions concerning products and sums of classes. They are concerned chiefly with classes of classes of classes, or with relations of relations of relations. These are required respectively in cardinal and in ordinal arithmetic. Thus $*42{\cdot}1$ is used in $*112$ and $*113$, which are concerned with cardinal addition and multiplication, while $*42{\cdot}12{\cdot}2$ are used in $*160$ and $*162$, which are concerned with ordinal addition. $*42{\cdot}22$, though not explicitly referred to, is useful in facilitating the comprehension of propositions on series of series of series, or rather on relations between relations between relations, which are required in connection with the associative law of multiplication in relation-arithmetic.

$*42{\cdot}1$. $\vdash . s{}^\backprime s{}^{\backprime\backprime}\kappa = s{}^\backprime s{}^\backprime\kappa$

Here κ must, for significance, be a class of classes of classes. The proposition states that if we take each member, α, of κ, and form $s{}^\backprime\alpha$, and then form the sum of all the classes so obtained, the result is the same as if we form the sum of the sum of κ. This is the associative law for s, and is (as will appear later) the source of the associative law of addition in cardinal arithmetic. The way in which this proposition comes to be the associative law for s may be seen as follows: Suppose κ consists of two classes, α and β; suppose α in turn consists of the two classes ξ and η, and β of the two classes ξ' and η'. Then $s{}^\backprime\alpha = \xi \cup \eta \,.\, s{}^\backprime\beta = \xi' \cup \eta'$. (This will be proved later.) Thus $s{}^{\backprime\backprime}\kappa$ has two members, one of which is $\xi \cup \eta$, while the other is $\xi' \cup \eta'$. Thus

$$s{}^\backprime s{}^{\backprime\backprime}\kappa = (\xi \cup \eta) \cup (\xi' \cup \eta').$$

But $s{}^\backprime\kappa$ has four members, namely ξ, η, ξ', η'. Thus $s{}^\backprime s{}^\backprime\kappa = \xi \cup \eta \cup \xi' \cup \eta'$. Thus our proposition leads to

$$(\xi \cup \eta) \cup (\xi' \cup \eta') = \xi \cup \eta \cup \xi' \cup \eta',$$

which is obviously a case of the associative law.

Our proposition states the associative law generally, including the case where the number of brackets, or of summands in any bracket, is infinite. The proof is as follows.

Dem.

$$\begin{aligned}
&\vdash . *40{\cdot}4 . \supset \vdash :: x \in s{}^\backprime s{}^{\backprime\backprime}\kappa . \equiv :. (\exists\alpha) . \alpha \in \kappa . x \in s{}^\backprime\alpha :. \\
&[*40{\cdot}11] \qquad \equiv :. (\exists\alpha) : \alpha \in \kappa : (\exists\xi) . \xi \in \alpha . x \in \xi :. \\
&[*11{\cdot}6] \qquad \equiv :. (\exists\xi) :. (\exists\alpha) . \alpha \in \kappa . \xi \in \alpha : x \in \xi :. \\
&[*40{\cdot}11] \qquad \equiv :. (\exists\xi) . \xi \in s{}^\backprime\kappa . x \in \xi :. \\
&[*40{\cdot}11] \qquad \equiv :. x \in s{}^\backprime s{}^\backprime\kappa :: \supset \vdash . \text{Prop}
\end{aligned}$$

***42·11.** $\vdash . p\text{‘}p\text{“}\kappa = p\text{‘}s\text{‘}\kappa$

Dem.

$\vdash . *40·41 . \supset \vdash :. x \in p\text{‘}p\text{“}\kappa . \equiv : \beta \in \kappa . \supset_\beta . x \in p\text{‘}\beta :$

$[*40·1.*11·62] \quad \equiv : \beta \in \kappa . \gamma \in \beta . \supset_{\beta,\gamma} . x \in \gamma :$

$[*11·2.*10·23] \quad \equiv : (\exists \beta) . \beta \in \kappa . \gamma \in \beta . \supset_\gamma . x \in \gamma :$

$[*40·11] \quad \equiv : \gamma \in s\text{‘}\kappa . \supset_\gamma . x \in \gamma :$

$[*40·1] \quad \equiv : x \in p\text{‘}s\text{‘}\kappa :. \supset \vdash . \text{Prop}$

This is the associative law for products. Supposing again, for illustration, that κ consists of the two classes α, β, while α consists of the two classes ξ, η and β of the two classes ξ', η', then $p\text{“}\kappa$ consists of the two classes $\xi \cap \eta$ and $\xi' \cap \eta'$, so that $p\text{‘}p\text{“}\kappa = (\xi \cap \eta) \cap (\xi' \cap \eta')$, while $p\text{‘}s\text{‘}\kappa = \xi \cap \eta \cap \xi' \cap \eta'$. Thus our proposition becomes

$$(\xi \cap \eta) \cap (\xi' \cap \eta') = \xi \cap \eta \cap \xi' \cap \eta'.$$

A descriptive function $R\text{‘}\kappa$ whose arguments are classes or classes of classes may be said to obey the associative law provided

$$R\text{‘}R\text{“}\kappa = R\text{‘}s\text{‘}\kappa.$$

This equation may be interpreted as follows: Given a class α, divide it into any number of subordinate classes, so that no member is left out, though one member may belong to two or more classes. Let the classes into which α is divided make up the class κ, so that κ is a class of classes, and $s\text{‘}\kappa = \alpha$. Then the above equation asserts that if we first form the R's of the various sub-classes of α, and then the R of the resulting class, the result is the same as if we formed the R of α directly.

In some cases—for example, that of arithmetical addition of cardinals—the above equation holds only when no two members of κ have a common term, *i.e.* when the parts into which α is divided are mutually exclusive.

For a descriptive function whose arguments are relations of relations, we shall find another form for the associative law; this form plays in ordinal arithmetic a part analogous to that played by the above form in cardinal arithmetic.

***42·12.** $\vdash . \dot{s}\text{‘}\dot{s}\text{“}\lambda = \dot{s}\text{‘}s\text{‘}\lambda$

Dem.

$\vdash . *41·11 . \supset \vdash : x (\dot{s}\text{‘}\dot{s}\text{“}\lambda) y . \equiv . (\exists \mu) . \mu \in \lambda . x (\dot{s}\text{‘}\mu) y .$

$[*41·11] \quad \equiv . (\exists \mu, P) . \mu \in \lambda . P \in \mu . xPy .$

$[*40·11] \quad \equiv . (\exists P) . P \in s\text{‘}\lambda . xPy .$

$[*41·11] \quad \equiv . x (\dot{s}\text{‘}s\text{‘}\lambda) y : \supset \vdash . \text{Prop}$

***42·13.** $\vdash . \dot{p}\text{‘}\dot{p}\text{“}\lambda = \dot{p}\text{‘}s\text{‘}\lambda$

Dem.

$\vdash . *41·1 . \supset \vdash :. x (\dot{p}\text{‘}\dot{p}\text{“}\lambda) y . \equiv : \mu \in \lambda . \supset_\mu . x (\dot{p}\text{‘}\mu) y :$

$[*41·1] \quad \equiv : \mu \in \lambda . R \in \mu . \supset_{\mu,R} . xRy :$

$$[*11\cdot2.*10\cdot23] \qquad \equiv : (\exists\mu) . \mu \epsilon \lambda . R \epsilon \mu . \supset_R . xRy :$$
$$[*40\cdot11] \qquad \equiv : R \epsilon s`\lambda . \supset_R . xRy :$$
$$[*41\cdot1] \qquad \equiv : x(\dot{p}`s`\lambda)y :. \supset \vdash . \text{Prop}$$

***42·2.** $\vdash . C`\dot{s}`C`P = s`C``C`P = F``C`P = \overrightarrow{F^2}`P$

This proposition assumes that P is a relation between relations. For example, suppose we have a series of series, whose generating relations are ordered by the relation P. Then $C`P$ is the class of these generating relations; $\dot{s}`C`P$ is the relation "one or other of the generating relations which compose $C`P$," and $C`\dot{s}`C`P$ is the class of all the terms occurring in any of the series. $C``C`P$ is the fields of the various series, and $s`C``C`P$ is again all the terms occurring in any of the series. $F``C`P$ is all the terms belonging to fields of series which are members of $C`P$, and $\overrightarrow{F^2}`P$ is all members of fields of members of the field of P; each of these again is all the terms occurring in any of the series. The proof is as follows:

Dem.

$$\vdash . *41\cdot45 . \supset \vdash . C`\dot{s}`C`P = s`C``C`P \quad (1)$$
$$\vdash . *40\cdot56 . \supset \vdash . s`C``C`P = F``C`P \quad (2)$$
$$\vdash . *33\cdot5 . \supset \vdash . F``C`P = F``\overrightarrow{F}`P$$
$$[*37\cdot38] \qquad = \overrightarrow{F^2}`P \quad (3)$$
$$\vdash . (1) . (2) . (3) . \supset \vdash . \text{Prop}$$

The following propositions apply to a relation of relations of relations. These propositions are useful for proving associative laws in ordinal arithmetic, since these laws deal with series of series of series, and series of series of series are most simply constituted by supposing the generating relations of the constituent series to be ordered by relations which are themselves ordered by a relation P.

***42·21.** $\vdash . s`C```C``C`P = C``s`C``C`P = C``C`\dot{s}`C`P = C``F``C`P = C``\overrightarrow{F^2}`P$

Dem.

$$\vdash . *40\cdot38 . \supset \vdash . s`C```C``C`P = C``s`C``C`P \quad (1)$$
$$\vdash . (1) . *42\cdot2 . \supset \vdash . \text{Prop}$$

***42·22.** $\vdash . s`s`C```C``C`P = s`C``s`C``C`P = s`C``C`\dot{s}`C`P$
$= C`\dot{s}`C`\dot{s}`C`P = s`C``F``C`P$
$= F``F``C`P = F``\overrightarrow{F^2}`P = \overrightarrow{F^3}`P$
$[*42\cdot21 . *41\cdot45 . *40\cdot56 . *42\cdot2 . *37\cdot3]$

If P, in the above proposition, is a relation which generates a series of series of series, the above gives various forms for the class of ultimate terms of these series. Thus suppose $Q \epsilon C`P$; then Q is a relation between generating

relations of series. If now $R \epsilon C'Q$, R is the generating relation of a series which we may regard as composed of individuals. The class of individuals so obtainable may be expressed in any of the above forms, as well as in others which are not given above.

***42·3.** $\vdash . s's''\overrightarrow{R}''\alpha = s'R''\alpha$

Dem.

$$\vdash . *42\cdot1 . \supset \vdash . s's''\overrightarrow{R}''\alpha = s's'\overrightarrow{R}''\alpha$$
$$[*40\cdot5] \qquad = s'R''\alpha . \supset \vdash . \text{Prop}$$

***42·31.** $\vdash . s's''\overleftarrow{R}''\alpha = s'\breve{R}''\alpha$ [Proof as in *42·3]

$*43$. THE RELATIONS OF A RELATIVE PRODUCT TO ITS FACTORS

Summary of $*43$.

The purpose of the present number is to give certain propositions on the relation which holds between P and Q whenever $P = Q \mid R$, or whenever $P = R \mid Q$, or whenever $P = R \mid Q \mid S$, where R and S are fixed. In virtue of the general definitions of $*38$, these relations are respectively $\mid R$, $R \mid$, and $(R \mid) \mid (\mid S)$. Such relations are of great utility both in cardinal and in ordinal arithmetic; they are also much used in the theory of induction (Part II, Section E). In place of the notation $(R \mid) \mid (\mid S)$, which is cumbrous, we adopt the more compact notation $R \parallel S$. If λ is a class of relations, $R \mid ``\lambda$ will be the class of relations $R \mid P$ where $P \in \lambda$, $\mid R``\lambda$ will be the class of relations $P \mid R$ where $P \in \lambda$, and $(R \parallel S)``\lambda$ will be the class of relations $R \mid P \mid S$ where $P \in \lambda$. These classes of relations are often required in subsequent work.

In virtue of our definitions, we have

$*43{\cdot}112.\ \vdash . (R \parallel S)`Q = R \mid Q \mid S$

The propositions most used in the present number (except such as merely embody definitions) are the following:

$*43{\cdot}302.\ \vdash . (P) . P \in \text{Ɑ}`(R \parallel S)$

$*43{\cdot}411.\ \vdash . \breve{R}```\text{Ɑ}``\lambda = \text{Ɑ}``\mid R``\lambda$

$*43{\cdot}421.\ \vdash . \dot{s}`\mid R``\lambda = (\dot{s}`\lambda) \mid R$

The remaining propositions are used seldom, but their uses, when they are used, are important.

$*43{\cdot}01.\quad R \parallel S = (R \mid) \mid (\mid S) \quad \text{Df}$

At a later stage (in $*150$) we shall introduce a simpler notation for the special case of $R \parallel \breve{R}$. The following propositions are for the most part immediate consequences of the definitions, and proofs are therefore usually omitted.

$*43{\cdot}1.\quad \vdash : P (R \mid) Q . \equiv . P = R \mid Q$

$*43{\cdot}101.\ \vdash : P (\mid R) Q . \equiv . P = Q \mid R$

$*43{\cdot}102.\ \vdash : P (R \parallel S) Q . \equiv . P = R \mid Q \mid S$

$*43{\cdot}11.\quad \vdash . R \mid `Q = R \mid Q$

$*43{\cdot}111.\ \vdash . \mid R`Q = Q \mid R$

$*43{\cdot}112.\ \vdash . (R \parallel S)`Q = R \mid Q \mid S$

$*43{\cdot}12.\quad \vdash . \text{E} ! \, R \mid `Q$

***43·121.** $\vdash . \mathrm{E}! \mid R`Q$

***43·122.** $\vdash . \mathrm{E}! (R \| S)`Q$

***43·2.** $\vdash . (R|) \mid (S|) = (R \mid S)|$

Dem.

$\vdash . *43·1 . \supset \vdash : L \{(R|) \mid (S|)\} N . \equiv . (\exists M) . L = R \mid M . M = S \mid N .$

$[*13·195 . *34·21] \qquad \equiv . L = R \mid S \mid N .$

$[*43·1] \qquad \equiv . L \{(R \mid S)|\} N : \supset \vdash . \mathrm{Prop}$

***43·201.** $\vdash . (|R) \mid (|S) = |(S \mid R)$ [Proof as in *43·2]

***43·202.** $\vdash . (|R) \mid (S|) = (S|) \mid (|R) = S \| R$ [Proof as in *43·2]

***43·21.** $\vdash . (P \| Q) \mid (R|) = (P \mid R) \| Q$

***43·211.** $\vdash . (R|) \mid (P \| Q) = (R \mid P) \| Q$

***43·212.** $\vdash . (P \| Q) \mid (|R) = P \| (R \mid Q)$

***43·213.** $\vdash . (|R) \mid (P \| Q) = P \| (Q \mid R)$

***43·22.** $\vdash . (P \| Q) \mid (R \| S) = (P \mid R) \| (S \mid Q)$

***43·3.** $\vdash . (P) . P \epsilon \mathrm{G}`R|$ [*43·12 . *33·43]

***43·301.** $\vdash . (P) . P \epsilon \mathrm{G}`|R$

***43·302.** $\vdash . (P) . P \epsilon \mathrm{G}`(R \| S)$

***43·31.** $\vdash . P \upharpoonright \mathrm{G}`R| = P \upharpoonright C`R| = P$

Dem.

$\vdash . *43·12 . *33·431 . \supset \vdash . \mathrm{G}`P \subset \mathrm{G}`R| \qquad (1)$

$[*33·161] \qquad \supset \vdash . \mathrm{G}`P \subset C`R| \qquad (2)$

$\vdash . (1) . (2) . *35·452 . \supset \vdash . \mathrm{Prop}$

***43·311.** $\vdash . P \upharpoonright \mathrm{G}`|R = P \upharpoonright C`|R = P$

***43·312.** $\vdash . P \upharpoonright \mathrm{G}`(R \| S) = P \upharpoonright C`(R \| S) = P$

***43·34.** $\vdash . R \mid `R = |R`R = R^2$ [*43·11·111]

***43·4.** $\vdash . R``\mathrm{D}`P = \mathrm{D}`R \mid `P$ [*37·32 . *43·1]

***43·401.** $\vdash . \breve{R}``\mathrm{G}`P = \mathrm{G}` \mid R`P$ [*37·32 . *43·101]

***43·41.** $\vdash . R```\mathrm{D}``\lambda = \mathrm{D}``R \mid ``\lambda$ [*43·4 . *37·355]

***43·411.** $\vdash . \breve{R}```\mathrm{G}``\lambda = \mathrm{G}`` \mid R``\lambda$ [*43·401 . *37·355]

***43·42.** $\vdash . \dot{s}`R \mid ``\lambda = R \mid \dot{s}`\lambda$

Dem.

$\vdash . *41·11 . *37·1 . *43·1 . \supset$

$\vdash :. x (\dot{s}`R \mid ``\lambda) z . \equiv : (\exists T) . T \epsilon \lambda . x (R \mid T) z :$

$[*34·1] \qquad \equiv : (\exists T) : T \epsilon \lambda : (\exists y) . xRy . yTz :$

$[*11·6] \qquad \equiv : (\exists y) : xRy : (\exists T) . T \epsilon \lambda . yTz :$

$[*41·11 . *34·1] \qquad \equiv : x (R \mid \dot{s}`\lambda) z :. \supset \vdash . \mathrm{Prop}$

***43·421.** $\vdash . \dot{s}\text{‘} \,|\, R\text{“}\lambda = (\dot{s}\text{‘}\lambda) \,|\, R$ [Proof as in *43·42]

***43·43.** $\vdash . \dot{s}\text{‘}(R \,\|\, S)\text{“}\lambda = (R \,\|\, S)\text{‘}\dot{s}\text{‘}\lambda$

Dem.

$$\begin{aligned}
&\vdash . *37{\cdot}33 . \supset \vdash . \dot{s}\text{‘}(R \,\|\, S)\text{“}\lambda = \dot{s}\text{‘}R \,|\, \text{“} \,|\, S\text{“}\lambda \\
&[*43{\cdot}42] \qquad\qquad\qquad = R \,|\, (\dot{s}\text{‘} \,|\, S\text{“}\lambda) \\
&[*43{\cdot}421] \qquad\qquad\quad = R \,|\, \dot{s}\text{‘}\lambda \,|\, S \\
&[*43{\cdot}112] \qquad\qquad\quad = (R \,\|\, S)\text{‘}\dot{s}\text{‘}\lambda . \supset \vdash . \text{Prop}
\end{aligned}$$

***43·48.** $\vdash : \text{D}\text{‘}P \subset \alpha . \supset . Q \,|\text{‘}P = (Q \upharpoonright \alpha) \,|\text{‘}P$ [*35·481]

***43·481.** $\vdash : \text{Ɑ}\text{‘}P \subset \beta . \supset . \,|R\text{‘}P = \,|(\beta \upharpoonleft R)\text{‘}P$ [*35·48]

***43·49.** $\vdash : s\text{‘}\text{D}\text{“}\lambda \subset \alpha . \supset . (Q \,|\,) \upharpoonright \lambda = \{(Q \upharpoonright \alpha) \,|\,\} \upharpoonright \lambda$

Dem.

$$\begin{aligned}
&\vdash . *40{\cdot}43 . \supset \vdash :. \text{Hp} . \supset : P \in \lambda . \supset . \text{D}\text{‘}P \subset \alpha . \\
&[*43{\cdot}48] \qquad\qquad \supset . Q \,|\text{‘}P = \{(Q \upharpoonright \alpha) \,|\,\}\text{‘}P \qquad (1) \\
&\vdash . (1) . *35{\cdot}71 . \supset \vdash . \text{Prop}
\end{aligned}$$

***43·491.** $\vdash : s\text{‘}\text{Ɑ}\text{“}\lambda \subset \beta . \supset . (\,|R) \upharpoonright \lambda = \{\,|(\beta \upharpoonleft R)\} \upharpoonright \lambda$ [Proof as in *43·49]

***43·5.** $\vdash : \text{D}\text{‘}P \subset \alpha . \text{Ɑ}\text{‘}P \subset \beta . \supset . (Q \,\|\, R)\text{‘}P = \{(Q \upharpoonright \alpha) \,\|\, (\beta \upharpoonleft R)\}\text{‘}P$
[*35·48·481 . *43·112]

***43·51.** $\vdash : s\text{‘}\text{D}\text{“}\lambda \subset \alpha . s\text{‘}\text{Ɑ}\text{“}\lambda \subset \beta . \supset . (Q \,\|\, R) \upharpoonright \lambda = \{(Q \upharpoonright \alpha) \,\|\, (\beta \upharpoonleft R)\} \upharpoonright \lambda$

Dem.

$$\begin{aligned}
&\vdash . *40{\cdot}43 . \supset \vdash :. \text{Hp} . \supset : P \in \lambda . \supset . \text{D}\text{‘}P \subset \alpha . \text{Ɑ}\text{‘}P \subset \beta . \\
&[*43{\cdot}5] \qquad\qquad \supset . (Q \,\|\, R)\text{‘}P = \{(Q \upharpoonright \alpha) \,\|\, (\beta \upharpoonleft R)\}\text{‘}P \qquad (1) \\
&\vdash . (1) . *35{\cdot}71 . \supset \vdash . \text{Prop}
\end{aligned}$$

The above proposition is used in the proof of *74·773.

PART II

PROLEGOMENA TO CARDINAL ARITHMETIC

SUMMARY OF PART II

THE objects to be studied in this Part are not sharply distinguished from those studied in Part I. The difference is one of degree, the objects in this Part being of somewhat less general importance than those of Part I, and being studied more on account of their bearing on cardinal arithmetic than on their own account. Although cardinal arithmetic is the goal which determines our course in Part II, all the objects studied will be found to be also required in ordinal arithmetic and the theory of series. As this Part advances, the approach to cardinal arithmetic becomes gradually more marked, until at last nothing is lacking except the definition of cardinal numbers, with which Part III opens.

Section A of this Part deals with unit classes and couples. A *unit* class is the class of terms identical with a given term, *i.e.* the class whose only member is the given term. (As explained in the Introduction, Chapter III, pp. 76 to 79, the class whose only member is x is not identical with x.) We define 1 as the class of all unit classes, leaving it to Part III to show that 1, so defined, is a cardinal number. In like manner, we define a (cardinal or ordinal) couple, and then define 2 as the class of all couples. The propositions on couples will not be much referred to in the remainder of the present Part, since their use belongs chiefly to arithmetic (Parts III and IV). On the other hand, the properties of unit classes are constantly required in Sections C, D, E of this Part.

Section B deals, first, with the class of sub-classes of a given class, *i.e.* of classes contained in a given class. The sub-classes of a given class are often important in arithmetic. Next we consider the class of sub-relations of a given relation, *i.e.* relations contained in a given relation. The propositions on this subject are analogous to those on sub-classes, but less important. Next we consider the question of "relative types," *i.e.* taking any object x, and calling its type $t\text{‘}x$, we give a notation for expressing in terms of $t\text{‘}x$ the type of classes of which x is a member, or of relations in which x may be either referent or relatum, and so on. The notations introduced in this connection are very useful in arithmetic, especially in connection with existence-theorems. But the propositions of Section B are very seldom required in the later sections of the present Part.

Section C, which deals with one-many, many-one and one-one relations, is very important, and is constantly relevant in the sequel. A relation is one-many when no term has more than one referent, many-one if no term has more than one relatum, and one-one if it is both one-many and many-one.

In this section, we define the notion of *similarity*, upon which all cardinal arithmetic is based: two classes are said to be *similar* when there is a one-one relation whose domain is the one and whose converse domain is the other. We prove the elementary properties of similarity, including the Schröder-Bernstein theorem, namely: If α is similar to part of β, and β is similar to part of α, then α is similar to β.

Section D deals with the notion of *selections*, upon which both cardinal and ordinal multiplication are based. A selection from a set of classes is a class consisting of one member from each class of the set. Thus a selective relation R may be defined as one which, for a given class of classes κ, makes $R\text{`}\alpha$ a member of α whenever α is a member of κ. More exactly, a selective relation for a class of classes κ is one which is one-many, which has κ for its converse domain, and is such that, if $xR\alpha$, then $x \epsilon \alpha$. Such a relation may be called an ϵ-selector from κ. More generally, we may define a P-selector from κ as a relation which is one-many, which has κ for its converse domain, and which is contained in P. The theory of selectors is very important in arithmetic. But until we come to cardinal multiplication in Part III, Section B, the propositions of this fourth section will seldom be relevant.

Section E deals with mathematical induction, not in the special form in which it applies to finite integers (this is considered in Part III, Section C), but in a general form in which it applies to all relations. The propositions of this section are of very great importance, primarily in the theory of finite and infinite (Part III, Section C, and Part V, Section E), but also in many other subjects, and especially in the derivation of series from one-many, many-one or one-one relations—for example, in ordering the "rational" points of a projective space by means of successive constructions of harmonic points. The ideas involved in this section are somewhat complicated, and we must refer the reader to the section itself for an account of them.

SECTION A

UNIT CLASSES AND COUPLES

Summary of Section A.

In this section we begin ($*50$) by introducing a notation for the *relation* of identity, as opposed to the *function* "$x = y$"; that is, calling the relation of identity I, we put

$$I = \hat{x}\hat{y}\,(x = y) \quad \text{Df.}$$

The purpose of this definition is chiefly convenience of notation. The definition enables us to speak of $\overrightarrow{I}$, $\text{D}`I$, $I \,|\, R$, $\alpha \rceil I$, $I``\alpha$, etc., which we could not otherwise do.

At the same time we introduce *diversity*, which is defined as the negation of identity, and denoted by the letter J. The properties of I and J result immediately from $*13$, since

$$xIy \,.\equiv.\, x = y.$$

We next introduce a very important notation, due to Peano, for the class whose only member is x. If we took a strictly and purely extensional view of classes, we should naturally suppose this class to be identical with x. But in view of the theory of classes explained in $*20$, it is plain that x can never be identical with a class of which it is a member, even when it is the only member of that class. Peano uses the notation "ιx" for the class whose only member is x; we shall alter this to "$\iota`x$," following our general notation for descriptive functions. Thus we are to have

$$\iota`x = \hat{y}\,(y = x) = \hat{y}\,(yIx) = \overrightarrow{I}`x.$$

Hence we take as our definition

$$\iota = \overrightarrow{I} \quad \text{Df,}$$

since this definition gives the desired value of $\iota`x$. The properties of ι are many and important.

It is important to observe that "$\breve{\iota}`\alpha$" means "the only member of $\alpha$." Thus it exists when, and only when, $\alpha$ has one member and no more, in which case $\alpha$ is of the form $\iota`x$, if x is its only member. Thus "$\breve{\iota}`\alpha$" means the same as "$(\imath x)\,(x \,\epsilon\, \alpha)$," and "$\breve{\iota}`\hat{z}\,(\phi z)$" means the same as "$(\imath x)\,(\phi x)$." What we call "$\breve{\iota}`\alpha$" is denoted, in Peano's notation, by "$\imath\alpha$."

Classes of the form $\iota`x$ are called *unit classes*, and the class of all such classes is called 1. This is the cardinal number 1, according to the definition of cardinal numbers which will be given in $*100$. The properties of 1, so far as they do not depend upon other cardinals, or upon the fact that 1 is a cardinal, will be studied in $*52$.

After a number (*53) containing various propositions involving 1 or ι, we pass to the consideration of cardinal couples (*54) and ordinal couples (*55). A cardinal couple is a class $\iota`x \cup \iota`y$, where $x \neq y$. The class of such couples is defined as 2, and will be shown at a later stage (*101) to be a cardinal number. An ordinal couple, which, unlike a cardinal couple, involves an order as between its members, is defined as a relation $\iota`x \uparrow \iota`y$ (cf. *35·04), where we may either add $x \neq y$ or not. The properties of ordinal couples are in part analogous to those of unit classes, in part to those of cardinal couples. In *56, we define the ordinal number 2 (which we denote by 2_r, to distinguish it from the cardinal 2) as the class of all ordinal couples $\iota`x \uparrow \iota`y$, where $x \neq y$. It will be shown at a later stage that this is an ordinal number according to our definition of ordinal numbers (*153 and *251).

$*50$. IDENTITY AND DIVERSITY AS RELATIONS

Summary of $*50$.

The purpose of the present number is primarily notational. For notational reasons, we must be able to express identity and diversity as relations, and not merely as propositional functions, *i.e.* we require a notation for $\hat{x}\hat{y}\,(x=y)$ and $\hat{x}\hat{y}\,(x \neq y)$. We therefore put

$$I = \hat{x}\hat{y}\,(x=y) \quad \text{Df},$$
$$J = \dot{-}\, I \qquad \text{Df}.$$

In spite of the fact that diversity is merely the negation of identity, the kinds of propositions that employ diversity are quite different from the kinds that employ identity. Identity as a relation is required, to begin with, in the theory of unit classes, which is our reason for treating of it at this stage. It is next required, constantly, in the theory of mathematical induction (Part II, Section E). It is required also in showing that cardinal and ordinal similarity are reflexive. These are its principal uses.

Diversity, on the other hand, is required almost exclusively in the theory of series (Part V), and the first number in that theory will be devoted to diversity. Until that stage, diversity will seldom be referred to, with one important exception, namely in proving the associative law of multiplication in relation-arithmetic ($*174$).

The most important propositions on identity in the present number are the following:

$*50{\cdot}16$. $\vdash . I``\alpha = \alpha$

$*50{\cdot}4$. $\vdash . R \,|\, I = I \,|\, R = R$

$*50{\cdot}5$. $\vdash . \alpha \rceil I = I \upharpoonright \alpha = \alpha \rceil I \upharpoonright \alpha$

$*50{\cdot}51$. $\vdash . \mathrm{Cnv}`(\alpha \rceil I) = \alpha \rceil I$

$*50{\cdot}52$. $\vdash . \mathrm{D}`(\alpha \rceil I) = \mathrm{Cl}`(\alpha \rceil I) = C`(\alpha \rceil I) = \alpha$

$*50{\cdot}62$. $\vdash : \mathrm{Cl}`R \subset \alpha . \supset . R \,|\, (I \upharpoonright \alpha) = R$

$*50{\cdot}63$. $\vdash : \mathrm{D}`R \subset \alpha . \supset . I \upharpoonright \alpha \,|\, R = R$

The most important propositions on diversity in the present number are the following:

$*50{\cdot}23$. $\vdash : R \in J . \equiv . \breve{R} \in J$

$*50{\cdot}24$. $\vdash : R \in J . \equiv . (x) . \sim (xRx)$

$*50{\cdot}43$. $\vdash : R^2 \in J . \equiv . R \mathbin{\dot{\cap}} \breve{R} = \dot{\Lambda}$

$*50{\cdot}45$. $\vdash : R^2 \in J . \supset . R \in J$

$*50{\cdot}47$. $\vdash :. R^2 \in R . \supset : R \in J . \equiv . R^2 \in J . \equiv . R \mathbin{\dot{\cap}} \breve{R} = \dot{\Lambda}$

It will be observed that all these propositions are concerned with $R \subset J$ or $R^2 \subset J$, both of which are satisfied if R is a *serial* relation. The hypothesis $R^2 \subset J$ or $R \mathbin{\dot\wedge} \breve{R} = \Lambda$ characterizes an *asymmetrical* relation, *i.e.* one which, if it holds between x and y, cannot hold between y and x.

***50·01.** $I = \hat{x}\hat{y}\,(x = y)$ Df

***50·02.** $J = \dot{-}\, I$ Df

Most of the propositions of this number are obvious, and call for no comment.

***50·1.** $\vdash : xIy \,.\equiv .\, x = y$ [*21·3 . (*50·01)]

***50·11.** $\vdash : xJy \,.\equiv .\, x \neq y$ [*23·35 . *50·1 . (*50·02)]

***50·12.** $\vdash .\, J = \hat{x}\hat{y}\,(x \neq y)$ [*50·11 . *21·33]

***50·13.** $\vdash .\, \dot{\exists} \,!\, I$ [*13·19 . *10·24·281 . *50·1]

***50·14.** $\vdash .\, I‘y = y$ [*30·3 . *50·1 . *10·11]

***50·15.** $\vdash .\, (y) \,.\, E \,!\, I‘y$ [*50·14 . *14·21 . *10·11]

***50·16.** $\vdash .\, I‘‘\alpha = \alpha$

Dem.

$\vdash . *37·1 . \supset \vdash : x \in I‘‘\alpha \,.\equiv .\, (\exists y) \,.\, y \in \alpha \,.\, xIy \,.$

[*50·1] $\equiv .\, (\exists y) \,.\, y \in \alpha \,.\, x = y \,.$

[*13·195] $\equiv .\, x \in \alpha : \supset \vdash .$ Prop

***50·17.** $\vdash :.\, x \in \alpha \,.\supset_x .\, R‘x = x : \supset .\, R‘‘\alpha = \alpha$

Dem.

$\vdash . *14·21 . \supset \vdash : \mathrm{Hp} \,.\supset .\, E \,!!\, R‘‘\alpha$ (1)

$\vdash . *50·14 . \supset \vdash :.\, \mathrm{Hp} \,.\supset : x \in \alpha \,.\supset_x .\, R‘x = I‘x :$

[*37·69.(1)] $\supset : R‘‘\alpha = I‘‘\alpha :$

[*50·16] $\supset : R‘‘\alpha = \alpha :. \supset \vdash .$ Prop

***50·2.** $\vdash .\, I = \breve{I}$

Dem.

$\vdash . *50·1 . \supset \vdash : xIy \,.\equiv .\, x = y \,.$

[*13·16] $\equiv .\, y = x \,.$

[*50·1] $\equiv .\, yIx \,.$

[*31·11] $\equiv .\, x\breve{I}y : \supset \vdash .$ Prop

***50·21.** $\vdash .\, J = \breve{J}$

Dem.

$\vdash . *21·2 . (*50·02) . \supset \vdash .\, J = \dot{-}\, I$ (1)

[*50·2.*23·83] $= \dot{-}\, \breve{I}$

[*31·16] $= \mathrm{Cnv}‘\dot{-}\, I$

[(1).*31·32] $= \breve{J} \,.\supset \vdash .$ Prop

***50·22.** $\vdash : R \subset I . \equiv . \breve{R} \subset I$ [*31·4 . *50·2]

***50·23.** $\vdash : R \subset J . \equiv . \breve{R} \subset J$ [*31·4 . *50·21]

***50·24.** $\vdash : R \subset J . \equiv . (x) . \sim (xRx)$

Dem.

$\vdash . *50 \cdot 11 . \supset \vdash :. R \subset J . \equiv : xRy . \supset_{x,y} . x \neq y :$

[Transp] $\equiv : x = y . \supset_{x,y} . \sim (xRy) :$

[*13·191] $\equiv : (x) . \sim (xRx) :. \supset \vdash . \text{Prop}$

***50·3.** $\vdash . (x) . xIx$ [*50·1 . *13·15]

***50·31.** $\vdash . D`I = V . \text{Ɑ}`I = V$

Dem.

$\vdash . *50 \cdot 3 . *10 \cdot 24 . \supset \vdash :. (x) : (\exists y) . xIy :. (x) : (\exists y) . yIx :.$

[*33·13·131] $\supset \vdash : (x) . x \in D`I : (x) . x \in \text{Ɑ}`I :$

[*24·14] $\supset \vdash . D`I = V . \text{Ɑ}`I = V . \supset \vdash . \text{Prop}$

***50·32.** $\vdash . C`I = V$ [*50·31 . *33·16 . *24·27]

***50·33.** $\vdash : \dot{\exists} ! J . \supset . D`J = V . \text{Ɑ}`J = V . C`J = V$

Dem.

$\vdash . *13 \cdot 171 . \text{Transp} . \supset \vdash :. y \neq z . \supset : x \neq y . \vee . x \neq z :.$

[*50·11] $\supset \vdash :. yJz . \supset : xJy . \vee . xJz :$

[*33·14] $\supset : x \in D`J$ (1)

$\vdash . (1) . *11 \cdot 11 \cdot 35 .$ $\supset \vdash : \dot{\exists} ! J . \supset . x \in D`J :$

[*10·11·21] $\supset \vdash : \dot{\exists} ! J . \supset . (x) . x \in D`J .$

[*24·14] $\supset . D`J = V$ (2)

$\vdash . (2) . *50 \cdot 21 . \supset \vdash . \text{Prop}$

In the above proposition (*50·33), the hypothesis $\dot{\exists} ! J$ is equivalent to the hypothesis that more than one object exists of the type in question. This can be proved for all except the lowest type. For the lowest type, we can only prove the existence of at least one object: this is proved in *24·52. For the next type, we can prove the existence of at least two objects, namely Λ and V; these are distinct, by *24·1. For the next type, we can prove the existence of 2^2 objects; for the next, 2^4, etc. But for the class of individuals we cannot prove, from our primitive propositions, that there is more than one object in the universe, and therefore we cannot prove $\dot{\exists} ! J$. We might, of course, have included among our primitive propositions the assumption that more than one individual exists, or some assumption from which this would follow, such as

$$(\exists \phi, x, y) . \phi ! x . \sim \phi ! y.$$

But very few of the propositions which we might wish to prove depend upon this assumption, and we have therefore excluded it. It should be observed that many philosophers, being monists, deny this assumption.

***50·34.** $\vdash . \dot{\exists} ! J \upharpoonright \mathrm{Cls}$

Dem.

$\vdash . *20{\cdot}41 . *22{\cdot}38 . (*24{\cdot}01{\cdot}02) . \supset \vdash . \Lambda, \mathrm{V} \in \mathrm{Cls} .$
$[*24{\cdot}1] \quad \supset \vdash . \Lambda \neq \mathrm{V} . \Lambda, \mathrm{V} \in \mathrm{Cls} .$
$[*36{\cdot}13 . *50{\cdot}11] \quad \supset \vdash . \Lambda \{J \upharpoonright \mathrm{Cls}\} \mathrm{V} .$
$[*10{\cdot}24] \quad \supset \vdash . \mathrm{Prop}$

***50·35.** $\vdash . \dot{\exists} ! J \upharpoonright \mathrm{Rel}$ [Proof as in *50·34]

***50·4.** $\vdash . R | I = I | R = R$

Dem.

$\vdash . *34{\cdot}1 . \supset \vdash : x (R | I) z . \equiv . (\exists y) . xRy . yIz .$
$[*50{\cdot}1] \quad \equiv . (\exists y) . xRy . y = z .$
$[*13{\cdot}195] \quad \equiv . xRz \quad (1)$
$\vdash . *34{\cdot}1 . \supset \vdash : x (I | R) z . \equiv . (\exists y) . xIy . yRz .$
$[*50{\cdot}1] \quad \equiv . (\exists y) . x = y . yRz .$
$[*13{\cdot}195] \quad \equiv . xRz \quad (2)$
$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***50·41.** $\vdash : R | \breve{P} \mathrel{\dot\subset} J . \equiv . \breve{R} | P \mathrel{\dot\subset} J . \equiv . R \mathbin{\dot\cap} P = \dot\Lambda$

Dem.

$\vdash . *34{\cdot}1 . *50{\cdot}11 . \supset \vdash :. R | \breve{P} \mathrel{\dot\subset} J . \equiv : (\exists y) . xRy . y\breve{P}z . \supset_{x,z} . x \neq z :$
$[*13{\cdot}196] \quad \equiv : (x) : \sim (\exists y) . xRy . y\breve{P}x :$
$[*10{\cdot}252] \quad \equiv : \sim (\exists x, y) . xRy . y\breve{P}x :$
$[*31{\cdot}11] \quad \equiv : \sim (\exists x, y) . xRy . xPy :$
$[*23{\cdot}33 . *25{\cdot}51] \quad \equiv : R \mathbin{\dot\cap} P = \dot\Lambda : \quad (1)$
$[*31{\cdot}14{\cdot}24] \quad \equiv : \breve{R} \mathbin{\dot\cap} \breve{P} = \dot\Lambda :$
$\left[(1) \dfrac{\breve{R}, \breve{P}}{R, P}\right] \quad \equiv : \breve{R} | \mathrm{Cnv}{}^{\backprime}\breve{P} \mathrel{\dot\subset} J :$
$[*34{\cdot}203] \quad \equiv : \breve{R} | P \mathrel{\dot\subset} J \quad (2)$
$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***50·42.** $\vdash . I^2 = I$

Dem.

$\vdash . *34{\cdot}5 . \supset \vdash : xI^2z . \equiv . (\exists y) . xIy . yIz .$
$[*50{\cdot}1] \quad \equiv . (\exists y) . xIy . y = z .$
$[*13{\cdot}195] \quad \equiv . xIz : \supset \vdash . \mathrm{Prop}$

***50·43.** $\vdash : R^2 \mathrel{\dot\subset} J . \equiv . R \mathbin{\dot\cap} \breve{R} = \dot\Lambda \quad \left[*50{\cdot}41 \dfrac{\breve{R}}{P}\right]$

This proposition is useful in the theory of series. "$R \mathbin{\dot\cap} \breve{R} = \dot\Lambda$" is the characteristic of an *asymmetrical* relation.

***50·44.** $\vdash : \dot{\exists} ! (R \dot{\cap} I) . \supset . \dot{\exists} ! (R^2 \dot{\cap} I)$

Dem.

$\vdash . *23{\cdot}33 . *50{\cdot}1 . \supset \vdash : \dot{\exists} ! (R \dot{\cap} I) . \equiv . (\exists x, y) . xRy . x = y .$

$[*13{\cdot}195] \qquad \equiv . (\exists x) . xRx .$

$[*34{\cdot}54] \qquad \supset . (\exists x) . xR^2x .$

$[*13{\cdot}195] \qquad \supset . (\exists x, y) . xR^2y . x = y .$

$[*23{\cdot}33 . *50{\cdot}1] \qquad \supset . \dot{\exists} ! (R^2 \dot{\cap} I) : \supset \vdash . \text{Prop}$

***50·45.** $\vdash : R^2 \dot{\subset} J . \supset . R \dot{\subset} J$ $\quad [*50{\cdot}44 . \text{Transp} . *25{\cdot}311]$

***50·46.** $\vdash : R \dot{\cap} \breve{R} = \dot{\Lambda} . \supset . R \dot{\subset} J$ $\quad [*50{\cdot}43{\cdot}45]$

***50·47.** $\vdash :. R^2 \dot{\subset} R . \supset : R \dot{\subset} J . \equiv . R^2 \dot{\subset} J . \equiv . R \dot{\cap} \breve{R} = \dot{\Lambda}$

Dem.

$\vdash . *23{\cdot}44 . \supset \vdash :. \text{Hp} . \supset : R \dot{\subset} J . \supset . R^2 \dot{\subset} J \qquad (1)$

$\vdash . (1) . *50{\cdot}45{\cdot}43 . \supset \vdash . \text{Prop}$

This proposition is used in the theory of series. If R is a serial relation, we shall have $R^2 \dot{\subset} R$ and $R \dot{\subset} J$.

***50·5.** $\vdash . \alpha \upharpoonleft I = I \upharpoonright \alpha = \alpha \upharpoonleft I \upharpoonright \alpha$

Dem.

$\vdash . *35{\cdot}1 . \supset \vdash : x (\alpha \upharpoonleft I) y . \equiv . x \in \alpha . xIy .$

$[*50{\cdot}1] \qquad \equiv . x \in \alpha . x = y .$

$[*13{\cdot}193] \qquad \equiv . y \in \alpha . x = y .$

$[*50{\cdot}1] \qquad \equiv . xIy . y \in \alpha .$

$[*35{\cdot}101] \qquad \equiv . x (I \upharpoonright \alpha) y \qquad (1)$

$\vdash . (1) . *23{\cdot}5 . \supset \vdash . \alpha \upharpoonleft I = \alpha \upharpoonleft I \dot{\cap} I \upharpoonright \alpha$

$[*35{\cdot}11] \qquad = \alpha \upharpoonleft I \upharpoonright \alpha \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***50·51.** $\vdash . \text{Cnv}\text{`}(\alpha \upharpoonleft I) = \alpha \upharpoonleft I$ $\quad [*35{\cdot}51 . *50{\cdot}2{\cdot}5]$

***50·52.** $\vdash . \text{D}\text{`}(\alpha \upharpoonleft I) = \text{Ɑ}\text{`}(\alpha \upharpoonleft I) = C\text{`}(\alpha \upharpoonleft I) = \alpha$

Dem.

$\vdash . *35{\cdot}61 . \supset \vdash . \text{D}\text{`}(\alpha \upharpoonleft I) = \alpha \cap \text{D}\text{`}I$

$[*50{\cdot}31] \qquad = \alpha \cap \text{V}$

$[*24{\cdot}26] \qquad = \alpha \qquad (1)$

Similarly $\quad \vdash . \text{Ɑ}\text{`}(\alpha \upharpoonleft I) = \alpha \qquad (2)$

$\vdash . (1) . (2) . *33{\cdot}18 . \supset \vdash . \text{Prop}$

***50·53.** $\vdash . \alpha \upharpoonleft I \upharpoonright \beta = (\alpha \cap \beta) \upharpoonleft I = I \upharpoonright (\alpha \cap \beta)$

Dem.

$\vdash . *35{\cdot}21 . *50{\cdot}5 . \supset \vdash . \alpha \upharpoonleft I \upharpoonright \beta = \alpha \upharpoonleft (\beta \upharpoonleft I)$

$[*35{\cdot}32] \qquad = (\alpha \cap \beta) \upharpoonleft I \qquad (1)$

$\vdash . (1) . *50{\cdot}5 . \supset \vdash . \text{Prop}$

***50·54.** $\vdash . (\alpha \upharpoonleft I)^2 = \alpha \upharpoonleft I$

Dem.

$\vdash . *50{\cdot}5 . \supset \vdash . (\alpha \upharpoonleft I)^2 = (\alpha \upharpoonleft I) | (I \upharpoonright \alpha)$

$[*35{\cdot}12] \quad = \alpha \upharpoonleft I^2 \upharpoonright \alpha$

$[*50{\cdot}42] \quad = \alpha \upharpoonleft I \upharpoonright \alpha$

$[*50{\cdot}5] \quad = \alpha \upharpoonleft I . \supset \vdash . \text{Prop}$

***50·55.** $\vdash : \alpha \cap \beta = \Lambda . \equiv . \alpha \uparrow \beta \subset J$

Dem.

$\vdash . *24{\cdot}37 . *50{\cdot}11 . \supset$

$\vdash :. \alpha \cap \beta = \Lambda . \equiv : x \in \alpha . y \in \beta . \supset_{x,y} . xJy :$

$[*35{\cdot}103] \quad \equiv : \alpha \uparrow \beta \subset J :. \supset \vdash . \text{Prop}$

***50·56.** $\vdash : \exists ! (\alpha \cap \beta) . \equiv . \dot{\exists} ! \{(\alpha \uparrow \beta) \dot{\cap} I\}$

Dem.

$\vdash . *50{\cdot}55 . \text{Transp} . *24{\cdot}54 . \supset$

$\vdash : \exists ! (\alpha \cap \beta) . \quad \equiv . \sim \{\alpha \uparrow \beta \subset J\} .$

$[*25{\cdot}55] \quad \equiv . \dot{\exists} (\alpha \uparrow \beta) \dot{-} J .$

$[*23{\cdot}831 . (*50{\cdot}02)] \equiv . \dot{\exists} ! \{(\alpha \uparrow \beta) \dot{\cap} I\} : \supset \vdash . \text{Prop}$

***50·57.** $\vdash . I \dot{\cap} \alpha \upharpoonleft R = I \dot{\cap} R \upharpoonright \alpha = I \dot{\cap} \alpha \upharpoonleft R \upharpoonright \alpha$

Dem.

$\vdash . *35{\cdot}16 . \supset \vdash . I \dot{\cap} \alpha \upharpoonleft R = \alpha \upharpoonleft I \dot{\cap} R$

$[*50{\cdot}5] \quad = I \upharpoonright \alpha \dot{\cap} R$

$[*35{\cdot}17] \quad = I \dot{\cap} R \upharpoonright \alpha \quad (1)$

$[*50{\cdot}5] \quad = \alpha \upharpoonleft I \upharpoonright \alpha \dot{\cap} R$

$[*35{\cdot}16{\cdot}17{\cdot}21] \quad = I \dot{\cap} \alpha \upharpoonleft R \upharpoonright \alpha \quad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***50·58.** $\vdash : \alpha \upharpoonleft R \subset J . \equiv . R \upharpoonright \alpha \subset J . \equiv . \alpha \upharpoonleft R \upharpoonright \alpha \subset J$

Dem.

$\vdash . *50{\cdot}57 . \supset \vdash : I \dot{\cap} \alpha \upharpoonleft R = \dot{\Lambda} . \equiv . I \dot{\cap} R \upharpoonright \alpha = \dot{\Lambda} . \equiv . I \dot{\cap} \alpha \upharpoonleft R \upharpoonright \alpha = \dot{\Lambda} \quad (1)$

$\vdash . (1) . *50{\cdot}41 . \supset \vdash . \text{Prop}$

***50·59.** $\vdash . (I \upharpoonright \alpha) \text{“} \beta = \alpha \cap \beta$

Dem.

$\vdash . *37{\cdot}412 . \supset \vdash . (I \upharpoonright \alpha) \text{“} \beta = I \text{“} (\alpha \cap \beta)$

$[*50{\cdot}16] \quad = \alpha \cap \beta . \supset \vdash . \text{Prop}$

***50·6.** $\vdash . R | (I \upharpoonright \alpha) = R \upharpoonright \alpha$

Dem.

$\vdash . *35{\cdot}23 . \supset \vdash . R | (I \upharpoonright \alpha) = (R | I) \upharpoonright \alpha$

$[*50{\cdot}4] \quad = R \upharpoonright \alpha . \supset \vdash . \text{Prop}$

***50·61.** $\vdash . I \upharpoonright \alpha | R = \alpha \upharpoonleft R$

Dem.

$\vdash . *35{\cdot}354 . \supset \vdash . I \upharpoonright \alpha | R = I | (\alpha \upharpoonleft R)$

$[*50{\cdot}4] \quad = \alpha \upharpoonleft R . \supset \vdash . \text{Prop}$

***50·62.** $\vdash : \text{Ɑ}`R \subset \alpha . \supset . R \mid (I \upharpoonright \alpha) = R$ [*50·6 . *35·452]

***50·63.** $\vdash : \text{D}`R \subset \alpha . \supset . I \upharpoonright \alpha \mid R = R$ [*50·61 . *35·451]

***50·64.** $\vdash . R \mid (I \upharpoonright \text{Ɑ}`R) = R \mid (I \upharpoonright C`R) = R$ [*50·62 . *22·42 . *33·161]

***50·65.** $\vdash . I \upharpoonright (\text{D}`R) \mid R = I \upharpoonright (C`R) \mid R = R$ [*50·63 . *22·42 . *33·161]

***50·7.** $\vdash : \text{Ɑ}`R \subset \alpha . \supset . R \mid `I \upharpoonright \alpha = R$ [*50·62 . *43·11]

***50·71.** $\vdash : \text{D}`R \subset \alpha . \supset . \mid R`I \upharpoonright \alpha = R$ [*50·63 . *43·111]

***50·72.** $\vdash . R \mid `(I \upharpoonright C`R) = \mid R`(I \upharpoonright C`R) = R$ [*50·7·71]

***50·73.** $\vdash . R \mid `I = \mid R`I = R$ [*50·4 . *43·11·111]

***50·74.** $\vdash . R \parallel I = R \mid$

Dem.

$\vdash . *43·112 . \supset \vdash . (R \parallel I)`Q = R \mid Q \mid I$

[*50·4] $= R \mid Q$

[*43·11] $= R \mid `Q$ (1)

$\vdash . (1) . *30·41 . \supset \vdash .$ Prop

***50·75.** $\vdash . I \parallel R = \mid R$ [Proof as in *50·74]

***50·76.** $\vdash : P \mid = R \mid . \equiv . P = R$

Dem.

$\vdash . *34·27 . *30·41 . \supset \vdash : P = R . \supset . P \mid = R \mid$ (1)

$\vdash . *50·73 . *30·36 . \supset \vdash : P \mid = R \mid . \supset . P = R$ (2)

$\vdash . (1) . (2) . \supset \vdash .$ Prop

***50·761.** $\vdash : \mid P = \mid R . \equiv . P = R$ [Proof as in *50·76]

$*51$. UNIT CLASSES

Summary of $*51$.

In this number we introduce a new descriptive function $\iota`x$, meaning "the class of terms which are identical with x," which is the same thing as "the class whose only member is x." We are thus to have

$$\iota`x = \hat{y}(y = x).$$

But $\hat{y}(y = x) = \overrightarrow{I}`x$. Hence we secure what we require by the following definition:

$*51\cdot01$. $\iota = \overrightarrow{I}$ Df

As a matter of notation, it might be thought that $\overrightarrow{I}$ would do as well as ι, and that this definition is superfluous. But we need also the converse of this relation, and "$\mathrm{Cnv}`\overrightarrow{I}$" is not a sufficiently convenient symbol.

The propositions of this number are constantly used in what follows. It should be observed that the class whose members are x and y is $\iota`x \cup \iota`y$, the class whose members are x, y, z is $\iota`x \cup \iota`y \cup \iota`z$, the class formed by adding $x$ to $\alpha$ is $\alpha \cup \iota`x$, and the class formed by taking x away from α is $\alpha - \iota`x$. (If x is not a member of α, this is equal to α.)

The distinction between x and $\iota`x$ is one of the merits of Peano's symbolic logic, as well as of Frege's. On the basis of our theory of classes, the necessity for the distinction is of course obvious. But apart from this, the following consideration makes the necessity apparent. Let α be a class; then the class whose only member is α has only one member, namely α, while α may have many members. Hence the class whose only member is α cannot be identical with α*.

The propositions of the present number which are most used are the following:

$*51\cdot15$. $\vdash : y \in \iota`x \,.\equiv.\, y = x$

$*51\cdot16$. $\vdash .\, x \in \iota`x$

$*51\cdot2$. $\vdash : x \in \alpha \,.\equiv.\, \iota`x \subset \alpha$

This proposition is useful because it enables us to replace membership of a class ($x \in \alpha$) by inclusion in the class ($\iota`x \subset \alpha$).

$*51\cdot211$. $\vdash : x \sim\in \alpha \,.\equiv.\, \iota`x \cap \alpha = \Lambda$

$*51\cdot221$. $\vdash : x \in \alpha \,.\equiv.\, (\alpha - \iota`x) \cup \iota`x = \alpha$

* This argument is due to Frege. See his article "Kritische Beleuchtung einiger Punkte in E. Schröder's Vorlesungen über die Algebra der Logik," *Archiv für Syst. Phil.*, vol. I. p. 444 (1895).

***51·222.** $\vdash : x \sim\epsilon\, \alpha \,.\equiv .\, \alpha - \iota\text{‘}x = \alpha$

***51·23.** $\vdash : \iota\text{‘}x = \iota\text{‘}y \,.\equiv .\, y \,\epsilon\, \iota\text{‘}x \,.\equiv .\, x \,\epsilon\, \iota\text{‘}y \,.\equiv .\, x = y$

***51·4.** $\vdash : \exists\,!\,\alpha \,.\, \alpha \subset \iota\text{‘}x \,.\equiv .\, \alpha = \iota\text{‘}x$

I.e. an existent class contained in a unit class must be identical with the unit class. From this proposition it will follow that 0 is the only cardinal which is less than 1.

***51·51.** $\vdash : \alpha = \iota\text{‘}x \,.\equiv .\, x = \breve{\iota}\text{‘}\alpha \,.\equiv .\, x \,\breve{\iota}\, \alpha$

For classes, $\breve{\iota}\text{‘}\alpha$ has the same uses that $(\imath x)(\phi x)$ has for functions; "$\breve{\iota}\text{‘}\alpha$" means "the only member of α." We have

***51·59.** $\vdash : \psi\{\breve{\iota}\text{‘}\hat{z}(\phi z)\} \,.\equiv .\, \psi(\imath x)(\phi x)$

***51·01.** $\iota = \overrightarrow{I}$ Df

***51·1.** $\vdash : \alpha\iota x \,.\equiv .\, \alpha = \hat{y}(y = x)$

Dem.

$\vdash . *4\cdot2 . (*51\cdot01) . \supset \vdash : \alpha\iota x \,.\equiv .\, \alpha\overrightarrow{I}x \,.$

$[*32\cdot1] \qquad \equiv .\, \alpha = \hat{y}(yIx) \,.$

$[*50\cdot1] \qquad \equiv .\, \alpha = \hat{y}(y = x) : \supset \vdash . \text{Prop}$

***51·11.** $\vdash .\, \iota\text{‘}x = \hat{y}(y = x)$ $\quad [*30\cdot3 . *51\cdot1]$

***51·12.** $\vdash .\, \text{E}\,!\,\iota\text{‘}x$ $\quad [*51\cdot11 . *14\cdot21]$

***51·13.** $\vdash : \alpha = \iota\text{‘}x \,.\equiv .\, \alpha = \hat{y}(y = x)$ $\quad [*20\cdot57\cdot2 . *51\cdot11]$

***51·131.** $\vdash : \alpha\iota x \,.\equiv .\, \alpha = \iota\text{‘}x$ $\quad [*51\cdot1\cdot13]$

***51·14.** $\vdash :. \alpha = \iota\text{‘}x \,.\equiv : y \,\epsilon\, \alpha \,.\equiv_y .\, y = x$ $\quad [*51\cdot13 . *20\cdot33]$

***51·141.** $\vdash :. \alpha = \iota\text{‘}x \,.\equiv : \exists\,!\,\alpha : y \,\epsilon\, \alpha \,.\supset_y .\, y = x : \equiv : x \,\epsilon\, \alpha : y \,\epsilon\, \alpha \,.\supset_y .\, y = x$

$[*51\cdot14 . *14\cdot122]$

***51·15.** $\vdash : y \,\epsilon\, \iota\text{‘}x \,.\equiv .\, y = x$ $\quad [*51\cdot11 . *20\cdot33]$

***51·16.** $\vdash .\, x \,\epsilon\, \iota\text{‘}x$ $\quad [*51\cdot15 . *13\cdot15]$

***51·161.** $\vdash .\, \exists\,!\,\iota\text{‘}x$ $\quad [*51\cdot16 . *10\cdot24]$

***51·17.** $\vdash .\, \text{Cl}\text{‘}\iota = \text{V}$

Dem.

$\vdash . *51\cdot1 . *20\cdot2 . \supset \vdash . \{\hat{y}(y = x)\}\, \iota\, x \,.$

$[*10\cdot24] \qquad \supset \vdash . (\exists\alpha) . \alpha\iota x \,.$

$[*33\cdot131] \qquad \supset \vdash . x \,\epsilon\, \text{Cl}\text{‘}\iota \,.$

$[*10\cdot11] \qquad \supset \vdash . (x) . x \,\epsilon\, \text{Cl}\text{‘}\iota \,.$

$[*24\cdot14] \qquad \supset \vdash . \text{Cl}\text{‘}\iota = \text{V}$

The above proposition is used in the theory of selections (*83·71).

$*51{\cdot}2$. $\vdash : x \in \alpha \,.\equiv.\, \iota`x \subset \alpha$

Dem.

$$\begin{aligned}
&\vdash . *13{\cdot}191 . \supset \vdash :. x \in \alpha . \equiv : y = x . \supset_y . y \in \alpha : \\
&[*51{\cdot}15] \qquad \equiv : y \in \iota`x . \supset_y . y \in \alpha : \\
&[*22{\cdot}1] \qquad \equiv : \iota`x \subset \alpha :. \supset \vdash . \text{Prop}
\end{aligned}$$

The above proposition shows how to replace membership of a class by inclusion in a class; thus for example it gives:

Socrates is a man $.\equiv.$ the class of terms identical with Socrates is included in the class of men.

Before Peano and Frege, the relation of membership (ϵ) was regarded as merely a particular case of the relation of inclusion ($\subset$). For this reason, the traditional formal logic treated such propositions as "Socrates is a man" as instances of the universal affirmative A, "All S is P," which is what we express by "$\alpha \subset \beta$." This involved a confusion of fundamentally different kinds of propositions, which greatly hindered the development and usefulness of symbolic logic. But by means of the above proposition ($*51{\cdot}2$), we can always obtain a proposition stating an inclusion (namely "$\iota`x \subset \alpha$") which is equivalent to a given proposition stating membership of a class (namely "$x \in \alpha$").

$*51{\cdot}21$. $\vdash . x \sim\in \alpha - \iota`x$

Dem.

$$\begin{aligned}
&\vdash . *22{\cdot}33{\cdot}35 . \supset \vdash : x \in \alpha - \iota`x . \equiv . x \in \alpha . x \sim\in \iota`x . \\
&[*3{\cdot}27] \qquad \supset . x \sim\in \iota`x \qquad (1) \\
&\vdash . (1) . \text{Transp} . *51{\cdot}16 . \supset \vdash . \text{Prop}
\end{aligned}$$

$*51{\cdot}211$. $\vdash : x \sim\in \alpha \,.\equiv.\, \iota`x \cap \alpha = \Lambda$

Dem.

$$\begin{aligned}
&\vdash . *24{\cdot}39 . \supset \vdash :. \iota`x \cap \alpha = \Lambda . \equiv : y \in \iota`x . \supset_y . y \sim\in \alpha : \\
&[*51{\cdot}15] \qquad \equiv : y = x . \supset_y . y \sim\in \alpha : \\
&[*13{\cdot}191] \qquad \equiv : x \sim\in \alpha :. \supset \vdash . \text{Prop}
\end{aligned}$$

$*51{\cdot}22$. $\vdash : \alpha \cap \iota`x = \Lambda . \alpha \cup \iota`x = \beta . \equiv . x \in \beta . \alpha = \beta - \iota`x$

Dem.

$$\begin{aligned}
&\vdash . *24{\cdot}47 . \supset \\
&\vdash : \alpha \cap \iota`x = \Lambda . \alpha \cup \iota`x = \beta . \equiv . \iota`x \subset \beta . \alpha = \beta - \iota`x . \\
&[*51{\cdot}2] \qquad \equiv . x \in \beta . \alpha = \beta - \iota`x : \supset \vdash . \text{Prop}
\end{aligned}$$

$*51{\cdot}221$. $\vdash : x \in \alpha . \equiv . (\alpha - \iota`x) \cup \iota`x = \alpha$

Dem.

$$\begin{aligned}
&\vdash . *51{\cdot}2 . \supset \vdash : x \in \alpha . \equiv . \iota`x \subset \alpha . \\
&[*22{\cdot}62] \qquad \equiv . \iota`x \cup \alpha = \alpha . \\
&[*22{\cdot}91] \qquad \equiv . (\alpha - \iota`x) \cup \iota`x = \alpha : \supset \vdash . \text{Prop}
\end{aligned}$$

***51·222.** $\vdash : x \sim\epsilon\, \alpha \,.\equiv .\, \alpha - \iota`x = \alpha$ [*51·211 . *24·313]

***51·23.** $\vdash : \iota`x = \iota`y \,.\equiv .\, y \,\epsilon\, \iota`x \,.\equiv .\, x \,\epsilon\, \iota`y \,.\equiv .\, x = y$

Dem.

$$\begin{array}{lll}
\vdash . *20{\cdot}31 . *51{\cdot}15 . \supset & & \\
\vdash :. \iota`x = \iota`y \,.\equiv : z = x \,.\equiv_z .\, z = y : & & \\
[*13{\cdot}183] & \equiv : x = y : & (1) \\
[*51{\cdot}15] & \equiv : x \,\epsilon\, \iota`y : & (2) \\
[(1).*13{\cdot}16] & \equiv : y \,\epsilon\, \iota`x & (3) \\
\vdash . (1) . (2) . (3) . \supset \vdash . \text{Prop} & &
\end{array}$$

***51·231.** $\vdash : \iota`x \cap \iota`y = \Lambda \,.\equiv .\, x \neq y$

Dem.

$$\begin{array}{ll}
\vdash . *24{\cdot}311 . \supset \vdash :. \iota`x \cap \iota`y = \Lambda \,.\equiv : \iota`x \subset -\iota`y : & \\
[*51{\cdot}15] & \equiv : z = x \,.\supset_z .\, z \neq y : \\
[*13{\cdot}191] & \equiv : x \neq y :. \supset \vdash . \text{Prop}
\end{array}$$

***51·232.** $\vdash :. z \,\epsilon\, (\iota`x \cup \iota`y) \,.\equiv : z = x \,.\vee .\, z = y$ [*22·34 . *51·15]

This proposition states that a member of $\iota`x \cup \iota`y$ must be either x or y, and vice versa, *i.e.* that $\iota`x \cup \iota`y$ is the class whose only members are x and y.

***51·233.** $\vdash :: \alpha = \iota`x \cup \iota`y \,.\supset :. (z) :. z \,\epsilon\, \alpha \,.\equiv : z = x \,.\vee .\, z = y$
[*51·232 . *10·11 . *20·18]

***51·234.** $\vdash :: \alpha = \iota`x \cup \iota`y \,.\supset :. z \,\epsilon\, \alpha \,.\supset_z .\, \phi z : \equiv .\, \phi x \,.\, \phi y$

Dem.

$$\begin{array}{ll}
\vdash . *51{\cdot}233 . \supset \vdash ::. \text{Hp} . \supset :: z \,\epsilon\, \alpha \,.\supset_z .\, \phi z : \equiv :. z = x \,.\vee .\, z = y : \supset_z .\, \phi z :. & \\
[*4{\cdot}77] & \equiv :. (z) :. z = x \,.\supset .\, \phi z : z = y \,.\supset .\, \phi z :. \\
[*10{\cdot}22] & \equiv :. z = x \,.\supset_z .\, \phi z : z = y \,.\supset_z .\, \phi z :. \\
[*13{\cdot}191] & \equiv :. \phi x \,.\, \phi y ::. \supset \vdash . \text{Prop}
\end{array}$$

***51·235.** $\vdash :: \alpha = \iota`x \cup \iota`y \,.\supset :. (\exists z) . z \,\epsilon\, \alpha \,.\, \phi z \,.\equiv : \phi x \,.\vee .\, \phi y$

Dem.

$$\begin{array}{ll}
\vdash . *51{\cdot}233 . \supset & \\
\vdash :: \text{Hp} . \supset :. (\exists z) . z \,\epsilon\, \alpha \,.\, \phi z \,.\equiv : (\exists z) : z = x \,.\vee .\, z = y : \phi z : & \\
[*4{\cdot}4] & \equiv : (\exists z) : z = x \,.\, \phi z \,.\vee .\, z = y \,.\, \phi z : \\
[*10{\cdot}42] & \equiv : (\exists z) . z = x \,.\, \phi z \,.\vee .\, (\exists z) . z = y \,.\, \phi z : \\
[*13{\cdot}195] & \equiv : \phi x \,.\vee .\, \phi y :: \supset \vdash . \text{Prop}
\end{array}$$

***51·236.** $\vdash :. z \,\epsilon\, \iota`x \cup \beta \,.\equiv : z = x \,.\vee .\, z \,\epsilon\, \beta$ [*22·34 . *51·15]

***51·237.** $\vdash :: \alpha = \iota`x \cup \beta \,.\supset :. (z) :. z \,\epsilon\, \alpha \,.\equiv : z = x \,.\vee .\, z \,\epsilon\, \beta$
[*51·236 . *10·11 . *20·18]

***51·238.** $\vdash :: \alpha = \iota`x \cup \beta \,.\supset :. z \,\epsilon\, \alpha \,.\supset_z .\, \phi z : \equiv : \phi x : z \,\epsilon\, \beta \,.\supset_z .\, \phi z$

Dem.

$$\begin{array}{ll}
\vdash . *51{\cdot}237 . \supset \vdash ::. \text{Hp} . \supset :: z \,\epsilon\, \alpha \,.\supset_z .\, \phi z : \equiv :. z = x \,.\vee .\, z \,\epsilon\, \beta : \supset_z .\, \phi z :. & \\
[*4{\cdot}77] & \equiv :. (z) :. z = x \,.\supset .\, \phi z : z \,\epsilon\, \beta \,.\supset .\, \phi z :. \\
[*10{\cdot}22] & \equiv :. z = x \,.\supset_z .\, \phi z : z \,\epsilon\, \beta \,.\supset_z .\, \phi z :. \\
[*13{\cdot}191] & \equiv :. \phi x : z \,\epsilon\, \beta \,.\supset_z .\, \phi z ::. \supset \vdash . \text{Prop}
\end{array}$$

***51·239.** $\vdash :: \alpha = \iota'x \cup \beta \,.\supset :. (\exists z) \,.\, z \in \alpha \,.\, \phi z \,.\equiv : \phi x \,.\vee.\, (\exists z) \,.\, z \in \beta \,.\, \phi z$

Dem.

$\vdash .\ *51{\cdot}237 \,.\supset$

$\vdash :: \mathrm{Hp} \,.\supset :. (\exists z) \,.\, z \in \alpha \,.\, \phi z \,.\equiv : (\exists z) : z = x \,.\vee.\, z \in \beta : \phi z :$

$[*4{\cdot}4] \qquad \equiv : (\exists z) : z = x \,.\, \phi z \,.\vee.\, z \in \beta \,.\, \phi z :$

$[*10{\cdot}42] \qquad \equiv : (\exists z) \,.\, z = x \,.\, \phi z \,.\vee.\, (\exists z) \,.\, z \in \beta \,.\, \phi z :$

$[*13{\cdot}195] \qquad \equiv : \phi x \,.\vee.\, (\exists z) \,.\, z \in \beta \,.\, \phi z :: \supset \vdash .\ \mathrm{Prop}$

***51·24.** $\vdash :. \iota'y \subset \iota'x \cup \beta \,.\equiv : y = x \,.\vee.\, y \in \beta$

Dem.

$\vdash .\ *51{\cdot}236 \,.\supset$

$\vdash :: \iota'y \subset \iota'x \cup \beta \,.\equiv :. z \in \iota'y \,.\supset_z : z = x \,.\vee.\, z \in \beta :.$

$[*51{\cdot}15] \qquad \equiv :. z = y \,.\supset_z : z = x \,.\vee.\, z \in \beta :.$

$[*13{\cdot}191] \qquad \equiv :. y = x \,.\vee.\, y \in \beta :: \supset \vdash .\ \mathrm{Prop}$

***51·25.** $\vdash : \alpha \subset \iota'x \cup \beta \,.\, x \sim\in \alpha \,.\supset.\, \alpha \subset \beta$ [*51·211 . *24·49]

***51·3.** $\vdash : y \in \alpha \,.\, y \neq x \,.\equiv.\, y \in \alpha - \iota'x$ [*51·15 . *22·33·35]

***51·31.** $\vdash : \exists ! \alpha \cap \iota'x \,.\equiv.\, \iota'x \subset \alpha \,.\equiv.\, \alpha \cap \iota'x = \iota'x \,.\equiv.\, x \in \alpha$

Dem.

$\vdash .\ *22{\cdot}33 \,.\, *51{\cdot}15 \,.\supset \vdash : \exists ! \alpha \cap \iota'x \,.\equiv.\, (\exists y) \,.\, y \in \alpha \,.\, y = x \,.$

$[*13{\cdot}195] \qquad \equiv .\, x \in \alpha \,. \qquad (1)$

$[*51{\cdot}2] \qquad \equiv .\, \iota'x \subset \alpha \,. \qquad (2)$

$[*22{\cdot}621] \qquad \equiv .\, \iota'x = \iota'x \cap \alpha \qquad (3)$

$\vdash .\ (1) \,.\, (2) \,.\, (3) \,.\supset \vdash .\ \mathrm{Prop}$

***51·34.** $\vdash : x \in \alpha \,.\equiv.\, -\alpha \subset -\iota'x$ [*51·2 . *22·81]

***51·35.** $\vdash : x \sim\in \alpha \,.\equiv.\, \iota'x \subset -\alpha$ [*51·2 . *22·35]

***51·36.** $\vdash : x \sim\in \alpha \,.\equiv.\, \alpha \subset -\iota'x$ [*51·35 . *22·811]

*51·36 is frequently used.

***51·37.** $\vdash .\ \alpha = \hat{x}(\iota'x \subset \alpha)$ [*51·2 . *20·33]

***51·4.** $\vdash : \exists ! \alpha \,.\, \alpha \subset \iota'x \,.\equiv.\, \alpha = \iota'x$

Dem.

$\vdash .\ *24{\cdot}5 \,.\, *51{\cdot}15 \,.\supset \vdash :. \exists ! \alpha \,.\, \alpha \subset \iota'x \,.\equiv : (\exists y) \,.\, y \in \alpha : y \in \alpha \,.\supset_y .\, y = x :$

$[*14{\cdot}122] \qquad \equiv : y \in \alpha \,.\equiv_y .\, y = x :$

$[*51{\cdot}11 . *20{\cdot}33] \qquad \equiv : \alpha = \iota'x :. \supset \vdash .\ \mathrm{Prop}$

***51·401.** $\vdash :. \alpha \subset \iota'x \,.\equiv : \alpha = \Lambda \,.\vee.\, \alpha = \iota'x$

Dem.

$\vdash .\ *51{\cdot}4 \,.\, *5{\cdot}6 \,. \quad \supset \vdash :. \alpha \subset \iota'x \,.\supset : \alpha = \Lambda \,.\vee.\, \alpha = \iota'x \qquad (1)$

$\vdash .\ *24{\cdot}12 \,.\, *22{\cdot}42 \,.\supset \vdash :. \alpha = \Lambda \,.\vee.\, \alpha = \iota'x : \supset .\, \alpha \subset \iota'x \qquad (2)$

$\vdash .\ (1) \,.\, (2) \,.\supset \vdash .\ \mathrm{Prop}$

This proposition shows that unit classes are the smallest existent classes.

***51·41.** $\vdash : \iota`x \cup \iota`y = \iota`x \cup \iota`z \,.\equiv.\, y = z$

Dem.

$\vdash . *20·2 . *13·13 . \supset \vdash : y = z . \supset . \iota`x \cup \iota`y = \iota`x \cup \iota`z$ (1)

$\vdash . *22·58 . \supset \vdash :. \iota`x \cup \iota`y = \iota`x \cup \iota`z . \supset : \iota`y \subset \iota`x \cup \iota`z . \iota`z \subset \iota`x \cup \iota`y :$

$[*51·16·232]$ $\supset : y = x . \vee . y = z : z = x . \vee . z = y :$

$[*13·16 . *4·41]$ $\supset : y = x . z = x . \vee . y = z :$

$[*13·172 . *2·621]$ $\supset : y = z$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

The two following propositions are lemmas for *51·43.

***51·42.** $\vdash :. \iota`x \cup \iota`y = \iota`z \cup \iota`w . \supset : x = z . y = w . \vee . x = w . y = z$

Dem.

$\vdash . *51·232 . \supset$

$\vdash :: \iota`x \cup \iota`y = \iota`z \cup \iota`w . \equiv :. a = x . \vee . a = y : \equiv_a : a = z . \vee . a = w :.$

$[*10·1]$ $\supset :. x = x . \vee . x = y : \equiv : x = z . \vee . x = w :.$

$[*13·15]$ $\supset :. x = z . \vee . x = w$ (1)

$\vdash . *20·2 . *13·13 . \supset \vdash : \iota`x \cup \iota`y = \iota`z \cup \iota`w . x = z . \supset . \iota`x \cup \iota`y = \iota`x \cup \iota`w .$

$[*51·41]$ $\supset . y = w$ (2)

Similarly $\vdash : \iota`x \cup \iota`y = \iota`z \cup \iota`w . x = w . \supset . y = z$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash . \text{Prop}$

***51·421.** $\vdash :. x = z . y = w . \vee . x = w . y = z : \supset . \iota`x \cup \iota`y = \iota`z \cup \iota`w$ [*51·41]

***51·43.** $\vdash :. \iota`x \cup \iota`y = \iota`z \cup \iota`w . \equiv : x = z . y = w . \vee . x = w . y = z$
[*51·42·421]

The following propositions are concerned with $\breve{\iota}$, *i.e.* with the relation of the only member of a unit class to that class. If α is a unit class, $\breve{\iota}`\alpha$ is its only member. $(\imath x)(\phi x)$ and $\breve{\iota}`\hat{z}(\phi z)$ are equal whenever either exists, and any proposition about the one is equivalent to the same proposition about the other.

***51·51.** $\vdash : \alpha = \iota`x . \equiv . x = \breve{\iota}`\alpha . \equiv . x \,\breve{\iota}\, \alpha$

Dem.

$\vdash . *51·131 . *31·11 . \supset \vdash : \alpha = \iota`x . \equiv . x \,\breve{\iota}\, \alpha$ (1)

$\vdash . (1) . \supset \vdash : x \,\breve{\iota}\, \alpha . y \,\breve{\iota}\, \alpha . \supset . \alpha = \iota`x . \alpha = \iota`y .$

$[*51·23 . *20·57·2]$ $\supset . x = y$ (2)

$\vdash . (2) . \text{Exp} . *10·11 . *4·71 . \supset \vdash :. x \,\breve{\iota}\, \alpha . \equiv : x \,\breve{\iota}\, \alpha : y \,\breve{\iota}\, \alpha . \supset_y . x = y :$

$[*30·31]$ $\equiv : x = \breve{\iota}`\alpha$ (3)

$\vdash . (1) . (3) . \supset \vdash . \text{Prop}$

***51·511.** $\vdash . \breve{\iota}\text{‘}\iota\text{‘}x = x$ $\left[*51\cdot51 \dfrac{\iota\text{‘}x}{\alpha} . *20\cdot2\right]$

***51·52.** $\vdash : \mathrm{E}\,!\,\breve{\iota}\text{‘}\alpha . \equiv . \alpha = \iota\text{‘}\breve{\iota}\text{‘}\alpha$ $\left[*51\cdot51 \dfrac{\breve{\iota}\text{‘}\alpha}{x} . *14\cdot21\cdot18\right]$

***51·53.** $\vdash : \mathrm{E}\,!\,\breve{\iota}\text{‘}\alpha . \equiv . \breve{\iota}\text{‘}\alpha \in \alpha$ $[*51\cdot52\cdot16 . *14\cdot21\cdot18]$

***51·54.** $\vdash : \mathrm{E}\,!\,\breve{\iota}\text{‘}\alpha . \equiv . (\exists x) . \alpha = \iota\text{‘}x$ $[*51\cdot51 . *14\cdot204]$

***51·55.** $\vdash : \mathrm{E}\,!\,\breve{\iota}\text{‘}\alpha . \equiv . \mathrm{E}\,!\,(\imath x)(x \in \alpha)$

Dem.

$\vdash . *51\cdot54\cdot14 . \supset \vdash :. \mathrm{E}\,!\,\breve{\iota}\text{‘}\alpha . \equiv : (\exists x) : y \in \alpha . \equiv_y . y = x :$

$[*14\cdot11]$ $\equiv : \mathrm{E}\,!\,(\imath x)(x \in \alpha) :. \supset \vdash . \mathrm{Prop}$

***51·56.** $\vdash : b = \breve{\iota}\text{‘}\hat{y}(\phi y) . \equiv . \hat{y}(\phi y) = \iota\text{‘}b . \equiv . b = (\imath x)(\phi x)$

Dem.

$\vdash . *51\cdot51 . \supset \vdash :. b = \breve{\iota}\text{‘}\hat{y}(\phi y) . \equiv : \hat{y}(\phi y) = \iota\text{‘}b :$ (1)

$[*20\cdot15 . *51\cdot11]$ $\equiv : \phi y . \equiv_y . y = b :$

$[*14\cdot202]$ $\equiv : b = (\imath x)(\phi x)$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***51·57.** $\vdash : \mathrm{E}\,!\,\breve{\iota}\text{‘}\hat{y}(\phi y) . \equiv . \breve{\iota}\text{‘}\hat{y}(\phi y) = (\imath x)(\phi x) . \equiv . \mathrm{E}\,!\,(\imath x)(\phi x)$

Dem.

$\vdash . *14\cdot204 . *51\cdot56 . \supset \vdash : \mathrm{E}\,!\,\breve{\iota}\text{‘}\hat{y}(\phi y) . \equiv . \mathrm{E}\,!\,(\imath x)(\phi x)$ (1)

$\vdash . *14\cdot205 . \supset \vdash : (\imath x)(\phi x) = \breve{\iota}\text{‘}\hat{y}(\phi y) . \equiv . (\exists b) . b = (\imath x)(\phi x) . b = \breve{\iota}\text{‘}\hat{y}(\phi y) .$

$[*51\cdot56 . *4\cdot71]$ $\equiv . (\exists b) . b = (\imath x)(\phi x) .$

$[*14\cdot204\cdot13]$ $\equiv . \mathrm{E}\,!\,(\imath x)(\phi x)$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***51·58.** $\vdash : \mathrm{E}\,!\,\breve{\iota}\text{‘}\alpha . \equiv . \breve{\iota}\text{‘}\alpha = (\imath x)(x \in \alpha)$ $[*51\cdot57 . *20\cdot3 . *14\cdot272]$

***51·59.** $\vdash : \psi\{\breve{\iota}\text{‘}\hat{z}(\phi z)\} . \equiv . \psi(\imath x)(\phi x)$ $[*51\cdot56 . *14\cdot205]$

$*52$. THE CARDINAL NUMBER 1

Summary of $*52$.

In this number, we introduce the cardinal number 1, defined as the class of all unit classes. The fact that 1 so defined is a cardinal number is not relevant at present, and cannot of course be proved until "cardinal number" has been defined. For the present, therefore, 1 is to be regarded simply as the class of all unit classes, unit classes being such classes as are of the form $\iota`x$ for some x.

Like Λ and V, 1 is ambiguous as to type: it means "all unit classes of the type in question." The symbol "$1\,(\alpha)$," where α is a type, will mean "all unit classes whose sole members belong to the type α" (cf. $*65$). Thus *e.g.* "$\xi \epsilon 1\,(\text{Indiv})$" will mean "$\xi$ is a class consisting of one individual," if "Indiv" stands for the class of individuals.

The properties of 1 to be proved in the present number are what we may call *logical* as opposed to *arithmetical* properties, *i.e.* they are not concerned with the arithmetical operations (addition, etc.) which can be performed with 1, but with the relations of 1 to unit classes. The arithmetical properties of 1 will be considered later, in Part III.

The propositions of the present number which are most used are the following:

$*52\cdot16$. $\vdash :. \alpha \epsilon 1 . \equiv : \exists ! \alpha : x, y \epsilon \alpha . \supset_{x,y} . x = y$

I.e. α is a unit class if, and only if, it is not null, and all its members are identical.

$*52\cdot22$. $\vdash . \iota`x \epsilon 1$

$*52\cdot4$. $\vdash :. \alpha \epsilon 1 \cup \iota`\Lambda . \equiv : x, y \epsilon \alpha . \supset_{x,y} . x = y$

We shall define 0 as $\iota`\Lambda$. Thus the above proposition states that a class has one member or none when, and only when, all its members are identical.

$*52\cdot41$. $\vdash : \exists ! \alpha . \alpha \sim\epsilon 1 . \equiv . (\exists x, y) . x, y \epsilon \alpha . x \neq y$

This proposition is obtainable from $*52\cdot4$ by transposition, *i.e.* by negating each side of the equivalence.

$*52\cdot46$. $\vdash :. \alpha, \beta \epsilon 1 . \supset : \alpha \subset \beta . \equiv . \alpha = \beta . \equiv . \exists ! (\alpha \cap \beta)$

I.e. two unit classes are identical when, and only when, one is contained in the other, and when and only when they have a common part.

$*52\cdot01$. $1 = \hat{\alpha} \{(\exists x) . \alpha = \iota`x\}$ Df

$*52\cdot1$. $\vdash : \alpha \epsilon 1 . \equiv . (\exists x) . \alpha = \iota`x$ $[*20\cdot3 . (*52\cdot01)]$

$*52 \cdot 11$. $\vdash :. \alpha \epsilon 1 . \equiv : (\exists x) : y \epsilon \alpha . \equiv_y . y = x$ [$*52 \cdot 1 . *51 \cdot 14$]

$*52 \cdot 12$. $\vdash : \hat{z}(\phi z) \epsilon 1 . \equiv . E ! (\imath x)(\phi x)$

Dem.

$\vdash . *52 \cdot 11 . \supset \vdash :. \hat{z}(\phi z) \epsilon 1 . \equiv : (\exists x) : y \epsilon \hat{z}(\phi z) . \equiv_y . y = x :$

[$*20 \cdot 3$] $\equiv : (\exists x) : \phi y . \equiv_y . y = x :$

[$*14 \cdot 11$] $\equiv : E ! (\imath x)(\phi x) :. \supset \vdash . \text{Prop}$

$*52 \cdot 13$. $\vdash . 1 = D ' \iota$

Dem.

$\vdash . *51 \cdot 131 . \supset \vdash : \alpha = \iota ' x . \equiv . \alpha \iota x :$

[$*10 \cdot 11 \cdot 281$] $\supset \vdash : (\exists x) . \alpha = \iota ' x . \equiv . (\exists x) . \alpha \iota x :$

[$*52 \cdot 1$] $\supset \vdash : \alpha \epsilon 1 . \equiv . (\exists x) . \alpha \iota x$

[$*33 \cdot 13$] $\equiv . \alpha \epsilon D ' \iota : \supset \vdash . \text{Prop}$

$*52 \cdot 14$. $\vdash . 1 = \iota `` V$ [$*52 \cdot 13 . *37 \cdot 28$]

$*52 \cdot 15$. $\vdash : \alpha \epsilon 1 . \equiv . E ! \breve{\iota} ' \alpha$ [$*51 \cdot 54 . *52 \cdot 1$]

$*52 \cdot 16$. $\vdash :. \alpha \epsilon 1 . \equiv : \exists ! \alpha : x, y \epsilon \alpha . \supset_{x,y} . x = y$ [$*52 \cdot 15 . *51 \cdot 55 . *14 \cdot 203$]

$*52 \cdot 17$. $\vdash : \alpha \epsilon 1 . \equiv . \breve{\iota} ' \alpha = (\imath x)(x \epsilon \alpha)$ [$*51 \cdot 58 . *52 \cdot 15$]

$*52 \cdot 171$. $\vdash : \alpha \epsilon 1 . \equiv . E ! (\imath x)(x \epsilon \alpha)$ [$*51 \cdot 55 . *52 \cdot 15$]

$*52 \cdot 172$. $\vdash : \alpha \epsilon 1 . \equiv . \iota ' \breve{\iota} ' \alpha = \alpha$ [$*51 \cdot 52 . *52 \cdot 15$]

$*52 \cdot 173$. $\vdash : \alpha \epsilon 1 . \equiv . \breve{\iota} ' \alpha \epsilon \alpha$ [$*51 \cdot 53 . *52 \cdot 15$]

$*52 \cdot 18$. $\vdash :. \alpha \epsilon 1 . \equiv : (\exists x) : x \epsilon \alpha : y \epsilon \alpha . \supset_y . y = x$

Dem.

$\vdash . *51 \cdot 141 . \supset \vdash :. (\exists x) . \alpha = \iota ' x . \equiv : (\exists x) : x \epsilon \alpha : y \epsilon \alpha . \supset_y . y = x$ (1)

$\vdash . (1) . *52 \cdot 1 . \supset \vdash . \text{Prop}$

$*52 \cdot 181$. $\vdash :. \alpha \sim \epsilon 1 . \equiv : x \epsilon \alpha . \supset_x . (\exists y) . y \epsilon \alpha . y \neq x$ [$*52 \cdot 18 . *10 \cdot 51$]

$*52 \cdot 2$. $\vdash . 1 \subset \text{Cls}$

Dem.

$\vdash . *52 \cdot 1 . \supset \vdash : \alpha \epsilon 1 . \supset . (\exists x) . \alpha = \iota ' x .$

[$*51 \cdot 11$] $\supset . (\exists x) . \alpha = \hat{z}(z = x) .$

[$*20 \cdot 54$] $\supset . (\exists x, \phi) . \hat{z}(\phi ! z) = \hat{z}(z = x) . \alpha = \hat{z}(\phi ! z) .$

[$*10 \cdot 5$] $\supset . (\exists \phi) . \alpha = \hat{z}(\phi ! z) .$

[$*20 \cdot 4$] $\supset . \alpha \epsilon \text{Cls} : \supset \vdash . \text{Prop}$

$*52 \cdot 21$. $\vdash . \Lambda \sim \epsilon 1$

Dem.

$\vdash . *52 \cdot 16 . \supset \vdash : \alpha \epsilon 1 . \supset_\alpha . \exists ! \alpha :$

[$*24 \cdot 63$] $\supset \vdash : \Lambda \sim \epsilon 1$

$*52 \cdot 22$. $\vdash . \iota ' x \epsilon 1$ [$*51 \cdot 12 . *14 \cdot 28 . *10 \cdot 24 . *52 \cdot 1$]

∗52·23. $\vdash . \exists ! 1 . \exists ! -1$

Dem.

$$\begin{aligned}
&\vdash . *52\cdot22 . *10\cdot24 . &&\supset \vdash . (\exists x) . \iota`x \epsilon 1 . \\
&[*20\cdot54] &&\supset \vdash . (\exists x, \alpha) . \alpha = \iota`x . \alpha \epsilon 1 . \\
&[*10\cdot5] &&\supset \vdash . (\exists \alpha) . \alpha \epsilon 1 \quad (1) \\
&\vdash . *52\cdot21 . *22\cdot35 . &&\supset \vdash . \Lambda \epsilon -1 . \\
&[*10\cdot24] &&\supset \vdash . (\exists \alpha) . \alpha \epsilon -1 \quad (2) \\
&\vdash . (1) . (2) . &&\supset \vdash . \text{Prop}
\end{aligned}$$

∗52·24. $\vdash . 1 \neq \Lambda \cap \text{Cls} . 1 \neq V \cap \text{Cls}$ [∗52·23 . ∗24·54 . ∗24·17 . Transp]

∗52·3. $\vdash . \iota``\alpha \subset 1$

Dem.

$$\begin{aligned}
&\vdash . *52\cdot22 . *2\cdot02 . &&\supset \vdash : y \epsilon \alpha . \supset . \iota`y \epsilon 1 : \\
&[*51\cdot12 . *10\cdot11 . *37\cdot61] &&\supset \vdash . \iota``\alpha \subset 1
\end{aligned}$$

∗52·31. $\vdash : \kappa \subset 1 . \equiv . (\exists \alpha) . \kappa = \iota``\alpha$

Dem.

$$\begin{aligned}
&\vdash . *52\cdot14 . \supset \vdash : \kappa \subset 1 . &&\equiv . \kappa \subset \iota``V . \\
&[*37\cdot66 . *51\cdot12] &&\equiv . (\exists \alpha) . \alpha \subset V . \kappa = \iota``\alpha . \\
&[*24\cdot11] &&\equiv . (\exists \alpha) . \kappa = \iota``\alpha : \supset \vdash . \text{Prop}
\end{aligned}$$

∗52·4. $\vdash :. \alpha \epsilon 1 \cup \iota`\Lambda . \equiv : x, y \epsilon \alpha . \supset_{x,y} . x = y$

Dem.

$$\begin{aligned}
&\vdash . *52\cdot16 . *24\cdot54 . \supset \\
&\qquad \vdash :. \alpha \epsilon 1 . &&\equiv : \alpha \neq \Lambda : x, y \epsilon \alpha . \supset_{x,y} . x = y :. \\
&[*4\cdot37] \ \supset \vdash :: \alpha \epsilon 1 . \vee . \alpha = \Lambda : &&\equiv :. \alpha = \Lambda :. \vee :. \alpha \neq \Lambda : x, y \epsilon \alpha . \supset_{x,y} . x = y :. \\
&[*5\cdot63] &&\equiv :. \alpha = \Lambda : \vee : x, y \epsilon \alpha . \supset_{x,y} . x = y \quad (1) \\
&\vdash . *24\cdot51 . *10\cdot53 . *11\cdot62 . \supset \vdash :. \alpha = \Lambda . \supset : x, y \epsilon \alpha . \supset_{x,y} . x = y && \quad (2) \\
&\vdash . (1) . (2) . *4\cdot72 . \quad \supset \vdash :: \alpha \epsilon 1 . \vee . \alpha = \Lambda : \equiv :. x, y \epsilon \alpha . \supset_{x,y} . x = y && \quad (3) \\
&\vdash . (3) . *51\cdot236 . \quad \supset \vdash . \text{Prop}
\end{aligned}$$

This proposition is frequently useful. We shall define the number 0 as $\iota`\Lambda$; thus the above proposition states that a class has one member or none when, and only when, all its members are identical. It will be seen that $x, y \epsilon \alpha . \supset_{x,y} . x = y$ does not imply $\exists ! \alpha$, and therefore allows the possibility of α having no members.

∗52·41. $\vdash : \exists ! \alpha . \alpha \sim\epsilon 1 . \equiv . (\exists x, y) . x, y \epsilon \alpha . x \neq y$

Dem.

$$\begin{aligned}
&\vdash . *24\cdot54 . \supset \vdash :. \exists ! \alpha . \alpha \sim\epsilon 1 . &&\equiv : \alpha \neq \Lambda . \alpha \sim\epsilon 1 : \\
&[*4\cdot56] &&\equiv : \sim\{\alpha \epsilon 1 . \vee . \alpha = \Lambda\} : \\
&[*51\cdot236] &&\equiv : \sim(\alpha \epsilon 1 \cup \iota`\Lambda) : \\
&[*52\cdot4 . \text{Transp}] &&\equiv : \sim\{x, y \epsilon \alpha . \supset_{x,y} . x = y\} \\
&[*11\cdot52] &&\equiv : (\exists x, y) . x, y \epsilon \alpha . x \neq y :. \supset \vdash . \text{Prop}
\end{aligned}$$

***52·42.** $\vdash :. \alpha \in 1 . \supset : \exists ! \alpha \cap \beta . \equiv . \alpha \cap \beta \in 1$

Dem.

$\vdash . *51·31 . \quad \supset \vdash :. \exists ! \iota`x \cap \beta . \equiv . \iota`x \cap \beta = \iota`x :.$

$[*20·53] \quad \supset \vdash :. \alpha = \iota`x . \supset : \exists ! \alpha \cap \beta . \equiv . \alpha \cap \beta = \iota`x :.$

$[*10·11·28] \quad \supset \vdash :. (\exists x) . \alpha = \iota`x . \supset : (\exists x) : \exists ! \alpha \cap \beta . \equiv . \alpha \cap \beta = \iota`x :$

$[*10·37] \quad \supset : \exists ! \alpha \cap \beta . \supset . (\exists x) . \alpha \cap \beta = \iota`x \quad (1)$

$\vdash . (1) . *52·1 . \supset \vdash :. \alpha \in 1 . \supset : \exists ! \alpha \cap \beta . \supset . \alpha \cap \beta \in 1 \quad (2)$

$\vdash . *52·16 . \quad \supset \vdash : \alpha \cap \beta \in 1 . \supset . \exists ! \alpha \cap \beta \quad (3)$

$\vdash . (2) . (3) . \quad \supset \vdash .$ Prop

***52·43.** $\vdash : \alpha \in 1 . \exists ! \alpha \cap \beta . \equiv . \alpha \in 1 . \alpha \cap \beta \in 1 \quad [*52·42 . *5·32]$

***52·44.** $\vdash :. \alpha \in 1 . \supset : \exists ! \alpha \cap \beta . \equiv . \alpha \subset \beta . \equiv . \alpha \cap \beta = \alpha$

Dem.

$\vdash . *51·31 . \quad \supset \vdash : \exists ! \iota`x \cap \beta . \equiv . \iota`x \subset \beta :$

$[*13·13.\text{Exp}] \quad \supset \vdash :. \alpha = \iota`x . \supset : \exists ! \alpha \cap \beta . \equiv . \alpha \subset \beta :.$

$[*10·11·23] \quad \supset \vdash :. (\exists x) . \alpha = \iota`x . \supset : \exists ! \alpha \cap \beta . \equiv . \alpha \subset \beta :.$

$[*52·1] \quad \supset \vdash :. \alpha \in 1 . \supset : \exists ! \alpha \cap \beta . \equiv . \alpha \subset \beta \quad (1)$

$\vdash . (1) . *22·621 . \supset \vdash .$ Prop

***52·45.** $\vdash :: \alpha, \beta \in 1 . \supset :. \alpha \subset \beta \cup \gamma . \equiv : \alpha = \beta . \vee . \alpha \subset \gamma$

Dem.

$\vdash . *51·236 \dfrac{x, y, \gamma}{z, x, \beta} . \supset$

$\vdash :. x \in \iota`y \cup \gamma . \equiv : x = y . \vee . x \in \gamma :.$

$[*51·2·23] \quad \supset \vdash :. \iota`x \subset \iota`y \cup \gamma . \equiv : \iota`x = \iota`y . \vee . \iota`x \subset \gamma :.$

$[*13·21] \quad \supset \vdash :: \alpha = \iota`x . \beta = \iota`y . \supset :. \alpha \subset \beta \cup \gamma . \equiv : \alpha = \beta . \vee . \alpha \subset \gamma ::$

$[*11·11·35] \quad \supset \vdash :: (\exists x, y) . \alpha = \iota`x . \beta = \iota`y . \supset :. \alpha \subset \beta \cup \gamma . \equiv : \alpha = \beta . \vee . \alpha \subset \gamma \quad (1)$

$\vdash . (1) . *52·1 . \supset \vdash .$ Prop

***52·46.** $\vdash :. \alpha, \beta \in 1 . \supset : \alpha \subset \beta . \equiv . \alpha = \beta . \equiv . \exists ! (\alpha \cap \beta)$

Dem.

$\vdash . *51·2·23 . \quad \supset \vdash : \iota`x \subset \iota`y . \equiv . \iota`x = \iota`y \quad (1)$

$\vdash . (1) . *13·21 . \quad \supset \vdash :. \alpha = \iota`x . \beta = \iota`y . \supset : \alpha \subset \beta . \equiv . \alpha = \beta \quad (2)$

$\vdash . (2) . *11·11·35 . *52·1 . \supset \vdash :. \alpha, \beta \in 1 . \supset : \alpha \subset \beta . \equiv . \alpha = \beta \quad (3)$

$\vdash . (3) . *52·44 . \quad \supset \vdash .$ Prop

***52·6.** $\vdash :. \alpha \in 1 . \supset : x \in \alpha . \equiv . \iota`x = \alpha . \equiv . x = \breve{\iota}`\alpha$

Dem.

$\vdash . *51·23 . \quad \supset \vdash : x \in \iota`y . \equiv . \iota`x = \iota`y :$

$[*13·13.\text{Exp}] \quad \supset \vdash :. \alpha = \iota`y . \supset : x \in \alpha . \equiv . \iota`x = \alpha :.$

$[*10·11·23 . *52·1] \supset \vdash :. \alpha \in 1 . \quad \supset : x \in \alpha . \equiv . \iota`x = \alpha . \quad (1)$

$[*51·51] \quad \equiv . x = \breve{\iota}`\alpha \quad (2)$

$\vdash . (1) . (2) . \supset \vdash .$ Prop

***52·601.** $\vdash :: \alpha \in 1 . \supset :. \phi(\breve{\iota}`\alpha) . \equiv : x \in \alpha . \supset_x . \phi x : \equiv : (\exists x) . x \in \alpha . \phi x$

Dem.

$\vdash . *52\cdot15 . \supset \vdash :. \mathrm{Hp} . \supset : E ! \breve{\iota}`\alpha :$ (1)

[*30·4] $\supset : x \breve{\iota} \alpha . \equiv . x = \breve{\iota}`\alpha .$

[*52·6] $\equiv . x \in \alpha$ (2)

$\vdash . (1) . *30\cdot33 . \supset$

$\vdash :: \mathrm{Hp} . \supset :. \phi(\breve{\iota}`\alpha) . \equiv : x \breve{\iota} \alpha . \supset_x . \phi x : \equiv : (\exists x) . x \breve{\iota} \alpha . \phi x$ (3)

$\vdash . (2) . (3) . \supset \vdash . \mathrm{Prop}$

***52·602.** $\vdash :. \hat{z}(\phi z) \in 1 . \supset : \psi(\imath x)(\phi x) . \equiv . \phi x \supset_x \psi x . \equiv . (\exists x) . \phi x . \psi x$ [*52·12 . *14·26]

***52·61.** $\vdash :. \alpha \in 1 . \supset : \breve{\iota}`\alpha \in \beta . \equiv . \alpha \subset \beta . \equiv . \exists ! (\alpha \cap \beta)$ $\left[*52\cdot601 \dfrac{x \in \beta}{\phi x}\right]$

***52·62.** $\vdash :. \alpha, \beta \in 1 . \supset : \alpha = \beta . \equiv . \breve{\iota}`\alpha = \breve{\iota}`\beta$

Dem.

$\vdash . *52\cdot601 . \supset \vdash :: \mathrm{Hp} . \supset :. \breve{\iota}`\alpha = \breve{\iota}`\beta . \equiv : x \in \alpha . \supset_x . x = \breve{\iota}`\beta :$

[*52·6] $\equiv : x \in \alpha . \supset_x . x \in \beta :$

[*52·46] $\equiv : \alpha = \beta :: \supset \vdash . \mathrm{Prop}$

***52·63.** $\vdash : \alpha, \beta \in 1 . \alpha \neq \beta . \supset . \alpha \cap \beta = \Lambda$ [*52·46 . Transp]

***52·64.** $\vdash : \alpha \in 1 . \supset . \alpha \cap \beta \in 1 \cup \iota`\Lambda$

Dem.

$\vdash . *52\cdot43 . \supset \vdash : \mathrm{Hp} . \exists ! \alpha \cap \beta . \supset . \alpha \cap \beta \in 1 :$

[*5·6.*24·54] $\supset \vdash :. \mathrm{Hp} . \supset : \alpha \cap \beta = \Lambda . \vee . \alpha \cap \beta \in 1 :$

[*51·236] $\supset : \alpha \cap \beta \in 1 \cup \iota`\Lambda :. \supset \vdash . \mathrm{Prop}$

***52·7.** $\vdash :. \beta - \alpha \in 1 . \alpha \subset \xi . \xi \subset \beta . \supset : \xi = \alpha . \vee . \xi = \beta$

Dem.

$\vdash . *22\cdot41 . \supset \vdash : \mathrm{Hp} . \xi \subset \alpha . \supset . \xi = \alpha$ (1)

$\vdash . *24\cdot55 . \supset \vdash : \sim(\xi \subset \alpha) . \supset . \exists ! \xi - \alpha$ (2)

$\vdash . *22\cdot48 . \supset \vdash : \mathrm{Hp} . \supset . \xi - \alpha \subset \beta - \alpha$ (3)

$\vdash . (2) . (3) . \supset \vdash : \mathrm{Hp} . \sim(\xi \subset \alpha) . \supset . \exists ! \xi - \alpha . \xi - \alpha \subset \beta - \alpha$ (4)

$\vdash . *52\cdot1 . \supset \vdash : \mathrm{Hp} . \supset . (\exists x) . \beta - \alpha = \iota`x$ (5)

$\vdash . (4) . (5) . *51\cdot4 . \supset \vdash : \mathrm{Hp} . \sim(\xi \subset \alpha) . \supset . \xi - \alpha = \beta - \alpha .$

[*24·411] $\supset . \xi = \beta$ (6)

$\vdash . (1) . (6) . \supset \vdash . \mathrm{Prop}$

$*53$. MISCELLANEOUS PROPOSITIONS INVOLVING UNIT CLASSES

Summary of $*53$.

The propositions to be given in this number are mostly such as would have come more naturally at an earlier stage, but could not be given sooner because they involved unit classes. It is to be observed that $\iota`x \cup \iota`y$ is the class consisting of the members x and y, while $\iota`x \uparrow \iota`y$ is the relation which holds only between x and y. If α and β are classes, $\iota`\alpha \cup \iota`\beta$ is a class of classes, its members being α and β. If R and S are relations, $\iota`R \uparrow \iota`S$ is a relation of relations; and so on.

The present number begins by connecting products and sums $p`\kappa$, $s`\kappa$, $\dot{p}`\lambda$, $\dot{s}`\lambda$, in cases where the members of κ or λ are specified, with the products or sums $\alpha \cap \beta$, $\alpha \cup \beta$, $R \dot{\cap} S$, $R \dot{\cup} S$. We have

$*53\cdot01$. $\vdash . p`\iota`\alpha = \alpha$

$*53\cdot1$. $\vdash . p`(\iota`\alpha \cup \iota`\beta) = \alpha \cap \beta$

$*53\cdot14$. $\vdash . p`(\kappa \cup \iota`\alpha) = p`\kappa \cap \alpha$

with similar propositions for s, $\dot{p}$ and $\dot{s}$.

We have next a set of propositions on sums and products of classes of unit classes. The most important of these is

$*53\cdot22$. $\vdash . s`\iota``\alpha = \alpha$

We have next a proposition showing that the sum of κ is null when, and only when, κ is either null or has the null-class for its only member, *i.e.*

$*53\cdot24$. $\vdash :. s`\kappa = \Lambda . \equiv : \kappa = \Lambda \cap \mathrm{Cls} . \vee . \kappa = \iota`\Lambda$

(Here we write "$\Lambda \cap \mathrm{Cls}$," to show that the "$\Lambda$" in question is of the next type above that of the other two Λ's.)

We have next various propositions on the relations of $\overrightarrow{R}`x$ and $R`x$ and $R``\alpha$ in various cases, first for a general relation R, and then for the particular relation s defined in $*40$. Three of these propositions are very frequently used, namely:

$*53\cdot3$. $\vdash : E ! R`x . \equiv . \overrightarrow{R}`x \in 1$

$*53\cdot301$. $\vdash . R``\iota`x = \overrightarrow{R}`x$

$*53\cdot31$. $\vdash : E ! R`x . \supset . R``\iota`x = \iota`R`x = \overrightarrow{R}`x$

The remaining propositions of this number are of less importance, and are seldom referred to.

***53·01.** $\vdash . p^{\backprime}\iota^{\backprime}\alpha = \alpha$

Dem.

$\vdash . *40\cdot1 . \supset \vdash :. x \epsilon p^{\backprime}\iota^{\backprime}\alpha . \equiv : \beta \epsilon \iota^{\backprime}\alpha . \supset_\beta . x \epsilon \beta :$

$[*51\cdot15] \qquad \equiv : \beta = \alpha . \supset_\beta . x \epsilon \beta :$

$[*13\cdot191] \qquad \equiv : x \epsilon \alpha :. \supset \vdash . \text{Prop}$

***53·02.** $\vdash . s^{\backprime}\iota^{\backprime}\alpha = \alpha$

Dem.

$\vdash . *40\cdot11 . \supset \vdash : x \epsilon s^{\backprime}\iota^{\backprime}\alpha . \equiv . (\exists\beta) . \beta \epsilon \iota^{\backprime}\alpha . x \epsilon \beta .$

$[*51\cdot15] \qquad \equiv . (\exists\beta) . \beta = \alpha . x \epsilon \beta .$

$[*13\cdot195] \qquad \equiv . x \epsilon \alpha : \supset \vdash . \text{Prop}$

***53·03.** $\vdash . \dot{p}^{\backprime}\iota^{\backprime}R = R$ [Proof as in *53·01]

***53·04.** $\vdash . \dot{s}^{\backprime}\iota^{\backprime}R = R$ [Proof as in *53·02]

***53·1.** $\vdash . p^{\backprime}(\iota^{\backprime}\alpha \cup \iota^{\backprime}\beta) = \alpha \cap \beta$

Dem.

$\vdash . *40\cdot18 . \supset \vdash . p^{\backprime}(\iota^{\backprime}\alpha \cup \iota^{\backprime}\beta) = p^{\backprime}\iota^{\backprime}\alpha \cap p^{\backprime}\iota^{\backprime}\beta$

$[*53\cdot01] \qquad = \alpha \cap \beta . \supset \vdash . \text{Prop}$

This proposition can be extended to $\iota^{\backprime}\alpha \cup \iota^{\backprime}\beta \cup \iota^{\backprime}\gamma$, etc. It shows the connection (for finite classes of classes) between the product $p^{\backprime}\kappa$ and the product of the members $\alpha \cap \beta \cap \gamma \cap \ldots$.

***53·11.** $\vdash . s^{\backprime}(\iota^{\backprime}\alpha \cup \iota^{\backprime}\beta) = \alpha \cup \beta$

Dem.

$\vdash . *40\cdot171 . \supset \vdash . s^{\backprime}(\iota^{\backprime}\alpha \cup \iota^{\backprime}\beta) = s^{\backprime}\iota^{\backprime}\alpha \cup s^{\backprime}\iota^{\backprime}\beta$

$[*53\cdot02] \qquad = \alpha \cup \beta . \supset \vdash . \text{Prop}$

Similar remarks apply to this proposition as to *53·1.

***53·12.** $\vdash . \dot{p}^{\backprime}(\iota^{\backprime}R \cup \iota^{\backprime}S) = R \mathbin{\dot{\cap}} S$ [*41·18 . *53·03]

This proposition shows the connection between the product $\dot{p}^{\backprime}\kappa$ for a class κ consisting of two relations R and S, and the product $R \mathbin{\dot{\cap}} S$. The proposition can be extended to the product of any given finite class of relations.

***53·13.** $\vdash . \dot{s}^{\backprime}(\iota^{\backprime}R \cup \iota^{\backprime}S) = R \mathbin{\dot{\cup}} S$ [*41·171 . *53·04]

Similar remarks apply to this proposition as to *53·12.

***53·14.** $\vdash . p^{\backprime}(\kappa \cup \iota^{\backprime}\alpha) = p^{\backprime}\kappa \cap \alpha$

Dem.

$\vdash . *40\cdot18 . \supset \vdash . p^{\backprime}(\kappa \cup \iota^{\backprime}\alpha) = p^{\backprime}\kappa \cap p^{\backprime}\iota^{\backprime}\alpha$

$[*53\cdot01] \qquad = p^{\backprime}\kappa \cap \alpha$

***53·15.** $\vdash . s^{\backprime}(\kappa \cup \iota^{\backprime}\alpha) = s^{\backprime}\kappa \cup \alpha$ [Proof as in *53·14]

***53·16.** $\vdash . \dot{p}^{\backprime}(\lambda \cup \iota^{\backprime}R) = \dot{p}^{\backprime}\lambda \mathbin{\dot{\cap}} R$ [Proof as in *53·14]

***53·17.** $\vdash . \dot{s}^{\backprime}(\lambda \cup \iota^{\backprime}R) = \dot{s}^{\backprime}\lambda \mathbin{\dot{\cup}} R$ [Proof as in *53·14]

The above proposition and the next are both used in connection with mathematical induction (*91·55 and *97·46 respectively).

***53·18.** $\vdash . s`(\alpha - \iota`\Lambda) = s`\alpha$

Dem.

$$\begin{array}{ll} \vdash . *51\cdot221 . \supset \vdash : \Lambda \epsilon \alpha . & \supset . (\alpha - \iota`\Lambda) \cup \iota`\Lambda = \alpha . \\ [*53\cdot15] & \supset . s`(\alpha - \iota`\Lambda) \cup \Lambda = s`\alpha . \\ [*24\cdot24] & \supset . s`(\alpha - \iota`\Lambda) = s`\alpha \quad (1) \\ \vdash . *51\cdot222 . \supset \vdash : \Lambda \sim\epsilon \alpha . \supset . & \alpha - \iota`\Lambda = \alpha . \\ [*30\cdot37] & \supset . s`(\alpha - \iota`\Lambda) = s`\alpha \quad (2) \\ \vdash . (1) . (2) . \supset \vdash . \text{Prop} & \end{array}$$

***53·181.** $\vdash . \dot{s}`(\lambda - \iota`\dot{\Lambda}) = \dot{s}`\lambda$ [Proof as in *53·18]

***53·2.** $\vdash : \kappa \epsilon 1 . \supset . \breve{\iota}`\kappa = p`\kappa = s`\kappa$

This proposition requires, for significance, that κ should be a class of classes. It is used in *88·47, in the number on the existence of selections and the multiplicative axiom.

Dem.

$$\vdash . *52\cdot601 . \supset \vdash :: \text{Hp} . \supset :. x \epsilon \breve{\iota}`\kappa : \equiv : \alpha \epsilon \kappa . \supset_\alpha . x \epsilon \alpha : \equiv : (\exists\alpha) . \alpha \epsilon \kappa . x \epsilon \alpha \quad (1)$$

$$\vdash . (1) . *40\cdot1\cdot11 . \quad \supset \vdash . \text{Prop}$$

***53·21.** $\vdash : \lambda \epsilon 1 . \supset . \breve{\iota}`\lambda = \dot{p}`\lambda = \dot{s}`\lambda$ [Similar proof]

This proposition requires, for significance, that λ should be a class of relations.

***53·22.** $\vdash . s`\iota``\alpha = \alpha$

Dem.

$$\begin{array}{ll} \vdash . *40\cdot11 . \supset \vdash : x \epsilon s`\iota``\alpha . & \equiv . (\exists\gamma) . \gamma \epsilon \iota``\alpha . x \epsilon \gamma . \\ [*37\cdot64 . *51\cdot12] & \equiv . (\exists y) . y \epsilon \alpha . x \epsilon \iota`y . \\ [*51\cdot15] & \equiv . (\exists y) . y \epsilon \alpha . x = y . \\ [*13\cdot195] & \equiv . x \epsilon \alpha : \supset \vdash . \text{Prop} \end{array}$$

***53·221.** $\vdash . \iota``(\iota`x \cup \iota`y) = \iota`\iota`x \cup \iota`\iota`y$

Dem.

$$\begin{array}{ll} \vdash . *37\cdot1 . \supset \vdash :. \alpha \epsilon \iota``(\iota`x \cup \iota`y) . & \equiv : (\exists z) . z \epsilon (\iota`x \cup \iota`y) . \alpha \iota z : \\ [*51\cdot131] & \equiv : (\exists z) . z \epsilon (\iota`x \cup \iota`y) . \alpha = \iota`z : \\ [*51\cdot235] & \equiv : \alpha = \iota`x . \vee . \alpha = \iota`y : \\ [*51\cdot232] & \equiv : \alpha \epsilon (\iota`\iota`x \cup \iota`\iota`y) :. \supset \vdash . \text{Prop} \end{array}$$

***53·222.** $\vdash : \kappa = \iota``\alpha . \supset . \alpha = \breve{\iota}``\kappa$

Dem.

$$\begin{array}{ll} \vdash . *13\cdot12 . *20\cdot2 . \supset \vdash : \text{Hp} . \supset . \breve{\iota}``\kappa = \breve{\iota}``\iota``\alpha & \\ [*51\cdot511 . *14\cdot21 . *37\cdot67] & = \hat{x} \{(\exists y) . y \epsilon \alpha . x = \breve{\iota}`\iota`y\} \\ [*51\cdot511] & = \hat{x} \{(\exists y) . y \epsilon \alpha . x = y\} \\ [*13\cdot195] & = \alpha : \supset \vdash . \text{Prop} \end{array}$$

***53·23.** $\vdash : \kappa \subset 1 . \supset . s{}^{\backprime}\kappa = \breve{\iota}{}^{\backprime\backprime}\kappa$

Dem.

$\vdash . *52·31 . \supset \vdash : \text{Hp} . \equiv . (\exists \alpha) . \kappa = \iota{}^{\backprime\backprime}\alpha$ (1)

$\vdash . *53·22 . \supset \vdash : \kappa = \iota{}^{\backprime\backprime}\alpha . \supset . s{}^{\backprime}\kappa = \alpha$

[*53·222] $= \breve{\iota}{}^{\backprime\backprime}\kappa$ (2)

$\vdash . (1) . (2) . *10·11·23 . \supset \vdash .$ Prop

***53·231.** $\vdash :. x \in \alpha . \supset_x . x = y : \equiv : \alpha = \Lambda . \vee . \alpha = \iota{}^{\backprime}y$

Dem.

$\vdash . *51·141 . \supset \vdash :. \exists ! \alpha : x \in \alpha . \supset_x . x = y : \equiv : \alpha = \iota{}^{\backprime}y$ (1)

$\vdash . *10·53 . \supset \vdash :. \sim\exists ! \alpha . \supset : x \in \alpha . \supset_x . x = y :.$

[*4·71] $\supset \vdash :. \sim\exists ! \alpha : x \in \alpha . \supset_x . x = y : \equiv . \sim\exists ! \alpha .$

[*24·51] $\equiv . \alpha = \Lambda$ (2)

$\vdash . (1) . (2) . *4·42·39 . \supset \vdash .$ Prop

***53·24.** $\vdash :. s{}^{\backprime}\kappa = \Lambda . \equiv : \kappa = \Lambda \cap \text{Cls} . \vee . \kappa = \iota{}^{\backprime}\Lambda$

Dem.

$\vdash . *24·15 . *40·11 . \supset$

$\vdash :. s{}^{\backprime}\kappa = \Lambda . \equiv : (x) : \sim\{(\exists \alpha) . \alpha \in \kappa . x \in \alpha\} :$

[*10·51] $\equiv : (x, \alpha) : x \in \alpha . \supset . \alpha \sim\in \kappa :$

[*11·2.*10·23] $\equiv : (\exists x) . x \in \alpha . \supset_\alpha . \alpha \sim\in \kappa :$

[*24·54] $\equiv : \alpha \neq \Lambda . \supset_\alpha . \alpha \sim\in \kappa :$

[Transp] $\equiv : \alpha \in \kappa . \supset_\alpha . \alpha = \Lambda :$

[*53·231] $\equiv : \kappa = \Lambda \cap \text{Cls} . \vee . \kappa = \iota{}^{\backprime}\Lambda :. \supset \vdash .$ Prop

In the enunciation and the last line of the proof of the above proposition, we write "$\kappa = \Lambda \cap \text{Cls}$" rather than "$\kappa = \Lambda$," because this Λ must be of the type next above that of the Λ in "$\kappa = \iota{}^{\backprime}\Lambda$."

The following proposition is used in the theory of selections (*83·731).

***53·25.** $\vdash :. s{}^{\backprime}\kappa \cap s{}^{\backprime}\lambda = \Lambda . \supset : \kappa \cap \lambda = \Lambda \cap \text{Cls} . \vee . \kappa \cap \lambda = \iota{}^{\backprime}\Lambda$

Dem.

$\vdash . *40·181 . \supset \vdash :. \text{Hp} . \supset : s{}^{\backprime}(\kappa \cap \lambda) = \Lambda :$

[*53·24] $\supset : \kappa \cap \lambda = \Lambda \cap \text{Cls} . \vee . \kappa \cap \lambda = \iota{}^{\backprime}\Lambda :. \supset \vdash .$ Prop

***53·3.** $\vdash : E ! R{}^{\backprime}x . \equiv . \overrightarrow{R}{}^{\backprime}x \in 1$

Dem.

$\vdash . *30·2 . \supset \vdash :. E ! R{}^{\backprime}x . \equiv : (\exists b) : yRx . \equiv_y . y = b :$

[*32·18.*51·15] $\equiv : (\exists b) : y \in \overrightarrow{R}{}^{\backprime}x . \equiv_y . y \in \iota{}^{\backprime}b :$

[*20·31] $\equiv : (\exists b) . \overrightarrow{R}{}^{\backprime}x = \iota{}^{\backprime}b :$

[*52·1] $\equiv : \overrightarrow{R}{}^{\backprime}x \in 1 :. \supset \vdash .$ Prop

The above proposition is very frequently used.

***53·301.** $\vdash . R``\iota`x = \overrightarrow{R}`x$

Dem.

$$\begin{aligned}
&\vdash . *37\cdot1 . *51\cdot15 . \supset \vdash : y \epsilon R``\iota`x . \equiv . (\exists z) . z = x . yRz . \\
&[*13\cdot195] \qquad \equiv . yRx . \\
&[*32\cdot18] \qquad \equiv . y \epsilon \overrightarrow{R}`x : \supset \vdash . \text{Prop}
\end{aligned}$$

***53·302.** $\vdash . R``(\iota`x \cup \iota`y) = \overrightarrow{R}`x \cup \overrightarrow{R}`y$ [*37·22 . *53·301]

The above proposition is used in the cardinal theory of exponentiation (*116·71).

***53·31.** $\vdash : \text{E} ! R`x . \supset . R``\iota`x = \iota`R`x = \overrightarrow{R}`x$

The above proposition is one of which the subsequent use is frequent.

Dem.

$$\begin{aligned}
&\vdash . *51\cdot11 . *14\cdot18 . \supset \vdash : \text{Hp} . \supset . \iota`R`x = \hat{y}(y = R`x) \\
&[*30\cdot4] \qquad = \hat{y}(yRx) \\
&[*32\cdot13] \qquad = \overrightarrow{R}`x \qquad (1) \\
&\vdash . (1) . *53\cdot301 . \quad \supset \vdash . \text{Prop}
\end{aligned}$$

***53·32.** $\vdash : \text{E} ! R`x . \text{E} ! R`y . \supset . R``(\iota`x \cup \iota`y) = \iota`R`x \cup \iota`R`y$

Dem.

$$\begin{aligned}
&\vdash . *37\cdot22 . \supset \vdash . R``(\iota`x \cup \iota`y) = R``\iota`x \cup R``\iota`y \qquad (1) \\
&\vdash . (1) . *53\cdot31 . \supset \vdash . \text{Prop}
\end{aligned}$$

***53·33.** $\vdash . s``\iota`\kappa = \iota`s`\kappa$ $\left[*53\cdot31 \frac{s}{R}\right]$

***53·34.** $\vdash . s``(\iota`\kappa \cup \iota`\lambda) = \iota`s`\kappa \cup \iota`s`\lambda$ $\left[*53\cdot32 \frac{s}{R}\right]$

***53·35.** $\vdash . s`s``(\iota`\kappa \cup \iota`\lambda) = s`\kappa \cup s`\lambda = s`(\kappa \cup \lambda)$

Dem.

$$\begin{aligned}
&\vdash . *53\cdot34 . \supset \vdash . s`s``(\iota`\kappa \cup \iota`\lambda) = s`(\iota`s`\kappa \cup \iota`s`\lambda) \\
&[*53\cdot11] \qquad = s`\kappa \cup s`\lambda \\
&[*40\cdot171] \qquad = s`(\kappa \cup \lambda) . \supset \vdash . \text{Prop}
\end{aligned}$$

The above proposition may also be proved as follows:

$$\begin{aligned}
&\vdash . *42\cdot1 . \supset \vdash . s`s``(\iota`\kappa \cup \iota`\lambda) = s`s`(\iota`\kappa \cup \iota`\lambda) \\
&[*53\cdot11] \qquad = s`(\kappa \cup \lambda) \\
&[*40\cdot171] \qquad = s`\kappa \cup s`\lambda . \supset \vdash . \text{Prop}
\end{aligned}$$

***53·4.** $\vdash : x = R`y . \equiv . \overrightarrow{R}`y \epsilon 1 . x \epsilon \overrightarrow{R}`y . \equiv . \iota`x = \overrightarrow{R}`y . \equiv . x = \breve{\iota}`\overrightarrow{R}`y$

Dem.

$$\begin{aligned}
&\vdash . *14\cdot21 . *4\cdot71 . \supset \vdash : x = R`y . \equiv . \text{E} ! R`y . x = R`y . \\
&[*30\cdot4 . *5\cdot32] \qquad \equiv . \text{E} ! R`y . xRy . \\
&[*53\cdot3 . *32\cdot18] \qquad \equiv . \overrightarrow{R}`y \epsilon 1 . x \epsilon \overrightarrow{R}`y . \qquad (1) \\
&[*52\cdot6 . *5\cdot32] \qquad \equiv . \overrightarrow{R}`y \epsilon 1 . \iota`x = \overrightarrow{R}`y .
\end{aligned}$$

[*52·22] $\equiv . \iota'x = \overrightarrow{R}'y .$ (2)

[*51·51] $\equiv . x = \iota'\overset{\smile}{\overrightarrow{R}}'y$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash .$ Prop

***53·5.** $\vdash : \exists ! \alpha . \equiv . \alpha \epsilon \text{Cls} - \iota'\Lambda$

Dem.

$\vdash . *20·41 . \supset \vdash : \exists ! \hat{z}(\phi z) . \equiv . \hat{z}(\phi z) \epsilon \text{Cls} . \exists ! \hat{z}(\phi z) .$

[*24·54] $\equiv . \hat{z}(\phi z) \epsilon \text{Cls} . \hat{z}(\phi z) \neq \Lambda .$

[*51·3] $\equiv . \hat{z}(\phi z) \epsilon \text{Cls} - \iota'\Lambda : \supset \vdash .$ Prop

In the above proof, as usually where "Cls" or other type-symbols occur, it is necessary to abandon the notation by Greek letters and revert to the explicit notation.

***53·51.** $\vdash : \dot{\exists} ! R . \equiv . R \epsilon \text{Rel} - \iota'\dot{\Lambda}$ [Proof as in *53·5]

***53·52.** $\vdash : \alpha \epsilon \kappa . \exists ! \alpha . \equiv . \alpha \epsilon \kappa - \iota'\Lambda$

Dem.

$\vdash . *24·54 . \supset \vdash : \alpha \epsilon \kappa . \exists ! \alpha . \equiv . \alpha \epsilon \kappa . \alpha \neq \Lambda .$

[*51·3] $\equiv . \alpha \epsilon \kappa - \iota'\Lambda : \supset \vdash .$ Prop

***53·53.** $\vdash : R \epsilon \lambda . \dot{\exists} ! R . \equiv . R \epsilon \lambda - \iota'\dot{\Lambda}$ [Proof as in *53·52]

The following propositions are inserted because of their connection with the definition of $\alpha \rightarrow \beta$ in *70. $\overrightarrow{R}``\text{Cl}'R$ and $\overrightarrow{R}``\text{V}$ are both important classes.

***53·6.** $\vdash : R = \dot{\Lambda} . \exists ! \alpha . \supset . \overrightarrow{R}``\alpha = \iota'\Lambda . \overleftarrow{R}``\alpha = \iota'\Lambda$

Dem.

$\vdash . *33·15·241 . *24·13 . \supset \vdash : \text{Hp} . \supset . \overrightarrow{R}'x = \Lambda$ (1)

$\vdash . (1) . *37·7 . \supset \vdash : \text{Hp} . \supset . \overrightarrow{R}``\alpha = \hat{\beta}\{(\exists x) . x \epsilon \alpha . \beta = \Lambda\}$

[*10·35] $= \hat{\beta}\{\exists ! \alpha . \beta = \Lambda\}$

[*4·73] $= \hat{\beta}(\beta = \Lambda)$

[*51·11] $= \iota'\Lambda$ (2)

Similarly $\vdash : \text{Hp} . \supset . \overleftarrow{R}``\alpha = \iota'\Lambda$ (3)

$\vdash . (2) . (3) . \supset \vdash .$ Prop

***53·601.** $\vdash : \exists ! \alpha . \alpha \cap \text{Cl}'R = \Lambda . \supset . \overrightarrow{R}``\alpha = \iota'\Lambda$

Dem.

$\vdash . *33·41 . \supset \vdash : \text{Hp} . x \epsilon \alpha . \supset . \overrightarrow{R}'x = \Lambda$ (1)

$\vdash . (1) . *37·7 . \supset \vdash : \text{Hp} . \supset . \overrightarrow{R}``\alpha = \hat{\beta}\{(\exists x) . x \epsilon \alpha . \beta = \Lambda\}$

[*10·35] $= \hat{\beta}\{\exists ! \alpha . \beta = \Lambda\}$

[*4·73.*51·11] $= \iota'\Lambda : \supset \vdash .$ Prop

***53·602.** $\vdash : \exists ! \alpha . \alpha \cap \text{D}'R = \Lambda . \supset . \overleftarrow{R}``\alpha = \iota'\Lambda$ [Proof as in *53·601]

***53·603.** $\vdash : \exists ! - \text{Cl}'R . \supset . \overrightarrow{R}``(-\text{Cl}'R) = \iota'\Lambda$ [*24·21 . *53·601]

∗53·604. $\vdash : \exists ! - \text{D}'R \,.\, \supset . \overleftarrow{R}``(-\text{D}'R) = \iota`\Lambda$ [∗24·21 . ∗53·602]

∗53·61. $\vdash : \text{C}'R \subset \alpha \,.\, \text{C}'R \neq \alpha \,.\, \supset . \overrightarrow{R}``\alpha = \overrightarrow{R}``\text{C}'R \cup \iota`\Lambda$

Dem.

$\vdash . *22\cdot92 . \quad \supset \vdash : \text{Hp} . \supset . \alpha = \text{C}'R \cup (\alpha - \text{C}'R)$ (1)

$\vdash . *24\cdot6 . \quad \supset \vdash : \text{Hp} . \supset . \exists ! \alpha - \text{C}'R .$

[∗24·21.∗53·601] $\supset . \overrightarrow{R}``(\alpha - \text{C}'R) = \iota`\Lambda$ (2)

$\vdash . (1) . *37\cdot22 . \supset \vdash : \text{Hp} . \supset . \overrightarrow{R}``\alpha = \overrightarrow{R}``\text{C}'R \cup \overrightarrow{R}``(\alpha - \text{C}'R)$

[(2)] $= \overrightarrow{R}``\text{C}'R \cup \iota`\Lambda : \supset \vdash . \text{Prop}$

∗53·611. $\vdash : \text{D}'R \subset \alpha \,.\, \text{D}'R \neq \alpha \,.\, \supset . \overleftarrow{R}``\alpha = \overleftarrow{R}``\text{D}'R \cup \iota`\Lambda$ [Proof as in ∗53·61]

∗53·612. $\vdash : \text{C}'R \neq \text{V} \,.\, \supset . \overrightarrow{R}``\text{V} = \overrightarrow{R}``\text{C}'R \cup \iota`\Lambda$ [∗53·61 . ∗24·11]

∗53·613. $\vdash : \text{D}'R \neq \text{V} \,.\, \supset . \overleftarrow{R}``\text{V} = \overleftarrow{R}``\text{D}'R \cup \iota`\Lambda$ [∗53·611 . ∗24·11]

∗53·614. $\vdash . \overrightarrow{R}``\text{C}'R = \overrightarrow{R}``\text{V} - \iota`\Lambda$

Dem.

$\vdash . *53\cdot612 . *22\cdot68 . *24\cdot21 . \supset$

$\vdash : \text{C}'R \neq \text{V} . \supset . \overrightarrow{R}``\text{V} - \iota`\Lambda = \overrightarrow{R}``\text{C}'R - \iota`\Lambda$ (1)

$\vdash . *22\cdot481 . \supset \vdash : \text{C}'R = \text{V} . \supset . \overrightarrow{R}``\text{V} - \iota`\Lambda = \overrightarrow{R}``\text{C}'R - \iota`\Lambda$ (2)

$\vdash . *37\cdot772 . *51\cdot36 . *22\cdot621 . \supset \vdash . \overrightarrow{R}``\text{C}'R - \iota`\Lambda = \overrightarrow{R}``\text{C}'R$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash . \text{Prop}$

∗53·615. $\vdash . \overleftarrow{R}``\text{D}'R = \overleftarrow{R}``\text{V} - \iota`\Lambda$ [Proof as in ∗53·614]

The two following propositions are used in ∗70·12.

∗53·62. $\vdash : \overrightarrow{R}``\text{C}'R \subset \gamma . \equiv . \overrightarrow{R}``\text{V} \subset \gamma \cup \iota`\Lambda$

Dem.

$\vdash . *53\cdot614 . \supset \vdash : \overrightarrow{R}``\text{C}'R \subset \gamma . \equiv . \overrightarrow{R}``\text{V} - \iota`\Lambda \subset \gamma .$

[∗24·43] $\equiv . \overrightarrow{R}``\text{V} \subset \gamma \cup \iota`\Lambda : \supset \vdash . \text{Prop}$

∗53·621. $\vdash : \overleftarrow{R}``\text{D}'R \subset \gamma . \equiv . \overleftarrow{R}``\text{V} \subset \gamma \cup \iota`\Lambda$ [Proof as in ∗53·62]

∗53·63. $\vdash : \text{C}'R \neq \text{V} . \supset . \text{D}'\overrightarrow{R} = \overrightarrow{R}``\text{C}'R \cup \iota`\Lambda$ [∗37·78 . ∗53·612]

∗53·631. $\vdash : \text{D}'R \neq \text{V} . \supset . \text{D}'\overleftarrow{R} = \overleftarrow{R}``\text{D}'R \cup \iota`\Lambda$ [∗37·781 . ∗53·613]

∗53·64. $\vdash : \text{C}'R = \text{V} . \supset . \text{D}'\overrightarrow{R} = \overrightarrow{R}``\text{C}'R$ [∗37·78]

∗53·641. $\vdash : \text{D}'R = \text{V} . \supset . \text{D}'\overleftarrow{R} = \overleftarrow{R}``\text{D}'R$ [∗37·781]

$*54$. CARDINAL COUPLES

Summary of $*54$.

Couples are of two kinds, namely (1) $\iota`x \cup \iota`y$, in which there is no order as between x and y, and (2) $\iota`x \uparrow \iota`y$, in which there is an order. We may distinguish these two kinds of couples as cardinal and ordinal respectively, since (as will be shown hereafter) the class of all couples of the form $\iota`x \cup \iota`y$ (where $x \neq y$) is the cardinal number 2, while the class of all couples of the form $\iota`x \uparrow \iota`y$ (where $x \neq y$) is the ordinal number 2, to which, for the sake of distinction, we assign the symbol "2_r," where the suffix "r" stands for "relational," because the ordinal 2 is a class of relations. In the present and the following numbers, we shall define 2 and 2_r as the classes of cardinal and ordinal couples respectively, leaving it to a later stage to show that 2 and 2_r, so defined, are respectively a cardinal and an ordinal number. An ordinal couple will also be called an *ordered* couple or a *couple with sense*. Thus a couple with sense is a couple of which one comes first and the other second.

We introduce here the cardinal number 0, defined as $\iota`\Lambda$. That 0 so defined is a cardinal number, will be proved at a later stage; for the present, we postpone the proof that 0 so defined has the arithmetical properties of zero.

Cardinal couples are much less important, even in cardinal arithmetic, than ordinal couples, which will be considered in the two following numbers ($*55$ and $*56$). It is necessary, however, to prove some of the properties of cardinal couples, and this will be done in the present number. Some properties of cardinal couples which have been already proved are here repeated for convenience of reference. The definitions of 0 and 2 are:

$*54\cdot01$. $0 = \iota`\Lambda$ Df

$*54\cdot02$. $2 = \hat{\alpha}\{(\exists x, y) \,.\, x \neq y \,.\, \alpha = \iota`x \cup \iota`y\}$ Df

Most of the propositions of the present number, except those that merely embody the definitions ($*54\cdot1\cdot101\cdot102$), are used very seldom. The following are among the most important.

$*54\cdot26$. $\vdash : \iota`x \cup \iota`y \in 2 \,.\equiv.\, x \neq y$

$*54\cdot3$. $\vdash . 2 = \hat{\alpha}\{(\exists x) \,.\, x \in \alpha \,.\, \alpha - \iota`x \in 1\}$

$*54\cdot4$. $\vdash :. \beta \subset \iota`x \cup \iota`y \,.\equiv:\, \beta = \Lambda \,.\vee.\, \beta = \iota`x \,.\vee.\, \beta = \iota`y \,.\vee.\, \beta = \iota`x \cup \iota`y$

$*54\cdot53$. $\vdash : \alpha \in 2 \,.\, x, y \in \alpha \,.\, x \neq y \,.\supset.\, \alpha = \iota`x \cup \iota`y$

$*54\cdot56$. $\vdash : \alpha \sim\in 0 \cup 1 \cup 2 \,.\equiv.\, (\exists x, y, z) \,.\, x, y, z \in \alpha \,.\, x \neq y \,.\, x \neq z \,.\, y \neq z$

***54·01.** $0 = \iota`\Lambda$ Df

***54·02.** $2 = \hat{\alpha}\{(\exists x, y) . x \neq y . \alpha = \iota`x \cup \iota`y\}$ Df

***54·1.** $\vdash . 0 = \iota`\Lambda$ [(*54·01)]

***54·101.** $\vdash : \alpha \in 2 . \equiv . (\exists x, y) . x \neq y . \alpha = \iota`x \cup \iota`y$ [(*54·02)]

***54·102.** $\vdash : \alpha \in 0 . \equiv . \alpha = \Lambda$ [*54·1]

The two following propositions have already occurred in *51, but are here repeated, because they belong to the subject of the present number.

***54·21.** $\vdash : \iota`x \cup \iota`y = \iota`x \cup \iota`z . \equiv . y = z$ [*51·41]

***54·22.** $\vdash :. \iota`x \cup \iota`y = \iota`z \cup \iota`w . \equiv : x = z . y = w . \vee . x = w . y = z$ [*51·43]

***54·25.** $\vdash : \iota`x \cup \iota`y \in 1 . \equiv . x = y$

Dem.

$\vdash . *52·46·1 . *22·58 . \supset \vdash : \iota`x \cup \iota`y \in 1 . \supset . \iota`x \cup \iota`y = \iota`x . \iota`x \cup \iota`y = \iota`y .$

[*20·23] $\supset . \iota`x = \iota`y$ (1)

$\vdash . *22·56 . \supset \vdash : \iota`x = \iota`y . \supset . \iota`x \cup \iota`y = \iota`x .$

[*52·22] $\supset . \iota`x \cup \iota`y \in 1$ (2)

$\vdash . (1) . (2) . \supset \vdash : \iota`x \cup \iota`y \in 1 . \equiv . \iota`x = \iota`y .$

[*51·23] $\equiv . x = y : \supset \vdash .$ Prop

***54·26.** $\vdash : \iota`x \cup \iota`y \in 2 . \equiv . x \neq y$

Dem.

$\vdash . *54·101 . \supset \vdash :: \iota`x \cup \iota`y \in 2 .$

$\equiv :. (\exists z, w) . z \neq w . \iota`x \cup \iota`y = \iota`z \cup \iota`w :.$

[*54·22] $\equiv :. (\exists z, w) : z \neq w : x = z . y = w . \vee . x = w . y = z :.$

[*4·4.*11·41] $\equiv :. (\exists z, w) . z \neq w . x = z . y = w . \vee . (\exists z, w) . z \neq w . x = w . y = z :.$

[*13·22] $\equiv :. x \neq y . \vee . y \neq x :.$

[*13·16] $\equiv :. x \neq y :: \supset \vdash .$ Prop

***54·27.** $\vdash . \iota`x \cup \iota`y \in 1 \cup 2$ [*54·25·26]

***54·271.** $\vdash . 1 \cup 2 = \hat{\alpha}\{(\exists x, y) . \alpha = \iota`x \cup \iota`y\}$

Dem.

$\vdash . *4·42 . \supset$

$\vdash :. \alpha = \iota`x \cup \iota`y . \equiv : x = y . \alpha = \iota`x \cup \iota`y . \vee . x \neq y . \alpha = \iota`x \cup \iota`y$ (1)

$\vdash . (1) . *11·11·341·41 . \supset \vdash :. (\exists x, y) . \alpha = \iota`x \cup \iota`y .$

$\equiv : (\exists x, y) . x = y . \alpha = \iota`x \cup \iota`y . \vee . (\exists x, y) . x \neq y . \alpha = \iota`x \cup \iota`y :$

[*13·195] $\equiv : (\exists x) . \alpha = \iota`x \cup \iota`x . \vee . (\exists x, y) . x \neq y . \alpha = \iota`x \cup \iota`y :$

[*22·56] $\equiv : (\exists x) . \alpha = \iota`x . \vee . (\exists x, y) . x \neq y . \alpha = \iota`x \cup \iota`y :$

[*52·1.*54·101] $\equiv : \alpha \in 1 . \vee . \alpha \in 2 :$

[*22·34] $\equiv : \alpha \in 1 \cup 2 :. \supset \vdash .$ Prop

***54·3.** $\vdash . 2 = \hat{\alpha}\{(\exists x) . x \in \alpha . \alpha - \iota`x \in 1\}$

Dem.

$\vdash . *52·1 . *10·35 . \supset$

$\vdash : (\exists x) . x \in \alpha . \alpha - \iota`x \in 1 . \equiv . (\exists x, y) . x \in \alpha . \alpha - \iota`x = \iota`y .$

$\left[*51·22 \frac{\iota`y, \alpha}{\alpha, \beta}\right]$ $\equiv . (\exists x, y) . \iota`x \cap \iota`y = \Lambda . \iota`x \cup \iota`y = \alpha .$

[*51·231.*54·101] $\equiv . \alpha \in 2 : \supset \vdash . \text{Prop}$

***54·4.** $\vdash :. \beta \subset \iota`x \cup \iota`y . \equiv : \beta = \Lambda . \vee . \beta = \iota`x . \vee . \beta = \iota`y . \vee . \beta = \iota`x \cup \iota`y$

Dem.

$\vdash . *51·2 . \supset \vdash : x, y \in \beta . \supset . \iota`x \cup \iota`y \subset \beta :$

[Fact] $\supset \vdash : \beta \subset \iota`x \cup \iota`y . x, y \in \beta . \supset . \beta \subset \iota`x \cup \iota`y . \iota`x \cup \iota`y \subset \beta .$

[*22·41] $\supset . \beta = \iota`x \cup \iota`y$ (1)

$\vdash . *51·25 . \supset \vdash :. \beta \subset \iota`x \cup \iota`y . y \sim\in \beta . \supset : \beta \subset \iota`x :$

[*51·401] $\supset : \beta = \Lambda . \vee . \beta = \iota`x$ (2)

Similarly $\vdash :. \beta \subset \iota`x \cup \iota`y . x \sim\in \beta . \supset : \beta = \Lambda . \vee . \beta = \iota`y$ (3)

$\vdash . (2) . (3) . *3·48 . \supset$

$\vdash :. \beta \subset \iota`x \cup \iota`y . \sim(x, y \in \beta) . \supset : \beta = \Lambda . \vee . \beta = \iota`x . \vee . \beta = \iota`y$ (4)

$\vdash . (1) . (4) . *34·8 . \supset$

$\vdash :. \beta \subset \iota`x \cup \iota`y . \supset : \beta = \Lambda . \vee . \beta = \iota`x . \vee . \beta = \iota`y . \vee . \beta = \iota`x \cup \iota`y$ (5)

$\vdash . *24·12 . *22·58·42 . \supset$

$\vdash :. \beta = \Lambda . \vee . \beta = \iota`x . \vee . \beta = \iota`y . \vee . \beta = \iota`x \cup \iota`y : \supset . \beta \subset \iota`x \cup \iota`y$ (6)

$\vdash . (5) . (6) . \supset \vdash . \text{Prop}$

This proposition shows that a class contained in a couple is either the null-class or a unit class or the couple itself, whence it will follow that 0 and 1 are the only numbers which are less than 2.

***54·41.** $\vdash :: \alpha \in 2 . \supset :. \beta \subset \alpha . \supset : \beta = \Lambda . \vee . \beta \in 1 . \vee . \beta \in 2$

Dem.

$\vdash . *52·1 . \supset \vdash :. \beta = \iota`x . \vee . \beta = \iota`y : \supset . \beta \in 1$ (1)

$\vdash . *54·26 . \supset \vdash :. x \neq y . \supset : \beta = \iota`x \cup \iota`y . \supset . \beta \in 2$ (2)

$\vdash . (1) . (2) . *54·4 . \supset$

$\vdash :: x \neq y . \supset :. \beta \subset \iota`x \cup \iota`y . \supset : \beta = \Lambda . \vee . \beta \in 1 . \vee . \beta \in 2 ::$

[*13·12] $\supset \vdash :: \alpha = \iota`x \cup \iota`y . x \neq y . \supset :. \beta \subset \alpha . \supset : \beta = \Lambda . \vee . \beta \in 1 . \vee . \beta \in 2 ::$

[11·11·35] $\supset$

$\vdash :. (\exists x, y) . \alpha = \iota`x \cup \iota`y . x \neq y . \supset :. \beta \subset \alpha : \beta = \Lambda . \vee . \beta \in 1 . \vee . \beta \in 2$ (3)

$\vdash . (3) . *54·101 . \supset \vdash . \text{Prop}$

***54·411.** $\vdash :. \alpha \in 2 . \supset : \beta \subset \alpha . \supset . \beta \in 0 \cup 1 \cup 2$ [*54·41·102]

***54·42.** $\vdash :: \alpha \epsilon 2 . \supset :. \beta \subset \alpha . \exists ! \beta . \beta \neq \alpha . \equiv . \beta \epsilon \iota``\alpha$

Dem.

$\vdash . *54·4 . \quad \supset \vdash :: \alpha = \iota`x \cup \iota`y . \supset :.$

$$\beta \subset \alpha . \exists ! \beta . \equiv : \beta = \Lambda . \vee . \beta = \iota`x . \vee . \beta = \iota`y . \vee . \beta = \alpha : \exists ! \beta :$$

[*24·53·56.*51·161] $\quad \equiv : \beta = \iota`x . \vee . \beta = \iota`y . \vee . \beta = \alpha \quad (1)$

$\vdash . *54·25 . \text{Transp} . *52·22 . \supset \vdash : x \neq y . \supset . \iota`x \cup \iota`y \neq \iota`x . \iota`x \cup \iota`y \neq \iota`y :$

[*13·12] $\quad \supset \vdash : \alpha = \iota`x \cup \iota`y . x \neq y . \supset . \alpha \neq \iota`x . \alpha \neq \iota`y \quad (2)$

$\vdash . (1) . (2) . \supset \vdash :: \alpha = \iota`x \cup \iota`y . x \neq y . \supset :.$

$$\beta \subset \alpha . \exists ! \beta . \beta \neq \alpha . \equiv : \beta = \iota`x . \vee . \beta = \iota`y :$$

[*51·235] $\quad \equiv : (\exists z) . z \epsilon \alpha . \beta = \iota`z :$

[*37·6] $\quad \equiv : \beta \epsilon \iota``\alpha \quad (3)$

$\vdash . (3) . *11·11·35 . *54·101 . \supset \vdash . \text{Prop}$

***54·43.** $\vdash :. \alpha, \beta \epsilon 1 . \supset : \alpha \cap \beta = \Lambda . \equiv . \alpha \cup \beta \epsilon 2$

Dem.

$\vdash . *54·26 . \supset \vdash :. \alpha = \iota`x . \beta = \iota`y . \supset : \alpha \cup \beta \epsilon 2 . \equiv . x \neq y .$

[*51·231] $\quad \equiv . \iota`x \cap \iota`y = \Lambda .$

[*13·12] $\quad \equiv . \alpha \cap \beta = \Lambda \quad (1)$

$\vdash . (1) . *11·11·35 . \supset$

$\quad \vdash :. (\exists x, y) . \alpha = \iota`x . \beta = \iota`y . \supset : \alpha \cup \beta \epsilon 2 . \equiv . \alpha \cap \beta = \Lambda \quad (2)$

$\vdash . (2) . *11·54 . *52·1 . \supset \vdash . \text{Prop}$

From this proposition it will follow, when arithmetical addition has been defined, that $1 + 1 = 2$.

***54·44.** $\vdash :. z, w \epsilon \iota`x \cup \iota`y . \supset_{z,w} . \phi(z, w) : \equiv . \phi(x, x) . \phi(x, y) . \phi(y, x) . \phi(y, y)$

Dem.

$\vdash . *51·234 . *11·62 . \supset \vdash :. z, w \epsilon \iota`x \cup \iota`y . \supset_{z,w} . \phi(z, w) : \equiv :$

$$z \epsilon \iota`x \cup \iota`y . \supset_z . \phi(z, x) . \phi(z, y) :$$

[*51·234.*10·29] $\equiv : \phi(x, x) . \phi(x, y) . \phi(y, x) . \phi(y, y) :. \supset \vdash . \text{Prop}$

***54·441.** $\vdash :: z, w \epsilon \iota`x \cup \iota`y . z \neq w . \supset_{z,w} . \phi(z, w) : \equiv :. x = y : \vee : \phi(x, y) . \phi(y, x)$

Dem.

$\vdash . *5·6 . \supset \vdash :: z, w \epsilon \iota`x \cup \iota`y . z \neq w . \supset_{z,w} . \phi(z, w) : \equiv :.$

$$z, w \epsilon \iota`x \cup \iota`y . \supset_{z,w} : z = w . \vee . \phi(z, w) :.$$

[*54·44] $\quad \equiv : x = x . \vee . \phi(x, x) : x = y . \vee . \phi(x, y) :$

$$y = x . \vee . \phi(y, x) : y = y . \vee . \phi(y, y) :$$

[*13·15] $\quad \equiv : x = y . \vee . \phi(x, y) : y = x . \vee . \phi(y, x) :$

[*13·16.*4·41] $\equiv : x = y . \vee . \phi(x, y) . \phi(y, x)$

This proposition is used in *163·42, in the theory of relations of mutually exclusive relations.

***54·442.** $\vdash :: x \neq y . \supset :. z, w \epsilon \iota`x \cup \iota`y . z \neq w . \supset_{z,w} . \phi(z, w) : \equiv . \phi(x, y) . \phi(y, x)$

[*54·441]

$*54\cdot443$. $\vdash :: x \neq y : \phi(x,y) . \equiv . \phi(y,x) : \supset :.$
$z, w \in \iota'x \cup \iota'y . z \neq w . \supset_{z,w} . \phi(z,w) : \equiv . \phi(x,y)$ [$*54\cdot442$]

$*54\cdot45$. $\vdash :. (\exists z, w) . z, w \in \iota'x \cup \iota'y . \phi(z,w) .$
$\equiv : \phi(x,x) . \vee . \phi(x,y) . \vee . \phi(y,x) . \vee . \phi(y,y)$ [$*51\cdot235$]

$*54\cdot451$. $\vdash :: \sim\phi(x,x) . \sim\phi(y,y) . \supset :. (\exists z, w) . z, w \in \iota'x \cup \iota'y . \phi(z,w) .$
$\equiv : \phi(x,y) . \vee . \phi(y,x)$ [$*54\cdot45$]

$*54\cdot452$. $\vdash :: \sim\phi(x,x) . \sim\phi(y,y) : \phi(x,y) . \equiv . \phi(y,x) : \supset :$
$(\exists z, w) . z, w \in \iota'x \cup \iota'y . \phi(z,w) . \equiv . \phi(x,y)$ [$*54\cdot451$]

$*54\cdot46$. $\vdash : (\exists z, w) . z, w \in \iota'x \cup \iota'y . z \neq w . \equiv . x \neq y$ [$*54\cdot452 . *13\cdot15\cdot16$]

$*54\cdot5$. $\vdash :. \alpha \in 2 . \supset : \alpha \subset \iota'z \cup \iota'w . \equiv . \alpha = \iota'z \cup \iota'w$

Dem.

$\vdash . *54\cdot4 . \supset$

$\vdash :. \alpha \subset \iota'z \cup \iota'w . \supset : \alpha = \Lambda . \vee . \alpha = \iota'z . \vee . \alpha = \iota'w . \vee . \alpha = \iota'z \cup \iota'w$ (1)

$\vdash . *54\cdot3 . *24\cdot54 . \quad \supset \vdash : \mathrm{Hp} . \supset . \alpha \neq \Lambda$ (2)

$\vdash . *54\cdot26 \dfrac{z, z}{x, y} . *13\cdot15 . \quad \supset \vdash : \mathrm{Hp} . \supset . \alpha \neq \iota'z$ (3)

$\vdash . (3) \dfrac{w}{z} . \quad \supset \vdash : \mathrm{Hp} . \supset . \alpha \neq \iota'w$ (4)

$\vdash . (1) . (2) . (3) . (4) . *2\cdot53 . \supset \vdash :. \mathrm{Hp} . \supset : \alpha \subset \iota'z \cup \iota'w . \supset . \alpha = \iota'z \cup \iota'w$ (5)

$\vdash . *22\cdot42 . \quad \supset \vdash : \alpha = \iota'z \cup \iota'w . \supset . \alpha \subset \iota'z \cup \iota'w$ (6)

$\vdash . (5) . (6) . \supset \vdash . \mathrm{Prop}$

$*54\cdot51$. $\vdash :. \alpha \in 2 . \beta \in 1 \cup 2 . \supset : \alpha \subset \beta . \equiv . \alpha = \beta$

Dem.

$\vdash . *54\cdot5 . \supset \vdash :. \alpha \in 2 . \beta = \iota'z \cup \iota'w . \supset : \alpha \subset \beta . \equiv . \alpha = \beta$ (1)

$\vdash . (1) . *11\cdot11\cdot35\cdot45 . \supset$

$\vdash :. \alpha \in 2 : (\exists z, w) . \beta = \iota'z \cup \iota'w : \supset : \alpha \subset \beta . \equiv . \alpha = \beta$ (2)

$\vdash . (2) . *54\cdot271 . \supset \vdash . \mathrm{Prop}$

$*54\cdot52$. $\vdash :. \alpha, \beta \in 2 . \supset : \alpha \subset \beta . \equiv . \alpha = \beta . \equiv . \beta \subset \alpha$ [$*54\cdot51$]

$*54\cdot53$. $\vdash : \alpha \in 2 . x, y \in \alpha . x \neq y . \supset . \alpha = \iota'x \cup \iota'y$

Dem.

$\vdash . *51\cdot2 . \quad \supset \vdash : \mathrm{Hp} . \supset . \iota'x \subset \alpha . \iota'y \subset \alpha .$

[$*22\cdot59$] $\quad \supset . \iota'x \cup \iota'y \subset \alpha$ (1)

$\vdash . *54\cdot26 . \quad \supset \vdash : \mathrm{Hp} . \supset . \iota'x \cup \iota'y \in 2$ (2)

$\vdash . (1) . (2) . *54\cdot52 . \supset \vdash . \mathrm{Prop}$

$*54\cdot531$. $\vdash :. \alpha \in 2 . \supset : x, y \in \alpha . x \neq y . \equiv . \alpha = \iota'x \cup \iota'y$

Dem.

$\vdash . *54\cdot53 . \mathrm{Exp} . \supset \vdash :. \alpha \in 2 . \supset : x, y \in \alpha . x \neq y . \supset . \alpha = \iota'x \cup \iota'y$ (1)

$\vdash . *54\cdot26 . \quad \supset \vdash :. \alpha \in 2 . \supset : \alpha = \iota'x \cup \iota'y . \supset . x \neq y$ (2)

$\vdash . *51\cdot16 . \quad \supset \vdash : \alpha = \iota'x \cup \iota'y . \supset . x, y \in \alpha$ (3)

$\vdash . (2) . (3) . \quad \supset \vdash :. \alpha \in 2 . \supset : \alpha = \iota'x \cup \iota'y . \supset . x, y \in \alpha . x \neq y$ (4)

$\vdash . (1) . (4) . \quad \supset \vdash . \mathrm{Prop}$

***54·54.** $\vdash :. \alpha \in 2 . \equiv : x, y \in \alpha . x \neq y . \supset_{x,y} . \alpha = \iota`x \cup \iota`y : (\exists x, y) . x, y \in \alpha . x \neq y$

Dem.

$\vdash . *54 \cdot 531 . *11 \cdot 11 \cdot 3 . \supset \vdash :. \alpha \in 2 . \supset : x, y \in \alpha . x \neq y . \supset_{x,y} . \alpha = \iota`x \cup \iota`y \quad (1)$

$\vdash . *51 \cdot 16 . *54 \cdot 101 . \quad \supset \vdash : \alpha \in 2 . \supset . (\exists x, y) . x, y \in \alpha . x \neq y \quad (2)$

$\vdash . *5 \cdot 3 . *3 \cdot 27 . \quad \supset \vdash :. x, y \in \alpha . x \neq y . \supset . \alpha = \iota`x \cup \iota`y : \supset :$
$x, y \in \alpha . x \neq y . \supset . x \neq y . \alpha = \iota`x \cup \iota`y :.$

$[*11 \cdot 11 \cdot 32 \cdot 34] \quad \supset \vdash :. x, y \in \alpha . x \neq y . \supset_{x,y} . \alpha = \iota`x \cup \iota`y : \supset :$
$(\exists x, y) . x, y \in \alpha . x \neq y . \supset . (\exists x, y) . x \neq y . \alpha = \iota`x \cup \iota`y \quad (3)$

$\vdash . (3) . \mathrm{Imp} . *54 \cdot 101 . \supset \vdash :. x, y \in \alpha . x \neq y . \supset_{x,y} . \alpha = \iota`x \cup \iota`y :$
$(\exists x, y) . x, y \in \alpha . x \neq y : \supset . \alpha \in 2 \quad (4)$

$\vdash . (1) . (2) . (4) . \quad \supset \vdash . \mathrm{Prop}$

In the above proposition, "$x, y \in \alpha . x \neq y . \supset_{x,y} . \alpha = \iota`x \cup \iota`y$" secures that α has not *more* than two members, while "$(\exists x, y) . x, y \in \alpha . x \neq y$" secures that α has not *fewer* than two members.

***54·55.** $\vdash . 0 \cup 1 \cup 2 = \hat{\alpha} \{x, y \in \alpha . x \neq y . \supset_{x,y} . \alpha = \iota`x \cup \iota`y\}$

Dem.

$\vdash . *4 \cdot 42 . \supset \vdash :: x, y \in \alpha . x \neq y . \supset_{x,y} . \alpha = \iota`x \cup \iota`y : \equiv :.$
$x, y \in \alpha . x \neq y . \supset_{x,y} . \alpha = \iota`x \cup \iota`y : \sim (\exists x, y) . x, y \in \alpha . x \neq y :.$
$\vee :. x, y \in \alpha . x \neq y . \supset_{x,y} . \alpha = \iota`x \cup \iota`y : (\exists x, y) . x, y \in \alpha . x \neq y \quad (1)$

$\vdash . *11 \cdot 63 . \supset \vdash :. \sim (\exists x, y) . x, y \in \alpha . x \neq y . \supset : x, y \in \alpha . x \neq y . \supset_{x,y} . \alpha = \iota`x \cup \iota`y :.$

$[*4 \cdot 71] \quad \supset \vdash :. x, y \in \alpha . x \neq y . \supset_{x,y} . \alpha = \iota`x \cup \iota`y : \sim (\exists x, y) . x, y \in \alpha . x \neq y : \equiv :$
$\sim (\exists x, y) . x, y \in \alpha . x \neq y :$

$[*11 \cdot 521] \quad \equiv : x, y \in \alpha . \supset_{x,y} . x = y :$

$[*52 \cdot 4] \quad \equiv : \alpha \in 0 \cup 1 \quad (2)$

$\vdash . (1) . (2) . *54 \cdot 54 . \supset$

$\vdash :. x, y \in \alpha . x \neq y . \supset_{x,y} . \alpha = \iota`x \cup \iota`y : \equiv : \alpha \in 0 \cup 1 . \vee . \alpha \in 2 :$

$[*22 \cdot 34] \quad \equiv : \alpha \in 0 \cup 1 \cup 2 :. \supset \vdash . \mathrm{Prop}$

***54·56.** $\vdash : \alpha \sim \in 0 \cup 1 \cup 2 . \equiv . (\exists x, y, z) . x, y, z \in \alpha . x \neq y . x \neq z . y \neq z$

Dem.

$\vdash . *54 \cdot 55 . *11 \cdot 52 . \supset$

$\vdash :. \alpha \sim \in 0 \cup 1 \cup 2 . \equiv : (\exists x, y) . x, y \in \alpha . x \neq y . \alpha \neq \iota`x \cup \iota`y :$

$[*51 \cdot 2 . *22 \cdot 59] \quad \equiv : (\exists x, y) . \iota`x \cup \iota`y \subset \alpha . x \neq y . \alpha \neq \iota`x \cup \iota`y :$

$[*24 \cdot 6] \quad \equiv : (\exists x, y) . \iota`x \cup \iota`y \subset \alpha . x \neq y . \exists ! \alpha - (\iota`x \cup \iota`y) :$

$[*51 \cdot 232 . \mathrm{Transp}] \quad \equiv : (\exists x, y) : \iota`x \cup \iota`y \subset \alpha . x \neq y : (\exists z) . z \in \alpha . z \neq x . z \neq y :$

$[*51 \cdot 2 . *22 \cdot 59] \quad \equiv : (\exists x, y, z) . x, y, z \in \alpha . x \neq y . x \neq z . y \neq z :. \supset \vdash . \mathrm{Prop}$

In virtue of this proposition, a class which is neither null nor a unit class nor a couple contains at least three distinct members. Hence it will follow that any cardinal number other than 0 or 1 or 2 is equal to or greater than 3. The above proposition is used in *104·43, which is an existence-theorem of considerable importance in cardinal arithmetic.

$*54\cdot6$. $\vdash :. \alpha \cap \beta = \Lambda . x, x' \epsilon \alpha . y, y' \epsilon \beta . \supset :$

$$\iota`x \cup \iota`y = \iota`x' \cup \iota`y' . \equiv . x = x' . y = y'$$

Dem.

$\vdash . *51\cdot2 . \supset \vdash :. \mathrm{Hp} . \supset : \iota`x \subset \alpha . \iota`x' \subset \alpha . \iota`y \subset \beta . \iota`y' \subset \beta . \alpha \cap \beta = \Lambda :$

$[*24\cdot48]$ $\supset : \iota`x \cup \iota`y = \iota`x' \cup \iota`y' . \equiv . \iota`x = \iota`x' . \iota`y = \iota`y' .$

$[*51\cdot23]$ $\equiv . x = x' . y = y' :. \supset \vdash . \mathrm{Prop}$

The above proposition is useful in dealing with sets of couples formed of one member of a class α and one member of a class β, where α and β have no members in common. It is used in the theory of cardinal multiplication ($*113\cdot148$).

$*55$. ORDINAL COUPLES

Summary of $*55$.

Ordinal couples, which are now to be considered, are much more important, even in cardinal arithmetic, than cardinal couples. Their properties are in part analogous to those of cardinal couples, but in part also to those of unit classes; for they are the smallest existent relations, just as unit classes are the smallest existent classes. The properties which are analogous to those of unit classes do not demand that the two terms of the couple should be distinct, *i.e.* they hold for $\iota\text{‘}x \uparrow \iota\text{‘}x$ as well as for $\iota\text{‘}x \uparrow \iota\text{‘}y$ (where $x \neq y$); on the other hand, the properties which are analogous to those of cardinal couples do in general demand that the two terms of the ordinal couple should be distinct.

The notation $\iota\text{‘}x \uparrow \iota\text{‘}y$ is cumbrous, and does not readily enable us to exhibit the couple as a descriptive function of x for the argument y, or vice versa. We therefore introduce a new symbol, "$x \downarrow y$," for the couple. In a couple $x \downarrow y$, we shall call x the referent of the couple, and y the relatum. In virtue of the definitions in $*38$, this gives rise to two relations $x \downarrow$ and $\downarrow y$; hence we obtain the notations $x \downarrow \text{‘‘}\beta$, $\downarrow y\text{‘‘}\alpha$, $\alpha \underset{\text{,,}}{\downarrow} y$, $\alpha \underset{\text{,,}}{\downarrow} \text{‘‘}\beta$ and so on, which will be much used in the sequel. It should be observed that $x \downarrow \text{‘‘}\beta$ means the class of ordinal couples in which x is referent and a member of β is relatum, while $\downarrow y\text{‘‘}\alpha$ or $\alpha \underset{\text{,,}}{\downarrow} y$ denotes the class of couples having y as relatum and a member of α as referent; $\alpha \underset{\text{,,}}{\downarrow} \text{‘‘}\beta$ denotes all such classes of couples as $\downarrow y\text{‘‘}\alpha$, where y is any member of β; and in virtue of $*40{\cdot}7$, $s\text{‘}\alpha \underset{\text{,,}}{\downarrow} \text{‘‘}\beta$ denotes all ordinal couples of which the referent is a member of α, while the relatum is a member of β. This is a very important class, which will be used to define the product of two cardinal numbers; for it is evident that the number of members of $s\text{‘}\alpha \underset{\text{,,}}{\downarrow} \text{‘‘}\beta$ is the product of the number of members of α and the number of members of β.

The first few propositions of the present number are immediate consequences of the definition of $x \downarrow y$ and the notations introduced in $*38$. We then proceed to various elementary properties of the relation $x \downarrow y$, of which the most used are the following:

$*55{\cdot}13$. $\vdash : z\,(x \downarrow y)\, w \,.\equiv.\, z = x \,.\, w = y$

$*55{\cdot}15$. $\vdash .\, \mathrm{D}\text{‘}(x \downarrow y) = \iota\text{‘}x \,.\, \text{Ꮖ}\text{‘}(x \downarrow y) = \iota\text{‘}y \,.\, C\text{‘}(x \downarrow y) = \iota\text{‘}x \cup \iota\text{‘}y$

$*55{\cdot}16$. $\vdash : \mathrm{D}\text{‘}R = \iota\text{‘}x \,.\, \text{Ꮖ}\text{‘}R = \iota\text{‘}y \,.\equiv.\, R = x \downarrow y$

$*55{\cdot}202$. $\vdash : x \downarrow y = z \downarrow w \,.\equiv.\, x = z \,.\, y = w \,.\equiv.\, y \downarrow x = w \downarrow z$

This proposition should be contrasted with $*54\cdot22$, as giving one reason why ordinal couples are more useful in arithmetic than cardinal couples. In virtue of the above proposition, when two ordinal couples are identical, their referents are identical, and their relata are identical.

We proceed next to various properties of the relations $x \downarrow$ and $\downarrow x$. These relations play a great part in arithmetic. It will be observed that if two terms have the relation $x \downarrow$, the referent is a couple whose relatum is the relatum in the relation $x \downarrow$, *i.e.* when we have $R\,(x \downarrow)\,y$, we have $R = x \downarrow y$ (cf. $*55\cdot122$). Similar remarks apply to the relation $\downarrow x$. The class $\downarrow x``\alpha$, consisting of all couples whose referent is a member of α, while the relatum is x, is important. We have

$*55\cdot232$. $\vdash : \exists ! \downarrow x``\alpha \cap \downarrow y``\beta \,.\equiv.\, x = y \,.\, \exists ! \alpha \cap \beta$

This proposition is frequently useful.

We proceed next ($*55\cdot3$—$\cdot51$) to give various properties of $x \downarrow y$ which are analogous to the properties of unit classes. Among the more important of these properties are the following:

$*55\cdot3$. $\vdash : xRy \,.\equiv.\, x \downarrow y \subset R \,.\equiv.\, \dot{\exists} ! (x \downarrow y) \mathbin{\dot{\cap}} R$

This is the analogue of $*51\cdot31$.

$*55\cdot34$. $\vdash : \dot{\exists} ! R \,.\, R \subset x \downarrow y \,.\equiv.\, R = x \downarrow y$

This is the analogue of $*51\cdot4$.

$*55\cdot5$. $\vdash :. R \subset x \downarrow y \mathbin{\dot{\cup}} z \downarrow w \,.\equiv:$
$R = \dot{\Lambda} \,.\vee.\, R = x \downarrow y \,.\vee.\, R = z \downarrow w \,.\vee.\, R = x \downarrow y \mathbin{\dot{\cup}} z \downarrow w$

This is the analogue of $*54\cdot4$.

We then proceed to such properties of ordinal couples as are not analogous to those of unit classes. For connecting the cardinal number 2 with the ordinal number 2_r, we have the proposition

$*55\cdot54$. $\vdash :: x \neq y \,.\supset:.\, C`R = \iota`x \cup \iota`y \,.\, R \mathbin{\dot{\cap}} \breve{R} = \dot{\Lambda} \,.\equiv:\, R = x \downarrow y \,.\vee.\, R = y \downarrow x$

This proposition shows that the only asymmetrical relations which have a given cardinal couple $\iota`x \cup \iota`y$ for their field are the two corresponding ordinal couples $x \downarrow y$ and $y \downarrow x$. We have next a set of propositions on the relative products of couples and other relations, *i.e.* on $R \,|\, (x \downarrow y)$, $(x \downarrow y) \,|\, S$, and $R \,|\, (x \downarrow y) \,|\, S$. These propositions are very useful in arithmetic. The chief of them is

$*55\cdot61$. $\vdash : E ! R`z \,.\, E ! S`w \,.\supset.\, (R \parallel \breve{S})`(z \downarrow w) = (R`z) \downarrow (S`w)$

Finally we have four propositions which belong, by their subject, to $*43$, but could not be given there, because the proofs make use of ordinal couples.

***55·01.** $x \downarrow y = \iota\text{‘}x \uparrow \iota\text{‘}y$ Df

***55·02.** $R\text{‘}x \downarrow y = R\text{‘}(x \downarrow y)$ Df

This definition serves merely for the avoidance of brackets.

***55·1.** $\vdash . x \downarrow y = (\iota\text{‘}x) \uparrow (\iota\text{‘}y)$ [(*55·01)]

***55·11.** $\vdash . x \downarrow \text{‘}y = \downarrow y\text{‘}x = x \downarrow y = \iota\text{‘}x \uparrow \iota\text{‘}y$ [*38·11 . *55·1]

***55·12.** $\vdash . \mathrm{E}\,!\, x \downarrow \text{‘}y$ [*55·11 . *14·21]

***55·121.** $\vdash . \mathrm{E}\,!\downarrow y\text{‘}x$

***55·122.** $\vdash : R(x \downarrow) y . \equiv . R = x \downarrow y$ [*55·11]

***55·123.** $\vdash : R(\downarrow y) x . \equiv . R = x \downarrow y$ [*55·11]

***55·13.** $\vdash : z(x \downarrow y) w . \equiv . z = x . w = y$

Dem.

$\vdash . *35\cdot103 . *55\cdot1 . \supset \vdash : z(x \downarrow y) w . \equiv . z \,\epsilon\, \iota\text{‘}x . w \,\epsilon\, \iota\text{‘}y .$

[*51·15] $\equiv . z = x . w = y : \supset \vdash . \mathrm{Prop}$

***55·132.** $\vdash . x(x \downarrow y) y$ [*55·13]

***55·134.** $\vdash . \dot{\exists}\,!\,(x \downarrow y)$ [*55·132]

***55·14.** $\vdash . x \downarrow y = \mathrm{Cnv}\text{‘}y \downarrow x$ [*55·13 . *31·131]

***55·15.** $\vdash . \mathrm{D}\text{‘}x \downarrow y = \iota\text{‘}x . \mathrm{Q}\text{‘}x \downarrow y = \iota\text{‘}y . C\text{‘}x \downarrow y = \iota\text{‘}x \cup \iota\text{‘}y$
[*35·85·86 . *51·161]

***55·16.** $\vdash : \mathrm{D}\text{‘}R = \iota\text{‘}x . \mathrm{Q}\text{‘}R = \iota\text{‘}y . \equiv . R = x \downarrow y$

Dem.

$\vdash . *33\cdot13\cdot131 . *51\cdot15 . \supset$

$\vdash :: \mathrm{D}\text{‘}R = \iota\text{‘}x . \mathrm{Q}\text{‘}R = \iota\text{‘}y . \equiv :. (\exists w) . zRw . \equiv_z . z = x : (\exists z) . zRw . \equiv_w . w = y :.$

[*14·122] $\equiv :. (\exists z, w) . zRw : (\exists w) . zRw . \supset_z . z = x :$
$(\exists w, z) . zRw : (\exists z) . zRw . \supset_w . w = y :.$

[*11·23.*4·71] $\equiv :. (\exists z, w) . zRw : (\exists w) . zRw . \supset_z . z = x : (\exists z) . zRw . \supset_w . w = y :.$

[*10·23] $\equiv :. (\exists z, w) . zRw : zRw . \supset_{z,w} . z = x : zRw . \supset_{z,w} . w = y :.$

[*11·391] $\equiv :. (\exists z, w) . zRw : zRw . \supset_{z,w} . z = x . w = y :.$

[*14·123] $\equiv :. zRw . \equiv_{z,w} . z = x . w = y :.$

[*55·13] $\equiv :. zRw . \equiv_{z,w} . z(x \downarrow y) w :.$

[*21·43] $\equiv :. R = x \downarrow y :: \supset \vdash . \mathrm{Prop}$

The above proposition is important, and will be frequently used.

***55·161.** $\vdash . x \downarrow y = \breve{\iota}\text{‘}\hat{R}(\mathrm{D}\text{‘}R = \iota\text{‘}x . \mathrm{Q}\text{‘}R = \iota\text{‘}y)$

Dem.

$\vdash . *55\cdot16 . *20\cdot15 . \supset$

$\vdash . \hat{R}(\mathrm{D}\text{‘}R = \iota\text{‘}x . \mathrm{Q}\text{‘}R = \iota\text{‘}y) = \hat{R}(R = x \downarrow y)$

[*51·11] $= \iota\text{‘}(x \downarrow y)$ (1)

$\vdash . (1) . *51\cdot51 . \supset \vdash . \mathrm{Prop}$

***55·17.** $\vdash . x \downarrow y = \breve{\iota}\text{‘}(\overleftarrow{\mathrm{D}}\text{‘}\iota\text{‘}x \cap \overleftarrow{\text{ᗡ}}\text{‘}\iota\text{‘}y)$ [*55·161 . *33·6·61]

***55·2.** $\vdash : x \downarrow y = x \downarrow z . \equiv . y = z$

Dem.

$\vdash . *30·37 . *55·11·12 . \supset \vdash : y = z . \supset . x \downarrow y = x \downarrow z$ (1)

$\vdash . *30·37 . *33·121 . \supset$

$\vdash : x \downarrow y = x \downarrow z . \supset . \text{ᗡ}\text{‘}x \downarrow y = \text{ᗡ}\text{‘}x \downarrow z .$

[*55·15] $\supset . \iota\text{‘}y = \iota\text{‘}z .$

[*51·23] $\supset . y = z$ (2)

$\vdash . (1) . (2) . \supset \vdash .$ Prop

***55·201.** $\vdash : x \downarrow z = y \downarrow z . \equiv . x = y$

***55·202.** $\vdash : x \downarrow y = z \downarrow w . \equiv . x = z . y = w . \equiv . y \downarrow x = w \downarrow z$

Dem.

$\vdash . *55·2·201 . \supset$

$\vdash : x = z . y = w . \supset . x \downarrow y = z \downarrow y . z \downarrow y = z \downarrow w .$

[*13·17] $\supset . x \downarrow y = z \downarrow w$ (1)

$\vdash . *30·37 . *33·12·121 . \supset$

$\vdash : x \downarrow y = z \downarrow w . \supset . \mathrm{D}\text{‘}x \downarrow y = \mathrm{D}\text{‘}z \downarrow w . \text{ᗡ}\text{‘}x \downarrow y = \text{ᗡ}\text{‘}z \downarrow w .$

[*55·15] $\supset . \iota\text{‘}x = \iota\text{‘}z . \iota\text{‘}y = \iota\text{‘}w .$

[*51·23] $\supset . x = z . y = w$ (2)

$\vdash . (1) . (2) . \supset$

$\vdash : x \downarrow y = z \downarrow w . \equiv . x = z . y = w$ (3)

Similarly

$\vdash : y \downarrow x = w \downarrow z . \equiv . x = z . y = w$ (4)

$\vdash . (3) . (4) . \supset \vdash .$ Prop

The above proposition is important.

***55·21.** $\vdash . \text{ᗡ}\text{‘}x \downarrow = \mathrm{V} . \text{ᗡ}\text{‘} \downarrow x = \mathrm{V}$ [*33·432 . *55·12·121]

***55·22.** $\vdash . \mathrm{D}\text{‘}x \downarrow = \hat{R}\, \{(\exists y) \; R = x \downarrow y\}$ [*55·122]

***55·221.** $\vdash . \mathrm{D}\text{‘} \downarrow x = \hat{R}\, \{(\exists y) . R = y \downarrow x\}$ [*55·123]

***55·222.** $\vdash : R \in \mathrm{D}\text{‘}x \downarrow . \equiv . \mathrm{D}\text{‘}R = \iota\text{‘}x . \text{ᗡ}\text{‘}R \in 1$

Dem.

$\vdash . *55·22·16 . \supset \vdash :. R \in \mathrm{D}\text{‘}x \downarrow . \equiv : (\exists y) . \mathrm{D}\text{‘}R = \iota\text{‘}x . \text{ᗡ}\text{‘}R = \iota\text{‘}y :$

[*10·35] $\equiv : \mathrm{D}\text{‘}R = \iota\text{‘}x : (\exists y) . \text{ᗡ}\text{‘}R = \iota\text{‘}y :$

[*52·1] $\equiv : \mathrm{D}\text{‘}R = \iota\text{‘}x . \text{ᗡ}\text{‘}R \in 1 :. \supset \vdash .$ Prop

***55·223.** $\vdash : R \in \mathrm{D}\text{‘} \downarrow x . \equiv . \text{ᗡ}\text{‘}R = \iota\text{‘}x . \mathrm{D}\text{‘}R \in 1$ [Proof as in *55·222]

***55·224.** $\vdash . \mathrm{D}\text{‘}x \downarrow \cap \mathrm{D}\text{‘} \downarrow y = \iota\text{‘}(x \downarrow y)$

Dem.

$\vdash . *55·222·223 . \supset$

$\vdash : R \in \mathrm{D}\text{‘}x \downarrow \cap \mathrm{D}\text{‘} \downarrow y . \equiv . \mathrm{D}\text{‘}R = \iota\text{‘}x . \text{ᗡ}\text{‘}R \in 1 . \text{ᗡ}\text{‘}R = \iota\text{‘}y . \mathrm{D}\text{‘}R \in 1 .$

$[*52{\cdot}22.*4{\cdot}71] \quad \equiv . \mathrm{D}'R = \iota'x . \mathrm{d}'R = \iota'y .$
$[*55{\cdot}16] \quad \equiv . R = x \downarrow y .$
$[*51{\cdot}15] \quad \equiv . R \in \iota'(x \downarrow y) : \supset \vdash . \mathrm{Prop}$

***55·23.** $\vdash . x \downarrow ``\alpha = \hat{R}\{(\exists y) . y \in \alpha . R = x \downarrow y\}$ [*38·13]

***55·231.** $\vdash . \downarrow x``\alpha = \hat{R}\{(\exists y) . y \in \alpha . R = y \downarrow x\}$ [*38·131]

***55·232.** $\vdash : \exists ! \downarrow x``\alpha \cap \downarrow y``\beta . \equiv . x = y . \exists ! \alpha \cap \beta$

Dem.

$\vdash . *55{\cdot}231 . *11{\cdot}55 . \supset$
$\vdash :. \exists ! \downarrow x``\alpha \cap \downarrow y``\beta . \equiv : (\exists R) : (\exists z, w) . z \in \alpha . R = z \downarrow x . w \in \beta . R = w \downarrow y :$
$[*13{\cdot}195] \quad \equiv : (\exists z, w) . z \in \alpha . w \in \beta . z \downarrow x = w \downarrow y :$
$[*55{\cdot}202] \quad \equiv : (\exists z, w) . z \in \alpha . w \in \beta . x = y . z = w :$
$[*13{\cdot}195] \quad \equiv : (\exists z) . z \in \alpha \cap \beta . x = y :$
$[*10{\cdot}35] \quad \equiv : \exists ! \alpha \cap \beta . x = y :. \supset \vdash . \mathrm{Prop}$

***55·233.** $\vdash : x \neq y . \supset . \downarrow x``\alpha \cap \downarrow y``\beta = \Lambda$ [*55·232 . Transp]

The above two propositions are frequently useful in arithmetic.

***55·24.** $\vdash . \dot{s}'x \downarrow ``\alpha = \iota'x \uparrow \alpha$

Dem.

$\vdash . *41{\cdot}11 . \supset$
$\vdash :. z(\dot{s}'x \downarrow ``\alpha) w . \equiv . (\exists R) . R \in x \downarrow ``\alpha . zRw .$
$[*55{\cdot}23] \quad \equiv . (\exists R, y) . y \in \alpha . R = x \downarrow y . zRw .$
$[*13{\cdot}195] \quad \equiv . (\exists y) . y \in \alpha . z(x \downarrow y) w .$
$[*55{\cdot}13] \quad \equiv . (\exists y) . y \in \alpha . z = x . w = y .$
$[*13{\cdot}195] \quad \equiv . z = x . w \in \alpha .$
$[*51{\cdot}15 . *35{\cdot}103] \quad \equiv . z(\iota'x \uparrow \alpha) w :. \supset \vdash . \mathrm{Prop}$

***55·241.** $\vdash . \dot{s}'\downarrow x``\alpha = \alpha \uparrow \iota'x$ [Proof as in *55·24]

***55·25.** $\vdash : \exists ! \alpha . \supset . \mathrm{D}``x \downarrow ``\alpha = \iota'\iota'x$

Dem.

$\vdash . *37{\cdot}67 . *33{\cdot}12 . *55{\cdot}12 . \supset$
$\vdash : \beta \in \mathrm{D}``x \downarrow ``\alpha . \equiv . (\exists y) . y \in \alpha . \beta = \mathrm{D}'x \downarrow y .$
$[*55{\cdot}15] \quad \equiv . (\exists y) . y \in \alpha . \beta = \iota'x .$
$[*10{\cdot}35] \quad \equiv . \exists ! \alpha . \beta = \iota'x \qquad (1)$
$\vdash . (1) . \supset \vdash :. \mathrm{Hp} . \supset : \beta \in \mathrm{D}``x \downarrow ``\alpha . \equiv . \beta = \iota'x .$
$[*51{\cdot}15] \quad \equiv . \beta \in \iota'\iota'x :. \supset \vdash . \mathrm{Prop}$

***55·251.** $\vdash : \exists ! \alpha . \supset . \mathrm{d}`` \downarrow x``\alpha = \iota'\iota'x$ [Proof as in *55·25]

This proposition is used in the theory of cardinal multiplication (*113·142).

***55·26.** $\vdash . \mathrm{d}``x \downarrow ``\alpha = \iota``\alpha$ [*55·15 . *37·35]

***55·261.** $\vdash . \mathrm{D}`` \downarrow x``\alpha = \iota``\alpha$ [*55·15 . *37·35]

***55·262.** $\vdash : \downarrow x``\alpha = \downarrow y``\beta . \supset . \alpha = \beta$ [*55·261 . *53·22]

***55·27.** $\vdash . C``\downarrow x``\alpha = C``x\downarrow``\alpha = \hat{\beta}\{(\exists y) . y \in \alpha . \beta = \iota`x \cup \iota`y\}$ [*55·15]

***55·28.** $\vdash : \text{C}`x \downarrow y = \text{C}`x \downarrow z . \equiv . y = z . \equiv . x \downarrow y = x \downarrow z$
[*55·15 . *51·23 . *55·2]

***55·281.** $\vdash : \text{D}`y \downarrow x = \text{D}`z \downarrow x . \equiv . y = z . \equiv . y \downarrow x = z \downarrow x$

***55·282.** $\vdash : C`x \downarrow y = C`x \downarrow z . \equiv . y = z . \equiv . x \downarrow y = x \downarrow z$
[*55·15·2 . *54·21]

***55·283.** $\vdash : C`y \downarrow x = C`z \downarrow x . \equiv . y = z . \equiv . y \downarrow x = z \downarrow x$

***55·29.** $\vdash . \text{C} | (x \downarrow) = \iota$ [*55·15 . *34·42]

***55·291.** $\vdash . \text{D} | (\downarrow x) = \iota$ [*55·15 . *34·42]

***55·292.** $\vdash . C | (x \downarrow) = C | (\downarrow x) = \hat{\alpha}\hat{y}(\alpha = \iota`x \cup \iota`y)$ [*55·15 . *34·41]

The following propositions, down to *55·51 inclusive, give properties of ordinal couples which are analogous to the properties of unit classes.

***55·3.** $\vdash : xRy . \equiv . x \downarrow y \subset R . \equiv . \dot{\exists} ! (x \downarrow y) \dot{\cap} R$ [*13·21·22 . *55·13]

The first half of this proposition is the analogue of *51·2; like that proposition, it gives a means of reducing propositions to the form of inclusions. For the second half, compare *51·31.

***55·31.** $\vdash : x \downarrow y = z \downarrow w . \equiv . z (x \downarrow y) w . \equiv . x (z \downarrow w) y . \equiv . x = z . y = w$

This proposition is the analogue of *51·23.

Dem.

$\vdash . *55·16 . \supset \vdash : x \downarrow y = z \downarrow w . \equiv . \text{D}`x \downarrow y = \iota`z . \text{C}`x \downarrow y = \iota`w .$

[*55·15] $\equiv . \iota`x = \iota`z . \iota`y = \iota`w .$

[*51·23] $\equiv . x = z . y = w .$ (1)

[*55·13] $\equiv . x (z \downarrow w) y .$ (2)

[(1).*13·16] $\equiv . z = x . w = y .$

[*55·13] $\equiv . z (x \downarrow y) w$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash . \text{Prop}$

***55·32.** $\vdash :. x \downarrow y \dot{\cap} z \downarrow w = \dot{\Lambda} . \equiv : x \neq z . \vee . y \neq w$

Dem.

$\vdash . *55·3 . \supset \vdash : \dot{\exists} ! x \downarrow y \dot{\cap} z \downarrow w . \equiv . x (z \downarrow w) y .$

[*55·13] $\equiv . x = z . y = w$ (1)

$\vdash . (1) . \text{Transp} . \supset \vdash . \text{Prop}$

***55·33.** $\vdash : xRy . \equiv . x \downarrow y \dot{\cap} R = x \downarrow y$ [*55·3 . *23·621]

***55·34.** $\vdash : \dot{\exists} ! R . R \subset x \downarrow y . \equiv . R = x \downarrow y$

Dem.

$\vdash . *55·13 . \supset \vdash :. \dot{\exists} ! R . R \subset x \downarrow y . \equiv : (\exists z, w) . zRw : zRw . \supset_{z,w} . z = x . w = y :$

[*14·123] $\equiv : zRw . \equiv_{z,w} . z = x . w = y :$

[*55·13] $\equiv : zRw . \equiv_{z,w} . z (x \downarrow y) w :. \supset \vdash . \text{Prop}$

***55·341.** $\vdash :. R \subset x \downarrow y . \equiv : R = \dot{\Lambda} . \vee . R = x \downarrow y$

Dem.

$\vdash . *4·42 . \supset \vdash :. R \subset x \downarrow y . \equiv : R \subset x \downarrow y . R = \dot{\Lambda} . \vee . R \subset x \downarrow y . R \neq \dot{\Lambda} :$

[*25·54] $\equiv : R \subset x \downarrow y . R = \dot{\Lambda} . \vee . R \subset x \downarrow y . \dot{\exists} ! R :$

[*55·34] $\equiv : R \subset x \downarrow y . R = \dot{\Lambda} . \vee . R = x \downarrow y :$

[*25·12] $\equiv : R = \dot{\Lambda} . \vee . R = x \downarrow y :. \supset \vdash . \text{Prop}$

***55·35.** $\vdash : R \dot{\cap} x \downarrow y = \dot{\Lambda} . R \dot{\cup} x \downarrow y = S . \equiv . xSy . R = S \dot{-} x \downarrow y$

Dem.

$\vdash . *25·47 . \supset$

$\vdash : R \dot{\cap} x \downarrow y = \dot{\Lambda} . R \dot{\cup} x \downarrow y = S . \equiv . x \downarrow y \subset S . R = S \dot{-} x \downarrow y .$

[*55·3] $\equiv . xSy . R = S \dot{-} x \downarrow y : \supset \vdash . \text{Prop}$

***55·36.** $\vdash : xRy . \equiv . (R \dot{-} x \downarrow y) \dot{\cup} x \downarrow y = R$

Dem.

$\vdash . *55·3 . \supset \vdash : xRy . \equiv . x \downarrow y \subset R .$

[*23·62] $\equiv . x \downarrow y \dot{\cup} R = R .$

[*23·91] $\equiv . (R \dot{-} x \downarrow y) \dot{\cup} x \downarrow y = R : \supset \vdash . \text{Prop}$

***55·37.** $\vdash : x \epsilon \alpha . y \epsilon \beta . \equiv . x \downarrow y \subset \alpha \uparrow \beta$

Dem.

$\vdash . *35·103 . \supset \vdash : x \epsilon \alpha . y \epsilon \beta . \equiv . x (\alpha \uparrow \beta) y .$

[*55·3] $\equiv . x \downarrow y \subset \alpha \uparrow \beta : \supset \vdash . \text{Prop}$

The following proposition is the analogue of *51·232.

***55·4.** $\vdash :. a \{x \downarrow y \dot{\cup} z \downarrow w\} b . \equiv : a = x . b = y . \vee . a = z . b = w$ [*55·13 . *23·34]

***55·41.** $\vdash :: R = x \downarrow y \dot{\cup} z \downarrow w . \supset :. aRb . \supset_{a,b} . \phi(a,b) : \equiv . \phi(x,y) . \phi(z,w)$

Dem.

$\vdash . *55·4 . \supset \vdash ::. \text{Hp} . \supset :: aRb . \supset_{a,b} . \phi(a,b) : \equiv :.$

$a = x . b = y . \vee . a = z . b = w : \supset_{a,b} . \phi(a,b) :.$

[*4·77] $\equiv :. (a,b) :. a = x . b = y . \supset . \phi(a,b) : a = z . b = w . \supset . \phi(a,b) :.$

[*11·31] $\equiv :. (a,b) : a = x . b = y . \supset . \phi(a,b) :. (a,b) : a = z . b = w . \supset . \phi(a,b) :.$

[*13·21] $\equiv :. \phi(x,y) . \phi(z,w) ::. \supset \vdash . \text{Prop}$

The above proposition is the analogue of *51·234. The following proposition (*55·42) is the analogue of *51·235.

***55·42.** $\vdash :: R = x \downarrow y \dot{\cup} z \downarrow w . \supset :. (\exists a,b) . aRb . \phi(a,b) . \equiv : \phi(x,y) . \vee . \phi(z,w)$

Dem.

$\vdash . *55·4 . \supset \vdash ::. \text{Hp} . \supset :: (\exists a,b) . aRb . \phi(a,b) . \equiv :.$

$(\exists a,b) :. a = x . b = y . \vee . a = z . b = w : \phi(a,b) :.$

[*4·4] $\equiv :. (\exists a,b) : a = x . b = y . \phi(a,b) : \vee : a = z . b = w . \phi(a,b) :.$

[*11·41] $\equiv :. (\exists a,b) . a = x . b = y . \phi(a,b) . \vee . (\exists a,b) . a = z . b = w . \phi(a,b) :.$

[*13·22] $\equiv :. \phi(x,y) . \vee . \phi(z,w) ::. \supset \vdash . \text{Prop}$

$*55{\cdot}43$. $\vdash : x \downarrow y \cup z \downarrow w = x \downarrow y \cup c \downarrow d . \equiv . z = c . w = d . \equiv . z \downarrow w = c \downarrow d$

This proposition is the analogue of $*51{\cdot}41$.

Dem.

$\vdash . *55{\cdot}202 . \supset \vdash : z = c . w = d . \supset . z \downarrow w = c \downarrow d .$

$[*23{\cdot}551] \qquad \supset . x \downarrow y \cup z \downarrow w = x \downarrow y \cup c \downarrow d \qquad (1)$

$\vdash . *23{\cdot}58 . \supset \vdash :. x \downarrow y \cup z \downarrow w = x \downarrow y \cup c \downarrow d . \supset :$

$z \downarrow w \in x \downarrow y \cup c \downarrow d . c \downarrow d \in x \downarrow y \cup z \downarrow w :$

$[*55{\cdot}313 . *23{\cdot}34] \supset : z = x . w = y . \vee . z = c . w = d : c = x . d = y . \vee . c = z . d = w :$

$[*13{\cdot}16] \qquad \supset : z = x . w = y . \vee . z = c . w = d : c = x . d = y . \vee . z = c . w = d :$

$[*4{\cdot}41] \qquad \supset : z = x . w = y . c = x . d = y . \vee . z = c . w = d :$

$[*13{\cdot}172] \qquad \supset : z = c . w = d \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash : x \downarrow y \cup z \downarrow w = x \downarrow y \cup c \downarrow d . \equiv . z = c . w = d \qquad (3)$

$\vdash . (3) . *55{\cdot}202 . \supset \vdash . \text{Prop}$

$*55{\cdot}431$. $\vdash :. x \downarrow y \cup z \downarrow w = a \downarrow b \cup c \downarrow d . \supset :$

$x = a . y = b . z = c . w = d . \vee . x = c . y = d . z = a . w = b$

Dem.

$\vdash . *55{\cdot}4 . \supset \vdash :: \text{Hp} . \equiv :. u = x . v = y . \vee . u = z . v = w :$

$\equiv_{u,v} : u = a . v = b . \vee . u = c . v = d :.$

$[*11{\cdot}1] \qquad \supset :. x = x . y = y . \vee . x = z . y = w :$

$\equiv : x = a . y = b . \vee . x = c . y = d :.$

$[*13{\cdot}15] \qquad \supset :. x = a . y = b . \vee . x = c . y = d \qquad (1)$

$\vdash . *55{\cdot}43 . \supset \vdash :. x = a . y = b . \supset : x \downarrow y \cup z \downarrow w = a \downarrow b \cup z \downarrow w :$

$[*13{\cdot}171] \qquad \supset : \text{Hp} . \supset . a \downarrow b \cup z \downarrow w = a \downarrow b \cup c \downarrow d .$

$[*55{\cdot}43] \qquad \supset . z = c . w = d \qquad (2)$

$\vdash . (2) . \text{Comm} . *4{\cdot}7 . \supset \vdash :. \text{Hp} . \supset : x = a . y = b . \supset . x = a . y = b . z = c . w = d \quad (3)$

Similarly $\qquad \vdash :. \text{Hp} . \supset : x = c . y = d . \supset . x = c . y = d . z = a . w = b \quad (4)$

$\vdash . (1) . (3) . (4) . \supset \vdash . \text{Prop}$

$*55{\cdot}44$. $\vdash :. x \downarrow y \cup z \downarrow w = a \downarrow b \cup c \downarrow d .$

$\equiv : x = a . y = b . z = c . w = d . \vee . x = c . y = d . z = a . w = b :$

$\equiv : x \downarrow y = a \downarrow b . z \downarrow w = c \downarrow d . \vee . x \downarrow y = c \downarrow d . z \downarrow w = a \downarrow b$

Dem.

$\vdash . *55{\cdot}43 . \qquad \supset \vdash : x = a . y = b . \supset . x \downarrow y \cup z \downarrow w = a \downarrow b \cup z \downarrow w :$

$z = c . w = d . \supset . a \downarrow b \cup z \downarrow w = a \downarrow b \cup c \downarrow d :$

$[*3{\cdot}47 . *13{\cdot}17] \supset \vdash : x = a . y = b . z = c . w = d .$

$\supset . x \downarrow y \cup z \downarrow w = a \downarrow b \cup c \downarrow d \quad (1)$

Similarly $\qquad \vdash : x = c . y = d . z = a . w = b .$

$\supset . x \downarrow y \cup z \downarrow w = a \downarrow b \cup c \downarrow d \quad (2)$

$\vdash . (1) . (2) . *55{\cdot}431{\cdot}202 . \supset \vdash . \text{Prop}$

The above proposition is the analogue of $*51{\cdot}43$

***55·5.** $\vdash :. R \subset x \downarrow y \,\dot{\cup}\, z \downarrow w \,.$
$$\equiv : R = \dot{\Lambda} \,.\vee.\, R = x \downarrow y \,.\vee.\, R = z \downarrow w \,.\vee.\, R = x \downarrow y \,\dot{\cup}\, z \downarrow w$$

Dem.

$\vdash . *25{\cdot}12 . *23{\cdot}58{\cdot}42 . \supset$

$\vdash :: R = \dot{\Lambda} \,.\vee.\, R = x \downarrow y \,.\vee.\, R = z \downarrow w \,.\vee.\, R = x \downarrow y \,\dot{\cup}\, z \downarrow w :$

$\supset . R \subset x \downarrow y \,\dot{\cup}\, z \downarrow w$ (1)

$\vdash . *25{\cdot}49 . \quad \supset \vdash :. R \subset x \downarrow y \,\dot{\cup}\, z \downarrow w \,.\, R \,\dot{\cap}\, x \downarrow y = \dot{\Lambda} . \supset : R \subset z \downarrow w :$

[*55·341] $\supset : R = \dot{\Lambda} \,.\vee.\, R = z \downarrow w$ (2)

$\vdash . *25{\cdot}43 . \quad \supset \vdash :. R \subset x \downarrow y \,\dot{\cup}\, z \downarrow w . \supset : R \,\dot{-}\, x \downarrow y \subset z \downarrow w :$

[*55·341] $\supset : R \,\dot{-}\, x \downarrow y = \dot{\Lambda} \,.\vee.\, R \,\dot{-}\, x \downarrow y = z \downarrow w :$

[*25·24.*23·551] $\supset : (R \,\dot{-}\, x \downarrow y) \,\dot{\cup}\, x \downarrow y = x \downarrow y \,.\vee.$

$(R \,\dot{-}\, x \downarrow y) \,\dot{\cup}\, x \downarrow y = x \downarrow y \,\dot{\cup}\, z \downarrow w$ (3)

$\vdash . *55{\cdot}3{\cdot}36 . \supset \vdash : \dot{\exists} ! (R \,\dot{\cap}\, x \downarrow y) . \supset . (R \,\dot{-}\, x \downarrow y) \,\dot{\cup}\, x \downarrow y = R$ (4)

$\vdash . (3) . (4) . \quad \supset \vdash :. R \subset x \downarrow y \,\dot{\cup}\, z \downarrow w \,.\, \dot{\exists} ! (R \,\dot{\cap}\, x \downarrow y) . \supset :$

$R = x \downarrow y \,.\vee.\, R = x \downarrow y \,\dot{\cup}\, z \downarrow w$ (5)

$\vdash . (2) . (5) . \quad \supset \vdash :. R \subset x \downarrow y \,\dot{\cup}\, z \downarrow w . \supset :$

$R = \dot{\Lambda} \,.\vee.\, R = x \downarrow y \,.\vee.\, R = z \downarrow w \,.\vee.\, R = x \downarrow y \,\dot{\cup}\, z \downarrow w$ (6)

$\vdash . (1) . (6) . \quad \supset \vdash .$ Prop

The above proposition is the analogue of *54·4.

***55·51.** $\vdash :. R \subset x \downarrow y \,\dot{\cup}\, S . \supset : xRy \,.\vee.\, R \subset S$

Dem.

$\vdash . *55{\cdot}3 . \quad \supset \vdash : \dot{\exists} ! (R \,\dot{\cap}\, x \downarrow y) . \supset . xRy$ (1)

$\vdash . *25{\cdot}49 . \quad \supset \vdash : \mathrm{Hp} . \sim \dot{\exists} ! (R \,\dot{\cap}\, x \downarrow y) . \supset . R \subset S$ (2)

$\vdash . (1) . (2) . \supset \vdash .$ Prop

In the remainder of the present number, we are concerned with properties of ordinal couples which have no analogues for unit classes.

***55·52.** $\vdash . (\iota`x \cup \iota`y) \uparrow (\iota`z \cup \iota`w) = x \downarrow z \,\dot{\cup}\, x \downarrow w \,\dot{\cup}\, y \downarrow z \,\dot{\cup}\, y \downarrow w$ [*35·82·413]

***55·521.** $\vdash : x \neq y . \equiv . x \downarrow y \subset J$ [*55·3 . *50·11]

***55·53.** $\vdash :. x \neq y . \supset : C`R = \iota`x \cup \iota`y \,.\, R \subset J . \equiv . \dot{\exists} ! R \,.\, R \subset x \downarrow y \,\dot{\cup}\, y \downarrow x$

Dem.

$\vdash . *55{\cdot}5 . \supset \vdash :. \dot{\exists} ! R \,.\, R \subset x \downarrow y \,\dot{\cup}\, y \downarrow x . \equiv :$

$R = x \downarrow y \,.\vee.\, R = y \downarrow x \,.\vee.\, R = x \downarrow y \,\dot{\cup}\, y \downarrow x$ (1)

$\vdash . *55{\cdot}15 . \quad \supset \vdash . C`x \downarrow y = \iota`x \cup \iota`y \,.\, C`y \downarrow x = \iota`x \cup \iota`y$ (2)

$\vdash . (2) . *33{\cdot}262 . \supset \vdash . C`(x \downarrow y \,\dot{\cup}\, y \downarrow x) = \iota`x \cup \iota`y$ (3)

$\vdash . *55{\cdot}521 . \quad \supset \vdash : x \neq y . \supset . x \downarrow y \subset J \,.\, y \downarrow x \subset J .$ (4)

[*23·59] $\supset . x \downarrow y \,\dot{\cup}\, y \downarrow x \subset J$ (5)

$\vdash . (1) . (2) . (3) . (4) . (5) . \supset \vdash :.$

$x \neq y . \supset : \dot{\exists} ! R \,.\, R \subset x \downarrow y \,\dot{\cup}\, y \downarrow x . \supset . C`R = \iota`x \cup \iota`y \,.\, R \subset J$ (6)

$\vdash . *35{\cdot}91 . \supset \vdash : C`R = \iota`x \cup \iota`y . \supset . R \subset (\iota`x \cup \iota`y) \uparrow (\iota`x \cup \iota`y) .$

[*55·52] $\supset . R \subset x \downarrow x \,\dot{\cup}\, x \downarrow y \,\dot{\cup}\, y \downarrow x \,\dot{\cup}\, y \downarrow y$ (7)

$\vdash . *50{\cdot}24 . \supset \vdash : R \mathrel{\dot{\subset}} J . \supset . \sim(xRx) . \sim(yRy) .$

[*55·3.Transp] $\supset . R \mathbin{\dot{\wedge}} x \downarrow x = \dot{\Lambda} . R \mathbin{\dot{\wedge}} y \downarrow y = \dot{\Lambda}$ (8)

$\vdash . (7) . (8) . *25{\cdot}49 . \supset \vdash : C{`}R = \iota{`}x \cup \iota{`}y . R \mathrel{\dot{\subset}} J . \supset . R \mathrel{\dot{\subset}} x \downarrow y \mathbin{\dot{\cup}} y \downarrow x$ (9)

$\vdash . *33{\cdot}24 . *51{\cdot}161 . \supset \vdash : C{`}R = \iota{`}x \cup \iota{`}y . \supset . \dot{\exists} ! R$ (10)

$\vdash . (9) . (10) . \supset \vdash : C{`}R = \iota{`}x \cup \iota{`}y . R \mathrel{\dot{\subset}} J . \supset . \dot{\exists} ! R . R \mathrel{\dot{\subset}} x \downarrow y \mathbin{\dot{\cup}} y \downarrow x$ (11)

$\vdash . (6) . (11) . \supset \vdash .$ Prop

***55·54.** $\vdash :: x \neq y . \supset :. C{`}R = \iota{`}x \cup \iota{`}y . R \mathbin{\dot{\wedge}} \breve{R} = \dot{\Lambda} . \equiv : R = x \downarrow y . \vee . R = y \downarrow x$

Dem.

$\vdash . *50{\cdot}46 . *4{\cdot}71 . \supset \vdash : R \mathbin{\dot{\wedge}} \breve{R} = \dot{\Lambda} . \equiv . R \mathrel{\dot{\subset}} J . R \mathbin{\dot{\wedge}} \breve{R} = \dot{\Lambda}$ (1)

$\vdash . (1) . *55{\cdot}53 . \supset \vdash :: x \neq y . \supset :. C{`}R = \iota{`}x \cup \iota{`}y . R \mathbin{\dot{\wedge}} \breve{R} = \dot{\Lambda} .$

$\equiv : \dot{\exists} ! R . R \mathrel{\dot{\subset}} x \downarrow y \mathbin{\dot{\cup}} y \downarrow x . R \mathbin{\dot{\wedge}} \breve{R} = \dot{\Lambda} :$

[*55·5·134] $\equiv : R = x \downarrow y . \vee . R = y \downarrow x . \vee . R = x \downarrow y \mathbin{\dot{\cup}} y \downarrow x : R \mathbin{\dot{\wedge}} \breve{R} = \dot{\Lambda}$ (2)

$\vdash . *55{\cdot}32 . \supset \vdash :. x \neq y . \supset : x \downarrow y \mathbin{\dot{\wedge}} y \downarrow x = \dot{\Lambda} :$

[*55·14] $\supset : R = x \downarrow y . \supset . R \mathbin{\dot{\wedge}} \breve{R} = \dot{\Lambda} :$

$R = y \downarrow x . \supset . R \mathbin{\dot{\wedge}} \breve{R} = \dot{\Lambda}$ (3)

$\vdash . *55{\cdot}14 . *31{\cdot}15{\cdot}33 . \supset \vdash : R = x \downarrow y \mathbin{\dot{\cup}} y \downarrow x . \supset . R = \breve{R} .$

[*23·5] $\supset . R \mathbin{\dot{\wedge}} \breve{R} = R .$

[*55·134] $\supset . \dot{\exists} ! R \mathbin{\dot{\wedge}} \breve{R}$ (4)

$\vdash . (3) . (4) . *4{\cdot}71 . *5{\cdot}71 . \supset$

$\vdash :: x \neq y . \supset :. R = x \downarrow y . \vee . R = y \downarrow x . \vee . R = x \downarrow y \mathbin{\dot{\cup}} y \downarrow x : R \mathbin{\dot{\wedge}} \breve{R} = \dot{\Lambda} :$

$\equiv : R = x \downarrow y . \vee . R = y \downarrow x$ (5)

$\vdash . (2) . (5) . \supset \vdash .$ Prop

***55·57.** $\vdash . R | (x \downarrow y) = \overrightarrow{R}{`}x \uparrow \iota{`}y$ [*37·81 . *55·1 . *53·301]

***55·571.** $\vdash . (x \downarrow y) | S = \iota{`}x \uparrow \overleftarrow{S}{`}y$

***55·572.** $\vdash . R | (x \downarrow y) | S = \overrightarrow{R}{`}x \uparrow \overleftarrow{S}{`}y$ [*55·571 . *37·81]

***55·573.** $\vdash . R | (x \downarrow y) | \breve{S} = \overrightarrow{R}{`}x \uparrow \overrightarrow{S}{`}y$ $\left[*55{\cdot}572 \frac{\breve{S}}{S}\right]$

***55·58.** $\vdash : E ! R{`}x . \supset . R | (x \downarrow y) = (R{`}x) \downarrow y$ [*55·57 . *53·31 . *55·1]

***55·581.** $\vdash : E ! \breve{S}{`}y . \supset . (x \downarrow y) | S = x \downarrow (\breve{S}{`}y)$

***55·582.** $\vdash : E ! R{`}x . E ! \breve{S}{`}y . \supset . R | (x \downarrow y) | S = (R{`}x) \downarrow (\breve{S}{`}y)$ [*55·58·581]

***55·583.** $\vdash : E ! R{`}x . E ! S{`}y . \supset . R | (x \downarrow y) | \breve{S} = (R{`}x) \downarrow (S{`}y)$ $\left[*55{\cdot}582 \frac{\breve{S}}{S}\right]$

The above propositions are frequently useful in arithmetic. Their use arises as follows. Let α, β, γ, δ be classes of which α is correlated with γ by the relation R, and β with δ by the relation S. Then if $x \in \gamma . y \in \delta$, the

couple consisting of the correlate of x and the correlate of y is $(R`x) \downarrow (S`y)$, *i.e.*, by the above, $R \mid (x \downarrow y) \mid \breve{S}$, *i.e.* $(R \parallel \breve{S})`(x \downarrow y)$. Thus the relation $R \parallel \breve{S}$ correlates the couples, in α and β, composed of the correlates of terms in γ and δ. The most useful form, in practice, of *55·583, is that given below in *55·61.

***55·6.** $\vdash . (R \parallel \breve{S})`(z \downarrow w) = \overrightarrow{R}`z \uparrow \overrightarrow{S}`w$ [*55·573 . *43·112]

***55·61.** $\vdash : \text{E} ! R`z . \text{E} ! S`w . \supset . (R \parallel \breve{S})`(z \downarrow w) = (R`z) \downarrow (S`w)$
[*55·583 . *43·112]

***55·62.** $\vdash : z \neq w . S = x \downarrow z \cup y \downarrow w . \supset . S`z = x . S`w = y$

Dem.

$\vdash . *55{\cdot}13 . \supset \vdash :: \text{Hp} . \supset :. uSz . \equiv : u = x . z = z . \vee . u = y . z = w$ (1)

$\vdash . (1) . *13{\cdot}15 . \supset \vdash :. \text{Hp} . \supset : uSz . \equiv . u = x$ (2)

Similarly $\vdash :. \text{Hp} . \supset : uSw . \equiv . u = y$ (3)

$\vdash . (2) . (3) . *30{\cdot}3 . \supset \vdash . \text{Prop}$

***55·621.** $\vdash : x \neq y . S = x \downarrow z \cup y \downarrow w . \supset . \breve{S}`x = z . \breve{S}`y = w$
[Proof as in *55·62]

The four following propositions belong to *43, but are inserted here because the proof uses *55·13.

***55·63.** $\vdash : \dot{\exists} ! Q \dot{\wedge} S . P \parallel Q = R \parallel S . \supset . P = R$

Dem.

$\vdash . *43{\cdot}112 . \supset \vdash :: \text{Hp} . \supset :. P \mid (y \downarrow z) \mid Q = R \mid (y \downarrow z) \mid S :.$

[*34·1] $\supset :. (\exists u, v) . xPu . u (y \downarrow z) v . vQw . \equiv_{x,w} .$
$(\exists u, v) . xRu . u (y \downarrow z) v . vSw :.$

[*55·13.*13·22] $\supset :. xPy . zQw . \equiv_{x,w} . xRy . zSw :.$

[*4·73] $\supset :. zQw . zSw . \supset_w : xPy . \equiv_x . xRy$ (1)

$\vdash . (1) . *10{\cdot}11 . *11{\cdot}35 . \supset \vdash :. \text{Hp} . \supset : xPy . \equiv_x . xRy$ (2)

$\vdash . (2) . *10{\cdot}11{\cdot}21 . \supset \vdash . \text{Prop}$

***55·631.** $\vdash : \dot{\exists} ! P \dot{\wedge} R . P \parallel Q = R \parallel S . \supset . Q = S$ [Proof as in *55·63]

***55·632.** $\vdash : P \parallel Q = R \parallel S . \dot{\exists} ! P . \dot{\exists} ! Q . \supset . \dot{\exists} ! P \dot{\wedge} R . \dot{\exists} ! Q \dot{\wedge} S$

Dem.

$\vdash . *55{\cdot}13 . \supset \vdash : xPy . zQw . \supset . x \{P \mid (y \downarrow z) \mid Q\} w .$

[*43·112] $\supset . x \{(P \parallel Q)`(y \downarrow z)\} w$ (1)

$\vdash . (1) . \supset \vdash :. \text{Hp} . \supset : xPy . zQw . \supset . x \{(R \parallel S)`(y \downarrow z)\} w .$

[*43·112] $\supset . x \{R \mid (y \downarrow z) \mid S\} w .$

[*34·1] $\supset . (\exists u, v) . xRu . u (y \downarrow z) v . vSw .$

[*55·13.*13·22] $\supset . xRy . zSw .$

[*4·7] $\supset . x (P \dot{\wedge} R) y . z (Q \dot{\wedge} S) w :. \supset \vdash . \text{Prop}$

***55·64.** $\vdash :. \dot{\exists} ! P . \dot{\exists} ! Q . \vee . \dot{\exists} ! R . \dot{\exists} ! S : \supset : P \parallel Q = R \parallel S . \equiv . P = R . Q = S$
[*55·63·631·632]

$*56$. THE ORDINAL NUMBER 2_r.

Summary of $*56$.

In this number, we have to consider the class of those relations which are each constituted by a single couple. In case the two members of this couple are not identical, the class of such relations is (as will be shown later) the ordinal number 2, which, to distinguish it from the cardinal number 2, we denote by "2_r." (Here the suffix is intended to suggest "relational.") The class of all relations consisting of a single couple, without the restriction that the two members of the couple are to be distinct, will be denoted by "$\dot{2}$." This is not an ordinal number. It will be observed that there is no ordinal number 1, because ordinal numbers apply to series, and series must have more than one member if they have any members. This will appear more fully when we come to deal with series.

The properties of $\dot{2}$ are largely analogous to those of 1, while the properties of 2_r are more analogous to those of 2.

Most of the propositions of the present number are seldom referred to in the sequel, but such references as occur are important. The most useful propositions in the present number are the following:

$*56 \cdot 111$. $\vdash : R \in 2_r . \equiv . \mathrm{D}`R, \mathrm{Q}`R \in 1 . \mathrm{D}`R \cap \mathrm{Q}`R = \Lambda$

$*56 \cdot 112$. $\vdash : R \in 2_r . \equiv . \mathrm{D}`R, \mathrm{Q}`R \in 1 . C`R \in 2$

$*56 \cdot 113$. $\vdash . 2_r = \dot{2} \cap \breve{C}``2$

Observe that "$\breve{C}``2$" means "relations whose fields have two terms."

$*56 \cdot 13$. $\vdash . \dot{2} - 2_r = \hat{R}\{(\exists a) . R = a \downarrow a\}$

$*56 \cdot 37$. $\vdash : R \in 2_r . \equiv . C`R \in 2 . R \,\dot{\cap}\, \breve{R} = \dot{\Lambda}$

I.e. 2_r is the class of asymmetrical relations whose fields have two terms.

$*56 \cdot 381$. $\vdash : C`R = \iota`x . \equiv . R = x \downarrow x$

$*56 \cdot 39$. $\vdash . \dot{2} - 2_r = \breve{C}``1$

I.e. the relations which are couples whose referent and relatum are identical are the relations whose fields consist of a single term.

$*56 \cdot 01$. $\dot{2} = \hat{R}\{(\exists x, y) . R = x \downarrow y\}$ Df

$*56 \cdot 02$. $2_r = \hat{R}\{(\exists x, y) . x \neq y . R = x \downarrow y\}$ Df

$*56 \cdot 03$. $0_r = \iota`\dot{\Lambda}$ Df

$*56 \cdot 1$. $\vdash : R \in \dot{2} . \equiv . (\exists x, y) . R = x \downarrow y$ $[*20 \cdot 3 . (*56 \cdot 01)]$

***56·101.** $\vdash : R \in \dot{2} . \equiv . \mathrm{D}\text{‘}R, \text{Ɑ}\text{‘}R \in 1$

Dem.

$\vdash . *55\cdot16 . *11\cdot11\cdot341 . \supset$

$\vdash :. (\exists x, y) . R = x \downarrow y . \equiv : (\exists x, y) . \mathrm{D}\text{‘}R = \iota\text{‘}x . \text{Ɑ}\text{‘}R = \iota\text{‘}y :$

$[*11\cdot54] \quad \equiv : (\exists x) . \mathrm{D}\text{‘}R = \iota\text{‘}x : (\exists y) . \text{Ɑ}\text{‘}R = \iota\text{‘}y :$

$[*52\cdot1] \quad \equiv : \mathrm{D}\text{‘}R, \text{Ɑ}\text{‘}R \in 1 \qquad (1)$

$\vdash . (1) . *56\cdot1 . \supset \vdash . \mathrm{Prop}$

***56·102.** $\vdash . \dot{2} = \breve{\mathrm{D}}\text{“}1 \cap \breve{\text{Ɑ}}\text{“}1$

Dem.

$\vdash . *56\cdot101 . *37\cdot106 . \supset$

$\vdash : R \in \dot{2} . \equiv . R \in \breve{\mathrm{D}}\text{“}1 . R \in \breve{\text{Ɑ}}\text{“}1 .$

$[*22\cdot33] \quad \equiv . R \in \breve{\mathrm{D}}\text{“}1 \cap \breve{\text{Ɑ}}\text{“}1 : \supset \vdash . \mathrm{Prop}$

***56·103.** $\vdash : R \in \dot{2} . \supset . \dot{\exists} ! R$

Dem.

$\vdash . *56\cdot101 . \supset \vdash : R \in \dot{2} . \supset . \mathrm{D}\text{‘}R \in 1 .$

$[*52\cdot16] \quad \supset . \exists ! \mathrm{D}\text{‘}R .$

$[*33\cdot24] \quad \supset . \dot{\exists} ! R : \supset \vdash . \mathrm{Prop}$

***56·104.** $\vdash : R \in 0_r . \equiv . R = \dot{\Lambda} \quad [(*56\cdot03)]$

***56·11.** $\vdash : R \in 2_r . \equiv . (\exists x, y) . x \neq y . R = x \downarrow y \quad [*20\cdot3 . (*56\cdot02)]$

***56·111.** $\vdash : R \in 2_r . \equiv . \mathrm{D}\text{‘}R, \text{Ɑ}\text{‘}R \in 1 . \mathrm{D}\text{‘}R \cap \text{Ɑ}\text{‘}R = \Lambda$

Dem.

$\vdash . *51\cdot231 . *55\cdot16 . \supset$

$\vdash : x \neq y . R = x \downarrow y . \equiv . \iota\text{‘}x \cap \iota\text{‘}y = \Lambda . \mathrm{D}\text{‘}R = \iota\text{‘}x . \text{Ɑ}\text{‘}R = \iota\text{‘}y .$

$[*13\cdot193] \quad \equiv . \mathrm{D}\text{‘}R \cap \text{Ɑ}\text{‘}R = \Lambda . \mathrm{D}\text{‘}R = \iota\text{‘}x . \text{Ɑ}\text{‘}R = \iota\text{‘}y \qquad (1)$

$\vdash . (1) . *56\cdot11 . *11\cdot11\cdot341 . \supset$

$\vdash :. R \in 2_r . \equiv : (\exists x, y) . \mathrm{D}\text{‘}R \cap \text{Ɑ}\text{‘}R = \Lambda . \mathrm{D}\text{‘}R = \iota\text{‘}x . \text{Ɑ}\text{‘}R = \iota\text{‘}y :$

$[*11\cdot45] \quad \equiv : \mathrm{D}\text{‘}R \cap \text{Ɑ}\text{‘}R = \Lambda : (\exists x, y) . \mathrm{D}\text{‘}R = \iota\text{‘}x . \text{Ɑ}\text{‘}R = \iota\text{‘}y :$

$[*11\cdot54] \quad \equiv : \mathrm{D}\text{‘}R \cap \text{Ɑ}\text{‘}R = \Lambda : (\exists x) . \mathrm{D}\text{‘}R = \iota\text{‘}x : (\exists y) . \text{Ɑ}\text{‘}R = \iota\text{‘}y :$

$[*52\cdot1] \quad \equiv : \mathrm{D}\text{‘}R \cap \text{Ɑ}\text{‘}R = \Lambda . \mathrm{D}\text{‘}R, \text{Ɑ}\text{‘}R \in 1 :. \supset \vdash . \mathrm{Prop}$

***56·112.** $\vdash : R \in 2_r . \equiv . \mathrm{D}\text{‘}R, \text{Ɑ}\text{‘}R \in 1 . C\text{‘}R \in 2$

Dem.

$\vdash . *56\cdot111 . *54\cdot43 . \supset$

$\vdash : R \in 2_r . \equiv . \mathrm{D}\text{‘}R, \text{Ɑ}\text{‘}R \in 1 . \mathrm{D}\text{‘}R \cup \text{Ɑ}\text{‘}R \in 2 .$

$[*33\cdot16] \quad \equiv . \mathrm{D}\text{‘}R, \text{Ɑ}\text{‘}R \in 1 . C\text{‘}R \in 2 : \supset \vdash . \mathrm{Prop}$

***56·113.** $\vdash . 2_r = \dot{2} \cap \breve{C}\text{“}2$

Dem.

$\vdash . *56\cdot112\cdot101 . \supset \vdash : R \in 2_r . \equiv . R \in \dot{2} . C\text{‘}R \in 2 .$

$[*37\cdot106 . *33\cdot122] \quad \equiv . R \in \dot{2} . R \in \breve{C}\text{“}2 .$

$[*22\cdot33] \quad \equiv . R \in \dot{2} \cap \breve{C}\text{“}2 : \supset \vdash . \mathrm{Prop}$

***56·114.** $\vdash . 2_r = \breve{\mathrm{D}}``1 \cap \breve{\mathrm{C}}``1 \cap \breve{C}``2$ [*56·113·102]

***56·12.** $\vdash : R \epsilon 2_r . \equiv . R \epsilon \dot{2} . R \mathrel{\mathsf{G}} J$

Dem.

$\vdash . *55·3 . *50·11 . \supset \vdash : x \neq y . \equiv . x \downarrow y \mathrel{\mathsf{G}} J :$

[Fact] $\supset \vdash : R = x \downarrow y . x \neq y . \equiv . R = x \downarrow y . x \downarrow y \mathrel{\mathsf{G}} J .$

[*13·193] $\equiv . R = x \downarrow y . R \mathrel{\mathsf{G}} J$ (1)

$\vdash . (1) . *11·11·341 . \supset$

$\vdash :. (\exists x, y) . R = x \downarrow y . x \neq y . \equiv : (\exists x, y) . R = x \downarrow y . R \mathrel{\mathsf{G}} J :$

[*11·45] $\equiv : (\exists x, y) . R = x \downarrow y : R \mathrel{\mathsf{G}} J :$

[*56·1] $\equiv : R \epsilon \dot{2} . R \mathrel{\mathsf{G}} J$ (2)

$\vdash . (2) . *56·11 . \supset \vdash .$ Prop

***56·121.** $\vdash . 2_r \mathrel{\mathsf{G}} \dot{2}$ [*56·113]

***56·122.** $\vdash : R \epsilon 2_r . \supset . \dot{\exists} ! R$ [*56·121·103]

***56·13.** $\vdash . \dot{2} - 2_r = \hat{R} \{(\exists a) . R = a \downarrow a\}$

Dem.

$\vdash . *56·11 . *11·52 .$ Transp $. \supset$

$\vdash : R \sim \epsilon 2_r . \equiv : R = x \downarrow y . \supset_{x,y} . x = y$ (1)

$\vdash . (1) . *56·1 . \supset$

$\vdash :. R \epsilon \dot{2} - 2_r . \equiv : (\exists a, b) . R = a \downarrow b : R = x \downarrow y . \supset_{x,y} . x = y :$

[*11·45] $\equiv : (\exists a, b) : R = a \downarrow b : R = x \downarrow y . \supset_{x,y} . x = y :$

[*13·193] $\equiv : (\exists a, b) : R = a \downarrow b : a \downarrow b = x \downarrow y . \supset_{x,y} . x = y :$

[*55·202] $\equiv : (\exists a, b) : R = a \downarrow b : a = x . b = y . \supset_{x,y} . x = y :$

[*13·21] $\equiv : (\exists a, b) . R = a \downarrow b . a = b :$

[*13·195] $\equiv : (\exists a) . R = a \downarrow a :. \supset \vdash .$ Prop

$\dot{2} - 2_r$ might be defined as the ordinal number 1, since it is what we shall call a relation number (cf. *153). But we wish our ordinal numbers to be classes of *serial* relations, and such relations have the property of being contained in diversity. Hence if we were to define $\dot{2} - 2_r$ as the ordinal number 1, we should introduce a tiresome exception, from which trivial complications would be introduced into ordinal arithmetic. We have, therefore, not adopted this course.

***56·14.** $\vdash . \mathrm{D}`(x \downarrow) = \dot{2} \cap \overleftarrow{\mathrm{D}}`\iota`x$

Dem.

$\vdash . *33·6 . \supset \vdash : \mathrm{D}`R = \iota`x . \equiv . R \epsilon \overleftarrow{\mathrm{D}}`\iota`x$ (1)

$\vdash . (1) . *56·1 . \supset$

$\vdash :. R \epsilon \dot{2} \cap \overleftarrow{\mathrm{D}}`\iota`x . \equiv : (\exists z, y) . R = z \downarrow y : \mathrm{D}`R = \iota`x :$

[*55·16] $\equiv : (\exists z, y) . \mathrm{D}`R = \iota`z . \mathrm{C}`R = \iota`y : \mathrm{D}`R = \iota`x :$

[*11·45] $\equiv : (\exists z, y) . \mathrm{D}`R = \iota`z . \mathrm{C}`R = \iota`y . \mathrm{D}`R = \iota`x :$

[*13·193] $\equiv : (\exists z, y) . \text{D}'R = \iota'z . \text{Ɑ}'R = \iota'y . \iota'z = \iota'x :$
[*51·23] $\equiv : (\exists z, y) . \text{D}'R = \iota'z . \text{Ɑ}'R = \iota'y . z = x :$
[*13·195] $\equiv : (\exists y) . \text{D}'R = \iota'x . \text{Ɑ}'R = \iota'y :$
[*55·16] $\equiv : (\exists y) . R = x \downarrow y :$
[*55·22] $\equiv : R \in \text{D}'(x \downarrow) :. \supset \vdash . \text{Prop}$

***56·141.** $\vdash . \text{D}'\downarrow x = \dot{2} \cap \overleftarrow{\text{Ɑ}}'\iota'x$ [Proof as in *56·14]

***56·15.** $\vdash . \text{D}'(x \downarrow) - \iota'(x \downarrow x) = 2_r \cap \overleftarrow{\text{D}}'\iota'x$

Dem.

$\vdash . *55·22·16 . \supset \vdash :. R \in \{\text{D}'(x \downarrow)\} - \iota'(x \downarrow x) .$
$\equiv : (\exists y) . \text{D}'R = \iota'x . \text{Ɑ}'R = \iota'y : \sim(\text{D}'R = \iota'x . \text{Ɑ}'R = \iota'x) :$
[*10·35.*4·51.*5·61] $\equiv : (\exists y) . \text{D}'R = \iota'x . \text{Ɑ}'R = \iota'y . \sim(\text{Ɑ}'R = \iota'x) :$
[*13·193] $\equiv : (\exists y) . \text{D}'R = \iota'x . \text{Ɑ}'R = \iota'y . \sim(\iota'y = \iota'x) :$
[*51·23] $\equiv : (\exists y) . \text{D}'R = \iota'x . \text{Ɑ}'R = \iota'y . x \neq y :$
[*13·195.*51·23] $\equiv : (\exists z, y) . z \neq y . \text{D}'R = \iota'z . \text{Ɑ}'R = \iota'y . \iota'z = \iota'x :$
[*13·193] $\equiv : (\exists z, y) . z \neq y . \text{D}'R = \iota'z . \text{Ɑ}'R = \iota'y . \text{D}'R = \iota'x :$
[*11·45] $\equiv : (\exists z, y) . z \neq y . \text{D}'R = \iota'z . \text{Ɑ}'R = \iota'y : \text{D}'R = \iota'x :$
[*55·16.*33·6] $\equiv : (\exists z, y) . z \neq y . R = z \downarrow y : R \in \overleftarrow{\text{D}}'\iota'x :$
[*56·11.*22·33] $\equiv : R \in 2_r \cap \overleftarrow{\text{D}}'\iota'x :. \supset \vdash . \text{Prop}$

***56·151.** $\vdash . \text{D}'(\downarrow x) - \iota'(x \downarrow x) = 2_r \cap \overleftarrow{\text{Ɑ}}'\iota'x$ [Proof as in *56·15]

***56·16.** $\vdash . x \downarrow y \in \dot{2}$

Dem.

$\vdash . *21·2 . \supset \vdash . x \downarrow y = x \downarrow y .$
[*11·36] $\supset \vdash . (\exists z, w) . x \downarrow y = z \downarrow w .$
[*56·1] $\supset \vdash . x \downarrow y \in \dot{2} . \supset \vdash . \text{Prop}$

***56·17.** $\vdash : x \downarrow y \in 2_r . \equiv . y \downarrow x \in 2_r . \equiv . x \neq y$

Dem.

$\vdash . *56·11 . \supset$
$\vdash :. x \downarrow y \in 2_r . \equiv : (\exists z, w) . z \neq w . x \downarrow y = z \downarrow w :$
[*55·202] $\equiv : (\exists z, w) . z \neq w . x = z . y = w :$
[*13·22] $\equiv : x \neq y$ (1)
Similarly
$\vdash : y \downarrow x \in 2_r . \equiv . x \neq y$ (2)
$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***56·18.** $\vdash : x \sim\in \alpha . \equiv . x \downarrow``\alpha \subset 2_r . \equiv . \downarrow x``\alpha \subset 2_r$

Dem.

$\vdash . *13·196 . \supset \vdash :. x \sim\in \alpha . \equiv : y \in \alpha . \supset_y . y \neq x :$
[*56·17] $\equiv : y \in \alpha . \supset_y . x \downarrow y \in 2_r :$
[*37·61.*38·12·11] $\equiv : x \downarrow ``\alpha \subset 2_r$ (1)
Similarly $\vdash : x \sim\in \alpha . \equiv . \downarrow x``\alpha \subset 2_r$ (2)
$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

$*56{\cdot}19$. $\vdash : R \in 2_r \,.\, x \in \mathrm{D}'R \,.\equiv .\, (\exists y) \,.\, x \neq y \,.\, R = x \downarrow y \,.\equiv .\, R \in x \downarrow{}``-\iota'x$

Dem.

$\vdash . *56{\cdot}11 . *11{\cdot}45 . \supset \vdash :. R \in 2_r \,.\, x \in \mathrm{D}'R \,.\equiv : (\exists y, z) \,.\, y \neq z \,.\, R = y \downarrow z \,.\, x \in \mathrm{D}'R :$

$[*55{\cdot}15] \qquad \equiv : (\exists y, z) \,.\, y \neq z \,.\, R = y \downarrow z \,.\, x \in \iota' y :$

$[*51{\cdot}23] \qquad \equiv : (\exists y, z) \,.\, y \neq z \,.\, R = y \downarrow z \,.\, x = y :$

$[*13{\cdot}195] \qquad \equiv : (\exists z) \,.\, x \neq z \,.\, R = x \downarrow z : \qquad (1)$

$[*51{\cdot}15] \qquad \equiv : (\exists z) \,.\, z \in -\iota'x \,.\, R = x \downarrow z :$

$[*38{\cdot}13] \qquad \equiv : R \in x \downarrow{}``-\iota'x \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

$*56{\cdot}191$. $\vdash : R \in 2_r \,.\, x \in \text{Ɑ}'R \,.\equiv .\, (\exists y) \,.\, x \neq y \,.\, R = y \downarrow x \,.\equiv .\, R \in {\downarrow} x``-\iota'x$

[Proof as in $*56{\cdot}19$]

$*56{\cdot}2$. $\vdash :. R \in \dot{2} \,.\equiv : (\exists x, y) : zRw \,.\equiv_{z,w} .\, z = x \,.\, w = y \qquad [*55{\cdot}13 . *56{\cdot}1]$

$*56{\cdot}21$. $\vdash :. R \in \dot{2} \,.\equiv : \dot{\exists} \,!\, R : xRy \,.\, zRw \,.\supset_{x,y,z,w} .\, x = z \,.\, y = w \qquad [*56{\cdot}2 . *14{\cdot}124]$

$*56{\cdot}22$. $\vdash . \dot{\Lambda} \sim\in \dot{2} \qquad [*56{\cdot}103 . *25{\cdot}53]$

$*56{\cdot}24$. $\vdash . \exists \,!\, \dot{2} \,.\, \exists \,!\, -\dot{2} \qquad [*56{\cdot}22{\cdot}16 . *10{\cdot}24]$

$*56{\cdot}25$. $\vdash . \dot{2} \neq \Lambda \cap \mathrm{Rel} \,.\, \dot{2} \neq V \cap \mathrm{Rel} \qquad [*56{\cdot}24 . *24{\cdot}54{\cdot}17]$

$*56{\cdot}26$. $\vdash :. R \in \dot{2} \cup \iota'\dot{\Lambda} \,.\equiv : xRy \,.\, zRw \,.\supset_{x,y,z,w} .\, x = z \,.\, y = w$

This proposition is the analogue of $*52{\cdot}4$.

Dem.

$\vdash . *51{\cdot}236 . \supset \vdash :: R \in \dot{2} \cup \iota'\dot{\Lambda} \,.$

$\qquad \equiv :. R \in \dot{2} \,.\vee .\, R = \dot{\Lambda} :.$

$[*25{\cdot}51] \qquad \equiv :. R \in \dot{2} \,.\vee .\, \sim\dot{\exists} \,!\, R :.$

$[*56{\cdot}21] \qquad \equiv :. \dot{\exists} \,!\, R : xRy \,.\, zRw \,.\supset_{x,y,z,w} .\, x = z \,.\, y = w :.\vee:. \sim\dot{\exists} \,!\, R :.$

$[*5{\cdot}62] \qquad \equiv :. xRy \,.\, zRw \,.\supset_{x,y,z,w} .\, x = z \,.\, y = w \,.\vee .\, \sim\dot{\exists} \,!\, R \qquad (1)$

$\vdash . *11{\cdot}36 . \mathrm{Transp} . \supset \vdash :. \sim\dot{\exists} \,!\, R \,.\supset : \sim(xRy) \,.\, \sim(zRw) :$

$[*2{\cdot}21] \qquad \supset : xRy \,.\supset .\, x = z : zRw \,.\supset .\, y = w :$

$[*3{\cdot}47] \qquad \supset : xRy \,.\, zRw \,.\supset .\, x = z \,.\, y = w \qquad (2)$

$\vdash . (2) . *11{\cdot}11{\cdot}3 . \quad \supset \vdash :. \sim\dot{\exists} \,!\, R \,.\supset : xRy \,.\, zRw \,.\supset_{x,y,z,w} .\, x = z \,.\, y = w \quad (3)$

$\vdash . (1) . (3) . *4{\cdot}72 . \quad \supset \vdash . \mathrm{Prop}$

$*56{\cdot}261$. $\vdash :: R \in \dot{2} \,.\supset :. S \mathrel{\dot{\subset}} R \,.\equiv : S = \dot{\Lambda} \,.\vee .\, S = R$

Dem.

$\vdash . *55{\cdot}341 . \supset \vdash :: R = x \downarrow y \supset :. S \mathrel{\dot{\subset}} R \,.\equiv : S = \dot{\Lambda} \,.\vee .\, S = R \qquad (1)$

$\vdash . (1) . *11{\cdot}11{\cdot}35 . *56{\cdot}1 . \supset \vdash . \mathrm{Prop}$

$*56{\cdot}262$. $\vdash :. R \in \dot{2} \,.\supset : S \mathrel{\dot{\subset}} R \,.\, \dot{\exists} \,!\, S \,.\equiv .\, S = R$

Dem.

$\vdash . *56{\cdot}22 . \supset \vdash :. R \in \dot{2} \,.\supset : S = R \,.\supset .\, S \neq \dot{\Lambda} \qquad (1)$

$\vdash . (1) . *5{\cdot}75 . *56{\cdot}261 . \supset$

$\vdash :. R \in \dot{2} \,.\supset : S \mathrel{\dot{\subset}} R \,.\, S \neq \dot{\Lambda} \,.\equiv .\, S = R \qquad (2)$

$\vdash . (2) . *25{\cdot}54 . \supset \vdash . \mathrm{Prop}$

***56·27.** $\vdash :. R \in \dot{2} . \supset : \dot{\exists} ! R \dot{\cap} S . \equiv . R \dot{\cap} S \in \dot{2}$

Dem.

$\vdash . *55·34 . *23·43 . \supset$

$\vdash :. R = x \downarrow y . \supset : \dot{\exists} ! R \dot{\cap} S . \equiv . R \dot{\cap} S = R .$

$[*56·16] \qquad \supset . R \dot{\cap} S \in \dot{2}$ (1)

$\vdash . *56·103 . \supset \vdash : R \dot{\cap} S \in \dot{2} . \supset . \dot{\exists} ! R \dot{\cap} S$ (2)

$\vdash . (1) . (2) . \supset \vdash :. R = x \downarrow y . \supset : \dot{\exists} ! R \dot{\cap} S . \equiv . R \dot{\cap} S \in \dot{2}$ (3)

$\vdash . (3) . *11·11·35 . *56·1 . \supset \vdash . \mathrm{Prop}$

***56·28.** $\vdash :. R \in \dot{2} . \supset : \dot{\exists} ! R \dot{\cap} S . \equiv . R \mathrel{\dot{\subset}} S . \equiv . R \dot{\cap} S = R$

Dem.

$\vdash . *55·3 . \supset \vdash :. R = x \downarrow y . \supset : \dot{\exists} ! R \dot{\cap} S . \equiv . R \mathrel{\dot{\subset}} S .$ (1)

$[*23·621] \qquad \equiv . R \dot{\cap} S = R$ (2)

$\vdash . (1) . (2) . *11·11·35 . *56·1 . \supset \vdash . \mathrm{Prop}$

***56·281.** $\vdash :. R \in 2_r . \supset : \dot{\exists} ! R \dot{\cap} S . \equiv . R \mathrel{\dot{\subset}} S . \equiv . R \dot{\cap} S = R . \equiv . R \dot{\cap} S \in 2_r$

Dem.

$\vdash . *56·121 . \supset \vdash :. \mathrm{Hp} . \supset : R \in \dot{2} :$

$[*56·28] \qquad \supset : \dot{\exists} ! R \dot{\cap} S . \equiv . R \mathrel{\dot{\subset}} S . \equiv . R \dot{\cap} S = R$ (1)

$\vdash . *13·13 . \supset \vdash :. \mathrm{Hp} . \supset : R \dot{\cap} S = R . \supset . R \dot{\cap} S \in 2_r :$

$[(1)] \qquad \supset : \dot{\exists} ! R \dot{\cap} S . \supset . R \dot{\cap} S \in 2_r$ (2)

$\vdash . *56·122 . \supset \vdash : R \dot{\cap} S \in 2_r . \supset . \dot{\exists} ! R \dot{\cap} S$ (3)

$\vdash . (2) . (3) . \supset \vdash :. \mathrm{Hp} . \supset : \dot{\exists} ! R \dot{\cap} S . \equiv . R \dot{\cap} S \in 2_r$ (4)

$\vdash . (1) . (4) . \supset \vdash . \mathrm{Prop}$

***56·29.** $\vdash :: P, Q \in \dot{2} . \supset :. P \mathrel{\dot{\subset}} Q \dot{\cup} R . \equiv : P = Q . \vee . P \mathrel{\dot{\subset}} R$

Dem.

$\vdash . *55·51 . \supset$

$\vdash :. x \downarrow y \mathrel{\dot{\subset}} z \downarrow w \dot{\cup} R . \supset : x (z \downarrow w) y . \vee . x \downarrow y \mathrel{\dot{\subset}} R :$

$[*55·31] \qquad \supset : x \downarrow y = z \downarrow w . \vee . x \downarrow y \mathrel{\dot{\subset}} R$ (1)

$\vdash . (1) . *13·12 . \supset$

$\vdash ::. P = x \downarrow y . \supset :: Q = z \downarrow w . \supset :. P \mathrel{\dot{\subset}} Q \dot{\cup} R . \supset : P = Q . \vee . P \mathrel{\dot{\subset}} R$ (2)

$\vdash . (2) . *11·11·35 . *56·1 . \supset$

$\vdash ::. P \in \dot{2} . \supset :: Q = z \downarrow w . \supset :. P \mathrel{\dot{\subset}} Q \dot{\cup} R . \supset : P = Q . \vee . P \mathrel{\dot{\subset}} R$ (3)

$\vdash . (3) . *11·11·3·35 . *56·1 . \supset$

$\vdash ::. P \in \dot{2} . \supset :: Q \in \dot{2} . \supset :. P \mathrel{\dot{\subset}} Q \dot{\cup} R . \supset : P = Q . \vee . P \mathrel{\dot{\subset}} R$ (4)

$\vdash . *23·58·61 . \supset \vdash :. P = Q . \vee . P \mathrel{\dot{\subset}} R : \supset . P \mathrel{\dot{\subset}} Q \dot{\cup} R$ (5)

$\vdash . (4) . \mathrm{Imp} . (5) . \supset \vdash . \mathrm{Prop}$

***56·3.** $\vdash :. P, Q \in \dot{2} . \supset : P \mathrel{\dot{\subset}} Q . \equiv . P = Q . \equiv . \dot{\exists} ! P \dot{\cap} Q$

Dem.

$\vdash . *55·3·31 . \supset$

$\vdash : x \downarrow y \mathrel{\dot{\subset}} z \downarrow w . \equiv . x \downarrow y = z \downarrow w . \equiv . \dot{\exists} ! (x \downarrow y) \dot{\cap} (z \downarrow w)$ (1)

$\vdash . (1) . *13{\cdot}12 . \supset$

$\vdash :. P = x \downarrow y . Q = z \downarrow w . \supset : P \subseteq Q . \equiv . P = Q . \equiv . \dot{\exists} ! P \dot{\cap} Q$ (2)

$\vdash . (2) . *11{\cdot}11{\cdot}35 . *56{\cdot}1 . \supset \vdash . \text{Prop}$

The steps from (2) to the conclusion are analogous to those from (2) of $*56{\cdot}29$ to the conclusion of $*56{\cdot}29$. Analogous steps in succeeding proofs will be merely indicated as above.

$*56{\cdot}31$. $\vdash :. P, Q \in \dot{2} . \supset : P \neq Q . \equiv . P \dot{\cap} Q = \dot{\Lambda}$ [$*56{\cdot}3$. Transp]

$*56{\cdot}32$. $\vdash : P \in \dot{2} . \supset . P \dot{\cap} Q \in \dot{2} \cup \iota`\dot{\Lambda}$

Dem.

$\vdash . *56{\cdot}27 . \supset \vdash :. \text{Hp} . \supset : \dot{\exists} ! P \dot{\cap} Q . \supset . P \dot{\cap} Q \in \dot{2} :$

[$*2{\cdot}54 . *25{\cdot}54$] $\supset : P \dot{\cap} Q = \dot{\Lambda} . \mathbf{v} . P \dot{\cap} Q \in \dot{2} :$

[$*51{\cdot}236$] $\supset : P \dot{\cap} Q \in \dot{2} \cup \iota`\dot{\Lambda} :. \supset \vdash . \text{Prop}$

$*56{\cdot}33$. $\vdash :: P, Q \in \dot{2} . \supset :. R \subseteq P \cup Q . \equiv : R = \dot{\Lambda} . \mathbf{v} . R = P . \mathbf{v} . R = Q . \mathbf{v} . R = P \cup Q$

Dem.

$\vdash . *55{\cdot}5 . *13{\cdot}12 . \supset \vdash :: P = x \downarrow y . Q = z \downarrow w . \supset :.$

$R \subseteq P \cup Q . \equiv : R = \dot{\Lambda} . \mathbf{v} . R = P . \mathbf{v} . R = Q . \mathbf{v} . R = P \cup Q$ (1)

$\vdash . (1) . *11{\cdot}11{\cdot}35 . *56{\cdot}1 . \supset \vdash . \text{Prop}$

$*56{\cdot}34$. $\vdash :: P, Q \in \dot{2} . P \neq Q . \supset :. R \subseteq P \cup Q . \dot{\exists} ! R . R \neq P \cup Q . \equiv : R = P . \mathbf{v} . R = Q$

Dem.

$\vdash . *56{\cdot}33{\cdot}103 . *5{\cdot}75 . *25{\cdot}54 . \supset$

$\vdash :: P, Q \in \dot{2} . \supset :. R \subseteq P \cup Q . \dot{\exists} ! R . \equiv : R = P . \mathbf{v} . R = Q . \mathbf{v} . R = P \cup Q$ (1)

$\vdash . *23{\cdot}62 . \quad \supset \vdash : P = P \cup Q . \equiv . Q \subseteq P :$

[$*56{\cdot}3$] $\supset \vdash :. P, Q \in \dot{2} . \supset : P = P \cup Q . \equiv . P = Q :$

[Transp] $\supset : P \neq Q . \supset . P \neq P \cup Q :.$

[$*13{\cdot}181$] $\supset \vdash :. P, Q \in \dot{2} . P \neq Q . \supset : R = P . \supset . R \neq P \cup Q$ (2)

$\vdash . (2) \dfrac{Q, P}{P, Q} . \supset \vdash :. P, Q \in \dot{2} . P \neq Q . \supset : R = Q . \supset . R \neq P \cup Q$ (3)

$\vdash . (2) . (3) . \supset \vdash :: P, Q \in \dot{2} . P \neq Q . \supset :. R = P . \mathbf{v} . R = Q : \supset . R \neq P \cup Q$ (4)

$\vdash . (1) . (4) . *5{\cdot}75 . \supset \vdash . \text{Prop}$

$*56{\cdot}35$. $\vdash : C`R \in 2 . R \dot{\cap} \breve{R} = \dot{\Lambda} . \supset . R \in 2_r$

Dem.

$\vdash . *55{\cdot}54 . \supset$

$\vdash :. x \neq y . C`R = \iota`x \cup \iota`y . R \dot{\cap} \breve{R} = \dot{\Lambda} . \supset : R = x \downarrow y . \mathbf{v} . R = y \downarrow x :$

[$*56{\cdot}17$] $\supset : R \in 2_r$ (1)

$\vdash . (1) . *11{\cdot}11{\cdot}35 . *54{\cdot}101 . \supset \vdash . \text{Prop}$

***56·36.** $\vdash : R \epsilon 2_r . \supset . C\text{‘}R \epsilon 2 . R \dot{\wedge} \breve{R} = \dot{\Lambda}$

Dem.

$\vdash . *55\cdot54 . \supset$

$\vdash : x \neq y . R = x \downarrow y . \supset . x \neq y . C\text{‘}R = \iota\text{‘}x \cup \iota\text{‘}y . R \dot{\wedge} \breve{R} = \dot{\Lambda}$ (1)

$\vdash . (1) . *11\cdot11\cdot34 . *56\cdot11 . \supset$

$\vdash :. R \epsilon 2_r . \supset : (\exists x, y) . x \neq y . C\text{‘}R = \iota\text{‘}x \cup \iota\text{‘}y . R \dot{\wedge} \breve{R} = \dot{\Lambda} :$

$[*54\cdot101 . *11\cdot45] \supset : C\text{‘}R \epsilon 2 . R \dot{\wedge} \breve{R} = \dot{\Lambda} :. \supset \vdash .$ Prop

The following proposition, in addition to being used in *56·38, is used in the elementary theory of series (*204·463).

***56·37.** $\vdash : R \epsilon 2_r . \equiv . C\text{‘}R \epsilon 2 . R \dot{\wedge} \breve{R} = \dot{\Lambda}$ [*56·35·36]

***56·38.** $\vdash . 2_r = \breve{C}\text{“}2 \cap \hat{R}(R \dot{\wedge} \breve{R} = \dot{\Lambda})$

Dem.

$\vdash . *37\cdot106 . *33\cdot122 . \supset \vdash : C\text{‘}R \epsilon 2 . \equiv . R \epsilon \breve{C}\text{“}2$ (1)

$\vdash . *20\cdot3 . \supset \vdash : R \dot{\wedge} \breve{R} = \dot{\Lambda} . \equiv . R \epsilon \hat{R}(R \dot{\wedge} \breve{R} = \dot{\Lambda})$ (2)

$\vdash . (1) . (2) . *56\cdot37 . \supset \vdash : R \epsilon 2_r . \equiv . R \epsilon \breve{C}\text{“}2 . R \epsilon \hat{R}(R \dot{\wedge} \breve{R} = \dot{\Lambda}) .$

$[*22\cdot33] \equiv . R \epsilon \breve{C}\text{“}2 \cap \hat{R}(R \dot{\wedge} \breve{R} = \dot{\Lambda}) : \supset \vdash .$ Prop

This proposition is important as establishing the connection between the cardinal and ordinal 2. It shows that the ordinal 2 consists of those asymmetrical relations whose fields have (cardinal) 2 terms. It is used in the theory of well-ordered series (*250·44).

The following proposition, in addition to being used in *56·39, is used in relation-arithmetic (*165·38) and in the theory of series (*205·4).

***56·381.** $\vdash : C\text{‘}R = \iota\text{‘}x . \equiv . R = x \downarrow x$

Dem.

$\vdash . *33\cdot24\cdot161 . *51\cdot161 . \supset \vdash : C\text{‘}R = \iota\text{‘}x . \supset . \exists ! \text{D}\text{‘}R . \text{D}\text{‘}R \subset \iota\text{‘}x .$

$[*51\cdot4] \supset . \text{D}\text{‘}R = \iota\text{‘}x$ (1)

Similarly $\vdash : C\text{‘}R = \iota\text{‘}x . \supset . \text{Ꙅ}\text{‘}R = \iota\text{‘}x$ (2)

$\vdash . (1) . (2) . *55\cdot16 . \supset \vdash : C\text{‘}R = \iota\text{‘}x . \supset . R = x \downarrow x$ (3)

$\vdash . *55\cdot15 . \supset \vdash : R = x \downarrow x . \supset . C\text{‘}R = \iota\text{‘}x$ (4)

$\vdash . (3) . (4) . \supset \vdash .$ Prop

***56·39.** $\vdash . \dot{2} - 2_r = \breve{C}\text{“}1$

Dem.

$\vdash . *56\cdot381 . \supset \vdash : C\text{‘}R \epsilon 1 . \equiv . (\exists x) . R = x \downarrow x .$

$[*56\cdot13] \equiv . R \epsilon \dot{2} - 2_r$ (1)

$\vdash . (1) . *37\cdot106 . \supset \vdash .$ Prop

This proposition establishes the connection between $\dot{2} - 2_r$ and 1, showing that $\dot{2} - 2_r$ is the class of those relations whose fields consist of a single term. It is used in the discussion of 0_r and 2_r and $\dot{2} - 2_r$ as relation-numbers (*153·301).

***56·4.** $\vdash :. \mu \subset \dot{2} . \supset : x \downarrow y \in \mu . \equiv . x (\dot{s}`\mu) y$

Dem.

$\vdash . *41{\cdot}11 . \supset \vdash :. \text{Hp} . \supset : x (\dot{s}`\mu) y . \equiv . (\exists R) . R \in \dot{2} . R \in \mu . xRy .$

[*56·1] $\equiv . (\exists z, w) . z \downarrow w \in \mu . x (z \downarrow w) y .$

[*55·13] $\equiv . (\exists z, w) . z \downarrow w \in \mu . z = x . w = y .$

[*13·22] $\equiv . x \downarrow y \in \mu :. \supset \vdash . \text{Prop}$

This proposition is the analogue of *53·23. It is used in the number on exponentiation in relation-arithmetic (*176·19).

SECTION B

SUB-CLASSES, SUB-RELATIONS, AND RELATIVE TYPES

Summary of Section B.

In this section, we consider first the classes contained in a given class and the relations contained in a given relation. If α is any class, the classes contained in α are the members of $\hat{\beta}(\beta \subset \alpha)$; these are also called the sub-classes of α, or (sometimes) the "parts" of α. In this last usage, they are called "proper parts" when they are not coextensive with α, this phrase being formed on the analogy of "proper fractions." The sub-classes of α are all the classes that can be formed from members of α; they are the same thing as the "combinations" of members of α taken any number at a time. If n is the number of members of α, 2^n is the number of sub-classes of α, whether n be finite or infinite. The number of sub-classes of α is always greater than the number of members of α. On account of these and other propositions, the class of sub-classes of a given class is an important function of the class. If the class is α, we denote the class of its sub-classes by "$\text{Cl}`\alpha$." This is a descriptive function, derived from the relation "Cl," defined as follows:

$$\text{Cl} = \hat{\kappa}\hat{\alpha}\{\kappa = \hat{\beta}(\beta \subset \alpha)\} \quad \text{Df.}$$

The sub-relations of a given relation are all the relations contained in the given relation, *i.e.* all relations which imply the given relation for all possible arguments. That is, if P is the given relation, R is a sub-relation of P if $R \mathrel{\subset\!\!\!\cdot} P$. Thus denoting the class of sub-relations of P by "$\text{Rl}`P$," we are to have

$$\text{Rl}`P = \hat{R}(R \mathrel{\subset\!\!\!\cdot} P);$$

hence we take as the definition of "Rl" the following:

$$\text{Rl} = \hat{\lambda}\hat{P}\{\lambda = \hat{R}(R \mathrel{\subset\!\!\!\cdot} P)\} \quad \text{Df.}$$

Sub-relations have properties analogous to those of sub-classes, but they are of somewhat less importance. It should, however, be observed that when one series is contained in another, *i.e.* is obtained by selecting some of the terms of the other series without changing their order, then the generating relation of the one series is a sub-relation of the generating relation of the other series. (It is not the case that a sub-relation of the generating relation of a series must generate a contained series, for its field may fall apart into detached portions, or otherwise fail of being serial.)

We shall also consider in this section (∗62) the relation of membership of a class, *i.e.* the relation which x has to α when $x \epsilon \alpha$. This relation bears the same relation to "$x \epsilon \alpha$" as "I" bears to "$x = y$." Strictly speaking, we ought to introduce a new notation for it, putting (say)

$$A = \hat{x}\hat{\alpha}(x \epsilon \alpha) \quad \text{Df.}$$

But as ϵ, unlike "$=$," is a letter, and capable of being conveniently used alone, it seems more desirable, from the point of view of avoiding unnecessary duplication of symbols, to put

$$\epsilon = \hat{x}\hat{\alpha}(x \epsilon \alpha) \quad \text{Df.}$$

Strictly speaking, this definition is faulty, since it gives two different meanings to "ϵ." But practically this does not matter, since the above definition gives

$$\vdash : x \epsilon \alpha \,.\equiv.\, x \epsilon \alpha,$$

where the first ϵ has the meaning just defined, while the second has the old meaning. Thus all that is really required of the above definition, namely to give a meaning to formulae in which ϵ occurs without referent or relatum, is effected without the danger of any confusion that could lead to errors.

The chief importance of ϵ as a relation arises from the fact that relations contained in ϵ play a very important part in arithmetic. Take, for example, the problem of selecting one term out of each member of a class of classes: in this case we require a selecting relation R which is such that whenever $xR\alpha$, x is a member of α, *i.e.* such that $R \subseteq \epsilon$. (This condition is only part of the definition of a selecting relation; the complete definition is given in ∗80.)

Three numbers in this section (∗63, ∗64, ∗65) are devoted to the discussion of *relative types*. Given a variable x, we often want to define the relative types of other variables, or of ambiguous symbols, occurring in the same context; that is, we wish to express the types of these other symbols in terms of that of x. We use "$t\text{‘}x$" for the type of x, "$t_0\text{‘}\alpha$" for the type in which α is contained. Then $t_0\text{‘}\alpha = \alpha \cup -\alpha$, $t\text{‘}x = \iota\text{‘}x \cup -\iota\text{‘}x = t_0\text{‘}\iota\text{‘}x$, and $t\text{‘}\alpha = t_0\text{‘}\mathrm{Cl}\text{‘}\alpha = \mathrm{Cl}\text{‘}t_0\text{‘}\alpha$. Also we introduce a notation (∗65) for giving typical definiteness, relatively to x, to typically ambiguous symbols. This notation is very useful in cardinal and ordinal arithmetic, since numbers are typically ambiguous, and the failure to take account of this fact has led to the contradictions concerning the greatest cardinal and the greatest ordinal.

$*60$. THE SUB-CLASSES OF A GIVEN CLASS

Summary of $*60$.

Our definitions in this number are as follows:

$*60{\cdot}01$. $\mathrm{Cl} = \hat{\kappa}\hat{\alpha}\{\kappa = \hat{\beta}(\beta \subset \alpha)\}$ Df

This defines the relation to a class α of the class of all its sub-classes.

$*60{\cdot}02$. $\mathrm{Cl\,ex} = \hat{\kappa}\hat{\alpha}\{\kappa = \hat{\beta}(\beta \subset \alpha \,.\, \exists ! \beta)\}$ Df

This defines the relation to a class α of the class of all its *existent* sub-classes, *i.e.* of all its sub-classes except Λ. This is often required, as, for example, in the statement of Zermelo's axiom: "Given any class α, there is a relation R such that, if β is any existent sub-class of α, $R`\beta$ is a member of β," *i.e.*

$$\text{“}(\exists R) : \beta \in \mathrm{Cl\,ex}`\alpha \,.\, \supset_\beta \,.\, R`\beta \in \beta.\text{”}$$

This axiom, or its equivalent the multiplicative axiom, plays (as will appear hereafter) an important part as the hypothesis to many propositions in cardinal arithmetic.

$*60{\cdot}03$. $\mathrm{Cls}^2 = \mathrm{Cl}`\mathrm{Cls}$ Df

A Cls^2 is a class whose members are classes.

$*60{\cdot}04$. $\mathrm{Cls}^3 = \mathrm{Cl}`\mathrm{Cls}^2$ Df

A Cls^3 is a class whose members are classes whose members are classes, *i.e.* a Cls^3 is a class of classes of classes.

Apart from propositions which merely embody the definitions, the most useful propositions in this number are the following:

$*60{\cdot}3$. $\vdash . \Lambda \in \mathrm{Cl}`\alpha$

$*60{\cdot}32$. $\vdash . \mathrm{Cl}`\Lambda = \iota`\Lambda$

$*60{\cdot}34$. $\vdash . \alpha \in \mathrm{Cl}`\alpha$

$*60{\cdot}362$. $\vdash . \mathrm{Cl}`\iota`x = \iota`\Lambda \cup \iota`\iota`x$

I.e. Λ and $\iota`x$ are the only sub-classes of a unit class $\iota`x$.

$*60{\cdot}5$. $\vdash . s`\mathrm{Cl}`\alpha = \alpha$

$*60{\cdot}57$. $\vdash . \kappa \subset \mathrm{Cl}`s`\kappa$

$*60{\cdot}6$. $\vdash : x \in \alpha \,.\, \supset . \, \iota`x \in \mathrm{Cl\,ex}`\alpha$

The propositions of this number are chiefly useful in cardinal and ordinal arithmetic, but uses also occur in the theory of series; hardly any uses occur before cardinal arithmetic.

***60·01.** $\mathrm{Cl} = \hat{\kappa}\hat{\alpha}\{\kappa = \hat{\beta}(\beta \subset \alpha)\}$ Df

***60·02.** $\mathrm{Cl\,ex} = \hat{\kappa}\hat{\alpha}\{\kappa = \hat{\beta}(\beta \subset \alpha \,.\, \exists ! \beta)\}$ Df

***60·03.** $\mathrm{Cls}^2 = \mathrm{Cl}`\mathrm{Cls}$ Df

***60·04.** $\mathrm{Cls}^3 = \mathrm{Cl}`\mathrm{Cls}^2$ Df

***60·1.** $\vdash : \kappa \,\mathrm{Cl}\, \alpha \,.\equiv .\, \kappa = \hat{\beta}(\beta \subset \alpha)$ [*21·3 . (*60·01)]

***60·11.** $\vdash : \kappa \,\mathrm{Cl\,ex}\, \alpha \,.\equiv .\, \kappa = \hat{\beta}(\beta \subset \alpha \,.\, \exists ! \beta)$ [*21·3 . (*60·02)]

***60·12.** $\vdash .\, \mathrm{Cl}`\alpha = \hat{\beta}(\beta \subset \alpha)$ [*30·3 . *60·1]

***60·13.** $\vdash .\, \mathrm{Cl\,ex}`\alpha = \hat{\beta}(\beta \subset \alpha \,.\, \exists ! \beta)$ [*30·3 . *60·11]

***60·14.** $\vdash .\, \mathrm{E} ! \,\mathrm{Cl}`\alpha$ [*60·12 . *14·21]

***60·15.** $\vdash .\, \mathrm{E} ! \,\mathrm{Cl\,ex}`\alpha$ [*60·13 . *14·21]

***60·2.** $\vdash : \beta \in \mathrm{Cl}`\alpha \,.\equiv .\, \beta \subset \alpha$ [*60·12 . *20·33]

***60·21.** $\vdash : \beta \in \mathrm{Cl\,ex}`\alpha \,.\equiv .\, \beta \subset \alpha \,.\, \exists ! \beta$ [*60·13 . *20·33]

***60·22.** $\vdash : \beta \in \mathrm{Cl\,ex}`\alpha \,.\equiv .\, \beta \in \mathrm{Cl}`\alpha \,.\, \exists ! \beta$ [*60·2·21]

***60·23.** $\vdash : \beta \in \mathrm{Cl\,ex}`\alpha \,.\equiv .\, \beta \in \mathrm{Cl}`\alpha - \iota`\Lambda$ [*60·22 . *53·52]

***60·24.** $\vdash .\, \mathrm{Cl\,ex}`\alpha = \mathrm{Cl}`\alpha - \iota`\Lambda$ [*60·23 . *20·43]

***60·3.** $\vdash .\, \Lambda \in \mathrm{Cl}`\alpha$ [*24·12 . *60·2]

***60·31.** $\vdash .\, \exists ! \,\mathrm{Cl}`\alpha$ [*60·3 . *10·24]

***60·32.** $\vdash .\, \mathrm{Cl}`\Lambda = \iota`\Lambda$

Dem.

$\vdash .\, *60·2 \,.\, *24·13 \,.\supset \vdash : \alpha \in \mathrm{Cl}`\Lambda \,.\equiv .\, \alpha = \Lambda \,.$

[*51·15] $\equiv .\, \alpha \in \iota`\Lambda : \supset \vdash .\, \mathrm{Prop}$

***60·321.** $\vdash : \alpha = \Lambda \,.\equiv .\, \mathrm{Cl}`\alpha = \iota`\alpha$

Dem.

$\vdash .\, *60·32 \,.\supset \vdash : \alpha = \Lambda \,.\supset .\, \mathrm{Cl}`\alpha = \iota`\alpha$ (1)

$\vdash .\, *60·2 \,.\, *51·15 \,.\supset$

$\vdash :.\, \mathrm{Cl}`\alpha = \iota`\alpha \,.\equiv : \beta \subset \alpha \,.\equiv_\beta .\, \beta = \alpha :$

[*10·1] $\supset : \Lambda \subset \alpha \,.\equiv .\, \Lambda = \alpha :$

[*24·12] $\supset : \Lambda = \alpha$ (2)

$\vdash .\, (1) \,.\, (2) \,.\supset \vdash .\, \mathrm{Prop}$

***60·33.** $\vdash .\, \mathrm{Cl\,ex}`\Lambda = \Lambda \cap \mathrm{Cls}$

We write "$\Lambda \cap \mathrm{Cls}$" on the right, to indicate that the Λ concerned is of higher type than the Λ on the left.

Dem.

$\vdash .\, *60·22·32 \,.\supset \vdash : \beta \in \mathrm{Cl\,ex}`\Lambda \,.\equiv .\, \beta \in \iota`\Lambda \,.\, \exists ! \beta \,.$

[*51·15.*24·54] $\equiv .\, \beta = \Lambda \,.\, \beta \neq \Lambda$ (1)

$\vdash .\, (1) \,.\, *3·24 \,.\supset \vdash .\, \beta \sim\in \mathrm{Cl\,ex}`\Lambda$ (2)

$\vdash .\, (2) \,.\, *10·11 \,.\, *24·15 \,.\supset \vdash .\, \mathrm{Prop}$

∗60·34. $\vdash . \alpha \epsilon \mathrm{Cl}`\alpha$ [∗22·42 . ∗60·2]

∗60·35. $\vdash : \exists ! \alpha . \supset . \alpha \epsilon \mathrm{Cl\,ex}`\alpha$ [∗60·22·34]

∗60·36. $\vdash : \exists ! \alpha . \supset . \exists ! \mathrm{Cl\,ex}`\alpha$ [∗60·35 . ∗10·24]

∗60·361. $\vdash : \exists ! \alpha . \equiv . \exists ! \mathrm{Cl\,ex}`\alpha$ [∗60·36·33]

∗60·362. $\vdash . \mathrm{Cl}`\iota`x = \iota`\Lambda \cup \iota`\iota`x$ [∗51·401 . ∗60·2]

∗60·37. $\vdash . \mathrm{Cl\,ex}`\iota`x = \iota`\iota`x$

Dem.

$\vdash . ∗60·21 . \supset \vdash : \beta \epsilon \mathrm{Cl\,ex}`\iota`x . \equiv . \beta \subset \iota`x . \exists ! \beta .$

[∗51·4] $\equiv . \beta = \iota`x .$

[∗51·15] $\equiv . \beta \epsilon \iota`\iota`x : \supset \vdash . \mathrm{Prop}$

∗60·371. $\vdash : \alpha \epsilon 1 . \supset . \mathrm{Cl}`\alpha \subset 0 \cup 1$

Dem.

$\vdash . ∗51·401 . \supset \vdash :: \alpha = \iota`x . \supset :. \beta \subset \alpha . \equiv : \beta = \Lambda . \vee . \beta = \iota`x :$

[∗54·102.∗52·22] $\supset : \beta \epsilon 0 . \vee . \beta \epsilon 1 :.$

[∗60·2.∗22·34] $\supset :. \beta \epsilon \mathrm{Cl}`\alpha . \supset . \beta \epsilon 0 \cup 1$ (1)

$\vdash . (1) . ∗10·11·23 . ∗52·1 . \supset \vdash . \mathrm{Prop}$

∗60·38. $\vdash : \alpha \epsilon 1 . \equiv . \mathrm{Cl\,ex}`\alpha = \iota`\alpha$

Dem.

$\vdash . ∗60·37 .$ $\supset \vdash : \alpha = \iota`x . \supset . \mathrm{Cl\,ex}`\alpha = \iota`\alpha :$

[∗10·11·23] $\supset \vdash : (\exists x) . \alpha = \iota`x . \supset . \mathrm{Cl\,ex}`\alpha = \iota`\alpha :$

[∗52·1] $\supset \vdash : \alpha \epsilon 1 . \supset . \mathrm{Cl\,ex}`\alpha = \iota`\alpha$ (1)

$\vdash . ∗60·361 . ∗51·161 .$ $\supset \vdash : \mathrm{Cl\,ex}`\alpha = \iota`\alpha . \supset . \exists ! \alpha$ (2)

$\vdash . ∗60·21 . ∗10·1 .$ $\supset \vdash :. \mathrm{Cl\,ex}`\alpha = \iota`\alpha . \supset : \iota`x \subset \alpha . \exists ! \iota`x . \equiv . \iota`x = \alpha :$

[∗51·161] $\supset : \iota`x \subset \alpha . \equiv . \iota`x = \alpha :$

[∗51·2] $\supset : x \epsilon \alpha . \equiv . \iota`x = \alpha$ (3)

$\vdash . (3) . ∗10·11·21·281 . \supset \vdash :. \mathrm{Cl\,ex}`\alpha = \iota`\alpha . \supset : \exists ! \alpha . \equiv . (\exists x) . \iota`x = \alpha .$

[∗52·1] $\equiv . \alpha \epsilon 1 :$

[(2)] $\supset : \alpha \epsilon 1$ (4)

$\vdash . (1) . (4) . \supset \vdash . \mathrm{Prop}$

∗60·39. $\vdash . \mathrm{Cl}`(\iota`x \cup \iota`y) = \iota`\Lambda \cup \iota`\iota`x \cup \iota`\iota`y \cup \iota`(\iota`x \cup \iota`y)$ [∗54·4 . ∗60·2]

∗60·391. $\vdash : \alpha \epsilon 2 . \supset . \mathrm{Cl}`\alpha \subset 0 \cup 1 \cup 2$ [∗54·411 . ∗60·2]

This proposition is used in the theory of the continuity of functions (∗234·202).

∗60·4. $\vdash : \beta \epsilon \mathrm{Cl}`\alpha . \gamma \subset \beta . \supset . \gamma \epsilon \mathrm{Cl}`\alpha$ [∗60·2 . ∗22·44]

∗60·41. $\vdash : \beta \epsilon \mathrm{Cl}`\alpha . \supset . \beta \cap \gamma \epsilon \mathrm{Cl}`\alpha$ [∗60·4 . ∗22·43]

The following proposition is used in the theory of well-ordered series (∗250·14).

∗60·42. $\vdash : \beta \epsilon \mathrm{Cl}`\alpha . \gamma \subset \beta . \exists ! \gamma . \supset . \gamma \epsilon \mathrm{Cl\,ex}`\alpha$ [∗60·4·22]

***60·43.** $\vdash : \beta, \gamma \epsilon \text{Cl}‘\alpha . \equiv . \beta \cup \gamma \epsilon \text{Cl}‘\alpha$ [*22·59 . *60·2]

***60·44.** $\vdash : \beta \epsilon \text{Cl}‘\alpha . \gamma \epsilon \text{Cl ex}‘\alpha . \supset . \beta \cup \gamma \epsilon \text{Cl ex}‘\alpha$ [*60·43 . *24·56 . *60·22]

The following proposition is required in the theory of "first differences" (*170·65).

***60·45.** $\vdash : \rho \epsilon \text{Cl}‘(\alpha \cup \beta) . \equiv . (\exists \gamma, \delta) . \gamma \epsilon \text{Cl}‘\alpha . \delta \epsilon \text{Cl}‘\beta . \rho = \gamma \cup \delta$

Dem.

$\vdash . *60·2 . *22·621·68 . \supset$

$\vdash : \rho \epsilon \text{Cl}‘(\alpha \cup \beta) . \supset . \rho = (\rho \cap \alpha) \cup (\rho \cap \beta)$ (1)

$\vdash . *60·2 . *22·43 . \supset \vdash . \rho \cap \alpha \epsilon \text{Cl}‘\alpha . \rho \cap \beta \epsilon \text{Cl}‘\beta$ (2)

$\vdash . (1) . (2) . *10·24 . \supset$

$\vdash : \rho \epsilon \text{Cl}‘(\alpha \cup \beta) . \supset . (\exists \gamma, \delta) . \gamma \epsilon \text{Cl}‘\alpha . \delta \epsilon \text{Cl}‘\beta . \rho = \gamma \cup \delta$ (3)

$\vdash . *60·2 . \supset$

$\vdash : (\exists \gamma, \delta) . \gamma \epsilon \text{Cl}‘\alpha . \delta \epsilon \text{Cl}‘\beta . \rho = \gamma \cup \delta . \supset . (\exists \gamma, \delta) . \gamma \subset \alpha . \delta \subset \beta . \rho = \gamma \cup \delta .$

[*22·72] $\supset . \rho \subset \alpha \cup \beta .$

[*60·2] $\supset . \rho \epsilon \text{Cl}‘(\alpha \cup \beta)$ (4)

$\vdash . (3) . (4) . \supset \vdash .$ Prop

***60·5.** $\vdash . s‘\text{Cl}‘\alpha = \alpha$

Dem.

$\vdash . *40·11 . *60·2 . \supset \vdash : x \epsilon s‘\text{Cl}‘\alpha . \equiv . (\exists \beta) . \beta \subset \alpha . x \epsilon \beta .$ (1)

[*22·441] $\supset . x \epsilon \alpha$ (2)

$\vdash . *22·42 . \supset \vdash : x \epsilon \alpha . \supset . \alpha \subset \alpha . x \epsilon \alpha .$

[*10·24] $\supset . (\exists \beta) . \beta \subset \alpha . x \epsilon \beta .$

[(1)] $\supset . x \epsilon s‘\text{Cl}‘\alpha$ (3)

$\vdash . (2) . (3) . \supset \vdash .$ Prop

***60·501.** $\vdash . s‘\text{Cl ex}‘\alpha = \alpha$

Dem.

$\vdash . *40·11 . *60·21 . \supset \vdash : x \epsilon s‘\text{Cl ex}‘\alpha . \equiv . (\exists \beta) . \beta \subset \alpha . \exists ! \beta . x \epsilon \beta .$ (1)

[*22·441] $\supset . x \epsilon \alpha$ (2)

$\vdash . *22·42 . \supset \vdash : x \epsilon \alpha . \supset . \alpha \subset \alpha . x \epsilon \alpha .$

[*10·24.*24·5.*4·7] $\supset . \alpha \subset \alpha . \exists ! \alpha . x \epsilon \alpha .$

[*10·24] $\supset . (\exists \beta) . \beta \subset \alpha . \exists ! \beta . x \epsilon \beta .$

[(1)] $\supset . x \epsilon s‘\text{Cl ex}‘\alpha$ (3)

$\vdash . (2) . (3) . \supset \vdash .$ Prop

The above proposition is used in the theory of cardinal multiplication (*115·17).

***60·51.** $\vdash . p‘\text{Cl}‘\alpha = \Lambda$ [*40·22 . *60·3]

The following proposition is used in the cardinal theory of finite and infinite (*124·541).

***60·52.** $\vdash : s‘\kappa \subset \beta . \equiv . \kappa \subset \text{Cl}‘\beta$ [*40·151 . *60·2]

***60·53.** $\vdash : \beta \subset p\text{‘}\kappa . \equiv . \beta \in p\text{‘}\mathrm{Cl}\text{‘‘}\kappa$

Dem.

$\vdash . *40\cdot15 . *60\cdot2 . \supset \vdash :. \beta \subset p\text{‘}\kappa . \equiv : \gamma \in \kappa . \supset_\gamma . \beta \in \mathrm{Cl}\text{‘}\gamma :$

$[*40\cdot41 . *60\cdot14] \quad \equiv : \beta \in p\text{‘}\mathrm{Cl}\text{‘‘}\kappa :. \supset \vdash . \mathrm{Prop}$

***60·54.** $\vdash . \mathrm{Cl}\text{‘}p\text{‘}\kappa = p\text{‘}\mathrm{Cl}\text{‘‘}\kappa$ [*60·53·2]

***60·55.** $\vdash : \mathrm{Cl}\text{‘}\alpha = \mathrm{Cl}\text{‘}\beta . \equiv . \alpha = \beta$

Dem.

$\vdash . *30\cdot37 . *60\cdot14 . \supset \vdash : \alpha = \beta . \supset . \mathrm{Cl}\text{‘}\alpha = \mathrm{Cl}\text{‘}\beta$ (1)

$\vdash . *30\cdot37 . \quad \supset \vdash : \mathrm{Cl}\text{‘}\alpha = \mathrm{Cl}\text{‘}\beta . \supset . s\text{‘}\mathrm{Cl}\text{‘}\alpha = s\text{‘}\mathrm{Cl}\text{‘}\beta .$

$[*60\cdot5] \quad \supset . \alpha = \beta$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***60·56.** $\vdash : \mathrm{Cl}\,\mathrm{ex}\text{‘}\alpha = \mathrm{Cl}\,\mathrm{ex}\text{‘}\beta . \equiv . \alpha = \beta$ [Proof as in *60·55]

The following proposition is used frequently.

***60·57.** $\vdash . \kappa \subset \mathrm{Cl}\text{‘}s\text{‘}\kappa$

Dem.

$\vdash . *40\cdot13 . *60\cdot2 . \quad \supset \vdash : \alpha \in \kappa . \supset . \alpha \in \mathrm{Cl}\text{‘}s\text{‘}\kappa$ (1)

$\vdash . (1) . *10\cdot11 . *22\cdot1 . \supset \vdash . \mathrm{Prop}$

***60·6.** $\vdash : x \in \alpha . \supset . \iota\text{‘}x \in \mathrm{Cl}\,\mathrm{ex}\text{‘}\alpha$ [*51·2·161 . *60·21]

The following proposition is used in connection with cardinal multiplication and with greater and less (*115·17 and *117·66).

***60·61.** $\vdash . \iota\text{‘‘}\alpha \subset \mathrm{Cl}\,\mathrm{ex}\text{‘}\alpha$ [*37·61 . *51·12 . *60·6]

***60·62.** $\vdash : x, y \in \alpha . \supset . \iota\text{‘}x \cup \iota\text{‘}y \in \mathrm{Cl}\,\mathrm{ex}\text{‘}\alpha$ [*60·6·44]

***60·7.** $\vdash . \mathrm{Cl}\text{‘}\alpha \in \mathrm{Cls}^2$

Dem.

$\vdash . *60\cdot2 . \supset \vdash : \beta \in \mathrm{Cl}\text{‘}\alpha . \equiv . \beta \subset \alpha .$

$[*22\cdot1 . *20\cdot1\cdot3] \quad \equiv . (\exists \phi, \psi) . \alpha = \hat{z}(\phi ! z) . \beta = \hat{z}(\psi ! z) . \psi ! x \supset_x \phi ! x .$

$[*10\cdot5] \quad \supset . (\exists \psi) . \beta = \hat{z}(\psi ! z) .$

$[*20\cdot4] \quad \supset . \beta \in \mathrm{Cls}$ (1)

$\vdash . (1) . *60\cdot2 . (*60\cdot03) . \supset \vdash . \mathrm{Prop}$

***60·71.** $\vdash . \mathrm{Cls}^2 = \mathrm{Cl}\text{‘}\mathrm{Cls}$ [(*60·03)]

***60·72.** $\vdash . \mathrm{Cls}^3 = \mathrm{Cl}\text{‘}\mathrm{Cls}^2$ [(*60·04)]

$*61$. THE SUB-RELATIONS OF A GIVEN RELATION

Summary of $*61$.

The propositions of this number (except that $*61\cdot371\cdot372\cdot373$ imperfectly correspond to $*60\cdot371$) are the analogues of those with the same decimal part in $*60$. Proofs are omitted, as they are exactly analogous to those in $*60$. There are very few subsequent references to the propositions of this number.

$*61\cdot01$. $\mathrm{Rl} = \hat{\lambda}\hat{P}\{\lambda = \hat{R}(R \Subset P)\}$ Df

$*61\cdot02$. $\mathrm{Rl\,ex} = \hat{\lambda}\hat{P}\{\lambda = \hat{R}(R \Subset P \,.\, \dot{\exists}\,!\,R)\}$ Df

$*61\cdot03$. $\mathrm{Rel}^2 = \mathrm{Rl}\text{‘}(\mathrm{Rel} \uparrow \mathrm{Rel})$ Df

$*61\cdot04$. $\mathrm{Rel}^3 = \mathrm{Rl}\text{‘}(\mathrm{Rel}^2 \uparrow \mathrm{Rel}^2)$ Df

$*61\cdot1$. $\vdash : \lambda \,\mathrm{Rl}\, P \,.\equiv.\, \lambda = \hat{R}(R \Subset P)$

$*61\cdot11$. $\vdash : \lambda \,\mathrm{Rl\,ex}\, P \,.\equiv.\, \lambda = \hat{R}(R \Subset P \,.\, \dot{\exists}\,!\,R)$

$*61\cdot12$. $\vdash .\, \mathrm{Rl}\text{‘}P = \hat{R}(R \Subset P)$

$*61\cdot13$. $\vdash .\, \mathrm{Rl\,ex}\text{‘}P = \hat{R}(R \Subset P \,.\, \dot{\exists}\,!\,R)$

$*61\cdot14$. $\vdash .\, \mathrm{E}\,!\,\mathrm{Rl}\text{‘}P$

$*61\cdot15$. $\vdash .\, \mathrm{E}\,!\,\mathrm{Rl\,ex}\text{‘}P$

$*61\cdot2$. $\vdash : R \in \mathrm{Rl}\text{‘}P \,.\equiv.\, R \Subset P$

$*61\cdot21$. $\vdash : R \in \mathrm{Rl\,ex}\text{‘}P \,.\equiv.\, R \Subset P \,.\, \dot{\exists}\,!\,R$

$*61\cdot22$. $\vdash : R \in \mathrm{Rl\,ex}\text{‘}P \,.\equiv.\, R \in \mathrm{Rl}\text{‘}P \,.\, \dot{\exists}\,!\,R$

$*61\cdot23$. $\vdash : R \in \mathrm{Rl\,ex}\text{‘}P \,.\equiv.\, R \in \mathrm{Rl}\text{‘}P - \iota\text{‘}\dot{\Lambda}$

$*61\cdot24$. $\vdash .\, \mathrm{Rl\,ex}\text{‘}P = \mathrm{Rl}\text{‘}P - \iota\text{‘}\dot{\Lambda}$

$*61\cdot3$. $\vdash .\, \dot{\Lambda} \in \mathrm{Rl}\text{‘}P$

$*61\cdot31$. $\vdash .\, \exists\,!\,\mathrm{Rl}\text{‘}P$

$*61\cdot32$. $\vdash .\, \mathrm{Rl}\text{‘}\dot{\Lambda} = \iota\text{‘}\dot{\Lambda}$

$*61\cdot321$. $\vdash : P = \dot{\Lambda} \,.\equiv.\, \mathrm{Rl}\text{‘}P = \iota\text{‘}P$

$*61\cdot33$. $\vdash .\, \mathrm{Rl\,ex}\text{‘}\dot{\Lambda} = \Lambda \cap \mathrm{Rel}$

$*61\cdot34$. $\vdash .\, P \in \mathrm{Rl}\text{‘}P$

$*61\cdot35$. $\vdash : \dot{\exists}\,!\,P \,.\supset.\, P \in \mathrm{Rl\,ex}\text{‘}P$

$*61\cdot36$. $\vdash : \dot{\exists}\,!\,P \,.\supset.\, \exists\,!\,\mathrm{Rl\,ex}\text{‘}P$

$*61\cdot361$. $\vdash : \dot{\exists}\,!\,P \,.\equiv.\, \exists\,!\,\mathrm{Rl\,ex}\text{‘}P$

$*61\cdot362$. $\vdash .\, \mathrm{Rl}\text{‘}(x \downarrow y) = \iota\text{‘}\dot{\Lambda} \cup \iota\text{‘}(x \downarrow y)$

$*61\cdot37$. $\vdash .\, \mathrm{Rl\,ex}\text{‘}(x \downarrow y) = \iota\text{‘}(x \downarrow y)$

$*61\cdot371$. $\vdash : R \in \dot{2} \,.\supset.\, \mathrm{Rl}\text{‘}R = \iota\text{‘}\dot{\Lambda} \cup \iota\text{‘}R$

$*61{\cdot}372.\ \vdash : R \in \dot{2} . \supset . \mathrm{Rl}{}^{\backprime}R \subset 0_r \cup \dot{2}$

$*61{\cdot}373.\ \vdash : R \in 2_r . \supset . \mathrm{Rl}{}^{\backprime}R \subset 0_r \cup 2_r$

$*61{\cdot}38.\ \vdash : R \in \dot{2} . \equiv . \mathrm{Rl\,ex}{}^{\backprime}R = \iota{}^{\backprime}R$

$*61{\cdot}39.\ \vdash . \mathrm{Rl}{}^{\backprime}(x \downarrow y \,\dot{\cup}\, z \downarrow w) = \iota{}^{\backprime}\dot{\Lambda} \cup \iota{}^{\backprime}(x \downarrow y) \cup \iota{}^{\backprime}(z \downarrow w) \cup \iota{}^{\backprime}(x \downarrow y \,\dot{\cup}\, z \downarrow w)$

$*61{\cdot}391.\ \vdash : P, Q \in \dot{2} . \supset . \mathrm{Rl}{}^{\backprime}(P \,\dot{\cup}\, Q) = \iota{}^{\backprime}\dot{\Lambda} \cup \iota{}^{\backprime}P \cup \iota{}^{\backprime}Q \cup \iota{}^{\backprime}(P \,\dot{\cup}\, Q)$

$*61{\cdot}4.\ \vdash : Q \in \mathrm{Rl}{}^{\backprime}P . R \mathrel{\dot{\subset}} Q . \supset . R \in \mathrm{Rl}{}^{\backprime}P$

$*61{\cdot}41.\ \vdash : Q \in \mathrm{Rl}{}^{\backprime}P . \supset . Q \mathbin{\dot{\cap}} R \in \mathrm{Rl}{}^{\backprime}P$

$*61{\cdot}42.\ \vdash : Q \in \mathrm{Rl}{}^{\backprime}P . R \mathrel{\dot{\subset}} Q . \dot{\exists} ! R . \supset . R \in \mathrm{Rl\,ex}{}^{\backprime}P$

$*61{\cdot}43.\ \vdash : Q, R \in \mathrm{Rl}{}^{\backprime}P . \equiv . Q \,\dot{\cup}\, R \in \mathrm{Rl}{}^{\backprime}P$

$*61{\cdot}44.\ \vdash : Q \in \mathrm{Rl}{}^{\backprime}P . R \in \mathrm{Rl\,ex}{}^{\backprime}P . \supset . Q \,\dot{\cup}\, R \in \mathrm{Rl\,ex}{}^{\backprime}P$

$*61{\cdot}5.\ \vdash . \dot{s}{}^{\backprime}\mathrm{Rl}{}^{\backprime}P = P$

$*61{\cdot}501.\ \vdash . \dot{s}{}^{\backprime}\mathrm{Rl\,ex}{}^{\backprime}P = P$

$*61{\cdot}51.\ \vdash . \dot{p}{}^{\backprime}\mathrm{Rl}{}^{\backprime}P = \dot{\Lambda}$

$*61{\cdot}52.\ \vdash : \dot{s}{}^{\backprime}\lambda \mathrel{\dot{\subset}} Q . \equiv . \lambda \subset \mathrm{Rl}{}^{\backprime}Q$

$*61{\cdot}53.\ \vdash : Q \mathrel{\dot{\subset}} \dot{p}{}^{\backprime}\lambda . \equiv . Q \in p{}^{\backprime}\mathrm{Rl}{}^{\backprime\backprime}\lambda$

$*61{\cdot}54.\ \vdash . \mathrm{Rl}{}^{\backprime}\dot{p}{}^{\backprime}\lambda = p{}^{\backprime}\mathrm{Rl}{}^{\backprime\backprime}\lambda$

$*61{\cdot}55.\ \vdash . \mathrm{Rl}{}^{\backprime}P = \mathrm{Rl}{}^{\backprime}Q . \equiv . P = Q$

$*61{\cdot}56.\ \vdash . \mathrm{Rl\,ex}{}^{\backprime}P = \mathrm{Rl\,ex}{}^{\backprime}Q . \equiv . P = Q$

$*61{\cdot}6.\ \vdash : xPy . \supset . x \downarrow y \in \mathrm{Rl\,ex}{}^{\backprime}P$

The analogue of $*60{\cdot}61$ is not given, because we have no suitable notation for expressing it.

$*61{\cdot}62.\ \vdash : xPy . zPw . \supset . x \downarrow y \,\dot{\cup}\, z \downarrow w \in \mathrm{Rl\,ex}{}^{\backprime}P$

$*61{\cdot}7.\ \vdash . \mathrm{Rl}{}^{\backprime}P \in \mathrm{Cl}{}^{\backprime}\mathrm{Rel}$

*62. THE RELATION OF MEMBERSHIP OF A CLASS

Summary of *62.

When "$x \in \alpha$" was defined, in *20, it was defined as a propositional function; and this mode of definition was necessary, because we had to treat of this function before treating of relations. But for many purposes it is desirable to regard ϵ as a relation, so that "$x \in \alpha$" becomes an instance of the notation "uRv." This requires, strictly speaking, a change in the meaning of "$x \in \alpha$," but it is a change which does not falsify any of the previous propositions in which "$x \in \alpha$" occurs; for if we call the new meaning "$x \,\epsilon'\, \alpha$," *i.e.* if we put

$$\epsilon' = \hat{x}\hat{\alpha}(x \in \alpha) \quad \text{Df},$$

we have

$$\vdash : x \,\epsilon'\, \alpha \,.\equiv.\, x \in \alpha .$$

Hence it is unnecessary in practice to have a new notation for the new meaning, and we put simply

$$\epsilon = \hat{x}\hat{\alpha}(x \in \alpha) \quad \text{Df}.$$

This definition, though strictly incorrect, is recommended by its convenience, and by the fact that it cannot lead to any harmful confusions. The new meaning of ϵ may be taken as replacing the old throughout the remainder of this work.

The uses of the propositions of the present number occur almost exclusively in the theory of selections from a class of classes (*83, *84, *85 and *88). Such selections are effected by means of selective relations, part of whose definition is that they are contained in ϵ. Hence the uses of the present number. If κ is the class of classes from which a selection is to be made, a selective relation will in fact be contained in $\epsilon \upharpoonright \kappa$; hence the properties of $\epsilon \upharpoonright \kappa$ become important. Some of these properties are given in *62·4 ff.

The most important propositions of the present number are the following:

*62·2. $\vdash . \overrightarrow{\epsilon}\,{}^{\backprime}\alpha = \alpha$

*62·231. $\vdash : \kappa \subset \text{C}\!\text{I}{}^{\backprime}\epsilon \,.\equiv.\, \Lambda \sim \in \kappa$

*62·26. $\vdash . R = \epsilon \,|\, \overrightarrow{R}$

*62·3. $\vdash . \epsilon{}^{\backprime\backprime}\kappa = s{}^{\backprime}\kappa$

*62·42. $\vdash : \Lambda \sim \in \kappa \,.\supset.\, \text{C}\!\text{I}{}^{\backprime}\epsilon \upharpoonright \kappa = \kappa$

*62·43. $\vdash . \text{D}{}^{\backprime}\epsilon \upharpoonright \kappa = s{}^{\backprime}\kappa$

*62·55. $\vdash : \kappa \subset 1 \,.\supset.\, \epsilon \upharpoonright \kappa = \breve{\iota} \upharpoonright \kappa$

***62·01.** $\epsilon = \hat{x}\hat{\alpha}(x \mathbin{\dot\epsilon} \alpha)$ Df

***62·1.** $\vdash : x \epsilon \alpha . \equiv . x \epsilon \alpha$ [*21·3 . (*62·01)]

In the above proposition, the first ϵ has the newly-defined meaning, while the second has the old meaning. In virtue of the above proposition, the new meaning may be substituted for the old in all propositions hitherto proved concerning ϵ, and may take the place of the old meaning in all that follows.

***62·2.** $\vdash . \overrightarrow{\epsilon}{}^{\backprime}\alpha = \alpha$

Dem.

$$\begin{aligned} &\vdash . *32{\cdot}13 . \supset \vdash . \overrightarrow{\epsilon}{}^{\backprime}\alpha = \hat{x}(x \epsilon \alpha) \\ &[*20{\cdot}42] \qquad\qquad = \alpha . \supset \vdash . \text{Prop} \end{aligned}$$

***62·21.** $\vdash . \overleftarrow{\epsilon}{}^{\backprime}x = \hat{\alpha}(x \epsilon \alpha)$ [*32·131]

Thus $\overleftarrow{\epsilon}{}^{\backprime}x$ consists of the classes of which x is a member.

***62·22.** $\vdash . \text{D}{}^{\backprime}\epsilon = \text{V}$

Dem.

$$\begin{aligned} &\vdash . *24{\cdot}104 . \supset \vdash . (x) . x \epsilon \text{V} . \\ &[*10{\cdot}24] \quad \supset \vdash : (x) : (\exists \alpha) . x \epsilon \alpha : \\ &[*33{\cdot}13] \quad \supset \vdash . (x) . x \epsilon \text{D}{}^{\backprime}\epsilon : \\ &[*24{\cdot}14] \quad \supset \vdash . \text{D}{}^{\backprime}\epsilon = \text{V} \end{aligned}$$

***62·23.** $\vdash . \text{Cl}{}^{\backprime}\epsilon = \text{Cls} - \iota{}^{\backprime}\Lambda$

Dem.

$$\begin{aligned} &\vdash . *53{\cdot}5 . \supset \vdash : \alpha \epsilon \text{Cls} - \iota{}^{\backprime}\Lambda . \equiv . \exists ! \alpha . \\ &[*33{\cdot}131] \qquad\qquad \equiv . \alpha \epsilon \text{Cl}{}^{\backprime}\epsilon : \supset \vdash . \text{Prop} \end{aligned}$$

***62·231.** $\vdash : \kappa \subset \text{Cl}{}^{\backprime}\epsilon . \equiv . \Lambda \sim\epsilon \kappa$ [*24·63 . *33·131]

***62·24.** $\vdash . \epsilon \,|\, \breve{\epsilon} = \dot{\text{V}}$

Dem.

$$\begin{aligned} &\vdash . *24{\cdot}104 . *11{\cdot}57 . \supset \vdash . (x, y) . x \epsilon \text{V} . y \epsilon \text{V} . \\ &[*31{\cdot}11] \qquad \supset \vdash . (x, y) . x \epsilon \text{V} . \text{V} \,\breve{\epsilon}\, y . \\ &[*10{\cdot}24] \qquad \supset \vdash : (x, y) : (\exists \alpha) . x \epsilon \alpha . \alpha \,\breve{\epsilon}\, y : \\ &[*34{\cdot}1] \qquad \supset \vdash : (x, y) : x \,\epsilon \,|\, \breve{\epsilon}\, y : \\ &[*25{\cdot}14] \qquad \supset \vdash . \epsilon \,|\, \breve{\epsilon} = \dot{\text{V}} \end{aligned}$$

***62·25.** $\vdash . \breve{\epsilon} \,|\, \epsilon = \hat{\alpha}\hat{\beta}\{\exists ! (\alpha \cap \beta)\}$

Dem.

$$\begin{aligned} &\vdash . *34{\cdot}1 . *31{\cdot}11 . \supset \vdash : \alpha (\breve{\epsilon} \,|\, \epsilon) \beta . \equiv . (\exists x) . x \epsilon \alpha . x \epsilon \beta . \\ &[*22{\cdot}33] \qquad\qquad \equiv . \exists ! (\alpha \cap \beta) : \supset \vdash . \text{Prop} \end{aligned}$$

***62·26.** $\vdash . R = \epsilon \,|\, \overrightarrow{R}$

Dem.

$$\vdash . *32·18 . \supset \vdash : xRy . \equiv . x \epsilon \overrightarrow{R}‘y .$$
$$[*30·33 . *32·12] \qquad \equiv . (\exists\alpha) . x \epsilon \alpha . \alpha \overrightarrow{R} y .$$
$$[*34·1] \qquad \equiv . x (\epsilon \,|\, \overrightarrow{R}) y : \supset \vdash . \text{Prop}$$

***62·3.** $\vdash . \epsilon‘‘\kappa = s‘\kappa$

Dem.

$$\vdash . *37·1 . \supset \vdash . \epsilon‘‘\kappa = \hat{x} \{(\exists\alpha) . \alpha \epsilon \kappa . x \epsilon \alpha\}$$
$$[(*40·02)] \qquad = s‘\kappa . \supset \vdash . \text{Prop}$$

***62·31.** $\vdash . \overrightarrow{\epsilon^2}‘\kappa = s‘\kappa$

Note that, since ϵ is not a homogeneous relation, *i.e.* not one in which referent and relatum belong to the same type, ϵ^2 is strictly meaningless. For if we have $x \epsilon \alpha . \alpha \epsilon \kappa$, the two ϵ's have different meanings, and do not therefore properly give $x \epsilon^2 \kappa$. But it is convenient to allow ϵ^2, on the understanding that the ambiguity of ϵ is to be differently determined for the two factors in the product $\epsilon \,|\, \epsilon$, namely the second ϵ must make both referent and relatum belong to the next type above that to which they respectively belong for the first ϵ.

Dem.

$$\vdash . *32·13 . \supset \vdash . \overrightarrow{\epsilon^2}‘\kappa = \hat{x} (x \epsilon^2 \kappa)$$
$$[*34·5] \qquad = \hat{x} \{(\exists\alpha) . x \epsilon \alpha . \alpha \epsilon \kappa\}$$
$$[(*40·02)] \qquad = s‘\kappa$$

***62·32.** $\vdash . s = \epsilon_\epsilon = \overrightarrow{\epsilon^2}$ [*30·41 . *62·3·31 . *37·11]

***62·33.** $\vdash . \dot{\overrightarrow{\epsilon}} = I \upharpoonright \text{Cls}$

Dem.

$$\vdash . *62·2 . *30·3 . \supset \vdash : \beta \overrightarrow{\epsilon} \alpha . \equiv_\beta . \beta = \alpha .$$
$$[*20·41] \qquad \equiv_\beta . \beta = \alpha . \alpha \epsilon \text{Cls} .$$
$$[*50·1 . *35·101] \qquad \equiv_\beta . \beta (I \upharpoonright \text{Cls}) \alpha : \supset \vdash . \text{Prop}$$

The use of *20·41 in the above proof depends upon the fact that α is merely an abbreviation for an expression of the form $\hat{z} (\psi z)$.

***62·34.** $\vdash . P_\epsilon = \text{sg}‘(P \,|\, \epsilon)$

Dem.

$$\vdash . *37·101 . (*37·01) . \supset \vdash :. \alpha P_\epsilon \beta . \equiv : \alpha = \hat{x} \{(\exists y) . y \epsilon \beta . xPy\}$$
$$[*34·1] \qquad = \hat{x} \{x (P \,|\, \epsilon) \beta\} :$$
$$[*32·1·23] \qquad \equiv : \alpha \{\text{sg}‘(P \,|\, \epsilon)\} \beta :. \supset \vdash . \text{Prop}$$

***62·4.** $\vdash . \epsilon \upharpoonright \kappa = \hat{x}\hat{\alpha}(x \epsilon \alpha . \alpha \epsilon \kappa)$ [*21·2.(*35·02)]

The relation $\epsilon \upharpoonright \kappa$ is very important in cardinal arithmetic, in connection with the problem of selection from the members of κ, *i.e.* of extracting one term out of each of the members of κ. A relation which is to effect this selection must be contained in $\epsilon \upharpoonright \kappa$.

***62·41.** $\vdash . \text{Ɑ}\text{‘}\epsilon \upharpoonright \kappa = \kappa - \iota\text{‘}\Lambda$

Dem.

$$\vdash . *35\cdot101 . \supset \vdash : x(\epsilon \upharpoonright \kappa)\alpha . \equiv . x \epsilon \alpha . \alpha \epsilon \kappa :$$
$$[*10\cdot11\cdot281] \supset \vdash :. (\exists x) . x(\epsilon \upharpoonright \kappa)\alpha . \equiv : (\exists x) . x \epsilon \alpha . \alpha \epsilon \kappa :$$
$$[*10\cdot35] \qquad \equiv : (\exists x) . x \epsilon \alpha : \alpha \epsilon \kappa :$$
$$[*24\cdot5] \qquad \equiv : \exists ! \alpha . \alpha \epsilon \kappa :$$
$$[*53\cdot52] \qquad \equiv : \alpha \epsilon \kappa - \iota\text{‘}\Lambda \qquad (1)$$
$$\vdash . (1) . *33\cdot131 . \supset \vdash . \text{Prop}$$

***62·42.** $\vdash : \Lambda \sim\epsilon \kappa . \supset . \text{Ɑ}\text{‘}\epsilon \upharpoonright \kappa = \kappa$

Dem.

$$\vdash . *51\cdot36 . \supset \vdash : \text{Hp} . \supset . \kappa \subset -\iota\text{‘}\Lambda .$$
$$[*22\cdot621] \qquad \supset . \kappa = \kappa - \iota\text{‘}\Lambda .$$
$$[*62\cdot41] \qquad \supset . \text{Ɑ}\text{‘}\epsilon \upharpoonright \kappa = \kappa : \supset \vdash . \text{Prop}$$

***62·43.** $\vdash . \text{D}\text{‘}\epsilon \upharpoonright \kappa = s\text{‘}\kappa$

Dem.

$$\vdash . *33\cdot11 . \supset \vdash . \text{D}\text{‘}\epsilon \upharpoonright \kappa = \hat{x}\{(\exists \alpha) . x(\epsilon \upharpoonright \kappa)\alpha\}$$
$$[*35\cdot101] \qquad = \hat{x}\{(\exists \alpha) . x \epsilon \alpha . \alpha \epsilon \kappa\}$$
$$[(*40\cdot02)] \qquad = s\text{‘}\kappa . \supset \vdash . \text{Prop}$$

***62·44.** $\vdash : R \mathrel{G} \epsilon . \equiv . (\alpha) . \overrightarrow{R}\text{‘}\alpha \subset \alpha$

Dem.

$$\vdash . *23\cdot1 . \supset \vdash :. R \mathrel{G} \epsilon . \equiv : xR\alpha . \supset_{x,\alpha} . x \epsilon \alpha :$$
$$[*32\cdot18] \qquad \equiv : x \epsilon \overrightarrow{R}\text{‘}\alpha . \supset_{x,\alpha} . x \epsilon \alpha :$$
$$[*11\cdot2 . *22\cdot1] \qquad \equiv : (\alpha) . \overrightarrow{R}\text{‘}\alpha \subset \alpha :. \supset \vdash . \text{Prop}$$

***62·45.** $\vdash :. R \mathrel{G} \epsilon . \text{E}!! R\text{‘‘}\text{Ɑ}\text{‘}R . \equiv : \alpha \epsilon \text{Ɑ}\text{‘}R . \supset_\alpha . R\text{‘}\alpha \epsilon \alpha$

Dem.

$$\vdash . *14\cdot21 . *4\cdot71 . \supset \vdash :. R\text{‘}\alpha \epsilon \alpha . \equiv : \text{E} ! R\text{‘}\alpha . R\text{‘}\alpha \epsilon \alpha :$$
$$[*30\cdot33 . *5\cdot32] \qquad \equiv : \text{E} ! R\text{‘}\alpha : xR\alpha . \supset_x . x \epsilon \alpha \qquad (1)$$
$$\vdash . (1) . *10\cdot413 . \supset \vdash :: \alpha \epsilon \text{Ɑ}\text{‘}R . \supset_\alpha . R\text{‘}\alpha \epsilon \alpha : \equiv :.$$
$$\alpha \epsilon \text{Ɑ}\text{‘}R . \supset_\alpha : \text{E} ! R\text{‘}\alpha : xR\alpha . \supset_x . x \epsilon \alpha :.$$
$$[*10\cdot29 . *11\cdot62] \equiv :. \alpha \epsilon \text{Ɑ}\text{‘}R . \supset_\alpha . \text{E} ! R\text{‘}\alpha : \alpha \epsilon \text{Ɑ}\text{‘}R . xR\alpha . \supset_{\alpha,x} . x \epsilon \alpha :.$$
$$[*33\cdot14 . *4\cdot71] \equiv :. \alpha \epsilon \text{Ɑ}\text{‘}R . \supset_\alpha . \text{E} ! R\text{‘}\alpha : xR\alpha . \supset_{\alpha,x} . x \epsilon \alpha :.$$
$$[*37\cdot104 . *11\cdot2] \equiv :. \text{E} !! R\text{‘‘}\text{Ɑ}\text{‘}R . R \mathrel{G} \epsilon :: \supset \vdash . \text{Prop}$$

This proposition is useful in the theory of selections. It is used in the proof of *83·27, and thence of *83·28.

***62·5.** $\vdash . \breve{\iota} \mathrel{\dot{\subset}} \epsilon$

Dem. $\vdash . *33{\cdot}21 . *52{\cdot}13 . \supset \vdash . \text{Ɑ}`\breve{\iota} = 1 .$

[*52·173] $\supset \vdash : \alpha \epsilon \text{Ɑ}`\breve{\iota} . \supset_\alpha . \breve{\iota}`\alpha \epsilon \alpha :$

[*62·45] $\supset \vdash . \breve{\iota} \mathrel{\dot{\subset}} \epsilon$

***62·51.** $\vdash : \mathrm{E}\,!\,\breve{\iota}`\alpha . \supset . \breve{\iota}`\alpha = \epsilon`\alpha$

Dem. $\vdash . *52{\cdot}15{\cdot}172 . \supset \vdash :. \mathrm{Hp} . \supset : \iota`\breve{\iota}`\alpha = \alpha :$

[*51·15] $\supset : x = \breve{\iota}`\alpha . \equiv_x . x \epsilon \alpha :$

[*30·3] $\supset : \breve{\iota}`\alpha = \epsilon`\alpha :. \supset \vdash . \mathrm{Prop}$

***62·52.** $\vdash : \mathrm{E}\,!\,\epsilon`\alpha . \equiv . \alpha \epsilon 1 . \equiv . \mathrm{E}\,!\,\breve{\iota}`\alpha$

Dem. $\vdash . *30{\cdot}2 . \supset \vdash :. \mathrm{E}\,!\,\epsilon`\alpha . \equiv : (\exists b) : x \epsilon \alpha . \equiv_x . x = b :$

[*52·11] $\equiv : \alpha \epsilon 1 :$

[*52·15] $\equiv : \mathrm{E}\,!\,\breve{\iota}`\alpha :. \supset \vdash . \mathrm{Prop}$

***62·53.** $\vdash : \mathrm{E}\,!\,\epsilon`\alpha . \supset . \epsilon`\alpha = \breve{\iota}`\alpha$ [*62·51·52]

***62·54.** $\vdash : \alpha \epsilon 1 . \supset . \epsilon`\alpha = \breve{\iota}`\alpha$ [*62·51·52]

***62·55.** $\vdash : \kappa \subset 1 . \supset . \epsilon \restriction \kappa = \breve{\iota} \restriction \kappa$

Dem. $\vdash . *62{\cdot}54 . \supset \vdash :. \mathrm{Hp} . \supset : \alpha \epsilon \kappa . \supset_\alpha . \epsilon`\alpha = \breve{\iota}`\alpha :$

[*35·71] $\supset : \epsilon \restriction \kappa = \breve{\iota} \restriction \kappa : \supset \vdash . \mathrm{Prop}$

***62·56.** $\vdash . \epsilon \restriction \iota``\alpha = \breve{\iota} \restriction \iota``\alpha = \alpha \mathbin{\rceil} \breve{\iota}$

Dem.

$\vdash . *52{\cdot}3 . *62{\cdot}55 . \supset \vdash . \epsilon \restriction \iota``\alpha = \breve{\iota} \restriction \iota``\alpha$ (1)

$\vdash . *35{\cdot}101 . *37{\cdot}6 . \supset \vdash :. x (\breve{\iota} \restriction \iota``\alpha) \beta . \equiv : x \breve{\iota} \beta : (\exists y) . y \epsilon \alpha . \beta = \iota`y :$

[*51·51] $\equiv : \beta = \iota`x : (\exists y) . y \epsilon \alpha . \beta = \iota`y :$

[*10·35] $\equiv : (\exists y) . \beta = \iota`x . y \epsilon \alpha . \beta = \iota`y :$

[*13·193] $\equiv : (\exists y) . \beta = \iota`x . y \epsilon \alpha . \iota`x = \iota`y :$

[*51·23] $\equiv : (\exists y) . \beta = \iota`x . y \epsilon \alpha . x = y :$

[*13·195] $\equiv : \beta = \iota`x . x \epsilon \alpha :$

[*51·51] $\equiv : x \breve{\iota} \beta . x \epsilon \alpha :$

[*35·1] $\equiv : x (\alpha \mathbin{\rceil} \breve{\iota}) \beta$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***62·57.** $\vdash . \breve{\iota} = \epsilon \restriction 1$

Dem. $\vdash . *62{\cdot}55 . \supset \vdash . \epsilon \restriction 1 = \breve{\iota} \restriction 1$

[*52·13] $= \breve{\iota} \restriction \text{Ɑ}`\breve{\iota}$

[*35·452] $= \breve{\iota} . \supset \vdash . \mathrm{Prop}$

$*63$. RELATIVE TYPES OF CLASSES

Summary of $*63$.

The notations introduced in this and the two following numbers serve to express the type of one variable in terms of the type of another. They are very useful in arithmetic, where it is necessary to take account of types in order to avoid contradictions. The two chief notations are "t_0'α," for the type in which α is contained, and "t'x," for the type of which x is a member. We put

$*63 \cdot 02$. $t_0{}^{\backprime}\alpha = \alpha \cup - \alpha$ Df

This defines "the type of members of α," or "the type which is of the same type as α." The characteristic of a type is that if τ is a type, we have

$$(x) . x \epsilon \tau,$$

and conversely, if $(x) . x \epsilon \tau$, then τ is a type. For in that case, "$x \epsilon \tau$" is true whenever it is significant, *i.e.* whenever x belongs to the type which is the range of significance of x in "$x \epsilon \tau$." Consequently τ is this range of significance, *i.e.* is a type.

Since we have $(x) . x \epsilon (\alpha \cup - \alpha)$, it follows that $\alpha \cup - \alpha$ is a type. It is not "the type of α" but "the type of the members of α." (In case α is null, "the type of the members of α" may be interpreted as meaning "the type to which x belongs when '$x \epsilon \alpha$' is significant.") "The type of x," *i.e.* the type of which x is a member, is defined as follows:

$*63 \cdot 01$. $t{}^{\backprime}x = \iota{}^{\backprime}x \cup - \iota{}^{\backprime}x$ Df

By what was said above, "$t_0{}^{\backprime}\iota{}^{\backprime}x$" is the type of the members of $\iota{}^{\backprime}x$, *i.e.* the type of x. By combining the definitions of $t{}^{\backprime}x$ and $t_0{}^{\backprime}\alpha$, we obtain

$$\vdash . t{}^{\backprime}x = t_0{}^{\backprime}\iota{}^{\backprime}x.$$

Thus $\vdash . x \epsilon t{}^{\backprime}x$ and $\vdash : y \neq x . \supset . y \epsilon t{}^{\backprime}x.$

In short, $t{}^{\backprime}x$ consists of everything either identical or not identical with x, that is, every y for which there is such a proposition, whether true or false, as "$y = x$." We put "$t{}^{\backprime}x$" here instead of "$t{}^{\backprime}\alpha$," because x need not be a class, and is in fact subject to no limitation whatever, whereas "$t_0{}^{\backprime}x$" is not significant unless x is a class, and therefore we write "$t_0{}^{\backprime}\alpha$" rather than "$t_0{}^{\backprime}x$."

We put also

$*63 \cdot 011$. $t^1{}^{\backprime}x = t{}^{\backprime}x$ Df

This definition serves merely to bring $t{}^{\backprime}x$ notationally into line with $t_0{}^{\backprime}x$ and the types $t^2{}^{\backprime}x, t^3{}^{\backprime}x, \ldots t_2{}^{\backprime}x, t_3{}^{\backprime}x, \ldots$ defined below.

In virtue of $*20 \cdot 8$, we have

$$\vdash : \phi a \vee \sim \phi a . \supset . \hat{x}(\phi x \vee \sim \phi x) = t{}^{\backprime}a,$$

i.e. if "ϕa" is significant, then the range of significance of the function $\phi \hat{z}$ is the type of a. It follows that two ranges of significance which overlap are identical, and two different ranges of significance have no member in common.

It will be seen that $\iota`x$ is always of the next type above that of $x$, and $s`\kappa$ (if κ is a class of classes) is of the next type below that of κ. We put

***63·03.** $t_1`\kappa = t_0`s`\kappa$ Df

so that $t_1`\kappa$ is the type next below that in which $\kappa$ is contained. Thus if $\kappa$ is a class of classes of individuals, $t_1`\kappa$ is the class of individuals. We put also

***63·04.** $t^2`x = t`t`x$ Df

***63·041.** $t^3`x = t`t^2`x$ Df and so on

***63·05.** $t_2`\kappa = t_1`t_1`\kappa$ Df

***63·051.** $t_3`\kappa = t_1`t_2`\kappa$ Df and so on

Thus given any two objects which are members of any one of the following: the type of x, the type of the classes to which x belongs, the type of the classes to which these classes belong, and so on, we can express the type of either of our two objects by means of its relation to the other object.

The propositions of this and the two following numbers will hardly ever be used until we come to cardinal arithmetic. They are used constantly in the first section on cardinal arithmetic, and they are constantly relevant in the first section on relation-arithmetic. Moreover they are usually required for cardinal and ordinal existence-theorems.

Among the most useful propositions of the present number are the following:

***63·103.** $\vdash . x \epsilon t`x$

***63·105.** $\vdash . \alpha \subset t_0`\alpha$

***63·11.** $\vdash : x \epsilon t_0`\alpha . \supset . t`x = \alpha \cup - \alpha = t_0`\alpha$

I.e. if x either is or is not a member of α, then the type of x is the type which contains α. This proposition uses *20·8.

***63·13.** $\vdash : \phi x . \phi y . \supset . y \epsilon t`x$

I.e. if there is any function satisfied by both x and y, then y is of the type of x. It is necessary to the use of this proposition that, if $\phi \hat{z}$ is a typically ambiguous function, it should receive the same typical determination for x and for y. For example, we have always $x = x$ and $y = y$; but we must not regard these as values of one function $\hat{z} = \hat{z}$, because such a function is typically ambiguous. On the other hand, $x = a$ and $y = a$ are values of one function $\hat{z} = a$, because here the presence of a renders the function typically determinate.

***63·15.** $\vdash . t_0`t`x = t`x$

***63·19.** $\vdash . t`t_0`\alpha = t`\alpha$

***63·16.** $\vdash : x \in t`y . \equiv . y \in t`x . \equiv . \exists ! t`x \cap t`y . \equiv . t`x = t`y$

This proposition, which depends upon *63·11, and thence upon *20·8 and *13·3, and thence upon *9·14·15, is vital to the whole theory of types.

***63·32.** $\vdash . t_1`\kappa = s`t_0`\kappa$

***63·371.** $\vdash : \beta \subset t_0`\alpha . \equiv . \beta \in t`\alpha$

***63·383.** $\vdash . t`t_1`\kappa = t_0`\kappa$

We shall have generally $t^m`t^n`\kappa = t^{m+n}`\kappa$, where we may count suffixes as negative indices, so that $t^m`t_n`\kappa = t^{m-n}`\kappa$ or $t_{n-m}`\kappa$ according as m or n is the greater.

***63·5.** $\vdash : x \in t_0`\alpha . \equiv . \alpha \in t^2`x . \equiv . \alpha \subset t`x . \equiv . t`x = t_0`\alpha$

This proposition is used constantly.

***63·51.** $\vdash : \alpha \in t_0`\kappa . \equiv . \alpha \subset t_1`\kappa . \equiv . \kappa \subset t`\alpha . \equiv . t`\alpha = t_0`\kappa$

***63·52.** $\vdash : \alpha \in t_1`\lambda . \equiv . \alpha \subset t_2`\lambda . \equiv . \lambda \subset t^2`\alpha . \equiv . t`\alpha = t_1`\lambda . \equiv . t^2`\alpha = t_0`\lambda$

***63·53.** $\vdash : x \in t_0`\alpha . \equiv . t^2`x = t`\alpha . \equiv . t`x = t_0`\alpha$

The above four propositions, together with four similar ones (*63·54·55·56·57), give transformations which enable us to express any relation of type, as between class and members or members of members or etc., that is likely to occur in practice.

***63·64.** $\vdash . t`\beta = t_0`\iota``\beta$

This proposition is often used in the first section on cardinal arithmetic.

***63·66.** $\vdash . \mathrm{Cl}`t`x = t^2`x$

***63·01.** $t`x = \iota`x \cup - \iota`x$ Df

***63·011.** $t^1`x = t`x$ Df

***63·02.** $t_0`\alpha = \alpha \cup - \alpha$ Df

***63·03.** $t_1`\kappa = t_0`s`\kappa$ Df

***63·04.** $t^2`x = t`t`x$ Df

***63·041.** $t^3`x = t`t^2`x$ Df

***63·05.** $t_2`\kappa = t_1`t_1`\kappa$ Df

***63·051.** $t_3`\kappa = t_1`t_2`\kappa$ Df

***63·1.** $\vdash . (x) . x \in t_0`\alpha$ [*22·88]

***63·101.** $\vdash . t`x = t_0`\iota`x = \iota`x \cup - \iota`x$ [*20·2 . (*63·01·02)]

***63·102.** $\vdash . (y) . y \in t`x$ [*63·1·101]

***63·103.** $\vdash . x \in t`x$ [*63·101 . *51·16]

***63·104.** $\vdash : \phi x . \sim \phi y . \supset . y \in t`x$ [*63·101 . *13·14]

***63·105.** $\vdash . \alpha \subset t_0`\alpha$ [*22·58]

***63·106.** $\vdash . t_0`\alpha = t_0` - \alpha$ [*22·8]

***63·107.** $\vdash :. (x) . \phi x : f(\phi y) : \supset . \phi y$

Dem.

$\vdash . *2·11 . *10·11 . \supset \vdash . (y) . f(\phi y) \lor \sim f(\phi y)$ (1)

$\vdash . (1) . *10·13·221 . \supset \vdash :. (x) . \phi x . \supset : \phi y . f(\phi y) \lor \sim f(\phi y) :$

$[*5·1] \quad \supset : \phi y . \equiv . f(\phi y) \lor \sim f(\phi y) :$

$[*2·2] \quad \supset : f(\phi y) . \supset . \phi y :. \supset \vdash . \text{Prop}$

***63·108.** $\vdash : f(y \in t^{\backprime}x) . \supset . y \in t^{\backprime}x$ [*63·107·102]

***63·109.** $\vdash : f(y \in t_0{}^{\backprime}\alpha) . \supset . y \in t_0{}^{\backprime}\alpha$ [*63·107·1]

***63·11.** $\vdash : x \in t_0{}^{\backprime}\alpha . \supset . t^{\backprime}x = \alpha \cup - \alpha = t_0{}^{\backprime}\alpha$

Dem.

$\vdash . *22·34 . (*63·02) . \supset \vdash :. \text{Hp} . \supset : x \in \alpha . \lor . x \sim \in \alpha :$

$[*20·8] \quad \supset : \hat{y}(y \in \alpha . \lor . y \sim \in \alpha) = \hat{y}(y = x . \lor . y \neq x) :$

$[*22·3·31.*51·15] \quad \supset : \alpha \cup - \alpha = \iota^{\backprime}x \cup - \iota^{\backprime}x$ (1)

$\vdash . (1) . (*63·01·02) . \supset \vdash . \text{Prop}$

***63·12.** $\vdash :. \phi x \lor \sim \phi x . \supset : \phi y \lor \sim \phi y . \equiv_y . y \in t^{\backprime}x$

Dem.

$\vdash . *63·11 . *20·8 . \supset \vdash :. \text{Hp} . \supset : t^{\backprime}x = \hat{z}(\phi z) \cup - \hat{z}(\phi z) :$

$[*20·31.*22·391·392] \quad \supset : y \in t^{\backprime}x . \equiv_y . \phi y \lor \sim \phi y :. \supset \vdash . \text{Prop}$

***63·13.** $\vdash : \phi x . \phi y . \supset . y \in t^{\backprime}x$ [*63·12 . Imp . Add]

***63·14.** $\vdash : (x) . x \in \alpha . \supset . t_0{}^{\backprime}\alpha = \alpha$ [*24·14·17·24 . (*63·02)]

***63·15.** $\vdash . t_0{}^{\backprime}t^{\backprime}x = t^{\backprime}x$ [*63·14·102]

***63·151.** $\vdash . t_0{}^{\backprime}t_0{}^{\backprime}\alpha = t_0{}^{\backprime}\alpha$ [*63·14·1]

***63·152.** $\vdash . x \in t_0{}^{\backprime}t^{\backprime}x$ [*63·103·15]

***63·16.** $\vdash : x \in t^{\backprime}y . \equiv . y \in t^{\backprime}x . \equiv . \exists ! t^{\backprime}x \cap t^{\backprime}y . \equiv . t^{\backprime}x = t^{\backprime}y$

Dem.

$\vdash . *63·101 . *51·23 . \supset \vdash : x \in t^{\backprime}y . \equiv . y \in t^{\backprime}x$ (1)

$\vdash . *63·13 . \supset \vdash : (\exists z) . z \in t^{\backprime}x . z \in t^{\backprime}y . \supset . y \in t^{\backprime}x$ (2)

$\vdash . *63·103 . \supset \vdash : y \in t^{\backprime}x . \supset . y \in t^{\backprime}x . y \in t^{\backprime}y .$

$[*10·24] \quad \supset . \exists ! t^{\backprime}x \cap t^{\backprime}y$ (3)

$\vdash . (2) . (3) . \supset \vdash : y \in t^{\backprime}x . \equiv . \exists ! t^{\backprime}x \cap t^{\backprime}y$ (4)

$\vdash . *63·103 . \supset \vdash : t^{\backprime}x = t^{\backprime}y . \supset . y \in t^{\backprime}x$ (5)

$\vdash . *63·13 . \supset \vdash : y \in t^{\backprime}x . z \in t^{\backprime}x . \supset . z \in t^{\backprime}y$ (6)

$\vdash . *63·13 . \supset \vdash : x \in t^{\backprime}y . z \in t^{\backprime}y . \supset . z \in t^{\backprime}x :$

$[(1)] \quad \supset \vdash : y \in t^{\backprime}x . z \in t^{\backprime}y . \supset . z \in t^{\backprime}x$ (7)

$\vdash . (6) . (7) . \supset \vdash :. y \in t^{\backprime}x . \supset : z \in t^{\backprime}x . \equiv . z \in t^{\backprime}y$ (8)

$\vdash . (5) . (8) . \supset \vdash :. y \in t^{\backprime}x . \equiv . t^{\backprime}x = t^{\backprime}y$ (9)

$\vdash . (1) . (4) . (9) . \supset \vdash . \text{Prop}$

***63·17.** $\vdash : y \in t\text{‘}x \,.\, z \in t\text{‘}y \,.\, \supset \,.\, z \in t\text{‘}x$ [*63·16]

***63·18.** $\vdash \,.\, \exists !\, t_0\text{‘}\alpha$ [*10·25 . *63·1]

***63·181.** $\vdash : \alpha \subset t_0\text{‘}\beta \,.\equiv .\, \beta \subset t_0\text{‘}\alpha \,.\equiv .\, \exists !\, t_0\text{‘}\alpha \cap t_0\text{‘}\beta \,.\equiv .\, t_0\text{‘}\alpha = t_0\text{‘}\beta$

Dem.

$\vdash .\, *63\cdot105 \,.\quad \supset \vdash : t_0\text{‘}\alpha = t_0\text{‘}\beta \,.\supset .\, \alpha \subset t_0\text{‘}\beta$ (1)

$\vdash .\, *24\cdot6 \,.\quad \supset \vdash :.\, \alpha \subset t_0\text{‘}\beta \,.\supset : \alpha = t_0\text{‘}\beta \,.\vee .\, \exists !\, t_0\text{‘}\beta - \alpha$ (2)

$\vdash .\, *63\cdot151 \,.\quad \supset \vdash : \alpha = t_0\text{‘}\beta \,.\supset .\, t_0\text{‘}\alpha = t_0\text{‘}\beta$ (3)

$\vdash .\, *63\cdot11 \,.\quad \supset \vdash : x \in t_0\text{‘}\beta \,.\, x \in -\alpha \,.\supset .\, t\text{‘}x = t_0\text{‘}\beta \,.\, t\text{‘}x = t_0\text{‘}-\alpha \,.$

[*63·106] $\quad \supset .\, t_0\text{‘}\alpha = t_0\text{‘}\beta$ (4)

$\vdash .\, (2).(3).(4) \,.\supset \vdash : \alpha \subset t_0\text{‘}\beta \,.\supset .\, t_0\text{‘}\alpha = t_0\text{‘}\beta$ (5)

$\vdash .\, (1).(5) \,.\quad \supset \vdash : \alpha \subset t_0\text{‘}\beta \,.\equiv .\, t_0\text{‘}\alpha = t_0\text{‘}\beta$ (6)

$\vdash .\, (6)\dfrac{\beta, \alpha}{\alpha, \beta} \,.\quad \supset \vdash : \beta \subset t_0\text{‘}\alpha \,.\equiv .\, t_0\text{‘}\alpha = t_0\text{‘}\beta$ (7)

$\vdash .\, *63\cdot11 \,.\quad \supset \vdash : x \in t_0\text{‘}\alpha \cap t_0\text{‘}\beta \,.\supset .\, t\text{‘}x = t_0\text{‘}\alpha \,.\, t\text{‘}x = t_0\text{‘}\beta \,.$

[*13·171] $\quad \supset .\, t_0\text{‘}\alpha = t_0\text{‘}\beta$ (8)

$\vdash .\, *63\cdot18 \,.\quad \supset \vdash : t_0\text{‘}\alpha = t_0\text{‘}\beta \,.\supset .\, \exists !\, t_0\text{‘}\alpha \cap t_0\text{‘}\beta$ (9)

$\vdash .\, (8).(9) \,.\quad \supset \vdash : \exists !\, t_0\text{‘}\alpha \cap t_0\text{‘}\beta \,.\equiv .\, t_0\text{‘}\alpha = t_0\text{‘}\beta$ (10)

$\vdash .\, (6).(7).(10) \,.\supset \vdash .$ Prop

***63·182.** $\vdash : \alpha \subset t_0\text{‘}\beta \,.\, \beta \subset t_0\text{‘}\gamma \,.\supset .\, \alpha \subset t_0\text{‘}\gamma$ [*63·181]

***63·19.** $\vdash .\, t\text{‘}t_0\text{‘}\alpha = t\text{‘}\alpha$

Dem.

$\vdash .\, *63\cdot105 \,.\, *22\cdot42 \,.\supset \vdash .\, \alpha \subset t_0\text{‘}\alpha \,.\, t_0\text{‘}\alpha \subset t_0\text{‘}\alpha \,.$

[*63·13] $\quad \supset \vdash .\, \alpha \in t\text{‘}t_0\text{‘}\alpha \,.$

[*63·16] $\quad \supset \vdash .$ Prop

***63·191.** $\vdash .\, t_0\text{‘}\alpha \in t\text{‘}\alpha$ [*63·103·19]

***63·2.** $\vdash : x \in t_0\text{‘}\alpha \,.\, \alpha \in t_0\text{‘}\kappa \,.\supset .\, t^2\text{‘}x = t\text{‘}\alpha = t_0\text{‘}\kappa$

Dem.

$\vdash .\, *63\cdot11 \,.\quad \supset \vdash : \text{Hp} \,.\supset .\, t\text{‘}x = t_0\text{‘}\alpha \,.\, t\text{‘}\alpha = t_0\text{‘}\kappa$ (1)

$\vdash .\, (1) \,.\, *63\cdot19 \,.\, (*63\cdot04) \,.\supset \vdash : \text{Hp} \,.\supset .\, t^2\text{‘}x = t\text{‘}\alpha = t_0\text{‘}\kappa : \supset \vdash .$ Prop

***63·21.** $\vdash : \alpha \subset t\text{‘}x \,.\equiv .\, t_0\text{‘}\alpha = t\text{‘}x$

Dem.

$\vdash .\, *63\cdot181\cdot15 \,.\supset \vdash : \alpha \subset t\text{‘}x \,.\equiv .\, t_0\text{‘}\alpha = t_0\text{‘}t\text{‘}x$

[*63·15] $\quad = t\text{‘}x : \supset \vdash .$ Prop

***63·22.** $\vdash : \alpha \subset t\text{‘}x \,.\equiv .\, x \in t_0\text{‘}\alpha \,.\equiv .\, t\text{‘}x = t_0\text{‘}\alpha$

Dem.

$\vdash .\, *63\cdot103 \,.\quad \supset \vdash : t\text{‘}x = t_0\text{‘}\alpha \,.\supset .\, x \in t_0\text{‘}\alpha$ (1)

$\vdash .\, (1) \,.\, *63\cdot11 \,.\supset \vdash : x \in t_0\text{‘}\alpha \,.\equiv .\, t\text{‘}x = t_0\text{‘}\alpha$ (2)

$\vdash .\, (2) \,.\, *63\cdot21 \,.\supset \vdash .$ Prop

***63·23.** $\vdash : \alpha \subset t`x \,.\, \kappa \subset t`\alpha \,.\, \supset \,.\, t^2`x = t`\alpha = t_0`\kappa$ [*63·2·22]

Propositions of the same kind as the above can obviously be extended to $t^3`x$, etc.

***63·3.** $\vdash : (\alpha) \,.\, \alpha \in \kappa \,.\, \supset \,.\, (x) \,.\, x \in s`\kappa$

Dem.

$\vdash \,.\, *10·1 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, \mathrm{V} \in \kappa \,.$

[*40·221] $\supset \,.\, s`\kappa = \mathrm{V} \,.$

[*24·14] $\supset \,.\, (x) \,.\, x \in s`\kappa : \supset \vdash \,.\, \mathrm{Prop}$

***63·31.** $\vdash \,.\, s`(\kappa \cup -\kappa) = s`\kappa \cup -s`\kappa$

Dem.

$\vdash \,.\, *40·171 \,.\, \supset \vdash :. \, x \in s`(\kappa \cup -\kappa) \,.\, \equiv : x \in s`\kappa \,.\, \mathrm{v} \,.\, x \in s`-\kappa$ (1)

$\vdash \,.\, (1) \,.\, *22·88 \,.\, *63·3 \,.\, \supset \vdash : x \in s`\kappa \,.\, \mathrm{v} \,.\, x \in s`-\kappa$ (2)

$\vdash \,.\, *22·88 \,.\, \supset \vdash : x \in s`\kappa \,.\, \mathrm{v} \,.\, x \in -s`\kappa$ (3)

$\vdash \,.\, (2) \,.\, (3) \,.\, *10·221·13 \,.\, \supset$

$\vdash :. \, x \in s`\kappa \,.\, \mathrm{v} \,.\, x \in s`-\kappa : x \in s`\kappa \,.\, \mathrm{v} \,.\, x \in -s`\kappa :.$

[(1).*5·1] $\supset \vdash :. \, x \in s`(\kappa \cup -\kappa) \,.\, \equiv : x \in s`\kappa \,.\, \mathrm{v} \,.\, x \in -s`\kappa :. \supset \vdash \,.\, \mathrm{Prop}$

Note that the use of *10·221 in the above proof depends upon the fact that $x \in s`\kappa$ occurs both in (2) and in (3), so that these are both of the form $f(x \in s`\kappa)$.

***63·32.** $\vdash \,.\, t_1`\kappa = s`t_0`\kappa$ [*63·31 . (*63·02·03)]

***63·321.** $\vdash \,.\, t_1`\kappa = t_1`t_0`\kappa = t_0`t_1`\kappa$

Dem.

$\vdash \,.\, *20·2 \,.\, (*63·03) \,.\, \supset \vdash \,.\, t_1`t_0`\kappa = t_0`s`t_0`\kappa$

[*63·32] $= t_0`t_1`\kappa$ (1)

[*20·2.(*63·03)] $= t_0`t_0`s`\kappa$

[*63·151] $= t_0`s`\kappa$

[*20·2.(*63·03)] $= t_1`\kappa$ (2)

$\vdash \,.\, (1) \,.\, (2) \,.\, \supset \vdash \,.\, \mathrm{Prop}$

***63·33.** $\vdash : t_0`\kappa = t_0`\lambda \,.\, \supset \,.\, t_1`\kappa = t_1`\lambda$ [*30·37 . *63·32]

***63·34.** $\vdash \,.\, t_1`t`\alpha = t_0`\alpha = s`t`\alpha$

Dem.

$\vdash \,.\, *63·32 \,.\, \supset \vdash \,.\, t_1`t`\alpha = s`t_0`t`\alpha$

[*63·15] $= s`t`\alpha$ (1)

[*63·101] $= s`(\iota`\alpha \cup -\iota`\alpha)$

[*63·31] $= s`\iota`\alpha \cup -s`\iota`\alpha$

[*53·02] $= \alpha \cup -\alpha$

[(*63·02)] $= t_0`\alpha$ (2)

$\vdash \,.\, (1) \,.\, (2) \,.\, \supset \vdash \,.\, \mathrm{Prop}$

***63·35.** $\vdash : t`\alpha = t`\beta . \supset . t_0`\alpha = t_0`\beta$ [*30·37 . *63·34]

***63·36.** $\vdash : t`\kappa = t`\lambda . \supset . t_1`\kappa = t_1`\lambda$ [*63·35·33]

***63·361.** $\vdash : t_0`\alpha = t_0`\beta . \supset . t`\alpha = t`\beta$ [*30·37 . *63·19]

***63·37.** $\vdash : t_0`\alpha = t_0`\beta . \equiv . t`\alpha = t`\beta$ [*63·35·361]

***63·371.** $\vdash : \beta \subset t_0`\alpha . \equiv . \beta \epsilon t`\alpha$

Dem.

$\vdash .$ *63·181 $. \supset \vdash : \beta \subset t_0`\alpha . \equiv . t_0`\alpha = t_0`\beta .$
[*63·37] $\equiv . t`\alpha = t`\beta .$
[*63·16] $\equiv . \beta \epsilon t`\alpha : \supset \vdash .$ Prop

***63·38.** $\vdash : \alpha \epsilon t_0`\kappa . x \epsilon t_0`\alpha . \supset . t`x = t_0`\alpha = t_1`\kappa$

Dem.

$\vdash .$ *63·11 $. \supset \vdash :$ Hp $. \supset . t`x = t_0`\alpha . t`\alpha = t_0`\kappa$ (1)
$\vdash . (1) .$ *63·34 $. \supset \vdash :$ Hp $. \supset . t_0`\alpha = t_1`t_0`\kappa$
[*63·151·33] $= t_1`\kappa$ (2)
$\vdash . (1) . (2) . \supset \vdash .$ Prop

***63·381.** $\vdash : x \epsilon t_1`\kappa . \supset . t`x = t_1`\kappa$

Dem.

$\vdash .$ *63·38·105 $. \supset \vdash : \alpha \epsilon t_0`\kappa . x \epsilon \alpha . \supset . t`x = t_1`\kappa :$
[*10·11·23.*40·11] $\supset \vdash : x \epsilon s`t_0`\kappa . \supset . t`x = t_1`\kappa$ (1)
$\vdash . (1) .$ *63·32 $. \supset \vdash .$ Prop

***63·382.** $\vdash . \exists ! t_1`\kappa$ [*63·18 . (*63·03)]

***63·383.** $\vdash . t`t_1`\kappa = t_0`\kappa$

Dem.

$\vdash .$ *63·38·18 . *10·11·23·35 $. \supset \vdash : \alpha \epsilon t_0`\kappa . \supset . t`t_1`\kappa = t`t_0`\alpha$
[*63·19] $= t`\alpha$
[*63·11] $= t_0`\kappa$ (1)
$\vdash . (1) .$ *10·11·23 . *63·18 $. \supset \vdash .$ Prop

***63·384.** $\vdash : t_1`\kappa = t_1`\lambda . \supset . t_0`\kappa = t_0`\lambda . t`\kappa = t`\lambda$ [*63·383·37]

***63·39.** $\vdash : t_1`\kappa = t_1`\lambda . \equiv . t_0`\kappa = t_0`\lambda . \equiv . t`\kappa = t`\lambda$ [*63·33·384·37]

***63·391.** $\vdash : t`x = t`y . \equiv . t^2`x = t^2`y$

Dem.

$\vdash .$ *63·39 $. \supset \vdash : t^2`x = t^2`y . \equiv . t_0`t`x = t_0`t`y .$
[*63·15] $\equiv . t`x = t`y : \supset \vdash .$ Prop

***63·392.** $\vdash : t_2`\kappa = t_2`\lambda . \equiv . t_1`\kappa = t_1`\lambda . \equiv . t_0`\kappa = t_0`\lambda$

Dem.

$\vdash .$ *63·39 $. \supset \vdash : t_2`\kappa = t_2`\lambda . \equiv . t_0`t_1`\kappa = t_0`t_1`\lambda .$
[*63·321] $\equiv . t_1`\kappa = t_1`\lambda$ (1)
$\vdash . (1) .$ *63·39 $. \supset \vdash .$ Prop

***63·4.** $\vdash : \alpha \epsilon t_0{}^{\backprime}\kappa \,.\, \kappa \epsilon t_0{}^{\backprime}\lambda \,.\, \supset .\, t_0{}^{\backprime}\alpha = t_1{}^{\backprime}\kappa = t_2{}^{\backprime}\lambda$

Dem.

$\vdash .\, *63{\cdot}38{\cdot}18 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset .\, t_0{}^{\backprime}\alpha = t_1{}^{\backprime}\kappa \,.\, t_0{}^{\backprime}\kappa = t_1{}^{\backprime}\lambda \,.$

$[*30{\cdot}37.(*63{\cdot}05)] \quad \supset .\, t_0{}^{\backprime}\alpha = t_1{}^{\backprime}\kappa \,.\, t_1{}^{\backprime}t_0{}^{\backprime}\kappa = t_2{}^{\backprime}\lambda \,.$

$[*63{\cdot}321] \quad \supset .\, t_0{}^{\backprime}\alpha = t_1{}^{\backprime}\kappa \,.\, t_1{}^{\backprime}\kappa = t_2{}^{\backprime}\lambda : \supset \vdash .\, \mathrm{Prop}$

***63·41.** $\vdash .\, t^{\backprime}t_2{}^{\backprime}\lambda = t_1{}^{\backprime}\lambda$

Dem.

$\vdash .\, *63{\cdot}4{\cdot}18 \,.\, *10{\cdot}11{\cdot}23{\cdot}35 \,.\, \supset \vdash : \kappa \epsilon t_0{}^{\backprime}\lambda \,.\, \supset .\, t^{\backprime}t_2{}^{\backprime}\lambda = t^{\backprime}t_1{}^{\backprime}\kappa$

$[*63{\cdot}383] \quad = t_0{}^{\backprime}\kappa$

$[*63{\cdot}38{\cdot}18.*10{\cdot}11{\cdot}23{\cdot}35] \quad = t_1{}^{\backprime}\lambda \quad (1)$

$\vdash .\, (1) \,.\, *63{\cdot}18 \,.\, \supset \vdash .\, \mathrm{Prop}$

***63·42.** $\vdash .\, t^{2\backprime}t_2{}^{\backprime}\lambda = t_0{}^{\backprime}\lambda \quad [*30{\cdot}37 \,.\, *63{\cdot}41{\cdot}383]$

***63·43.** $\vdash .\, t_1{}^{\backprime}t^{2\backprime}x = t^{\backprime}x \quad [*63{\cdot}34{\cdot}15]$

***63·44.** $\vdash .\, t_2{}^{\backprime}t^{2\backprime}\alpha = t_0{}^{\backprime}\alpha \quad [*63{\cdot}43{\cdot}34]$

It is obvious that the analogues of the above propositions will hold for t^3 and t_3, t^4 and t_4, etc. We shall not prove these analogues, but if occasion arises we shall assume them, referring to the corresponding propositions for t^2 and t_2.

***63·5.** $\vdash : x \epsilon t_0{}^{\backprime}\alpha \,.\, \equiv .\, \alpha \epsilon t^{2\backprime}x \,.\, \equiv .\, \alpha \subset t^{\backprime}x \,.\, \equiv .\, t^{\backprime}x = t_0{}^{\backprime}\alpha$

Dem.

$\vdash .\, *63{\cdot}15 \,.\, \supset \vdash : \alpha \subset t^{\backprime}x \,.\, \equiv .\, \alpha \subset t_0{}^{\backprime}t^{\backprime}x \,.$

$[*63{\cdot}371] \quad \equiv .\, \alpha \epsilon t^{2\backprime}x \quad (1)$

$\vdash .\, (1) \,.\, *63{\cdot}22 \,.\, \supset \vdash .\, \mathrm{Prop}$

***63·51.** $\vdash : \alpha \epsilon t_0{}^{\backprime}\kappa \,.\, \equiv .\, \alpha \subset t_1{}^{\backprime}\kappa \,.\, \equiv .\, \kappa \subset t^{\backprime}\alpha \,.\, \equiv .\, t^{\backprime}\alpha = t_0{}^{\backprime}\kappa$

Dem.

$\vdash .\, *4{\cdot}2 \,.\, (*63{\cdot}03) \,.\, \supset \vdash : \alpha \subset t_1{}^{\backprime}\kappa \,.\, \equiv .\, \alpha \subset t_0{}^{\backprime}s^{\backprime}\kappa \,.$

$[*63{\cdot}371{\cdot}19] \quad \equiv .\, \alpha \epsilon t^{\backprime}t_0{}^{\backprime}s^{\backprime}\kappa \,.$

$[*4{\cdot}2.(*63{\cdot}03)] \quad \equiv .\, \alpha \epsilon t^{\backprime}t_1{}^{\backprime}\kappa \,.$

$[*63{\cdot}383] \quad \equiv .\, \alpha \epsilon t_0{}^{\backprime}\kappa \quad (1)$

$\vdash .\, (1) \,.\, *63{\cdot}5{\cdot}22 \,.\, \supset \vdash .\, \mathrm{Prop}$

***63·52.** $\vdash : \alpha \epsilon t_1{}^{\backprime}\lambda \,.\, \equiv .\, \alpha \subset t_2{}^{\backprime}\lambda \,.\, \equiv .\, \lambda \subset t^{2\backprime}\alpha \,.\, \equiv .\, t^{\backprime}\alpha = t_1{}^{\backprime}\lambda \,.\, \equiv .\, t^{2\backprime}\alpha = t_0{}^{\backprime}\lambda$

Dem.

$\vdash .\, *63{\cdot}51 \dfrac{s^{\backprime}\lambda}{\kappa} \,.\, (*63{\cdot}03) \,.\, \supset$

$\vdash : \alpha \epsilon t_1{}^{\backprime}\lambda \,. \quad \equiv .\, \alpha \subset t_1{}^{\backprime}s^{\backprime}\lambda \,.$

$[*63{\cdot}321] \quad \equiv .\, \alpha \subset t_1{}^{\backprime}t_0{}^{\backprime}s^{\backprime}\lambda \,.$

$[(*63{\cdot}03{\cdot}05)] \quad \equiv .\, \alpha \subset t_2{}^{\backprime}\lambda \quad (1)$

$\vdash . *63{\cdot}321 . \supset$

$\vdash : \alpha \epsilon t_1{}^{\backprime}\lambda . \equiv . \alpha \epsilon t_0{}^{\backprime}t_1{}^{\backprime}\lambda .$

[*63·22] $\equiv . t^{\backprime}\alpha = t_0{}^{\backprime}t_1{}^{\backprime}\lambda$

[*63·321] $= t_1{}^{\backprime}\lambda .$ (2)

[*63·391·41·42] $\equiv . t^{2\backprime}\alpha = t_0{}^{\backprime}\lambda .$ (3)

[*63·15·181] $\equiv . \lambda \subset t_0{}^{\backprime}t^{2\backprime}\alpha .$

[*63·15] $\equiv . \lambda \subset t^{2\backprime}\alpha$ (4)

$\vdash . (1) . (2) . (3) . (4) . \supset \vdash .$ Prop

***63·53.** $\vdash : x \epsilon t_0{}^{\backprime}\alpha . \equiv . t^{2\backprime}x = t^{\backprime}\alpha . \equiv . t^{\backprime}x = t_0{}^{\backprime}\alpha$

Dem.

$\vdash . *30{\cdot}37 . \supset \vdash : t^{2\backprime}x = t^{\backprime}\alpha . \supset . t_1{}^{\backprime}t^{2\backprime}x = t_1{}^{\backprime}t^{\backprime}\alpha .$

[*63·43·34] $\supset . t^{\backprime}x = t_0{}^{\backprime}\alpha$ (1)

$\vdash . *63{\cdot}19 . \supset \vdash : t^{\backprime}x = t_0{}^{\backprime}\alpha . \supset . t^{2\backprime}x = t^{\backprime}\alpha$ (2)

$\vdash . (1) . (2) . *63{\cdot}5 . \supset \vdash .$ Prop

***63·54.** $\vdash : \alpha \epsilon t_0{}^{\backprime}\kappa . \equiv . t_0{}^{\backprime}\alpha = t_1{}^{\backprime}\kappa . \equiv . t^{\backprime}\alpha = t_0{}^{\backprime}\kappa . \equiv . t^{2\backprime}\alpha = t^{\backprime}\kappa$

Dem.

$\vdash . *30{\cdot}37 . \supset \vdash : t^{\backprime}\alpha = t_0{}^{\backprime}\kappa . \supset . t_1{}^{\backprime}t^{\backprime}\alpha = t_1{}^{\backprime}t_0{}^{\backprime}\kappa .$

[*63·34·321] $\supset . t_0{}^{\backprime}\alpha = t_1{}^{\backprime}\kappa$ (1)

$\vdash . *30{\cdot}37 . \supset \vdash : t_0{}^{\backprime}\alpha = t_1{}^{\backprime}\kappa . \supset . t^{\backprime}t_0{}^{\backprime}\alpha \equiv t^{\backprime}t_1{}^{\backprime}\kappa .$

[*63·19·383] $\supset . t^{\backprime}\alpha = t_0{}^{\backprime}\kappa$ (2)

$\vdash . (1) . (2) . *63{\cdot}51{\cdot}53 . \supset \vdash .$ Prop

***63·55.** $\vdash : \kappa \epsilon t_0{}^{\backprime}\lambda . \equiv . t_1{}^{\backprime}\kappa = t_2{}^{\backprime}\lambda . \equiv . t_0{}^{\backprime}\kappa = t_1{}^{\backprime}\lambda . \equiv . t^{\backprime}\kappa = t_0{}^{\backprime}\lambda . \equiv . t^{2\backprime}\kappa = t^{\backprime}\lambda$

[Proof as in *63·54]

***63·56.** $\vdash : x \epsilon t_1{}^{\backprime}\kappa . \equiv . t^{\backprime}x = t_1{}^{\backprime}\kappa . \equiv . t^{2\backprime}x = t_0{}^{\backprime}\kappa$

Dem.

$\vdash . *63{\cdot}321 . \supset \vdash : x \epsilon t_1{}^{\backprime}\kappa . \equiv . x \epsilon t_0{}^{\backprime}t_1{}^{\backprime}\kappa .$

[*63·53] $\equiv . t^{2\backprime}x = t^{\backprime}t_1{}^{\backprime}\kappa$ (1)

[*63·383] $= t_0{}^{\backprime}\kappa$ (2)

$\vdash . (1) . *63{\cdot}53 . \supset \vdash : x \epsilon t_1{}^{\backprime}\kappa . \equiv . t^{\backprime}x = t_0{}^{\backprime}t_1{}^{\backprime}\kappa$

[*63·321] $= t_1{}^{\backprime}\kappa$ (3)

$\vdash . (2) . (3) . \supset \vdash .$ Prop

***63·57.** $\vdash : \alpha \epsilon t_1{}^{\backprime}\lambda . \equiv . t_0{}^{\backprime}\alpha = t_2{}^{\backprime}\lambda . \equiv . t^{\backprime}\alpha = t_1{}^{\backprime}\lambda . \equiv . t^{2\backprime}\alpha = t_0{}^{\backprime}\lambda$

[Proof as in *63·56]

***63·61.** $\vdash . t^{2\backprime}x = t^{\backprime}\iota^{\backprime}x$ [*63·19·101]

***63·62.** $\vdash : x \epsilon t_0{}^{\backprime}\alpha . \supset . \iota^{\backprime}x \epsilon t^{\backprime}\alpha . t^{\backprime}\iota^{\backprime}x = t^{\backprime}\alpha$

Dem.

$\vdash . *63{\cdot}53 . \supset \vdash : \text{Hp} . \supset . t^{2\backprime}x = t^{\backprime}\alpha .$

[*63·61] $\supset . t^{\backprime}\iota^{\backprime}x = t^{\backprime}\alpha .$

[*63·16] $\supset . \iota^{\backprime}x \epsilon t^{\backprime}\alpha : \supset \vdash .$ Prop

***63·621.** $\vdash : x \in \alpha . \supset . \iota`x \in t`\alpha . t`\iota`x = t`\alpha$ [*63·62 . *63·105]

***63·63.** $\vdash : x \in t_0`\alpha . \supset . \iota`\iota`x \in t^{2}`\alpha . t`\iota`\iota`x = t^{2}`\alpha$

Dem.

$\vdash . *63·101 . \supset \vdash . t`\iota`x = t_0`\iota`\iota`x .$
[*63·62] $\supset \vdash : \mathrm{Hp} . \supset . t`\alpha \equiv t_0`\iota`\iota`x .$
[*63·19] $\supset . t^{2}`\alpha = t`\iota`\iota`x$ (1)
$\vdash . (1) . *63·103 . \supset \vdash . \mathrm{Prop}$

***63·64.** $\vdash . t`\beta = t_0`\iota``\beta$

Dem.

$\vdash . *51·16 . *37·62 . \supset$
$\vdash : x \in \beta . \supset . x \in \iota`x . \iota`x \in \iota``\beta .$
[*63·105·38] $\supset . x \in t_0`\iota`x . t_0`\iota`x = t_1`\iota``\beta .$
[*13·13] $\supset . x \in t_1`\iota``\beta$ (1)
$\vdash . (1) . *63·51 . \supset \vdash . \mathrm{Prop}$

***63·65.** $\vdash . \mathrm{Cl}`t_0`\alpha = t`\alpha$ [*63·371 . *60·2]

***63·66.** $\vdash . \mathrm{Cl}`t`x = t^{2}`x$ [*63·5 . *60·2]

***63·661.** $\vdash . t`\mathrm{Cl}`\alpha = t^{2}`\alpha$ [*60·34 . *63·105·53]

***63·67.** $\vdash . \mathrm{Cl}`t_1`\kappa = t_0`\kappa$ [*63·51 . *60·2]

***63·68.** $\vdash . \mathrm{Cl}`t_2`\kappa = t_1`\kappa$ [*63·52 . *60·2]

∗64. RELATIVE TYPES OF RELATIONS

Summary of ∗64.

In the present number, we introduce notations defining the type of a relation relatively to the types of its domain and converse domain, when these types are given relatively to some fixed class α. If R is any relation, it is of the same type as $t_0\text{‘}\mathrm{D}\text{‘}R \uparrow t_0\text{‘}\text{Ɑ}\text{‘}R$. If $\mathrm{D}\text{‘}R$ and $\text{Ɑ}\text{‘}R$ are both of the same type as α, R is of the same type as $t_0\text{‘}\alpha \uparrow t_0\text{‘}\alpha$, which is of the same type as $\alpha \uparrow \alpha$. The type of $t_0\text{‘}\alpha \uparrow t_0\text{‘}\alpha$ we call $t_{00}\text{‘}\alpha$, and the type of $t^m\text{‘}\alpha \uparrow t^n\text{‘}\alpha$ we call $t^{mn}\text{‘}\alpha$, and the type of $t_m\text{‘}\alpha \uparrow t_n\text{‘}\alpha$ we call $t_{mn}\text{‘}\alpha$, and the type of $t_m\text{‘}\alpha \uparrow t^n\text{‘}\alpha$ we call $t_m{}^n\text{‘}\alpha$, and the type of $t^m\text{‘}\alpha \uparrow t_n\text{‘}\alpha$ we call ${}^m t_n\text{‘}\alpha$. We thus have a means of expressing the type of any relation R in terms of the type of α, provided the types of the domain and converse domain of R are given relatively to α.

The most useful propositions of the present number are the following:

∗64·16. $\vdash : R \mathrel{\unicode{x2AE}} t_0\text{‘}\alpha \uparrow t_0\text{‘}\beta \,.\equiv.\, R \in t\text{‘}(t_0\text{‘}\alpha \uparrow t_0\text{‘}\beta)$

∗64·201. $\vdash : R \mathrel{\unicode{x2AE}} S \,.\supset.\, R \in t\text{‘}S \,.\, t\text{‘}R = t\text{‘}S$

∗64·231. $\vdash : R \in t\text{‘}Q \,.\supset.\, \mathrm{D}\text{‘}R \in t\text{‘}\mathrm{D}\text{‘}Q \,.\, \text{Ɑ}\text{‘}R \in t\text{‘}\text{Ɑ}\text{‘}Q \,.\, C\text{‘}R \in t\text{‘}C\text{‘}Q$

Here "$C\text{‘}R \in t\text{‘}C\text{‘}Q$" will only be significant if R and Q are homogeneous relations, which is not required by the rest of the proposition. When R and Q are homogeneous relations we have

∗64·24. $\vdash : R \in t\text{‘}Q \,.\equiv.\, C\text{‘}R \in t\text{‘}C\text{‘}Q \,.\equiv.\, t_0\text{‘}C\text{‘}R = t_0\text{‘}C\text{‘}Q$

This proposition is useful in connecting ordinal and cardinal existence-theorems.

∗64·312. $\vdash .\, t^{22}\text{‘}x = t^{11}\text{‘}t\text{‘}x = t_{00}\text{‘}t^2\text{‘}x$

∗64·5. $\vdash .\, \mathrm{Rl}\text{‘}(t_0\text{‘}\alpha \uparrow t_0\text{‘}\beta) = t\text{‘}(t_0\text{‘}\alpha \uparrow t_0\text{‘}\beta) = t\text{‘}(\alpha \uparrow \beta)$

This proposition is frequently used. It states that the class of relations whose referents are of the type of members of α while their relata are of the type of members of β (*i.e.* the class of all relations contained in $t_0\text{‘}\alpha \uparrow t_0\text{‘}\beta$) is the type of $t_0\text{‘}\alpha \uparrow t_0\text{‘}\beta$ and is also the type of $\alpha \uparrow \beta$.

∗64·55. $\vdash : C\text{‘}P \subset t_0\text{‘}\alpha \,.\equiv.\, P \in t_{00}\text{‘}\alpha$

∗64·57. $\vdash : C\text{‘}P \subset t\text{‘}x \,.\equiv.\, P \in t^{11}\text{‘}x$

The propositions of the present number are mostly obvious, though formal proofs are sometimes not very easily found. The use of the propositions of this number occurs chiefly in the first section on relation-arithmetic and in the proofs of existence-theorems in ordinal arithmetic and the theory of ratio.

***64·01.** $t_{00}{}^\backprime\alpha = t^\backprime(t_0{}^\backprime\alpha \uparrow t_0{}^\backprime\alpha)$ Df

***64·011.** $t^{11}{}^\backprime x = t^\backprime(t^\backprime x \uparrow t^\backprime x)$ Df

***64·012.** $t^{12}{}^\backprime x = t^\backprime(t^\backprime x \uparrow t^{2}{}^\backprime x)$ Df

***64·013.** $t^{21}{}^\backprime x = t^\backprime(t^{2}{}^\backprime x \uparrow t^\backprime x)$ Df

***64·014.** $t^{22}{}^\backprime x = t^\backprime(t^{2}{}^\backprime x \uparrow t^{2}{}^\backprime x)$ Df

etc.

***64·02.** $t_{01}{}^\backprime\alpha = t^\backprime(t_0{}^\backprime\alpha \uparrow t_1{}^\backprime\alpha)$ Df

***64·021.** $t_{10}{}^\backprime\alpha = t^\backprime(t_1{}^\backprime\alpha \uparrow t_0{}^\backprime\alpha)$ Df

***64·022.** $t_{11}{}^\backprime\alpha = t^\backprime(t_1{}^\backprime\alpha \uparrow t_1{}^\backprime\alpha)$ Df

etc.

***64·03.** $t_0{}^{1}{}^\backprime\alpha = t^\backprime(t_0{}^\backprime\alpha \uparrow t^\backprime\alpha)$ Df

***64·031.** $t_1{}^{1}{}^\backprime\alpha = t^\backprime(t_1{}^\backprime\alpha \uparrow t^\backprime\alpha)$ Df

etc.

***64·04.** ${}^1t_0{}^\backprime\alpha = t^\backprime(t^\backprime\alpha \uparrow t_0{}^\backprime\alpha)$ Df

***64·041.** ${}^1t_1{}^\backprime\alpha = t^\backprime(t^\backprime\alpha \uparrow t_1{}^\backprime\alpha)$ Df

etc.

***64·1.** $\vdash . \alpha \uparrow \alpha \in t_{00}{}^\backprime\alpha$

Dem.

$\vdash . *21{\cdot}2 . \supset \vdash : \alpha = t_0{}^\backprime\alpha . \supset . \alpha \uparrow \alpha = t_0{}^\backprime\alpha \uparrow t_0{}^\backprime\alpha$ (1)

$\vdash . *35{\cdot}9 . \supset \vdash : \alpha \uparrow \alpha = t_0{}^\backprime\alpha \uparrow t_0{}^\backprime\alpha . \supset . \alpha = t_0{}^\backprime\alpha :$

[Transp] $\supset \vdash : \alpha \neq t_0{}^\backprime\alpha . \supset . \alpha \uparrow \alpha \neq t_0{}^\backprime\alpha \uparrow t_0{}^\backprime\alpha$ (2)

$\vdash . (1) . (2) . \supset \vdash :. \alpha = t_0{}^\backprime\alpha . \vee . \alpha \neq t_0{}^\backprime\alpha : \supset : \alpha \uparrow \alpha = t_0{}^\backprime\alpha \uparrow t_0{}^\backprime\alpha . \vee . \alpha \uparrow \alpha \neq t_0{}^\backprime\alpha \uparrow t_0{}^\backprime\alpha$ (3)

$\vdash . (3) . *51{\cdot}15 . *63{\cdot}101{\cdot}191 . \supset \vdash : \alpha \uparrow \alpha = t_0{}^\backprime\alpha \uparrow t_0{}^\backprime\alpha . \vee . \alpha \uparrow \alpha \neq t_0{}^\backprime\alpha \uparrow t_0{}^\backprime\alpha$ (4)

$\vdash . (4) . *51{\cdot}15 . *63{\cdot}101 . (*64{\cdot}01) . \supset \vdash .$ Prop

***64·11.** $\vdash . t_{00}{}^\backprime\alpha = t^\backprime(\alpha \uparrow \alpha)$ [*64·1 . *63·16]

***64·12.** $\vdash . \alpha \uparrow \beta \in t^\backprime(t_0{}^\backprime\alpha \uparrow t_0{}^\backprime\beta)$

Dem.

$\vdash . *35{\cdot}85{\cdot}86 . *63{\cdot}18 . \supset \vdash : \alpha \uparrow \beta = t_0{}^\backprime\alpha \uparrow t_0{}^\backprime\beta . \equiv . \alpha = t_0{}^\backprime\alpha . \beta = t_0{}^\backprime\beta$ (1)

$\vdash . (1) . \text{Transp} . \supset \vdash : \alpha = t_0{}^\backprime\alpha . \beta = t_0{}^\backprime\beta . \supset . \alpha \uparrow \beta = t_0{}^\backprime\alpha \uparrow t_0{}^\backprime\beta :$

$\alpha = t_0{}^\backprime\alpha . \beta \neq t_0{}^\backprime\beta . \supset . \alpha \uparrow \beta \neq t_0{}^\backprime\alpha \uparrow t_0{}^\backprime\beta :$

[*63·101.*51·15] $\supset \vdash : \alpha = t_0{}^\backprime\alpha . \supset . \alpha \uparrow \beta \in t^\backprime(t_0{}^\backprime\alpha \uparrow t_0{}^\backprime\beta)$ (2)

$\vdash . (1) . \text{Transp} . \supset \vdash : \alpha \neq t_0{}^\backprime\alpha . \supset . \alpha \uparrow \beta \neq (t_0{}^\backprime\alpha \uparrow t_0{}^\backprime\beta) .$

[*63·101.*51·15.Transp] $\supset . \alpha \uparrow \beta \in t^\backprime(t_0{}^\backprime\alpha \uparrow t_0{}^\backprime\beta)$ (3)

$\vdash . (2) . (3) . \supset \vdash .$ Prop

***64·13.** $\vdash . t^\backprime(t_0{}^\backprime\alpha \uparrow t_0{}^\backprime\beta) = t^\backprime(\alpha \uparrow \beta)$ [*64·12 . *63·16]

***64·14.** $\vdash . (x, y) . x\,(t_0{}^\backprime\alpha \uparrow t_0{}^\backprime\beta)\,y$ [*63·1 . *35·103]

***64·15.** $\vdash . (R) . R \in t_0{}^\backprime\alpha \uparrow t_0{}^\backprime\beta$ [*64·14 . *25·14·11]

∗64·16. $\vdash : R \subset t_0{}^{\backprime}\alpha \uparrow t_0{}^{\backprime}\beta . \equiv . R \in t^{\backprime}(t_0{}^{\backprime}\alpha \uparrow t_0{}^{\backprime}\beta)$

Dem.

$\vdash . *2{\cdot}11 . \supset \vdash : R = t_0{}^{\backprime}\alpha \uparrow t_0{}^{\backprime}\beta . \mathrm{v} . R \neq t_0{}^{\backprime}\alpha \uparrow t_0{}^{\backprime}\beta :$

$[*23{\cdot}42] \quad \supset \vdash : R = t_0{}^{\backprime}\alpha \uparrow t_0{}^{\backprime}\beta . R \subset t_0{}^{\backprime}\alpha \uparrow t_0{}^{\backprime}\beta . \mathrm{v} . R \neq t_0{}^{\backprime}\alpha \uparrow t_0{}^{\backprime}\beta \quad (1)$

$\vdash . (1) . *64{\cdot}15 . *10{\cdot}221{\cdot}13 . \supset$

$\vdash : R \subset t_0{}^{\backprime}\alpha \uparrow t_0{}^{\backprime}\beta : R = t_0{}^{\backprime}\alpha \uparrow t_0{}^{\backprime}\beta . R \subset t_0{}^{\backprime}\alpha \uparrow t_0{}^{\backprime}\beta . \mathrm{v} . R \neq t_0{}^{\backprime}\alpha \uparrow t_0{}^{\backprime}\beta \quad (2)$

$\vdash . (2) . *5{\cdot}1 . \supset$

$\vdash :. R \subset t_0{}^{\backprime}\alpha \uparrow t_0{}^{\backprime}\beta . \equiv : R = t_0{}^{\backprime}\alpha \uparrow t_0{}^{\backprime}\beta . R \subset t_0{}^{\backprime}\alpha \uparrow t_0{}^{\backprime}\beta . \mathrm{v} . R \neq t_0{}^{\backprime}\alpha \uparrow t_0{}^{\backprime}\beta :$

$[*23{\cdot}42] \quad \equiv : R = t_0{}^{\backprime}\alpha \uparrow t_0{}^{\backprime}\beta . \mathrm{v} . R \neq t_0{}^{\backprime}\alpha \uparrow t_0{}^{\backprime}\beta :. \supset \vdash . \mathrm{Prop}$

By putting $t_s{}^{i\backprime}\alpha$ (where i and s are some index and suffix which have been defined) for α and $t_{s'}{}^{i'\backprime}\alpha$ for β, the above propositions give results applicable to any of the types defined at the beginning of this number, because of $t_0{}^{\backprime}t_s{}^{i\backprime}\alpha = t_s{}^{i\backprime}\alpha$.

∗64·2. $\vdash : \dot{\exists} ! R \dot{\frown} S . \supset . S \in t^{\backprime}R . t^{\backprime}R = t^{\backprime}S \quad [*63{\cdot}13{\cdot}16]$

∗64·201. $\vdash : R \subset S . \supset . R \in t^{\backprime}S . t^{\backprime}R = t^{\backprime}S$

Dem.

$\vdash . *25{\cdot}6 . \supset \vdash :. \mathrm{Hp} . \supset : R = S . \mathrm{v} . \dot{\exists} ! S \dot{-} R :$

$[*13{\cdot}14] \quad \supset : R = S . \mathrm{v} . R \neq S :. \supset \vdash . \mathrm{Prop}$

∗64·21. $\vdash : xRy . \supset . R \in t^{\backprime}(t^{\backprime}x \uparrow t^{\backprime}y)$

Dem.

$\vdash . *63{\cdot}103 . *35{\cdot}103 . \supset \vdash . x (t^{\backprime}x \uparrow t^{\backprime}y) y \quad (1)$

$\vdash . (1) . \quad \supset \vdash : \mathrm{Hp} . \supset . \dot{\exists} ! R \dot{\frown} (t^{\backprime}x \uparrow t^{\backprime}y) \quad (2)$

$\vdash . (2) . *64{\cdot}2 . \quad \supset \vdash . \mathrm{Prop}$

∗64·22. $\vdash . R \in t^{\backprime}(t_0{}^{\backprime}\mathrm{D}^{\backprime}R \uparrow t_0{}^{\backprime}\mathrm{G}^{\backprime}R) \quad [*64{\cdot}16 . *63{\cdot}105 . *35{\cdot}83]$

∗64·23. $\vdash . t^{\backprime}R = t^{\backprime}\dot{s}^{\backprime}t^{\backprime}R$

Dem.

$\vdash . *63{\cdot}103 . *41{\cdot}13 . \supset \vdash . R \subset \dot{s}^{\backprime}t^{\backprime}R \quad (1)$

$\vdash . (1) . *64{\cdot}201 . \quad \supset \vdash . \mathrm{Prop}$

∗64·231. $\vdash : R \in t^{\backprime}Q . \supset . \mathrm{D}^{\backprime}R \in t^{\backprime}\mathrm{D}^{\backprime}Q . \mathrm{G}^{\backprime}R \in t^{\backprime}\mathrm{G}^{\backprime}Q . C^{\backprime}R \in t^{\backprime}C^{\backprime}Q$

Dem.

$\vdash . *63{\cdot}12 . \supset \vdash ::. \mathrm{Hp} . \supset :: xRy . \supset_{x,y} :. xQy . \mathrm{v} . \sim(xQy) ::$

$[*10{\cdot}28] \quad \supset :: (\exists y) . xRy . \supset_x :. (\exists y) . xQy . \mathrm{v} . (\exists y) . \sim(xQy) :.$

$[*5{\cdot}63] \quad \supset_x :. (\exists y) . xQy :. \mathrm{v} :. \sim(\exists y) . xQy : (\exists y) . \sim xQy :.$

$[*3{\cdot}26] \quad \supset_x :. (\exists y) . xQy . \mathrm{v} . \sim(\exists y) . xQy \quad (1)$

$\vdash . (1) . *33{\cdot}13 . \supset \vdash :. \mathrm{Hp} . \supset : x \in \mathrm{D}^{\backprime}R . \supset_x . x \in \mathrm{D}^{\backprime}Q \cup -\mathrm{D}^{\backprime}Q :$

$[(*63{\cdot}02)] \quad \supset : \mathrm{D}^{\backprime}R \subset t_0{}^{\backprime}\mathrm{D}^{\backprime}Q :$

$[*63{\cdot}371] \quad \supset : \mathrm{D}^{\backprime}R \in t^{\backprime}\mathrm{D}^{\backprime}Q \quad (2)$

Similarly $\quad \vdash : \mathrm{Hp} . \supset . \mathrm{G}^{\backprime}R \in t^{\backprime}\mathrm{G}^{\backprime}Q . C^{\backprime}R \in t^{\backprime}C^{\backprime}Q \quad (3)$

$\vdash . (2) . (3) . \quad \supset \vdash . \mathrm{Prop}$

***64·24.** $\vdash : R \epsilon t'Q . \equiv . C'R \epsilon t'C'Q . \equiv . t_0'C'R = t_0'C'Q$

This proposition is only significant when R and Q are homogeneous relations.

Dem.

$\vdash . *64·22 . *63·181 . \supset \vdash . R \epsilon t'(t_0'C'R \uparrow t_0'C'R) .$

$[*13·12] \qquad \supset \vdash : t_0'C'R = t_0'C'Q . \supset . R \epsilon t'(t_0'C'Q \uparrow t_0'C'Q) \quad (1)$

$\vdash . *64·22 . *63·181 . \supset \vdash . Q \epsilon t'(t_0'C'Q \uparrow t_0'C'Q) \quad (2)$

$\vdash . (1) . (2) . *63·16 . \supset \vdash : t_0'C'R = t_0'C'Q . \supset . R \epsilon t'Q \quad (3)$

$\vdash . (3) . *64·231 . *63·16·37 . \supset$

$\vdash : R \epsilon t'Q . \equiv . t_0'C'R = t_0'C'Q . \equiv . C'R \epsilon t'C'Q : \supset \vdash . \text{Prop}$

***64·3.** $\vdash : t_{00}'\alpha = t_{00}'\beta . \equiv . \alpha \epsilon t'\beta . \equiv . t'\alpha = t'\beta . \equiv . t_0'\alpha = t_0'\beta$

Dem.

$\vdash . *30·37 . (*64·01) . \quad \supset \vdash : t_0'\alpha = t_0'\beta . \supset . t_{00}'\alpha = t_{00}'\beta \quad (1)$

$\vdash . *64·1 . \quad \supset \vdash : t_{00}'\alpha = t_{00}'\beta . \supset . \alpha \uparrow \alpha \epsilon t_{00}'\beta .$

$[*64·16] \qquad \supset . \alpha \uparrow \alpha \in t_0'\beta \uparrow t_0'\beta .$

$[*35·9·91] \qquad \supset . \alpha \subset t_0'\beta .$

$[*63·181] \qquad \supset . t_0'\alpha = t_0'\beta \quad (2)$

$\vdash . (1) . (2) . *63·16·37 . \supset \vdash . \text{Prop}$

***64·31.** $\vdash . t^{11}'x = t_{00}'t'x \qquad [*63·15 . (*64·01·011)]$

***64·311.** $\vdash . t_{11}'\alpha = t_{00}'t_1'\alpha \qquad [*63·321 . (*64·022·01)]$

***64·312.** $\vdash . t^{22}'x = t^{11}'t'x = t_{00}'t^{2}'x \qquad [*63·15 . (*63·04) . (*64·014·011·01)]$

***64·313.** $\vdash . t_{22}'\alpha = t_{11}'t_1'\alpha = t_{00}'t_2'\alpha \qquad [*63·321 . (*63·05)]$

***64·32.** $\vdash : t_{22}'\alpha = t_{22}'\beta . \equiv . t_{11}'\alpha = t_{11}'\beta . \equiv . t_{00}'\alpha = t_{00}'\beta . \equiv . t^{11}'\alpha = t^{11}'\beta .$
$\equiv . t^{22}'\alpha = t^{22}'\beta . \equiv . \alpha \epsilon t'\beta . \equiv . t'\alpha = t'\beta$

Dem.

$\vdash . *64·313·3 . \supset \vdash : t_{22}'\alpha = t_{22}'\beta . \equiv . t't_2'\alpha = t't_2'\beta .$

$[*63·41·39] \qquad \equiv . t'\alpha = t'\beta$

Similarly the other equivalences are proved.

***64·33.** $\vdash : \alpha \epsilon t_0'\mu . \equiv . t_{11}'\alpha = t_{22}'\mu . \equiv . t_{00}'\alpha = t_{11}'\mu . \equiv . t^{11}'\alpha = t_{00}'\mu .$
$\equiv . t^{22}'\alpha = t^{11}'\mu . \equiv . t'\alpha = t_0'\mu$

Dem.

$\vdash . *64·311·313 . \supset \vdash : t_{11}'\alpha = t_{22}'\mu . \equiv . t_{00}'t_1'\alpha = t_{00}'t_2'\mu .$

$[*64·3] \qquad \equiv . t't_1'\alpha = t't_2'\mu .$

$[*63·383·41·55] \qquad \equiv . t'\alpha = t_0'\mu \quad (1)$

Similarly the other equivalences are proved.

***64·34.** $\vdash : \alpha \epsilon t_1'\mu . \equiv . t_{00}'\alpha = t_{22}'\mu . \equiv . t^{11}'\alpha = t_{11}'\mu . \equiv . t^{22}'\alpha = t_{00}'\mu . \equiv . t^{2}'\alpha = t_0'\mu$
[Proof as in *64·33]

***64·5.** $\vdash . \mathrm{Rl}\text{‘}(t_0\text{‘}\alpha \uparrow t_0\text{‘}\beta) = t\text{‘}(t_0\text{‘}\alpha \uparrow t_0\text{‘}\beta) = t\text{‘}(\alpha \uparrow \beta)$ [*64·13·16 . *61·2]

***64·51.** $\vdash . x \downarrow y \in t\text{‘}(t\text{‘}x \uparrow t\text{‘}y)$ [*64·21 . *55·132]

***64·52.** $\vdash : x \in t_0\text{‘}\alpha . y \in t_0\text{‘}\beta . \supset . x \downarrow y \in t\text{‘}(t_0\text{‘}\alpha \uparrow t_0\text{‘}\beta)$ [*63·11 . *64·51]

***64·53.** $\vdash : x \in t_0\text{‘}\alpha . \delta \subset t_0\text{‘}\beta . \supset . (\iota\text{‘}x) \downarrow \delta \in t\text{‘}(t\text{‘}\alpha \uparrow t\text{‘}\beta)$

Dem.

$\vdash . *64·51 . \supset \vdash . (\iota\text{‘}x) \downarrow \delta \in t\text{‘}(t\text{‘}\iota\text{‘}x \uparrow t\text{‘}\delta)$ (1)

$\vdash . *63·62 . \supset \vdash : \mathrm{Hp} . \supset . t\text{‘}\iota\text{‘}x = t\text{‘}\alpha$ (2)

$\vdash . *63·181·37 . \supset \vdash : \mathrm{Hp} . \supset . t\text{‘}\delta = t\text{‘}\beta$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash . \mathrm{Prop}$

This proposition is used in connection with cardinal addition (*110·18).

***64·54.** $\vdash . \mathrm{Rl}\text{‘}(t_0\text{‘}\alpha \uparrow t_0\text{‘}\alpha) = t_{00}\text{‘}\alpha = t\text{‘}(\alpha \uparrow \alpha) = t_0\text{‘}\mathrm{Rl}\text{‘}(\alpha \uparrow \alpha)$
[*64·5 . *61·34 . *63·105·11 . (*64·01)]

***64·55.** $\vdash : C\text{‘}P \subset t_0\text{‘}\alpha . \equiv . P \in t_{00}\text{‘}\alpha$

Dem.

$\vdash . *35·91 . \supset \vdash : C\text{‘}P \subset t_0\text{‘}\alpha . \equiv . P \mathrel{\overset{\cdot}{\subset}} t_0\text{‘}\alpha \uparrow t_0\text{‘}\alpha .$

[*64·54] $\equiv . P \in t_{00}\text{‘}\alpha : \supset \vdash . \mathrm{Prop}$

***64·56.** $\vdash . \mathrm{Rl}\text{‘}(t\text{‘}x \uparrow t\text{‘}x) = t^{11}\text{‘}x$

Dem.

$\vdash . *64·5 . *63·15 . \supset \vdash . \mathrm{Rl}\text{‘}(t\text{‘}x \uparrow t\text{‘}x) = t\text{‘}(t\text{‘}x \uparrow t\text{‘}x)$

[(*64·011)] $= t^{11}\text{‘}x . \supset \vdash . \mathrm{Prop}$

***64·57.** $\vdash : C\text{‘}P \subset t\text{‘}x . \equiv . P \in t^{11}\text{‘}x$ [*64·56 . *35·91 . *61·2]

***64·6.** $\vdash . t\text{‘}P = \mathrm{Rl}\text{‘}(t_0\text{‘}\mathrm{D}\text{‘}P \uparrow t_0\text{‘}\mathrm{G}\text{‘}P)$

Dem.

$\vdash . *35·83 . *63·105 . \supset \vdash . P \mathrel{\overset{\cdot}{\subset}} t_0\text{‘}\mathrm{D}\text{‘}P \uparrow t_0\text{‘}\mathrm{G}\text{‘}P .$

[*64·201] $\supset \vdash . t\text{‘}P = t\text{‘}(t_0\text{‘}\mathrm{D}\text{‘}P \uparrow t_0\text{‘}\mathrm{G}\text{‘}P)$

[*64·5] $= \mathrm{Rl}\text{‘}(t_0\text{‘}\mathrm{D}\text{‘}P \uparrow t_0\text{‘}\mathrm{G}\text{‘}P) . \supset \vdash . \mathrm{Prop}$

***64·61.** $\vdash : \mathrm{D}\text{‘}P \in t\text{‘}\alpha . \mathrm{G}\text{‘}P \in t\text{‘}\beta . \supset . t\text{‘}P = t\text{‘}(\alpha \uparrow \beta)$

Dem.

$\vdash . *63·16·35 . \supset \vdash : \mathrm{Hp} . \supset . t_0\text{‘}\mathrm{D}\text{‘}P = t_0\text{‘}\alpha . t_0\text{‘}\mathrm{G}\text{‘}P = t_0\text{‘}\beta .$

[*64·6] $\supset . t\text{‘}P = t\text{‘}(t_0\text{‘}\alpha \uparrow t_0\text{‘}\beta)$

[*64·5] $= t\text{‘}(\alpha \uparrow \beta) : \supset \vdash . \mathrm{Prop}$

***64·62.** $\vdash : \mathrm{D}\text{‘}P \in t\text{‘}\mathrm{D}\text{‘}Q . \mathrm{G}\text{‘}P \in t\text{‘}\mathrm{G}\text{‘}Q . \equiv . P \in t\text{‘}Q . \equiv . t\text{‘}P = t\text{‘}Q$

Dem.

$\vdash . *64·61 . \supset \vdash : \mathrm{Hp} . \supset . t\text{‘}P = t\text{‘}(\mathrm{D}\text{‘}Q \uparrow \mathrm{G}\text{‘}Q)$

[*64·5·22 . *63·16] $= t\text{‘}Q$ (1)

$\vdash . (1) . *64·231 . \supset \vdash . \mathrm{Prop}$

***64·63.** $\vdash : \mathrm{D}\text{‘}P \in t\text{‘}\alpha . \mathrm{G}\text{‘}P \in t\text{‘}\beta . \equiv . t\text{‘}P = t\text{‘}(\alpha \uparrow \beta) . \equiv . P \in t\text{‘}(\alpha \uparrow \beta)$

Dem.

$\vdash . *64·5 . \supset \vdash : t\text{‘}P = t\text{‘}(\alpha \uparrow \beta) . \supset . t\text{‘}P = t\text{‘}(t_0\text{‘}\alpha \uparrow t_0\text{‘}\beta) .$

[*64·231 . *35·85·86] $\supset . \mathrm{D}\text{‘}P \in t\text{‘}t_0\text{‘}\alpha . \mathrm{G}\text{‘}P \in t\text{‘}t_0\text{‘}\beta .$

[*63·19] $\supset . \mathrm{D}\text{‘}P \in t\text{‘}\alpha . \mathrm{G}\text{‘}P \in t\text{‘}\beta$ (1)

$\vdash . (1) . *64·61 . *63·16 . \supset \vdash . \mathrm{Prop}$

$*65$. ON THE TYPICAL DEFINITION OF AMBIGUOUS SYMBOLS

Summary of $*65$.

In this number we are concerned with definitions and propositions in which an ambiguous symbol is determined as belonging to some assigned type. If "α" is an ambiguous symbol representing a class (such as Λ or V for example), "α_x" is to denote what α becomes when its members are determined as belonging to the type of x, while "$\alpha(x)$" denotes what α becomes when its members are determined as belonging to the type of $t'x$. Thus *e.g.* "V_x" will be everything of the same type as x, *i.e.* $t'x$; $V(x)$ will be $t't'x$. Similarly if "R" stands for a relation of ambiguous type, such as $\dot{\Lambda}$ or $\dot{V}$, R_x will denote what R becomes when its domain is confined within the type of x; $R_{(x,y)}$ will denote what R becomes when its domain and converse domain are confined respectively within the types of x and y; $R(x,y)$ will have the domain and converse domain confined respectively to the types of $t'x$ and $t'y$; with analogous meanings for $R(x)$ and $R(x_y)$. Throughout this number, R and α do not stand for proper variables, but for typically ambiguous symbols.

The notations of the present number are used in the elementary parts of the theory of cardinals and ordinals, *i.e.* in Part III, Section A, and in Part IV, Section A. The only *proposition*, however, which is much used, is

$*65 \cdot 13$. $\vdash : \alpha = \beta_x . \equiv . \alpha = t'x \cap \beta . \equiv . \alpha \subset t'x . \alpha = \beta$

Here β is supposed to be a typically ambiguous symbol. The first equivalence, "$\alpha = \beta_x . \equiv . \alpha = t'x \cap \beta$," merely embodies the definition of β_x ($*65 \cdot 01$). It is the second equivalence that is important. Let us, for the sake of illustration, put 1 in place of β. Then we are to have

$$\alpha = t'x \cap 1 . \equiv . \alpha \subset t'x . \alpha = 1.$$

(Since 1 is a class of classes, we shall have to suppose that x is a class.) Consider $y \in \alpha$. If $\alpha = t'x \cap 1$, $y \in \alpha . \equiv . y \in t'x . y \in 1$. But we have $(y) . y \in t'x$. Hence $y \in \alpha . \equiv . y \in 1$, whence $\alpha = 1$. Also if $\alpha = t'x \cap 1$, of course $\alpha \subset t'x$. Thus $\alpha = t'x \cap 1 . \supset . \alpha \subset t'x . \alpha = 1$. The converse implication follows from $*22 \cdot 621$. The reason for the proposition is that a symbol such as "1," if it occurs in such a proposition as $\alpha = t'x \cap 1$, must, for significance, be determined as meaning that 1 which is of the same type as α, *i.e.* the class of all unit classes which are of the same type as members of α. And similarly, when we put $\alpha = 1$, that does not mean that α is the class of all unit classes, but only that it is the class of all unit classes of the appropriate type, which

if $\alpha \subset t`x$, will be $t`x \cap 1$. The proposition "$t`x \cap 1 = 1$" is true whenever it is significant, but $t`x \cap 1$ is typically definite when x is given, whereas 1 is typically ambiguous. The use of the above proposition lies in its enabling us to substitute typically definite symbols for such as are typically ambiguous.

Another useful proposition is

***65·2.** $\vdash . \text{sg}`\{R_{(x,y)}\} = \overrightarrow{R}(x_y)$

Here R is supposed to be a typically ambiguous symbol; the proposition states that if R is typically defined as going from objects of type x to objects of type y, then $\overrightarrow{R}$ must go from objects of type $t`x$ to objects of type y. This proposition is only used twice (*102·3 and *154·2), but both uses are of great importance, the one in cardinal and the other in ordinal arithmetic.

The only other proposition of this number which is subsequently used is

***65·3.** $\vdash . R_\beta``\mu = (R``\mu)_\beta = R``\mu \cap t`\beta$

This proposition is used in *102·84.

***65·01.** $\alpha_x = \alpha \cap t`x$ Df

***65·02.** $\alpha(x) = \alpha \cap t`t`x$ Df

***65·03.** $R_x = (t`x) \upharpoonleft R$ Df

***65·04.** $R(x) = (t^2`x) \upharpoonleft R$ Df

***65·1.** $R_{(x,y)} = (t`x) \upharpoonleft R \upharpoonright (t`y)$ Df

***65·11.** $R(x_y) = (t^2`x) \upharpoonleft R \upharpoonright (t`y)$ Df

***65·12.** $R(x,y) = (t^2`x) \upharpoonleft R \upharpoonright (t^2`y)$ Df

***65·13.** $\vdash : \alpha = \beta_x . \equiv . \alpha = t`x \cap \beta . \equiv . \alpha \subset t`x . \alpha = \beta$

Dem.

$\vdash . *4·2 . (*65·01) . \quad \supset \vdash : \alpha = \beta_x . \equiv . \alpha = t`x \cap \beta \quad (1)$

$\vdash . *22·621 . *13·13 . \supset \vdash : \alpha \subset t`x . \alpha = \beta . \supset . \alpha = t`x \cap \beta \quad (2)$

$\vdash . *22·43 . \quad \supset \vdash : \alpha = t`x \cap \beta . \supset . \alpha \subset t`x . \alpha \subset \beta . \quad (3)$

$[*63·13] \quad \supset . \beta \in t`t`x .$

$[*63·371·15] \quad \supset . \beta \subset t`x .$

$[*22·621] \quad \supset . \beta = t`x \cap \beta \quad (4)$

$\vdash . (3) . (4) . \quad \supset \vdash : \alpha = t`x \cap \beta . \supset . \alpha \subset t`x . \alpha = \beta \quad (5)$

$\vdash . (1) . (2) . (5) . \quad \supset \vdash . \text{Prop}$

***65·14.** $\vdash : x \in t_0`\alpha . \supset . \gamma(x) = \gamma_\alpha$ [*63·53 . (*65·01·02)]

***65·15.** $\vdash : x \in t_0`\alpha . \supset . R(x) = R_\alpha . R(x_y) = R_{(\alpha,y)}$ [*63·53 . (*65·03·04·1·11)]

***65·16.** $\vdash : x \in t_0`\alpha . y \in t_0`\beta . \supset . R(x,y) = R(x_\beta) = R_{(\alpha,\beta)}$ [*63·53 . (*65·1·11·12)]

∗65·2. $\vdash . \mathrm{sg}`\{R_{(x,y)}\} = \overrightarrow{R}(x_y)$

Dem.

$\vdash . *32{\cdot}1{\cdot}23 . (*65{\cdot}1) . \supset$

$\vdash : \alpha\,[\mathrm{sg}`\{R_{(x,y)}\}]\,w . \equiv . \alpha = \hat{z}\,\{z \,\epsilon\, t`x . w \,\epsilon\, t`y . zRw\} .$

$[*22{\cdot}39 . *20{\cdot}42] \quad \equiv . \alpha = t`x \cap \hat{z}\,(w \,\epsilon\, t`y . zRw) .$

$[*65{\cdot}13] \quad \equiv . \alpha \subset t`x . \alpha = \hat{z}\,(w \,\epsilon\, t`y . zRw) \qquad (1)$

$\vdash . *20{\cdot}33 . \supset \vdash :: \alpha = \hat{z}\,(w \,\epsilon\, t`y . zRw) . \equiv :. z \,\epsilon\, \alpha . \equiv_z . w \,\epsilon\, t`y . zRw :.$

$[*63{\cdot}108] \quad \equiv :. w \,\epsilon\, t`y : z \,\epsilon\, \alpha . \equiv_z . w \,\epsilon\, t`y . zRw :.$

$[*4{\cdot}73] \quad \equiv :. w \,\epsilon\, t`y : z \,\epsilon\, \alpha . \equiv_z . zRw :.$

$[*20{\cdot}33 . *32{\cdot}1] \quad \equiv :. w \,\epsilon\, t`y . \alpha\overrightarrow{R}w \qquad (2)$

$\vdash . (1) . (2) . *63{\cdot}5 . \supset \vdash : \alpha\,[\mathrm{sg}`\{R_{(x,y)}\}]\,w . \equiv . \alpha \,\epsilon\, t^{2}`x . w \,\epsilon\, t`y . \alpha\overrightarrow{R}w .$

$[*35{\cdot}102 . (*65{\cdot}11)] \quad \equiv . \alpha\,\{\overrightarrow{R}(x_y)\}\,w : \supset \vdash . \mathrm{Prop}$

∗65·21. $\vdash . R_{(x,y)} = \{R_{(x,y)}\}_{(x,y)}$

Dem.

$\vdash . *21{\cdot}2 . (*65{\cdot}1) . \supset \vdash . \{R_{(x,y)}\}_{(x,y)} = t`x \,\urcorner\, \{t`x \,\urcorner\, R \restriction t`y\} \restriction t`y$

$[*35{\cdot}33{\cdot}34] \quad = t`x \,\urcorner\, R \restriction t`y$

$[(*65{\cdot}1)] \quad = R_{(x,y)} . \supset \vdash . \mathrm{Prop}$

∗65·22. $\vdash . R(x,y) = \{R(x,y)\}(x,y)$

This and the following three propositions are proved as ∗65·21 is proved.

∗65·23. $\vdash . R(x_y) = \{R(x_y)\}(x_y)$

∗65·24. $\vdash . R_x = (R_x)_x$

∗65·25. $\vdash . R(x) = \{R(x)\}(x)$

∗65·3. $\vdash . R_\beta``\mu = (R``\mu)_\beta = R``\mu \cap t`\beta$

Dem.

$\vdash . *37{\cdot}1 . (*65{\cdot}03) . \supset \vdash . R_\beta``\mu = \hat{x}\,\{(\exists y) . y \,\epsilon\, \mu . xRy . x \,\epsilon\, t`\beta\}$

$[*22{\cdot}39 . (*37{\cdot}01)] \quad = R``\mu \cap t`\beta \qquad (1)$

$[(*65{\cdot}01)] \quad = (R``\mu)_\beta \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

SECTION C

ONE-MANY, MANY-ONE, AND ONE-ONE RELATIONS

Summary of Section C.

In the present section we have to consider three very important classes of relations, of which the use in arithmetic is constant. A *one-many* relation is a relation R such that, if y is any member of $\mathrm{G}'R$, there is one, and only one, term x which has the relation R to y, *i.e.* $\overrightarrow{R}'y \in 1$. Thus the relation of father to son is one-many, because every son has one father and no more. The relation of husband to wife is one-many except in countries which practise polyandry. (It is one-many in monogamous as well as in polygamous countries, because, according to the definition, nothing is fixed as to the number of relata for a given referent, and there *may* be only one relatum for each given referent without the relation ceasing to be one-many according to the definition.) The relation in algebra of x^2 to x is one-many, but that of x to x^2 is not, because there are two different values of x that give the same value of x^2.

When a relation R is one-many, $R'y$ exists whenever $y \in \mathrm{G}'R$, and vice versa; *i.e.* we have

$$R \in \text{one-many} \,.\equiv :\, y \in \mathrm{G}'R \,.\supset_y .\, \mathrm{E}\,!\, R'y.$$

Thus relations which give descriptive functions that are existent whenever their arguments belong to the converse domains of the relations in question are one-many relations. Hence Cnv, D, G, C, $\overrightarrow{R}$, $\overleftarrow{R}$, sg, gs, R_ϵ, p, s, $\dot{p}$, $\dot{s}$, I, ι, $\breve{\iota}$, Cl, Rl are all of them one-many relations.

When R is a one-many relation, $R'y$ is a one-valued function; conversely, every one-valued function is derivable from a one-many relation. A *many-valued* function of y is a member of $\overrightarrow{R}'y$, where $\overrightarrow{R}'y$ is not a unit class, and any one of its members is regarded as a value of the function for the argument y; but a *one*-valued function of y is the single term $R'y$ which is obtained when R is one-many. Thus for example the sine would, in our notation, appear as a relation, *i.e.* we should put

$$\sin = \hat{x}\hat{y}\,\{x = y - y^3/3\,! + y^5/5\,! - \ldots\} \quad \mathrm{Df},$$

whence
$$\sin'y = y - y^3/3\,! + y^5/5\,! - \ldots,$$

so that "$\sin'y$" has the usual meaning of $\sin y$. Then instead of $\sin^{-1} x$, we should have $\overleftarrow{\sin}'x$, which would be the class of values of $\sin^{-1} x$; and instead of "$y = \sin^{-1} x$," which is a misleading notation because $y = \sin^{-1} x$ and $z = \sin^{-1} x$ do not imply $y = z$, we should have $y \in \overleftarrow{\sin}'x$. Similar remarks would apply to any of the other functions that occur in analysis.

A relation R is called *many-one* when, if x is any member of $\text{D}'R$, there is one, and only one, term y to which x has the relation R, *i.e.* $\overleftarrow{R}'x \in 1$. Thus many-one relations are the converses of one-many relations. When a relation R is many-one, $\breve{R}'x$ exists whenever $x \in \text{D}'R$.

A relation is called *one-one* when it is both one-many and many-one, or, what comes to the same, when both it and its converse are one-many. Of the one-many relations above enumerated, Cnv, sg, gs, I, ι, $\breve{\iota}$, Cl, Rl are one-one.

Two classes α, β are said to be *similar* when there is a one-one relation R such that $\text{D}'R = \alpha \,.\, \text{Ɑ}'R = \beta$, *i.e.* when their terms can be connected one to one, so that no term of either is omitted or repeated. We write "$\alpha \,\text{sm}\, \beta$" for "$\alpha$ is similar to β." When two classes are similar, the cardinal numbers of their terms are the same; it is this fact chiefly that makes one-one relations of fundamental importance in cardinal arithmetic.

According to the above, a relation is one-many when

$$y \in \text{Ɑ}'R \,.\, \supset_y .\, \overrightarrow{R}'y \in 1,$$

i.e. when

$$\overrightarrow{R}``\text{Ɑ}'R \subset 1.$$

Similarly a relation is many-one when

$$\overleftarrow{R}``\text{D}'R \subset 1,$$

and a relation is one-one when both conditions are fulfilled. The classes $\overrightarrow{R}``\text{Ɑ}'R$, $\overleftarrow{R}``\text{D}'R$, which appear here, are often important; some of their properties have already been given in *37·77·771·772·773 and in *53·61 to *53·641.

It is convenient to regard one-many, many-one and one-one relations as particular cases of relations which, for some given α and β, have

$$\overrightarrow{R}``\text{Ɑ}'R \subset \alpha \,.\, \overleftarrow{R}``\text{D}'R \subset \beta.$$

We put $\quad \alpha \rightarrow \beta = \hat{R}\{\overrightarrow{R}``\text{Ɑ}'R \subset \alpha \,.\, \overleftarrow{R}``\text{D}'R \subset \beta\} \quad \text{Df.}$

Hence, without a new definition, "$1 \rightarrow 1$" becomes the class of one-one relations; also, as will be shown, "$1 \rightarrow \text{Cls}$" becomes the class of one-many relations, and "$\text{Cls} \rightarrow 1$" becomes the class of many-one relations. Although it is chiefly these three special values of $\alpha \rightarrow \beta$ that are important, we shall begin by a general study of classes of relations of the form $\alpha \rightarrow \beta$.

$*70$. RELATIONS WHOSE CLASSES OF REFERENTS AND OF RELATA BELONG TO GIVEN CLASSES

Summary of $*70$.

If α and β are two given classes of classes, a relation R is said to belong to the class $\alpha \rightarrow \beta$ if $\overrightarrow{R}‘y \in \alpha$ whenever $y \in \text{Cl}‘R$, and $\overleftarrow{R}‘x \in \beta$ whenever $x \in \text{D}‘R$. If only one of these conditions is to be imposed, this result is secured by replacing the class involved in the other condition by "Cls," since "$\overrightarrow{R}‘y \in \text{Cls}$" always holds, and so does "$\overleftarrow{R}‘x \in \text{Cls}$," and therefore neither imposes any limitation on R. In the most important cases, α and β are either both cardinal numbers, or one is a cardinal number while the other is Cls.

In virtue of $*37{\cdot}702{\cdot}703$, the conditions above mentioned as imposed upon R by membership of $\alpha \rightarrow \beta$ are equivalent to

$$\overrightarrow{R}‘‘\text{Cl}‘R \subset \alpha \,.\, \overleftarrow{R}‘‘\text{D}‘R \subset \beta.$$

This form is used in the definition ($*70{\cdot}01$).

The propositions of the present number are hardly ever used except in $*71$, where α and β are both replaced by 1 or Cls. The most useful propositions are

$*70{\cdot}1$. $\vdash : R \in \alpha \rightarrow \beta \,.\equiv.\, \overrightarrow{R}‘‘\text{Cl}‘R \subset \alpha \,.\, \overleftarrow{R}‘‘\text{D}‘R \subset \beta$

(This merely embodies the definition.)

$*70{\cdot}13$. $\vdash :. R \in \alpha \rightarrow \beta \,.\equiv:\, (y) \,.\, \overrightarrow{R}‘y \in \alpha \cup \iota‘\Lambda : (x) \,.\, \overleftarrow{R}‘x \in \beta \cup \iota‘\Lambda$

$*70{\cdot}22$. $\vdash .\, \beta \rightarrow \alpha = \text{Cnv}‘‘(\alpha \rightarrow \beta)$

$*70{\cdot}4$. $\vdash .\, \alpha \rightarrow \text{Cls} = \hat{R}\,(\overrightarrow{R}‘‘\text{Cl}‘R \subset \alpha)$

$*70{\cdot}41$. $\vdash .\, \text{Cls} \rightarrow \beta = \hat{R}\,(\overleftarrow{R}‘‘\text{D}‘R \subset \beta)$

$*70{\cdot}42$. $\vdash .\, \alpha \rightarrow \beta = (\alpha \rightarrow \text{Cls}) \cap (\text{Cls} \rightarrow \beta)$

$*70{\cdot}54$. $\vdash : \text{Cl}‘R \cap \text{Cl}‘S = \Lambda \,.\, R, S \in \alpha \rightarrow \text{Cls} \,.\supset.\, R \cup S \in \alpha \rightarrow \text{Cls}$

with similar propositions for $\text{Cls} \rightarrow \beta$ and $\alpha \rightarrow \beta$.

$*70{\cdot}62$. $\vdash : R \in \alpha \rightarrow \text{Cls} \,.\supset.\, R \upharpoonright \gamma \in \alpha \rightarrow \text{Cls}$

with a similar proposition for $\text{Cls} \rightarrow \beta$.

$*70{\cdot}01$. $\alpha \rightarrow \beta = \hat{R}\,(\overrightarrow{R}‘‘\text{Cl}‘R \subset \alpha \,.\, \overleftarrow{R}‘‘\text{D}‘R \subset \beta)$ Df

$*70{\cdot}1$. $\vdash : R \in \alpha \rightarrow \beta \,.\equiv.\, \overrightarrow{R}‘‘\text{Cl}‘R \subset \alpha \,.\, \overleftarrow{R}‘‘\text{D}‘R \subset \beta$ [$*20{\cdot}3 \,.\, (*70{\cdot}01)$]

$*70{\cdot}11$. $\vdash :. R \in \alpha \rightarrow \beta \,.\equiv:\, y \in \text{Cl}‘R \,.\supset_y.\, \overrightarrow{R}‘y \in \alpha : x \in \text{D}‘R \,.\supset_x.\, \overleftarrow{R}‘x \in \beta$
[$*37{\cdot}702{\cdot}703 \,.\, *70{\cdot}1$]

***70·12.** $\vdash : R \epsilon \alpha \rightarrow \beta . \equiv . \overrightarrow{R}``\mathrm{V} \subset \alpha \cup \iota`\Lambda . \overleftarrow{R}``\mathrm{V} \subset \beta \cup \iota`\Lambda$ [*70·1 . *53·62·621]

***70·13.** $\vdash :. R \epsilon \alpha \rightarrow \beta . \equiv : (y) . \overrightarrow{R}`y \epsilon \alpha \cup \iota`\Lambda : (x) . \overleftarrow{R}`x \epsilon \beta \cup \iota`\Lambda$

Dem.

$\vdash . *37·702 . \supset \vdash :. \overrightarrow{R}``\mathrm{V} \subset \alpha \cup \iota`\Lambda . \equiv : y \epsilon \mathrm{V} . \supset_y . \overrightarrow{R}`y \epsilon \alpha \cup \iota`\Lambda :$

[*24·104.*5·5] $\equiv : (y) . \overrightarrow{R}`y \epsilon \alpha \cup \iota`\Lambda$ (1)

Similarly $\vdash :. \overleftarrow{R}``\mathrm{V} \subset \beta \cup \iota`\Lambda . \equiv : (x) . \overleftarrow{R}`x \epsilon \beta \cup \iota`\Lambda$ (2)

$\vdash . (1) . (2) . *70·12 . \supset \vdash .$ Prop

***70·14.** $\vdash :: R \epsilon \alpha \rightarrow \beta . \equiv :. (y) : \overrightarrow{R}`y \epsilon \alpha . \mathrm{v} . \overrightarrow{R}`y = \Lambda :. (x) : \overleftarrow{R}`x \epsilon \beta . \mathrm{v} . \overleftarrow{R}`x = \Lambda$
[*70·13 . *51·236]

***70·15.** $\vdash :. R \epsilon \alpha \rightarrow \beta . \equiv : \exists ! \overrightarrow{R}`y . \supset_y . \overrightarrow{R}`y \epsilon \alpha : \exists ! \overleftarrow{R}`x . \supset_x . \overleftarrow{R}`x \epsilon \beta$
[*24·51 . *4·6 . *70·14]

***70·16.** $\vdash : R \epsilon \alpha \rightarrow \beta . \equiv . \mathrm{D}`\overrightarrow{R} \subset \alpha \cup \iota`\Lambda . \mathrm{D}`\overleftarrow{R} \subset \beta \cup \iota`\Lambda$ [*37·78·781 . *70·12]

***70·17.** $\vdash :: \Lambda \epsilon \alpha . \supset :. R \epsilon \alpha \rightarrow \beta . \equiv : (y) . \overrightarrow{R}`y \epsilon \alpha : \exists ! \overleftarrow{R}`x . \supset_x . \overleftarrow{R}`x \epsilon \beta$

Dem.

$\vdash . *51·2 . *22·62 . \supset \vdash : \mathrm{Hp} . \supset . \alpha = \alpha \cup \iota`\Lambda$ (1)

$\vdash . (1) . *70·13 . \supset$

$\vdash :: \mathrm{Hp} . \supset :. R \epsilon \alpha \rightarrow \beta . \equiv : (y) . \overrightarrow{R}`y \epsilon \alpha : (x) . \overleftarrow{R}`x \epsilon \beta \cup \iota`\Lambda$ (2)

$\vdash . *51·236 . \supset \vdash :. \overleftarrow{R}`x \epsilon \beta \cup \iota`\Lambda . \equiv : \overleftarrow{R}`x \epsilon \beta . \mathrm{v} . \overleftarrow{R}`x = \Lambda :$

[*24·51.*4·6] $\equiv : \exists ! \overleftarrow{R}`x . \supset . \overleftarrow{R}`x \epsilon \beta$ (3)

$\vdash . (2) . (3) . \supset \vdash .$ Prop

***70·171.** $\vdash :: \Lambda \epsilon \beta . \supset :. R \epsilon \alpha \rightarrow \beta . \equiv : \exists ! \overrightarrow{R}`y . \supset_y . \overrightarrow{R}`y \epsilon \alpha : (x) . \overleftarrow{R}`x \epsilon \beta$
[Proof as in *70·17]

***70·18.** $\vdash :: \Lambda \epsilon \alpha . \Lambda \epsilon \beta . \supset :. R \epsilon \alpha \rightarrow \beta . \equiv : (y) . \overrightarrow{R}`y \epsilon \alpha : (x) . \overleftarrow{R}`x \epsilon \beta$
[Proof as in *70·17]

***70·2.** $\vdash . \alpha \rightarrow \beta = (\alpha \cup \iota`\Lambda) \rightarrow \beta = \alpha \rightarrow (\beta \cup \iota`\Lambda) = (\alpha \cup \iota`\Lambda) \rightarrow (\beta \cup \iota`\Lambda)$

Dem.

$\vdash . *22·58·62 . \supset \vdash . (\alpha \cup \iota`\Lambda) \cup \iota`\Lambda = \alpha \cup \iota`\Lambda . (\beta \cup \iota`\Lambda) \cup \iota`\Lambda = \beta \cup \iota`\Lambda$ (1)

$\vdash . *70·12 . (1) . \supset \vdash : R \epsilon \alpha \rightarrow \beta . \equiv . \overrightarrow{R}``\mathrm{V} \subset (\alpha \cup \iota`\Lambda) \cup \iota`\Lambda . \overleftarrow{R}``\mathrm{V} \subset \beta \cup \iota`\Lambda .$

[*70·12] $\equiv . R \epsilon (\alpha \cup \iota`\Lambda) \rightarrow \beta .$ (2)

[*70·12.(1)] $\equiv . \overrightarrow{R}``\mathrm{V} \subset (\alpha \cup \iota`\Lambda) \cup \iota`\Lambda . \overleftarrow{R}``\mathrm{V} \subset (\beta \cup \iota`\Lambda) \cup \iota`\Lambda .$

[*70·12] $\equiv . R \epsilon (\alpha \cup \iota`\Lambda) \rightarrow (\beta \cup \iota`\Lambda) .$ (3)

[*70·12.(1)] $\equiv . \overrightarrow{R}``\mathrm{V} \subset \alpha \cup \iota`\Lambda . \overleftarrow{R}``\mathrm{V} \subset (\beta \cup \iota`\Lambda) \cup \iota`\Lambda .$

[*70·12] $\equiv . R \epsilon \alpha \rightarrow (\beta \cup \iota`\Lambda)$ (4)

$\vdash . (2) . (3) . (4) . \supset \vdash .$ Prop

***70·21.** $\vdash . \alpha \rightarrow \beta = (\alpha - \iota`\Lambda) \rightarrow \beta = \alpha \rightarrow (\beta - \iota`\Lambda) = (\alpha - \iota`\Lambda) \rightarrow (\beta - \iota`\Lambda)$

Dem.

$\vdash . *51{\cdot}222 . \supset \vdash : \Lambda \sim\epsilon\, \alpha . \supset . \alpha - \iota`\Lambda = \alpha : \Lambda \sim\epsilon\, \beta . \supset . \beta - \iota`\Lambda = \beta$ (1)

$\vdash . *51{\cdot}221 . \supset \vdash : \Lambda \epsilon \alpha . \supset . (\alpha - \iota`\Lambda) \cup \iota`\Lambda = \alpha : \Lambda \epsilon \beta . \supset . (\beta - \iota`\Lambda) \cup \beta = \beta$ (2)

$\vdash . (1) . \supset$

$\vdash : \Lambda \sim\epsilon\, \alpha . \supset . (\alpha - \iota`\Lambda) \rightarrow \beta = \alpha \rightarrow \beta . (\alpha - \iota`\Lambda) \rightarrow (\beta - \iota`\Lambda) = \alpha \rightarrow (\beta - \iota`\Lambda)$ (3)

$\vdash . (2) . *70{\cdot}2 . \supset$

$\vdash : \Lambda \epsilon \alpha . \supset . (\alpha - \iota`\Lambda) \rightarrow \beta = \alpha \rightarrow \beta . (\alpha - \iota`\Lambda) \rightarrow (\beta - \iota`\Lambda) = \alpha \rightarrow (\beta - \iota`\Lambda)$ (4)

$\vdash . (3) . (4) . *4{\cdot}83 . \supset$

$\vdash . (\alpha - \iota`\Lambda) \rightarrow \beta = \alpha \rightarrow \beta . (\alpha - \iota`\Lambda) \rightarrow (\beta - \iota`\Lambda) = \alpha \rightarrow (\beta - \iota`\Lambda)$ (5)

Similarly $\vdash . \alpha \rightarrow (\beta - \iota`\Lambda) = \alpha \rightarrow \beta . (\alpha - \iota`\Lambda) \rightarrow (\beta - \iota`\Lambda) = (\alpha - \iota`\Lambda) \rightarrow \beta$ (6)

$\vdash . (5) . (6) . \supset \vdash .$ Prop

***70·22.** $\vdash . \beta \rightarrow \alpha = \text{Cnv}``(\alpha \rightarrow \beta)$

Dem.

$\vdash . *37{\cdot}6 . *31{\cdot}13 . \supset$

$\vdash :. Q \epsilon \text{Cnv}``(\alpha \rightarrow \beta) . \equiv : (\exists R) . R \epsilon \alpha \rightarrow \beta . Q = \text{Cnv}`R :$

[*70·12] $\equiv : (\exists R) . \overrightarrow{R}``\text{V} \subset \alpha \cup \iota`\Lambda . \overleftarrow{R}``\text{V} \subset \beta \cup \iota`\Lambda . Q = \text{Cnv}`R :$

[*32·24·241] $\equiv : (\exists R) . (\text{gs}`\text{Cnv}`R)``\text{V} \subset \alpha \cup \iota`\Lambda .$
$(\text{sg}`\text{Cnv}`R)``\text{V} \subset \beta \cup \iota`\Lambda . Q = \text{Cnv}`R :$

[*13·193] $\equiv : (\exists R) . (\text{gs}`Q)``\text{V} \subset \alpha \cup \iota`\Lambda .$
$(\text{sg}`Q)``\text{V} \subset \beta \cup \iota`\Lambda . Q = \text{Cnv}`R :$

[*32·23·231.*10·35] $\equiv : \overleftarrow{Q}``\text{V} \subset \alpha \cup \iota`\Lambda . \overrightarrow{Q}``\text{V} \subset \beta \cup \iota`\Lambda : (\exists R) . Q = \text{Cnv}`R :$

[*31·33.*10·24] $\equiv : \overleftarrow{Q}``\text{V} \subset \alpha \cup \iota`\Lambda . \overrightarrow{Q}``\text{V} \subset \beta \cup \iota`\Lambda :$

[*70·12] $\equiv : Q \epsilon \beta \rightarrow \alpha :. \supset \vdash .$ Prop

***70·3.** $\vdash . \alpha \subset \gamma . \beta \subset \delta . \supset . \alpha \rightarrow \beta \subset \gamma \rightarrow \delta$

Dem.

$\vdash . *70{\cdot}1 . \supset \vdash : \text{Hp} . R \epsilon \alpha \rightarrow \beta . \supset . \overrightarrow{R}``\text{C}`R \subset \alpha . \overleftarrow{R}``\text{D}`R \subset \beta . \alpha \subset \gamma . \beta \subset \delta .$

[*22·44] $\supset . \overrightarrow{R}``\text{C}`R \subset \gamma . \overleftarrow{R}``\text{D}`R \subset \delta .$

[*70·1] $\supset . R \epsilon \gamma \rightarrow \delta$ (1)

$\vdash . (1) . \text{Exp} . *10{\cdot}11{\cdot}21 . \supset \vdash .$ Prop

***70·31.** $\vdash . (\alpha \rightarrow \beta) \cap (\gamma \rightarrow \delta) = (\alpha \cap \gamma) \rightarrow (\beta \cap \delta)$

Dem.

$\vdash . *70{\cdot}1 . \supset \vdash : R \epsilon (\alpha \rightarrow \beta) \cap (\gamma \rightarrow \delta) . \equiv .$
$\overrightarrow{R}``\text{C}`R \subset \alpha . \overrightarrow{R}``\text{C}`R \subset \gamma . \overleftarrow{R}``\text{D}`R \subset \beta . \overleftarrow{R}``\text{D}`R \subset \delta .$

[*22·45] $\equiv . \overrightarrow{R}``\text{C}`R \subset \alpha \cap \gamma . \overleftarrow{R}``\text{D}`R \subset \beta \cap \delta .$

[*70·1] $\equiv . R \epsilon (\alpha \cap \gamma) \rightarrow (\beta \cap \delta) : \supset \vdash .$ Prop

***70·32.** $\vdash . (\alpha \rightarrow \beta) \cup (\gamma \rightarrow \delta) \subset (\alpha \cup \gamma) \rightarrow (\beta \cup \delta)$

Dem.

$\vdash . *70{\cdot}1 . \supset \vdash :. R \in (\alpha \rightarrow \beta) \cup (\gamma \rightarrow \delta) . \equiv :$

$\qquad \overrightarrow{R}\text{“}\text{Ɑ}\text{‘}R \subset \alpha . \overleftarrow{R}\text{“}\mathrm{D}\text{‘}R \subset \beta . \mathrm{v} . \overrightarrow{R}\text{“}\text{Ɑ}\text{‘}R \subset \gamma . \overleftarrow{R}\text{“}\mathrm{D}\text{‘}R \subset \delta :$

$[*3{\cdot}26{\cdot}27{\cdot}48] \supset : \overrightarrow{R}\text{“}\text{Ɑ}\text{‘}R \subset \alpha . \mathrm{v} . \overrightarrow{R}\text{“}\text{Ɑ}\text{‘}R \subset \gamma : \overleftarrow{R}\text{“}\mathrm{D}\text{‘}R \subset \beta . \mathrm{v} . \overleftarrow{R}\text{“}\mathrm{D}\text{‘}R \subset \delta :$

$[*22{\cdot}65] \qquad \supset : \overrightarrow{R}\text{“}\text{Ɑ}\text{‘}R \subset \alpha \cup \gamma . \overleftarrow{R}\text{“}\mathrm{D}\text{‘}R \subset \beta \cup \delta :$

$[*70{\cdot}1] \qquad \supset : R \in (\alpha \cup \gamma) \rightarrow (\beta \cup \delta) :. \supset \vdash . \text{Prop}$

***70·4.** $\vdash . \alpha \rightarrow \text{Cls} = \hat{R}\, (\overrightarrow{R}\text{“}\text{Ɑ}\text{‘}R \subset \alpha)$

Dem.

$\vdash . *70{\cdot}1 . \supset \vdash : R \in \alpha \rightarrow \text{Cls} . \equiv . \overrightarrow{R}\text{“}\text{Ɑ}\text{‘}R \subset \alpha . \overleftarrow{R}\text{“}\mathrm{D}\text{‘}R \subset \text{Cls} .$

$[*37{\cdot}761] \qquad \equiv . \overrightarrow{R}\text{“}\text{Ɑ}\text{‘}R \subset \alpha : \supset \vdash . \text{Prop}$

***70·41.** $\vdash . \text{Cls} \rightarrow \beta = \hat{R}\, (\overleftarrow{R}\text{“}\mathrm{D}\text{‘}R \subset \beta)$ [Proof as in *70·4]

***70·42.** $\vdash . \alpha \rightarrow \beta = (\alpha \rightarrow \text{Cls}) \cap (\text{Cls} \rightarrow \beta)$ [*70·4·41]

***70·43.** $\vdash :. R \in \alpha \rightarrow \text{Cls} . \equiv : y \in \text{Ɑ}\text{‘}R . \supset_y . \overrightarrow{R}\text{‘}y \in \alpha$ [As in *70·11]

***70·431.** $\vdash :. R \in \text{Cls} \rightarrow \beta . \equiv : x \in \mathrm{D}\text{‘}R . \supset_x . \overleftarrow{R}\text{‘}x \in \beta$ [As in *70·11]

***70·44.** $\vdash : R \in \alpha \rightarrow \text{Cls} . \equiv . \overrightarrow{R}\text{“}\mathrm{V} \subset \alpha \cup \iota\text{‘}\Lambda$ [As in *70·12]

***70·441.** $\vdash : R \in \text{Cls} \rightarrow \beta . \equiv . \overleftarrow{R}\text{“}\mathrm{V} \subset \beta \cup \iota\text{‘}\Lambda$ [As in *70·12]

***70·45.** $\vdash : R \in \alpha \rightarrow \text{Cls} . \equiv . (y) . \overrightarrow{R}\text{‘}y \in \alpha \cup \iota\text{‘}\Lambda$ [As in *70·13]

***70·451.** $\vdash : R \in \text{Cls} \rightarrow \beta . \equiv . (x) . \overleftarrow{R}\text{‘}x \in \beta \cup \iota\text{‘}\Lambda$ [As in *70·13]

***70·46.** $\vdash :. R \in \alpha \rightarrow \text{Cls} . \equiv : (y) : \overrightarrow{R}\text{‘}y \in \alpha . \mathrm{v} . \overrightarrow{R}\text{‘}y = \Lambda$ [As in *70·14]

***70·461.** $\vdash :. R \in \text{Cls} \rightarrow \beta . \equiv : (x) : \overleftarrow{R}\text{‘}x \in \beta . \mathrm{v} . \overleftarrow{R}\text{‘}x = \Lambda$ [As in *70·14]

***70·47.** $\vdash :. R \in \alpha \rightarrow \text{Cls} . \equiv : \exists ! \overrightarrow{R}\text{‘}y . \supset_y . \overrightarrow{R}\text{‘}y \in \alpha$ [As in *70·15]

***70·471.** $\vdash :. R \in \text{Cls} \rightarrow \beta . \equiv : \exists ! \overleftarrow{R}\text{‘}x . \supset_x . \overleftarrow{R}\text{‘}x \in \beta$ [As in *70·15]

***70·48.** $\vdash : R \in \alpha \rightarrow \text{Cls} . \equiv . \mathrm{D}\text{‘}\overrightarrow{R} \subset \alpha \cup \iota\text{‘}\Lambda$ [As in *70·16]

***70·481.** $\vdash : R \in \text{Cls} \rightarrow \beta . \equiv . \mathrm{D}\text{‘}\overleftarrow{R} \subset \beta \cup \iota\text{‘}\Lambda$ [As in *70·16]

***70·5.** $\vdash . \text{Cls} \rightarrow \alpha = \text{Cnv}\text{“}(\alpha \rightarrow \text{Cls}) . \alpha \rightarrow \text{Cls} = \text{Cnv}\text{“}(\text{Cls} \rightarrow \alpha)$ [*70·22]

***70·51.** $\vdash :. \xi, \eta \in \alpha . \supset_{\xi, \eta} . \xi \cap \eta \in \alpha \cup \iota\text{‘}\Lambda : \supset : R, S \in \alpha \rightarrow \text{Cls} . \supset . R \dot{\cap} S \in \alpha \rightarrow \text{Cls}$

Dem.

$\vdash . *32{\cdot}3 . \supset \qquad \vdash :. \text{Hp} . \supset : \overrightarrow{R}\text{‘}y \in \alpha . \overrightarrow{S}\text{‘}y \in \alpha . \supset . \{\text{sg}\text{‘}(R \dot{\cap} S)\}\text{‘}y \in \alpha \cup \iota\text{‘}\Lambda \quad (1)$

$\vdash . *32{\cdot}3 . *51{\cdot}15 . *24{\cdot}34 . \supset$

$\qquad \vdash : \overrightarrow{R}\text{‘}y \in \alpha . \overrightarrow{S}\text{‘}y \in \iota\text{‘}\Lambda . \supset . \{\text{sg}\text{‘}(R \dot{\cap} S)\}\text{‘}y = \Lambda .$

$[*51{\cdot}236] \qquad \supset . \{\text{sg}\text{‘}(R \dot{\cap} S)\}\text{‘}y \in \alpha \cup \iota\text{‘}\Lambda \quad (2)$

$\vdash . (1) . (2) . *4\cdot4 . \supset \vdash :. \text{Hp} . \supset : \overrightarrow{R}`y \in \alpha . \overrightarrow{S}`y \in \alpha \cup \iota`\Lambda . \supset . \{\text{sg}`(R \mathbin{\dot\frown} S)\}`y \in \alpha \cup \iota`\Lambda$ (3)

$\vdash . *32\cdot3 . *51\cdot15 . *24\cdot34 . *51\cdot236 . \supset$

$\vdash : \overrightarrow{R}`y \in \iota`\Lambda . \overrightarrow{S}`y \in \alpha \cup \iota`\Lambda . \supset . \{\text{sg}`(R \mathbin{\dot\frown} S)\}`y \in \alpha \cup \iota`\Lambda$ (4)

$\vdash . (3) . (4) . *4\cdot4 . \supset \vdash :. \text{Hp} . \supset : \overrightarrow{R}`y, \overrightarrow{S}`y \in \alpha \cup \iota`\Lambda . \supset . \{\text{sg}`(R \mathbin{\dot\frown} S)\}`y \in \alpha \cup \iota`\Lambda :$

[$*10\cdot11\cdot21\cdot27 . *70\cdot45$] $\supset : R, S \in \alpha \rightarrow \text{Cls} . \supset . (y) . \{\text{sg}`(R \mathbin{\dot\frown} S)\}`y \in \alpha \cup \iota`\Lambda .$

[$*70\cdot45 . *32\cdot23$] $\supset . R \mathbin{\dot\frown} S \in \alpha \rightarrow \text{Cls} :. \supset \vdash . \text{Prop}$

$*70\cdot52$. $\vdash :. \xi, \eta \in \beta . \supset_{\xi, \eta} . \xi \frown \eta \in \beta \cup \iota`\Lambda : \supset : R, S \in \text{Cls} \rightarrow \beta . \supset . R \mathbin{\dot\frown} S \in \text{Cls} \rightarrow \beta$

[Proof as in $*70\cdot51$]

$*70\cdot53$. $\vdash :. \xi, \eta \in \alpha . \supset_{\xi, \eta} . \xi \frown \eta \in \alpha \cup \iota`\Lambda : \xi, \eta \in \beta . \supset_{\xi, \eta} . \xi \frown \eta \in \beta \cup \iota`\Lambda : \supset :$

$R, S \in \alpha \rightarrow \beta . \supset . R \mathbin{\dot\frown} S \in \alpha \rightarrow \beta$

Dem.

$\vdash . *70\cdot5\cdot51 . \supset \vdash :. \text{Hp} . \supset : R, S \in \alpha \rightarrow \text{Cls} . R, S \in \text{Cls} \rightarrow \beta . \supset .$

$R \mathbin{\dot\frown} S \in \alpha \rightarrow \text{Cls} . R \mathbin{\dot\frown} S \in \text{Cls} \rightarrow \beta$ (1)

$\vdash . (1) . *70\cdot42 . \supset \vdash . \text{Prop}$

$*70\cdot54$. $\vdash : \text{Ꮰ}`R \cap \text{Ꮰ}`S = \Lambda . R, S \in \alpha \rightarrow \text{Cls} . \supset . R \mathbin{\dot\cup} S \in \alpha \rightarrow \text{Cls}$

Dem.

$\vdash . *24\cdot15 . *22\cdot33 . \supset$

$\vdash :. \text{Ꮰ}`R \cap \text{Ꮰ}`S = \Lambda . \supset : (y) : \sim \{y \in \text{Ꮰ}`R . y \in \text{Ꮰ}`S\} :$

[$*33\cdot41$] $\supset : (y) : \sim \{\exists ! \overrightarrow{R}`y . \exists ! \overrightarrow{S}`y\} :$

[$*4\cdot51 . *24\cdot51$] $\supset : (y) : \overrightarrow{R}`y = \Lambda . \vee . \overrightarrow{S}`y = \Lambda :$

[$*24\cdot36$] $\supset : (y) : \overrightarrow{R}`y \cup \overrightarrow{S}`y = \overrightarrow{S}`y . \vee . \overrightarrow{R}`y \cup \overrightarrow{S}`y = \overrightarrow{R}`y$ (1)

$\vdash . *70\cdot45 . \supset$

$\vdash :. R, S \in \alpha \rightarrow \text{Cls} . \supset : (y) . \overrightarrow{R}`y \in \alpha \cup \iota`\Lambda : (y) . \overrightarrow{S}`y \in \alpha \cup \iota`\Lambda$ (2)

$\vdash . (1) . (2) . \supset \vdash :. \text{Hp} . \supset : (y) . \overrightarrow{R}`y \cup \overrightarrow{S}`y \in \alpha \cup \iota`\Lambda :$

[$*32\cdot32$] $\supset : (y) . \{\text{sg}`(R \mathbin{\dot\cup} S)\}`y \in \alpha \cup \iota`\Lambda :$

[$*70\cdot45$] $\supset : R \mathbin{\dot\cup} S \in \alpha \rightarrow \text{Cls} :. \supset \vdash . \text{Prop}$

$*70\cdot55$. $\vdash : \text{D}`R \cap \text{D}`S = \Lambda . R, S \in \text{Cls} \rightarrow \beta . \supset . R \mathbin{\dot\cup} S \in \text{Cls} \rightarrow \beta$

[Proof as in $*70\cdot54$]

$*70\cdot56$. $\vdash : \text{D}`R \cap \text{D}`S = \Lambda . \text{Ꮰ}`R \cap \text{Ꮰ}`S = \Lambda . R, S \in \alpha \rightarrow \beta . \supset . R \mathbin{\dot\cup} S \in \alpha \rightarrow \beta$

[$*70\cdot54\cdot55\cdot42$]

$*70\cdot57$. $\vdash : C`R \cap C`S = \Lambda . R, S \in \alpha \rightarrow \beta . \supset . R \mathbin{\dot\cup} S \in \alpha \rightarrow \beta$

Dem.

$\vdash . *33\cdot161 . \supset \vdash . \text{D}`R \cap \text{D}`S \subset C`R \cap C`S . \text{Ꮰ}`R \cap \text{Ꮰ}`S \subset C`R \cap C`S .$

[$*24\cdot13$] $\supset \vdash : C`R \cap C`S = \Lambda . \supset . \text{D}`R \cap \text{D}`S = \Lambda . \text{Ꮰ}`R \cap \text{Ꮰ}`S = \Lambda$ (1)

$\vdash . (1) . *70\cdot56 . \supset \vdash . \text{Prop}$

***70·6.** $\vdash : S \epsilon \alpha \rightarrow \mathrm{Cls} . R``\text{`}\alpha \subset \alpha \cup \iota`\Lambda \mathbin{\dot{:}} \supset . R \mid S \epsilon \alpha \rightarrow \mathrm{Cls}$

Dem.

$\vdash . *37\cdot31 . \quad \supset \vdash . \{\mathrm{sg}`(R \mid S)\}``\mathrm{V} = (R_\epsilon \mid \overrightarrow{S})``\mathrm{V}$

$[*37\cdot33] \quad = R_\epsilon``\overrightarrow{S}``\mathrm{V} \quad (1)$

$\vdash . (1) . *70\cdot44 . \supset \vdash : S \epsilon \alpha \rightarrow \mathrm{Cls} . \supset . \{\mathrm{sg}`(R \mid S)\}``\mathrm{V} \subset R_\epsilon``(\alpha \cup \iota`\Lambda) \quad (2)$

$\vdash . *37\cdot22 . \quad \supset \vdash . R_\epsilon``(\alpha \cup \iota`\Lambda) = R_\epsilon``\alpha \cup R_\epsilon``\iota`\Lambda$

$[*53\cdot31] \quad = R_\epsilon``\alpha \cup \iota`R_\epsilon`\Lambda$

$[(*37\cdot04).*37\cdot11\cdot29] \quad = R```\alpha \cup \iota`\Lambda \quad (3)$

$\vdash . (3) . *22\cdot66 . \supset \vdash : R```\alpha \subset \alpha \cup \iota`\Lambda . \supset . R_\epsilon``(\alpha \cup \iota`\Lambda) \subset \alpha \cup \iota`\Lambda \cup \iota`\Lambda .$

$[*22\cdot56] \quad \supset . R_\epsilon``(\alpha \cup \iota`\Lambda) \subset \alpha \cup \iota`\Lambda \quad (4)$

$\vdash . (2) . (4) . \quad \supset \vdash : \mathrm{Hp} . \supset . \{\mathrm{sg}`(R \mid S)\}``\mathrm{V} \subset \alpha \cup \iota`\Lambda .$

$[*70\cdot44] \quad \supset . R \mid S \epsilon \alpha \rightarrow \mathrm{Cls} : \supset \vdash . \mathrm{Prop}$

***70·61.** $\vdash : R \epsilon \mathrm{Cls} \rightarrow \beta . \breve{S}```\beta \subset \beta \cup \iota`\Lambda . \supset . R \mid S \epsilon \mathrm{Cls} \rightarrow \beta$ [As in *70·6]

***70·62.** $\vdash : R \epsilon \alpha \rightarrow \mathrm{Cls} . \supset . R \upharpoonright \gamma \epsilon \alpha \rightarrow \mathrm{Cls}$

Dem.

$\vdash . *35\cdot64 . \mathrm{Transp} . \supset \vdash : y \sim \epsilon \gamma . \supset . y \sim \epsilon \text{ɑ}`(R \upharpoonright \gamma) .$

$[*33\cdot41.*24\cdot51] \quad \supset . \{\mathrm{sg}`(R \upharpoonright \gamma)\}`y = \Lambda .$

$[*51\cdot236] \quad \supset . \{\mathrm{sg}`(R \upharpoonright \gamma)\}`y \epsilon \alpha \cup \iota`\Lambda \quad (1)$

$\vdash . *35\cdot101 . *4\cdot73 . \supset \vdash :. y \epsilon \gamma . \supset : x (R \upharpoonright \gamma) y . \equiv_x . xRy :$

$[*20\cdot15.*32\cdot13\cdot23] \quad \supset : \{\mathrm{sg}`(R \upharpoonright \gamma)\}`y = \overrightarrow{R}`y \quad (2)$

$\vdash . *70\cdot45 . \quad \supset \vdash : \mathrm{Hp} . \supset . \overrightarrow{R}`y \epsilon \alpha \cup \iota`\Lambda \quad (3)$

$\vdash . (2) . (3) . \quad \supset \vdash :. \mathrm{Hp} . \supset : y \epsilon \gamma . \supset . \{\mathrm{sg}`(R \upharpoonright \gamma)\}`y \epsilon \alpha \cup \iota`\Lambda \quad (4)$

$\vdash . (1) . (4) . *4\cdot83 . \supset \vdash : \mathrm{Hp} . \supset . \{\mathrm{sg}`(R \upharpoonright \gamma)\}`y \epsilon \alpha \cup \iota`\Lambda \quad (5)$

$\vdash . (5) . *10\cdot11\cdot21 . *70\cdot45 . \supset \vdash . \mathrm{Prop}$

***70·63.** $\vdash : R \epsilon \mathrm{Cls} \rightarrow \beta . \supset . \delta \upharpoonleft R \epsilon \mathrm{Cls} \rightarrow \beta$ [As in *70·62]

*71. ONE-MANY, MANY-ONE, AND ONE-ONE RELATIONS

Summary of *71.

In this number we shall be concerned with the more elementary properties of one-many, many-one, and one-one relations. These properties are very numerous and very important. The properties of many-one relations (*i.e.* of relations belonging to the class $\text{Cls} \rightarrow 1$) result from those of one-many relations by means of *70·5, whence it follows that many-one relations are the converses of one-many relations. It is thus only necessary to interchange R and $\breve{R}$, D and Ɑ, $\overrightarrow{R}$ and $\overleftarrow{R}$ in order to obtain a property of a many-one relation from a property of a one-many relation. Or we may repeat the various steps of any proof, making the above interchanges at every step, and the analogous proposition will result. For this reason, in what follows, we shall omit all proofs of properties of many-one relations, confining ourselves to proving the analogous properties of one-many relations.

In virtue of *70·42, one-one relations (*i.e.* relations belonging to the class $1 \rightarrow 1$) are the relations which are both one-many and many-one; hence their properties result from combining the properties of one-many and many-one relations. We shall omit the proofs when they consist merely in such combinations.

A one-many relation gives rise to a descriptive function which is existent whenever its argument belongs to the converse domain of the relation. That is, if $R \epsilon 1 \rightarrow \text{Cls}$, we have $\text{E} ! R \text{‘} y$ whenever $y \epsilon \text{Ɑ‘}R$. Conversely, if a descriptive function $R\text{‘}y$ exists for the argument y, then R is one-many so far as that argument is concerned, *i.e.* $\overrightarrow{R}\text{‘}y \epsilon 1$. Thus we find

$$R \epsilon 1 \rightarrow \text{Cls} . \equiv . \text{E} !! R\text{“Ɑ‘}R.$$

The descriptive function $R\text{‘}y$ derived from a one-many relation R has thus a definite value whenever $y \epsilon \text{Ɑ‘}R$, and not otherwise. Thus the class of arguments for which such a function exists is the converse domain of the relation which gives rise to the function, *i.e.*

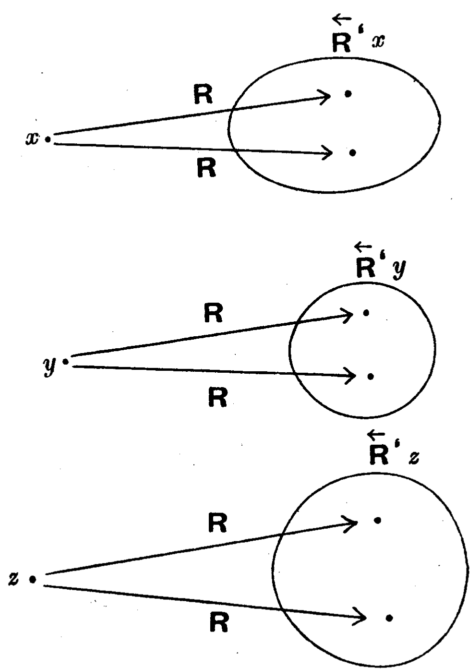
$$R \epsilon 1 \rightarrow \text{Cls} . \supset . \hat{y} \{\text{E} ! R\text{‘}y\} = \text{Ɑ‘}R,$$

and the converse implication also holds.

It often happens that a relation which is not in general one-many becomes so when its domain, converse domain, or field is subjected to some limitation. For example, let R be the relation of parent to child, α the class of males, and β the class of females. Then R is not one-many, but $\alpha \upharpoonleft R$ and $\beta \upharpoonleft R$ are one-many, and in fact $(\alpha \upharpoonleft R)\text{‘}y$ = the father of y, $(\beta \upharpoonleft R)\text{‘}y$ = the mother of y. We shall often have occasion to deal with relations obtained by limitations imposed on D or Ɑ; thus $\alpha (\text{D} \upharpoonright \lambda) R . \equiv . R$ belongs to the class λ, and has α for its

domain. The class λ may be so constituted that only one relation R fulfils this condition; in that case, $\mathrm{D} \upharpoonleft \lambda \in \mathrm{Cls} \rightarrow 1$. Since $\mathrm{D} \in 1 \rightarrow \mathrm{Cls}$, we find $\mathrm{D} \upharpoonleft \lambda \in \mathrm{Cls} \rightarrow 1 \,.\equiv.\, \mathrm{D} \upharpoonleft \lambda \in 1 \rightarrow 1$. This type of condition, $\mathrm{D} \upharpoonleft \lambda \in 1 \rightarrow 1$ or $\text{Ɑ} \upharpoonleft \lambda \in 1 \rightarrow 1$ or $C \upharpoonleft \lambda \in 1 \rightarrow 1$, is one which frequently occurs in subsequent work. Another condition which often occurs is $F \upharpoonleft \lambda \in \mathrm{Cls} \rightarrow 1$. When this condition is realized, a term x which belongs to the field of one relation of the class λ does not belong to the field of any other relation of this class, *i.e.* the fields of relations of this class are mutually exclusive.

For purposes of realizing imaginatively the properties of one-many relations, it is often convenient to picture their structure as in the accompanying figure. Here $x, y, z, \ldots$ form the domain of R, and all the points



in the oval marked $\overleftarrow{R}\text{‘}x$ are such that x has the relation R to each of them, with similar conditions for y and z. What characterizes R as a $1 \rightarrow \mathrm{Cls}$ is the absence of overlapping in the ovals. For if $\overleftarrow{R}\text{‘}x$ and $\overleftarrow{R}\text{‘}y$ had a point in common, this would be a relatum both to x and y, and both x and y would be referents to it; whereas in a $1 \rightarrow \mathrm{Cls}$, no term has more than one referent.

The above figure illustrates a very important property of one-many relations, namely

$$R \in 1 \rightarrow \mathrm{Cls} \,.\equiv.\, R \,|\, \breve{R} = I \upharpoonleft \mathrm{D}\text{‘}R.$$

In the above figure, $I \upharpoonright \text{D}`R$ is the relation of identity confined to $x, y, z, \ldots$. If $R$ were not a $1 \rightarrow \text{Cls}$, we could sometimes go from $x$ to some term of $\overleftarrow{R}`x \cap \overleftarrow{R}`y$ by the relation R, and thence back to y by the relation $\breve{R}$. But when $R \epsilon 1 \rightarrow \text{Cls}$, $R \,|\, \breve{R}$ must bring us back to the point from which we started.

When $R \epsilon 1 \rightarrow 1$, each of the ovals $\overleftarrow{R}`x, \overleftarrow{R}`y, \overleftarrow{R}`z, \ldots$ in the above figure shrinks to a single point, so that $\overleftarrow{R}`x = \iota`\breve{R}`x$. Thus when R is given as a $1 \rightarrow \text{Cls}$, it will be a $1 \rightarrow 1$ if $R`y = R`z \,.\, \supset_{y,z} . \, y = z$. This proposition is constantly used, and so is the consequence that $R \upharpoonright \beta$ is a $1 \rightarrow 1$ if $y, z \epsilon \beta \,.\, R`y = R`z \,.\, \supset_{y,z} . \, y = z$. (These propositions are $*71{\cdot}54{\cdot}55$ below.)

The hypothesis $R \epsilon 1 \rightarrow \text{Cls}$ is equivalent to the hypothesis

$$xRz \,.\, yRz \,.\, \supset_{x,y,z} . \, x = y$$

(cf. $*71{\cdot}17$, below), and the hypothesis $R \epsilon \text{Cls} \rightarrow 1$ is equivalent to

$$xRy \,.\, xRz \,.\, \supset_{x,y,z} . \, y = z.$$

These are for many purposes the most convenient hypotheses to use.

The most useful propositions in the present number are the following. (We omit here propositions concerning $\text{Cls} \rightarrow 1$ or $1 \rightarrow 1$ which are mere analogues of propositions concerning $1 \rightarrow \text{Cls}$.)

$*71{\cdot}16$. $\vdash : R \epsilon 1 \rightarrow \text{Cls} \,.\, \equiv . \, \text{E} !! \, R``\text{Ɑ}`R$

This gives the connection of one-many relations with descriptive functions. We have also

$*71{\cdot}163$. $\vdash :. \, R \epsilon 1 \rightarrow \text{Cls} \,.\, \equiv : y \epsilon \text{Ɑ}`R \,.\, \equiv_y . \, \text{E} ! \, R`y$

For many of the constant relations defined from time to time, such as Cnv or D, the following proposition is useful:

$*71{\cdot}166$. $\vdash : (y) . \, \text{E} ! \, R`y \,.\, \supset . \, R \epsilon 1 \rightarrow \text{Cls}$

$*71{\cdot}17$. $\vdash :. \, R \epsilon 1 \rightarrow \text{Cls} \,.\, \equiv : xRz \,.\, yRz \,.\, \supset_{x,y,z} . \, x = y$

This might have been taken as the definition of one-many relations, if we had not wished to derive them from the more general notion of $\alpha \rightarrow \beta$. In proving that a relation is one-many, $*71{\cdot}17$ is more often employed than any other proposition.

$*71{\cdot}22$. $\vdash : R \epsilon 1 \rightarrow \text{Cls} \,.\, S \subset R \,.\, \supset . \, S \epsilon 1 \rightarrow \text{Cls}$

$*71{\cdot}25$. $\vdash . \, R, S \epsilon 1 \rightarrow \text{Cls} \,.\, \supset . \, R \,|\, S \epsilon 1 \rightarrow \text{Cls}$

$*71{\cdot}36$. $\vdash :. \, R \epsilon 1 \rightarrow \text{Cls} \,.\, \supset : x = R`y \,.\, \equiv . \, xRy$

$*71{\cdot}381$. $\vdash : R \epsilon \text{Cls} \rightarrow 1 \,.\, \supset . \, R``(\alpha - \beta) = R``\alpha - R``\beta$

(This proposition is more useful than the corresponding property of $1 \rightarrow \text{Cls}$.)

$*71{\cdot}55.\quad \vdash :: R \epsilon 1 \to \mathrm{Cls} \,.\supset :. R \upharpoonright \beta \epsilon 1 \to 1 \,.\equiv : y, z \epsilon \beta \,.\, R`y = R`z \,.\supset_{y,z} .\, y = z$

This proposition is constantly used. For example, putting $\mathrm{C\!\!\!I}$ for R, it gives

$$\vdash :. \mathrm{C\!\!\!I} \upharpoonright \beta \epsilon 1 \to 1 \,.\equiv : P, Q \epsilon \beta \,.\, \mathrm{C\!\!\!I}`P = \mathrm{C\!\!\!I}`Q \,.\supset_{P,Q} .\, P = Q.$$

Most of the relations used to establish correlations in arithmetic are obtained from a one-many relation, such as $\mathrm{C\!\!\!I}$, by imposing some limitation on the converse domain which makes the relation one-one.

$*71{\cdot}571.\quad \vdash :. y \epsilon \beta \,.\supset_y .\, \mathrm{E}!\, R`y :\equiv .\, R \upharpoonright \beta \epsilon 1 \to \mathrm{Cls} \,.\, \beta \subset \mathrm{C\!\!\!I}`R$

Here "$y \epsilon \beta \,.\supset_y .\, \mathrm{E}!\, R`y$" is $\mathrm{E}!!\, R``\beta$, which has already played a large part as a hypothesis, *e.g.* in $*37{\cdot}6$ ff.

$*71{\cdot}7.\quad \vdash :. Q \epsilon 1 \to \mathrm{Cls} \,.\supset : xP \,|\, Qz \,.\equiv .\, xP(Q`z)$

Thus for example we shall have $x(P \,|\, \mathrm{Cnv})\, R \,.\equiv .\, xP(\mathrm{Cnv}`R)$.

$*71{\cdot}01.\quad \vdash .\, 1 \to \mathrm{Cls} = \hat{R}(\overrightarrow{R}``\mathrm{C\!\!\!I}`R \subset 1) \qquad [*70{\cdot}4]$

$*71{\cdot}02.\quad \vdash .\, \mathrm{Cls} \to 1 = \hat{R}(\overleftarrow{R}``\mathrm{D}`R \subset 1) \qquad [*70{\cdot}41]$

$*71{\cdot}03.\quad \vdash .\, 1 \to 1 = \hat{R}(\overrightarrow{R}``\mathrm{C\!\!\!I}`R \subset 1 \,.\, \overleftarrow{R}``\mathrm{D}`R \subset 1) \qquad [*20{\cdot}2 \,.\, (*70{\cdot}01)]$

$*71{\cdot}04.\quad \vdash .\, 1 \to 1 = (1 \to \mathrm{Cls}) \cap (\mathrm{Cls} \to 1) \qquad [*70{\cdot}42]$

$*71{\cdot}1.\quad \vdash : R \epsilon 1 \to \mathrm{Cls} \,.\equiv .\, \overrightarrow{R}``\mathrm{C\!\!\!I}`R \subset 1 \qquad [*20{\cdot}33 \,.\, *71{\cdot}01]$

$*71{\cdot}101.\quad \vdash : R \epsilon \mathrm{Cls} \to 1 \,.\equiv .\, \overleftarrow{R}``\mathrm{D}`R \subset 1 \qquad [*20{\cdot}33 \,.\, *71{\cdot}02]$

$*71{\cdot}102.\quad \vdash : R \epsilon 1 \to 1 \,.\equiv .\, \overrightarrow{R}``\mathrm{C\!\!\!I}`R \subset 1 \,.\, \overleftarrow{R}``\mathrm{D}`R \subset 1 \qquad [*20{\cdot}33 \,.\, *71{\cdot}03]$

$*71{\cdot}103.\quad \vdash : R \epsilon 1 \to 1 \,.\equiv .\, R \epsilon 1 \to \mathrm{Cls} \,.\, R \epsilon \mathrm{Cls} \to 1 \qquad [*22{\cdot}33 \,.\, *71{\cdot}04]$

$*71{\cdot}11.\quad \vdash : R \epsilon 1 \to \mathrm{Cls} \,.\equiv .\, \overrightarrow{R}``V \subset 1 \cup \iota`\Lambda \qquad [*70{\cdot}44]$

$*71{\cdot}111.\quad \vdash : R \epsilon \mathrm{Cls} \to 1 \,.\equiv .\, \overleftarrow{R}``V \subset 1 \cup \iota`\Lambda \qquad [*70{\cdot}441]$

$*71{\cdot}112.\quad \vdash : R \epsilon 1 \to 1 \,.\equiv .\, \overrightarrow{R}``V \subset 1 \cup \iota`\Lambda \,.\, \overleftarrow{R}``V \subset 1 \cup \iota`\Lambda \qquad [*70{\cdot}12]$

$*71{\cdot}12.\quad \vdash : R \epsilon 1 \to \mathrm{Cls} \,.\equiv .\, (y) \,.\, \overrightarrow{R}`y \epsilon 1 \cup \iota`\Lambda \qquad [*70{\cdot}45]$

$*71{\cdot}121.\quad \vdash : R \epsilon \mathrm{Cls} \to 1 \,.\equiv .\, (x) \,.\, \overleftarrow{R}`x \epsilon 1 \cup \iota`\Lambda \qquad [*70{\cdot}451]$

$*71{\cdot}122.\quad \vdash :. R \epsilon 1 \to 1 \,.\equiv : (y) \,.\, \overrightarrow{R}`y \epsilon 1 \cup \iota`\Lambda : (x) \,.\, \overleftarrow{R}`x \epsilon 1 \cup \iota`\Lambda \qquad [*70{\cdot}13]$

$*71{\cdot}13.\quad \vdash :. R \epsilon 1 \to \mathrm{Cls} \,.\equiv : (y) : \overrightarrow{R}`y \epsilon 1 \,.\vee .\, \overrightarrow{R}`y = \Lambda \qquad [*70{\cdot}46]$

$*71{\cdot}131.\quad \vdash :. R \epsilon \mathrm{Cls} \to 1 \,.\equiv : (x) : \overleftarrow{R}`x \epsilon 1 \,.\vee .\, \overleftarrow{R}`x = \Lambda \qquad [*70{\cdot}461]$

$*71{\cdot}132.\quad \vdash :: R \epsilon 1 \to 1 \,.\equiv :. (y) : \overrightarrow{R}`y \epsilon 1 \,.\vee .\, \overrightarrow{R}`y = \Lambda :. (x) : \overleftarrow{R}`x \epsilon 1 \,.\vee .\, \overleftarrow{R}`x = \Lambda$
$[*70{\cdot}14]$

$*71{\cdot}14.\quad \vdash :. R \epsilon 1 \to \mathrm{Cls} \,.\equiv : \exists !\, \overrightarrow{R}`y \,.\supset_y .\, \overrightarrow{R}`y \epsilon 1 \qquad [*70{\cdot}47]$

$*71{\cdot}141.\quad \vdash :. R \epsilon \mathrm{Cls} \to 1 \,.\equiv : \exists !\, \overleftarrow{R}`x \,.\supset_x .\, \overleftarrow{R}`x \epsilon 1 \qquad [*70{\cdot}471]$

***71·142.** $\vdash :. R \epsilon 1 \rightarrow 1 . \equiv : \exists ! \overrightarrow{R}\text{‘}y . \supset_y . \overrightarrow{R}\text{‘}y \epsilon 1 : \exists ! \overleftarrow{R}\text{‘}x . \supset_x . \overleftarrow{R}\text{‘}x \epsilon 1$ [*70·15]

***71·15.** $\vdash : R \epsilon 1 \rightarrow \text{Cls} . \equiv . \text{D}\text{‘}\overrightarrow{R} \subset 1 \cup \iota\text{‘}\Lambda$ [*70·48]

***71·151.** $\vdash : R \epsilon \text{Cls} \rightarrow 1 . \equiv . \text{D}\text{‘}\overleftarrow{R} \subset 1 \cup \iota\text{‘}\Lambda$ [*70·481]

***71·152.** $\vdash : R \epsilon 1 \rightarrow 1 . \equiv . \text{D}\text{‘}\overrightarrow{R} \subset 1 \cup \iota\text{‘}\Lambda . \text{D}\text{‘}\overleftarrow{R} \subset 1 \cup \iota\text{‘}\Lambda$ [*70·16]

***71·16.** $\vdash : R \epsilon 1 \rightarrow \text{Cls} . \equiv . \text{E} !! R\text{‘‘}\text{Ɑ}\text{‘}R$

Dem.

$\vdash . *37·702 . *71·1 . \supset$

$\vdash :. R \epsilon 1 \rightarrow \text{Cls} . \equiv : y \epsilon \text{Ɑ}\text{‘}R . \supset_y . \overrightarrow{R}\text{‘}y \epsilon 1 :$

[*53·3] $\equiv : y \epsilon \text{Ɑ}\text{‘}R . \supset_y . \text{E} ! R\text{‘}y :$

[*37·104] $\equiv : \text{E} !! R\text{‘‘}\text{Ɑ}\text{‘}R :. \supset \vdash . \text{Prop}$

This proposition is very important; it exhibits the connection of descriptive functions with one-many relations.

***71·161.** $\vdash : R \epsilon \text{Cls} \rightarrow 1 . \equiv . \text{E} !! \breve{R}\text{‘‘}\text{D}\text{‘}R$

***71·162.** $\vdash : R \epsilon 1 \rightarrow 1 . \equiv . \text{E} !! R\text{‘‘}\text{Ɑ}\text{‘}R . \text{E} !! \breve{R}\text{‘‘}\text{D}\text{‘}R$

***71·163.** $\vdash :. R \epsilon 1 \rightarrow \text{Cls} . \equiv : y \epsilon \text{Ɑ}\text{‘}R . \equiv_y . \text{E} ! R\text{‘}y$

Dem.

$\vdash . *33·43 .$ $\supset \vdash : \text{E} ! R\text{‘}y . \supset . y \epsilon \text{Ɑ}\text{‘}R :$

[*4·73] $\supset \vdash :. y \epsilon \text{Ɑ}\text{‘}R . \supset . \text{E} ! R\text{‘}y : \equiv : y \epsilon \text{Ɑ}\text{‘}R . \equiv . \text{E} ! R\text{‘}y :.$

[*10·11·271.*37·104] $\supset \vdash :. \text{E} !! R\text{‘‘}\text{Ɑ}\text{‘}R . \equiv : y \epsilon \text{Ɑ}\text{‘}R . \equiv_y . \text{E} ! R\text{‘}y$ (1)

$\vdash . (1) . *71·16 . \supset \vdash . \text{Prop}$

***71·164.** $\vdash :. R \epsilon \text{Cls} \rightarrow 1 . \equiv : x \epsilon \text{D}\text{‘}R . \equiv_x . \text{E} ! \breve{R}\text{‘}x$

***71·165.** $\vdash :. R \epsilon 1 \rightarrow 1 . \equiv : y \epsilon \text{Ɑ}\text{‘}R . \equiv_y . \text{E} ! R\text{‘}y : x \epsilon \text{D}\text{‘}R . \equiv_x . \text{E} ! \breve{R}\text{‘}x$

***71·166.** $\vdash : (y) . \text{E} ! R\text{‘}y . \supset . R \epsilon 1 \rightarrow \text{Cls}$

Dem.

$\vdash . *2·02 . *10·1 .$ $\supset \vdash :. \text{Hp} . \supset : y \epsilon \text{Ɑ}\text{‘}R . \supset . \text{E} ! R\text{‘}y :.$

[*10·11·21.*37·104] $\supset \vdash : \text{Hp} . \supset . \text{E} !! R\text{‘‘}\text{Ɑ}\text{‘}R .$

[*71·16] $\supset . R \epsilon 1 \rightarrow \text{Cls} : \supset \vdash . \text{Prop}$

***71·167.** $\vdash : (x) . \text{E} ! \breve{R}\text{‘}x . \supset . R \epsilon \text{Cls} \rightarrow 1$

***71·168.** $\vdash :. (y) . \text{E} ! R\text{‘}y : (x) . \text{E} ! \breve{R}\text{‘}x : \supset . R \epsilon 1 \rightarrow 1$

***71·17.** $\vdash :. R \epsilon 1 \rightarrow \text{Cls} . \equiv : xRz . yRz . \supset_{x,y,z} . x = y$

This proposition is constantly used in the sequel.

Dem.

$\vdash . *52·4 .$ $\supset \vdash :. \overrightarrow{R}\text{‘}z \epsilon 1 \cup \iota\text{‘}\Lambda .$ $\equiv : x, y \epsilon \overrightarrow{R}\text{‘}z . \supset_{x,y} . x = y :$

[*32·18] $\equiv : xRz . yRz . \supset_{x,y} . x = y :.$

[*10·11·271.*11·21] $\supset \vdash :. (z) . \overrightarrow{R}\text{‘}z \epsilon 1 \cup \iota\text{‘}\Lambda . \equiv : xRz . yRz . \supset_{x,y,z} . x = y$ (1)

$\vdash . (1) . *71·12 .$ $\supset \vdash . \text{Prop}$

***71·171.** $\vdash :. R \epsilon \mathrm{Cls} \to 1 . \equiv : xRy . xRz . \supset_{x,y,z} . y = z$

***71·172.** $\vdash :. R \epsilon 1 \to 1 . \equiv : xRz . yRz . \supset_{x,y,z} . x = y : xRy . xRz . \supset_{x,y,z} . y = z$

***71·18.** $\vdash :. R \epsilon 1 \to \mathrm{Cls} . \equiv : \exists ! \overleftarrow{R}`x \cap \overleftarrow{R}`y . \supset_{x,y} . x = y$

Dem.

$\vdash . *32·181 . *22·33 . \supset$

$\vdash :. \exists ! \overleftarrow{R}`x \cap \overleftarrow{R}`y . \supset_{x,y} . x = y : \equiv : (\exists z) . xRz . yRz . \supset_{x,y} . x = y :$

[*10·23] $\equiv : xRz . yRz . \supset_{x,y,z} . x = y :$

[*71·17] $\equiv : R \epsilon 1 \to \mathrm{Cls} :. \supset \vdash . \mathrm{Prop}$

***71·181.** $\vdash :. R \epsilon \mathrm{Cls} \to 1 . \equiv : \exists ! \overrightarrow{R}`y \cap \overrightarrow{R}`z . \supset_{y,z} . y = z$

***71·182.** $\vdash :: R \epsilon 1 \to 1 . \equiv :. \exists ! \overleftarrow{R}`x \cap \overleftarrow{R}`y . \vee . \exists ! \overrightarrow{R}`x \cap \overrightarrow{R}`y : \supset_{x,y} . x = y$

***71·19.** $\vdash : R \epsilon 1 \to \mathrm{Cls} . \equiv . R | \breve{R} = I \upharpoonright \mathrm{D}`R$

Dem.

$\vdash . *34·1 . *31·11 . \supset \vdash . x (R | \breve{R}) y . \equiv . (\exists z) . xRz . yRz$ (1)

$\vdash . *50·1 . *35·101 . \supset \vdash . x (I \upharpoonright \mathrm{D}`R) y . \equiv . x = y . y \epsilon \mathrm{D}`R$ (2)

$\vdash . (1) . (2) . *21·43 . \supset$

$\vdash :: R | \breve{R} = I \upharpoonright \mathrm{D}`R . \equiv :. (\exists z) . xRz . yRz . \equiv_{x,y} : x = y . y \epsilon \mathrm{D}`R :$

[*33·13.*10·35] $\equiv_{x,y} : (\exists z) . x = y . yRz :$

[*13·194] $\equiv_{x,y} : (\exists z) . x = y . xRz . yRz :$

[*10·35] $\equiv_{x,y} : x = y : (\exists z) . xRz . yRz :.$

[*4·71] $\equiv :. (\exists z) . xRz . yRz . \supset_{x,y} . x = y :.$

[*10·23] $\equiv :. xRz . yRz . \supset_{x,y,z} . x = y :.$

[*71·17] $\equiv :. R \epsilon 1 \to \mathrm{Cls} :: \supset \vdash . \mathrm{Prop}$

***71·191.** $\vdash : R \epsilon \mathrm{Cls} \to 1 . \equiv . \breve{R} | R = I \upharpoonright \mathrm{G}`R$

***71·192.** $\vdash : R \epsilon 1 \to 1 . \equiv . R | \breve{R} = I \upharpoonright \mathrm{D}`R . \breve{R} | R = I \upharpoonright \mathrm{G}`R$

***71·2.** $\vdash . \mathrm{Cls} \to 1 = \mathrm{Cnv}``(1 \to \mathrm{Cls}) .$
$1 \to \mathrm{Cls} = \mathrm{Cnv}``(\mathrm{Cls} \to 1) . 1 \to 1 = \mathrm{Cnv}``(1 \to 1)$ [*70·22]

***71·21.** $\vdash : R \epsilon 1 \to \mathrm{Cls} . \equiv . \breve{R} \epsilon \mathrm{Cls} \to 1$

Dem.

$\vdash . *37·62 . *31·13 . \supset \vdash : R \epsilon 1 \to \mathrm{Cls} . \supset . \mathrm{Cnv}`R \epsilon \mathrm{Cnv}``(1 \to \mathrm{Cls}) .$

[*31·12.*71·2] $\supset . \breve{R} \epsilon \mathrm{Cls} \to 1$ (1)

$\vdash . *37·62 . *31·13 . \supset \vdash : \breve{R} \epsilon \mathrm{Cls} \to 1 . \supset . \mathrm{Cnv}`\breve{R} \epsilon \mathrm{Cnv}``(\mathrm{Cls} \to 1) .$

[*31·33.*71·2] $\supset . R \epsilon 1 \to \mathrm{Cls}$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***71·211.** $\vdash : R \epsilon \text{Cls} \to 1 . \equiv . \breve{R} \epsilon 1 \to \text{Cls}$

***71·212.** $\vdash : R \epsilon 1 \to 1 . \equiv . \breve{R} \epsilon 1 \to 1$

***71·22.** $\vdash : R \epsilon 1 \to \text{Cls} . S \Subset R . \supset . S \epsilon 1 \to \text{Cls}$

Dem.

$\vdash . *23·1 . \supset$

$\vdash :. S \Subset R . \supset : xSz . ySz . \supset_{x,y,z} . xRz . yRz$ (1)

$\vdash . *71·17 . \supset$

$\vdash :. R \epsilon 1 \to \text{Cls} . \supset : xRz . yRz . \supset_{x,y,z} . x = y$ (2)

$\vdash . (1) . (2) . *11·37 . \supset$

$\vdash :. \text{Hp} . \supset : xSz . ySz . \supset_{x,y,z} . x = y :$

$[*71·17] \supset : S \epsilon 1 \to \text{Cls} :. \supset \vdash . \text{Prop}$

***71·221.** $\vdash : R \epsilon \text{Cls} \to 1 . S \Subset R . \supset . S \epsilon \text{Cls} \to 1$

***71·222.** $\vdash : R \epsilon 1 \to 1 . S \Subset R . \supset . S \epsilon 1 \to 1$

***71·223.** $\vdash : R \epsilon 1 \to \text{Cls} . \supset . \text{Rl}‘R \subset 1 \to \text{Cls}$ [*71·22 . *61·2]

***71·224.** $\vdash : R \epsilon \text{Cls} \to 1 . \supset . \text{Rl}‘R \subset \text{Cls} \to 1$

***71·225.** $\vdash : R \epsilon 1 \to 1 . \supset . \text{Rl}‘R \subset 1 \to 1$

***71·23.** $\vdash : R \epsilon 1 \to \text{Cls} . \supset . R \dot{\cap} S \epsilon 1 \to \text{Cls}$ [*71·22 . *23·43]

***71·231.** $\vdash : R \epsilon \text{Cls} \to 1 . \supset . R \dot{\cap} S \epsilon \text{Cls} \to 1$

***71·232.** $\vdash : R \epsilon 1 \to 1 . \supset . R \dot{\cap} S \epsilon 1 \to 1$

***71·233.** $\vdash : R, S \epsilon 1 \to \text{Cls} . \supset . R \dot{\cap} \breve{S} \epsilon 1 \to 1$

Dem.

$\vdash . *71·23 . \supset \vdash : \text{Hp} . \supset . R \dot{\cap} \breve{S} \epsilon 1 \to \text{Cls}$ (1)

$\vdash . *71·21 . \supset \vdash : \text{Hp} . \supset . \breve{S} \epsilon \text{Cls} \to 1 .$

$[*71·231] \qquad \supset . R \dot{\cap} \breve{S} \epsilon \text{Cls} \to 1$ (2)

$\vdash . (1) . (2) . *71·103 . \supset \vdash . \text{Prop}$

***71·234.** $\vdash : R, S \epsilon \text{Cls} \to 1 . \supset . R \dot{\cap} \breve{S} \epsilon 1 \to 1$

***71·235.** $\vdash : R \epsilon 1 \to \text{Cls} . S \epsilon \text{Cls} \to 1 . \supset . R \dot{\cap} S \epsilon 1 \to 1$

***71·24.** $\vdash : R, S \epsilon 1 \to \text{Cls} . \text{Ꮧ}‘R \cap \text{Ꮧ}‘S = \Lambda . \supset . R \dot{\cup} S \epsilon 1 \to \text{Cls}$ [*70·54]

***71·241.** $\vdash : R, S \epsilon \text{Cls} \to 1 . \text{D}‘R \cap \text{D}‘S = \Lambda . \supset . R \dot{\cup} S \epsilon \text{Cls} \to 1$ [*70·55]

***71·242.** $\vdash : R, S \epsilon 1 \to 1 . \text{D}‘R \cap \text{D}‘S = \Lambda . \text{Ꮧ}‘R \cap \text{Ꮧ}‘S = \Lambda . \supset . R \dot{\cup} S \epsilon 1 \to 1$ [*70·56]

***71·243.** $\vdash : R, S \epsilon 1 \to 1 . C‘R \cap C‘S = \Lambda . \supset . R \dot{\cup} S \epsilon 1 \to 1$ [*70·57]

***71·244.** $\vdash : R, S \epsilon 1 \to \text{Cls} . R \upharpoonright \text{Ꮧ}‘S \Subset S . \supset . R \dot{\cup} S \epsilon 1 \to \text{Cls}$

Dem.

$\vdash . *23·34 . *4·4 . \supset$

$\vdash :. x(R \dot{\cup} S)z . y(R \dot{\cup} S)z . \equiv : xRz . yRz . \vee . xRz . ySz . \vee . xSz . yRz . \vee . xSz . ySz$ (1)

$\vdash . *71\cdot17 . \supset \vdash :. R, S \epsilon 1 \rightarrow \text{Cls} . \supset : xRz . yRz . \supset . x = y : xSz . ySz . \supset . x = y$ (2)

$\vdash . *33\cdot14 . *4\cdot7 . \supset \vdash : xRz . ySz . \supset . xRz . ySz . z \epsilon \text{Œ}`S .$

[$*35\cdot101$] $\supset . x (R \upharpoonright \text{Œ}`S) z . ySz$ (3)

$\vdash . (3) . \quad \supset \vdash :. R \upharpoonright \text{Œ}`S \subseteq S . \supset : xRz . ySz . \supset . xSz . ySz$ (4)

$\vdash . (4) \dfrac{y, x}{x, y} . \quad \supset \vdash :. R \upharpoonright \text{Œ}`S \subseteq S . \supset : xSz . yRz . \supset . xSz . ySz$ (5)

$\vdash . (2) . (4) . (5) . \supset \vdash :. \text{Hp} . \supset : xRz . ySz . \supset . x = y : xSz . yRz . \supset . x = y$ (6)

$\vdash . (1) . (2) . (6) . *4\cdot77 . \supset \vdash :. \text{Hp} . \supset : x (R \cup S) z . y (R \cup S) z . \supset . x = y$ (7)

$\vdash . (7) . *10\cdot11\cdot21 . *71\cdot17 . \supset \vdash . \text{Prop}$

$*71\cdot245$. $\vdash : R, S \epsilon \text{Cls} \rightarrow 1 . (\text{D}`S) \upharpoonright R \subseteq S . \supset . R \cup S \epsilon \text{Cls} \rightarrow 1$

$*71\cdot25$. $\vdash : R, S \epsilon 1 \rightarrow \text{Cls} . \supset . R | S \epsilon 1 \rightarrow \text{Cls}$

Dem.

$\vdash . *71\cdot17 . \supset \vdash :. \text{Hp} . \supset : ySx . zSx . \supset . y = z :$

[Fact] $\supset : uRy . ySx . vRz . zSx . \supset . y = z . uRy . vRz .$

[$*13\cdot13$] $\supset . uRy . vRy .$

[$*71\cdot17$] $\supset . u = v$ (1)

$\vdash . (1) . *11\cdot11\cdot35\cdot4 . \supset$

$\vdash :: \text{Hp} . \supset :. (\exists y) . uRy . ySx : (\exists z) . vRz . zSx : \supset . u = v :.$

[$*34\cdot1$] $\supset :. u (R | S) x . v (R | S) x . \supset . u = v$ (2)

$\vdash . (2) . *71\cdot17 . \supset \vdash . \text{Prop}$

$*71\cdot251$. $\vdash : R, S \epsilon \text{Cls} \rightarrow 1 . \supset . R | S \epsilon \text{Cls} \rightarrow 1$

$*71\cdot252$. $\vdash : R, S \epsilon 1 \rightarrow 1 . \supset . R | S \epsilon 1 \rightarrow 1$

$*71\cdot25$ may also be deduced from $*70\cdot6$, as follows:

Alternative Dem. of $*71\cdot25$.

$\vdash . *53\cdot301 . *71\cdot12 . \supset \vdash : R \epsilon 1 \rightarrow \text{Cls} . \supset . R``\iota`x \epsilon 1 \cup \iota`\Lambda :$

[$*52\cdot1$] $\supset \vdash : R \epsilon 1 \rightarrow \text{Cls} . \alpha \epsilon 1 . \supset . R``\alpha \epsilon 1 \cup \iota`\Lambda :$

[$*37\cdot61\cdot11\cdot103$] $\supset \vdash : R \epsilon 1 \rightarrow \text{Cls} . \supset . R```1 \subset 1 \cup \iota`\Lambda$ (1)

$\vdash . (1) . *70\cdot6 . \supset \vdash . \text{Prop}$

Similarly $*71\cdot251$ may be deduced from $*70\cdot61$.

$*71\cdot26$. $\vdash : R \epsilon 1 \rightarrow \text{Cls} . \supset . R \upharpoonright \gamma \epsilon 1 \rightarrow \text{Cls}$ [$*70\cdot62$]

$*71\cdot261$. $\vdash : R \epsilon \text{Cls} \rightarrow 1 . \supset . \beta \upharpoonleft R \epsilon \text{Cls} \rightarrow 1$ [$*70\cdot63$]

$*71\cdot27$. $\vdash : R \epsilon 1 \rightarrow \text{Cls} . \supset . \beta \upharpoonleft R \epsilon 1 \rightarrow \text{Cls}$ [$*35\cdot44 . *71\cdot22$]

$*71\cdot271$. $\vdash : R \epsilon \text{Cls} \rightarrow 1 . \supset . R \upharpoonright \gamma \epsilon \text{Cls} \rightarrow 1$

$*71\cdot28$. $\vdash : R \epsilon 1 \rightarrow \text{Cls} . \supset . \beta \upharpoonleft R \upharpoonright \gamma \epsilon 1 \rightarrow \text{Cls}$ [$*35\cdot442 . *71\cdot22$]

$*71\cdot281$. $\vdash : R \epsilon \text{Cls} \rightarrow 1 . \supset . \beta \upharpoonleft R \upharpoonright \gamma \epsilon \text{Cls} \rightarrow 1$

$*71\cdot29$. $\vdash : R \epsilon 1 \rightarrow 1 . \supset . \beta \upharpoonleft R, R \upharpoonright \gamma, \beta \upharpoonleft R \upharpoonright \gamma \epsilon 1 \rightarrow 1$

$*71\cdot31$. $\vdash : R \epsilon 1 \rightarrow \text{Cls} . y \epsilon \text{Œ}`R . \supset . (R`y) R y$ [$*30\cdot32 . *71\cdot163$]

***71·311.** $\vdash : R \in \mathrm{Cls} \to 1 \,.\, x \in \mathrm{D}'R \,.\, \supset .\, xR(\breve{R}'x)$

***71·312.** $\vdash : R \in 1 \to 1 \,.\, x \in \mathrm{D}'R \,.\, y \in \text{Ɑ}'R \,.\, \supset .\, xR(\breve{R}'x) \,.\, (R'y)\,Ry$

***71·32.** $\vdash :: R \in 1 \to \mathrm{Cls} \,.\, y \in \text{Ɑ}'R \,.\, \supset :.\, \psi(R'y) \,.\, \equiv : (\exists x) \,.\, xRy \,.\, \psi x : \equiv : xRy \,.\, \supset_x .\, \psi x$
[*30·33 . *71·163]

***71·321.** $\vdash :: R \in \mathrm{Cls} \to 1 \,.\, x \in \mathrm{D}'R \,.\, \supset :.\, \psi(\breve{R}'x) \,.\, \equiv : (\exists y) \,.\, xRy \,.\, \psi y : \equiv : xRy \,.\, \supset_y .\, \psi y$

***71·33.** $\vdash :: R \in 1 \to \mathrm{Cls} \,.\, \supset :.\, \psi(R'y) : \equiv : (\exists x) \,.\, xRy \,.\, \psi x : \equiv : y \in \text{Ɑ}'R : xRy \,.\, \supset_x .\, \psi x$

Dem.

$\vdash .\, *71{\cdot}32 \,.\, *5{\cdot}32 \,.\, \supset$

$\vdash :: \mathrm{Hp} \,.\, \supset :.\, y \in \text{Ɑ}'R \,.\, \psi(R'y) \,.\, \equiv : y \in \text{Ɑ}'R : (\exists x) \,.\, xRy \,.\, \psi x :$

$\equiv : y \in \text{Ɑ}'R : xRy \,.\, \supset_x .\, \psi x \qquad (1)$

$\vdash .\, *14{\cdot}21 \,.\, \supset \vdash : \psi(R'y) \,.\, \supset .\, \mathrm{E}\,!\, R'y \,.$

[*33·43] $\supset .\, y \in \text{Ɑ}'R :$

[*4·71] $\supset \vdash :.\, y \in \text{Ɑ}'R \,.\, \psi(R'y) \,.\, \equiv .\, \psi(R'y) \qquad (2)$

$\vdash .\, *10{\cdot}5 \,.\, \supset \vdash : (\exists x) \,.\, xRy \,.\, \psi x \,.\, \supset .\, (\exists x) \,.\, xRy \,.$

[*33·131] $\supset .\, y \in \text{Ɑ}'R :$

[*4·71] $\supset \vdash :.\, y \in \text{Ɑ}'R : (\exists x) \,.\, xRy \,.\, \psi x : \equiv .\, (\exists x) \,.\, xRy \,.\, \psi x \qquad (3)$

$\vdash .\, (1) \,.\, (2) \,.\, (3) \,.\, \supset \vdash .\, \mathrm{Prop}$

***71·331.** $\vdash :: R \in \mathrm{Cls} \to 1 \,.\, \supset :.\, \psi(\breve{R}'x) \,.\, \equiv : (\exists y) \,.\, xRy \,.\, \psi y : \equiv :$
$x \in \mathrm{D}'R : xRy \,.\, \supset_y .\, \psi y$

***71·332.** $\vdash :.\, R \in 1 \to \mathrm{Cls} \,.\, \supset : R'y \in \alpha \,.\, \equiv .\, \exists\,!\, \overrightarrow{R}'y \cap \alpha \,.\, \equiv .\, y \in \text{Ɑ}'R \,.\, \overrightarrow{R}'y \subset \alpha$
$\left[*71{\cdot}33 \dfrac{x \in \alpha}{\psi x}\right]$

***71·333.** $\vdash :.\, R \in \mathrm{Cls} \to 1 \,.\, \supset : \breve{R}'x \in \alpha \,.\, \equiv .\, \exists\,!\, \overleftarrow{R}'x \cap \alpha \,.\, \equiv .\, x \in \mathrm{D}'R \,.\, \overleftarrow{R}'x \subset \alpha$

***71·34.** $\vdash : R \in 1 \to \mathrm{Cls} \,.\, R = S \,.\, y \in \text{Ɑ}'R \,.\, \supset .\, R'y = S'y$ [*30·36 . *71·163]

***71·341.** $\vdash : R \in \mathrm{Cls} \to 1 \,.\, R = S \,.\, x \in \mathrm{D}'R \,.\, \supset .\, \breve{R}'x = \breve{S}'x$

***71·35.** $\vdash :: R \in 1 \to \mathrm{Cls} \,.\, \supset :.\, y \in \text{Ɑ}'R \cup \text{Ɑ}'S \,.\, \supset_y .\, R'y = S'y : \equiv .\, R = S$

Dem.

$\vdash .\, *21{\cdot}18 \,.\, \supset \vdash :.\, R = S \,.\, \supset : y \in \text{Ɑ}'R \cup \text{Ɑ}'S \,.\, \equiv .\, y \in \text{Ɑ}'R \cup \text{Ɑ}'R \,.$

[*22·56] $\equiv .\, y \in \text{Ɑ}'R \qquad (1)$

$\vdash .\, (1) \,.\, *71{\cdot}34 \,.\, \supset \vdash :: \mathrm{Hp} \,.\, R = S \,.\, \supset : y \in \text{Ɑ}'R \cup \text{Ɑ}'S \,.\, \supset_y .\, R'y = S'y \qquad (2)$

$\vdash .\, (2) \,.\, *33{\cdot}45 \,.\, \supset \vdash .\, \mathrm{Prop}$

***71·351.** $\vdash :: R \in \mathrm{Cls} \to 1 \,.\, \supset :.\, x \in \mathrm{D}'R \cup \mathrm{D}'S \,.\, \supset_x .\, \breve{R}'x = \breve{S}'x : \equiv .\, R = S$

***71·352.** $\vdash :: R \in 1 \to 1 \,.\, \supset :.\, y \in \text{Ɑ}'R \cup \text{Ɑ}'S \,.\, \supset_y .\, R'y = S'y : \equiv : R = S :$
$\equiv : x \in \mathrm{D}'R \cup \mathrm{D}'S \,.\, \supset_x .\, \breve{R}'x = \breve{S}'x$

***71·36.** $\vdash :. R \in 1 \rightarrow \mathrm{Cls} . \supset : x = R\text{‘}y . \equiv . xRy$

Dem.

$$\vdash . *30{\cdot}4 . *71{\cdot}163 . \supset$$
$$\vdash :. \mathrm{Hp} . y \in \text{Ɑ}\text{‘}R . \supset : x = R\text{‘}y . \equiv . xRy \qquad (1)$$
$$\vdash . *71{\cdot}163 . \mathrm{Transp} . \supset$$
$$\vdash :. \mathrm{Hp} . y \sim\in \text{Ɑ}\text{‘}R . \supset . \sim E ! R\text{‘}y .$$
$$[*14{\cdot}21 . \mathrm{Transp}] \quad \supset . \sim (x = R\text{‘}y) \qquad (2)$$
$$\vdash . *33{\cdot}14 . \mathrm{Transp} . \supset \vdash : y \sim\in \text{Ɑ}\text{‘}R . \supset . \sim (xRy) \qquad (3)$$
$$\vdash . (2) . (3) . *5{\cdot}21 . \supset$$
$$\vdash :. \mathrm{Hp} . y \sim\in \text{Ɑ}\text{‘}R . \supset : x = R\text{‘}y . \equiv . xRy \qquad (4)$$
$$\vdash . (1) . (4) . *4{\cdot}83 . \supset \vdash . \mathrm{Prop}$$

***71·361.** $\vdash :. R \in \mathrm{Cls} \rightarrow 1 . \supset :. y = \breve{R}\text{‘}x . \equiv . xRy$

***71·362.** $\vdash :. R \in 1 \rightarrow 1 . \supset : x = R\text{‘}y . \equiv . xRy . \equiv . y = \breve{R}\text{‘}x$

***71·37.** $\vdash :. R \in 1 \rightarrow \mathrm{Cls} . \supset : y \in \breve{R}\text{‘‘}\alpha . \equiv . R\text{‘}y \in \alpha$

Dem.

$$\vdash . *71{\cdot}33 . \supset \vdash :. \mathrm{Hp} . \supset : R\text{‘}y \in \alpha . \equiv . (\exists x) . xRy . x \in \alpha .$$
$$[*37{\cdot}105] \qquad \equiv . y \in \breve{R}\text{‘‘}\alpha :. \supset \vdash . \mathrm{Prop}$$

***71·371.** $\vdash :. R \in \mathrm{Cls} \rightarrow 1 . \supset : x \in R\text{‘‘}\alpha . \equiv . \breve{R}\text{‘}x \in \alpha$

***71·38.** $\vdash : R \in 1 \rightarrow \mathrm{Cls} . \supset . \breve{R}\text{‘‘}(\alpha - \beta) = \breve{R}\text{‘‘}\alpha - \breve{R}\text{‘‘}\beta$

Dem.

$$\vdash . *71{\cdot}37 . \supset \vdash :. \mathrm{Hp} . \supset : y \in \breve{R}\text{‘‘}(\alpha - \beta) . \equiv . R\text{‘}y \in \alpha - \beta .$$
$$[*22{\cdot}32 . *14{\cdot}21] \qquad \equiv . R\text{‘}y \in \alpha . \sim (R\text{‘}y \in \beta) .$$
$$[*71{\cdot}37] \qquad \equiv . y \in \breve{R}\text{‘‘}\alpha . \sim (y \in \breve{R}\text{‘‘}\beta) .$$
$$[*22{\cdot}32] \qquad \equiv . y \in \breve{R}\text{‘‘}\alpha - \breve{R}\text{‘‘}\beta :. \supset \vdash . \mathrm{Prop}$$

***71·381.** $\vdash : R \in \mathrm{Cls} \rightarrow 1 . \supset . R\text{‘‘}(\alpha - \beta) = R\text{‘‘}\alpha - R\text{‘‘}\beta$

***71·4.** $\vdash : R \in 1 \rightarrow \mathrm{Cls} . \supset . R\text{‘‘}\beta = \hat{x} \{(\exists y) . y \in \beta . x = R\text{‘}y\}$ $[*37{\cdot}1 . *71{\cdot}36]$

***71·401.** $\vdash : R \in \mathrm{Cls} \rightarrow 1 . \supset . \breve{R}\text{‘‘}\beta = \hat{y} \{(\exists x) . x \in \beta . y = \breve{R}\text{‘}x\}$

***71·41.** $\vdash : R \in 1 \rightarrow \mathrm{Cls} . \supset . \mathrm{D}\text{‘}R = \hat{x} \{(\exists y) . x = R\text{‘}y\}$ $[*33{\cdot}11 . *71{\cdot}36]$

***71·411.** $\vdash : R \in \mathrm{Cls} \rightarrow 1 . \supset . \text{Ɑ}\text{‘}R = \hat{y} \{(\exists x) . y = \breve{R}\text{‘}x\}$

***71·42.** $\vdash :: R \in 1 \rightarrow \mathrm{Cls} . \beta \subset \text{Ɑ}\text{‘}R . \supset :. R\text{‘‘}\beta \subset \alpha . \equiv : y \in \beta . \supset_y . R\text{‘}y \in \alpha$
$[*37{\cdot}61 . *71{\cdot}16]$

***71·421.** $\vdash :: R \in \mathrm{Cls} \rightarrow 1 . \alpha \subset \mathrm{D}\text{‘}R . \supset :. \breve{R}\text{‘‘}\alpha \subset \beta . \equiv : x \in \alpha . \supset_x . \breve{R}\text{‘}x \in \beta$

***71·43.** $\vdash : R \in 1 \rightarrow \mathrm{Cls} . y \in \alpha \cap \text{Ɑ}\text{‘}R . \supset . R\text{‘}y \in R\text{‘‘}\alpha$ $[*37{\cdot}62 . *71{\cdot}16]$

***71·431.** $\vdash : R \in \mathrm{Cls} \rightarrow 1 . x \in \alpha \cap \mathrm{D}\text{‘}R . \supset . \breve{R}\text{‘}x \in \breve{R}\text{‘‘}\alpha$

***71·44.** $\vdash :: R \epsilon 1 \rightarrow \mathrm{Cls} . \alpha \subset \mathrm{G}`R . \supset :. x \epsilon R``\alpha . \supset_x . \psi x : \equiv : y \epsilon \alpha . \supset_y . \psi(R`y)$
[*37·63 . *71·16]

***71·441.** $\vdash :: R \epsilon \mathrm{Cls} \rightarrow 1 . \alpha \subset \mathrm{D}`\breve{R} . \supset :. y \epsilon \breve{R}``\alpha . \supset_y . \psi y : \equiv : x \epsilon \alpha . \supset_x . \psi(\breve{R}`x)$

***71·45.** $\vdash :. R \epsilon 1 \rightarrow \mathrm{Cls} . \supset : (\exists x) . x \epsilon R``\alpha . \psi x . \equiv . (\exists y) . y \epsilon \alpha . \psi(R`y)$

Dem.

$\vdash . *37\cdot64 . *71\cdot16 . \supset$

$\vdash :. \mathrm{Hp} . \supset : (\exists x) . x \epsilon R``(\alpha \cap \mathrm{G}`R) . \psi x . \equiv . (\exists y) . y \epsilon \alpha \cap \mathrm{G}`R . \psi(R`y)$ (1)

$\vdash . *37\cdot26 . \quad \supset \vdash . R``(\alpha \cap \mathrm{G}`R) = R``\alpha$ (2)

$\vdash . *14\cdot21 . \quad \supset \vdash : y \epsilon \alpha . \psi(R`y) . \supset . \mathrm{E} ! R`y .$

$[*33\cdot43] \quad \supset . y \epsilon \mathrm{G}`R :$

$[*4\cdot71 . *22\cdot33] \supset \vdash : y \epsilon \alpha . \psi(R`y) . \equiv . y \epsilon \alpha \cap \mathrm{G}`R . \psi(R`y) :$

$[*10\cdot11\cdot281] \quad \supset \vdash : (\exists y) . y \epsilon \alpha . \psi(R`y) . \equiv . (\exists y) . y \epsilon \alpha \cap \mathrm{G}`R . \psi(R`y)$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash . \mathrm{Prop}$

***71·451.** $\vdash :. R \epsilon \mathrm{Cls} \rightarrow 1 . \supset : (\exists y) . y \epsilon \breve{R}``\alpha . \psi y . \equiv . (\exists x) . x \epsilon \alpha . \psi(\breve{R}`x)$

***71·46.** $\vdash : R \epsilon 1 \rightarrow \mathrm{Cls} . \alpha \subset R``\beta . \supset . \alpha = R``(\breve{R}``\alpha \cap \beta)$

Dem.

$\vdash . *37\cdot26 . \supset \vdash : R``\beta = R``(\beta \cap \mathrm{G}`R) . R``(\breve{R}``\alpha \cap \beta) = R``(\breve{R}``\alpha \cap \beta \cap \mathrm{G}`R)$ (1)

$\vdash . *37\cdot65 . *71\cdot16 . \supset$

$\vdash : R \epsilon 1 \rightarrow \mathrm{Cls} . \alpha \subset R``(\beta \cap \mathrm{G}`R) . \supset . \alpha = R``(\breve{R}``\alpha \cap \beta \cap \mathrm{G}`R)$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***71·461.** $\vdash : R \epsilon \mathrm{Cls} \rightarrow 1 . \beta \subset \breve{R}``\alpha . \supset . \beta = \breve{R}``(R``\beta \cap \alpha)$

***71·47.** $\vdash :. R \epsilon 1 \rightarrow \mathrm{Cls} . \supset : \alpha \subset R``\beta . \equiv . (\exists \gamma) . \gamma \subset \beta . \alpha = R``\gamma$

Dem.

$\vdash . *71\cdot46 . *10\cdot24 . *22\cdot43 . \supset \vdash :. \mathrm{Hp} . \supset : \alpha \subset R``\beta . \supset . (\exists \gamma) . \gamma \subset \beta . \alpha = R``\gamma$ (1)

$\vdash . *37\cdot2 . *10\cdot11\cdot23 . \quad \supset \vdash : (\exists \gamma) . \gamma \subset \beta . \alpha = R``\gamma . \supset . \alpha \subset R``\beta$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***71·471.** $\vdash :. R \epsilon \mathrm{Cls} \rightarrow 1 . \supset : \beta \subset \breve{R}``\alpha . \equiv . (\exists \gamma) . \gamma \subset \alpha . \beta = \breve{R}``\gamma$

***71·48.** $\vdash : R \epsilon 1 \rightarrow \mathrm{Cls} . \supset . \mathrm{D}`R_\epsilon = \mathrm{Cl}`\mathrm{D}`R$

Dem.

$\vdash . *37\cdot24 . *60\cdot2 . \quad \supset \vdash . \mathrm{D}`R_\epsilon \subset \mathrm{Cl}`\mathrm{D}`R$ (1)

$\vdash . *37\cdot25 . *71\cdot47 . *60\cdot2 . \supset \vdash : \mathrm{Hp} . \alpha \epsilon \mathrm{Cl}`\mathrm{D}`R . \supset . (\exists \gamma) . \gamma \subset \mathrm{G}`R . \alpha = R``\gamma .$

$[*10\cdot5 . *37\cdot23] \quad \supset . \alpha \epsilon \mathrm{D}`R_\epsilon :$

$[\mathrm{Exp} . *10\cdot11\cdot21] \quad \supset \vdash : \mathrm{Hp} . \supset . \mathrm{Cl}`\mathrm{D}`R \subset \mathrm{D}`R_\epsilon$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***71·481.** $\vdash : R \epsilon \mathrm{Cls} \rightarrow 1 . \supset . \mathrm{D}`(\breve{R})_\epsilon = \mathrm{Cl}`\mathrm{G}`R$

The following proposition is used in the theory of derivatives of a series (*216·411).

***71·49.** $\vdash : R \epsilon 1 \to \mathrm{Cls} . \alpha \subset \mathrm{G}`R . \supset . R```\mathrm{Cl}`\alpha = \mathrm{Cl}`R``\alpha . R```\mathrm{Cl\,ex}`\alpha = \mathrm{Cl\,ex}`R``\alpha$

Dem.

$\vdash . *71{\cdot}47 . *60{\cdot}2 . \supset \vdash :. \mathrm{Hp} . \supset : \gamma \epsilon \mathrm{Cl}`R``\alpha . \equiv . (\exists\beta) . \beta \subset \alpha . \gamma = R``\beta .$

$[*37{\cdot}103] \qquad \equiv . \gamma \epsilon R```\mathrm{Cl}`\alpha \qquad (1)$

$\vdash . *37{\cdot}43 . \qquad \supset \vdash :. \mathrm{Hp} . \beta \epsilon \mathrm{Cl}`\alpha . \supset : \exists ! \beta . \equiv . \exists ! R``\beta \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***71·491.** $\vdash : R \epsilon \mathrm{Cls} \to 1 . \alpha \subset \mathrm{D}`R . \supset . \breve{R}```\mathrm{Cl}`\alpha = \mathrm{Cl}`\breve{R}``\alpha . \breve{R}```\mathrm{Cl\,ex}`\alpha = \mathrm{Cl\,ex}`\breve{R}``\alpha$

This proposition is used in the theory of derivatives of a series (*216·4) and in the theory of ordinal numbers (*251·11).

***71·5.** $\vdash :. R \epsilon 1 \to \mathrm{Cls} . \supset : xRy . \equiv . x = \breve{\iota}`\overrightarrow{R}`y$

Dem.

$\vdash . *71{\cdot}36 . *30{\cdot}1 . \supset \vdash :. \mathrm{Hp} . \supset : xRy . \equiv . x = (\imath x)(xRy) .$

$[*51{\cdot}56 . *32{\cdot}13] \qquad \equiv . x = \breve{\iota}`\overrightarrow{R}`y :. \supset \vdash . \mathrm{Prop}$

***71·501.** $\vdash :. R \epsilon \mathrm{Cls} \to 1 . \supset : xRy . \equiv . y = \breve{\iota}`\overleftarrow{R}`x$

***71·51.** $\vdash : R \epsilon 1 \to \mathrm{Cls} . y \epsilon \mathrm{G}`R . \supset . R`y = \breve{\iota}`\overrightarrow{R}`y$

Dem.

$\vdash . *53{\cdot}31 . *71{\cdot}163 . \supset \vdash : \mathrm{Hp} . \supset . \iota`R`y = \overrightarrow{R}`y .$

$[*51{\cdot}51] \qquad \supset . R`y = \breve{\iota}`\overrightarrow{R}`y : \supset \vdash . \mathrm{Prop}$

***71·511.** $\vdash : R \epsilon \mathrm{Cls} \to 1 . x \epsilon \mathrm{D}`R . \supset . \breve{R}`x = \breve{\iota}`\overleftarrow{R}`x$

***71·52.** $\vdash : R \epsilon 1 \to \mathrm{Cls} . \supset . R``\alpha = \breve{\iota}``\overrightarrow{R}``\alpha$

Dem.

$\vdash . *37{\cdot}1 . \qquad \supset \vdash . \breve{\iota}``\overrightarrow{R}``\alpha = \hat{x}\{(\exists\beta) . \beta \epsilon \overrightarrow{R}``\alpha . x \breve{\iota} \beta\}$

$[*51{\cdot}51] \qquad = \hat{x}\{(\exists\beta) . \beta \epsilon \overrightarrow{R}``\alpha . x = \breve{\iota}`\beta\}$

$[*37{\cdot}7] \qquad = \hat{x}\{(\exists\beta, y) . y \epsilon \alpha . \beta = \overrightarrow{R}`y . x = \breve{\iota}`\beta\}$

$[*11{\cdot}23 . *13{\cdot}195] \qquad = \hat{x}\{(\exists y) . y \epsilon \alpha . x = \breve{\iota}`\overrightarrow{R}`y\} \qquad (1)$

$\vdash . (1) . *71{\cdot}5 . \supset \vdash : \mathrm{Hp} . \supset . \breve{\iota}``\overrightarrow{R}``\alpha = \hat{x}\{(\exists y) . y \epsilon \alpha . xRy\}$

$[*37{\cdot}1] \qquad = R``\alpha : \supset \vdash . \mathrm{Prop}$

***71·521.** $\vdash : R \epsilon \mathrm{Cls} \to 1 . \supset . \breve{R}``\alpha = \breve{\iota}``\overleftarrow{R}``\alpha$

***71·53.** $\vdash : R \epsilon 1 \to \mathrm{Cls} . \breve{R}`x = \breve{R}`y . \supset . x = y$

Dem.

$\vdash . *14{\cdot}21 . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{E} ! \breve{R}`x . \mathrm{E} ! \breve{R}`y .$

$[*30{\cdot}32] \qquad \supset . xR(\breve{R}`x) . yR(\breve{R}`y) .$

$[*14{\cdot}16] \qquad \supset . xR(\breve{R}`y) . yR(\breve{R}`y) .$

$[*71{\cdot}17] \qquad \supset . x = y : \supset \vdash . \mathrm{Prop}$

***71·531.** $\vdash : R \in \mathrm{Cls} \to 1 \,.\, R\text{‘}y = R\text{‘}z \,.\, \supset \,.\, y = z$

***71·532.** $\vdash :. R \in 1 \to 1 \,.\, \supset : R\text{‘}y = R\text{‘}z \,.\, \supset \,.\, y = z : \breve{R}\text{‘}x = \breve{R}\text{‘}y \,.\, \supset \,.\, x = y$

***71·54.** $\vdash :: R \in 1 \to \mathrm{Cls} \,.\, \supset :. R \in 1 \to 1 \,.\, \equiv : R\text{‘}y = R\text{‘}z \,.\, \supset_{y,z} \,.\, y = z$

This proposition and the next (*71·55) are very often used.

Dem.

$\vdash \,.\, *71{\cdot}36 \,.\, \supset \vdash :. \mathrm{Hp} \,.\, \supset : (\exists x) \,.\, xRy \,.\, xRz \,.\, \equiv_{y,z} \,.\, (\exists x) \,.\, x = R\text{‘}y \,.\, x = R\text{‘}z \,.$

$[*14{\cdot}205] \qquad \equiv_{y,z} \,.\, R\text{‘}y = R\text{‘}z \qquad (1)$

$\vdash \,.\, (1) \,.\, \supset \vdash :: \mathrm{Hp} \,.\, \supset :. R\text{‘}y = R\text{‘}z \,.\, \supset_{y,z} \,.\, y = z :\equiv: (\exists x) \,.\, xRy \,.\, xRz \,.\, \supset_{y,z} \,.\, y = z :$

$[*10{\cdot}23] \qquad \equiv : xRy \,.\, xRz \,.\, \supset_{x,y,z} \,.\, y = z :$

$[*71{\cdot}171] \qquad \equiv : R \in \mathrm{Cls} \to 1 \qquad (2)$

$\vdash \,.\, *71{\cdot}103 \,.\, *4{\cdot}73 \,.\, \supset \vdash :. \mathrm{Hp} \,.\, \supset : R \in \mathrm{Cls} \to 1 \,.\, \equiv \,.\, R \in 1 \to 1 \qquad (3)$

$\vdash \,.\, (2) \,.\, (3) \,.\, \supset \vdash \,.\, \mathrm{Prop}$

***71·55.** $\vdash :: R \in 1 \to \mathrm{Cls} \,.\, \supset :. R \upharpoonright \beta \in 1 \to 1 \,.\, \equiv : y, z \in \beta \,.\, R\text{‘}y = R\text{‘}z \,.\, \supset_{y,z} \,.\, y = z$

Dem.

$\vdash \,.\, *71{\cdot}26 \,.\, \supset \vdash :: \mathrm{Hp} \,.\, \supset :. R \upharpoonright \beta \in 1 \to \mathrm{Cls} :.$

$[*71{\cdot}54] \qquad \supset :. R \upharpoonright \beta \in 1 \to 1 \,.\, \equiv : (R \upharpoonright \beta)\text{‘}y = (R \upharpoonright \beta)\text{‘}z \,.\, \supset_{y,z} \,.\, y = z :$

$[*35{\cdot}7] \qquad \equiv : y, z \in \beta \,.\, R\text{‘}y = R\text{‘}z \,.\, \supset_{y,z} \,.\, y = z :: \supset \vdash \,.\, \mathrm{Prop}$

***71·56.** $\vdash :. R \in 1 \to 1 \,.\, y \in \text{ᗡ}\text{‘}R \,.\, \supset : R\text{‘}y = R\text{‘}z \,.\, \equiv \,.\, y = z$

Dem.

$\vdash \,.\, *71{\cdot}532 \,. \qquad \supset \vdash : \mathrm{Hp} \,.\, R\text{‘}y = R\text{‘}z \,.\, \supset \,.\, y = z \qquad (1)$

$\vdash \,.\, *71{\cdot}165 \,.\, *30{\cdot}37 \,.\, \supset \vdash : \mathrm{Hp} \,.\, y = z \,.\, \supset \,.\, R\text{‘}y = R\text{‘}z \qquad (2)$

$\vdash \,.\, (1) \,.\, (2) \,.\, \supset \vdash \,.\, \mathrm{Prop}$

***71·561.** $\vdash :. R \in 1 \to 1 \,.\, x \in \mathrm{D}\text{‘}R \,.\, \supset : \breve{R}\text{‘}x = \breve{R}\text{‘}y \,.\, \equiv \,.\, x = y$

***71·57.** $\vdash :. R\text{‘}y = R\text{‘}z \,.\, \equiv_{y,z} \,.\, y = z :\equiv: R \in 1 \to 1 : (y) \,.\, \mathrm{E} \,!\, R\text{‘}y$

Dem.

$\vdash \,.\, *10{\cdot}1 \,.\, \supset \vdash :. R\text{‘}y = R\text{‘}z \,.\, \equiv_{y,z} \,.\, y = z : \supset : R\text{‘}y = R\text{‘}y \,.\, \equiv_y \,.\, y = y :$

$[*13{\cdot}15] \qquad \supset : (y) \,.\, R\text{‘}y = R\text{‘}y :$

$[*14{\cdot}28] \qquad \supset : (y) \,.\, \mathrm{E} \,!\, R\text{‘}y \qquad (1)$

$[*71{\cdot}166] \qquad \supset : R \in 1 \to \mathrm{Cls} \qquad (2)$

$\vdash \,.\, (2) \,.\, \supset \vdash :. \mathrm{Hp}(2) \,.\, \supset : R \in 1 \to \mathrm{Cls} : R\text{‘}y = R\text{‘}z \,.\, \supset_{y,z} \,.\, y = z :$

$[*71{\cdot}54] \qquad \supset : R \in 1 \to 1 \qquad (3)$

$\vdash \,.\, (1) \,.\, (3) \,.\, *71{\cdot}56 \,.\, \supset \vdash \,.\, \mathrm{Prop}$

***71·571.** $\vdash :. y \in \beta \,.\, \supset_y \,.\, \mathrm{E} \,!\, R\text{‘}y :\equiv \,.\, R \upharpoonright \beta \in 1 \to \mathrm{Cls} \,.\, \beta \subset \text{ᗡ}\text{‘}R$

Dem.

$\vdash \,.\, *71{\cdot}16 \,.\, \supset \vdash :. R \upharpoonright \beta \in 1 \to \mathrm{Cls} \,.\, \equiv : y \in \text{ᗡ}\text{‘}(R \upharpoonright \beta) \,.\, \supset_y \,.\, \mathrm{E} \,!\, (R \upharpoonright \beta)\text{‘}y :$

$[*35{\cdot}64{\cdot}7] \qquad \equiv : y \in \beta \cap \text{ᗡ}\text{‘}R \,.\, \supset_y \,.\, y \in \beta \,.\, \mathrm{E} \,!\, R\text{‘}y :$

$[*22{\cdot}33 . *5{\cdot}3] \qquad \equiv : y \in \beta \cap \text{ᗡ}\text{‘}R \,.\, \supset_y \,.\, \mathrm{E} \,!\, R\text{‘}y \qquad (1)$

$\vdash . (1) . *22 \cdot 621 . \supset$

$\vdash :. R \upharpoonright \beta \in 1 \to \mathrm{Cls} . \beta \subset \text{Ɑ}`R . \equiv : y \in \beta \cap \text{Ɑ}`R . \supset_y . \mathrm{E} ! R`y : \beta \cap \text{Ɑ}`R = \beta :$

$[*13 \cdot 193] \qquad \equiv : y \in \beta . \supset_y . \mathrm{E} ! R`y : \beta \cap \text{Ɑ}`R = \beta \qquad (2)$

$\vdash . *33 \cdot 43 . \supset \vdash :. y \in \beta . \supset_y . \mathrm{E} ! R`y : \supset . \beta \subset \text{Ɑ}`R .$

$[*22 \cdot 621] \qquad \supset . \beta \cap \text{Ɑ}`R = \beta \qquad (3)$

$\vdash . (2) . (3) . *4 \cdot 71 . \supset \vdash . \mathrm{Prop}$

***71·572.** $\vdash :. y \in \beta \cap \text{Ɑ}`R . \supset_y . \mathrm{E} ! R`y : \equiv . R \upharpoonright \beta \in 1 \to \mathrm{Cls}$

$[*71 \cdot 571 . *35 \cdot 351 . *22 \cdot 43]$

***71·58.** $\vdash :: y, z \in \beta . \supset_{y,z} : R`y = R`z . \equiv . y = z :. \supset . R \upharpoonright \beta \in 1 \to 1 . \beta \subset \text{Ɑ}`R$

Dem.

$\vdash . *10 \cdot 1 . \supset \vdash :: \mathrm{Hp} . \supset :. y \in \beta . \supset_y : R`y = R`y . \equiv . y = y :$

$[*13 \cdot 15 . *14 \cdot 28] \qquad \supset_y : \mathrm{E} ! R`y :.$

$[*71 \cdot 571] \qquad \supset :. R \upharpoonright \beta \in 1 \to \mathrm{Cls} . \beta \subset \text{Ɑ}`R \qquad (1)$

$\vdash . *3 \cdot 26 . \mathrm{Imp} . *11 \cdot 11 \cdot 32 . \supset$

$\vdash :. \mathrm{Hp} . \supset : y, z \in \beta . R`y = R`z . \supset_{y,z} . y = z :$

$[*35 \cdot 7] \qquad \supset : (R \upharpoonright \beta)`y = (R \upharpoonright \beta)`z . \supset_{y,z} . y = z :$

$[*71 \cdot 54 . (1)] \qquad \supset : R \upharpoonright \beta \in 1 \to 1 \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***71·59.** $\vdash :: y, z \in \beta . \supset_{y,z} : R`y = R`z . \equiv . y = z :. \equiv . R \upharpoonright \beta \in 1 \to 1 . \beta \subset \text{Ɑ}`R$

Dem.

$\vdash . *71 \cdot 56 . \supset \vdash :: R \upharpoonright \beta \in 1 \to 1 . \supset :. y \in \text{Ɑ}`(R \upharpoonright \beta) . \supset : (R \upharpoonright \beta)`y = (R \upharpoonright \beta)`z . \equiv . y = z :.$

$[*35 \cdot 64 \cdot 7] \qquad \supset :. y \in \beta \cap \text{Ɑ}`R . \supset : y, z \in \beta . R`y = R`z . \equiv . y = z \qquad (1)$

$\vdash . (1) . *22 \cdot 621 . \supset \vdash :: R \upharpoonright \beta \in 1 \to 1 . \beta \subset \text{Ɑ}`R . \supset :.$

$y \in \beta . \supset : y, z \in \beta . R`y = R`z . \equiv . y = z :.$

$[*4 \cdot 73] \qquad \supset :. y, z \in \beta . \supset : R`y = R`z . \equiv . y = z \qquad (2)$

$\vdash . (2) . *11 \cdot 11 \cdot 3 . \supset \vdash :: R \upharpoonright \beta \in 1 \to 1 . \beta \subset \text{Ɑ}`R . \supset :.$

$y, z \in \beta . \supset_{y,z} : R`y = R`z . \equiv . y = z \qquad (3)$

$\vdash . (3) . *71 \cdot 58 . \supset \vdash . \mathrm{Prop}$

The following proposition is used in the theory of selections (*80·91).

***71·6.** $\vdash : R \in 1 \to \mathrm{Cls} . \supset . R = \dot{s}`\hat{P} \{(\exists y) . y \in \text{Ɑ}`R . P = (R`y) \downarrow y\}$

Dem.

$\vdash . *41 \cdot 11 . *13 \cdot 195 . \supset$

$\vdash : x [\dot{s}`\hat{P} \{(\exists y) . y \in \text{Ɑ}`R . P = (R`y) \downarrow y\}] z . \equiv .$

$(\exists y) . y \in \text{Ɑ}`R . x \{(R`y) \downarrow y\} z .$

$[*55 \cdot 13] \equiv . (\exists y) . y \in \text{Ɑ}`R . x = R`y . z = y .$

$[*13 \cdot 195] \equiv . z \in \text{Ɑ}`R . x = R`z \qquad (1)$

$\vdash . *71 \cdot 36 . *33 \cdot 43 . \supset \vdash :. \mathrm{Hp} . \supset : z \in \text{Ɑ}`R . x = R`z . \equiv . xRz \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***71·61.** $\vdash : T \epsilon 1 \rightarrow \mathrm{Cls} \,.\supset .\, Q\text{‘‘‘}\overrightarrow{T}\text{‘‘}(\text{Ꮖ}\text{‘}T \cap \alpha) = \overrightarrow{Q}\text{‘‘}T\text{‘‘}\alpha$

Dem.

$\vdash .\, *37\cdot103\cdot67\cdot111 \,.\, *32\cdot12 \,.\supset$

$\vdash : \beta \,\epsilon\, Q\text{‘‘‘}\overrightarrow{T}\text{‘‘}(\text{Ꮖ}\text{‘}T \cap \alpha) \,.\equiv .\, (\exists x) \,.\, x \,\epsilon\, \text{Ꮖ}\text{‘}T \cap \alpha \,.\, \beta = Q\text{‘‘}\overrightarrow{T}\text{‘}x$ (1)

$\vdash .\, *53\cdot31 \,.\, *71\cdot16 \,.\supset \vdash : \mathrm{Hp} \,.\, x \,\epsilon\, \text{Ꮖ}\text{‘}T \cap \alpha \,.\supset .\, Q\text{‘‘}\overrightarrow{T}\text{‘}x = \overrightarrow{Q}\text{‘}T\text{‘}x$ (2)

$\vdash .\, (1) \,.\, (2) \,.\supset \vdash :.\, \mathrm{Hp} \,.\supset : \beta \,\epsilon\, Q\text{‘‘‘}\overrightarrow{T}\text{‘‘}(\text{Ꮖ}\text{‘}T \cap \alpha) \,.\equiv .\, (\exists x) \,.\, x \,\epsilon\, \text{Ꮖ}\text{‘}T \cap \alpha \,.\, \beta = \overrightarrow{Q}\text{‘}T\text{‘}x \,.$

[*37·67.*71·16] $\equiv .\, \beta \,\epsilon\, \overrightarrow{Q}\text{‘‘}T\text{‘‘}(\text{Ꮖ}\text{‘}T \cap \alpha) \,.$

[*37·26] $\equiv .\, \beta \,\epsilon\, \overrightarrow{Q}\text{‘‘}T\text{‘‘}\alpha :. \supset \vdash .\, \mathrm{Prop}$

***71·611.** $\vdash : T \epsilon \mathrm{Cls} \rightarrow 1 \,.\supset .\, Q\text{‘‘‘}\overleftarrow{T}\text{‘‘}(\mathrm{D}\text{‘}T \cap \alpha) = \overrightarrow{Q}\text{‘‘}\breve{T}\text{‘‘}\alpha$

***71·612.** $\vdash : T \epsilon 1 \rightarrow \mathrm{Cls} \,.\supset .\, \breve{Q}\text{‘‘‘}\overrightarrow{T}\text{‘‘}(\text{Ꮖ}\text{‘}T \cap \alpha) = \overleftarrow{Q}\text{‘‘}T\text{‘‘}\alpha$

***71·613.** $\vdash : T \epsilon \mathrm{Cls} \rightarrow 1 \,.\supset .\, \breve{Q}\text{‘‘‘}\overleftarrow{T}\text{‘‘}(\mathrm{D}\text{‘}T \cap \alpha) = \overleftarrow{Q}\text{‘‘}\breve{T}\text{‘‘}\alpha$

*71·613 is used in the theory of series (*206·6), and in the theory of "similarity of position" (*272·131).

***71·7.** $\vdash :.\, Q \,\epsilon\, 1 \rightarrow \mathrm{Cls} \,.\supset : xP \,|\, Qz \,.\equiv .\, xP\,(Q\text{‘}z)$

Dem.

$\vdash .\, *71\cdot36 \,.\supset \vdash :.\, \mathrm{Hp} \,.\supset : yQz \,.\equiv .\, y = Q\text{‘}z :$

[Fact] $\supset : xPy \,.\, yQz \,.\equiv .\, xPy \,.\, y = Q\text{‘}z :$

[*10·281] $\supset : (\exists y) \,.\, xPy \,.\, yQz \,.\equiv .\, (\exists y) \,.\, xPy \,.\, y = Q\text{‘}z :$

[*34·1.*13·195] $\supset : xP \,|\, Qz \,.\equiv .\, xP\,(Q\text{‘}z) :. \supset \vdash .\, \mathrm{Prop}$

***71·701.** $\vdash :.\, Q \,\epsilon\, \mathrm{Cls} \rightarrow 1 \,.\supset : xQ \,|\, Pz \,.\equiv .\, (\breve{Q}\text{‘}x)\, Pz$

$*72$. MISCELLANEOUS PROPOSITIONS CONCERNING ONE-MANY, MANY-ONE, AND ONE-ONE RELATIONS

Summary of $*72$.

In this number we shall prove various propositions involving $1 \rightarrow \mathrm{Cls}$, $\mathrm{Cls} \rightarrow 1$, or $1 \rightarrow 1$, but not embodying fundamental properties of these classes of relations.

The present number begins with various propositions ($*72{\cdot}1$—$\cdot191$) showing that various special relations are one-many or one-one. The most useful of these are

$*72{\cdot}182$. $\vdash . x \downarrow y \in 1 \rightarrow 1$

$*72{\cdot}184$. $\vdash . x \downarrow , \downarrow x \in 1 \rightarrow 1$

We have next a set of propositions concerning $R`S`z$ when R and S are one-many, or $R`\breve{R}`z$ when R is one-one, and kindred matters. The most useful of these is

$*72{\cdot}241$. $\vdash :. R \in 1 \rightarrow 1 . \supset : y \in \text{Ɑ}`R . \equiv . y = \breve{R}`R`y$

We have next a set of propositions ($*72{\cdot}3$—$\cdot341$) concerning products and sums of classes of relations; of these the one most used is

$*72{\cdot}32$. $\vdash :. \lambda \subset 1 \rightarrow \mathrm{Cls} : P, Q \in \lambda . \exists ! \text{Ɑ}`P \cap \text{Ɑ}`Q . \supset_{P,Q} . P = Q : \supset . \dot{s}`\lambda \in 1 \rightarrow \mathrm{Cls}$

which is an extension of $*71{\cdot}24$.

We have next a set of propositions ($*72{\cdot}4$—$\cdot481$) giving various relations of $\breve{R}``\alpha$ and $\breve{R}``\beta$ when $R \in 1 \rightarrow \mathrm{Cls}$, or of $R``\alpha$ and $R``\beta$ when $R \in \mathrm{Cls} \rightarrow 1$. The more useful propositions of this set are those that have the hypothesis $R \in \mathrm{Cls} \rightarrow 1$; these are occasionally useful in arithmetic. We have

$*72{\cdot}401$. $\vdash : R \in \mathrm{Cls} \rightarrow 1 . \supset . R``\alpha \cap R``\beta = R``(\alpha \cap \beta)$

$*72{\cdot}411$. $\vdash : R \in \mathrm{Cls} \rightarrow 1 . \alpha \cap \beta = \Lambda . \supset . R``\alpha \cap R``\beta = \Lambda$

For example, the relation of son to father is many-one. Let $\alpha =$ Cabinet Ministers, $\beta =$ fools; then assuming $\alpha \cap \beta = \Lambda$, it will follow that the sons of Cabinet Ministers and the sons of (male) fools have no common member. If we make R the relation of son to parent (which is not many-one), it no longer follows that the sons of Cabinet Ministers and the sons of fools have no common member.

We have

$*72{\cdot}451$. $\vdash : R \in \mathrm{Cls} \rightarrow 1 . \supset . R_\epsilon \upharpoonright \mathrm{Cl}`\text{Ɑ}`R \in 1 \rightarrow 1$

The effect of this proposition is that if α and β are both contained in $\text{Ɑ}`R$, and $R``\alpha = R``\beta$, then $\alpha = \beta$ (using $R_\epsilon`\alpha = R``\alpha$).

We next have a set of propositions concerned with the relations of R_ϵ and $(\breve{R})_\epsilon$, or, what comes to the same thing, with the circumstances under which $\alpha = R``\beta . \equiv . \beta = \breve{R}``\alpha$ and under which $R``\breve{R}``\alpha = \alpha$. We have

$*72{\cdot}502.\ \vdash : R \epsilon 1 \to \mathrm{Cls} . \alpha \subset \mathrm{D}`R . \supset . R``\breve{R}``\alpha = \alpha$

Thus for example the fathers of the children of wise fathers are the class of wise fathers; but the fathers of the children of wise parents are not all wise, and the parents of the children of wise parents are not all wise—the first because "$\alpha \subset \mathrm{D}`R$" fails, the second because "$R \epsilon 1 \to \mathrm{Cls}$" fails.

We have also

$*72{\cdot}52.\ \vdash :. R \epsilon 1 \to 1 . \alpha \subset \mathrm{D}`R . \beta \subset \text{Ɑ}`R . \supset : \alpha = R``\beta . \equiv . \beta = \breve{R}``\alpha$

We have next a set of propositions ($*72{\cdot}59$—$\cdot66$) in which the relative product $R \,|\, \breve{R}$ occurs if $R \epsilon 1 \to \mathrm{Cls}$, or $\breve{R} \,|\, R$ if $R \epsilon \mathrm{Cls} \to 1$. The most useful propositions in this set are

$*72{\cdot}591.\ \vdash : R \epsilon \mathrm{Cls} \to 1 . \supset . S \,|\, \breve{R} \,|\, R = S \upharpoonright \text{Ɑ}`R$

$*72{\cdot}601.\ \vdash : R \epsilon \mathrm{Cls} \to 1 . \text{Ɑ}`S \subset \text{Ɑ}`R . \supset . S \,|\, \breve{R} \,|\, R = S$

$*72{\cdot}66.\ \vdash : S^2 \mathrel{\text{Ⓒ}} S . S = \breve{S} . \equiv . (\exists R) . R \epsilon \mathrm{Cls} \to 1 . S = R \,|\, \breve{R}$

This is the "principle of abstraction." It shows that every relation which has the formal properties of equality, *i.e.* which is transitive and symmetrical, is equal to the relative product of a many-one relation into its converse; *i.e.* whenever the relation S holds between x and y, there is a term α such that $xR\alpha . yR\alpha$, where R is a many-one relation; and $*72{\cdot}64$ shows that this term α may be taken to be $\overleftarrow{S}`x$, which is equal to $\overleftarrow{S}`y$. This principle embodies a great part of the reasons for our definitions of the various kinds of numbers; in seeking these definitions, we always have, to begin with, some transitive symmetrical relation which we regard as sameness of number; thus by $*72{\cdot}64$, the desired properties of the numbers of the kind in question are secured by taking the number of an object to be the class of objects to which the said object has the transitive symmetrical relation in question. It is in this way that we are led to define cardinal numbers as classes of classes, and ordinal numbers as classes of relations.

The remaining propositions of this number are of less importance, with the exception of

$*72{\cdot}92.\ \vdash : R \epsilon 1 \to \mathrm{Cls} . S \mathrel{\text{Ⓒ}} R . \supset . S = R \upharpoonright \text{Ɑ}`S$

This proposition shows that every relation contained in a one-many relation is obtainable by a limitation of the converse domain. Thus *e.g.* every relation contained in that of father to son can be specified by specifying the class of sons who are to be its converse domain; for then all the fathers of these sons must be included to provide referents. But if we take the relation

of parent and child, which is not one-many or many-one, a contained relation is not determinate even when both its domain and its converse domain are given; for the relation may relate some of the children in any one family to the father and some to the mother, and so long as all the children and both parents are each related to some one by the relation, the domain and converse domain remain unchanged by permutations within the family.

***72·1.** $\vdash . \dot\Lambda \epsilon 1 \to 1$

Dem.

$\vdash . *25\cdot105 . \supset \vdash . \sim(x \dot\Lambda z . y \dot\Lambda z) .$

$[*2\cdot21] \supset \vdash : x\dot\Lambda z . y\dot\Lambda z . \supset . x = y :$

$[*11\cdot11 . *71\cdot17] \supset \vdash . \dot\Lambda \epsilon 1 \to \mathrm{Cls}$ (1)

Similarly $\vdash . \dot\Lambda \epsilon \mathrm{Cls} \to 1$ (2)

$\vdash . (1) . (2) . *71\cdot103 . \supset \vdash .$ Prop

***72·11.** $\vdash . \mathrm{Cnv} \epsilon 1 \to 1$

Dem.

$\vdash . *31\cdot13 . *71\cdot166 . \supset \vdash . \mathrm{Cnv} \epsilon 1 \to \mathrm{Cls}$ (1)

$\vdash . (1) . *71\cdot54 . *31\cdot32\cdot12 . \supset \vdash .$ Prop

***72·12.** $\vdash . \overrightarrow{R}, \overleftarrow{R} \epsilon 1 \to \mathrm{Cls}$ [*32·12·121 . *71·166]

***72·121.** $\vdash . \mathrm{sg}, \mathrm{gs} \epsilon 1 \to 1$

Dem.

$\vdash . *32\cdot22\cdot221 . *71\cdot166 . \supset \vdash . \mathrm{sg}, \mathrm{gs} \epsilon 1 \to \mathrm{Cls}$ (1)

$\vdash . (1) . *32\cdot14\cdot15\cdot21\cdot211 . *71\cdot54 . \supset \vdash .$ Prop

***72·13.** $\vdash . \mathrm{D} \epsilon 1 \to \mathrm{Cls}$ [*33·12 . *71·166]

***72·131.** $\vdash .$ ᗡ $\epsilon 1 \to \mathrm{Cls}$ [*33·121 . *71·166]

***72·132.** $\vdash : C \epsilon 1 \to \mathrm{Cls}$ [*33·122 . *71·166]

***72·14.** $\vdash . x$♀, ♀$x \epsilon 1 \to \mathrm{Cl}$ [*38·12 . *71·166]

This proposition applies to a great many of the relations we have to deal with, for example ↿P, P↾, P⊏, $P|$, $|P$, $x\downarrow$, $\downarrow x$, etc.

***72·15.** $\vdash . P_\epsilon \epsilon 1 \to \mathrm{Cls}$ [*37·111 . *71·166]

In *72·16 below, p has the meaning defined in *40·01, and does not represent a variable proposition. Similarly s in *72·161 has the meaning defined in *40·02.

***72·16.** $\vdash . p \epsilon 1 \to \mathrm{Cls}$

Dem.

$\vdash . *20\cdot2 . (*40\cdot01) . \supset \vdash . p\text{‘}\kappa = \hat{x}(\alpha \epsilon \kappa . \supset_\alpha . x \epsilon \alpha) .$

$[*14\cdot21] \supset \vdash . \mathrm{E} ! \, p\text{‘}\kappa$ (1)

$\vdash . (1) . *71\cdot166 . \supset \vdash .$ Prop

***72·161.** $\vdash . s \epsilon 1 \rightarrow \mathrm{Cls}$ [Proof as in *72·16]

***72·162.** $\vdash . \dot{p} \epsilon 1 \rightarrow \mathrm{Cls}$ [Proof as in *72·16]

***72·163.** $\vdash . \dot{s} \epsilon 1 \rightarrow \mathrm{Cls}$ [Proof as in *72·16]

***72·17.** $\vdash . I \epsilon 1 \rightarrow 1$

Dem.

$$\begin{array}{ll} \vdash . *52\cdot22 . (*51\cdot01) . \supset \vdash . (x) . \overrightarrow{I}`x \epsilon 1 . & \\ [*71\cdot12] \qquad \supset \vdash . I \epsilon 1 \rightarrow \mathrm{Cls} & (1) \\ \vdash . (1) . *71\cdot21 . *50\cdot2 . \supset \vdash . I \epsilon \mathrm{Cls} \rightarrow 1 & (2) \\ \vdash . (1) . (2) . \supset \vdash . \mathrm{Prop} & \end{array}$$

***72·18.** $\vdash . \iota \epsilon 1 \rightarrow 1$ [*51·23 . *71·57]

***72·181.** $\vdash . \breve{\iota} \epsilon 1 \rightarrow 1$ [*72·18 . *71·212]

***72·182.** $\vdash . x \downarrow y \epsilon 1 \rightarrow 1$

Dem.

$$\begin{array}{ll} \vdash . *55\cdot13 . \quad \supset \vdash : z (x \downarrow y) w . \equiv . z = x . w = y : & (1) \\ [*3\cdot47] \quad \supset \vdash : z (x \downarrow y) w . z' (x \downarrow y) w . \supset . z = x . z' = x . & \\ [*13\cdot172] \qquad \supset . z = z' & (2) \\ \vdash . (1) . *3\cdot47 . \supset \vdash : z (x \downarrow y) w . z (x \downarrow y) w' . \supset . w = y . w' = y . & \\ [*13\cdot172] \qquad \supset . w = w' & (3) \\ \vdash . (2) . (3) . *71\cdot172 . \supset \vdash . \mathrm{Prop} & \end{array}$$

***72·184.** $\vdash . x \downarrow, \downarrow x \epsilon 1 \rightarrow 1$ [*55·2 . *71·57]

***72·185.** $\vdash . (\downarrow x)_\epsilon \epsilon 1 \rightarrow 1$ [*55·262 . *37·11 . *72·15 . *71·54]

***72·19.** $\vdash . \mathrm{Cl} \epsilon 1 \rightarrow 1$ [*60·55 . *71·57]

***72·191.** $\vdash . \mathrm{Rl} \epsilon 1 \rightarrow 1$ [*61·55 . *71·57]

***72·192.** $\vdash . \mathrm{Cl\,ex} \epsilon 1 \rightarrow 1$ [*60·56 . *71·57]

***72·193.** $\vdash . \mathrm{Rl\,ex} \epsilon 1 \rightarrow 1$ [*61·56 . *71·57]

***72·2.** $\vdash :. R, S \epsilon 1 \rightarrow \mathrm{Cls} . \supset : x = R`S`z . \equiv . x (R \mid S) z . \equiv . x = (R \mid S)`z$

Dem.

$$\begin{array}{ll} \vdash . *71\cdot36 . \quad \supset \vdash :. \mathrm{Hp} . \supset : x = R`S`z . \equiv . xR (S`z) . & \\ [*71\cdot7] \qquad \equiv . x (R \mid S) z & (1) \\ \vdash . *71\cdot36\cdot25 . \supset \vdash :. \mathrm{Hp} . \supset : x (R \mid S) z . \equiv . x = (R \mid S)`z & (2) \\ \vdash . (1) . (2) . \supset \vdash . \mathrm{Prop} & \end{array}$$

***72·201.** $\vdash :. R, S \epsilon \mathrm{Cls} \rightarrow 1 . \supset : z = \breve{S}`\breve{R}`x . \equiv . x (R \mid S) z . \equiv . z = (\breve{S} \mid \breve{R})`x$

***72·202.** $\vdash :. R, S \epsilon 1 \rightarrow 1 . \supset : x = R`S`z . \equiv . x (R|S) z . \equiv . z = \breve{S}`\breve{R}`x$ [*72·2·201]

***72·21.** $\vdash :. R, S \epsilon 1 \rightarrow \mathrm{Cls} . \supset : z \epsilon \breve{S}``\mathrm{G}`R . \equiv . \mathrm{E} ! R`S`z . \equiv . \mathrm{E} ! (R \mid S)`z$

Dem.

$$\begin{array}{ll} \vdash . *71\cdot25\cdot163 . \supset \vdash :. \mathrm{Hp} . \supset : z \epsilon \mathrm{G}`(R \mid S) . \equiv . \mathrm{E} ! (R \mid S)`z & (1) \\ \vdash . (1) . *37\cdot32 . \supset \vdash :. \mathrm{Hp} . \supset : z \epsilon \breve{S}``\mathrm{G}`R . \equiv . \mathrm{E} ! (R \mid S)`z & (2) \end{array}$$

$\vdash . *72{\cdot}2 . *10{\cdot}11{\cdot}21{\cdot}281 . \supset$

$\vdash :. \mathrm{Hp} . \supset : (\exists x) . x = R‘S‘z . \equiv . (\exists x) . x = (R \mid S)‘z :$

[*14·204] $\supset : \mathrm{E} ! \, R‘S‘z . \quad \equiv . \mathrm{E} ! \, (R \mid S)‘z$ (3)

$\vdash . (2) . (3) . \supset \vdash . \mathrm{Prop}$

***72·211.** $\vdash :. R, S \in \mathrm{Cls} \rightarrow 1 . \supset : x \in R‘‘\mathrm{D}‘S . \equiv . \mathrm{E} ! \, \breve{S}‘\breve{R}‘x . \equiv . \mathrm{E} ! \, (\breve{S} \mid \breve{R})‘x$

***72·22.** $\vdash : R, S \in 1 \rightarrow \mathrm{Cls} . z \in \breve{S}‘‘\mathrm{Cl}‘R . \supset . R‘S‘z = (R \mid S)‘z$

Dem.

$\vdash . *72{\cdot}21 . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{E} ! \, R‘S‘z .$

[*34·41] $\supset . R‘S‘z = (R \mid S)‘z : \supset \vdash . \mathrm{Prop}$

***72·221.** $\vdash : R, S \in \mathrm{Cls} \rightarrow 1 . x \in R‘‘\mathrm{D}‘S . \supset . \breve{S}‘\breve{R}‘x = (\breve{S} \mid \breve{R})‘x$

***72·23.** $\vdash : R, S \in 1 \rightarrow \mathrm{Cls} . \supset . R‘‘S‘‘\gamma = \hat{x} \{(\exists z) . z \in \gamma . x = R‘S‘\gamma\}$

Dem.

$\vdash . *37{\cdot}33 . \quad \supset \vdash . R‘‘S‘‘\gamma = (R \mid S)‘‘\gamma$ (1)

$\vdash . *71{\cdot}25{\cdot}4 . \supset \vdash : \mathrm{Hp} . \supset . (R \mid S)‘‘\gamma = \hat{x} \{(\exists z) . z \in \gamma . x = (R \mid S)‘\gamma\}$

[*72·2] $= \hat{x} \{(\exists z) . z \in \gamma . x = R‘S‘\gamma\}$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***72·24.** $\vdash :. R \in 1 \rightarrow 1 . \supset : x \in \mathrm{D}‘R . \equiv . x = R‘\breve{R}‘x$

Dem.

$\vdash . *72{\cdot}202 . *71{\cdot}212 . \supset \vdash :. \mathrm{Hp} . \supset : x = R‘\breve{R}‘x . \equiv . x (R \mid \breve{R}) x .$

[*71·192] $\equiv . x (I \upharpoonright \mathrm{D}‘R) x .$

[*35·101.*50·1] $\equiv . x = x . x \in \mathrm{D}‘R .$

[*13·15.*4·73] $\equiv . x \in \mathrm{D}‘R :. \supset \vdash . \mathrm{Prop}$

***72·241.** $\vdash :. R \in 1 \rightarrow 1 . \supset : y \in \mathrm{Cl}‘R . \equiv . y = \breve{R}‘R‘y$

***72·242.** $\vdash :. R \in 1 \rightarrow 1 . \supset : \phi (R‘\breve{R}‘z) . \equiv . z \in \mathrm{D}‘R . \phi z : \phi (\breve{R}‘R‘z) . \equiv . z \in \mathrm{Cl}‘R . \phi z$

Dem.

$\vdash . *30{\cdot}501{\cdot}51 . \quad \supset \vdash : \phi (R‘\breve{R}‘z) . \equiv . (\exists x) . x = R‘\breve{R}‘z . \phi x$ (1)

$\vdash . (1) . *72{\cdot}2 . \quad \supset \vdash :. \mathrm{Hp} . \supset : \phi (R‘\breve{R}‘z) . \equiv . (\exists x) . x (R \mid \breve{R}) z . \phi x .$

[*71·192] $\equiv . (\exists x) . x = z . z \in \mathrm{D}‘R . \phi x .$

[*13·195] $\equiv . z \in \mathrm{D}‘R . \phi z$ (2)

$\vdash . (2) \dfrac{\breve{R}}{R} . *71{\cdot}212 . \supset \vdash :. \mathrm{Hp} . \supset : \phi (\breve{R}‘R‘z) . \equiv . z \in \mathrm{Cl}‘R . \phi z$ (3)

$\vdash . (2) . (3) . \supset \vdash . \mathrm{Prop}$

***72·243.** $\vdash :: R \in 1 \rightarrow 1 . \supset :. z \in \mathrm{D}‘R . \phi z . \equiv_z . \psi (\breve{R}‘z) : \equiv : \phi (R‘w) . \equiv_w . w \in \mathrm{Cl}‘R . \psi w$

Dem.

$\vdash . *72{\cdot}242 . \supset \vdash :: \mathrm{Hp} . \supset :. z \in \mathrm{D}‘R . \phi z . \equiv_z . \psi (\breve{R}‘z) : \supset :$

$\phi (R‘\breve{R}‘z) . \equiv_z . \psi (\breve{R}‘z) :$

[Fact]	$\supset : \phi(R`\breve{R}`z) . w = \breve{R}`z . \equiv_{z,w} . \psi(\breve{R}`z) . w = \breve{R}`z :$
[*14·15]	$\supset : \phi(R`w) . w = \breve{R}`z . \equiv_{z,w} . \psi w . w = \breve{R}`z :$
[*10·281]	$\supset : (\exists z) . \phi(R`w) . w = \breve{R}`z . \equiv_w . (\exists z) . \psi w . w = \breve{R}`z :$
[*71·411]	$\supset : \phi(R`w) . w \epsilon \text{G}`R . \equiv_w . \psi w . w \epsilon \text{G}`R :$
[*14·21.*71·163]	$\supset : \phi(R`w) . \equiv_w . \psi w . w \epsilon \text{G}`R$ (1)

$\vdash . (1) \frac{\breve{R}}{R} . \supset \vdash :: \text{Hp} . \supset :. w \epsilon \text{G}`R . \psi w . \equiv_w . \psi(R`w) : \supset : \psi(\breve{R}`z) . \equiv_z . \phi z . z \epsilon \text{D}`R$ (2)

$\vdash . (1) . (2) . \supset \vdash .$ Prop

The above proposition is used in *272·4·41, which are used in the theory of "rational series," *i.e.* series ordinally similar to the series of rationals.

***72·25.** $\vdash :. R \epsilon 1 \to 1 : (y) . \text{E} ! R`y : \supset . (y) . y = \breve{R}`R`y$

Dem.

$\vdash . *71·165 . \supset \vdash :. R \epsilon 1 \to 1 . \supset : (y) . \text{E} ! R`y . \equiv . (y) . y \epsilon \text{G}`R$ (1)

$\vdash . *72·241 . \supset \vdash :. R \epsilon 1 \to 1 . \supset : (y) . y \epsilon \text{G}`R . \equiv . (y) . y = \breve{R}`R`y$ (2)

$\vdash . (1) . (2) . \text{Imp} . \supset \vdash .$ Prop

The propositions $\text{Cnv}`\text{Cnv}`P = P$ and $\breve{\iota}`\iota`x = x$, which have been previously proved, are particular cases of the above; the former is a particular case because $\text{Cnv} = \text{Cnv}`\text{Cnv}$.

***72·26.** $\vdash : (y) . \text{E} ! R`y . \supset . R = \epsilon | \overrightarrow{R}$

In this proposition, the conditions of significance require that the domain of R should consist of classes. This proposition is used in *72·27.

Dem.

$\vdash . *37·31 . \supset \vdash . \epsilon | \overrightarrow{R} = \epsilon_\epsilon | \overrightarrow{R}$

$[*62·32] \quad = s | \overrightarrow{R}$ (1)

$\vdash . *53·31 . \supset \vdash : \text{Hp} . \supset . (y) . s`\overrightarrow{R}`y = s`\iota`R`y$

$[*53·02] \quad = R`y .$

$[*34·42] \quad \supset . s | \overrightarrow{R} = R$ (2)

$\vdash . (1) . (2) . \supset \vdash .$ Prop

***72·27.** $\vdash . \text{D} = \epsilon | \overrightarrow{\text{D}} . \text{G} = \epsilon | \overrightarrow{\text{G}}$ [*72·26 . *33·12·121]

*72·27 is used in *74·63·631 and again in *163·15.

***72·3.** $\vdash : \exists ! \lambda \cap (1 \to \text{Cls}) . \supset . \dot{p}`\lambda \epsilon 1 \to \text{Cls}$

Dem.

$\vdash . *41·12 . \text{Fact} . \supset \vdash : R \epsilon \lambda . R \epsilon 1 \to \text{Cls} . \supset . \dot{p}`\lambda \subset R . R \epsilon 1 \to \text{Cls} .$

$[*71·22] \quad \supset . \dot{p}`\lambda \epsilon 1 \to \text{Cls}$ (1)

$\vdash . (1) . *10·11·23 . \supset \vdash : (\exists R) . R \epsilon \lambda . R \epsilon 1 \to \text{Cls} . \supset . \dot{p}`\lambda \epsilon 1 \to \text{Cls}$ (2)

$\vdash . (2) . *22·33 . \supset \vdash .$ Prop

***72·301.** $\vdash : \exists ! \lambda \cap (\mathrm{Cls} \rightarrow 1) . \supset . \dot{p}`\lambda \epsilon \mathrm{Cls} \rightarrow 1$

***72·302.** $\vdash : \exists ! \lambda \cap (1 \rightarrow 1) . \supset . \dot{p}`\lambda \epsilon 1 \rightarrow 1$

***72·303.** $\vdash : \exists ! \lambda \cap (1 \rightarrow \mathrm{Cls}) . \exists ! \lambda \cap (\mathrm{Cls} \rightarrow 1) . \supset . \dot{p}`\lambda \epsilon 1 \rightarrow 1$ [*72·3·301]

***72·31.** $\vdash : \dot{s}`\lambda \epsilon 1 \rightarrow \mathrm{Cls} . \supset . \lambda \subset 1 \rightarrow \mathrm{Cls}$

Dem.

$\vdash . *41·13 . \supset \vdash : \dot{s}`\lambda \epsilon 1 \rightarrow \mathrm{Cls} . P \epsilon \lambda . \supset . \dot{s}`\lambda \epsilon 1 \rightarrow \mathrm{Cls} . P \mathrel{\dot{\subset}} \dot{s}`\lambda .$

[*71·22] $\supset . P \epsilon 1 \rightarrow \mathrm{Cls}$ (1)

$\vdash . (1) . \mathrm{Exp} . *10·11·21 . \supset \vdash . \mathrm{Prop}$

***72·311.** $\vdash : \dot{s}`\lambda \epsilon \mathrm{Cls} \rightarrow 1 . \supset . \lambda \subset \mathrm{Cls} \rightarrow 1$

***72·312.** $\vdash : \dot{s}`\lambda \epsilon 1 \rightarrow 1 . \supset . \lambda \subset 1 \rightarrow 1$

***72·32.** $\vdash :. \lambda \subset 1 \rightarrow \mathrm{Cls} : P, Q \epsilon \lambda . \exists ! \text{Ɑ}`P \cap \text{Ɑ}`Q . \supset_{P,Q} . P = Q : \supset . \dot{s}`\lambda \epsilon 1 \rightarrow \mathrm{Cls}$

Dem.

$\vdash . *41·11 . *11·54 . \supset \vdash : x (\dot{s}`\lambda) z . y (\dot{s}`\lambda) z . \equiv .$

$(\exists P, Q) . P, Q \epsilon \lambda . xPz . yQz .$

[*33·14.*4·71] $\equiv . (\exists P, Q) . P, Q \epsilon \lambda . xPz . yQz . z \epsilon \text{Ɑ}`P \cap \text{Ɑ}`Q$ (1)

$\vdash . (1) . *4·71 . \supset \vdash :. \mathrm{Hp} . \supset : x (\dot{s}`\lambda) z . y (\dot{s}`\lambda) z . \equiv .$

$(\exists P, Q) . P, Q \epsilon \lambda . xPz . yQz . z \epsilon \text{Ɑ}`P \cap \text{Ɑ}`Q . P = Q .$

[*13·195] $\supset . (\exists P) . P \epsilon \lambda . xPz . yPz .$

[*71·17.Hp] $\supset . x = y$ (2)

$\vdash . (2) . *11·11·3 . *71·17 . \supset \vdash . \mathrm{Prop}$

***72·321.** $\vdash :. \lambda \subset \mathrm{Cls} \rightarrow 1 : P, Q \epsilon \lambda . \exists ! \mathrm{D}`P \cap \mathrm{D}`Q . \supset_{P,Q} . P = Q : \supset . \dot{s}`\lambda \epsilon \mathrm{Cls} \rightarrow 1$
[Proof as in *72·32]

***72·322.** $\vdash :. \lambda \subset 1 \rightarrow 1 : P, Q \epsilon \lambda . \exists ! \text{Ɑ}`P \cap \text{Ɑ}`Q . \supset_{P,Q} . P = Q :$

$P, Q \epsilon \lambda . \exists ! \mathrm{D}`P \cap \mathrm{D}`Q . \supset_{P,Q} . P = Q : \supset . \dot{s}`\lambda \epsilon 1 \rightarrow 1$

[*72·32·321]

***72·323.** $\vdash :. \lambda \subset 1 \rightarrow 1 : P, Q \epsilon \lambda . \exists ! C`P \cap C`Q . \supset_{P,Q} . P = Q : \supset . \dot{s}`\lambda \epsilon 1 \rightarrow 1$

Dem.

$\vdash . *33·161 . *22·49 . \supset \vdash . \text{Ɑ}`P \cap \text{Ɑ}`Q \subset C`P \cap C`Q . \mathrm{D}`P \cap \mathrm{D}`Q \subset C`P \cap C`Q .$

[*24·58] $\supset \vdash : \exists ! \text{Ɑ}`P \cap \text{Ɑ}`Q . \supset . \exists ! C`P \cap C`Q :$

$\exists ! \mathrm{D}`P \cap \mathrm{D}`Q . \supset . \exists ! C`P \cap C`Q$ (1)

$\vdash . (1) . \mathrm{Syll} . \supset \vdash :. \mathrm{Hp} . \supset : P, Q \epsilon \lambda . \exists ! \text{Ɑ}`P \cap \text{Ɑ}`Q . \supset_{P,Q} . P = Q :$

$P, Q \epsilon \lambda . \exists ! \mathrm{D}`P \cap \mathrm{D}`Q . \supset_{P,Q} . P = Q$ (2)

$\vdash . (2) . *72·322 . \supset \vdash . \mathrm{Prop}$

***72·34.** $\vdash : R \epsilon 1 \rightarrow \mathrm{Cls} . \exists ! \kappa . \supset . p`\breve{R}'''\kappa = \breve{R}''p`\kappa$

Dem.

$\vdash . *40·35 . \supset \vdash :. y \epsilon p`\breve{R}'''\kappa . \equiv : \beta \epsilon \kappa . \supset_\beta . y \epsilon \breve{R}''\beta$ (1)

$\vdash . (1) . *71·37 . \supset \vdash :: \mathrm{Hp} . \supset :. y \epsilon p`\breve{R}'''\kappa . \equiv : \beta \epsilon \kappa . \supset_\beta . R`y \epsilon \beta$ (2)

$\vdash . *14\cdot21 . \quad \supset \vdash :. \beta \in \kappa . \supset . R'y \in \beta : \supset : \beta \in \kappa . \supset . E ! R'y :.$

$[*10\cdot52] \quad \supset \vdash :: \text{Hp} . \supset :. \beta \in \kappa . \supset_\beta . R'y \in \beta : \supset . E ! R'y \qquad (3)$

$\vdash . *14\cdot28 . *40\cdot1 . \quad \supset \vdash :: E ! R'y . \supset :. \beta \in \kappa . \supset_\beta . R'y \in \beta : \equiv . R'y \in p'\kappa ::$

$[(2).(3).*5\cdot32.*14\cdot21] \supset \vdash :. \text{Hp} . \supset : y \in p'\breve{R}'''\kappa . \equiv . R'y \in p'\kappa .$

$[*71\cdot37] \qquad \equiv . y \in \breve{R}''p'\kappa :. \supset \vdash . \text{Prop}$

$*72\cdot341$. $\vdash : R \in \text{Cls} \rightarrow 1 . \exists ! \kappa . \supset . p'R'''\kappa = R''p'\kappa$

This proposition should be compared with $*40\cdot37$ and $*40\cdot38$.

$*72\cdot4$. $\vdash : R \in 1 \rightarrow \text{Cls} . \supset . \breve{R}''\alpha \cap \breve{R}''\beta = \breve{R}''(\alpha \cap \beta)$

Dem.

$\vdash . *71\cdot37 . \supset \vdash :. \text{Hp} . \supset : y \in \breve{R}''\alpha \cap \breve{R}''\beta . \equiv . R'y \in \alpha . R'y \in \beta .$

$[*22\cdot33] \qquad \equiv . R'y \in \alpha \cap \beta .$

$[*71\cdot37] \qquad \equiv . y \in \breve{R}''(\alpha \cap \beta) :. \supset \vdash . \text{Prop}$

When R is not a $1 \rightarrow \text{Cls}$, we only have in general (cf. $*37\cdot21$)

$$\breve{R}''(\alpha \cap \beta) \subset \breve{R}''\alpha \cap \breve{R}''\beta .$$

$*72\cdot401$. $\vdash : R \in \text{Cls} \rightarrow 1 . \supset . R''\alpha \cap R''\beta = R''(\alpha \cap \beta)$

$*72\cdot41$. $\vdash : R \in 1 \rightarrow \text{Cls} . \alpha \cap \beta = \Lambda . \supset . \breve{R}''\alpha \cap \breve{R}''\beta = \Lambda \quad [*72\cdot4 . *37\cdot29]$

$*72\cdot411$. $\vdash : R \in \text{Cls} \rightarrow 1 . \alpha \cap \beta = \Lambda . \supset . R''\alpha \cap R''\beta = \Lambda$

$*72\cdot42$. $\vdash : R \in 1 \rightarrow \text{Cls} . \exists ! \breve{R}''\alpha \cap \breve{R}''\beta . \supset . \exists ! \alpha \cap \beta \quad [*72\cdot41 . \text{Transp}]$

$*72\cdot421$. $\vdash : R \in \text{Cls} \rightarrow 1 . \exists ! R''\alpha \cap R''\beta . \supset . \exists ! \alpha \cap \beta$

$*72\cdot43$. $\vdash : R \in 1 \rightarrow \text{Cls} . \breve{R}''\alpha = \breve{R}''\beta . \supset . \alpha \cap D'R = \beta \cap D'R$

Dem.

$\vdash . *71\cdot37 . \supset :. \text{Hp} . \supset : R'y \in \alpha . \equiv_y . R'y \in \beta :$

$[\text{Fact}] \qquad \supset : z = R'y . R'y \in \alpha . \equiv_y . z = R'y . R'y \in \beta :$

$[*14\cdot15] \qquad \supset : z = R'y . z \in \alpha . \equiv_y . z = R'y . z \in \beta :$

$[*10\cdot281] \qquad \supset : (\exists y) . z = R'y . z \in \alpha . \equiv . (\exists y) . z = R'y . z \in \beta :$

$[*71\cdot41.*10\cdot35] \qquad \supset : z \in D'R . z \in \alpha . \equiv . z \in D'R . z \in \beta :$

$[*22\cdot33] \qquad \supset : z \in D'R \cap \alpha . \equiv . z \in D'R \cap \beta :. \supset \vdash . \text{Prop}$

$*72\cdot431$. $\vdash : R \in \text{Cls} \rightarrow 1 . R''\alpha = R''\beta . \supset . \alpha \cap \text{Cl}'R = \beta \cap \text{Cl}'R$

$*72\cdot44$. $\vdash : R \in 1 \rightarrow \text{Cls} . \alpha \subset D'R . \beta \subset D'R . \breve{R}''\alpha = \breve{R}''\beta . \supset . \alpha = \beta$
$[*72\cdot43 . *22\cdot621]$

$*72\cdot441$. $\vdash : R \in \text{Cls} \rightarrow 1 . \alpha \subset \text{Cl}'R . \beta \subset \text{Cl}'R . R''\alpha = R''\beta . \supset . \alpha = \beta$

$*72\cdot441$ is used in the theory of cardinal exponentiation ($*116\cdot659$).

***72·45.** $\vdash : R \in 1 \to \mathrm{Cls} \,.\supset .\, (\breve{R})_\epsilon \restriction \mathrm{Cl}\text{‘}\mathrm{D}\text{‘}R \in 1 \to 1$

Dem.

$\vdash . *60{\cdot}2 . \quad \supset \vdash : \alpha \subset \mathrm{D}\text{‘}R \,.\, \beta \subset \mathrm{D}\text{‘}R \,.\equiv .\, \alpha, \beta \in \mathrm{Cl}\text{‘}\mathrm{D}\text{‘}R$ (1)

$\vdash . *37{\cdot}11 . \supset \vdash : \breve{R}\text{“}\alpha = \breve{R}\text{“}\beta \,.\equiv .\, (\breve{R})_\epsilon\text{‘}\alpha = (\breve{R})_\epsilon\text{‘}\beta$ (2)

$\vdash . (1) . (2) . *72{\cdot}44 . \supset$

$\vdash :. R \in 1 \to \mathrm{Cls} \,.\supset : \alpha, \beta \in \mathrm{Cl}\text{‘}\mathrm{D}\text{‘}R \,.\, (\breve{R})_\epsilon\text{‘}\alpha = (\breve{R})_\epsilon\text{‘}\beta \,.\supset_{\alpha,\beta} .\, \alpha = \beta :$

$[*71{\cdot}55 . *72{\cdot}15] \quad \supset : (\breve{R})_\epsilon \restriction \mathrm{Cl}\text{‘}\mathrm{D}\text{‘}R \in 1 \to 1 :. \supset \vdash . \mathrm{Prop}$

***72·451.** $\vdash : R \in \mathrm{Cls} \to 1 \,.\supset .\, R_\epsilon \restriction \mathrm{Cl}\text{‘}\text{ᗡ}\text{‘}R \in 1 \to 1$

***72·46.** $\vdash :. R \in 1 \to \mathrm{Cls} \,.\supset : \breve{R}\text{“}\alpha = \breve{R}\text{“}\beta \,.\equiv .\, \alpha \cap \mathrm{D}\text{‘}R = \beta \cap \mathrm{D}\text{‘}R$

$[*72{\cdot}43 . *37{\cdot}263]$

***72·461.** $\vdash :. R \in \mathrm{Cls} \to 1 \,.\supset : R\text{“}\alpha = R\text{“}\beta \,.\equiv .\, \alpha \cap \text{ᗡ}\text{‘}R = \beta \cap \text{ᗡ}\text{‘}R$

***72·47.** $\vdash :. R \in 1 \to \mathrm{Cls} \,.\supset : \breve{R}\text{“}\alpha = \text{ᗡ}\text{‘}R \,.\equiv .\, \mathrm{D}\text{‘}R \subset \alpha$

Dem.

$\vdash . *37{\cdot}25 . *72{\cdot}46 . \supset$

$\vdash :. \mathrm{Hp} \,.\supset : \breve{R}\text{“}\alpha = \text{ᗡ}\text{‘}R \,.\equiv .\, \alpha \cap \mathrm{D}\text{‘}R = \mathrm{D}\text{‘}R \cap \mathrm{D}\text{‘}R \,.$

$[*22{\cdot}5{\cdot}621] \qquad \equiv .\, \mathrm{D}\text{‘}R \subset \alpha : \supset \vdash . \mathrm{Prop}$

***72·471.** $\vdash :. R \in \mathrm{Cls} \to 1 \,.\supset : R\text{“}\alpha = \mathrm{D}\text{‘}R \,.\equiv .\, \text{ᗡ}\text{‘}R \subset \alpha$

***72·48.** $\vdash :. R \in 1 \to \mathrm{Cls} \,.\, \alpha, \beta \in \mathrm{Cl}\text{‘}\mathrm{D}\text{‘}R \,.\supset : \breve{R}\text{“}\alpha = \breve{R}\text{“}\beta \,.\equiv .\, \alpha = \beta$

Dem.

$\vdash . *22{\cdot}621 . \supset \vdash :. \mathrm{Hp} \,.\supset : \alpha = \beta \,.\equiv .\, \alpha \cap \mathrm{D}\text{‘}R = \beta \cap \mathrm{D}\text{‘}R \,.$

$[*72{\cdot}46] \qquad \equiv .\, \breve{R}\text{“}\alpha = \breve{R}\text{“}\beta :. \supset \vdash . \mathrm{Prop}$

***72·481.** $\vdash :. R \in \mathrm{Cls} \to 1 \,.\, \alpha, \beta \in \mathrm{Cl}\text{‘}\text{ᗡ}\text{‘}R \,.\supset : R\text{“}\alpha = R\text{“}\beta \,.\equiv .\, \alpha = \beta$

***72·49.** $\vdash :. Q \in 1 \to \mathrm{Cls} \,.\supset : \text{ᗡ}\text{‘}(P \mid Q) = \text{ᗡ}\text{‘}Q \,.\equiv .\, \mathrm{D}\text{‘}Q \subset \text{ᗡ}\text{‘}P$

Dem.

$\vdash . *72{\cdot}47 . \supset \vdash :. \mathrm{Hp} \,.\supset : \breve{Q}\text{“}\text{ᗡ}\text{‘}P = \text{ᗡ}\text{‘}Q \,.\equiv .\, \mathrm{D}\text{‘}Q \subset \text{ᗡ}\text{‘}P$ (1)

$\vdash . (1) . *37{\cdot}32 . \supset \vdash . \mathrm{Prop}$

***72·491.** $\vdash :. P \in \mathrm{Cls} \to 1 \,.\supset : \mathrm{D}\text{‘}(P \mid Q) = \mathrm{D}\text{‘}P \,.\equiv .\, \text{ᗡ}\text{‘}P \subset \mathrm{D}\text{‘}Q$

***72·492.** $\vdash :. P \in \mathrm{Cls} \to 1 \,.\, Q \in 1 \to \mathrm{Cls} \,.\supset :$

$\mathrm{D}\text{‘}(P \mid Q) = \mathrm{D}\text{‘}P \,.\, \text{ᗡ}\text{‘}(P \mid Q) = \text{ᗡ}\text{‘}Q \,.\equiv .\, \text{ᗡ}\text{‘}P = \mathrm{D}\text{‘}Q \quad [*72{\cdot}49{\cdot}491]$

***72·5.** $\vdash : R \in 1 \to \mathrm{Cls} \,.\supset .\, R\text{“}\breve{R}\text{“}\alpha = \alpha \cap \mathrm{D}\text{‘}R$

Dem.

$\vdash . *37{\cdot}33 . \qquad \supset \vdash . R\text{“}\breve{R}\text{“}\alpha = (R \mid \breve{R})\text{“}\alpha$ (1)

$\vdash . (1) . *71{\cdot}19 . \supset \vdash : \mathrm{Hp} \,.\supset .\, R\text{“}\breve{R}\text{“}\alpha = (I \restriction \mathrm{D}\text{‘}R)\text{“}\alpha$

$[*50{\cdot}59] \qquad = \alpha \cap \mathrm{D}\text{‘}R : \supset \vdash . \mathrm{Prop}$

$*72\cdot501.\ \vdash : R \epsilon \text{Cls} \to 1 \,.\supset .\, \breve{R}``R``\alpha = \alpha \cap \text{Ɑ}`R$

$*72\cdot502.\ \vdash : R \epsilon 1 \to \text{Cls} \,.\, \alpha \subset \text{D}`R \,.\supset .\, R``\breve{R}``\alpha = \alpha$ [$*72\cdot5 \,.\, *22\cdot621$]

$*72\cdot503.\ \vdash : R \epsilon \text{Cls} \to 1 \,.\, \alpha \subset \text{Ɑ}`R \,.\supset .\, \breve{R}``R``\alpha = \alpha$

$*72\cdot504.\ \vdash : \lambda \subset \text{D}`R_\epsilon \,.\supset .\, R_\epsilon``\breve{R}_\epsilon``\lambda = \lambda$ [$*72\cdot502\cdot15$]

Note that $\breve{R}_\epsilon$ means $\text{Cnv}`R_\epsilon$, not $(\breve{R})_\epsilon$. $*72\cdot504$ is used in the theory of segments of a series ($*211\cdot64$).

$*72\cdot51.\ \vdash : R \epsilon 1 \to \text{Cls} \,.\, \alpha \subset \text{D}`R \,.\, \beta = \breve{R}``\alpha \,.\supset .\, \alpha = R``\beta$ [$*72\cdot502 \,.\, *20\cdot18$]

$*72\cdot511.\ \vdash : R \epsilon \text{Cls} \to 1 \,.\, \beta \subset \text{Ɑ}`R \,.\, \alpha = R``\beta \,.\supset .\, \beta = \breve{R}``\alpha$ [$*72\cdot503 \,.\, *20\cdot18$]

$*72\cdot512.\ \vdash :. R \epsilon 1 \to 1 \,.\, \beta \subset \text{Ɑ}`R \,.\supset : y \epsilon \beta \,.\equiv .\, R`y \epsilon R``\beta$

Dem.

$\vdash . *71\cdot37 \,.\supset \vdash :. R \epsilon 1 \to \text{Cls} \,.\supset : y \epsilon \breve{R}``R``\beta \,.\equiv .\, R`y \epsilon R``\beta$ (1)

$\vdash . *72\cdot503 \,.\supset \vdash :. R \epsilon \text{Cls} \to 1 \,.\, \beta \subset \text{Ɑ}`R \,.\supset : y \epsilon \breve{R}``R``\beta \,.\equiv .\, y \epsilon \beta$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

$*72\cdot513.\ \vdash :. R \epsilon 1 \to 1 : (y) . \text{E}!R`y : \supset : y \epsilon \beta \,.\equiv .\, R`y \epsilon R``\beta$ [$*72\cdot512 . *33\cdot431$]

$*72\cdot52.\ \vdash :. R \epsilon 1 \to 1 . \alpha \subset \text{D}`R . \beta \subset \text{Ɑ}`R . \supset : \alpha = R``\beta . \equiv . \beta = \breve{R}``\alpha$ [$*72\cdot51\cdot511$]

$*72\cdot53.\ \vdash :. R \epsilon 1 \to 1 \,.\supset : \beta \subset \text{Ɑ}`R \,.\, \alpha = R``\beta \,.\equiv .\, \alpha \subset \text{D}`R \,.\, \beta = \breve{R}``\alpha$

Dem.

$\vdash . *72\cdot52 . *5\cdot32 . \supset$

$\vdash :. R \epsilon 1 \to 1 . \supset : \alpha \subset \text{D}`R . \beta \subset \text{Ɑ}`R . \alpha = R``\beta . \equiv . \alpha \subset \text{D}`R . \beta \subset \text{Ɑ}`R . \beta = \breve{R}``\alpha$ (1)

$\vdash . *37\cdot15 . \supset \vdash : \alpha = R``\beta . \supset . \alpha \subset \text{D}`R :$

[$*4\cdot71$] $\supset \vdash : \alpha \subset \text{D}`R . \beta \subset \text{Ɑ}`R . \alpha = R``\beta . \equiv . \beta \subset \text{Ɑ}`R . \alpha = R``\beta$ (2)

$\vdash . *37\cdot16 . \supset \vdash : \beta = \breve{R}``\alpha . \supset . \beta \subset \text{Ɑ}`R :$

[$*4\cdot71$] $\supset \vdash : \alpha \subset \text{D}`R . \beta \subset \text{Ɑ}`R . \beta = \breve{R}``\alpha . \equiv . \alpha \subset \text{D}`R . \beta = \breve{R}``\alpha$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash . \text{Prop}$

$*72\cdot54.\ \vdash : R \epsilon 1 \to 1 \,.\supset .\, \text{Cnv}`(R_\epsilon \upharpoonright \text{Cl}`\text{Ɑ}`R) = (\breve{R})_\epsilon \upharpoonright \text{Cl}`\text{D}`R$

Dem.

$\vdash . *31\cdot131 . \supset$

$\vdash : \beta \{\text{Cnv}`(R_\epsilon \upharpoonright \text{Cl}`\text{Ɑ}`R)\} \alpha . \equiv . \alpha (R_\epsilon \upharpoonright \text{Cl}`\text{Ɑ}`R) \beta .$

[$*37\cdot101 . *35\cdot101 . *60\cdot2$] $\equiv . \alpha = R``\beta . \beta \subset \text{Ɑ}`R$ (1)

$\vdash . *37\cdot102 . *35\cdot101 . *60\cdot2 . \supset$

$\vdash : \beta \{(\breve{R})_\epsilon \upharpoonright \text{Cl}`\text{D}`R\} \alpha . \equiv . \beta = \breve{R}``\alpha . \alpha \subset \text{D}`R$ (2)

$\vdash . (1) . (2) . *72\cdot53 . \supset \vdash . \text{Prop}$

$*72\cdot541.\ \vdash : R \epsilon 1 \to 1 \,.\, S = \breve{R} \,.\supset .\, \text{Cnv}`(R_\epsilon \upharpoonright \text{D}`S_\epsilon) = S_\epsilon \upharpoonright \text{D}`R_\epsilon$
[$*71\cdot48\cdot481 . *72\cdot54$]

***72·55.** $\vdash : R \in 1 \to \mathrm{Cls} . \supset . \alpha \rceil R = R \upharpoonright \breve{R}``\alpha = \alpha \rceil R \upharpoonright \breve{R}``\alpha$

Dem.

$\vdash . *35\cdot1 . *71\cdot36 . \supset \vdash :. \mathrm{Hp} . \supset : x (\alpha \rceil R) y . \equiv . x \in \alpha . x = R`y .$

$[*14\cdot15] \qquad \equiv . R`y \in \alpha . x = R`y .$

$[*71\cdot37] \qquad \equiv . y \in \breve{R}``\alpha . x = R`y .$

$[*71\cdot36 . *35\cdot101] \qquad \equiv . x (R \upharpoonright \breve{R}``\alpha) y \qquad (1)$

$\vdash . (1) . *35\cdot11 . \supset \vdash . \mathrm{Prop}$

***72·551.** $\vdash : R \in \mathrm{Cls} \to 1 . \supset . R \upharpoonright \beta = (R``\beta) \rceil R = (R``\beta) \rceil R \upharpoonright \beta$

***72·57.** $\vdash : Q \upharpoonright \lambda \in 1 \to \mathrm{Cls} . \lambda = \breve{Q}``\mu . \supset . \mu \cap \mathrm{D}`Q = Q``\lambda$

Dem.

$\vdash . *37\cdot42 . \qquad \supset \vdash : \lambda = \breve{Q}``\mu . \supset . (\lambda \rceil \breve{Q})``\mu = \breve{Q}``\mu \qquad (1)$

$\vdash . *37\cdot421 . \qquad \supset \vdash : \lambda = \breve{Q}``\mu . \supset . (Q \upharpoonright \lambda)``\breve{Q}``\mu = Q``\lambda \qquad (2)$

$\vdash . (1) . (2) . \qquad \supset \vdash : \lambda = \breve{Q}``\mu . \supset . (Q \upharpoonright \lambda)``(\lambda \rceil \breve{Q})``\mu = Q``\lambda \qquad (3)$

$\vdash . *72\cdot5 . *35\cdot52 . \supset \vdash : Q \upharpoonright \lambda \in 1 \to \mathrm{Cls} . \supset . (Q \upharpoonright \lambda)``(\lambda \rceil \breve{Q})``\mu = \mu \cap \mathrm{D}`Q \qquad (4)$

$\vdash . (3) . (4) . \supset \vdash . \mathrm{Prop}$

***72·59.** $\vdash : R \in 1 \to \mathrm{Cls} . \supset . S | R | \breve{R} = S \upharpoonright \mathrm{D}`R$

Dem.

$\vdash . *71\cdot19 . \supset \vdash : \mathrm{Hp} . \supset . S | R | \breve{R} = S | (I \upharpoonright \mathrm{D}`R)$

$[*50\cdot6] \qquad = S \upharpoonright \mathrm{D}`R : \supset \vdash . \mathrm{Prop}$

***72·591.** $\vdash : R \in \mathrm{Cls} \to 1 . \supset . S | \breve{R} | R = S \upharpoonright \mathrm{Cl}`R$

***72·6.** $\vdash : R \in 1 \to \mathrm{Cls} . \mathrm{Cl}`S \subset \mathrm{D}`R . \supset . S | R | \breve{R} = S$ $\qquad [*72\cdot59 . *35\cdot452]$

***72·601.** $\vdash : R \in \mathrm{Cls} \to 1 . \mathrm{Cl}`S \subset \mathrm{Cl}`R . \supset . S | \breve{R} | R = S$

***72·61.** $\vdash : R \in 1 \to \mathrm{Cls} . \mathrm{Cl}`S \subset \mathrm{D}`R . \supset . S | R | \breve{R} | \breve{S} = S | \breve{S}$ $\qquad [*72\cdot6 . *34\cdot27]$

***72·611.** $\vdash : R \in \mathrm{Cls} \to 1 . \mathrm{Cl}`S \subset \mathrm{Cl}`R . \supset . S | \breve{R} | R | \breve{S} = S | \breve{S}$

The following propositions lead up to the "principle of abstraction" (*72·66), which, though not explicitly referred to in the sequel, has a certain intrinsic interest, and generalizes a type of reasoning frequently employed by us.

***72·62.** $\vdash : R \in 1 \to \mathrm{Cls} . S = R | \breve{R} . \supset . S^2 = S . S = \breve{S}$

Dem.

$\vdash . *34\cdot21 . \qquad \supset \vdash : S = R | \breve{R} . \supset . S^2 = R | (\breve{R} | R | \breve{R}) \qquad (1)$

$\vdash . *72\cdot6 . *33\cdot21 . \supset \vdash : R \in 1 \to \mathrm{Cls} . \supset . \breve{R} | R | \breve{R} = \breve{R} \qquad (2)$

$\vdash . (1) . (2) . \qquad \supset \vdash : \mathrm{Hp} . \supset . S^2 = R | \breve{R}$

$[\mathrm{Hp}] \qquad = S \qquad (3)$

$\vdash . (3) . *34\cdot7 . \supset \vdash . \mathrm{Prop}$

***72·621.** $\vdash :. R \in 1 \to \mathrm{Cls} . \supset : y (\breve{R} \mid R) z . \equiv . R`y = R`z$

Dem.

$\vdash . *71·33 . \supset \vdash :. \mathrm{Hp} . \supset : R`y = R`z . \equiv . (\exists x) . xRy . x = R`z .$

$[*71·36] \qquad \equiv . (\exists x) . xRy . xRz .$

$[*31·11] \qquad \equiv . (\exists x) . y\breve{R}x . xRz .$

$[*34·1] \qquad \equiv . y (\breve{R} \mid R) z :. \supset \vdash . \mathrm{Prop}$

***72·622.** $\vdash :. R \in \mathrm{Cls} \to 1 . \supset : y (R \mid \breve{R}) z . \equiv . \breve{R}`y = \breve{R}`z$

***72·63.** $\vdash : R \in \mathrm{Cls} \to 1 . S = R \mid \breve{R} . \supset . S^2 = S . S = \breve{S}$

Dem.

$\vdash . *34·21 . \supset \vdash : S = R \mid \breve{R} . \supset . S^2 = (R \mid \breve{R} \mid R) \mid \breve{R}$ (1)

$\vdash . *72·601 . \supset \vdash : R \in \mathrm{Cls} \to 1 . \supset . R \mid \breve{R} \mid R = R$ (2)

$\vdash . (1) . (2) . \supset \vdash : \mathrm{Hp} . \supset . S^2 = R \mid \breve{R}$

$[\mathrm{Hp}] \qquad = S$ (3)

$\vdash . (3) . *34·7 . \supset \vdash . \mathrm{Prop}$

***72·64.** $\vdash : S^2 = S . S = \breve{S} . R = \mathrm{Cnv}`(\overleftarrow{S} \restriction D`S) . \supset . R \in \mathrm{Cls} \to 1 . S = R \mid \breve{R}$

Dem.

$\vdash . *72·12 . *71·26 . \supset \vdash . \overleftarrow{S} \restriction D`S \in 1 \to \mathrm{Cls} .$

$[*71·21] \qquad \supset \vdash : \mathrm{Hp} . \supset . R \in \mathrm{Cls} \to 1$ (1)

$\vdash . (1) . *72·622 . \supset$

$\vdash :. \mathrm{Hp} . \supset : y (R \mid \breve{R}) z . \equiv . \breve{R}`y = \breve{R}`z .$

$[*31·34 . \mathrm{Hp}] \qquad \equiv . (\overleftarrow{S} \restriction D`S)`y = (\overleftarrow{S} \restriction D`S)`z .$

$[*35·7] \qquad \equiv . y, z \in D`S . \overleftarrow{S}`y = \overleftarrow{S}`z .$

$[*34·85] \qquad \equiv . z \in D`S . ySz$ (2)

$\vdash . *31·11 . \supset \vdash :. \mathrm{Hp} . \supset : ySz . \supset . zSy .$

$[*33·14] \qquad \supset . z \in D`S :$

$[*4·71] \qquad \supset : ySz . \equiv . z \in D`S . ySz$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash . \mathrm{Prop}$

***72·65.** $\vdash : S^2 = S . S = \breve{S} . \equiv . (\exists R) . R \in \mathrm{Cls} \to 1 . S = R \mid \breve{R}$ [*72·63·64]

***72·66.** $\vdash : S^2 \subset S . S = \breve{S} . \equiv . (\exists R) . R \in \mathrm{Cls} \to 1 . S = R \mid \breve{R}$ [*72·65 . *34·81]

***72·7.** $\vdash : R \in 1 \to \mathrm{Cls} . \supset . \overleftarrow{R} \restriction D`R \in 1 \to 1$

Dem.

$\vdash . *33·4 . *22·5 . \supset \vdash : y, z \in D`R . \overleftarrow{R}`y = \overleftarrow{R}`z . \supset . \exists ! \overleftarrow{R}`y \cap \overleftarrow{R}`z$ (1)

$\vdash . (1) . *71·18 . \supset \vdash : y, z \in D`R . \overleftarrow{R}`y = \overleftarrow{R}`z . \supset . y = z$ (2)

$\vdash . (2) . *72·12 . *71·55 . \supset \vdash . \mathrm{Prop}$

***72·71.** $\vdash : R \in \text{Cls} \to 1 . \supset . \overrightarrow{R} \upharpoonright \text{G}\text{'}R \in 1 \to 1$

***72·72.** $\vdash : R \in 1 \to 1 . \supset . \overrightarrow{R} \upharpoonright \text{G}\text{'}R, \overleftarrow{R} \upharpoonright \text{D}\text{'}R \in 1 \to 1$

***72·8.** $\vdash : \lambda \subset \text{D}\text{'}x \downarrow . \supset . \text{G} \upharpoonright \lambda \in 1 \to 1$ [*55·28·22 . *71·58]

The above proposition is used in *73·62.

***72·81.** $\vdash : \lambda \subset \text{D}\text{'} \downarrow x . \supset . \text{D} \upharpoonright \lambda \in 1 \to 1$ [*55·281·221 . *71·58]

***72·9.** $\vdash :. R \in 1 \to \text{Cls} . S \Subset R . \supset : \text{E} ! S\text{'}y . \equiv . R\text{'}y = S\text{'}y . \equiv . y \in \text{G}\text{'}S$

Dem.

$\vdash . *71{\cdot}22 . \quad \supset \vdash :. \text{Hp} . \supset : S \in 1 \to \text{Cls} :$

$[*71{\cdot}163] \quad \supset : \text{E} ! S\text{'}y . \equiv . y \in \text{G}\text{'}S$ (1)

$\vdash . *14{\cdot}21 . \quad \supset \vdash : R\text{'}y = S\text{'}y . \supset . \text{E} ! S\text{'}y$ (2)

$\vdash . *30{\cdot}32 . (1) . \supset \vdash :. \text{Hp} . \supset : y \in \text{G}\text{'}S . \supset . (S\text{'}y) Sy .$

$[\text{Hp}] \quad \supset . (S\text{'}y) Ry .$

$[*71{\cdot}36] \quad \supset . S\text{'}y = R\text{'}y$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash . \text{Prop}$

***72·91.** $\vdash : R \in 1 \to \text{Cls} . S \Subset R . \supset . \text{G}\text{'}(R \dot{-} S) = \text{G}\text{'}R - \text{G}\text{'}S$

Dem.

$\vdash . *33{\cdot}131 . *23{\cdot}33{\cdot}35 . \supset$

$\vdash : y \in \text{G}\text{'}(R \dot{-} S) . \equiv . (\exists x) . xRy . \sim (xSy)$ (1)

$\vdash . (1) . *71{\cdot}36 . \supset$

$\vdash :. \text{Hp} . \supset : y \in \text{G}\text{'}(R \dot{-} S) . \equiv . (\exists x) . x = R\text{'}y . \sim (x = S\text{'}y) .$

$[*14{\cdot}15 . *5{\cdot}32] \quad \equiv . (\exists x) . x = R\text{'}y . \sim (R\text{'}y = S\text{'}y) .$

$[*10{\cdot}35 . *14{\cdot}204 . *72{\cdot}9] \quad \equiv . \text{E} ! R\text{'}y . \sim (y \in \text{G}\text{'}S) .$

$[*71{\cdot}163] \quad \equiv . y \in \text{G}\text{'}R - \text{G}\text{'}S :. \supset \vdash . \text{Prop}$

***72·911.** $\vdash : R \in \text{Cls} \to 1 . S \Subset R . \supset . \text{D}\text{'}(R \dot{-} S) = \text{D}\text{'}R - \text{D}\text{'}S$

***72·92.** $\vdash : R \in 1 \to \text{Cls} . S \Subset R . \supset . S = R \upharpoonright \text{G}\text{'}S$

Dem.

$\vdash . *23{\cdot}1 . *33{\cdot}14 . \quad \supset \vdash :. \text{Hp} . \supset : xSy . \supset_{x,y} . xRy . y \in \text{G}\text{'}S .$

$[*35{\cdot}101] \quad \supset_{x,y} . x (R \upharpoonright \text{G}\text{'}S) y :$

$[*23{\cdot}1] \quad \supset : S \Subset R \upharpoonright \text{G}\text{'}S$ (1)

$\vdash . *35{\cdot}101 . *71{\cdot}36 . \supset \vdash :. \text{Hp} . \supset : x (R \upharpoonright \text{G}\text{'}S) y . \equiv . x = R\text{'}y . y \in \text{G}\text{'}S .$

$[*72{\cdot}9] \quad \equiv . x = R\text{'}y . R\text{'}y = S\text{'}y .$

$[*14{\cdot}142] \quad \supset . x = S\text{'}y .$

$[*30{\cdot}31] \quad \supset . xSy$ (2)

$\vdash . (2) . *11{\cdot}11{\cdot}3 . \quad \supset \vdash : \text{Hp} . \supset . R \upharpoonright \text{G}\text{'}S \Subset S$ (3)

$\vdash . (1) . (3) . \supset \vdash . \text{Prop}$

***72·921.** $\vdash : R \in \text{Cls} \to 1 . S \Subset R \ \supset . S = (\text{D}\text{'}S) \upharpoonleft R$

***72·93.** $\vdash :. R \epsilon 1 \rightarrow \mathrm{Cls} . R \Subset S . \equiv : y \epsilon \text{ᗡ}'R . \supset_y . (R'y) S y$

Dem.

$\vdash . *14{\cdot}21 . *4{\cdot}71 . \supset \vdash :: y \epsilon \text{ᗡ}'R . \supset_y . (R'y) S y : \equiv :.$

$y \epsilon \text{ᗡ}'R . \supset_y . E ! R'y . (R'y) S y :.$

[*14·25] $\equiv :. y \epsilon \text{ᗡ}'R . \supset_y . E ! R'y : xRy . \supset_x . xSy :.$

[*10·29.*11·62] $\equiv :. y \epsilon \text{ᗡ}'R . \supset_y . E ! R'y : y \epsilon \text{ᗡ}'R . xRy . \supset_{x,y} . xSy :.$

[*71·16.*33·14] $\equiv :. R \epsilon 1 \rightarrow \mathrm{Cls} . R \Subset S :: \supset \vdash . \mathrm{Prop}$

***72·931.** $\vdash :. R \epsilon \mathrm{Cls} \rightarrow 1 . R \Subset S . \equiv : x \epsilon D'R . \supset_x . xS(\breve{R}'x)$

***72·94.** $\vdash :. R, S \epsilon 1 \rightarrow \mathrm{Cls} . \supset : \dot{\exists} ! R \dot{\cap} S . \equiv . (\exists y) . R'y = S'y$

Dem.

$\vdash . *71{\cdot}36 . \supset \vdash :. \mathrm{Hp} . \supset : \dot{\exists} ! R \dot{\cap} S . \equiv . (\exists x, y) . x = R'y . x = S'y .$

[*14·205] $\equiv . (\exists y) . R'y = S'y :. \supset \vdash . \mathrm{Prop}$

*73. SIMILARITY OF CLASSES

Summary of *73.

Two classes α and β are said to be *similar* when there is a one-one relation whose domain is α and whose converse domain is β. We express "α is similar to β" by the notation "$\alpha \operatorname{sm} \beta$." When two classes are similar, they have the same cardinal number of terms: it is this fact which gives importance to the relation of similarity.

We have

$$\alpha \operatorname{sm} \beta \,.\equiv.\, (\exists R) \,.\, R \in 1 \rightarrow 1 \,.\, \alpha = \mathrm{D}‘R \,.\, \beta = \text{Ɑ}‘R.$$

The relation of similarity is that of the domain of a $1 \rightarrow 1$ to the converse domain, *i.e.* it is the relative product of $\mathrm{D} \upharpoonright (1 \rightarrow 1)$ and $(1 \rightarrow 1) \upharpoonleft \breve{\text{Ɑ}}$, or, what comes to the same thing, it is the relative product of $\mathrm{D} \upharpoonright (1 \rightarrow 1)$ and $\breve{\text{Ɑ}}$.

Most of the properties of similarity result immediately from those of one-one relations and offer no difficulty of any kind.

When there are relations which correlate α's with β's so as to make α similar to β, we denote the class of such relations by "$\alpha \,\overline{\operatorname{sm}}\, \beta$." Thus we have

$$\alpha \,\overline{\operatorname{sm}}\, \beta = (1 \rightarrow 1) \cap \overleftarrow{\mathrm{D}}‘\alpha \cap \overleftarrow{\text{Ɑ}}‘\beta \quad \mathrm{Df}$$

and

$$\operatorname{sm} = \hat{\alpha}\hat{\beta}\,(\exists \,!\, \alpha \,\overline{\operatorname{sm}}\, \beta) \quad \mathrm{Df}$$

When, as in this case, we have a descriptive double function closely connected with a relation, we shall make it a practice to distinguish the descriptive double function by a bar.

It is to be observed that "sm," like Λ and V and 1 and $1 \rightarrow 1$, is ambiguous as to type, and only acquires a definite meaning when the types of its domain and converse domain are specified. The domain and the converse domain may or may not be of the same type, *i.e.* "sm" may or may not be a homogeneous relation. This enables us to speak of two classes of different types as having the same number of terms. We shall return to this point in connection with cardinal numbers (cf. especially *102—*106).

The propositions of the present number are important, and are very frequently referred to throughout cardinal arithmetic. In order to prove that two classes α and β have the same cardinal number of terms, it is generally necessary, in the fundamental arithmetical propositions with which we are concerned, actually to construct a relation R such that $R \in \alpha \,\overline{\operatorname{sm}}\, \beta$. Such a relation will be called a *correlator* of α and β. It will usually be obtained by taking some relation S for which we have $(y) \,.\, \mathrm{E} \,!\, S‘y$, and

limiting the converse domain to β, so that $S \upharpoonright \beta$ is the required correlator. Very frequently we shall have $S \epsilon 1 \rightarrow \mathrm{Cls}$, not $S \epsilon 1 \rightarrow 1$, but β will be such that $S \upharpoonright \beta \epsilon 1 \rightarrow 1$.

Among the more important propositions of the present number are the following:

***73·142.** $\vdash : R \upharpoonright \beta \epsilon \alpha \overline{\mathrm{sm}} \beta . \equiv . R \upharpoonright \beta \epsilon 1 \rightarrow 1 . \beta \subset \mathrm{Ꮖ}'R . \alpha = R``\beta$

I.e $R \upharpoonright \beta$ is a correlator of α and β if (1) $R \upharpoonright \beta$ is one-one, (2) β is contained in the converse domain of R, (3) α is the class of those terms which have the relation R to members of β.

***73·2.** $\vdash : R \epsilon 1 \rightarrow 1 . \supset . \mathrm{D}'R \,\mathrm{sm}\, \mathrm{Ꮖ}'R . \mathrm{Ꮖ}'R \,\mathrm{sm}\, \mathrm{D}'R$

This results immediately from the definition.

***73·22.** $\vdash : R \epsilon 1 \rightarrow 1 . \beta \subset \mathrm{Ꮖ}'R . \supset . R``\beta \,\mathrm{sm}\, \beta . R \upharpoonright \beta \epsilon (R``\beta) \overline{\mathrm{sm}} \beta$

***73·3.** $\vdash . \alpha \,\mathrm{sm}\, \alpha . I \upharpoonright \alpha \epsilon \alpha \overline{\mathrm{sm}} \alpha$

***73·31.** $\vdash : \alpha \,\mathrm{sm}\, \beta . \equiv . \beta \,\mathrm{sm}\, \alpha$

***73·32.** $\vdash : \alpha \,\mathrm{sm}\, \beta . \beta \,\mathrm{sm}\, \gamma . \supset . \alpha \,\mathrm{sm}\, \gamma$

The above three propositions show that similarity is reflexive, symmetrical, and transitive.

***73·36.** $\vdash :. \alpha \,\mathrm{sm}\, \beta . \supset : \exists ! \alpha . \equiv . \exists ! \beta$

***73·41.** $\vdash . \iota``\alpha \,\mathrm{sm}\, \alpha . \iota \upharpoonright \alpha \epsilon (\iota``\alpha) \overline{\mathrm{sm}} \alpha$

Thus every class α is similar to a class $\iota``\alpha$ of higher type, and consisting wholly of unit classes.

***73·45.** $\vdash . 1 = \hat{\beta} (\beta \,\mathrm{sm}\, \iota'x)$

Thus 1 is the class of all classes similar to any unit class.

***73·48.** $\vdash . 0 = \hat{\beta} (\beta \,\mathrm{sm}\, \Lambda)$

Thus 0 is the class of all classes similar to the null-class.

***73·611.** $\vdash . \downarrow x``\alpha \,\mathrm{sm}\, \alpha . (\downarrow x) \upharpoonright \alpha \epsilon (\downarrow x``\alpha) \overline{\mathrm{sm}} \alpha$

This proposition is very often useful. For arithmetical purposes, we often wish to obtain mutually exclusive classes. Now whether or not α and β be mutually exclusive, $\downarrow x``\alpha$ and $\downarrow y``\beta$ are mutually exclusive provided $x \neq y$. Thus by means of the above proposition we can always construct mutually exclusive classes each similar to a given class, *i.e.* each having some assigned number of members.

***73·71.** $\vdash : \alpha \,\mathrm{sm}\, \beta . \gamma \,\mathrm{sm}\, \delta . \alpha \cap \gamma = \Lambda . \beta \cap \delta = \Lambda . \supset . (\alpha \cup \gamma) \,\mathrm{sm}\, (\beta \cup \delta)$

This proposition is fundamental in the theory of addition.

***73·88.** $\vdash : \alpha \,\mathrm{sm}\, \gamma . \beta \,\mathrm{sm}\, \delta . \gamma \subset \beta . \delta \subset \alpha . \supset . \alpha \,\mathrm{sm}\, \beta$

I.e. "if α is similar to a part of β, and β is similar to a part of α, then α is similar to β." This is the Schröder-Bernstein theorem. The proof given below is due to Zermelo.

$*73{\cdot}01.\quad \alpha\,\overline{\mathrm{sm}}\,\beta = (1\rightarrow 1) \cap \overleftarrow{\mathrm{D}}\text{‘}\alpha \cap \overleftarrow{\text{Ɑ}}\text{‘}\beta \quad \mathrm{Df}$

$*73{\cdot}02.\quad \mathrm{sm} = \hat{\alpha}\hat{\beta}\,(\exists\,!\,\alpha\,\overline{\mathrm{sm}}\,\beta) \quad \mathrm{Df}$

$*73{\cdot}03.\quad \vdash : R \in \alpha\,\overline{\mathrm{sm}}\,\beta\,.\equiv .\, R \in 1\rightarrow 1\,.\,\alpha = \mathrm{D}\text{‘}R\,.\,\beta = \text{Ɑ}\text{‘}R \quad [*33{\cdot}6{\cdot}61\,.\,(*73{\cdot}01)]$

$*73{\cdot}04.\quad \vdash : \alpha\,\mathrm{sm}\,\beta\,.\equiv .\,\exists\,!\,\alpha\,\overline{\mathrm{sm}}\,\beta \quad [(*73{\cdot}02)]$

$*73{\cdot}1.\quad \vdash : \alpha\,\mathrm{sm}\,\beta\,.\equiv .\,(\exists R)\,.\,R \in 1\rightarrow 1\,.\,\alpha = \mathrm{D}\text{‘}R\,.\,\beta = \text{Ɑ}\text{‘}R \quad [*73{\cdot}03{\cdot}04]$

$*73{\cdot}11.\quad \vdash : \alpha\,\mathrm{sm}\,\beta\,.\equiv .\,(\exists R)\,.\,R \in 1\rightarrow 1\,.\,\alpha \subset \mathrm{D}\text{‘}R\,.\,\beta = \breve{R}\text{‘‘}\alpha$

Dem.

$\vdash .\,*22{\cdot}42\,.\,*37{\cdot}25\,.\supset$

$\vdash : R \in 1\rightarrow 1\,.\,\alpha = \mathrm{D}\text{‘}R\,.\,\beta = \text{Ɑ}\text{‘}R\,.\supset .\,R \in 1\rightarrow 1\,.\,\alpha \subset \mathrm{D}\text{‘}R\,.\,\beta = \breve{R}\text{‘‘}\alpha :$

$[*10{\cdot}11{\cdot}28] \supset \vdash : (\exists R)\,.\,R \in 1\rightarrow 1\,.\,\alpha = \mathrm{D}\text{‘}R\,.\,\beta = \text{Ɑ}\text{‘}R\,.\supset .$

$\quad (\exists R)\,.\,R \in 1\rightarrow 1\,.\,\alpha \subset \mathrm{D}\text{‘}R\,.\,\beta = \breve{R}\text{‘‘}\alpha :$

$[*73{\cdot}1] \quad \supset \vdash : \alpha\,\mathrm{sm}\,\beta\,.\supset .\,(\exists R)\,.\,R \in 1\rightarrow 1\,.\,\alpha \subset \mathrm{D}\text{‘}R\,.\,\beta = \breve{R}\text{‘‘}\alpha \quad (1)$

$\vdash .\,*71{\cdot}29\,.\,*37{\cdot}4\,.\,*35{\cdot}62\,.\supset$

$\vdash : R \in 1\rightarrow 1\,.\,\alpha \subset \mathrm{D}\text{‘}R\,.\,\beta = \breve{R}\text{‘‘}\alpha\,.\supset .\,\alpha \rceil R \in 1\rightarrow 1\,.\,\alpha = \mathrm{D}\text{‘}(\alpha \rceil R)\,.\,\beta = \text{Ɑ}\text{‘}(\alpha \rceil R)\,.$

$[*10{\cdot}24] \quad \supset .\,(\exists S)\,.\,S \in 1\rightarrow 1\,.\,\alpha = \mathrm{D}\text{‘}S\,.\,\beta = \text{Ɑ}\text{‘}S\,.$

$[*73{\cdot}1] \quad \supset .\,\alpha\,\mathrm{sm}\,\beta \quad (2)$

$\vdash .\,(2)\,.\,*10{\cdot}11{\cdot}23\,.\supset$

$\vdash : (\exists R)\,.\,R \in 1\rightarrow 1\,.\,\alpha \subset \mathrm{D}\text{‘}R\,.\,\beta = \breve{R}\text{‘‘}\alpha\,.\supset .\,\alpha\,\mathrm{sm}\,\beta \quad (3)$

$\vdash .\,(1)\,.\,(3)\,.\supset \vdash .\,\mathrm{Prop}$

$*73{\cdot}12.\quad \vdash : \alpha\,\mathrm{sm}\,\beta\,.\equiv .\,(\exists R)\,.\,R \in 1\rightarrow 1\,.\,\beta \subset \text{Ɑ}\text{‘}R\,.\,\alpha = R\text{‘‘}\beta$

[Proof as in $*73{\cdot}11$]

$*73{\cdot}13.\quad \vdash : \alpha\,\mathrm{sm}\,\beta\,.\equiv .\,(\exists R)\,.\,R \in 1\rightarrow \mathrm{Cls}\,.\,R \upharpoonright \beta \in \mathrm{Cls}\rightarrow 1\,.\,\beta \subset \text{Ɑ}\text{‘}R\,.\,\alpha = R\text{‘‘}\beta$

Dem.

$\vdash .\,*71{\cdot}103{\cdot}271\,.\quad \supset \vdash : R \in 1\rightarrow 1\,.\supset .\,R \in 1\rightarrow \mathrm{Cls}\,.\,R \upharpoonright \beta \in \mathrm{Cls}\rightarrow 1 :$

$[\mathrm{Fact}] \quad \supset \vdash : R \in 1\rightarrow 1\,.\,\beta \subset \text{Ɑ}\text{‘}R\,.\,\alpha = R\text{‘‘}\beta\,.\supset .$

$\quad R \in 1\rightarrow \mathrm{Cls}\,.\,R \upharpoonright \beta \in \mathrm{Cls}\rightarrow 1\,.\,\beta \subset \text{Ɑ}\text{‘}R\,.\,\alpha = R\text{‘‘}\beta :$

$[*10{\cdot}11{\cdot}28\,.\,*73{\cdot}12] \supset \vdash : \alpha\,\mathrm{sm}\,\beta\,.\supset .$

$\quad (\exists R)\,.\,R \in 1\rightarrow \mathrm{Cls}\,.\,R \upharpoonright \beta \in \mathrm{Cls}\rightarrow 1\,.\,\beta \subset \text{Ɑ}\text{‘}R\,.\,\alpha = R\text{‘‘}\beta \quad (1)$

$\vdash .\,*71{\cdot}26\,.\supset \vdash : R \in 1\rightarrow \mathrm{Cls}\,.\,R \upharpoonright \beta \in \mathrm{Cls}\rightarrow 1\,.\supset .\,R \upharpoonright \beta \in 1\rightarrow \mathrm{Cls}\,.\,R \upharpoonright \beta \in \mathrm{Cls}\rightarrow 1\,.$

$[*71{\cdot}103] \quad \supset .\,R \upharpoonright \beta \in 1\rightarrow 1 \quad (2)$

$\vdash .\,*35{\cdot}65\,.\,*37{\cdot}401\,.\supset$

$\quad \vdash : \beta \subset \text{Ɑ}\text{‘}R\,.\,\alpha = R\text{‘‘}\beta\,.\supset .\,\beta = \text{Ɑ}\text{‘}(R \upharpoonright \beta)\,.\,\alpha = \mathrm{D}\text{‘}(R \upharpoonright \beta) \quad (3)$

$\vdash .\,(2)\,.\,(3)\,.\supset \vdash : R \in 1\rightarrow \mathrm{Cls}\,.\,R \upharpoonright \beta \in \mathrm{Cls}\rightarrow 1\,.\,\beta \subset \text{Ɑ}\text{‘}R\,.\,\alpha = R\text{‘‘}\beta\,.\supset .$

$\quad R \upharpoonright \beta \in 1\rightarrow 1\,.\,\alpha = \mathrm{D}\text{‘}(R \upharpoonright \beta)\,.\,\beta = \text{Ɑ}\text{‘}(R \upharpoonright \beta)\,.$

$[*10{\cdot}24\,.\,*73{\cdot}1] \quad \supset .\,\alpha\,\mathrm{sm}\,\beta \quad (4)$

$\vdash .\,(4)\,.\,*10{\cdot}11{\cdot}23\,.\supset$

$\vdash : (\exists R)\,.\,R \in 1\rightarrow \mathrm{Cls}\,.\,R \upharpoonright \beta \in \mathrm{Cls}\rightarrow 1\,.\,\beta \subset \text{Ɑ}\text{‘}R\,.\,\alpha = R\text{‘‘}\beta\,.\supset .\,\alpha\,\mathrm{sm}\,\beta \quad (5)$

$\vdash .\,(1)\,.\,(5)\,.\supset \vdash .\,\mathrm{Prop}$

***73·131.** $\vdash : \alpha \operatorname{sm} \beta . \equiv . (\exists R) . R \in \mathrm{Cls} \to 1 . \alpha \rceil R \in 1 \to \mathrm{Cls} . \alpha \subset \mathrm{D}`R . \beta = \breve{R}``\alpha$

[Proof as in *73·13]

***73·14.** $\vdash :. \alpha \operatorname{sm} \beta . \equiv : (\exists R) : R \in 1 \to \mathrm{Cls} . \beta \subset \text{Ɑ}`R . \alpha = R``\beta :$

$y, z \in \beta . R`y = R`z . \supset_{y,z} . y = z$

Dem.

$\vdash . *71·55 . *5·32 . \supset$

$\vdash :. R \in 1 \to \mathrm{Cls} . R \upharpoonright \beta \in 1 \to 1 . \equiv :$

$R \in 1 \to \mathrm{Cls} : y, z \in \beta . R`y = R`z . \supset_{y,z} . y = z$ (1)

$\vdash . *71·26 . \supset \vdash :. R \in 1 \to \mathrm{Cls} . \supset : R \upharpoonright \beta \in 1 \to \mathrm{Cls} :$

[*4·73.*71·103] $\supset : R \upharpoonright \beta \in 1 \to 1 . \equiv . R \upharpoonright \beta \in \mathrm{Cls} \to 1 :.$

[*5·32] $\supset \vdash :. R \in 1 \to \mathrm{Cls} . R \upharpoonright \beta \in 1 \to 1 . \equiv . R \in 1 \to \mathrm{Cls} . R \upharpoonright \beta \in \mathrm{Cls} \to 1$ (2)

$\vdash . (1) . (2) . \supset \vdash :. (\exists R) . R \in 1 \to \mathrm{Cls} . R \upharpoonright \beta \in 1 \to 1 . \beta \subset \text{Ɑ}`R . \alpha = R``\beta . \equiv :$

$(\exists R) : R \in 1 \to \mathrm{Cls} . \beta \subset \text{Ɑ}`R . \alpha = R``\beta :$

$y, z \in \beta . R`y = R`z . \supset_{y,z} . y = z$ (3)

$\vdash . (3) . *73·13 . \supset \vdash . \mathrm{Prop}$

The use of this proposition in proving similarity is very frequent.

***73·141.** $\vdash :. \alpha \operatorname{sm} \beta . \equiv : (\exists R) : R \in \mathrm{Cls} \to 1 . \alpha \subset \mathrm{D}`R . \beta = \breve{R}``\alpha :$

$y, z \in \alpha . \breve{R}`y = \breve{R}`z . \supset_{y,z} . y = z$

[Proof as in *73·14]

***73·142.** $\vdash : R \upharpoonright \beta \in \alpha \,\overline{\operatorname{sm}}\, \beta . \equiv . R \upharpoonright \beta \in 1 \to 1 . \beta \subset \text{Ɑ}`R . \alpha = R``\beta$

Dem.

$\vdash . *73·03 . \supset$

$\vdash : R \upharpoonright \beta \in \alpha \,\overline{\operatorname{sm}}\, \beta . \equiv . R \upharpoonright \beta \in 1 \to 1 . \alpha = \mathrm{D}`(R \upharpoonright \beta) . \beta = \text{Ɑ}`(R \upharpoonright \beta) .$

[*37·401.*35·64] $\equiv . R \upharpoonright \beta \in 1 \to 1 . \alpha = R``\beta . \beta = \beta \cap \text{Ɑ}`R .$

[*22·621] $\equiv . R \upharpoonright \beta \in 1 \to 1 . \alpha = R``\beta . \beta \subset \text{Ɑ}`R : \supset \vdash . \mathrm{Prop}$

***73·15.** $\vdash : \alpha \operatorname{sm} \beta . \equiv . (\exists R) . R \upharpoonright \beta \in 1 \to 1 . \beta \subset \text{Ɑ}`R . \alpha = R``\beta$

Dem.

$\vdash . *73·12 . *71·29 . \supset \vdash : \alpha \operatorname{sm} \beta . \supset . (\exists R) . R \upharpoonright \beta \in 1 \to 1 . \beta \subset \text{Ɑ}`R . \alpha = R``\beta$ (1)

$\vdash . *73·142·04 . \supset \vdash : (\exists R) . R \upharpoonright \beta \in 1 \to 1 . \beta \subset \text{Ɑ}`R . \alpha = R``\beta . \supset . \alpha \operatorname{sm} \beta$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***73·2.** $\vdash : R \in 1 \to 1 . \supset . \mathrm{D}`R \operatorname{sm} \text{Ɑ}`R . \text{Ɑ}`R \operatorname{sm} \mathrm{D}`R$

Dem.

$\vdash . *20·2 . *3·21 . \supset$

$\vdash : R \in 1 \to 1 . \supset . R \in 1 \to 1 . \mathrm{D}`R = \mathrm{D}`R . \text{Ɑ}`R = \text{Ɑ}`R .$

[*10·24] $\supset . (\exists S) . S \in 1 \to 1 . \mathrm{D}`R = \mathrm{D}`S . \text{Ɑ}`R = \text{Ɑ}`S .$

[*73·1] $\supset . \mathrm{D}`R \operatorname{sm} \text{Ɑ}`R$ (1)

$\vdash . (1) . *71·212 . \supset \vdash : R \in 1 \to 1 . \supset . \mathrm{D}`\breve{R} \operatorname{sm} \text{Ɑ}`\breve{R}$

[*33·2·21] $\supset . \text{Ɑ}`R \operatorname{sm} \mathrm{D}`R$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

The following propositions, down to ∗73·241, are deduced from preceding propositions of this number just as "$\mathrm{D}‘R \,\mathrm{sm}\, \text{Ɑ}‘R$" was deduced in ∗73·2 from ∗73·1. The proofs are therefore merely indicated by references to the previous propositions of this number which are used.

∗73·21. $\vdash : R \epsilon 1 \to 1 \,.\, \alpha \subset \mathrm{D}‘R \,.\, \supset \,.\, \alpha \,\mathrm{sm}\, \breve{R}“\alpha \,.\, \alpha \rceil R \epsilon \alpha \,\overline{\mathrm{sm}}\, (\breve{R}“\alpha)$ [∗73·11]

∗73·22. $\vdash : R \epsilon 1 \to 1 \,.\, \beta \subset \text{Ɑ}‘R \,.\, \supset \,.\, R“\beta \,\mathrm{sm}\, \beta \,.\, R \upharpoonright \beta \,\epsilon\, (R“\beta) \,\overline{\mathrm{sm}}\, \beta$ [∗73·12]

∗73·23. $\vdash : R \epsilon 1 \to \mathrm{Cls} \,.\, \beta \subset \text{Ɑ}‘R \,.\, R \upharpoonright \beta \,\epsilon\, \mathrm{Cls} \to 1 \,.\, \supset \,.$
$R“\beta \,\mathrm{sm}\, \beta \,.\, R \upharpoonright \beta \,\epsilon\, (R“\beta) \,\overline{\mathrm{sm}}\, \beta$ [∗73·13]

∗73·231. $\vdash : R \epsilon \mathrm{Cls} \to 1 \,.\, \alpha \subset \mathrm{D}‘R \,.\, \alpha \rceil R \,\epsilon\, 1 \to \mathrm{Cls} \,.\, \supset \,.$
$\alpha \,\mathrm{sm}\, \breve{R}“\alpha \,.\, \alpha \rceil R \epsilon \alpha \,\overline{\mathrm{sm}}\, (\breve{R}“\alpha)$ [∗73·131]

∗73·24. $\vdash :. R \epsilon 1 \to \mathrm{Cls} \,.\, \beta \subset \text{Ɑ}‘R : y, z \,\epsilon\, \beta \,.\, R‘y = R‘z \,.\, \supset_{y,z} \,.\, y = z : \supset \,.$
$R“\beta \,\mathrm{sm}\, \beta \,.\, R \upharpoonright \beta \,\epsilon\, (R“\beta) \,\overline{\mathrm{sm}}\, \beta$ [∗73·14·142]

∗73·241. $\vdash :. R \epsilon \mathrm{Cls} \to 1 \,.\, \alpha \subset \mathrm{D}‘R : y, z \,\epsilon\, \alpha \,.\, \breve{R}‘y = \breve{R}‘z \,.\, \supset_{y,z} \,.\, y = z : \supset \,.$
$\alpha \,\mathrm{sm}\, \breve{R}“\alpha \,.\, \alpha \rceil R \epsilon \alpha \,\overline{\mathrm{sm}}\, (\breve{R}“\alpha)$ [∗73·141·03]

∗73·25. $\vdash :. (y) \,.\, \mathrm{E}!\, R‘y : y, z \,\epsilon\, \beta \,.\, R‘y = R‘z \,.\, \supset_{y,z} \,.\, y = z : \supset \,.\, R“\beta \,\mathrm{sm}\, \beta$

Dem.

$\vdash \,.\, ∗71·166 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, R \epsilon 1 \to \mathrm{Cls}$ (1)

$\vdash \,.\, ∗33·431 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, \beta \subset \text{Ɑ}‘R$ (2)

$\vdash \,.\, (1) \,.\, (2) \,.\, \supset \vdash :. \mathrm{Hp} \,.\, \supset : R \epsilon 1 \to \mathrm{Cls} \,.\, \beta \subset \text{Ɑ}‘R : y, z \epsilon \beta \,.\, R‘y = R‘z \,.\, \supset_{y,z} \,.\, y = z :$

[∗73·24] $\supset : R“\beta \,\mathrm{sm}\, \beta :. \supset \vdash \,.\, \mathrm{Prop}$

This proposition will be convenient in such cases as the following: Let β be a class of relations whose domains are mutually exclusive, *i.e.* such that no two members of β have domains which have a member in common, and suppose we wish to prove that the class of these domains is similar to β. The class of domains is $\mathrm{D}“\beta$, and we have $(P) \,.\, \mathrm{E}!\, \mathrm{D}‘P$. Hence we have only to prove (putting D in place of the R of ∗73·25)

$$P, Q \,\epsilon\, \beta \,.\, \mathrm{D}‘P = \mathrm{D}‘Q \,.\, \supset_{P,Q} \,.\, P = Q,$$

which, in the case supposed, is proved immediately.

∗73·26. $\vdash :. (y) \,.\, \mathrm{E}!\, R‘y : R \epsilon 1 \to 1 : \supset \,.\, R“\beta \,\mathrm{sm}\, \beta \,.\, R \upharpoonright \beta \,\epsilon\, (R“\beta) \,\overline{\mathrm{sm}}\, \beta$

Dem.

$\vdash \,.\, ∗33·431 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, R \epsilon 1 \to 1 \,.\, \beta \subset \text{Ɑ}‘R \,.$

[∗73·22] $\supset \,.\, R“\beta \,\mathrm{sm}\, \beta \,.\, R \upharpoonright \beta \,\epsilon\, (R“\beta) \,\overline{\mathrm{sm}}\, \beta : \supset \vdash \,.\, \mathrm{Prop}$

∗73·27. $\vdash :. R‘y = R‘z \,.\, \equiv_{y,z} \,.\, y = z : \supset \,.\, R“\beta \,\mathrm{sm}\, \beta \,.\, R \upharpoonright \beta \,\epsilon\, (R“\beta) \,\overline{\mathrm{sm}}\, \beta$
[∗73·26 . ∗71·57]

∗73·28. $\vdash :: y, z \,\epsilon\, \beta \,.\, \supset_{y,z} : R‘y = R‘z \,.\, \equiv \,.\, y = z :. \supset \,.$
$R“\beta \,\mathrm{sm}\, \beta \,.\, R \upharpoonright \beta \,\epsilon\, (R“\beta) \,\overline{\mathrm{sm}}\, \beta$

Dem.

$\vdash \,.\, ∗71·58 \,.\, ∗73·03 \,.\, ∗37·421 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, R \upharpoonright \beta \,\epsilon\, (R“\beta) \,\overline{\mathrm{sm}}\, \beta : \supset \vdash \,.\, \mathrm{Prop}$

***73·3.** $\vdash . \alpha \,\mathrm{sm}\, \alpha . I \restriction \alpha \epsilon \alpha \,\overline{\mathrm{sm}}\, \alpha$

Dem.

$\vdash . *50·31 . *24·11 . \quad \supset \vdash . \alpha \subset \text{Ɑ}‘I$ (1)

$\vdash . (1) . *72·17 . *50·16 . \supset \vdash . I \epsilon 1 \rightarrow 1 . \alpha \subset \text{Ɑ}‘I . I“\alpha = \alpha$ (2)

$\vdash . (2) . *73·142·04 . \supset \vdash .$ Prop

This is the *reflexive* property of similarity. The conditions of significance require that α should be a class of some type, but impose no restriction as to the type of class.

***73·301.** $\vdash : R \epsilon \alpha \,\overline{\mathrm{sm}}\, \beta . \equiv . \breve{R} \epsilon \beta \,\overline{\mathrm{sm}}\, \alpha$

Dem.

$\vdash . *73·03 . *71·212 . *33·2·21 . \supset$

$\vdash : R \epsilon \alpha \,\overline{\mathrm{sm}}\, \beta . \equiv . \breve{R} \epsilon 1 \rightarrow 1 . \mathrm{D}‘\breve{R} = \beta . \text{Ɑ}‘\breve{R} = \alpha .$

$[*73·03] \quad \equiv . \breve{R} \epsilon \beta \,\overline{\mathrm{sm}}\, \alpha : \supset \vdash .$ Prop

***73·31.** $\vdash : \alpha \,\mathrm{sm}\, \beta . \equiv . \beta \,\mathrm{sm}\, \alpha \quad [*73·301·04 . *31·52]$

This proposition shows that similarity is a *symmetrical* relation.

***73·311.** $\vdash : R \epsilon \alpha \,\overline{\mathrm{sm}}\, \beta . S \epsilon \beta \,\overline{\mathrm{sm}}\, \gamma . \supset . R | S \epsilon \alpha \,\overline{\mathrm{sm}}\, \gamma$

Dem.

$\vdash . *73·03 . *71·252 . \supset \vdash : \mathrm{Hp} . \supset . R | S \epsilon 1 \rightarrow 1$ (1)

$\vdash . *73·03 . *37·32 . \quad \supset \vdash : \mathrm{Hp} . \supset . \mathrm{D}‘(R | S) = R“\beta . \text{Ɑ}‘(R | S) = \breve{S}“\beta .$
$\alpha = \mathrm{D}‘R . \beta = \text{Ɑ}‘R . \beta = \mathrm{D}‘S . \gamma = \text{Ɑ}‘S .$

$[*37·25] \quad \supset . \mathrm{D}‘(R | S) = \alpha . \text{Ɑ}‘(R | S) = \gamma$ (2)

$\vdash . (1) . (2) . *73·03 . \supset \vdash .$ Prop

***73·32.** $\vdash : \alpha \,\mathrm{sm}\, \beta . \beta \,\mathrm{sm}\, \gamma . \supset . \alpha \,\mathrm{sm}\, \gamma \quad [*73·311·04]$

This proposition shows that similarity is a *transitive* relation. Thus we have now proved that similarity is reflexive, symmetrical, and transitive.

***73·33.** $\vdash . \mathrm{Cnv}‘\mathrm{sm} = \mathrm{sm} \quad [*73·31 . *31·131]$

***73·34.** $\vdash . \mathrm{sm}^2 = \mathrm{sm}$

Dem.

$\vdash . *34·55 . *73·32 . \quad \supset \vdash . \mathrm{sm}^2 \mathrel{\subset\!\!\!\cdot} \mathrm{sm}$ (1)

$\vdash . (1) . *73·33 . *34·8 . \supset \vdash .$ Prop

***73·35.** $\vdash . \mathrm{D}‘\mathrm{sm} = \text{Ɑ}‘\mathrm{sm} = \mathrm{Cls}$

Dem.

$\vdash . *73·3 . \quad \supset \vdash . \hat{z}(\phi ! z) \,\mathrm{sm}\, \hat{z}(\phi ! z) .$

$[*20·18] \quad \supset \vdash : \alpha = \hat{z}(\phi ! z) . \supset . \alpha \,\mathrm{sm}\, \alpha :$

$[*10·11·23] \supset \vdash : (\exists \phi) . \alpha = \hat{z}(\phi ! z) . \supset . \alpha \,\mathrm{sm}\, \alpha .$

$[*33·14] \quad \supset . \alpha \epsilon \mathrm{D}‘\mathrm{sm} . \alpha \epsilon \text{Ɑ}‘\mathrm{sm} :$

$[*20·4] \quad \supset \vdash : \alpha \epsilon \mathrm{Cls} . \supset . \alpha \epsilon \mathrm{D}‘\mathrm{sm} . \alpha \epsilon \text{Ɑ}‘\mathrm{sm}$ (1)

$\vdash . *73\cdot1 . *10\cdot5 . \supset$

$\vdash :. \alpha \, \mathrm{sm} \, \beta . \supset : (\exists R) . \alpha = \mathrm{D}'R . \beta = \mathrm{Cl}'R :$

[$*10\cdot5 . *33\cdot11\cdot111$] $\supset : (\exists R) . \alpha = \hat{x}\{(\exists y) . xRy\} : (\exists R) . \beta = \hat{y}\{(\exists x) . xRy\} :$

[$*20\cdot41\cdot18$] $\supset : \alpha \epsilon \mathrm{Cls} . \beta \epsilon \mathrm{Cls}$ (2)

$\vdash . (2) . *10\cdot11\cdot23 . \supset$

$\vdash :. (\exists \beta) . \alpha \, \mathrm{sm} \, \beta . \supset . \alpha \epsilon \mathrm{Cls} : (\exists \alpha) . \alpha \, \mathrm{sm} \, \beta . \supset . \beta \epsilon \mathrm{Cls} :.$

[$*33\cdot13\cdot131$] $\supset \vdash :. \alpha \epsilon \mathrm{D}'\mathrm{sm} . \supset . \alpha \epsilon \mathrm{Cls} : \beta \epsilon \mathrm{Cl}'\mathrm{sm} . \supset . \beta \epsilon \mathrm{Cls}$ (3)

$\vdash . (1) . (3) . \supset \vdash$. Prop

$*73\cdot36$. $\vdash :. \alpha \, \mathrm{sm} \, \beta . \supset : \exists ! \alpha . \equiv . \exists ! \beta$

Dem.

$\vdash . *33\cdot24 . \supset \vdash :. \alpha = \mathrm{D}'R . \beta = \mathrm{Cl}'R . \supset : \exists ! \alpha . \equiv . \exists ! \beta :.$

[$*3\cdot42$] $\supset \vdash :. R \epsilon 1 \rightarrow 1 . \alpha = \mathrm{D}'R . \beta = \mathrm{Cl}'R . \supset : \exists ! \alpha . \equiv . \exists ! \beta :.$

[$*10\cdot11\cdot23$] $\supset \vdash :. (\exists R) . R \epsilon 1 \rightarrow 1 . \alpha = \mathrm{D}'R . \beta = \mathrm{Cl}'R . \supset : \exists ! \alpha . \equiv . \exists ! \beta$ (1)

$\vdash . (1) . *73\cdot1 . \supset \vdash$. Prop

$*73\cdot37$. $\vdash :. \alpha \, \mathrm{sm} \, \beta . \supset : \gamma \, \mathrm{sm} \, \alpha . \equiv . \gamma \, \mathrm{sm} \, \beta$

Dem.

$\vdash . *73\cdot32 . \supset \vdash : \alpha \, \mathrm{sm} \, \beta . \gamma \, \mathrm{sm} \, \alpha . \supset . \gamma \, \mathrm{sm} \, \beta$ (1)

$\vdash . *73\cdot31 . \supset \vdash : \alpha \, \mathrm{sm} \, \beta . \gamma \, \mathrm{sm} \, \beta . \supset . \beta \, \mathrm{sm} \, \alpha . \gamma \, \mathrm{sm} \, \beta .$

[$*73\cdot32$] $\supset . \gamma \, \mathrm{sm} \, \alpha$ (2)

$\vdash . (1) . (2) . \supset \vdash$. Prop

$*73\cdot4$. $\vdash . \mathrm{Cnv}``\lambda \, \mathrm{sm} \, \lambda . \mathrm{Cnv} \restriction \lambda \epsilon (\mathrm{Cnv}``\lambda) \, \overline{\mathrm{sm}} \, \lambda$ [$*73\cdot26 . *72\cdot11 . *31\cdot13$]

$*73\cdot41$. $\vdash . \iota``\alpha \, \mathrm{sm} \, \alpha . \iota \restriction \alpha \epsilon (\iota``\alpha) \, \overline{\mathrm{sm}} \, \alpha$ [$*73\cdot26 . *72\cdot18 . *51\cdot12$]

This proposition is useful, because it gives a class ($\iota``\alpha$) similar to α but of higher type. Thus if μ is a cardinal number, and it is known that in a certain type there are classes having μ terms, it follows that there will be classes having μ terms in the next higher type, and therefore in the next type above that, and so on. No corresponding means exist for lowering the type.

$*73\cdot42$. $\vdash : \alpha \subset 1 . \supset . \alpha \, \mathrm{sm} \, \breve{\iota}``\alpha$

Dem.

$\vdash . *52\cdot13 . \supset \vdash : \mathrm{Hp} . \supset . \alpha \subset \mathrm{D}'\iota$ (1)

$\vdash . (1) . *73\cdot21 . *72\cdot18 . \supset \vdash$. Prop

This proposition gives a means of lowering the type without altering the cardinal number, provided our class α is composed wholly of unit classes; for $\breve{\iota}``\alpha$ is of the type next below the type of α. But when α is not composed wholly of unit classes, this construction fails.

$*73\cdot43$. $\vdash . \iota'x \, \mathrm{sm} \, \iota'y . x \downarrow y \epsilon (\iota'x) \, \overline{\mathrm{sm}} \, (\iota'y)$ [$*55\cdot15 . *72\cdot182 . *73\cdot2$]

***73·44.** $\vdash :. \alpha \epsilon 1 . \supset : \beta \,\mathrm{sm}\, \alpha . \equiv . \beta \epsilon 1$

Dem.

$\vdash . *73·43 . \qquad \supset \vdash :. \alpha = \iota‘y . \supset : \beta = \iota‘x . \supset . \beta \,\mathrm{sm}\, \alpha :.$

$[*10·11·23] \qquad \supset \vdash :. (\exists y) . \alpha = \iota‘y . \supset : \beta = \iota‘x . \supset . \beta \,\mathrm{sm}\, \alpha :.$

$[*10·11·21·23] \qquad \supset \vdash :. (\exists y) . \alpha = \iota‘y . \supset : (\exists x) . \beta = \iota‘x . \supset . \beta \,\mathrm{sm}\, \alpha :.$

$[*52·1] \qquad \supset \vdash :. \alpha \epsilon 1 . \supset : \beta \epsilon 1 . \supset . \beta \,\mathrm{sm}\, \alpha \qquad (1)$

$\vdash . *37·25 . \qquad \supset \vdash :. R \epsilon 1 \rightarrow 1 . \mathrm{D}‘R = \iota‘x . \supset . \text{Ɑ}‘R = \breve{R}“\iota‘x$

$[*53·31.*71·165] \qquad = \iota‘\breve{R}‘x .$

$[*52·22] \qquad \supset . \text{Ɑ}‘R \epsilon 1 :.$

$[*20·18] \qquad \supset \vdash :. R \epsilon 1 \rightarrow 1 . \mathrm{D}‘R = \iota‘x . \text{Ɑ}‘R = \beta . \supset . \beta \epsilon 1 :.$

$[*10·11·23.*73·1] \supset \vdash : \iota‘x \,\mathrm{sm}\, \beta . \supset . \beta \epsilon 1 :$

$[*20·18] \qquad \supset \vdash :. \alpha = \iota‘x . \supset : \alpha \,\mathrm{sm}\, \beta . \supset . \beta \epsilon 1 :.$

$[*10·11·23] \qquad \supset \vdash :. (\exists x) . \alpha = \iota‘x . \supset : \alpha \,\mathrm{sm}\, \beta . \supset . \beta \epsilon 1 :.$

$[*73·31.*52·1] \qquad \supset \vdash :. \alpha \epsilon 1 . \supset : \beta \,\mathrm{sm}\, \alpha . \supset . \beta \epsilon 1 \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***73·45.** $\vdash . 1 = \hat{\beta} (\beta \,\mathrm{sm}\, \iota‘x)$

Dem.

$\vdash . *52·22 . *73·44 . \supset \vdash : \beta \,\mathrm{sm}\, \iota‘x . \equiv . \beta \epsilon 1 \qquad (1)$

$\vdash . (1) . *20·33 . \supset \vdash . \mathrm{Prop}$

***73·46.** $\vdash . \Lambda \,\mathrm{sm}\, \Lambda \quad [*72·1 . *33·29 . *73·2]$

***73·47.** $\vdash : \beta \,\mathrm{sm}\, \Lambda . \equiv . \beta = \Lambda$

Dem.

$\vdash . *73·46 . \supset \vdash : \beta = \Lambda . \supset . \beta \,\mathrm{sm}\, \Lambda \qquad (1)$

$\vdash . *73·12 . *10·5 . \supset$

$\vdash : \beta \,\mathrm{sm}\, \Lambda . \supset . (\exists R) . \beta = R“\Lambda .$

$[*37·29] \qquad \supset . \beta = \Lambda \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***73·48.** $\vdash . 0 = \hat{\beta} (\beta \,\mathrm{sm}\, \Lambda) \quad [*73·46 . *51·11 . (*54·01)]$

The following proposition is used in the theory of double similarity (*111·111).

***73·5.** $\vdash : R \epsilon 1 \rightarrow 1 . \equiv . R_\epsilon \upharpoonright \mathrm{Cl}‘\text{Ɑ}‘R \Subset \mathrm{sm}$

Dem.

$\vdash . *35·101 . *37·101 . *60·2 . \supset$

$\vdash :. R_\epsilon \upharpoonright \mathrm{Cl}‘\text{Ɑ}‘R \Subset \mathrm{sm} . \equiv : \beta \subset \text{Ɑ}‘R . \alpha = R“\beta . \supset_{\alpha, \beta} . \alpha \,\mathrm{sm}\, \beta \qquad (1)$

$\vdash . *73·22 . \mathrm{Exp} . \supset \vdash :. R \epsilon 1 \rightarrow 1 . \supset : \beta \subset \text{Ɑ}‘R . \alpha = R“\beta . \supset . \alpha \,\mathrm{sm}\, \beta :$

$[(1).*11·11·3] \qquad \supset : R_\epsilon \upharpoonright \mathrm{Cl}‘\text{Ɑ}‘R \Subset \mathrm{sm} \qquad (2)$

$\vdash . *30·18 . *51·12 . \supset$

$\vdash :. \beta \subset \text{Ɑ}‘R . \alpha = R“\beta . \supset_{\alpha, \beta} . \alpha \,\mathrm{sm}\, \beta : \supset : \iota‘y \subset \text{Ɑ}‘R . \alpha = R“\iota‘y . \supset_\alpha . \alpha \,\mathrm{sm}\, \iota‘y :$

$[*51·2.*53·301] \qquad \supset : y \epsilon \text{Ɑ}‘R . \alpha = \overrightarrow{R}‘y . \supset_\alpha . \alpha \,\mathrm{sm}\, \iota‘y :$

$[*20·53.*73·44] \qquad \supset : y \epsilon \text{Ɑ}‘R . \supset . \overrightarrow{R}‘y \epsilon 1 :$

$[*10·11·21.*37·702.*71·1] \qquad \supset : R \epsilon 1 \rightarrow \mathrm{Cls} : \qquad (3)$

[*72·51.*37·16] $\supset : \alpha \subset \text{D‘}R \,.\, \beta = \breve{R}\text{“}\alpha \,.\, \supset_{\alpha,\beta} \,.\, \beta \subset \text{Ɑ‘}R \,.\, \alpha = R\text{“}\beta$ (4)

$\vdash .\, (4) \,.\, {*4{\cdot}7} \,.\, {*11{\cdot}37} \,.\, \supset \vdash :.\, \text{Hp}(4) \,.\, \supset : \alpha \subset \text{D‘}R \,.\, \beta = \breve{R}\text{“}\alpha \,.\, \supset_{\alpha,\beta} \,.\, \alpha \,\text{sm}\, \beta :$

$\left[(3)\dfrac{\breve{R}}{R} \,.\, {*71{\cdot}211}.{*73{\cdot}31}\right]$ $\supset : R \in \text{Cls} \to 1$ (5)

$\vdash .\, (1) \,.\, (3) \,.\, (5) \,.\, {*71{\cdot}103} \,.\, \supset \vdash : R_\epsilon \upharpoonright \text{Cl‘Ɑ‘}R \mathrel{\dot\subset} \text{sm} \,.\, \supset .\, R \in 1 \to 1$ (6)

$\vdash .\, (2) \,.\, (6) \,.\, \supset \vdash .\, \text{Prop}$

***73·501.** $\vdash : R \in 1 \to 1 \,.\, \equiv .\, (\breve{R})_\epsilon \upharpoonright \text{Cl‘D‘}R \mathrel{\dot\subset} \text{sm}$

Dem.

$\vdash .\, {*71{\cdot}212} \,.\, \supset \vdash : R \in 1 \to 1 \,.\, \equiv .\, \breve{R} \in 1 \to 1 \,.$

[*73·5] $\equiv .\, (\breve{R})_\epsilon \upharpoonright \text{Cl‘Ɑ‘}\breve{R} \mathrel{\dot\subset} \text{sm} \,.$

[*33·21] $\equiv .\, (\breve{R})_\epsilon \upharpoonright \text{Cl‘D‘}R \mathrel{\dot\subset} \text{sm} : \supset \vdash .\, \text{Prop}$

***73·51.** $\vdash : R \in 1 \to \text{Cls} \,.\, \alpha \subset \text{D‘}R \,.\, \supset .\, \overleftarrow{R}\text{“}\alpha \,\text{sm}\, \alpha$

Dem.

$\vdash .\, {*72{\cdot}7} \,.$ $\supset \vdash : \text{Hp} \,.\, \supset .\, \overleftarrow{R} \upharpoonright \text{D‘}R \in 1 \to 1 \,.$

[*35·431.*71·222] $\supset .\, \overleftarrow{R} \upharpoonright \alpha \in 1 \to 1$ (1)

$\vdash .\, {*33{\cdot}431} \,.\, {*32{\cdot}121} \,.\, \supset \vdash .\, \alpha \subset \text{Ɑ‘}\overleftarrow{R}$ (2)

$\vdash .\, (1) \,.\, (2) \,.\, {*72{\cdot}12} \,.\, \supset \vdash : \text{Hp} \,.\, \supset .\, \overleftarrow{R} \in 1 \to \text{Cls} \,.\, \overleftarrow{R} \upharpoonright \alpha \in 1 \to 1 \,.\, \alpha \subset \text{Ɑ‘}\overleftarrow{R} \,.$

[*73·23] $\supset .\, \overleftarrow{R}\text{“}\alpha \,\text{sm}\, \alpha : \supset \vdash .\, \text{Prop}$

***73·511.** $\vdash : R \in \text{Cls} \to 1 \,.\, \alpha \subset \text{Ɑ‘}R \,.\, \supset .\, \overrightarrow{R}\text{“}\alpha \,\text{sm}\, \alpha$

$\left[{*73{\cdot}51}\dfrac{\breve{R}}{R} \,.\, {*71{\cdot}211} \,.\, {*33{\cdot}2} \,.\, {*32{\cdot}241}\right]$

***73·52.** $\vdash : R \in 1 \to \text{Cls} \,.\, \alpha \subset \text{Cl‘D‘}R \,.\, \supset .\, (\breve{R})_\epsilon\text{“}\alpha \,\text{sm}\, \alpha$

Dem.

$\vdash .\, {*72{\cdot}45} \,.\, \supset \vdash :.\, \text{Hp} \,.\, \supset : (\breve{R})_\epsilon \upharpoonright \text{Cl‘D‘}R \in 1 \to 1 :$

[*71·55.*72·15] $\supset : \xi, \eta \in \text{Cl‘D‘}R \,.\, (\breve{R})_\epsilon\text{‘}\xi = (\breve{R})_\epsilon\text{‘}\eta \,.\, \supset_{\xi,\eta} \,.\, \xi = \eta :$

[Hp] $\supset : \xi, \eta \in \alpha \,.\, (\breve{R})_\epsilon\text{‘}\xi = (\breve{R})_\epsilon\text{‘}\eta \,.\, \supset_{\xi,\eta} \,.\, \xi = \eta :$

[*73·25.*37·111] $\supset : (\breve{R})_\epsilon\text{“}\alpha \,\text{sm}\, \alpha :. \supset \vdash .\, \text{Prop}$

***73·521.** $\vdash : R \in \text{Cls} \to 1 \,.\, \beta \subset \text{Cl‘Ɑ‘}R \,.\, \supset .\, R_\epsilon\text{“}\beta \,\text{sm}\, \beta$ [Proof as in *73·52]

***73·53.** $\vdash : R \in 1 \to \text{Cls} \,.\, \alpha \subset \text{Cl‘D‘}R \,.\, \supset .\, \breve{R}\text{‘‘‘}\alpha \,\text{sm}\, \alpha$ [*73·52 . (*37·04)]

***73·531.** $\vdash : R \in \text{Cls} \to 1 \,.\, \beta \subset \text{Cl‘Ɑ‘}R \,.\, \supset .\, R\text{‘‘‘}\beta \,\text{sm}\, \beta$ [*73·521 . (*37·04)]

***73·61.** $\vdash .\, x\!\downarrow\text{“}\alpha \,\text{sm}\, \alpha \,.\, (x\!\downarrow) \upharpoonright \alpha \in (x\!\downarrow\text{“}\alpha) \,\overline{\text{sm}}\, \alpha$ [*73·27 . *55·2]

***73·611.** $\vdash .\, \downarrow\! x\text{“}\alpha \,\text{sm}\, \alpha \,.\, (\downarrow\! x) \upharpoonright \alpha \in (\downarrow\! x\text{“}\alpha) \,\overline{\text{sm}}\, \alpha$ [*73·27 . *55·201]

***73·62.** $\vdash : \lambda \subset \text{D‘}x\!\downarrow \,.\, \supset .\, \text{Ɑ“}\lambda \,\text{sm}\, \lambda \,.\, \text{Ɑ} \upharpoonright \lambda \in (\text{Ɑ“}\lambda) \,\overline{\text{sm}}\, \lambda$ [*73·23 . *72·131·8]

***73·621.** $\vdash : \lambda \subset \text{D‘}\!\downarrow\! x \,.\, \supset .\, \text{D“}\lambda \,\text{sm}\, \lambda \,.\, \text{D} \upharpoonright \lambda \in (\text{D“}\lambda) \,\overline{\text{sm}}\, \lambda$ [*73·23 . *72·13·81]

***73·63.** $\vdash : S \in \alpha\,\overline{\text{sm}}\,\beta \,.\, T \upharpoonright \alpha, T \upharpoonright \beta \in 1 \to 1 \,.\, \alpha \cup \beta \subset \text{Ɑ}‘T \,.\supset .\, T | S | \breve{T} \in (T“\alpha)\,\overline{\text{sm}}\,(T“\beta)$

Dem.

$\vdash . *73·03 . *35·452·453 . \supset \vdash : \text{Hp} . \supset . T|S|\breve{T} = T | \alpha \upharpoonleft S \upharpoonright \beta | \breve{T}$

$[*35·354] \qquad = T \upharpoonright \alpha | S | \beta \upharpoonleft \breve{T} .$

$[*35·52 . *71·252 . *73·03] \qquad \supset . T|S|\breve{T} \in 1 \to 1 \qquad (1)$

$\vdash . *37·32 . \qquad \supset \vdash . \text{D}‘(T|S|\breve{T}) = T“S“\text{Ɑ}‘T \qquad (2)$

$\vdash . (2) . *37·27 . *73·03 . \supset \vdash : \text{Hp} . \supset . \text{D}‘(T|S|\breve{T}) = T“\alpha \qquad (3)$

Similarly $\qquad \vdash : \text{Hp} . \supset . \text{Ɑ}‘(T|S|\breve{T}) = T“\beta \qquad (4)$

$\vdash . (1) . (3) . (4) . *73·03 . \supset \vdash . \text{Prop}$

The above proposition is used once in connection with cardinal addition (*112·231), and once in connection with cardinal multiplication (*114·561).

The following proposition (*73·69) is a lemma for *73·7.

***73·69.** $\vdash : R \in \alpha\,\overline{\text{sm}}\,\beta \,.\, \alpha \cap \gamma = \Lambda \,.\, \beta \cap \gamma = \Lambda \,.\supset .\, R \,\dot{\cup}\, I \upharpoonright \gamma \in (\alpha \cup \gamma)\,\overline{\text{sm}}\,(\beta \cup \gamma)$

Dem.

$\vdash . *33·26·261 . *50·5·52 . \supset$

$\vdash : \text{D}‘R = \alpha . \text{Ɑ}‘R = \beta . S = R \,\dot{\cup}\, I \upharpoonright \gamma . \supset . \text{D}‘S = \alpha \cup \gamma . \text{Ɑ}‘S = \beta \cup \gamma \qquad (1)$

$\vdash . *71·242 . *50·5·52 . \supset$

$\vdash : \text{Hp}(1) . R \in 1 \to 1 . \alpha \cap \gamma = \Lambda . \beta \cap \gamma = \Lambda . \supset . R \,\dot{\cup}\, I \upharpoonright \gamma \in 1 \to 1 \qquad (2)$

$\vdash . (1) . (2) . *73·03 . \supset \vdash . \text{Prop}$

***73·7.** $\vdash : \alpha \,\text{sm}\, \beta \,.\, \alpha \cap \gamma = \Lambda \,.\, \beta \cap \gamma = \Lambda \,.\supset .\, (\alpha \cup \gamma) \,\text{sm}\, (\beta \cup \gamma) \quad [*73·69·04]$

***73·701.** $\vdash : R \in \alpha\,\overline{\text{sm}}\,\beta \,.\, S \in \gamma\,\overline{\text{sm}}\,\delta \,.\, \alpha \cap \gamma = \Lambda \,.\, \beta \cap \delta = \Lambda \,.\supset .\, R \,\dot{\cup}\, S \in (\alpha \cup \gamma)\,\overline{\text{sm}}\,(\beta \cup \delta)$

Dem.

$\vdash . *73·03 . \supset \vdash : \text{Hp} . \supset . \text{D}‘R \cap \text{D}‘S = \Lambda . \text{Ɑ}‘R \cap \text{Ɑ}‘S = \Lambda . R, S \in 1 \to 1 .$

$[*71·242] \qquad \supset . R \,\dot{\cup}\, S \in 1 \to 1 \qquad (1)$

$\vdash . *33·26·261 . *73·03 . \supset \vdash : \text{Hp} . \supset . \text{D}‘(R \,\dot{\cup}\, S) = \alpha \cup \gamma . \text{Ɑ}‘(R \,\dot{\cup}\, S) = \beta \cup \delta \qquad (2)$

$\vdash . (1) . (2) . *73·03 . \supset \vdash . \text{Prop}$

***73·71.** $\vdash : \alpha \,\text{sm}\, \beta \,.\, \gamma \,\text{sm}\, \delta \,.\, \alpha \cap \gamma = \Lambda \,.\, \beta \cap \delta = \Lambda \,.\supset .\, (\alpha \cup \gamma) \,\text{sm}\, (\beta \cup \delta) \quad [*73·701·04]$

***73·72.** $\vdash : \alpha \cup \iota‘x \,\text{sm}\, \beta \cup \iota‘y \,.\, x \sim\in \alpha \,.\, y \sim\in \beta \,.\supset .\, \alpha \,\text{sm}\, \beta$

Dem.

$\vdash . *73·1 . \supset$

$\vdash : \text{Hp} . \supset . (\exists R) . R \in 1 \to 1 . \text{D}‘R = \alpha \cup \iota‘x . \text{Ɑ}‘R = \beta \cup \iota‘y . x \sim\in \alpha . y \sim\in \beta \qquad (1)$

$\vdash . *71·381 . \supset \vdash : R \in 1 \to 1 . x \in \text{D}‘R . y \in \text{Ɑ}‘R . \supset . R“(\text{Ɑ}‘R - \iota‘\breve{R}‘x - \iota‘y)$

$\qquad = R“\text{Ɑ}‘R - R“\iota‘\breve{R}‘x - R“\iota‘y$

$[*37·25 . *53·31] \qquad = \text{D}‘R - \iota‘R‘\breve{R}‘x - \iota‘R‘y$

$[*72·24] \qquad = \text{D}‘R - \iota‘x - \iota‘R‘y .$

$[*73·22] \qquad \supset . (\text{D}‘R - \iota‘x - \iota‘R‘y) \,\text{sm}\, (\text{Ɑ}‘R - \iota‘y - \iota‘\breve{R}‘x) \qquad (2)$

$\vdash . *71\cdot362 . *22\cdot5 . \supset \vdash : \text{Hp}(2) . x = R\text{‘}y . \supset .$

$$\text{D}\text{‘}R - \iota\text{‘}x - \iota\text{‘}R\text{‘}y = \text{D}\text{‘}R - \iota\text{‘}x . \text{G}\text{‘}R - \iota\text{‘}y - \iota\text{‘}\breve{R}\text{‘}x = \text{G}\text{‘}R - \iota\text{‘}y .$$

$[(2)] \quad \supset . (\text{D}\text{‘}R - \iota\text{‘}x) \,\text{sm}\, (\text{G}\text{‘}R - \iota\text{‘}y) \quad (3)$

$\vdash . *22\cdot92 . *33\cdot43 . \supset \vdash : \text{Hp}(2) . x \neq R\text{‘}y . \supset .$

$$(\text{D}\text{‘}R - \iota\text{‘}x - \iota\text{‘}R\text{‘}y) \cup \iota\text{‘}R\text{‘}y = \text{D}\text{‘}R - \iota\text{‘}x \quad (4)$$

$\vdash . *71\cdot362 . \quad \supset \vdash : \text{Hp}(4) . \supset . y \neq \breve{R}\text{‘}x .$

$[*22\cdot92 . *33\cdot44] \quad \supset . (\text{G}\text{‘}R - \iota\text{‘}y - \iota\text{‘}\breve{R}\text{‘}x) \cup \iota\text{‘}\breve{R}\text{‘}x = \text{G}\text{‘}R - \iota\text{‘}y \quad (5)$

$\vdash . (4) . (5) . *73\cdot71\cdot43 . (2) . \supset \vdash : \text{Hp}(4) . \supset . (\text{D}\text{‘}R - \iota\text{‘}x) \,\text{sm}\, (\text{G}\text{‘}R - \iota\text{‘}y) \quad (6)$

$\vdash . (3) . (6) . \quad \supset \vdash : \text{Hp}(2) . \supset . (\text{D}\text{‘}R - \iota\text{‘}x) \,\text{sm}\, (\text{G}\text{‘}R - \iota\text{‘}y) \quad (7)$

$\vdash . *51\cdot211\cdot22 . \supset \vdash : \text{D}\text{‘}R = \alpha \cup \iota\text{‘}x . \text{G}\text{‘}R = \beta \cup \iota\text{‘}y . x \sim\epsilon\, \alpha . y \sim\epsilon\, \beta .$

$$\supset . \text{D}\text{‘}R - \iota\text{‘}x = \alpha . \text{G}\text{‘}R - \iota\text{‘}y = \beta \quad (8)$$

$\vdash . (7) . (8) . \quad \supset \vdash : R \in 1 \to 1 . \text{Hp}(8) . \supset . \alpha \,\text{sm}\, \beta \quad (9)$

$\vdash . (1) . (9) . \quad \supset \vdash . \text{Prop}$

The following propositions give the proof of the Schröder-Bernstein theorem, namely: If one class is similar to part of another, and the other is similar to part of the one, then the two classes are similar. The proof here given is due to Zermelo*. An explanation of the following proof is given in connection with another proof in the summary of *94.

***73·8.** $\vdash : \text{G}\text{‘}R \subset \beta . \beta \subset \text{D}\text{‘}R . \kappa = \hat{\alpha}(\alpha \subset \text{D}\text{‘}R . \beta - \text{G}\text{‘}R \subset \alpha . \breve{R}\text{“}\alpha \subset \alpha) . \supset .$

$$\text{D}\text{‘}R \in \kappa . p\text{‘}\kappa \subset \text{D}\text{‘}R$$

Dem.

$\vdash . *22\cdot42\cdot43\cdot44 . \quad \supset \vdash : \text{Hp} . \supset . \text{D}\text{‘}R \subset \text{D}\text{‘}R . \beta - \text{G}\text{‘}R \subset \text{D}\text{‘}R \quad (1)$

$\vdash . *22\cdot44 . *37\cdot25 . \supset \vdash : \text{Hp} . \supset . \breve{R}\text{“}\text{D}\text{‘}R \subset \text{D}\text{‘}R \quad (2)$

$\vdash . (1) . (2) . \quad \supset \vdash : \text{Hp} . \supset . \text{D}\text{‘}R \in \kappa \quad (3)$

$\vdash . (3) . *40\cdot12 . \quad \supset \vdash . \text{Prop}$

***73·801.** $\vdash : \text{Hp}\, *73\cdot8 . \supset . \beta - \text{G}\text{‘}R \subset p\text{‘}\kappa$

Here "Hp *73·8" means "the hypothesis of *73·8."

Dem.

$\vdash . *20\cdot33 . \supset \vdash :. \text{Hp} . \supset : \alpha \in \kappa . \supset_\alpha . \beta - \text{G}\text{‘}R \subset \alpha :. \supset \vdash . \text{Prop}$

***73·802.** $\vdash : \text{Hp}\, *73\cdot8 . \supset . \breve{R}\text{“}p\text{‘}\kappa \subset p\text{‘}\kappa$

Dem.

$\vdash . *20\cdot33 . \supset \vdash :. \text{Hp} . \supset : \alpha \in \kappa . \supset_\alpha . \breve{R}\text{“}\alpha \subset \alpha \quad (1)$

$\vdash . (1) . *40\cdot81 . \supset \vdash . \text{Prop}$

***73·81.** $\vdash : \text{Hp}\, *73\cdot8 . \supset . p\text{‘}\kappa \in \kappa$

Dem.

$\vdash . *73\cdot8\cdot801\cdot802 . \supset \vdash : \text{Hp} . \supset . p\text{‘}\kappa \subset \text{D}\text{‘}R . \beta - \text{G}\text{‘}R \subset p\text{‘}\kappa . \breve{R}\text{“}p\text{‘}\kappa \subset p\text{‘}\kappa : \supset \vdash . \text{Prop}$

* *Math. Annalen*, vol. LXV. Heft 2, February 1908.

∗73·811. $\vdash : \text{Hp} ∗73·8 . \supset . \breve{R}``p`\kappa \subset p`\kappa - (\beta - \text{Cl}`R)$

Dem.

$$\begin{aligned} &\vdash . ∗37·16 . \supset \vdash . \breve{R}``p`\kappa \subset \text{Cl}`R \\ &[∗22·8] \qquad \subset -(-\text{Cl}`R) \\ &[∗22·81·43] \qquad \subset -(\beta - \text{Cl}`R) \qquad (1) \\ &\vdash . (1) . ∗73·802 . \supset \vdash . \text{Prop} \end{aligned}$$

∗73·812. $\vdash : \text{Hp} ∗73·8 . x \sim\epsilon (\beta - \text{Cl}`R) \cup \breve{R}``p`\kappa . \supset . \breve{R}``(p`\kappa - \iota`x) \subset p`\kappa - \iota`x$

Dem.

$$\begin{aligned} &\vdash . ∗22·87 . \quad \supset \vdash : \text{Hp} . \supset . x \sim\epsilon \breve{R}``p`\kappa . \\ &[∗51·36] \qquad \supset . \breve{R}``p`\kappa \subset -\iota`x \qquad (1) \\ &\vdash . (1) . ∗73·802 . \supset \vdash : \text{Hp} . \supset . \breve{R}``p`\kappa \subset p`\kappa - \iota`x . \\ &[∗37·2] \qquad \supset . \breve{R}``(p`\kappa - \iota`x) \subset p`\kappa - \iota`x : \supset \vdash . \text{Prop} \end{aligned}$$

∗73·82. $\vdash : \text{Hp} ∗73·812 . \supset . p`\kappa - \iota`x = p`\kappa . x \sim\epsilon p`\kappa$

Dem.

$$\begin{aligned} &\vdash . ∗22·87 . ∗51·36 . \quad \supset \vdash : \text{Hp} . \supset . \beta - \text{Cl}`R \subset -\iota`x . \\ &[∗73·801] \qquad \supset . \beta - \text{Cl}`R \subset p`\kappa - \iota`x \qquad (1) \\ &\vdash . ∗73·8 . \quad \supset \vdash : \text{Hp} . \supset . p`\kappa - \iota`x \subset \text{D}`R \qquad (2) \\ &\vdash . (1) . (2) . ∗73·812 . \supset \vdash : \text{Hp} . \supset . p`\kappa - \iota`x \,\epsilon\, \kappa . \\ &[∗40·12] \qquad \supset . p`\kappa \subset p`\kappa - \iota`x . \\ &[∗51·36 . ∗22·43] \qquad \supset . x \sim\epsilon p`\kappa . p`\kappa - \iota`x = p`\kappa : \supset \vdash . \text{Prop} \end{aligned}$$

∗73·821. $\vdash : \text{Hp} ∗73·8 . x \,\epsilon\, p`\kappa - (\beta - \text{Cl}`R) . \supset . x \,\epsilon\, \breve{R}``p`\kappa$

Dem.

$$\begin{aligned} &\vdash . ∗73·82 . \text{Transp} . \supset \vdash : \text{Hp} ∗73·8 . x \,\epsilon\, p`\kappa . \supset . x \,\epsilon\, (\beta - \text{Cl}`R) \cup \breve{R}``p`\kappa \qquad (1) \\ &\vdash . (1) . ∗5·6 . \supset \vdash . \text{Prop} \end{aligned}$$

∗73·83. $\vdash : \text{Hp} ∗73·8 . \supset . p`\kappa - (\beta - \text{Cl}`R) = \breve{R}``p`\kappa . p`\kappa = (\beta - \text{Cl}`R) \cup \breve{R}``p`\kappa$

Dem.

$$\begin{aligned} &\vdash . ∗73·821 . \quad \supset \vdash : \text{Hp} . \supset . p`\kappa - (\beta - \text{Cl}`R) \subset \breve{R}``p`\kappa \qquad (1) \\ &\vdash . (1) . ∗73·811 . \quad \supset \vdash : \text{Hp} . \supset . p`\kappa - (\beta - \text{Cl}`R) = \breve{R}``p`\kappa \qquad (2) \\ &\vdash . (2) . ∗24·47 . ∗73·801 . \supset \vdash : \text{Hp} . \supset . p`\kappa = (\beta - \text{Cl}`R) \cup \breve{R}``p`\kappa \qquad (3) \\ &\vdash . (2) . (3) . \supset \vdash . \text{Prop} \end{aligned}$$

∗73·84. $\vdash : \text{Hp} ∗73·8 . \supset . \beta = p`\kappa \cup (\text{Cl}`R - \breve{R}``p`\kappa)$

Dem.

$$\begin{aligned} &\vdash . ∗22·92 . \supset \vdash : \text{Hp} . \supset . \beta = (\beta - \text{Cl}`R) \cup \text{Cl}`R \\ &[∗22·92 . ∗37·16] \qquad = (\beta - \text{Cl}`R) \cup \breve{R}``p`\kappa \cup (\text{Cl}`R - \breve{R}``p`\kappa) \\ &[∗73·83] \qquad = p`\kappa \cup (\text{Cl}`R - \breve{R}``p`\kappa) : \supset \vdash . \text{Prop} \end{aligned}$$

***73·841.** $\vdash : \mathrm{Hp}\, *73{\cdot}8 \,.\, R \in 1 \rightarrow 1 \,.\, \supset .\, \beta \,\mathrm{sm}\, \text{Ɑ}`R \,.\, \beta \,\mathrm{sm}\, \mathrm{D}`R$

Dem.

$\vdash . *73{\cdot}8{\cdot}21 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset .\, p`\kappa \,\mathrm{sm}\, \breve{R}``p`\kappa$ (1)

$\vdash . *24{\cdot}21 \,.\, \supset \vdash .\, \breve{R}``p`\kappa \cap (\text{Ɑ}`R - \breve{R}``p`\kappa) = \Lambda$ (2)

$\vdash . *73{\cdot}83 \,.\, *24{\cdot}492 \,.\, *73{\cdot}801 \,.\, \supset$

$\vdash : \mathrm{Hp} \,.\, \supset .\, p`\kappa - \breve{R}``p`\kappa = \beta - \text{Ɑ}`R \,.$

$[*24{\cdot}21] \supset .\, p`\kappa \cap (\text{Ɑ}`R - \breve{R}``p`\kappa) = \Lambda$ (3)

$\vdash . (1) . (2) . (3) . *73{\cdot}7 \,.\, \supset$

$\vdash : \mathrm{Hp} \,.\quad \supset .\, p`\kappa \cup (\text{Ɑ}`R - \breve{R}``p`\kappa) \,\mathrm{sm}\, \breve{R}``p`\kappa \cup (\text{Ɑ}`R - \breve{R}``p`\kappa) \,.$

$[*73{\cdot}84] \quad \supset .\, \beta \,\mathrm{sm}\, \breve{R}``p`\kappa \cup (\text{Ɑ}`R - \breve{R}``p`\kappa) \,.$

$[*22{\cdot}92 . *37{\cdot}16] \supset .\, \beta \,\mathrm{sm}\, \text{Ɑ}`R$ (4)

$\vdash . (4) . *73{\cdot}2 \,.\, \supset \vdash .\, \mathrm{Prop}$

***73·85.** $\vdash : R \in 1 \rightarrow 1 \,.\, \text{Ɑ}`R \subset \beta \,.\, \beta \subset \mathrm{D}`R \,.\, \supset .\, \beta \,\mathrm{sm}\, \text{Ɑ}`R \,.\, \beta \,\mathrm{sm}\, \mathrm{D}`R$ [*73·841]

***73·86.** $\vdash : \text{Ɑ}`R \subset \mathrm{D}`S \,.\, \text{Ɑ}`S \subset \mathrm{D}`R \,.\, \supset .$

$\mathrm{D}`(R \mid S) = \mathrm{D}`R \,.\, \text{Ɑ}`(R \mid S) \subset \text{Ɑ}`S \,.\, \text{Ɑ}`S \subset \mathrm{D}`(R \mid S)$

Dem.

$\vdash . *37{\cdot}321 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset .\, \mathrm{D}`(R \mid S) = \mathrm{D}`R$ (1)

$\vdash . *34{\cdot}36 \,.\, \supset \vdash : \text{Ɑ}`(R \mid S) \subset \text{Ɑ}`S$ (2)

$\vdash . (1) \,.\quad \supset \vdash : \mathrm{Hp} \,.\, \supset .\, \text{Ɑ}`S \subset \mathrm{D}`(R \mid S)$ (3)

$\vdash . (1) . (2) . (3) \,.\, \supset \vdash .\, \mathrm{Prop}$

***73·87.** $\vdash : R, S \in 1 \rightarrow 1 \,.\, \text{Ɑ}`R \subset \mathrm{D}`S \,.\, \text{Ɑ}`S \subset \mathrm{D}`R \,.\, \supset .\, \mathrm{D}`R \,\mathrm{sm}\, \mathrm{D}`S$

Dem.

$\vdash . *71{\cdot}252 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset .\, R \mid S \in 1 \rightarrow 1 \,.$

$[*73{\cdot}86{\cdot}85] \quad \supset .\, \text{Ɑ}`S \,\mathrm{sm}\, \mathrm{D}`R \,.$

$[*73{\cdot}2] \quad \supset .\, \mathrm{D}`S \,\mathrm{sm}\, \mathrm{D}`R : \supset \vdash .\, \mathrm{Prop}$

***73·88.** $\vdash : \alpha \,\mathrm{sm}\, \gamma \,.\, \beta \,\mathrm{sm}\, \delta \,.\, \gamma \subset \beta \,.\, \delta \subset \alpha \,.\, \supset .\, \alpha \,\mathrm{sm}\, \beta$

Dem.

$\vdash . *73{\cdot}1 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset .\, (\exists R, S) \,.\, R, S \in 1 \rightarrow 1 \,.\, \mathrm{D}`R = \alpha \,.\, \text{Ɑ}`R = \gamma \,.$

$\mathrm{D}`S = \beta \,.\, \text{Ɑ}`S = \delta \,.\, \gamma \subset \beta \,.\, \delta \subset \alpha \,.$

$[*73{\cdot}87] \quad \supset .\, (\exists R, S) \,.\, \mathrm{D}`R = \alpha \,.\, \mathrm{D}`S = \beta \,.\, \mathrm{D}`R \,\mathrm{sm}\, \mathrm{D}`S \,.$

$[*13{\cdot}22] \quad \supset .\, \alpha \,\mathrm{sm}\, \beta : \supset \vdash .\, \mathrm{Prop}$

This is the Schröder-Bernstein theorem.

*74. ON ONE-MANY AND MANY-ONE RELATIONS WITH LIMITED FIELDS

Summary of *74.

The purpose of the present number is to collect together various propositions in which we have such hypotheses as

$$R \upharpoonright \lambda \in 1 \rightarrow \mathrm{Cls},\ \kappa \rceil R \in 1 \rightarrow \mathrm{Cls},\ \text{etc.}$$

or in which such hypotheses are shown to be deducible from others. Hypotheses of this kind occur very frequently, and it is important to be able to deal with them easily. For the sake of completeness, we shall here repeat propositions previously proved on this subject.

The propositions of this number are mostly of the nature of lemmas, to be used in the theory of selections (Part II, Section D), and in cardinal and ordinal arithmetic. The most useful of them are *74·772·773·774·775. These propositions are concerned with circumstances under which $Q \parallel \breve{R}$ or $|\breve{R}$, with or without some limitation of the converse domain, is a one-one relation. The reason they are important is that the correlators by means of which many of the fundamental theorems of cardinal and ordinal arithmetic are proved are such relations as $Q \parallel \breve{R}$ (with the converse domain limited) for suitable values of Q and R. The above-mentioned propositions are as follows:

***74·772.** $\vdash :. (x) . \mathrm{E} ! Q\text{‘}x : (y) . \mathrm{E} ! R\text{‘}y : Q, R \in \mathrm{Cls} \rightarrow 1 : \supset . Q \parallel \breve{R} \in 1 \rightarrow 1$

The hypothesis of this proposition will be verified if we put, for example, $Q = R = \downarrow x$. Thus $(\downarrow x) \parallel (\mathrm{Cnv}\text{‘} \downarrow x) \in 1 \rightarrow 1$. This proposition is used in *116·531, which is used in proving one of the formal laws of exponentiation, namely $\mu^{\varpi} \times \nu^{\varpi} = (\mu \times \nu)^{\varpi}$.

***74·773.** $\vdash : Q \upharpoonright \alpha, R \upharpoonright \beta \in \mathrm{Cls} \rightarrow 1 . \alpha \subset \text{ᗡ}\text{‘}Q . \beta \subset \text{ᗡ}\text{‘}R . s\text{‘}\mathrm{D}\text{“}\lambda \subset \alpha . s\text{‘}\text{ᗡ}\text{“}\lambda \subset \beta . \supset .$

$$(Q \parallel \breve{R}) \upharpoonright \lambda \in 1 \rightarrow 1 . (Q \parallel \breve{R}) \upharpoonright \lambda \in \{(Q \parallel \breve{R})\text{“}\lambda\}\ \overline{\mathrm{sm}}\ \lambda$$

This proposition is used in connection with both cardinal and ordinal multiplication and exponentiation. If $Q \upharpoonright \alpha$ and $R \upharpoonright \beta$ correlate γ with α and δ with β, then if we take for λ the class of all ordinal couples that can be formed of an α and a β, $(Q \parallel \breve{R})\text{“}\lambda$ will be the class of all couples that can be formed of a γ and a δ. Thus in virtue of the above proposition, if γ is similar to α and δ is similar to β, the class of ordinal couples formed of a γ and a δ is similar to the class of ordinal couples formed of an α and a β. This result is useful because we define the product of the number of members of α and the number of members of β as the number of ordinal couples formed of an α and a β.

***74·774.** $\vdash :. R \in \mathrm{Cls} \rightarrow 1 : (y) . E ! R\text{‘}y : \supset . \breve{R} \in 1 \rightarrow 1$

This proposition is useful when, for example, R is $\downarrow x$.

***74·775.** $\vdash : Q \upharpoonright s\text{‘}\mathrm{D}\text{“}\lambda, R \upharpoonright s\text{‘}\text{Ɑ}\text{“}\lambda \in \mathrm{Cls} \rightarrow 1 . s\text{‘}\mathrm{D}\text{“}\lambda \subset \text{Ɑ}\text{‘}Q . s\text{‘}\text{Ɑ}\text{“}\lambda \subset \text{Ɑ}\text{‘}R . \supset .$

$$(Q \parallel \breve{R}) \upharpoonright \lambda \in 1 \rightarrow 1 . (Q \parallel \breve{R}) \upharpoonright \lambda \in \{(Q \parallel \breve{R})\text{“}\lambda\} \,\overline{\mathrm{sm}}\, \lambda$$

This is a particular case of *74·773, and has similar uses.

***74·1.** $\vdash :: R \upharpoonright \beta \in 1 \rightarrow \mathrm{Cls} . \supset :. R \upharpoonright \beta \in 1 \rightarrow 1 . \equiv : y, z \in \beta . R\text{‘}y = R\text{‘}z . \supset_{y,z} . y = z$

Dem.

$\vdash . *71\cdot55 . \supset \vdash :: \mathrm{Hp} .$

$\supset :. (R \upharpoonright \beta) \upharpoonright \beta \in 1 \rightarrow 1 . \equiv : y, z \in \beta . (R \upharpoonright \beta)\text{‘}y = (R \upharpoonright \beta)\text{‘}z . \supset_{y,z} . y = z :.$

$[*35\cdot31\cdot7] \supset :. R \upharpoonright \beta \in 1 \rightarrow 1 . \equiv : y, z \in \beta . R\text{‘}y = R\text{‘}z . \supset_{y,z} . y = z :: \supset \vdash . \mathrm{Prop}$

***74·11.** $\vdash :. R \upharpoonright \beta \in 1 \rightarrow \mathrm{Cls} . \beta \subset \text{Ɑ}\text{‘}R . \equiv : E !! R\text{“}\beta$ [*71·571 . (*37·05)]

***74·12.** $\vdash :: R \upharpoonright \beta \in 1 \rightarrow 1 . \beta \subset \text{Ɑ}\text{‘}R . \equiv :. y, z \in \beta . \supset_{y,z} : R\text{‘}y = R\text{‘}z . \equiv . y = z$ [*71·59]

***74·13.** $\vdash : R \in 1 \rightarrow \mathrm{Cls} . \supset . (\breve{R})_\epsilon \upharpoonright \mathrm{Cl}\text{‘}\mathrm{D}\text{‘}R \in 1 \rightarrow 1$ [*72·45]

***74·131.** $\vdash : R \in \mathrm{Cls} \rightarrow 1 . \supset . R_\epsilon \upharpoonright \mathrm{Cl}\text{‘}\text{Ɑ}\text{‘}R \in 1 \rightarrow 1$ [*72·451]

***74·14.** $\vdash : R \in 1 \rightarrow \mathrm{Cls} . \beta = \breve{R}\text{“}\alpha . \supset . \alpha \rceil R = R \upharpoonright \beta = \alpha \rceil R \upharpoonright \beta$ [*72·55]

***74·141.** $\vdash : R \in \mathrm{Cls} \rightarrow 1 . \alpha = R\text{“}\beta . \supset . \alpha \rceil R = R \upharpoonright \beta = \alpha \rceil R \upharpoonright \beta$ [*72·551]

***74·15.** $\vdash : Q \upharpoonright \lambda \in 1 \rightarrow \mathrm{Cls} . \lambda = \breve{Q}\text{“}\kappa . \supset . \kappa \cap \mathrm{D}\text{‘}Q = Q\text{“}\lambda$ [*72·57]

***74·151.** $\vdash : \kappa \rceil Q \in \mathrm{Cls} \rightarrow 1 . \kappa = Q\text{“}\lambda . \supset . \lambda \cap \text{Ɑ}\text{‘}Q = \breve{Q}\text{“}\kappa$

***74·16.** $\vdash : Q \upharpoonright \lambda \in 1 \rightarrow \mathrm{Cls} . \kappa \subset \mathrm{D}\text{‘}Q . \lambda = \breve{Q}\text{“}\kappa . \supset . \kappa = Q\text{“}\lambda$ [*74·15 . *22·621]

***74·161.** $\vdash : \kappa \rceil Q \in \mathrm{Cls} \rightarrow 1 . \lambda \subset \text{Ɑ}\text{‘}Q . \kappa = Q\text{“}\lambda . \supset . \lambda = \breve{Q}\text{“}\kappa$

***74·17.** $\vdash : Q \upharpoonright \breve{Q}\text{“}\kappa \in 1 \rightarrow \mathrm{Cls} . \kappa \subset \mathrm{D}\text{‘}Q . \supset . \kappa = Q\text{“}\breve{Q}\text{“}\kappa$ [*74·16]

***74·171.** $\vdash : (Q\text{“}\lambda) \rceil Q \in \mathrm{Cls} \rightarrow 1 . \lambda \subset \text{Ɑ}\text{‘}Q . \supset . \lambda = \breve{Q}\text{“}Q\text{“}\lambda$

***74·2.** $\vdash : \breve{Q}\text{“}\alpha \subset \beta . \supset . \alpha \rceil Q = \alpha \rceil Q \upharpoonright \beta$

Dem.

$\vdash . *37\cdot4 . \supset \vdash : \mathrm{Hp} . \supset . \text{Ɑ}\text{‘}(\alpha \rceil Q) \subset \beta .$

$[*35\cdot454] \qquad \supset . \alpha \rceil Q = \alpha \rceil Q \upharpoonright \beta : \supset \vdash . \mathrm{Prop}$

***74·201.** $\vdash : Q\text{“}\beta \subset \alpha . \supset . Q \upharpoonright \beta = \alpha \rceil Q \upharpoonright \beta$ [Similar proof]

***74·21.** $\vdash . \alpha \rceil Q = \alpha \rceil Q \upharpoonright \breve{Q}\text{“}\alpha$ [*74·2]

***74·211.** $\vdash . Q \upharpoonright \beta = (Q\text{“}\beta) \rceil Q \upharpoonright \beta$ [*74·201]

***74·22.** $\vdash : \mathrm{D}\text{‘}Q \subset \alpha . \supset . Q = \alpha \rceil Q$ [*35·451]

***74·221.** $\vdash : \text{Ɑ}\text{‘}Q \subset \beta . \supset . Q = Q \upharpoonright \beta$ [*35·452]

***74·23.** $\vdash : \alpha = Q``\breve{Q}``\alpha . \supset . \alpha \rceil Q = Q \upharpoonright \breve{Q}``\alpha = \alpha \rceil Q \upharpoonright \breve{Q}``\alpha$ [*74·21·211]

***74·231.** $\vdash : \beta = \breve{Q}``Q``\beta . \supset . Q \upharpoonright \beta = (Q``\beta) \rceil Q = (Q``\beta) \rceil Q \upharpoonright \beta$ [*74·21·211]

***74·24.** $\vdash : \alpha = Q``\beta . \beta = \breve{Q}``\alpha . \supset . \alpha \rceil Q = Q \upharpoonright \beta = \alpha \rceil Q \upharpoonright \beta$ [*74·23]

***74·25.** $\vdash : Q \upharpoonright \beta \in 1 \rightarrow \mathrm{Cls} . \alpha \subset \mathrm{D}`Q . \beta = \breve{Q}``\alpha . \supset . \alpha \rceil Q = Q \upharpoonright \beta = \alpha \rceil Q \upharpoonright \beta$
[*74·16·24]

***74·251.** $\vdash : \alpha \rceil Q \in \mathrm{Cls} \rightarrow 1 . \beta \subset \text{Ɑ}`Q . \alpha = Q``\beta . \supset . \alpha \rceil Q = Q \upharpoonright \beta = \alpha \rceil Q \upharpoonright \beta$
[*74·161·24]

***74·26.** $\vdash : Q \upharpoonright \beta \in 1 \rightarrow 1 . \alpha \subset \mathrm{D}`Q . \beta = \breve{Q}``\alpha . \equiv . \alpha \rceil Q \in 1 \rightarrow 1 . \beta \subset \text{Ɑ}`Q . \alpha = Q``\beta$

Dem.

$\vdash . *74·25 . \supset \vdash : Q \upharpoonright \beta \in 1 \rightarrow 1 . \alpha \subset \mathrm{D}`Q . \beta = \breve{Q}``\alpha . \supset . \alpha \rceil Q = Q \upharpoonright \beta .$
[*13·12] $\supset . \alpha \rceil Q \in 1 \rightarrow 1$ (1)

$\vdash . *37·16 . \supset \vdash : \beta = \breve{Q}``\alpha . \supset . \beta \subset \text{Ɑ}`Q$ (2)

$\vdash . *74·16 . \supset \vdash : Q \upharpoonright \beta \in 1 \rightarrow 1 . \alpha \subset \mathrm{D}`Q . \beta = \breve{Q}``\alpha . \supset . \alpha = Q``\beta$ (3)

$\vdash . (1) . (2) . (3) . \supset$

$\vdash : Q \upharpoonright \beta \in 1 \rightarrow 1 . \alpha \subset \mathrm{D}`Q . \beta = \breve{Q}``\alpha . \supset . \alpha \rceil Q \in 1 \rightarrow 1 . \beta \subset \text{Ɑ}`Q . \alpha = Q``\beta$ (4)

Similarly

$\vdash : \alpha \rceil Q \in 1 \rightarrow 1 . \beta \subset \text{Ɑ}`Q . \alpha = Q``\beta . \supset . Q \upharpoonright \beta \in 1 \rightarrow 1 . \alpha \subset \mathrm{D}`Q . \beta = \breve{Q}``\alpha$ (5)

$\vdash . (4) . (5) . \supset \vdash . \mathrm{Prop}$

***74·27.** $\vdash : Q \upharpoonright \beta \in 1 \rightarrow 1 . \beta = \breve{Q}``Q``\beta . \equiv . (Q``\beta) \rceil Q \in 1 \rightarrow 1 . \beta \subset \text{Ɑ}`Q$

Dem.

$\vdash . *74·26 \dfrac{Q``\beta}{\alpha} . \supset$

$\vdash : Q \upharpoonright \beta \in 1 \rightarrow 1 . Q``\beta \subset \mathrm{D}`Q . \beta = \breve{Q}``Q``\beta . \equiv .$
$(Q``\beta) \rceil Q \in 1 \rightarrow 1 . \beta \subset \text{Ɑ}`Q . Q``\beta = Q``\beta$ (1)

$\vdash . (1) . *37·15 . *20·2 . \supset \vdash . \mathrm{Prop}$

***74·271.** $\vdash : \alpha \rceil Q \in 1 \rightarrow 1 . \alpha = Q``\breve{Q}``\alpha . \equiv . Q \upharpoonright \breve{Q}``\alpha \in 1 \rightarrow 1 . \alpha \subset \mathrm{D}`Q$

$\left[*74·26 \dfrac{\breve{Q}``\alpha}{\beta}\right]$

***74·3.** $\vdash :. Q \upharpoonright \beta \in 1 \rightarrow \mathrm{Cls} : (\exists \alpha) . \beta = \breve{Q}``\alpha : \supset . \breve{Q}``Q``\beta = \beta$

Dem.

$\vdash . *74·15 . \supset \vdash : Q \upharpoonright \beta \in 1 \rightarrow \mathrm{Cls} . \beta = \breve{Q}``\alpha . \supset . \breve{Q}``Q``\beta = \breve{Q}``(\alpha \cap \mathrm{D}`Q)$
[*37·261] $= \breve{Q}``\alpha$
[Hp] $= \beta$ (1)

$\vdash . (1) . *10·11·23·35 . \supset \vdash . \mathrm{Prop}$

***74·301.** $\vdash :. \alpha \upharpoonleft Q \in \text{Cls} \to 1 : (\exists \beta) . \alpha = Q``\beta : \supset . Q``\breve{Q}``\alpha = \alpha$ [Similar proof]

***74·31.** $\vdash : Q \upharpoonright \beta \in 1 \to \text{Cls} . \beta \in \text{D}`(\breve{Q})_\epsilon . \supset .$

$$\beta = \breve{Q}``Q``\beta . \beta \subset \text{ᗡ}`Q . Q \upharpoonright \beta = (Q``\beta) \upharpoonleft Q . (Q``\beta) \upharpoonleft Q \in 1 \to \text{Cls}$$

Dem.

$\vdash . *74{\cdot}3 . *37{\cdot}23 . \supset \vdash : \text{Hp} . \supset . \beta = \breve{Q}``Q``\beta$ (1)

$\vdash . *37{\cdot}23{\cdot}16 . \quad \supset \vdash : \text{Hp} . \supset . \beta \subset \text{ᗡ}`Q$ (2)

$\vdash . (1) . *74{\cdot}231 . \quad \supset \vdash : \text{Hp} . \supset . Q \upharpoonright \beta = (Q``\beta) \upharpoonleft Q$ (3)

$[*13{\cdot}12] \qquad \supset . (Q``\beta) \upharpoonleft Q \in 1 \to \text{Cls}$ (4)

$\vdash . (1) . (2) . (3) . (4) . \supset \vdash . \text{Prop}$

***74·311.** $\vdash : \alpha \upharpoonleft Q \in \text{Cls} \to 1 . \alpha \in \text{D}`Q_\epsilon . \supset .$

$$\alpha = Q``\breve{Q}``\alpha . \alpha \subset \text{D}`Q . \alpha \upharpoonleft Q = Q \upharpoonright \breve{Q}``\alpha . Q \upharpoonright \breve{Q}``\alpha \in \text{Cls} \to 1$$

[Similar proof]

***74·32.** $\vdash : \kappa \subset \text{ᗡ}`R . R \upharpoonright \kappa \in \text{Cls} \to 1 . \supset . \overrightarrow{R} \upharpoonright \kappa \in 1 \to 1$

Dem.

$\vdash . *33{\cdot}41 . \supset \vdash :. \text{Hp} . \supset : y, z \in \kappa . \overrightarrow{R}`y = \overrightarrow{R}`z . \supset . (\exists x) . xRy . xRz .$

$[*35{\cdot}101] \qquad \supset . (\exists x) . x (R \upharpoonright \kappa) y . x (R \upharpoonright \kappa) z .$

$[*71{\cdot}171.\text{Hp}] \qquad \supset . y = z$ (1)

$\vdash . (1) . *71{\cdot}55 . \supset \vdash . \text{Prop}$

***74·4.** $\vdash : P \mid (Q \upharpoonright \lambda) = P \mid Q . \equiv . \breve{Q}``\text{ᗡ}`P \subset \lambda$

Dem.

$\vdash . *35{\cdot}23 . \supset \vdash : P \mid (Q \upharpoonright \lambda) = P \mid Q . \equiv . (P \mid Q) \upharpoonright \lambda = P \mid Q .$

$[*35{\cdot}66] \qquad \equiv . \text{ᗡ}`(P \mid Q) \subset \lambda .$

$[*37{\cdot}32] \qquad \equiv . \breve{Q}``\text{ᗡ}`P \subset \lambda : \supset \vdash . \text{Prop}$

***74·41.** $\vdash : \text{ᗡ}`P \cap \text{D}`Q \subset \kappa . \supset . P \mid \kappa \upharpoonleft Q = P \mid Q$

Dem.

$\vdash . *33{\cdot}13{\cdot}131 . *10{\cdot}23 . \supset$

$\vdash :. \text{Hp} . \qquad \equiv : xPy . yQz . \supset_{x,y,z} . y \in \kappa :$

$[*4{\cdot}71] \qquad \equiv : xPy . yQz . \equiv_{x,y,z} . xPy . yQz . y \in \kappa :$

$[*10{\cdot}281] \qquad \supset : (\exists y) . xPy . yQz . \equiv_{x,z} . (\exists y) . xPy . yQz . y \in \kappa :$

$[*34{\cdot}1 . *35{\cdot}1] \supset : x (P \mid Q) z . \equiv_{x,z} . x (P \mid \kappa \upharpoonleft Q) z :. \supset \vdash . \text{Prop}$

***74·42.** $\vdash : \text{ᗡ}`P \subset Q``\lambda . \supset . \text{D}`(P \mid Q \upharpoonright \lambda) = \text{D}`P$ $[*37{\cdot}321{\cdot}401]$

***74·43.** $\vdash : Q``\lambda \subset \text{ᗡ}`P . \supset . \text{ᗡ}`(P \mid Q \upharpoonright \lambda) = \text{ᗡ}`Q \cap \lambda$ $[*37{\cdot}322{\cdot}401 . *35{\cdot}64]$

***74·44.** $\vdash : \text{ᗡ}`P = Q``\lambda . \supset . \text{D}`(P \mid Q \upharpoonright \lambda) = \text{D}`P . \text{ᗡ}`(P \mid Q \upharpoonright \lambda) = \text{ᗡ}`Q \cap \lambda$

$[*74{\cdot}42{\cdot}43]$

***74·5.** $\vdash : E\,!\,(P \upharpoonright \beta)`y \,.\equiv .\, y \in \beta \,.\, E\,!\,P`y \,.\equiv .\, (P \upharpoonright \beta)`y = P`y$

Dem.

$\vdash . *35·7 . \quad \supset \vdash : x = (P \upharpoonright \beta)`y \,.\equiv .\, y \in \beta \,.\, x = P`y \qquad (1)$

$\vdash . (1) . *10·11·281 . \supset \vdash :. (\exists x) . x = (P \upharpoonright \beta)`y \,.\equiv :\, y \in \beta : (\exists x) . x = P`y :.$

$[*14·204] \quad \supset \vdash :. E\,!\,(P \upharpoonright \beta)`y \,.\equiv .\, y \in \beta \,.\, E\,!\,P`y \qquad (2)$

$\vdash . *35·7 . \quad \supset \vdash : (P \upharpoonright \beta)`y = P`y \,.\equiv .\, y \in \beta \,.\, P`y = P`y \,.$

$[*14·28] \quad \equiv .\, y \in \beta \,.\, E\,!\,P`y \qquad (3)$

$\vdash . (2) . (3) . \supset \vdash . \text{Prop}$

***74·51.** $\vdash :. \overrightarrow{P}`y \subset \alpha \,.\supset : E\,!\,(\alpha \upharpoonleft P)`y \,.\equiv .\, E\,!\,P`y \,.\equiv .\, P`y = (\alpha \upharpoonleft P)`y$

Dem.

$\vdash . *32·18 . *35·1 . \supset \vdash :. \text{Hp} . \supset : xPy \,.\equiv_x .\, x(\alpha \upharpoonleft P)y : \qquad (1)$

$[*30·34] \quad \supset : E\,!\,(\alpha \upharpoonleft P)`y \,.\equiv .\, E\,!\,P`y \qquad (2)$

$\vdash . (1) . *30·341 . \supset \vdash :. \text{Hp} . \supset : E\,!\,P`y \,.\equiv .\, P`y = (\alpha \upharpoonleft P)`y \qquad (3)$

$\vdash . (2) . (3) . \supset \vdash . \text{Prop}$

***74·511.** $\vdash :. \overleftarrow{P}`x \subset \beta \,.\supset : E\,!\,(\beta \upharpoonleft \breve{P})`x \,.\equiv .\, E\,!\,\breve{P}`x \,.\equiv .\, \breve{P}`x = (\beta \upharpoonleft \breve{P})`x$

[Proof as in *74·51]

***74·52.** $\vdash : (S``\beta) \upharpoonleft S \in 1 \rightarrow \text{Cls} \,.\, \beta \subset \text{Ɑ}`S \,.\, y \in \beta \,.\supset .\, \{(S``\beta) \upharpoonleft S\}`y = S`y \,.\, E\,!\,S`y$

Dem.

$\vdash . *37·18 . \supset \vdash : \text{Hp} . \supset . \overrightarrow{S}`y \subset S``\beta \qquad (1)$

$\vdash . *37·1 . \supset \vdash : \text{Hp} . \supset . (\exists x) . xSy \,.\, x \in S``\beta \,.$

$[*33·131] \quad \supset . y \in \text{Ɑ}`\{(S``\beta) \upharpoonleft S\} \,.$

$[*71·16] \quad \supset . E\,!\,\{(S``\beta) \upharpoonleft S\}`y \qquad (2)$

$\vdash . (1) . (2) . *74·51 . \supset \vdash . \text{Prop}$

***74·521.** $\vdash : S \upharpoonright \breve{S}``\beta \in \text{Cls} \rightarrow 1 \,.\, \beta \subset D`S \,.\, y \in \beta \,.\supset .\, \{(\breve{S}``\beta) \upharpoonleft \breve{S}\}`y = \breve{S}`y \,.\, E\,!\,\breve{S}`y$

$\left[*74·52 \dfrac{\breve{S}}{S} \right]$

***74·53.** $\vdash : (S``\beta) \upharpoonleft S \in 1 \rightarrow 1 \,.\, \beta \subset \text{Ɑ}`S \,.\, y \in \beta \,.\supset .\, \breve{S}`S`y = y$

Dem.

$\vdash . *37·1 . *33·131 . \supset \vdash : \text{Hp} . \supset . y \in \text{Ɑ}`\{(S``\beta) \upharpoonleft S\} \,.$

$[*72·241 . *35·51] \quad \supset . (\breve{S} \upharpoonright S``\beta)`\{(S``\beta) \upharpoonleft S\}`y = y \qquad (1)$

$\vdash . *74·52 . \quad \supset \vdash : \text{Hp} . \supset . \{(S``\beta) \upharpoonleft S\}`y = S`y \qquad (2)$

$\vdash . (1) . (2) . \quad \supset \vdash : \text{Hp} . \supset . (\breve{S} \upharpoonright S``\beta)`S`y = y \,.$

$[*35·7] \quad \supset . \breve{S}`S`y = y : \supset \vdash . \text{Prop}$

***74·531.** $\vdash : S \upharpoonright \breve{S}``\beta \in 1 \rightarrow 1 \,.\, \beta \subset D`S \,.\, y \in \beta \,.\supset .\, S`\breve{S}`y = y$

$\left[*74·53 \dfrac{\breve{S}}{S} \right]$

***74·6.** $\vdash :. T \epsilon 1 \rightarrow 1 . \lambda \subset \text{Cl}'\text{Ɑ}'T . \kappa \subset \text{Cl}'\text{D}'T . \supset : \kappa = T_\epsilon{}''\lambda . \equiv . \lambda = (\breve{T})_\epsilon{}''\kappa$

Dem.

$\vdash . *37{\cdot}421 . \supset \vdash : \text{Hp} . \supset . T_\epsilon{}''\lambda = (T_\epsilon \upharpoonright \text{Cl}'\text{Ɑ}'T)''\lambda .$

$(\breve{T})_\epsilon{}''\kappa = \{(\breve{T})_\epsilon \upharpoonright \text{Cl}'\text{D}'T\}''\kappa \quad (1)$

$\vdash . *72{\cdot}451{\cdot}52 . \supset$

$\vdash :. \text{Hp} . \supset : \kappa = (T_\epsilon \upharpoonright \text{Cl}'\text{Ɑ}'T)''\lambda . \equiv . \lambda = \{\text{Cnv}'(T_\epsilon \upharpoonright \text{Cl}'\text{Ɑ}'T)\}''\kappa .$

[*72·54] $\equiv . \lambda = \{(\breve{T})_\epsilon \upharpoonright \text{Cl}'\text{D}'T\}''\kappa \quad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***74·61.** $\vdash :. T \epsilon 1 \rightarrow 1 . \supset : \lambda \subset \text{Cl}'\text{Ɑ}'T . \kappa = T'''\lambda . \equiv . \kappa \subset \text{Cl}'\text{D}'T . \lambda = \breve{T}'''\kappa$

Dem.

$\vdash . *74{\cdot}6 . *37{\cdot}103 . \supset \vdash :. \text{Hp} . \supset : \kappa \subset \text{Cl}'\text{D}'T . \lambda \subset \text{Cl}'\text{Ɑ}'T . \kappa = T'''\lambda . \equiv .$

$\kappa \subset \text{Cl}'\text{D}'T . \lambda \subset \text{Cl}'\text{Ɑ}'T . \lambda = \breve{T}'''\kappa \quad (1)$

$\vdash . *37{\cdot}15{\cdot}16 . \supset \vdash : \kappa = T'''\lambda . \supset . \kappa \subset \text{Cl}'\text{D}'T : \lambda = \breve{T}'''\kappa . \supset . \lambda \subset \text{Cl}'\text{Ɑ}'T \quad (2)$

$\vdash . (1) . (2) . *4{\cdot}71 . \supset \vdash . \text{Prop}$

***74·62.** $\vdash :. y, z \epsilon \beta . y \neq z . \supset_{y,z} . \overrightarrow{S}'y \cap \overrightarrow{S}'z = \Lambda : \equiv . S \upharpoonright \beta \epsilon \text{Cls} \rightarrow 1$

Dem.

$\vdash . \text{Transp} . \supset \vdash :. y, z \epsilon \beta . y \neq z . \supset_{y,z} . \overrightarrow{S}'y \cap \overrightarrow{S}'z = \Lambda : \equiv :$

$y, z \epsilon \beta . \exists ! \overrightarrow{S}'y \cap \overrightarrow{S}'z . \supset_{y,z} . y = z :$

[*32·18] $\equiv : y, z \epsilon \beta . xSy . xSz . \supset_{x,y,z} . y = z :$

[*35·101] $\equiv : x(S \upharpoonright \beta) y . x (S \upharpoonright \beta) z . \supset_{x,y,z} . y = z :$

[*71·171] $\equiv : S \upharpoonright \beta \epsilon \text{Cls} \rightarrow 1 :. \supset \vdash . \text{Prop}$

***74·63.** $\vdash :. P, Q \epsilon \lambda . P \neq Q . \supset_{P,Q} . \text{D}'P \cap \text{D}'Q = \Lambda : \equiv . \epsilon \,|\, \text{D} \upharpoonright \lambda \epsilon \text{Cls} \rightarrow 1$
[*74·62 . *72·27]

***74·631.** $\vdash :. P, Q \epsilon \lambda . P \neq Q . \supset_{P,Q} . \text{Ɑ}'P \cap \text{Ɑ}'Q = \Lambda : \equiv . \epsilon \,|\, \text{Ɑ} \upharpoonright \lambda \epsilon \text{Cls} \rightarrow 1$
[*74·62 . *72·27]

***74·632.** $\vdash :. P, Q \epsilon \lambda . P \neq Q . \supset_{P,Q} . C'P \cap C'Q = \Lambda : \equiv . F \upharpoonright \lambda \epsilon \text{Cls} \rightarrow 1$
[*74·62 . *33·5]

***74·7.** $\vdash : Q \epsilon 1 \rightarrow \text{Cls} . P \,|\, Q = P' \,|\, Q . \supset . P \upharpoonright \text{D}'Q = P' \upharpoonright \text{D}'Q$

Dem.

$\vdash . *34{\cdot}27 . \supset \vdash : \text{Hp} . \supset . P \,|\, Q \,|\, \breve{Q} = P' \,|\, Q \,|\, \breve{Q} .$

[*72·59] $\supset . P \upharpoonright \text{D}'Q = P' \upharpoonright \text{D}'Q : \supset \vdash . \text{Prop}$

***74·701.** $\vdash : Q \epsilon \text{Cls} \rightarrow 1 . Q \,|\, P = Q \,|\, P' . \supset . (\text{Ɑ}'Q) \upharpoonleft P = (\text{Ɑ}'Q) \upharpoonleft P'$

***74·71.** $\vdash :. Q \epsilon 1 \rightarrow \text{Cls} . \text{Ɑ}'P \subset \text{D}'Q . \text{Ɑ}'P' \subset \text{D}'Q . \supset : P \,|\, Q = P' \,|\, Q . \equiv . P = P'$
[*74·7 . *35·66 . *34·28]

***74·711.** $\vdash :. Q \epsilon \text{Cls} \rightarrow 1 . \text{D}'P \subset \text{Ɑ}'Q . \text{D}'P' \subset \text{Ɑ}'Q . \supset : Q \,|\, P = Q \,|\, P' . \equiv . P = P'$

***74·72.** $\vdash :. Q \epsilon 1 \rightarrow \mathrm{Cls} : P \epsilon \lambda \,.\supset_P .\, \text{Ɑ}'P \subset \mathrm{D}'Q : \supset .\, (|Q) \upharpoonright \lambda \,\epsilon\, (|Q``\lambda) \,\overline{\mathrm{sm}}\, \lambda$

Dem.

$\vdash .\, *74·71 \,.\supset \vdash :: \mathrm{Hp} \,.\supset :.\, P, P' \epsilon \lambda \,.\supset_{P,P'} : P | Q = P' | Q \,.\equiv .\, P = P'$ (1)

$\vdash .\, (1) \,.\, *73·28 \,.\supset \vdash .\,$ Prop

***74·721.** $\vdash :. Q \epsilon \mathrm{Cls} \rightarrow 1 : P \epsilon \lambda \,.\supset_P .\, \mathrm{D}'P \subset \text{Ɑ}'Q : \supset .\, (Q|) \upharpoonright \lambda \,\epsilon\, (Q|``\lambda) \,\overline{\mathrm{sm}}\, \lambda$

***74·73.** $\vdash : Q \epsilon 1 \rightarrow \mathrm{Cls} \,.\, s'\text{Ɑ}``\lambda \subset \mathrm{D}'Q \,.\supset .\, (|Q) \upharpoonright \lambda \,\epsilon\, (|Q``\lambda) \,\overline{\mathrm{sm}}\, \lambda$
[*74·72 . *40·43]

***74·731.** $\vdash : Q \epsilon \mathrm{Cls} \rightarrow 1 \,.\, s'\mathrm{D}``\lambda \subset \text{Ɑ}'Q \,.\supset .\, (Q|) \upharpoonright \lambda \,\epsilon\, (Q|``\lambda) \,\overline{\mathrm{sm}}\, \lambda$

***74·74.** $\vdash : Q \epsilon 1 \rightarrow \mathrm{Cls} \,.\, \text{Ɑ}'s'\lambda \subset \mathrm{D}'Q \,.\supset .\, (|Q) \upharpoonright \lambda \,\epsilon\, (|Q``\lambda) \,\overline{\mathrm{sm}}\, \lambda$
[*74·73 . *41·44]

***74·741.** $\vdash : Q \epsilon \mathrm{Cls} \rightarrow 1 \,.\, \mathrm{D}'s'\lambda \subset \text{Ɑ}'Q \,.\supset .\, (Q|) \upharpoonright \lambda \,\epsilon\, (Q|``\lambda) \,\overline{\mathrm{sm}}\, \lambda$

***74·75.** $\vdash : \alpha \rceil Q \epsilon 1 \rightarrow \mathrm{Cls} \,.\, \alpha \subset \mathrm{D}'Q \,.\, s'\text{Ɑ}``\lambda \subset \alpha \,.\supset .\, (|Q) \upharpoonright \lambda \,\epsilon\, (|Q``\lambda) \,\overline{\mathrm{sm}}\, \lambda$

Dem.

$\vdash .\, *40·43 \,.\quad \supset \vdash :.\, \mathrm{Hp} \,.\supset : P \epsilon \lambda \,.\supset_P .\, \text{Ɑ}'P \subset \alpha \,.$

[*43·481] $\quad \supset_P .\, |Q'P = |(\alpha \rceil Q)'P :$

[*37·69] $\quad \supset : |Q``\lambda = |(\alpha \rceil Q)``\lambda$ (1)

$\vdash .\, *43·491 \,.\quad \supset \vdash : \mathrm{Hp} \,.\supset .\, (|Q) \upharpoonright \lambda = \{|(\alpha \rceil Q)\} \upharpoonright \lambda$ (2)

$\vdash .\, *74·73 \,.\, *35·62 \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, \{|(\alpha \rceil Q)\} \upharpoonright \lambda \,\epsilon\, \{|(\alpha \rceil Q)``\lambda\} \,\overline{\mathrm{sm}}\, \lambda$ (3)

$\vdash .\, (1) \,.\, (2) \,.\, (3) \,.\quad \supset \vdash .\,$ Prop

***74·751.** $\vdash : Q \upharpoonright \alpha \,\epsilon\, \mathrm{Cls} \rightarrow 1 \,.\, \alpha \subset \text{Ɑ}'Q \,.\, s'\mathrm{D}``\lambda \subset \alpha \,.\supset .\, (Q|) \upharpoonright \lambda \,\epsilon\, (Q|``\lambda) \,\overline{\mathrm{sm}}\, \lambda$
[Proof as in *74·75, using *74·731, *43·48·49]

***74·76.** $\vdash : Q \epsilon \mathrm{Cls} \rightarrow 1 \,.\, R \epsilon 1 \rightarrow \mathrm{Cls} \,.\, Q | P | R = Q | P' | R \,.\supset .$
$(\text{Ɑ}'Q) \rceil P \upharpoonright \mathrm{D}'R = (\text{Ɑ}'Q) \rceil P' \upharpoonright \mathrm{D}'R$ [*74·7·701]

***74·761.** $\vdash :.\, \mathrm{Hp}\, *74·76 \,.\, \mathrm{D}'P \subset \text{Ɑ}'Q \,.\, \text{Ɑ}'P \subset \mathrm{D}'R \,.\, \mathrm{D}'P' \subset \text{Ɑ}'Q \,.\, \text{Ɑ}'P' \subset \mathrm{D}'R \,.\supset :$
$Q | P | R = Q | P' | R \,.\equiv .\, P = P'$ [*74·71·711]

***74·77.** $\vdash : Q, R \epsilon 1 \rightarrow \mathrm{Cls} \,.\, s'\mathrm{D}``\lambda \subset \mathrm{D}'Q \,.\, s'\text{Ɑ}``\lambda \subset \mathrm{D}'R \,.\supset .$
$(\breve{Q} \| R) \upharpoonright \lambda \,\epsilon\, 1 \rightarrow 1 \,.\, (\breve{Q} \| R) \upharpoonright \lambda \,\epsilon\, \{(\breve{Q} \| R)``\lambda\} \,\overline{\mathrm{sm}}\, \lambda$

Dem.

$\vdash .\, *74·761 \dfrac{\breve{Q}}{Q} \,.\, *40·43 \,.\supset$

$\vdash :: \mathrm{Hp} \,.\supset :.\, P, P' \epsilon \lambda \,.\supset : \breve{Q} | P | R = \breve{Q} | P' | R \,.\equiv .\, P = P' :$

[*43·112] $\quad \supset : (\breve{Q} \| R)'P = (\breve{Q} \| R)'P' \,.\equiv .\, P = P'$ (1)

$\vdash .\, (1) \,.\, *73·28 \,.\supset \vdash .\,$ Prop

***74·771.** $\vdash : Q, R \epsilon \mathrm{Cls} \rightarrow 1 \,.\, s'\mathrm{D}``\lambda \subset \text{Ɑ}'Q \,.\, s'\text{Ɑ}``\lambda \subset \text{Ɑ}'R \,.\supset .$
$(Q \| \breve{R}) \upharpoonright \lambda \,\epsilon\, 1 \rightarrow 1 \,.\, (Q \| \breve{R}) \upharpoonright \lambda \,\epsilon\, \{(Q \| \breve{R})``\lambda\} \,\overline{\mathrm{sm}}\, \lambda$

$\left[*74·77 \dfrac{\breve{Q}, \breve{R}}{Q, R}\right]$

$*74{\cdot}772$ and its immediate successors are of very great use in cardinal and ordinal arithmetic.

$*74{\cdot}772$. $\vdash :. (x) . \text{E} ! Q\text{‘}x : (y) . \text{E} ! R\text{‘}y : Q, R \in \text{Cls} \to 1 : \supset . Q \| \breve{R} \in 1 \to 1$
[$*74{\cdot}771 . *33{\cdot}431$]

$*74{\cdot}773$. $\vdash : Q \upharpoonright \alpha, R \upharpoonright \beta \in \text{Cls} \to 1 . \alpha \subset \text{Ɑ}\text{‘}Q . \beta \subset \text{Ɑ}\text{‘}R . s\text{‘}\text{D}\text{“}\lambda \subset \alpha . s\text{‘}\text{Ɑ}\text{“}\lambda \subset \beta . \supset .$
$(Q \| \breve{R}) \upharpoonright \lambda \in 1 \to 1 . (Q \| \breve{R}) \upharpoonright \lambda \in \{(Q \| \breve{R})\text{“}\lambda\} \, \overline{\text{sm}} \, \lambda$

Dem.

$\vdash . *35{\cdot}64 . \supset \vdash : \text{Hp} . \supset . s\text{‘}\text{D}\text{“}\lambda \subset \text{Ɑ}\text{‘}(Q \upharpoonright \alpha) . s\text{‘}\text{Ɑ}\text{“}\lambda \subset \text{Ɑ}\text{‘}(R \upharpoonright \beta)$ (1)

$\vdash . *43{\cdot}51 . \supset \vdash : \text{Hp} . \supset . \{(Q \upharpoonright \alpha) \| (\beta \rceil \breve{R})\} \upharpoonright \lambda = (Q \| \breve{R}) \upharpoonright \lambda$ (2)

$\vdash . (1) . (2) . *74{\cdot}771 . \supset \vdash . \text{Prop}$

$*74{\cdot}774$. $\vdash :. R \in \text{Cls} \to 1 : (y) . \text{E} ! R\text{‘}y : \supset . | \breve{R} \in 1 \to 1$

Dem.

$\vdash . *71{\cdot}166 . \quad \supset \vdash : \text{Hp} . \supset . \breve{R} \in \text{Cls} \to 1$ (1)

$\vdash . *33{\cdot}431 . \quad \supset \vdash : \text{Hp} . \supset . (P) . \text{Ɑ}\text{‘}P \subset \text{D}\text{‘}\breve{R}$ (2)

$\vdash . (1) . (2) . *74{\cdot}71 \dfrac{\breve{R}}{Q} . \supset \vdash :. \text{Hp} . \supset : P | \breve{R} = P' | \breve{R} . \equiv_{P, P'} . P = P'$ (3)

$\vdash . (3) . *71{\cdot}57 . \supset \vdash . \text{Prop}$

$*74{\cdot}775$. $\vdash : Q \upharpoonright s\text{‘}\text{D}\text{“}\lambda, R \upharpoonright s\text{‘}\text{Ɑ}\text{“}\lambda \in \text{Cls} \to 1 . s\text{‘}\text{D}\text{“}\lambda \subset \text{Ɑ}\text{‘}Q . s\text{‘}\text{Ɑ}\text{“}\lambda \subset \text{Ɑ}\text{‘}R . \supset .$
$(Q \| \breve{R}) \upharpoonright \lambda \in 1 \to 1 . (Q \| \breve{R}) \upharpoonright \lambda \in \{(Q \| \breve{R})\text{“}\lambda\} \, \overline{\text{sm}} \, \lambda$ [$*74{\cdot}773$]

$*74{\cdot}8$. $\vdash : R \upharpoonright (\beta \cup \gamma) \in 1 \to \text{Cls} . \equiv . R \upharpoonright \beta, R \upharpoonright \gamma \in 1 \to \text{Cls}$

Dem.

$\vdash . *71{\cdot}572 . \supset \vdash : R \upharpoonright (\beta \cup \gamma) \in 1 \to \text{Cls} . \equiv : y \in \text{Ɑ}\text{‘}R \cap (\beta \cup \gamma) . \supset_y . \text{E} ! R\text{‘}y :$
[$*22{\cdot}68 . *10{\cdot}41$] $\quad \equiv : y \in \text{Ɑ}\text{‘}R \cap \beta . \supset_y . \text{E} ! R\text{‘}y : y \in \text{Ɑ}\text{‘}R \cap \gamma . \supset_y . \text{E} ! R\text{‘}y :$
[$*71{\cdot}572$] $\quad \equiv : R \upharpoonright \beta, R \upharpoonright \gamma \in 1 \to \text{Cls} :. \supset \vdash . \text{Prop}$

$*74{\cdot}801$. $\vdash : (\beta \cup \gamma) \rceil R \in \text{Cls} \to 1 . \equiv . \beta \rceil R, \gamma \rceil R \in \text{Cls} \to 1$

$*74{\cdot}81$. $\vdash : R \upharpoonright s\text{‘}\kappa \in 1 \to \text{Cls} . \equiv . R \upharpoonright \text{“}\kappa \subset 1 \to \text{Cls}$

Dem.

$\vdash . *71{\cdot}572 . \supset \vdash :. R \upharpoonright s\text{‘}\kappa \in 1 \to \text{Cls} . \equiv : y \in \text{Ɑ}\text{‘}R \cap s\text{‘}\kappa . \supset_y . \text{E} ! R\text{‘}y :$
[$*40{\cdot}11 . *10{\cdot}35{\cdot}23$] $\quad \equiv : \alpha \in \kappa . y \in \text{Ɑ}\text{‘}R \cap \alpha . \supset_{\alpha, y} . \text{E} ! R\text{‘}y :$
[$*11{\cdot}62 . *71{\cdot}572$] $\quad \equiv : \alpha \in \kappa . \supset_\alpha . R \upharpoonright \alpha \in 1 \to \text{Cls} :$
[$*37{\cdot}61$] $\quad \equiv : R \upharpoonright \text{“}\kappa \subset 1 \to \text{Cls} :. \supset \vdash . \text{Prop}$

$*74{\cdot}811$. $\vdash : (s\text{‘}\kappa) \rceil R \in \text{Cls} \to 1 . \equiv . \rceil R\text{“}\kappa \subset \text{Cls} \to 1$

$*74{\cdot}82$. $\vdash : (\beta \cup \gamma) \rceil R \in 1 \to \text{Cls} . \equiv . \beta \rceil R, \gamma \rceil R \in 1 \to \text{Cls} . \breve{R}\text{“}(\beta - \gamma) \cap \breve{R}\text{“}\gamma = \Lambda$

Dem.

$\vdash . *35{\cdot}1 . *71{\cdot}17 . \supset$
$\vdash :: (\beta \cup \gamma) \rceil R \in 1 \to \text{Cls} . \equiv :. x, y \in \beta \cup \gamma . xRz . yRz . \supset_{x,y,z} . x = y :.$
[$*13{\cdot}12$] $\quad \supset :. x \in \beta . y \in \gamma . xRz . yRz . \supset_{x,y,z} . x \in \gamma :.$

[Transp] $\supset :. x \in \beta - \gamma . xRz . \supset_{x,y,z} . \sim(y \in \gamma . yRz) :.$

[*10·21·252] $\supset :. x \in \beta - \gamma . xRz . \supset_{x,z} . \sim(\exists y) . y \in \gamma . yRz :.$

[*10·28.*37·105] $\supset :. z \in \breve{R}``(\beta - \gamma) . \supset_z . z \sim\in \breve{R}``\gamma :.$

[*24·39] $\supset :. \breve{R}``(\beta - \gamma) \cap \breve{R}``\gamma = \Lambda$ (1)

$\vdash . (1) . *71·22 . \supset$

$\vdash : (\beta \cup \gamma) \rceil R \in 1 \rightarrow \mathrm{Cls} . \supset . \beta \rceil R, \gamma \rceil R \in 1 \rightarrow \mathrm{Cls} . \breve{R}``(\beta - \gamma) \cap \breve{R}``\gamma = \Lambda$ (2)

$\vdash . *71·22 . \supset \vdash : \beta \rceil R \in 1 \rightarrow \mathrm{Cls} . \supset . (\beta - \gamma) \rceil R \in 1 \rightarrow \mathrm{Cls}$ (3)

$\vdash . *37·4 . \supset \vdash : \breve{R}``(\beta - \gamma) \cap \breve{R}``\gamma = \Lambda . \supset . \text{ᗡ}`(\beta - \gamma) \rceil R \cap \text{ᗡ}`(\gamma \rceil R) = \Lambda$ (4)

$\vdash . (3) . (4) . *71·24 . \supset \vdash : \beta \rceil R, \gamma \rceil R \in 1 \rightarrow \mathrm{Cls} . \breve{R}``(\beta - \gamma) \cap \breve{R}``\gamma = \Lambda . \supset .$

$(\beta - \gamma) \rceil R \mathbin{\dot\cup} \gamma \rceil R \in 1 \rightarrow \mathrm{Cls} .$

[*35·41] $\supset . (\beta \cup \gamma) \rceil R \in 1 \rightarrow \mathrm{Cls}$ (5)

$\vdash . (2) . (5) . \supset \vdash .$ Prop

***74·821.** $\vdash : R \upharpoonleft (\beta \cup \gamma) \in \mathrm{Cls} \rightarrow 1 . \equiv .$

$R \upharpoonleft \beta, R \upharpoonleft \gamma \in \mathrm{Cls} \rightarrow 1 . R``(\beta - \gamma) \cap R``\gamma = \Lambda$

***74·822.** $\vdash : (\beta \cup \gamma) \rceil R \in 1 \rightarrow 1 . \equiv . \beta \rceil R, \gamma \rceil R \in 1 \rightarrow 1 . \breve{R}``(\beta - \gamma) \cap \breve{R}``\gamma = \Lambda$
[*74·82·801]

***74·823.** $\vdash : R \upharpoonleft (\beta \cup \gamma) \in 1 \rightarrow 1 . \equiv . R \upharpoonleft \beta, R \upharpoonleft \gamma \in 1 \rightarrow 1 . R``(\beta - \gamma) \cap R``\gamma = \Lambda$
[*74·8·821]

***74·83.** $\vdash :. \breve{R}``\beta \cap \breve{R}``\gamma = \Lambda . \supset : (\beta \cup \gamma) \rceil R \in 1 \rightarrow \mathrm{Cls} . \equiv . \beta \rceil R, \gamma \rceil R \in 1 \rightarrow \mathrm{Cls}$
[*74·82]

***74·831.** $\vdash :. R``\beta \cap R``\gamma = \Lambda . \supset : R \upharpoonleft (\beta \cup \gamma) \in \mathrm{Cls} \rightarrow 1 . \equiv . R \upharpoonleft \beta, R \upharpoonleft \gamma \in \mathrm{Cls} \rightarrow 1$

***74·832.** $\vdash :. \breve{R}``\beta \cap \breve{R}``\gamma = \Lambda . \supset : (\beta \cup \gamma) \rceil R \in 1 \rightarrow 1 . \equiv . \beta \rceil R, \gamma \rceil R \in 1 \rightarrow 1$
[*74·83·801]

***74·833.** $\vdash :. R``\beta \cap R``\gamma = \Lambda . \supset : R \upharpoonleft (\beta \cup \gamma) \in 1 \rightarrow 1 . \equiv . R \upharpoonleft \beta, R \upharpoonleft \gamma \in 1 \rightarrow 1$
[*74·8·831]

***74·84.** $\vdash :. (s`\kappa) \rceil R \in 1 \rightarrow \mathrm{Cls} . \equiv :$

$\rceil R``\kappa \subset 1 \rightarrow \mathrm{Cls} : \beta, \gamma \in \kappa . \supset_{\beta,\gamma} . \breve{R}``(\beta - \gamma) \cap \breve{R}``\gamma = \Lambda$

Dem.

$\vdash . *40·13 . *35·43 . \supset \vdash : \beta \in \kappa . \supset . \beta \rceil R \mathrel{\dot\subset} (s`\kappa) \rceil R :$

[*71·22] $\supset \vdash :. (s`\kappa) \rceil R \in 1 \rightarrow \mathrm{Cls} . \supset : \beta \in \kappa . \supset . \beta \rceil R \in 1 \rightarrow \mathrm{Cls} :$

[*37·61] $\supset : \rceil R``\kappa \subset 1 \rightarrow \mathrm{Cls}$ (1)

$\vdash . *72·41 . *37·421 . \supset \vdash :. (s`\kappa) \rceil R \in 1 \rightarrow \mathrm{Cls} . \supset :$

$\beta, \gamma \in \kappa . \supset_{\beta,\gamma} . \breve{R}``(\beta - \gamma) \cap \breve{R}``\gamma = \Lambda$ (2)

$\vdash . *37·105 . *24·39 . \supset$

$\vdash :. \beta, \gamma \in \kappa . \supset_{\beta,\gamma} . \breve{R}``(\beta - \gamma) \cap \breve{R}``\gamma = \Lambda : \equiv :$

$\beta, \gamma \in \kappa . x \in \beta - \gamma . xRz . \supset_{\beta,\gamma} . \sim(\exists y) . y \in \gamma . yRz :$

[Transp] $\supset : \beta, \gamma \in \kappa \,.\, x \in \beta \,.\, y \in \gamma \,.\, xRz \,.\, yRz \,.\, \supset_{\beta,\gamma} \,.\, x \in \gamma \,.$

[*4·7] $\supset_{\beta,\gamma} \,.\, x, y \in \gamma \,.\, xRz \,.\, yRz \,.$

[*35·1] $\supset_{\beta,\gamma} \,.\, x\,(\gamma \rceil R)\,z \,.\, y\,(\gamma \rceil R)\,z$ (3)

$\vdash . (3) . *71{\cdot}17 . \supset \vdash :. \beta, \gamma \in \kappa \,.\, \supset_{\beta,\gamma} .\, \breve{R}``(\beta - \gamma) \cap \breve{R}``\gamma = \Lambda : \rceil R``\kappa \subset 1 \rightarrow \mathrm{Cls} : \supset :$

$\beta, \gamma \in \kappa \,.\, x \in \beta \,.\, y \in \gamma \,.\, xRz \,.\, yRz \,.\, \supset_{\beta,\gamma,x,y,z} \,.\, x = y :$

[*10·23.*40·11.*37·1] $\supset : x\,\{(s`\kappa) \rceil R\}\,z \,.\, y\,\{(s`\kappa) \rceil R\}\,z \,.\, \supset_{x,y,z} \,.\, x = y :$

[*71·17] $\supset : s`\kappa \rceil R \in 1 \rightarrow \mathrm{Cls}$ (4)

$\vdash . (1) . (2) . (4) . \supset \vdash . \mathrm{Prop}$

***74·841.** $\vdash :. R \upharpoonright s`\kappa \in \mathrm{Cls} \rightarrow 1 \,.\, \equiv :$

$R \upharpoonright ``\kappa \subset \mathrm{Cls} \rightarrow 1 : \beta, \gamma \in \kappa \,.\, \supset_{\beta,\gamma} .\, R``(\beta - \gamma) \cap R``\gamma = \Lambda$

***74·842.** $\vdash :. (s`\kappa) \rceil R \in 1 \rightarrow 1 \,.\, \equiv :$

$\rceil R``\kappa \subset 1 \rightarrow 1 : \beta, \gamma \in \kappa \,.\, \supset_{\beta,\gamma} .\, \breve{R}``(\beta - \gamma) \cap \breve{R}``\gamma = \Lambda$ [*74·84·811]

***74·843.** $\vdash :. R \upharpoonright s`\kappa \in 1 \rightarrow 1 \,.\, \equiv :$

$R \upharpoonright ``\kappa \subset 1 \rightarrow 1 : \beta, \gamma \in \kappa \,.\, \supset_{\beta,\gamma} .\, R``(\beta - \gamma) \cap R``\gamma = \Lambda$ [*74·81·841]

SECTION D

SELECTIONS

Summary of Section D.

The subject to be considered in this section is important chiefly in connection with multiplication, both cardinal and ordinal. In order to get a definition of multiplication which is not confined to the case where the number of factors is finite, we have to seek a construction by which, from a given class of classes, κ say, we construct another class which, when κ is finite, has that number of terms which, in the usual elementary sense, is the product of the numbers of terms in the various classes which are members of κ, and which, whether κ is finite or not, obeys as many as possible of the formal laws of multiplication. The usual elementary sense of multiplication is derived from addition; that is to say, $\mu \times \nu$ is to be the number of terms in $s{\text{‘}}\kappa$, where κ is a class of μ mutually exclusive classes each having ν members, or vice versa. This sense can be extended to any finite number of factors, but not to an infinite number of factors; hence for a number of factors which may be infinite we require a different definition, and this is derived from the theory of *selections*.

Selections are of two kinds, selections from classes of classes, and selections from relations. The latter is the more general notion, from which the former is derived. But as the former is an easier notion, we will begin by explaining selections from classes of classes.

Given a class of classes κ, a class μ is called a *selected class* of κ when μ is formed by choosing one term out of each member of κ. For example, if κ consists of two members, α and β, and if $x \in \alpha$ and $y \in \beta$, then $\iota{\text{‘}}x \cup \iota{\text{‘}}y$ is a selected class of κ. If every constituency elects a local man, Parliament is a selected class of the constituencies. If κ is a class of mutually exclusive classes, *i.e.* a class no two of whose members have any member in common, then a selected class consists of only one term from each member of κ; *i.e.* μ is a selected class if

$$\mu \subset s{\text{‘}}\kappa : \alpha \in \kappa . \supset_\alpha . \mu \cap \alpha \in 1.$$

But if κ is not a class of mutually exclusive classes, this does not hold necessarily; for a term x which is a member of both α and β (where $\alpha, \beta \in \kappa$) may be chosen as the representative of α, while some other term may be chosen as the representative of β, so that two members of β may belong to the selected class. Again, if κ is a class of mutually exclusive classes, the relation of the representative to its class must be one-one, because, since no term belongs to two classes which are members of κ, no term can be the

representative of two classes. But when κ is not a class of mutually exclusive classes, a term which belongs to two classes α and β may be chosen as the representative of both. Thus the relation of the representative to its class may be only one-many, not one-one.

The relation of the representative to its class may be called a *selective relation*. A selective relation of κ is one which selects, from every class α which is a member of κ, a certain member x as the *representative* of α; that is, we have, if R is the selective relation,

$$\alpha \epsilon \kappa . \supset_\alpha . R`\alpha \epsilon \alpha : \text{Cl}`R = \kappa.$$

This condition is equivalent to

$$R \epsilon 1 \rightarrow \text{Cls} . R \subset \epsilon . \text{Cl}`R = \kappa.$$

If R is a selective relation, $\text{D}`R$ is a selected class; and if $\mu$ is a selected class, there is a selective relation $R$ such that $\mu = \text{D}`R$. Thus the study of selections from classes of classes is wholly contained in the study of selective relations.

The class of selective relations from a class κ is called $\epsilon_\Delta`\kappa$. Thus

$$R \epsilon \epsilon_\Delta`\kappa . \equiv . R \epsilon 1 \rightarrow \text{Cls} . R \subset \epsilon . \text{Cl}`R = \kappa,$$

and

$$\epsilon_\Delta`\kappa = (1 \rightarrow \text{Cls}) \cap \text{Rl}`\epsilon \cap \overleftarrow{\text{Cl}}`\kappa.$$

Then $\text{D}``\epsilon_\Delta`\kappa$ is the class of selected classes.

It will be seen that, if $\alpha \epsilon \kappa$, $R`\alpha$ may be any member of α, and we get a different R for each different member of α. Thus if we keep the representatives of all the other members of κ unchanged, the number of selective relations to be obtained by varying the representative of α is the number of members of α. Hence the number of selective relations altogether may be fitly defined as the product of the numbers of terms possessed by the various members of κ. In case κ is finite, this agrees with the usual definition of multiplication; and whether κ is finite or infinite, the product so defined obeys all the formal laws of multiplication.

To illustrate the notion of selective relations, let us take a very simple case, the case where κ consists of two classes α and β, each of which has two members. Let x and y be the members of α, z and w the members of β. We assume $\alpha \neq \beta$, $x \neq y$, $z \neq w$. Then the selective relations of κ are the following:

$$x \downarrow \alpha \cup z \downarrow \beta,$$
$$x \downarrow \alpha \cup w \downarrow \beta,$$
$$y \downarrow \alpha \cup z \downarrow \beta,$$
$$y \downarrow \alpha \cup w \downarrow \beta.$$

Thus they are four in number, *i.e.* the number of members of $\epsilon_\Delta`\kappa$ is the product of the number of members of α and the number of members of β. A similar process would show that our definition of the product agrees with the usual definition in any case in which all the numbers concerned are finite.

Selections from *relations* are an obvious generalization of selections from classes of classes. We had above

$$\epsilon_\Delta{}^{\backprime}\kappa = (1 \rightarrow \mathrm{Cls}) \cap \mathrm{Rl}{}^{\backprime}\epsilon \cap \overleftarrow{\mathrm{C\!I}}{}^{\backprime}\kappa.$$

We put, generally,

$$P_\Delta{}^{\backprime}\kappa = (1 \rightarrow \mathrm{Cls}) \cap \mathrm{Rl}{}^{\backprime}P \cap \overleftarrow{\mathrm{C\!I}}{}^{\backprime}\kappa,$$

which we derive from the definition

$$P_\Delta = \hat{\lambda}\hat{\kappa}\{\lambda = (1 \rightarrow \mathrm{Cls}) \cap \mathrm{Rl}{}^{\backprime}P \cap \overleftarrow{\mathrm{C\!I}}{}^{\backprime}\kappa\} \quad \mathrm{Df.}$$

This is the fundamental definition in the subject of selections. We have, in virtue of this definition,

$$\vdash : R \epsilon P_\Delta{}^{\backprime}\kappa \,.\equiv.\, R \epsilon 1 \rightarrow \mathrm{Cls} \,.\, R \subseteq P \,.\, \mathrm{C\!I}{}^{\backprime}R = \kappa.$$

When $\kappa = \mathrm{C\!I}{}^{\backprime}P$, we may call $P_\Delta{}^{\backprime}\kappa$ the class of selections from P. Thus generally, $P_\Delta{}^{\backprime}\kappa$ is the class of selections from $P \upharpoonright \kappa$ provided $\kappa \subset \mathrm{C\!I}{}^{\backprime}P$; and if this condition is not fulfilled, $P_\Delta{}^{\backprime}\kappa = \Lambda$. We may call the class $P_\Delta{}^{\backprime}\kappa$ the class of "P-selections from κ." The class of "ϵ-selections from κ" will be what we previously called the class of "selective relations of κ."

It will be observed that we have

$$R \epsilon P_\Delta{}^{\backprime}\kappa \,.\, y \epsilon \kappa \,.\supset.\, R{}^{\backprime}y \epsilon \overrightarrow{P}{}^{\backprime}y.$$

Thus if $\overrightarrow{P}{}^{\backprime\backprime}\kappa$ is a class of mutually exclusive classes, $\mathrm{D}{}^{\backprime}R$ selects one from each of these classes, and is therefore a selective class of $\overrightarrow{P}{}^{\backprime\backprime}\kappa$; hence in this case

$$\mathrm{D}{}^{\backprime\backprime}P_\Delta{}^{\backprime}\kappa = \mathrm{D}{}^{\backprime\backprime}\epsilon_\Delta{}^{\backprime}\overrightarrow{P}{}^{\backprime\backprime}\kappa.$$

In Cardinal Arithmetic, $\epsilon_\Delta{}^{\backprime}\kappa$ is the important notion, and the more general notion $P_\Delta{}^{\backprime}\kappa$ is seldom required. In Ordinal Arithmetic, $F_\Delta{}^{\backprime}\kappa$ is the important notion. It will be seen that

$$R \epsilon F_\Delta{}^{\backprime}\kappa \,.\equiv.\, R \epsilon 1 \rightarrow \mathrm{Cls} \,.\, R \subseteq F \,.\, \mathrm{C\!I}{}^{\backprime}R = \kappa.$$

Thus $F_\Delta{}^{\backprime}\kappa$ is only significant when κ is a class of relations; in this case we have

$$R \epsilon F_\Delta{}^{\backprime}\kappa \,.\, Q \epsilon \kappa \,.\supset.\, R{}^{\backprime}Q \epsilon C{}^{\backprime}Q.$$

Thus R chooses a representative member of the field of every member of κ. The most important case is when κ is of the form $C{}^{\backprime}P$, where P is a serial relation whose field consists of serial relations. Then $F_\Delta{}^{\backprime}C{}^{\backprime}P$ becomes the field of a relation which may be defined as the ordinal product of the relations composing $C{}^{\backprime}P$; in this way we get an infinite ordinal product analogous to the infinite cardinal product. This will be explained at a later stage (*172).

Although it is chiefly $\epsilon_\Delta{}^{\backprime}\kappa$ and $F_\Delta{}^{\backprime}\kappa$ that will be required in the sequel, we shall treat $P_\Delta{}^{\backprime}\kappa$ generally, because this introduces little extra complication, and most of the theorems which hold for $\epsilon_\Delta{}^{\backprime}\kappa$ or $F_\Delta{}^{\backprime}\kappa$ have exact analogues for $P_\Delta{}^{\backprime}\kappa$.

P_Δ'κ, as above defined, is the class of one-many relations contained in P and having κ for their converse domain. We know of no proof that there always are such relations when $\kappa \subset \mathrm{C}\text{'}P$. In fact, the proposition

$$\kappa \subset \mathrm{C}\text{'}P \,.\, \supset_{P,\kappa} .\, \exists !\, P_\Delta\text{'}\kappa$$

is equivalent to the "multiplicative axiom," *i.e.* to the axiom that, given any class of mutually exclusive classes, none of which is null, there is at least one class formed of one member from each of these classes. (This equivalence is proved in *88·36, below.) It is also equivalent to Zermelo's axiom*, which is

$$(\alpha) \,.\, \exists !\, \epsilon_\Delta\text{'}\mathrm{Cl\,ex}\text{'}\alpha\,;$$

hence also it is equivalent to the proposition that every class can be well-ordered. In the absence of evidence as to the truth or falsehood of these various propositions, we shall not assume their truth, but shall explicitly introduce them as hypotheses wherever they are relevant.

In the present section, we shall begin (*80) by considering such properties of P_Δ'κ as do not depend upon any hypothesis as to P. We shall then (*81) proceed to consider such further properties of P_Δ'κ as result from the hypothesis $P \upharpoonright \kappa \in \mathrm{Cls} \rightarrow 1$. This hypothesis is important, because it is verified in many of the applications we wish to make, and because it leads to important properties of P_Δ'κ which are not true in general when P is not subject to any hypothesis. These special properties are mostly due to the fact that when $P \upharpoonright \kappa$ is a many-one relation, P_Δ'κ consists of one-one relations (not merely of one-many relations, as it does in the general case). This is proved in *81·1. We then (*82) proceed to consider the case of relative products, *i.e.* $(P \mid Q)_\Delta$'λ. It will appear that, with a suitable hypothesis, $(P \mid Q)_\Delta\text{'}\lambda = \mid Q\text{''}P_\Delta\text{'}Q\text{''}\lambda$ and $\mathrm{D}\text{''}(P \mid Q)_\Delta\text{'}\lambda = \mathrm{D}\text{''}P_\Delta\text{'}Q\text{''}\lambda$. In the following number (*83) we apply the results of *80 to the particular case where P is replaced by ϵ, which is the important case for cardinal arithmetic. In *84 we apply the propositions of *81 to the case where P is replaced by ϵ, and where, therefore, we have the hypothesis $\epsilon \upharpoonright \kappa \in \mathrm{Cls} \rightarrow 1$. This hypothesis is equivalent to the hypothesis that no two members of κ have any members in common, *i.e.* that

$$\alpha, \beta \in \kappa \,.\, \alpha \neq \beta \,.\, \supset_{\alpha,\beta} .\, \alpha \cap \beta = \Lambda.$$

When κ fulfils this hypothesis, it is a class of mutually exclusive classes. For classes of mutually exclusive classes we adopt the notation "$\mathrm{Cls}^2\,\mathrm{excl}$." It is shown in *84·14 that a $\mathrm{Cls}^2\,\mathrm{excl}$ is one for which we have $\epsilon \upharpoonright \kappa \in \mathrm{Cls} \rightarrow 1$. When κ is a $\mathrm{Cls}^2\,\mathrm{excl}$, $\mathrm{D} \upharpoonright P_\Delta$'$\kappa$ is a one-one relation, and $\mathrm{D}\text{''}\epsilon_\Delta\text{'}\kappa \,\mathrm{sm}\, \epsilon_\Delta\text{'}\kappa$. Also in this case $\mathrm{D}\text{''}\epsilon_\Delta$'$\kappa$ consists of all classes formed of one member from each member of κ, *i.e.* all classes μ such that

$$\mu \subset s\text{'}\kappa : \alpha \in \kappa \,.\, \supset_\alpha .\, \mu \cap \alpha \in 1.$$

* See his "Beweis, dass jede Menge wohlgeordnet werden kann," *Math. Annalen*, Vol. LIX. pp. 514—516.

In $*85$, we prove various important propositions, of which the chief is a form of the associative law*, namely

$$\vdash : \kappa \in \mathrm{Cls}^2 \, \mathrm{excl} \, . \supset . \, \epsilon_\Delta{}^{\backprime} s^{\backprime}\kappa \; \mathrm{sm} \; \epsilon_\Delta{}^{\backprime}\epsilon_\Delta{}^{\backprime\backprime}\kappa .$$

Finally, in $*88$, we consider the question of the existence of selections. This cannot in general be proved when κ is an infinite class. The assumption that $\epsilon_\Delta{}^{\backprime}\kappa$ is never null unless one member of κ is null is equivalent to various other assumptions, for example to the assumption that every class can be well-ordered. One of these equivalent assumptions is called the "multiplicative axiom." This axiom is equivalent to the assumption that an arithmetical product cannot be zero unless one of its factors is zero, and is regarded by some mathematicians as a self-evident truth. This can be proved when the number of factors is finite, *i.e.* when κ is a finite class, but not when the number of factors is infinite. We have not assumed its truth in the general case where it cannot be proved, but have included it in the hypotheses of all propositions which depend upon it.

* Cf. notes to $*42 \cdot 1 \cdot 11$.

*80. ELEMENTARY PROPERTIES OF SELECTIONS

Summary of *80.

In this number, we shall give such properties of P_Δ as follow most directly from the definition, without any restrictive hypothesis as to P.

If $R \in P_\Delta\text{‘}\kappa$, R selects one member of $\overrightarrow{P}\text{‘}y$, whenever $y \in \kappa$, as the selected referent of y. For, since $R \in 1 \rightarrow \mathrm{Cls} \,.\, \text{Ɑ}\text{‘}R = \kappa$, we have $y \in \kappa \,.\supset.\, E\,!\,R\text{‘}y$; and since $R \mathrel{\dot{\subset}} P$, we have $y \in \kappa \,.\supset.\, (R\text{‘}y)\,Py$, *i.e.* $y \in \kappa \,.\supset.\, R\text{‘}y \in \overrightarrow{P}\text{‘}y$. Calling $R\text{‘}y$ the *selected referent* of y, it is evident that we may replace $R\text{‘}y$ by any other member of $\overrightarrow{P}\text{‘}y$, and still have a member of $P_\Delta\text{‘}\kappa$. (This is proved in *80·4.) Thus if $P_\Delta\text{‘}\kappa$ has any members at all, we can get as many members as there are members of $\overrightarrow{P}\text{‘}y$ by merely altering the selected referent of y, leaving the other selected referents unchanged.

In the present section, we first prove various simple properties of $P_\Delta\text{‘}\kappa$. Most of these are almost immediate consequences of

***80·14.** $\vdash : R \in P_\Delta\text{‘}\kappa \,.\equiv.\, R \in 1 \rightarrow \mathrm{Cls} \,.\, R \mathrel{\dot{\subset}} P \,.\, \text{Ɑ}\text{‘}R = \kappa$

The most useful of them are

***80·2.** $\vdash : \exists\,!\,P_\Delta\text{‘}\kappa \,.\supset.\, \kappa \subset \text{Ɑ}\text{‘}P$

***80·291.** $\vdash : R \in P_\Delta\text{‘}\kappa \,.\supset.\, R \mathrel{\dot{\subset}} P \upharpoonright \kappa$

***80·3.** $\vdash : R \in P_\Delta\text{‘}\kappa \,.\, y \in \kappa \,.\supset.\, E\,!\,R\text{‘}y$

***80·33.** $\vdash : R \in P_\Delta\text{‘}\kappa \,.\supset\dot{}.\, \mathrm{D}\text{‘}R \subset P\text{‘‘}\kappa$

We then have various propositions (*80·4—·46) concerned with $x \downarrow y$ when xPy. Of these the most important are the following:

***80·41.** $\vdash : R \in P_\Delta\text{‘}\kappa \,.\, y \in \kappa \,.\, x'Py \,.\supset.\, [\{R \mathbin{\dot{-}} (R\text{‘}y) \downarrow y\} \mathbin{\dot{\cup}} x' \downarrow y] \in P_\Delta\text{‘}\kappa$

I.e. given a selective relation R, the selected referent of y (where $y \in \text{Ɑ}\text{‘}P$) may be replaced by any other term having the relation P to y, and we shall still have a selective relation.

***80·45.** $\vdash .\, P_\Delta\text{‘}\iota\text{‘}y = \downarrow y\text{‘‘}\overrightarrow{P}\text{‘}y$

We then have a set of propositions (*80·5—·54) connecting $(P \mathbin{\dot{\cup}} Q)_\Delta\text{‘}(\kappa \cup \lambda)$ with $P_\Delta\text{‘}\kappa$ and $Q_\Delta\text{‘}\lambda$. These are chiefly useful as leading to the next set (*80·6—·69), connecting $P_\Delta\text{‘}(\kappa \cup \lambda)$ with $P_\Delta\text{‘}\kappa$ and $P_\Delta\text{‘}\lambda$. The most useful of these are the following:

***80·6.** $\vdash : R \in P_\Delta\text{‘}\kappa \,.\, \lambda \subset \kappa \,.\supset.\, R \upharpoonright \lambda \in P_\Delta\text{‘}\lambda$

***80·65.** $\vdash : \kappa \cap \lambda = \Lambda \,.\, R \in P_\Delta\text{‘}\kappa \,.\, S \in P_\Delta\text{‘}\lambda \,.\supset.\, R \mathbin{\dot{\cup}} S \in P_\Delta\text{‘}(\kappa \cup \lambda)$

***80·66.** $\vdash :. \kappa \cap \lambda = \Lambda \,.\supset:\, M \in P_\Delta\text{‘}(\kappa \cup \lambda) \,.\equiv.\, (\exists R, S) \,.\, R \in P_\Delta\text{‘}\kappa \,.\, S \in P_\Delta\text{‘}\lambda \,.\, M = R \mathbin{\dot{\cup}} S$

We have next a set of propositions (*80·7—·78) dealing with the relations of M and $M \dot{-} R$ when (*e.g.*) $M \in P_\Delta`(\kappa \cup \lambda)$ and $R \in P_\Delta`\kappa$. These propositions are seldom used, but they would be useful in considering division.

We next have a set of propositions (*80·8—·84) dealing with the relations of $P_\Delta`\alpha$ and $P_\Delta`\beta$. The most useful are

*80·81. $\vdash : \exists ! P_\Delta`\alpha . P_\Delta`\alpha = P_\Delta`\beta . \supset . \alpha = \beta$

*80·82. $\vdash : \alpha \neq \beta . \supset . P_\Delta`\alpha \cap P_\Delta`\beta = \Lambda$

Finally, we have four propositions (*80·9—·93) on $P_\Delta`(\iota`y \cup \iota`z)$ and one on $P_\Delta`(\beta \cup \iota`z)$. The most useful of these is

*80·9. $\vdash :. y \neq z . \supset : M \in P_\Delta`(\iota`y \cup \iota`z) . \equiv . (\exists u, v) . uPy . vPz . M = u \downarrow y \cup v \downarrow z$

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*80·01. $P_\Delta = \hat{\lambda} \hat{\kappa} \{\lambda = (1 \to \text{Cls}) \cap \text{Rl}`P \cap \overleftarrow{\text{Cl}}`\kappa\}$ Df

*80·1. $\vdash : \lambda P_\Delta \kappa . \equiv . \lambda = (1 \to \text{Cls}) \cap \text{Rl}`P \cap \overleftarrow{\text{Cl}}`\kappa$ [*21·3 . (*80·01)]

*80·11. $\vdash . P_\Delta`\kappa = (1 \to \text{Cls}) \cap \text{Rl}`P \cap \overleftarrow{\text{Cl}}`\kappa$ [*80·1 . *30·3]

*80·12. $\vdash . \text{E} ! P_\Delta`\kappa$ [*80·11 . *14·21]

*80·13. $\vdash : \lambda P_\Delta \kappa . \equiv . \lambda = P_\Delta`\kappa$ [*80·12 . *30·4]

*80·14. $\vdash : R \in P_\Delta`\kappa . \equiv . R \in 1 \to \text{Cls} . R \mathrel{\dot{\subset}} P . \text{Cl}`R = \kappa$
[*80·11 . *20·43 . *22·33 . *61·2 . *33·61]

*80·15. $\vdash : P \mathrel{\dot{\subset}} Q . \supset . P_\Delta`\kappa \subset Q_\Delta`\kappa$ [*80·14]

*80·16. $\vdash : R \in P_\Delta`\kappa . R \mathrel{\dot{\subset}} Q . \supset . R \in Q_\Delta`\kappa$

Dem.

$\vdash . *80·14 . \supset \vdash : R \in P_\Delta`\kappa . \supset . R \in 1 \to \text{Cls} . \text{Cl}`R = \kappa :$

[Fact] $\supset \vdash : R \in P_\Delta`\kappa . R \mathrel{\dot{\subset}} Q . \supset . R \in 1 \to \text{Cls} . \text{Cl}`R = \kappa . R \mathrel{\dot{\subset}} Q .$

[*80·14] $\supset . R \in Q_\Delta`\kappa : \supset \vdash . \text{Prop}$

*80·17. $\vdash : Q \mathrel{\dot{\subset}} P . \supset . Q_\Delta`\kappa = P_\Delta`\kappa \cap \text{Rl}`Q$

Dem.

$\vdash . *80·15 . \supset \vdash : \text{Hp} . \supset . Q_\Delta`\kappa \subset P_\Delta`\kappa$ (1)

$\vdash . *80·11 . \supset \vdash . Q_\Delta`\kappa \subset \text{Rl}`Q$ (2)

$\vdash . (1) . (2) . \supset \vdash : \text{Hp} . \supset . Q_\Delta`\kappa \subset P_\Delta`\kappa \cap \text{Rl}`Q$ (3)

$\vdash . *80·16 . \supset \vdash . P_\Delta`\kappa \cap \text{Rl}`Q \subset Q_\Delta`\kappa$ (4)

$\vdash . (3) . (4) . \supset \vdash . \text{Prop}$

This proposition is used in the theory of ordinal multiplication (*172·162).

*80·2. $\vdash : \exists ! P_\Delta`\kappa . \supset . \kappa \subset \text{Cl}`P$

Dem.

$\vdash . *80·14 . \supset \vdash : R \in P_\Delta`\kappa . \supset . R \mathrel{\dot{\subset}} P . \text{Cl}`R = \kappa .$

[*33·264] $\supset . \text{Cl}`R \subset \text{Cl}`P . \text{Cl}`R = \kappa .$

[*13·13] $\supset . \kappa \subset \text{Cl}`P$ (1)

$\vdash . (1) . *10·11·23 . \supset \vdash . \text{Prop}$

***80·21.** $\vdash : \sim(\kappa \subset \text{ᗡ}'P) . \supset . P_\Delta{}'\kappa = \Lambda$ [*80·2 . Transp]

***80·22.** $\vdash : P \upharpoonright \kappa = Q \upharpoonright \kappa . \supset . P_\Delta{}'\kappa = Q_\Delta{}'\kappa$

Dem.

$\vdash . *33\cdot14 . \supset \vdash :: \text{ᗡ}'R = \kappa . \supset :. xRy . \supset . y \in \kappa :.$

[*5·44] $\supset :. xRy . \supset . xPy : \equiv : xRy . \supset . xPy . y \in \kappa :$

[*35·101] $\equiv : xRy . \supset . x (P \upharpoonright \kappa) y$ (1)

$\vdash . (1) . *11\cdot11\cdot3\cdot33 . \supset$

$\vdash :. \text{ᗡ}'R = \kappa . \supset : R \mathrel{\text{Є}} P . \equiv . R \mathrel{\text{Є}} P \upharpoonright \kappa$ (2)

$\vdash . (2) \dfrac{Q}{P} . \supset \vdash :. \text{ᗡ}'R = \kappa . \supset : R \mathrel{\text{Є}} Q . \equiv . R \mathrel{\text{Є}} Q \upharpoonright \kappa$ (3)

$\vdash . (2) . (3) . *13\cdot12 . \supset \vdash :. \text{ᗡ}'R = \kappa . P \upharpoonright \kappa = Q \upharpoonright \kappa . \supset : R \mathrel{\text{Є}} P . \equiv . R \mathrel{\text{Є}} Q$ (4)

$\vdash . (4) . \text{Comm} . *5\cdot32 . \supset$

$\vdash :. \text{Hp} . \supset : R \mathrel{\text{Є}} P . \text{ᗡ}'R = \kappa . \equiv . R \mathrel{\text{Є}} Q . \text{ᗡ}'R = \kappa :$

[*80·14] $\supset : R \in P_\Delta{}'\kappa . \equiv . R \in Q_\Delta{}'\kappa :. \supset \vdash . \text{Prop}$

***80·23.** $\vdash . P_\Delta{}'\kappa = (P \upharpoonright \kappa)_\Delta{}'\kappa$

Dem.

$\vdash . *35\cdot31 . *22\cdot5 . \supset \vdash . P \upharpoonright \kappa = (P \upharpoonright \kappa) \upharpoonright \kappa$ (1)

$\vdash . (1) . *80\cdot22 . \supset \vdash . \text{Prop}$

***80·24.** $\vdash : \kappa \subset \text{ᗡ}'P . Q = P \upharpoonright \kappa . \supset . P_\Delta{}'\kappa = Q_\Delta{}'\text{ᗡ}'Q$ [*35·65 . *80·23]

***80·25.** $\vdash : \exists ! P_\Delta{}'\kappa . Q = P \upharpoonright \kappa . \supset . P_\Delta{}'\kappa = Q_\Delta{}'\text{ᗡ}'Q$ [*80·2·24]

***80·26.** $\vdash . P_\Delta{}'\Lambda = \iota'\dot\Lambda$

Dem.

$\vdash . *80\cdot14 . \supset \vdash : R \in P_\Delta{}'\Lambda . \equiv . R \in 1 \rightarrow \text{Cls} . R \mathrel{\text{Є}} P . \text{ᗡ}'R = \Lambda .$

[*33·241] $\equiv . R \in 1 \rightarrow \text{Cls} . R \mathrel{\text{Є}} P . R = \dot\Lambda .$

[*13·193] $\equiv . \dot\Lambda \in 1 \rightarrow \text{Cls} . \dot\Lambda \mathrel{\text{Є}} P . R = \dot\Lambda .$

[*72·1.*25·12] $\equiv . R = \dot\Lambda .$

[*51·15] $\equiv . R \in \iota'\dot\Lambda : \supset \vdash . \text{Prop}$

Note that $P_\Delta{}'\Lambda$ is a unit class, not the null-class. It is owing to this fact (as will appear later) that, if μ is any cardinal, $\mu^0 = 1$. See the note to *83·15.

***80·27.** $\vdash : \exists ! \kappa . \supset . \dot\Lambda_\Delta{}'\kappa = \Lambda$

Dem.

$\vdash . *80\cdot14 . \supset \vdash : R \in \dot\Lambda_\Delta{}'\kappa . \supset . R \mathrel{\text{Є}} \dot\Lambda . \text{ᗡ}'R = \kappa .$

[*25·13] $\supset . R = \dot\Lambda . \text{ᗡ}'R = \kappa .$

[*33·241] $\supset . \kappa = \Lambda$ (1)

$\vdash . (1) . \text{Transp} . *10\cdot11\cdot21 . \supset$

$\vdash : \exists ! \kappa . \supset . (R) . R \sim \in \dot\Lambda_\Delta{}'\kappa .$

[*24·15] $\supset . \dot\Lambda_\Delta{}'\kappa = \Lambda : \supset \vdash . \text{Prop}$

***80·28.** $\vdash : \exists ! \kappa . \supset . \dot{\Lambda} \sim \epsilon P_\Delta ‘\kappa$

Dem.

$\vdash . *80·14 . \supset \vdash :. \exists ! \kappa . \supset : R \epsilon P_\Delta ‘\kappa . \supset_R . \exists ! \text{Ɑ}‘R :$

$[*33·241] \quad \supset : R \epsilon P_\Delta ‘\kappa . \supset_R . \dot{\exists} ! R :$

$[*25·63] \quad \supset : \dot{\Lambda} \sim \epsilon P_\Delta ‘\kappa :. \supset \vdash . \text{Prop}$

***80·29.** $\vdash : R \epsilon P_\Delta ‘\kappa . \supset . R = R \upharpoonright \kappa$

Dem.

$\vdash . *80·14 . \supset \vdash : \text{Hp} . \supset . \text{Ɑ}‘R = \kappa .$

$[*35·452] \quad \supset . R = R \upharpoonright \kappa : \supset \vdash . \text{Prop}$

***80·291.** $\vdash : R \epsilon P_\Delta ‘\kappa . \supset . R \mathrel{\dot{\subset}} P \upharpoonright \kappa$

Dem.

$\vdash . *80·14 . *33·14 . \supset$

$\vdash :. \text{Hp} . \supset : xRy . \supset_{x,y} . xPy . y \epsilon \kappa .$

$[*35·101] \quad \supset_{x,y} . x (P \upharpoonright \kappa) y :. \supset \vdash . \text{Prop}$

***80·3.** $\vdash : R \epsilon P_\Delta ‘\kappa . y \epsilon \kappa . \supset . E ! R‘y$

Dem.

$\vdash . *80·14 . \supset \vdash : \text{Hp} . \supset . R \epsilon 1 \rightarrow \text{Cls} . y \epsilon \text{Ɑ}‘R .$

$[*71·163] \quad \supset . E ! R‘y : \supset \vdash . \text{Prop}$

***80·31.** $\vdash : R \epsilon P_\Delta ‘\kappa . y \epsilon \kappa . \supset . R‘y \epsilon \overrightarrow{P}‘y$

Dem.

$\vdash . *80·14 . \supset \vdash : \text{Hp} . \supset . R \epsilon 1 \rightarrow \text{Cls} . R \mathrel{\dot{\subset}} P . y \epsilon \text{Ɑ}‘R .$

$[*71·31] \quad \supset . R \mathrel{\dot{\subset}} P . (R‘y) Ry .$

$[*23·441] \quad \supset . (R‘y) Py .$

$[*32·18] \quad \supset . R‘y \epsilon \overrightarrow{P}‘y : \supset \vdash . \text{Prop}$

***80·32.** $\vdash :. R \epsilon P_\Delta ‘\kappa . \supset : y \epsilon \kappa . \equiv . E ! R‘y . \equiv . R‘y \epsilon \overrightarrow{P}‘y$

Dem.

$\vdash . *80·14 . \supset \vdash :. \text{Hp} . \supset : \text{Ɑ}‘R = \kappa :$

$[*33·43] \quad \supset : E ! R‘y . \supset . y \epsilon \kappa \quad (1)$

$\vdash . *14·21 . \supset \vdash : R‘y \epsilon \overrightarrow{P}‘y . \supset . E ! R‘y :$

$[(1)] \quad \supset \vdash :. \text{Hp} . \supset : R‘y \epsilon \overrightarrow{P}‘y . \supset . y \epsilon \kappa \quad (2)$

$\vdash . (1) . (2) . *80·3·31 . \supset \vdash . \text{Prop}$

***80·33.** $\vdash : R \epsilon P_\Delta ‘\kappa . \supset . D‘R \subset P‘‘\kappa$

Dem.

$\vdash . *80·14 . *37·25 . \supset \vdash : \text{Hp} . \supset . D‘R = R‘‘\kappa . R \mathrel{\dot{\subset}} P .$

$[*37·201] \quad \supset . D‘R \subset P‘‘\kappa : \supset \vdash . \text{Prop}$

***80·34.** $\vdash : R \epsilon P_\Delta ‘\kappa . \supset . E !! R‘‘\kappa . R‘‘\kappa = D‘R$

Dem.

$\vdash . *80·14 . \supset \vdash : \text{Hp} . \supset . R \epsilon 1 \rightarrow \text{Cls} . \text{Ɑ}‘R = \kappa .$

$[*71·16 . *37·25] \quad \supset . E !! R‘‘\kappa . R‘‘\kappa = D‘R : \supset \vdash . \text{Prop}$

***80·35.** $\vdash : R \epsilon P_\Delta{}^\backprime\kappa \,.\supset .\, \text{D}^\backprime R = \hat{x}\{(\exists y) \,.\, y \epsilon \kappa \,.\, x = R^\backprime y\}$ [*37·6 . *80·34]

***80·36.** $\vdash : R, S \epsilon P_\Delta{}^\backprime\kappa \,.\supset .\, R \upharpoonright \alpha \cup S \upharpoonright -\alpha \epsilon P_\Delta{}^\backprime\kappa$

Dem.

$\vdash . *71·26 . \quad \supset \vdash : \text{Hp} \,.\supset .\, R \upharpoonright \alpha, S \upharpoonright -\alpha \epsilon 1 \to \text{Cls}$ (1)

$\vdash . *35·64 . \quad \supset \vdash . \,\text{C}\!\text{I}^\backprime(R \upharpoonright \alpha) \cap \text{C}\!\text{I}^\backprime(S \upharpoonright -\alpha) = \Lambda$ (2)

$\vdash . (1) . (2) . *71·24 . \supset \vdash : \text{Hp} \,.\supset .\, R \upharpoonright \alpha \cup S \upharpoonright -\alpha \epsilon 1 \to \text{Cls}$ (3)

$\vdash . *35·64 . *80·14 . \quad \supset \vdash : \text{Hp} \,.\supset .\, \text{C}\!\text{I}^\backprime(R \upharpoonright \alpha) = \kappa \cap \alpha \,.\, \text{C}\!\text{I}^\backprime(S \upharpoonright -\alpha) = \kappa - \alpha \,.$

[*24·41] $\quad \supset .\, \text{C}\!\text{I}^\backprime(R \upharpoonright \alpha \cup S \upharpoonright -\alpha) = \kappa$ (4)

$\vdash . *35·441 . *80·14 . \supset \vdash : \text{Hp} \,.\supset .\, R \upharpoonright \alpha \subseteq P \,.\, S \upharpoonright -\alpha \subseteq P \,.$

[*23·59] $\quad \supset .\, R \upharpoonright \alpha \cup S \upharpoonright -\alpha \subseteq P$ (5)

$\vdash . (3) . (4) . (5) . *80·14 . \supset \vdash . \text{Prop}$

This proposition is used in dealing with greater and less among cardinals (*117·68).

***80·4.** $\vdash : R \epsilon P_\Delta{}^\backprime\kappa \,.\, y \epsilon \kappa \,.\, xRy \,.\, x'Py \,.\supset .\, \{(R \dot{-} x \downarrow y) \cup x' \downarrow y\} \epsilon P_\Delta{}^\backprime\kappa$

This proposition is important. It shows that, if $R \epsilon P_\Delta{}^\backprime\kappa$ and x is the selected referent of y (*i.e.* is $R^\backprime y$), then x may be replaced by any other member of $\overrightarrow{P}{}^\backprime y$ without our ceasing to have a member of $P_\Delta{}^\backprime\kappa$.

Dem.

$\vdash . *55·3 . \quad \supset \vdash :. \text{Hp} \,.\supset : x \downarrow y \subseteq R :$

[*72·91] $\quad \supset : \text{C}\!\text{I}^\backprime(R \dot{-} x \downarrow y) = \text{C}\!\text{I}^\backprime R - \text{C}\!\text{I}^\backprime(x \downarrow y)$

[*80·14.*55·15] $\quad = \kappa - \iota^\backprime y$ (1)

$\vdash . (1) . *33·261 . \supset \vdash : \text{Hp} . \supset . \,\text{C}\!\text{I}^\backprime\{(R \dot{-} x \downarrow y) \cup x' \downarrow y\} = (\kappa - \iota^\backprime y) \cup \text{C}\!\text{I}^\backprime x' \downarrow y$

[*55·15] $\quad = (\kappa - \iota^\backprime y) \cup \iota^\backprime y$

[*51·221] $\quad = \kappa$ (2)

$\vdash . (1) . *55·15 . \quad \supset \vdash : \text{Hp} \,.\supset .\, \text{C}\!\text{I}^\backprime(R \dot{-} x \downarrow y) \cap \text{C}\!\text{I}^\backprime(x' \downarrow y) = (\kappa - \iota^\backprime y) \cap \iota^\backprime y$

[*24·21] $\quad = \Lambda \,.$

[*71·24.*80·14] $\quad \supset .\, (R \dot{-} x \downarrow y) \cup x' \downarrow y \epsilon 1 \to \text{Cls}$ (3)

$\vdash . *80·14 . *55·3 . \supset \vdash : \text{Hp} \,.\supset .\, R \dot{-} x \downarrow y \subseteq P \,.\, x' \downarrow y \subseteq P \,.$

[*23·59] $\quad \supset .\, (R \dot{-} x \downarrow y) \cup x' \downarrow y \subseteq P$ (4)

$\vdash . (2) . (3) . (4) . *80·14 . \supset \vdash . \text{Prop}$

***80·41.** $\vdash : R \epsilon P_\Delta{}^\backprime\kappa \,.\, y \epsilon \kappa \,.\, x'Py \,.\supset\!\!\dot{}\; [\{R \dot{-} (R^\backprime y) \downarrow y\} \cup x' \downarrow y] \epsilon P_\Delta{}^\backprime\kappa$

Dem.

$\vdash . *80·3 . *30·32 . \supset \vdash : \text{Hp} \,.\supset .\, (R^\backprime y)\, Ry$ (1)

$\vdash . (1) . *80·4 . \supset \vdash . \text{Prop}$

***80·42.** $\vdash : \exists ! P_\Delta{}^\backprime\kappa \,.\supset .\, \dot{s}^\backprime P_\Delta{}^\backprime\kappa = P \upharpoonright \kappa$

Dem.

$\vdash . *41·11 . \supset \vdash : x (\dot{s}^\backprime P_\Delta{}^\backprime\kappa) y \,.\equiv .\, (\exists R) \,.\, R \epsilon P_\Delta{}^\backprime\kappa \,.\, xRy \,.$

[*80·14] $\quad \supset .\, xPy \,.\, y \epsilon \kappa \,.$

[*35·101] $\quad \supset .\, x (P \upharpoonright \kappa) y$ (1)

$\vdash . *80{\cdot}41 . *35{\cdot}101 . \supset$

$\vdash : R \in P_\Delta{}^\epsilon\kappa . x(P \upharpoonright \kappa)y . \supset . [\{R \dot{-} (R^\epsilon y) \downarrow y\} \cup x \downarrow y] \in P_\Delta{}^\epsilon\kappa .$

$[*55{\cdot}132] \supset . [\{R \dot{-} (R^\epsilon y) \downarrow y\} \cup x \downarrow y] \in P_\Delta{}^\epsilon\kappa . x[\{R \dot{-} (R^\epsilon y) \downarrow y\} \cup x \downarrow y] y .$

$[*41{\cdot}141] \supset . x(\dot{s}^\epsilon P_\Delta{}^\epsilon\kappa) y$ (2)

$\vdash . (2) . \text{Exp} . *11{\cdot}11{\cdot}3 . \supset \vdash : R \in P_\Delta{}^\epsilon\kappa . \supset . P \upharpoonright \kappa \subset \dot{s}^\epsilon P_\Delta{}^\epsilon\kappa$ (3)

$\vdash . (3) . *10{\cdot}11{\cdot}23 . \quad \supset \vdash : \exists ! P_\Delta{}^\epsilon\kappa . \supset . P \upharpoonright \kappa \subset \dot{s}^\epsilon P_\Delta{}^\epsilon\kappa$ (4)

$\vdash . (1) . (4) . \supset \vdash . \text{Prop}$

$*80{\cdot}43$. $\vdash : xPy . \equiv . x \downarrow y \in P_\Delta{}^\epsilon\iota^\epsilon y$

Dem.

$\vdash . *72{\cdot}182 . *55{\cdot}15 . \supset \vdash . x \downarrow y \in 1 \to \text{Cls} . \text{Œ}^\epsilon x \downarrow y = \iota^\epsilon y$ (1)

$\vdash . *55{\cdot}3 . \qquad \supset \vdash : xPy . \equiv . x \downarrow y \subset P$ (2)

$\vdash . (1) . (2) . *4{\cdot}73 . \supset \vdash : xPy . \equiv . x \downarrow y \subset P . x \downarrow y \in 1 \to \text{Cls} . \text{Œ}^\epsilon(x \downarrow y) = \iota^\epsilon y .$

$[*80{\cdot}14] \qquad \equiv . x \downarrow y \in P_\Delta{}^\epsilon\iota^\epsilon y : \supset \vdash . \text{Prop}$

$*80{\cdot}44$. $\vdash : R \in P_\Delta{}^\epsilon\iota^\epsilon y . \supset . R = (R^\epsilon y) \downarrow y$

Dem.

$\vdash . *80{\cdot}14 . \supset \vdash : \text{Hp} . \supset . R \in 1 \to \text{Cls} . \text{Œ}^\epsilon R = \iota^\epsilon y .$

$[*37{\cdot}25] \qquad \supset . R \in 1 \to \text{Cls} . \text{Œ}^\epsilon R = \iota^\epsilon y . \text{D}^\epsilon R = R^{\epsilon\epsilon}\iota^\epsilon y$

$[*53{\cdot}31 . *71{\cdot}163] \qquad = \iota^\epsilon R^\epsilon y .$

$[*55{\cdot}16] \qquad \supset . R = (R^\epsilon y) \downarrow y : \supset \vdash . \text{Prop}$

$*80{\cdot}45$. $\vdash . P_\Delta{}^\epsilon\iota^\epsilon y = \downarrow y^{\epsilon\epsilon}\overrightarrow{P}{}^\epsilon y$

Dem.

$\vdash . *38{\cdot}131 . \supset \vdash : R \in \downarrow y^{\epsilon\epsilon}\overrightarrow{P}{}^\epsilon y . \equiv . (\exists x) . x \in \overrightarrow{P}{}^\epsilon y . R = x \downarrow y .$

$[*32{\cdot}18] \qquad \equiv . (\exists x) . xPy . R = x \downarrow y .$

$[*80{\cdot}43] \qquad \supset . R \in P_\Delta{}^\epsilon\iota^\epsilon y$ (1)

$\vdash . *80{\cdot}44{\cdot}31 . \supset \vdash : R \in P_\Delta{}^\epsilon\iota^\epsilon y . \supset . R = (R^\epsilon y) \downarrow y . R^\epsilon y \in \overrightarrow{P}{}^\epsilon y .$

$[*14{\cdot}205] \qquad \supset . (\exists x) . R = x \downarrow y . x \in \overrightarrow{P}{}^\epsilon y .$

$[*38{\cdot}131] \qquad \supset . R \in \downarrow y^{\epsilon\epsilon}\overrightarrow{P}{}^\epsilon y$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

$*80{\cdot}46$. $\vdash : \exists ! P_\Delta{}^\epsilon\iota^\epsilon y . \equiv . \exists ! \overrightarrow{P}{}^\epsilon y . \equiv . y \in \text{Œ}^\epsilon P \quad [*80{\cdot}45 . *37{\cdot}45 . *33{\cdot}41]$

$*80{\cdot}5$. $\vdash : \kappa \cap \lambda = \Lambda . R \in P_\Delta{}^\epsilon\kappa . S \in Q_\Delta{}^\epsilon\lambda . \supset . R \cup S \in (P \cup Q)_\Delta{}^\epsilon(\kappa \cup \lambda)$

Dem.

$\vdash . *80{\cdot}14 . \supset \vdash : \text{Hp} . \supset . R, S \in 1 \to \text{Cls} . \text{Œ}^\epsilon R = \kappa . \text{Œ}^\epsilon S = \lambda . R \subset P . S \subset Q .$

$[\text{Hp} . *33{\cdot}261 . *23{\cdot}72] \supset . R, S \in 1 \to \text{Cls} . \text{Œ}^\epsilon R \cap \text{Œ}^\epsilon S = \Lambda . \text{Œ}^\epsilon(R \cup S) = \kappa \cup \lambda .$

$R \cup S \subset P \cup Q .$

$[*71{\cdot}24] \qquad \supset . R \cup S \in 1 \to \text{Cls} . \text{Œ}^\epsilon(R \cup S) = \kappa \cup \lambda . R \cup S \subset P \cup Q .$

$[*80{\cdot}14] \qquad \supset . R \cup S \in (P \cup Q)_\Delta{}^\epsilon(\kappa \cup \lambda) : \supset \vdash . \text{Prop}$

***80·51.** $\vdash : \lambda \cap \text{Ɑ}'P = \Lambda \,.\, R \in P_\Delta{}'\kappa \,.\, S \in Q_\Delta{}'\lambda \,.\supset .\, R \mathbin{\dot\cup} S \in (P \mathbin{\dot\cup} Q)_\Delta{}'(\kappa \cup \lambda)$

Dem.

$$
\begin{array}{lll}
 & \vdash .\, *10\cdot24 \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, \exists !\, P_\Delta{}'\kappa \,. & \\
{[*80\cdot2]} & \supset .\, \kappa \subset \text{Ɑ}'P \,. & \\
{[*22\cdot48]} & \supset .\, \kappa \cap \lambda \subset \text{Ɑ}'P \cap \lambda \,. & \\
{[\mathrm{Hp}.*24\cdot13]} & \supset .\, \kappa \cap \lambda = \Lambda & (1) \\
 & \vdash .\, (1) \,.\, *80\cdot5 \,.\supset \vdash .\, \mathrm{Prop} &
\end{array}
$$

***80·511.** $\vdash : \kappa \cap \text{Ɑ}'Q = \Lambda \,.\, \lambda \cap \text{Ɑ}'P = \Lambda \,.\, M \in (P \mathbin{\dot\cup} Q)_\Delta{}'(\kappa \cup \lambda) \,.\supset .$
$M \upharpoonright \kappa = M \mathbin{\dot\cap} P \,.\, M \upharpoonright \lambda = M \mathbin{\dot\cap} Q$

Dem.

$$
\begin{array}{lll}
\multicolumn{2}{l}{\vdash .\, *80\cdot14 \,.\, *23\cdot621 \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, M = M \mathbin{\dot\cap} (P \mathbin{\dot\cup} Q) \,.} & \\
{[*35\cdot17]} & \supset .\, M \upharpoonright \kappa = M \mathbin{\dot\cap} (P \mathbin{\dot\cup} Q) \upharpoonright \kappa & \\
{[*35\cdot644]} & = M \mathbin{\dot\cap} P \upharpoonright \kappa & \\
{[*35\cdot642.*25\cdot24]} & = M \mathbin{\dot\cap} (P \upharpoonright \kappa \mathbin{\dot\cup} P \upharpoonright \lambda) & \\
{[*35\cdot412\cdot17]} & = M \upharpoonright (\kappa \cup \lambda) \mathbin{\dot\cap} P & \\
{[*80\cdot29]} & = M \mathbin{\dot\cap} P & (1) \\
\vdash .\, (1) \dfrac{Q, P, \lambda, \kappa}{P, Q, \kappa, \lambda} \,. & \supset \vdash : \mathrm{Hp} \,.\supset .\, M \upharpoonright \lambda = M \mathbin{\dot\cap} Q & (2) \\
\vdash .\, (1) \,.\, (2) \,.\supset \vdash .\, \mathrm{Prop} & &
\end{array}
$$

***80·52.** $\vdash : \kappa \cap \text{Ɑ}'Q = \Lambda \,.\, \lambda \cap \text{Ɑ}'P = \Lambda \,.\, M \in (P \mathbin{\dot\cup} Q)_\Delta{}'(\kappa \cup \lambda) \,.\supset .$
$M \upharpoonright \kappa \in P_\Delta{}'\kappa \,.\, M \upharpoonright \lambda \in Q_\Delta{}'\lambda$

Dem.

$$
\begin{array}{lll}
\multicolumn{2}{l}{\vdash .\, *80\cdot14 \,.\, *71\cdot26 \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, M \upharpoonright \kappa, M \upharpoonright \lambda \in 1 \to \mathrm{Cls}} & (1) \\
\vdash .\, *80\cdot511 \,. & \supset \vdash : \mathrm{Hp} \,.\supset .\, M \upharpoonright \kappa = M \mathbin{\dot\cap} P \,.\, M \upharpoonright \lambda = M \mathbin{\dot\cap} Q \,. & \\
{[*23\cdot43]} & \supset .\, M \upharpoonright \kappa \mathbin{\dot\subset} P \,.\, M \upharpoonright \lambda \mathbin{\dot\subset} Q & (2) \\
\multicolumn{2}{l}{\vdash .\, *80\cdot14 \,.\, *22\cdot58 \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, \kappa \subset \text{Ɑ}'M \,.\, \lambda \subset \text{Ɑ}'M \,.} & \\
{[*35\cdot65]} & \supset .\, \text{Ɑ}'M \upharpoonright \kappa = \kappa \,.\, \text{Ɑ}'M \upharpoonright \lambda = \lambda & (3) \\
\multicolumn{2}{l}{\vdash .\, (1) \,.\, (2) \,.\, (3) \,.\, *80\cdot14 \,.\supset \vdash .\, \mathrm{Prop}} &
\end{array}
$$

***80·53.** $\vdash :.\, \kappa \cap \text{Ɑ}'Q = \Lambda \,.\, \lambda \cap \text{Ɑ}'P = \Lambda \,.\supset :$
$M \in (P \mathbin{\dot\cup} Q)_\Delta{}'(\kappa \cup \lambda) \,.\equiv .\, (\exists R, S) \,.\, R \in P_\Delta{}'\kappa \,.\, S \in Q_\Delta{}'\lambda \,.\, M = R \mathbin{\dot\cup} S$

Dem.

$$
\begin{array}{lll}
*80\cdot52 \,. & \supset \vdash : \mathrm{Hp} \,.\, M \in (P \mathbin{\dot\cup} Q)_\Delta{}'(\kappa \cup \lambda) \,.\supset .\, M \upharpoonright \kappa \in P_\Delta{}'\kappa \,.\, M \upharpoonright \lambda \in Q_\Delta{}'\lambda & (1) \\
*80\cdot29 \,. & \supset \vdash : \mathrm{Hp}(1) \,.\supset .\, M = M \upharpoonright (\kappa \cup \lambda) & \\
{[*35\cdot412]} & = M \upharpoonright \kappa \mathbin{\dot\cup} M \upharpoonright \lambda & (2) \\
\multicolumn{2}{l}{\vdash .\, (1) \,.\, (2) \,.\supset \vdash :.\, \mathrm{Hp} \,.\supset .\, M \in (P \mathbin{\dot\cup} Q)_\Delta{}'(\kappa \cup \lambda) \,.\supset .} & \\
 & (\exists R, S) \,.\, R \in P_\Delta{}'\kappa \,.\, S \in Q_\Delta{}'\lambda \,.\, M = R \mathbin{\dot\cup} S & (3) \\
\multicolumn{2}{l}{\vdash .\, *80\cdot51 \,.\supset \vdash :.\, \mathrm{Hp} \,.\supset : R \in P_\Delta{}'\kappa \,.\, S \in Q_\Delta{}'\lambda \,.\, M = R \mathbin{\dot\cup} S \,.\supset .} & \\
 & M \in (P \mathbin{\dot\cup} Q)_\Delta{}'(\kappa \cup \lambda) : & \\
{[*11\cdot11\cdot3\cdot35]} & \supset : (\exists R, S) \,.\, R \in P_\Delta{}'\kappa \,.\, S \in Q_\Delta{}'\lambda \,.\, M = R \mathbin{\dot\cup} S \,.\supset . & \\
 & M \in (P \mathbin{\dot\cup} Q)_\Delta{}'(\kappa \cup \lambda) & (4) \\
\multicolumn{2}{l}{\vdash .\, (3) \,.\, (4) \,.\supset \vdash .\, \mathrm{Prop}} &
\end{array}
$$

***80·54.** $\vdash :. \kappa \cap \text{Ɑ}\text{‘}Q = \Lambda . \lambda \cap \text{Ɑ}\text{‘}P = \Lambda . \supset :$

$$R \in P_\Delta\text{‘}\kappa . S \in Q_\Delta\text{‘}\lambda . \equiv . (\exists M) . M \in (P \dot\cup Q)_\Delta\text{‘}(\kappa \cup \lambda) . R = M \upharpoonright \kappa . S = M \upharpoonright \lambda$$

Dem.

$\vdash . *80{\cdot}51 . \supset \vdash : \mathrm{Hp} . R \in P_\Delta\text{‘}\kappa . S \in Q_\Delta\text{‘}\lambda . \supset . R \dot\cup S \in (P \dot\cup Q)_\Delta\text{‘}(\kappa \cup \lambda)$ (1)

$\vdash . *80{\cdot}14 . \supset \vdash : \mathrm{Hp}(1) . \supset . \kappa \cap \text{Ɑ}\text{‘}S = \Lambda . \lambda \cap \text{Ɑ}\text{‘}R = \Lambda .$

[*35·644] $\supset . (R \dot\cup S) \upharpoonright \kappa = R \upharpoonright \kappa . (R \dot\cup S) \upharpoonright \lambda = S \upharpoonright \lambda .$

[*80·29] $\supset . (R \dot\cup S) \upharpoonright \kappa = R . (R \dot\cup S) \upharpoonright \lambda = S$ (2)

$\vdash . (1) . (2) . \supset \vdash : \mathrm{Hp} . R \in P_\Delta\text{‘}\kappa . S \in Q_\Delta\text{‘}\lambda . \supset .$

$R \dot\cup S \in (P \dot\cup Q)_\Delta\text{‘}(\kappa \cup \lambda) . (R \dot\cup S) \upharpoonright \kappa = R . (R \dot\cup S) \upharpoonright \lambda = S .$

[*10·24] $\supset . (\exists M) . M \in (P \dot\cup Q)_\Delta\text{‘}(\kappa \cup \lambda) . M \upharpoonright \kappa = R . M \upharpoonright \lambda = S$ (3)

$\vdash . *80{\cdot}52 . \supset \vdash :. \mathrm{Hp} . \supset : M \in (P \dot\cup Q)_\Delta\text{‘}(\kappa \cup \lambda) . R = M \upharpoonright \kappa . S = M \upharpoonright \lambda . \supset .$

$R \in P_\Delta\text{‘}\kappa . S \in Q_\Delta\text{‘}\lambda :$

[*10·11·21·23] $\supset : (\exists M) . M \in (P \dot\cup Q)_\Delta\text{‘}(\kappa \cup \lambda) . R = M \upharpoonright \kappa . S = M \upharpoonright \lambda . \supset .$

$R \in P_\Delta\text{‘}\kappa . S \in Q_\Delta\text{‘}\lambda$ (4)

$\vdash . (3) . (4) . \supset \vdash . \mathrm{Prop}$

***80·6.** $\vdash : R \in P_\Delta\text{‘}\kappa . \lambda \subset \kappa . \supset . R \upharpoonright \lambda \in P_\Delta\text{‘}\lambda$

Dem.

$\vdash . *80{\cdot}14 . *71{\cdot}26 . \supset \vdash : \mathrm{Hp} . \supset . R \upharpoonright \lambda \in 1 \rightarrow \mathrm{Cls}$ (1)

$\vdash . *80{\cdot}14 . *35{\cdot}441 . \supset \vdash : \mathrm{Hp} . \supset . R \upharpoonright \lambda \,\dot\subset\, P$ (2)

$\vdash . *80{\cdot}14 . *35{\cdot}65 . \supset \vdash : \mathrm{Hp} . \supset . \text{Ɑ}\text{‘}R \upharpoonright \lambda = \lambda$ (3)

$\vdash . (1) . (2) . (3) . *80{\cdot}14 . \supset \vdash . \mathrm{Prop}$

***80·61.** $\vdash : M \upharpoonright \kappa \in P_\Delta\text{‘}\kappa . M \upharpoonright \lambda \in P_\Delta\text{‘}\lambda . \supset . M \upharpoonright (\kappa \cup \lambda) \in P_\Delta\text{‘}(\kappa \cup \lambda)$

Dem.

$\vdash . *80{\cdot}6 . \supset \vdash : M \upharpoonright \lambda \in P_\Delta\text{‘}\lambda . \supset . M \upharpoonright (\lambda - \kappa) \in P_\Delta\text{‘}(\lambda - \kappa) :$

[Fact] $\supset \vdash : \mathrm{Hp} . \supset . M \upharpoonright \kappa \in P_\Delta\text{‘}\kappa . M \upharpoonright (\lambda - \kappa) \in P_\Delta\text{‘}(\lambda - \kappa) .$

[*80·5.*24·21] $\supset . M \upharpoonright \kappa \dot\cup M \upharpoonright (\lambda - \kappa) \in P_\Delta\text{‘}\{\kappa \cup (\lambda - \kappa)\} .$

[*35·412.*22·91] $\supset . M \upharpoonright (\kappa \cup \lambda) \in P_\Delta\text{‘}(\kappa \cup \lambda) : \supset \vdash . \mathrm{Prop}$

***80·62.** $\vdash : M \in P_\Delta\text{‘}(\kappa \cup \lambda) . \supset . M \upharpoonright \kappa \in P_\Delta\text{‘}\kappa . M \upharpoonright \lambda \in P_\Delta\text{‘}\lambda$ [*80·6 . *22·58]

***80·621.** $\vdash : M \upharpoonright (\kappa \cup \lambda) \in P_\Delta\text{‘}(\kappa \cup \lambda) . \supset . M \upharpoonright \kappa \in P_\Delta\text{‘}\kappa . M \upharpoonright \lambda \in P_\Delta\text{‘}\lambda$

Dem.

$\vdash . *35{\cdot}31 . \supset \vdash . \{M \upharpoonright (\kappa \cup \lambda)\} \upharpoonright \kappa = M \upharpoonright \{(\kappa \cup \lambda) \cap \kappa\}$

[*22·631] $= M \upharpoonright \kappa$ (1)

Similarly $\vdash . \{M \upharpoonright (\kappa \cup \lambda)\} \upharpoonright \lambda = M \upharpoonright \lambda$ (2)

$\vdash . (1) . (2) . *80{\cdot}62 . \supset \vdash . \mathrm{Prop}$

***80·63.** $\vdash : M \upharpoonright \kappa \in P_\Delta\text{‘}\kappa . M \upharpoonright \lambda \in P_\Delta\text{‘}\lambda . \equiv . M \upharpoonright (\kappa \cup \lambda) \in P_\Delta\text{‘}(\kappa \cup \lambda)$ [*80·61·621]

***80·64.** $\vdash :. \text{Ɑ}\text{‘}M = \kappa \cup \lambda . \supset : M \upharpoonright \kappa \in P_\Delta\text{‘}\kappa . M \upharpoonright \lambda \in P_\Delta\text{‘}\lambda . \equiv . M \in P_\Delta\text{‘}(\kappa \cup \lambda)$

Dem.

$\vdash . *35{\cdot}452 . \supset \vdash : \mathrm{Hp} . \supset . M = M \upharpoonright (\kappa \cup \lambda)$ (1)

$\vdash . (1) . *80{\cdot}63 . \supset \vdash . \mathrm{Prop}$

***80·65.** $\vdash : \kappa \cap \lambda = \Lambda \,.\, R \in P_\Delta{}^\backprime\kappa \,.\, S \in P_\Delta{}^\backprime\lambda \,.\, \supset .\, R \cup\!\!\!\cdot\; S \in P_\Delta{}^\backprime(\kappa \cup \lambda)$

$\left[*80\cdot5 \dfrac{P}{Q} \,.\, *23\cdot56\right]$

***80·651.** $\vdash : R \in P_\Delta{}^\backprime\kappa \,.\, S \in P_\Delta{}^\backprime\lambda \,.\, \supset .\, R \cup\!\!\!\cdot\; S \upharpoonright (\lambda - \kappa) \in P_\Delta{}^\backprime(\kappa \cup \lambda)$

Dem.

$\vdash .\, *80\cdot6 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset .\, S \upharpoonright (\lambda - \kappa) \in P_\Delta{}^\backprime(\lambda - \kappa) \,.$

$[*80\cdot65] \qquad \supset .\, R \cup\!\!\!\cdot\; S \upharpoonright (\lambda - \kappa) \in P_\Delta{}^\backprime\{\kappa \cup (\lambda - \kappa)\} \,.$

$[*22\cdot91] \qquad \supset .\, R \cup\!\!\!\cdot\; S \upharpoonright (\lambda - \kappa) \in P_\Delta{}^\backprime(\kappa \cup \lambda) : \supset \vdash .\, \mathrm{Prop}$

***80·66.** $\vdash :.\, \kappa \cap \lambda = \Lambda \,.\, \supset :$

$M \in P_\Delta{}^\backprime(\kappa \cup \lambda) \,.\, \equiv .\, (\exists R, S) \,.\, R \in P_\Delta{}^\backprime\kappa \,.\, S \in P_\Delta{}^\backprime\lambda \,.\, M = R \cup\!\!\!\cdot\; S$

Dem.

$\vdash .\, *80\cdot62 \,.\, \supset \vdash : M \in P_\Delta{}^\backprime(\kappa \cup \lambda) \,.\, \supset .\, M \upharpoonright \kappa \in P_\Delta{}^\backprime\kappa \,.\, M \upharpoonright \lambda \in P_\Delta{}^\backprime\lambda \qquad (1)$

$\vdash .\, *35\cdot452 \,.\, \supset \vdash : M \in P_\Delta{}^\backprime(\kappa \cup \lambda) \,.\, \supset .\, M = M \upharpoonright (\kappa \cup \lambda)$

$[*35\cdot412] \qquad = M \upharpoonright \kappa \cup\!\!\!\cdot\; M \upharpoonright \lambda \qquad (2)$

$\vdash .\, (1) \,.\, (2) \,.\, \supset \vdash : M \in P_\Delta{}^\backprime(\kappa \cup \lambda) \,.\, \supset .\, M \upharpoonright \kappa \in P_\Delta{}^\backprime\kappa \,.\, M \upharpoonright \lambda \in P_\Delta{}^\backprime\lambda \,.\, M = M \upharpoonright \kappa \cup\!\!\!\cdot\; M \upharpoonright \lambda \,.$

$[*11\cdot36] \qquad \supset .\, (\exists R, S) \,.\, R \in P_\Delta{}^\backprime\kappa \,.\, S \in P_\Delta{}^\backprime\lambda \,.\, M = R \cup\!\!\!\cdot\; S \qquad (3)$

$\vdash .\, *80\cdot65 \,.\, \supset \vdash :.\, \mathrm{Hp} \,.\, \supset : R \in P_\Delta{}^\backprime\kappa \,.\, S \in P_\Delta{}^\backprime\lambda \,.\, M = R \cup\!\!\!\cdot\; S \,.\, \supset .\, M \in P_\Delta{}^\backprime(\kappa \cup \lambda) :$

$[*11\cdot11\cdot3\cdot35] \qquad \supset : (\exists R, S) \,.\, R \in P_\Delta{}^\backprime\kappa \,.\, S \in P_\Delta{}^\backprime\lambda \,.\, M = R \cup\!\!\!\cdot\; S \,.\, \supset .$

$M \in P_\Delta{}^\backprime(\kappa \cup \lambda) \qquad (4)$

$\vdash .\, (3) \,.\, (4) \,.\, \supset \vdash .\, \mathrm{Prop}$

***80·661.** $\vdash : \kappa \cap \lambda = \Lambda \,.\, R \in P_\Delta{}^\backprime\kappa \,.\, S \in P_\Delta{}^\backprime\lambda \,.\, \supset .\, R = (R \cup\!\!\!\cdot\; S) \upharpoonright \kappa \,.\, S = (R \cup\!\!\!\cdot\; S) \upharpoonright \lambda$

Dem.

$\vdash .\, *80\cdot14 \,. \qquad \supset \vdash : \mathrm{Hp} \,.\, \supset .\, \mathrm{G}^\backprime R = \kappa \,.\, \mathrm{G}^\backprime S \cap \kappa = \Lambda \,. \qquad (1)$

$[*35\cdot452] \qquad \supset .\, R \upharpoonright \kappa = R \qquad (2)$

$\vdash .\, (1) \,.\, (2) \,.\, *35\cdot644 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset .\, (R \cup\!\!\!\cdot\; S) \upharpoonright \kappa = R \qquad (3)$

Similarly $\qquad \vdash : \mathrm{Hp} \,.\, \supset .\, (R \cup\!\!\!\cdot\; S) \upharpoonright \lambda = S \qquad (4)$

$\vdash .\, (3) \,.\, (4) \,.\, \supset \vdash .\, \mathrm{Prop}$

***80·67.** $\vdash :.\, \kappa \cap \lambda = \Lambda \,.\, \supset : R \in P_\Delta{}^\backprime\kappa \,.\, S \in P_\Delta{}^\backprime\lambda \,.\, \equiv .$

$(\exists M) \,.\, M \in P_\Delta{}^\backprime(\kappa \cup \lambda) \,.\, R = M \upharpoonright \kappa \,.\, S = M \upharpoonright \lambda$

Dem.

$\vdash .\, *80\cdot65\cdot661 \,.\, \supset \vdash :.\, \mathrm{Hp} \,.\, \supset : R \in P_\Delta{}^\backprime\kappa \,.\, S \in P_\Delta{}^\backprime\lambda \,.\, \supset .$

$R \cup\!\!\!\cdot\; S \in P_\Delta{}^\backprime(\kappa \cup \lambda) \,.\, R = (R \cup\!\!\!\cdot\; S) \upharpoonright \kappa \,.\, S = (R \cup\!\!\!\cdot\; S) \upharpoonright \lambda \,.$

$[*10\cdot24] \qquad \supset .\, (\exists M) \,.\, M \in P_\Delta{}^\backprime(\kappa \cup \lambda) \,.\, R = M \upharpoonright \kappa \,.\, S = M \upharpoonright \lambda \qquad (1)$

$\vdash .\, *80\cdot62 \,.\, \supset \vdash : M \in P_\Delta{}^\backprime(\kappa \cup \lambda) \,.\, R = M \upharpoonright \kappa \,.\, S = M \upharpoonright \lambda \,.\, \supset .\, R \in P_\Delta{}^\backprime\kappa \,.\, S \in P_\Delta{}^\backprime\lambda :$

$[*10\cdot11\cdot23] \supset \vdash : (\exists M) \,.\, M \in P_\Delta{}^\backprime(\kappa \cup \lambda) \,.\, R = M \upharpoonright \kappa \,.\, S = M \upharpoonright \lambda \,.\, \supset .$

$R \in P_\Delta{}^\backprime\kappa \,.\, S \in P_\Delta{}^\backprime\lambda \qquad (2)$

$\vdash .\, (1) \,.\, (2) \,.\, \supset \vdash .\, \mathrm{Prop}$

***80·68.** $\vdash : R \epsilon P_\Delta‘(\kappa - \iota‘y) \,.\, y \epsilon \kappa \,.\, xPy \,.\, \supset \,.\, R \dot\cup x \downarrow y \epsilon P_\Delta‘\kappa$

Dem.

$\vdash .$ *80·43 . $\supset \vdash : \text{Hp} \,.\, \supset \,.\, x \downarrow y \epsilon P_\Delta‘\iota‘y$ (1)

$\vdash .$ *24·21 . $\supset \vdash . (\kappa - \iota‘y) \cap \iota‘y = \Lambda$ (2)

$\vdash . (1) . (2) .$ *80·65 . $\supset \vdash : \text{Hp} \,.\, \supset \,.\, R \dot\cup x \downarrow y \epsilon P_\Delta‘\{(\kappa - \iota‘y) \cup \iota‘y\} \,.$

[*51·221] $\supset \,.\, R \dot\cup x \downarrow y \epsilon P_\Delta‘\kappa : \supset \vdash . \text{Prop}$

***80·69.** $\vdash : \exists ! P_\Delta‘(\kappa \cup \lambda) \,.\, \equiv \,.\, \exists ! P_\Delta‘\kappa \,.\, \exists ! P_\Delta‘\lambda$

Dem.

$\vdash .$ *80·62 . $\supset \vdash : \exists ! P_\Delta‘(\kappa \cup \lambda) \,.\, \supset \,.\, \exists ! P_\Delta‘\kappa \,.\, \exists ! P_\Delta‘\lambda$ (1)

$\vdash .$ *80·6 . $\supset \vdash : \exists ! P_\Delta‘\lambda \,.\, \supset \,.\, \exists ! P_\Delta‘(\lambda - \kappa) :$

[Fact] $\supset \vdash : \exists ! P_\Delta‘\kappa \,.\, \exists ! P_\Delta‘\lambda \,.\, \supset \,.\, \exists ! P_\Delta‘\kappa \,.\, \exists ! P_\Delta‘(\lambda - \kappa)$ (2)

$\vdash .$ *80·65 . $\supset \vdash : R \epsilon P_\Delta‘\kappa \,.\, S \epsilon P_\Delta‘(\lambda - \kappa) \,.\, \supset \,.\, R \dot\cup S \epsilon P_\Delta‘(\kappa \cup \lambda) :$

[*10·11·23] $\supset \vdash : \exists ! P_\Delta‘\kappa \,.\, \exists ! P_\Delta‘(\lambda - \kappa) \,.\, \supset \,.\, \exists ! P_\Delta‘(\kappa \cup \lambda)$ (3)

$\vdash . (2) . (3) . \supset \vdash : \exists ! P_\Delta‘\kappa \,.\, \exists ! P_\Delta‘\lambda \,.\, \supset \,.\, \exists ! P_\Delta‘(\kappa \cup \lambda)$ (4)

$\vdash . (1) . (4) . \supset \vdash . \text{Prop}$

***80·7.** $\vdash : \text{Ɑ}‘P \cap \text{Ɑ}‘Q = \Lambda \,.\, \kappa \subset \text{Ɑ}‘P \,.\, \lambda \subset \text{Ɑ}‘Q \,.\, M \epsilon (P \dot\cup Q)_\Delta‘(\kappa \cup \lambda) \,.\, \supset .$

$M \dot- Q \epsilon P_\Delta‘\kappa \,.\, M \dot- P \epsilon Q_\Delta‘\lambda$

Dem.

$\vdash .$ *33·33 . *80·14 . $\supset \vdash : \text{Hp} \,.\, \supset \,.\, P \dot\cap Q = \dot\Lambda \,.\, M \mathrel{\dot\subset} P \dot\cup Q \,.$

[*25·491] $\supset \,.\, M \dot- Q = M \dot\cap P \,.\, M \dot- P = M \dot\cap Q$ (1)

$\vdash .$ *22·48 . *24·13 . $\supset \vdash : \text{Hp} \,.\, \supset \,.\, \kappa \cap \text{Ɑ}‘Q = \Lambda \,.\, \lambda \cap \text{Ɑ}‘P = \Lambda \,.$

[*80·511·52] $\supset \,.\, M \dot\cap P \epsilon P_\Delta‘\kappa \,.\, M \dot\cap Q \epsilon Q_\Delta‘\lambda$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***80·71.** $\vdash : \text{Ɑ}‘P \cap \text{Ɑ}‘Q = \Lambda \,.\, M \dot- Q \epsilon P_\Delta‘\kappa \,.\, M \dot- P \epsilon Q_\Delta‘\lambda \,.\, \supset \,.\, M \epsilon (P \dot\cup Q)_\Delta‘(\kappa \cup \lambda)$

Dem.

$\vdash .$ *33·33 . $\supset \vdash : \text{Hp} \,.\, \supset \,.\, P \dot\cap Q = \dot\Lambda \,.$

[*25·493] $\supset \,.\, M = (M \dot- P) \dot\cup (M \dot- Q)$ (1)

$\vdash .$ *80·2 . $\supset \vdash : \text{Hp} \,.\, \supset \,.\, \lambda \subset \text{Ɑ}‘Q \,.$

[*22·48.*24·13] $\supset \,.\, \lambda \cap \text{Ɑ}‘P = \Lambda \,.$

[*80·51] $\supset \,.\, (M \dot- Q) \dot\cup (M \dot- P) \epsilon (P \dot\cup Q)_\Delta‘(\kappa \cup \lambda)$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***80·72.** $\vdash :. \text{Ɑ}‘P \cap \text{Ɑ}‘Q = \Lambda \,.\, \kappa \subset \text{Ɑ}‘P \,.\, \lambda \subset \text{Ɑ}‘Q \,.\, \supset :$

$M \epsilon (P \dot\cup Q)_\Delta‘(\kappa \cup \lambda) \,.\, \equiv \,.\, M \dot- Q \epsilon P_\Delta‘\kappa \,.\, M \dot- P \epsilon Q_\Delta‘\lambda$ [*80·7·71]

***80·73.** $\vdash : Q = P \upharpoonright \kappa \,.\, R = P \upharpoonright \lambda \,.\, \supset \,.\, P_\Delta‘(\kappa \cup \lambda) = (Q \dot\cup R)_\Delta‘(\kappa \cup \lambda)$

Dem.

$\vdash .$ *35·412 . $\supset \vdash : \text{Hp} \,.\, \supset \,.\, Q \dot\cup R = P \upharpoonright (\kappa \cup \lambda) \,.$

[*80·23] $\supset \,.\, (Q \dot\cup R)_\Delta‘(\kappa \cup \lambda) = P_\Delta‘(\kappa \cup \lambda) : \supset \vdash . \text{Prop}$

***80·731.** $\vdash : Q = P \upharpoonright \kappa \,.\, R = P \upharpoonright \lambda \,.\, \kappa \cup \lambda \subset \text{Ɑ}‘P \,.\, \supset \,.\, \kappa = \text{Ɑ}‘Q \,.\, \lambda = \text{Ɑ}‘R$

Dem.

$\vdash .$ *22·59 . $\supset \vdash : \text{Hp} \,.\, \supset \,.\, \kappa \subset \text{Ɑ}‘P \,.\, \lambda \subset \text{Ɑ}‘P \,.$

[*35·65] $\supset \,.\, \kappa = \text{Ɑ}‘Q \,.\, \lambda = \text{Ɑ}‘R : \supset \vdash . \text{Prop}$

***80·732.** $\vdash : Q = P \upharpoonright \kappa \,.\, R = P \upharpoonright \lambda \,.\, \kappa \cap \lambda = \Lambda \,.\supset .\, \text{Ɑ}'Q \cap \text{Ɑ}'R = \Lambda$

Dem.

$\vdash .\, *35{\cdot}64 \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, \text{Ɑ}'Q \subset \kappa \,.\, \text{Ɑ}'R \subset \lambda \,.$

$[*22{\cdot}49] \qquad \supset .\, \text{Ɑ}'Q \cap \text{Ɑ}'R \subset \kappa \cap \lambda \,.$

$[*24{\cdot}13] \qquad \supset .\, \text{Ɑ}'Q \cap \text{Ɑ}'R = \Lambda : \supset \vdash .\, \mathrm{Prop}$

***80·74.** $\vdash : \kappa \cap \lambda = \Lambda \,.\, M \in P_\Delta{}'(\kappa \cup \lambda) \,.\supset .$

$$M \upharpoonright \kappa = M \upharpoonright -\lambda = M \dot{-} P \upharpoonright \lambda \,.\, M \upharpoonright \lambda = M \upharpoonright -\kappa = M \dot{-} P \upharpoonright \kappa$$

Dem.

$\vdash .\, *24{\cdot}4 \,. \qquad \supset \vdash : \mathrm{Hp} \,.\supset .\, M \upharpoonright \kappa = M \upharpoonright \{(\kappa \cup \lambda) - \lambda\}$

$[*35{\cdot}31] \qquad = \{M \upharpoonright (\kappa \cup \lambda)\} \upharpoonright -\lambda$

$[*80{\cdot}29] \qquad = M \upharpoonright -\lambda \qquad (1)$

$\vdash .\, *80{\cdot}732 \,. \qquad \supset \vdash : \mathrm{Hp} \,.\supset .\, \text{Ɑ}'(P \upharpoonright \kappa) \cap \text{Ɑ}'(P \upharpoonright \lambda) = \Lambda \,.$

$[*33{\cdot}33] \qquad \supset .\, P \upharpoonright \kappa \,\dot{\cap}\, P \upharpoonright \lambda = \dot{\Lambda} \qquad (2)$

$\vdash .\, *80{\cdot}291 \,. \qquad \supset \vdash : \mathrm{Hp} \,.\supset .\, M \,\dot{\subset}\, P \upharpoonright (\kappa \cup \lambda) \,.$

$[*35{\cdot}412] \qquad \supset .\, M \,\dot{\subset}\, P \upharpoonright \kappa \,\dot{\cup}\, P \upharpoonright \lambda \qquad (3)$

$\vdash .\, (2) \,.\, (3) \,.\, *25{\cdot}491 \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, M \dot{-} P \upharpoonright \lambda = M \,\dot{\cap}\, P \upharpoonright \kappa$

$[*35{\cdot}17] \qquad = (M \,\dot{\cap}\, P) \upharpoonright \kappa$

$[*80{\cdot}14 \,.\, *23{\cdot}621] \qquad = M \upharpoonright \kappa \qquad (4)$

$\vdash .\, (1) \,.\, (4) \,. \qquad \supset \vdash : \mathrm{Hp} \,.\supset .\, M \upharpoonright \kappa = M \upharpoonright -\lambda = M \dot{-} P \upharpoonright \lambda \quad (5)$

Similarly $\qquad \vdash : \mathrm{Hp} \,.\supset .\, M \upharpoonright \lambda = M \upharpoonright -\kappa = M \dot{-} P \upharpoonright \kappa \quad (6)$

$\vdash .\, (5) \,.\, (6) \,.\supset \vdash .\, \mathrm{Prop}$

***80·75.** $\vdash : \kappa \cap \lambda = \Lambda \,.\, M \in P_\Delta{}'(\kappa \cup \lambda) \,.\supset .\, M \dot{-} P \upharpoonright \lambda \in P_\Delta{}'\kappa \,.\, M \dot{-} P \upharpoonright \kappa \in P_\Delta{}'\lambda$

$[*80{\cdot}62{\cdot}74]$

***80·76.** $\vdash : M \in P_\Delta{}'\mu \,.\, R \in P_\Delta{}'\kappa \,.\, R \,\dot{\subset}\, M \,.\supset .\, M \dot{-} R \in P_\Delta{}'(\mu - \kappa)$

Dem.

$\vdash .\, *80{\cdot}14 \,. \qquad \supset \vdash : \mathrm{Hp} \,.\supset .\, \text{Ɑ}'R = \kappa \,.\, \text{Ɑ}'M = \mu \qquad (1)$

$\vdash .\, *80{\cdot}14 \,.\, *72{\cdot}91 \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, \text{Ɑ}'(M \dot{-} R) = \text{Ɑ}'M - \text{Ɑ}'R$

$[(1)] \qquad = \mu - \kappa \qquad (2)$

$\vdash .\, *80{\cdot}14 \,.\, *71{\cdot}22 \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, M \dot{-} R \in 1 \to \mathrm{Cls} \qquad (3)$

$\vdash .\, *80{\cdot}14 \,.\, *23{\cdot}47 \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, M \dot{-} R \,\dot{\subset}\, P \qquad (4)$

$\vdash .\, (2) \,.\, (3) \,.\, (4) \,.\, *80{\cdot}14 \,.\supset \vdash .\, \mathrm{Prop}$

***80·761.** $\vdash : \kappa \cap \lambda = \Lambda \,.\, M \in P_\Delta{}'(\kappa \cup \lambda) \,.\, R \in P_\Delta{}'\kappa \,.\, R \,\dot{\subset}\, M \,.\supset .\, M \dot{-} R \in P_\Delta{}'\lambda$

Dem.

$\vdash .\, *80{\cdot}76 \,. \quad \supset \vdash : \mathrm{Hp} \,.\supset .\, M \dot{-} R \in P_\Delta{}'\{(\kappa \cup \lambda) - \kappa\} \qquad (1)$

$\vdash .\, *24{\cdot}4 \,. \quad \supset \vdash : \mathrm{Hp} \,.\supset .\, (\kappa \cup \lambda) - \kappa = \lambda \qquad (2)$

$\vdash .\, (1) \,.\, (2) \,.\supset \vdash .\, \mathrm{Prop}$

***80·77.** $\vdash : M \in P_\Delta{}'\mu \,.\, M \dot{-} R \in P_\Delta{}'(\mu - \kappa) \,.\, R \,\dot{\subset}\, M \,.\, \kappa \subset \mu \,.\supset .\, R \in P_\Delta{}'\kappa$

Dem.

$\vdash .\, *80{\cdot}76 \,. \qquad \supset \vdash : \mathrm{Hp} \,.\supset .\, M \dot{-} (M \dot{-} R) \in P_\Delta{}'\{\mu - (\mu - \kappa)\} \quad (1)$

$\vdash .\, *25{\cdot}411 \,. \qquad \supset \vdash : \mathrm{Hp} \,.\supset .\, M = R \,\dot{\cup}\, (M \dot{-} R) \qquad (2)$

$\vdash . *25 \cdot 21 . \quad \supset \vdash . R \dot{\cap} (M \dot{-} R) = \dot{\Lambda}$ (3)

$\vdash . (2) . (3) . *25 \cdot 4 . \supset \vdash : \mathrm{Hp} . \supset . M \dot{-} (M \dot{-} R) = R$ (4)

$\vdash . *24 \cdot 411 \cdot 21 \cdot 4 \quad \supset \vdash : \mathrm{Hp} . \supset . \mu - (\mu - \kappa) = \kappa$ (5)

$\vdash . (1) . (4) . (5) . \quad \supset \vdash . \mathrm{Prop}$

***80·771.** $\vdash : \kappa \cap \lambda = \Lambda . M \in P_\Delta \text{‘}(\kappa \cup \lambda) . M \dot{-} R \in P_\Delta \text{‘} \lambda . R \subseteq M . \supset . R \in P_\Delta \text{‘} \kappa$

Dem.

$\vdash . *24 \cdot 4 . \supset \vdash : \mathrm{Hp} . \supset . \lambda = (\kappa \cup \lambda) - \kappa$ (1)

$\vdash . (1) . *80 \cdot 77 . \supset \vdash . \mathrm{Prop}$

***80·78.** $\vdash : M \in P_\Delta \text{‘} \mu . xMy . \supset . M \dot{-} x \downarrow y \in P_\Delta \text{‘}(\mu - \iota \text{‘} y)$

Dem.

$\vdash . *55 \cdot 3 . \quad \supset \vdash : \mathrm{Hp} . \supset . x \downarrow y \subseteq M$ (1)

$\vdash . *80 \cdot 14 . \supset \vdash : \mathrm{Hp} . \supset . xPy .$

$[*80 \cdot 43] \quad \supset . x \downarrow y \in P_\Delta \text{‘} \iota \text{‘} y$ (2)

$\vdash . (1) . (2) . *80 \cdot 76 . \supset \vdash . \mathrm{Prop}$

***80·8.** $\vdash : \exists ! P_\Delta \text{‘} \kappa . \supset . \text{C}\text{‘} \dot{s} \text{‘} P_\Delta \text{‘} \kappa = \kappa$

Dem.

$\vdash . *80 \cdot 42 . \supset \vdash : \mathrm{Hp} . \supset . \dot{s} \text{‘} P_\Delta \text{‘} \kappa = P \upharpoonright \kappa$ (1)

$\vdash . (1) . *80 \cdot 2 . *35 \cdot 65 . \supset \vdash . \mathrm{Prop}$

***80·81.** $\vdash : \exists ! P_\Delta \text{‘} \alpha . P_\Delta \text{‘} \alpha = P_\Delta \text{‘} \beta . \supset . \alpha = \beta$

Dem.

$\vdash . *30 \cdot 37 . \supset \vdash : \mathrm{Hp} . \supset . \text{C}\text{‘} \dot{s} \text{‘} P_\Delta \text{‘} \alpha = \text{C}\text{‘} \dot{s} \text{‘} P_\Delta \text{‘} \beta .$

$[*80 \cdot 8] \quad \supset . \alpha = \beta : \supset \vdash . \mathrm{Prop}$

***80·82.** $\vdash : \alpha \neq \beta . \supset . P_\Delta \text{‘} \alpha \cap P_\Delta \text{‘} \beta = \Lambda$

Dem.

$\vdash . *80 \cdot 14 . \supset \vdash : R \in P_\Delta \text{‘} \alpha . S \in P_\Delta \text{‘} \beta . \supset . \text{C}\text{‘} R = \alpha . \text{C}\text{‘} S = \beta :$

$[*13 \cdot 13] \quad \supset \vdash :. \mathrm{Hp} . \supset : R \in P_\Delta \text{‘} \alpha . S \in P_\Delta \text{‘} \beta . \supset . \text{C}\text{‘} R \neq \text{C}\text{‘} S .$

$[*30 \cdot 37 . *33 \cdot 121 . \mathrm{Transp}] \quad \supset . R \neq S$ (1)

$\vdash . (1) . *24 \cdot 37 . \supset \vdash . \mathrm{Prop}$

The following proposition is used in *80·84 and in the theory of double similarity (*111·3).

***80·83.** $\vdash . (-\iota \text{‘} \Lambda) \rceil P_\Delta \in 1 \to 1$

Dem.

$\vdash . *80 \cdot 12 . *71 \cdot 166 . \supset \vdash . P_\Delta \in 1 \to \mathrm{Cls} .$

$[*71 \cdot 27] \quad \supset \vdash . (-\iota \text{‘} \Lambda) \rceil P_\Delta \in 1 \to \mathrm{Cls}$ (1)

$\vdash . *35 \cdot 1 . *51 \cdot 15 . \supset$

$\vdash : \lambda \{(-\iota \text{‘} \Lambda) \rceil P_\Delta\} \alpha . \lambda \{(-\iota \text{‘} \Lambda) \rceil P_\Delta\} \beta .$

$\equiv . \lambda \neq \Lambda . \lambda P_\Delta \alpha . \lambda P_\Delta \beta .$

$[*24 \cdot 54 . *80 \cdot 13] \quad \equiv . \exists ! \lambda . \lambda = P_\Delta \text{‘} \alpha . \lambda = P_\Delta \text{‘} \beta .$

$[*80 \cdot 81] \quad \supset . \alpha = \beta$ (2)

$\vdash . (2) . *71 \cdot 171 . \supset \vdash . (-\iota \text{‘} \Lambda) \rceil P_\Delta \in \mathrm{Cls} \to 1$ (3)

$\vdash . (1) . (3) . \supset \vdash . \mathrm{Prop}$

***80·84.** $\vdash : \Lambda \sim\epsilon P_\Delta“\kappa \,.\supset .\, P_\Delta“\kappa \;\mathrm{sm}\; \kappa$

Dem.

$$\begin{array}{lll}
\vdash . *51{\cdot}36 . & \supset \vdash : \mathrm{Hp} . \supset . P_\Delta“\kappa \subset - \iota‘\Lambda . & (1) \\
[*37{\cdot}42] & \supset . P_\Delta“\kappa = \{(-\iota‘\Lambda) \upharpoonleft P_\Delta\}“\kappa & (2) \\
\vdash . *80{\cdot}12 . *33{\cdot}431 . & \supset \vdash . \kappa \subset C‘P_\Delta . & \\
[*37{\cdot}51] & \supset \vdash . \kappa \subset \breve{P}_\Delta“P_\Delta“\kappa & (3) \\
\vdash . (1) . *37{\cdot}2 . & \supset \vdash : \mathrm{Hp} . \supset . \breve{P}_\Delta“P_\Delta“\kappa \subset \breve{P}_\Delta“(-\iota‘\Lambda) & \\
[*37{\cdot}4] & \qquad \subset C‘\{(-\iota‘\Lambda) \upharpoonleft P_\Delta\} & (4) \\
\vdash . (3) . (4) . & \supset \vdash : \mathrm{Hp} . \supset . \kappa \subset C‘\{(-\iota‘\Lambda) \upharpoonleft P_\Delta\} & (5) \\
\vdash . (5) . *80{\cdot}83 . *73{\cdot}22 . & \supset \vdash : \mathrm{Hp} . \supset . \{(-\iota‘\Lambda) \upharpoonleft P_\Delta\}“\kappa \;\mathrm{sm}\; \kappa & (6) \\
\vdash . (2) . (6) . \supset \vdash . \mathrm{Prop} & &
\end{array}$$

The three following propositions are useful both in cardinal and in ordinal multiplication (*113 and *172).

***80·9.** $\vdash :. y \neq z . \supset : M \epsilon P_\Delta‘(\iota‘y \cup \iota‘z) . \equiv . (\exists u, v) . uPy . vPz . M = u \downarrow y \,\dot\cup\, v \downarrow z$

Dem.

$$\begin{array}{ll}
\vdash . *80{\cdot}45{\cdot}66 . \supset \vdash :. \mathrm{Hp} . \supset : M \epsilon P_\Delta‘(\iota‘y \cup \iota‘z) . \equiv . & \\
\qquad (\exists R, S) . R \epsilon \downarrow y“\overrightarrow{P}‘y . S \epsilon \downarrow z“\overrightarrow{P}‘z . M = R \,\dot\cup\, S . & \\
[*38{\cdot}131 . *32{\cdot}18] \quad \equiv . (\exists u, v) . uPy . vPz . M = u \downarrow y \,\dot\cup\, v \downarrow z :. \supset \vdash . \mathrm{Prop} &
\end{array}$$

***80·91.** $\vdash : M \epsilon P_\Delta‘(\iota‘y \cup \iota‘z) . \supset . M = (M‘y) \downarrow y \,\dot\cup\, (M‘z) \downarrow z$

Dem.

$$\begin{array}{ll}
\vdash . *71{\cdot}6 . *80{\cdot}14 . \supset & \\
\vdash : \mathrm{Hp} . \supset . M = \dot{s}‘\hat{Q}\{(\exists w) . w \epsilon \iota‘y \cup \iota‘z . Q = (M‘w) \downarrow w\} & \\
[*51{\cdot}235] \quad = \dot{s}‘\hat{Q}\{Q = (M‘y) \downarrow y . \vee . Q = (M‘z) \downarrow z\} & \\
[*51{\cdot}232] \quad = \dot{s}‘\{\iota‘(M‘y) \downarrow y \cup \iota‘(M‘z) \downarrow z\} & \\
[*53{\cdot}13] \quad = (M‘y) \downarrow y \,\dot\cup\, (M‘z) \downarrow z : \supset \vdash . \mathrm{Prop} &
\end{array}$$

*80·9·91 can be extended, by precisely similar proofs, to any finite number of variables $y, z, \ldots$. They will, on occasion, be assumed for three or four variables, without fresh proofs.

***80·92.** $\vdash : y \neq z . \supset . \mathrm{D}“P_\Delta‘(\iota‘y \cup \iota‘z) = \hat{\xi}\{(\exists u, v) . uPy . vPz . \xi = \iota‘u \cup \iota‘v\}$

Dem.

$$\begin{array}{lll}
\vdash . *55{\cdot}15 . *33{\cdot}26 . \quad \supset \vdash . \mathrm{D}‘(u \downarrow y \,\dot\cup\, v \downarrow z) = \iota‘u \cup \iota‘v & & (1) \\
\vdash . (1) . *80{\cdot}9 . *37{\cdot}6 . \supset \vdash :. \mathrm{Hp} . \supset : \xi \epsilon \mathrm{D}“P_\Delta‘(\iota‘y \cup \iota‘z) . \equiv . & & \\
\qquad (\exists u, v, M) . uPy . vPz . M = u \downarrow y \,\dot\cup\, v \downarrow z . \xi = \iota‘u \cup \iota‘v . & & \\
[*13{\cdot}19] \quad \equiv . (\exists u, v) . uPy . vPz . \xi = \iota‘u \cup \iota‘v :. \supset \vdash . \mathrm{Prop} & &
\end{array}$$

***80·93.** $\vdash : \exists ! P_\Delta‘(\iota‘y \cup \iota‘z) . \equiv . y, z \epsilon C‘P$ [*80·46·69]

***80·94.** $\vdash : \exists ! P_\Delta‘(\beta \cup \iota‘z) . \equiv . \exists ! P_\Delta‘\beta . z \epsilon C‘P$ [*80·46·69]

From this proposition, together with *80·26 (which gives $\exists ! P_\Delta‘\Lambda$), we shall obtain an inductive proof that $P_\Delta‘\beta$ exists whenever β is a finite class contained in $C‘P$ (cf. *120·611).

$*81$. SELECTIONS FROM MANY-ONE RELATIONS

Summary of $*81$.

When $P \upharpoonright \kappa$ is a many-one relation, $P_\Delta{}^\backprime\kappa$ has many important properties which do not hold in the general case. In the first place, $P_\Delta{}^\backprime\kappa$ consists wholly of one-one relations. In the second place, if $R \in P_\Delta{}^\backprime\kappa$, $\text{D}^\backprime R$ takes one term and no more out of each member of $\overrightarrow{P}{}^{\backprime\backprime}\kappa$. Again, if $R \in P_\Delta{}^\backprime\kappa$, R is determinate when $\text{D}^\backprime R$ is given; *i.e.* $R, S \in P_\Delta{}^\backprime\kappa \,.\, \text{D}^\backprime R = \text{D}^\backprime S \,.\supset.\, R = S$. It follows that $\text{D}^{\backprime\backprime}P_\Delta{}^\backprime\kappa$ is similar to $P_\Delta{}^\backprime\kappa$; hence the number of members of $P_\Delta{}^\backprime\kappa$ is the number of ways of choosing one member out of each class belonging to $\overrightarrow{P}{}^{\backprime\backprime}\kappa$. It should be remembered that when $P \upharpoonright \kappa$ is many-one, $\overrightarrow{P}{}^{\backprime\backprime}\kappa$ is a class of mutually exclusive classes, *i.e.* no two different members of $\overrightarrow{P}{}^{\backprime\backprime}\kappa$ have any common member. This follows immediately from $*71{\cdot}181$.

As explained in the introduction to this section, the propositions of this number are chiefly useful on account of their application to the case of ϵ. This application is made in $*84$. The most important propositions in this number are:

$*81{\cdot}1$. $\vdash : P \upharpoonright \kappa \in \text{Cls} \to 1 \,.\supset.\, P_\Delta{}^\backprime\kappa \subset 1 \to 1$

$*81{\cdot}14$. $\vdash : P \upharpoonright \kappa \in \text{Cls} \to 1 \,.\, R \in P_\Delta{}^\backprime\kappa \,.\supset.\, R = (\text{D}^\backprime R) \upharpoonleft P \upharpoonright \kappa = P \mathbin{\dot\cap} \text{D}^\backprime R \uparrow \kappa$

This proposition, by exhibiting R as a function of $\text{D}^\backprime R$, leads immediately to

$*81{\cdot}21$. $\vdash : P \upharpoonright \kappa \in \text{Cls} \to 1 \,.\supset.\, \text{D} \upharpoonright P_\Delta{}^\backprime\kappa \in 1 \to 1 \,.\, \text{D}^{\backprime\backprime}P_\Delta{}^\backprime\kappa \,\text{sm}\, P_\Delta{}^\backprime\kappa$

This is the principal proposition of this number. The following also is important:

$*81{\cdot}22$. $\vdash : P \upharpoonright \kappa \in \text{Cls} \to 1 \,.\supset.\, \text{D}^{\backprime\backprime}P_\Delta{}^\backprime\kappa = \hat\mu \{ y \in \kappa \,.\supset_y.\, \mu \cap \overrightarrow{P}{}^\backprime y \in 1 : \mu \subset P^{\backprime\backprime}\kappa \}$

$*81{\cdot}1$. $\vdash : P \upharpoonright \kappa \in \text{Cls} \to 1 \,.\supset.\, P_\Delta{}^\backprime\kappa \subset 1 \to 1$

Dem.

$$\begin{array}{lr}
\vdash . *80{\cdot}14 \,.\supset \vdash : R \in P_\Delta{}^\backprime\kappa \,.\supset.\, R \in 1 \to \text{Cls} & (1) \\
\vdash . *80{\cdot}291 \,.\supset \vdash :. R \in P_\Delta{}^\backprime\kappa \,.\supset : R \mathrel{\mathsf{G}} P \upharpoonright \kappa : & \\
[*71{\cdot}221] \qquad \supset : P \upharpoonright \kappa \in \text{Cls} \to 1 \,.\supset.\, R \in \text{Cls} \to 1 & (2) \\
\vdash . (1) . (2) \,.\supset \vdash . \text{Prop} &
\end{array}$$

$*81{\cdot}11$. $\vdash : P \upharpoonright \kappa \in \text{Cls} \to 1 \,.\, R \in P_\Delta{}^\backprime\kappa \,.\, x \in \text{D}^\backprime R \,.\supset.\, \text{E}! \breve{R}^\backprime x \,.\, x (P \upharpoonright \kappa) \breve{R}^\backprime x$

Dem.

$$\begin{array}{lr}
\vdash . *71{\cdot}165 . *81{\cdot}1 \,.\supset \vdash : \text{Hp} \,.\supset.\, \text{E}! \breve{R}^\backprime x \,. & (1) \\
[*30{\cdot}32 . *31{\cdot}11] \qquad \supset.\, x R (\breve{R}^\backprime x) \,. & \\
[*80{\cdot}291] \qquad \supset.\, x (P \upharpoonright \kappa) \breve{R}^\backprime x & (2) \\
\vdash . (1) . (2) \,.\supset \vdash . \text{Prop} &
\end{array}$$

∗81·12. $\vdash : P \upharpoonright \kappa \in \mathrm{Cls} \to 1 \,.\, R \in P_\Delta\text{‘}\kappa \,.\, x \in \mathrm{D}\text{‘}R \,.\, \supset .$

$$\breve{R}\text{‘}x = (\imath y)(y \in \kappa \,.\, xPy) = (\kappa \rceil \breve{P})\text{‘}x$$

Dem.

$\vdash . ∗71{\cdot}361 \,.\, \supset \vdash :. \mathrm{Hp} \,.\, \supset : x(P \upharpoonright \kappa)\breve{R}\text{‘}x \,.\, \equiv .\, \breve{R}\text{‘}x = \{\mathrm{Cnv}\text{‘}(P \upharpoonright \kappa)\}\text{‘}x :$

$[∗81{\cdot}11] \qquad \supset : \breve{R}\text{‘}x = \{\mathrm{Cnv}\text{‘}(P \upharpoonright \kappa)\}\text{‘}x$

$[∗35{\cdot}52] \qquad = (\kappa \rceil \breve{P})\text{‘}x \qquad (1)$

$[∗35{\cdot}1] \qquad = (\imath y)(y \in \kappa \,.\, xPy) \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

∗81·13. $\vdash :. P \upharpoonright \kappa \in \mathrm{Cls} \to 1 \,.\, R \in P_\Delta\text{‘}\kappa \,.\, \supset : xRy \,.\, \equiv .\, x \in \mathrm{D}\text{‘}R \,.\, xPy \,.\, y \in \kappa$

Dem.

$\vdash . ∗81{\cdot}12 \,.\, \supset \vdash :: \mathrm{Hp} \,.\, \supset :. x \in \mathrm{D}\text{‘}R \,.\, \supset : y = \breve{R}\text{‘}x \,.\, \equiv .\, y = (\kappa \rceil \breve{P})\text{‘}x :$

$[∗71{\cdot}361] \qquad \supset : xRy \,.\, \equiv .\, x(P \upharpoonright \kappa)y \,.$

$[∗35{\cdot}101] \qquad \equiv .\, xPy \,.\, y \in \kappa \qquad (1)$

$\vdash . (1) . ∗5{\cdot}32 \,.\, \supset$

$\vdash :. \mathrm{Hp} \,.\, \supset : x \in \mathrm{D}\text{‘}R \,.\, xRy \,.\, \equiv .\, x \in \mathrm{D}\text{‘}R \,.\, xPy \,.\, y \in \kappa :$

$[∗33{\cdot}14 . ∗4{\cdot}71] \qquad \supset : xRy \,.\, \equiv .\, x \in \mathrm{D}\text{‘}R \,.\, xPy \,.\, y \in \kappa :. \supset \vdash . \mathrm{Prop}$

∗81·14. $\vdash : P \upharpoonright \kappa \in \mathrm{Cls} \to 1 \,.\, R \in P_\Delta\text{‘}\kappa \,.\, \supset .\, R = (\mathrm{D}\text{‘}R) \rceil P \upharpoonright \kappa = P \mathbin{\dot{\cap}} \mathrm{D}\text{‘}R \uparrow \kappa$
$[∗81{\cdot}13 \,.\, ∗35{\cdot}102{\cdot}822]$

This proposition, by exhibiting R as a function of $\mathrm{D}\text{‘}R$, shows that a member of $P_\Delta\text{‘}\kappa$ is determinate when its domain is given, provided $P \upharpoonright \kappa \in \mathrm{Cls} \to 1$.

∗81·15. $\vdash : P \upharpoonright \kappa \in \mathrm{Cls} \to 1 \,.\, R \in P_\Delta\text{‘}\kappa \,.\, y \in \kappa \,.\, \supset .\, \iota\text{‘}R\text{‘}y = \mathrm{D}\text{‘}R \cap \overrightarrow{P}\text{‘}y$

Dem.

$\vdash . ∗81{\cdot}13 \,.\, \supset \vdash :. \mathrm{Hp} \,.\, \supset : xRy \,.\, \equiv_x .\, x \in \mathrm{D}\text{‘}R \,.\, xPy :$

$[∗32{\cdot}18] \qquad \supset : x \in \overrightarrow{R}\text{‘}y \,.\, \equiv_x .\, x \in \mathrm{D}\text{‘}R \,.\, x \in \overrightarrow{P}\text{‘}y :$

$[∗20{\cdot}43 . ∗22{\cdot}33] \qquad \supset : \overrightarrow{R}\text{‘}y = \mathrm{D}\text{‘}R \cap \overrightarrow{P}\text{‘}y :$

$[∗53{\cdot}31 . ∗71{\cdot}163 . ∗80{\cdot}14] \supset : \iota\text{‘}R\text{‘}y = \mathrm{D}\text{‘}R \cap \overrightarrow{P}\text{‘}y :. \supset \vdash . \mathrm{Prop}$

∗81·2. $\vdash :. P \upharpoonright \kappa \in \mathrm{Cls} \to 1 \,.\, R, S \in P_\Delta\text{‘}\kappa \,.\, \supset : \mathrm{D}\text{‘}R = \mathrm{D}\text{‘}S \,.\, \equiv .\, R = S$

Dem.

$\vdash . ∗30{\cdot}37 \,.\, ∗33{\cdot}12 \,.\, \supset \vdash : R = S \,.\, \supset .\, \mathrm{D}\text{‘}R = \mathrm{D}\text{‘}S \qquad (1)$

$\vdash . ∗81{\cdot}14 \,.\, ∗13{\cdot}12 \,.\, \supset \vdash :. \mathrm{Hp} \,.\, \supset : \mathrm{D}\text{‘}R = \mathrm{D}\text{‘}S \,.\, \supset .\, R = P \mathbin{\dot{\cap}} \mathrm{D}\text{‘}S \uparrow \kappa$

$[∗81{\cdot}14] \qquad = S \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

∗81·21. $\vdash : P \upharpoonright \kappa \in \mathrm{Cls} \to 1 \,.\, \supset .\, \mathrm{D} \upharpoonright P_\Delta\text{‘}\kappa \in 1 \to 1 \,.\, \mathrm{D}\text{‘‘}P_\Delta\text{‘}\kappa \;\mathrm{sm}\; P_\Delta\text{‘}\kappa$
$[∗81{\cdot}2 \,.\, ∗71{\cdot}59 \,.\, ∗73{\cdot}28]$

This proposition is very important. The class $\mathrm{D}``P_\Delta`\kappa$, when $P\restriction\kappa\in\mathrm{Cls}\to 1$, is formed, as we shall prove later, by making every possible selection of one term out of each member of $\overrightarrow{P}``\kappa$, each such selection giving us one member of $\mathrm{D}``P_\Delta`\kappa$. The fact that, with the above hypothesis, the class of classes $\mathrm{D}``P_\Delta`\kappa$ has the same number of terms as $P_\Delta`\kappa$ (which results from the above proposition), is of great utility in the theory of cardinal multiplication and exponentiation.

***81·211.** $\vdash : P\restriction\kappa\in\mathrm{Cls}\to 1 \,.\supset .\, \mathrm{D}``P_\Delta`\kappa \subset \hat{\mu}\{y\in\kappa \,.\supset_y.\, \mu\cap\overrightarrow{P}`y\in 1 : \mu\subset P``\kappa\}$

Dem.

$\vdash . *81·15 . *52·1 . \supset\vdash :. \mathrm{Hp} . R\in P_\Delta`\kappa . \mu=\mathrm{D}`R . \supset : y\in\kappa . \supset_y . \mu\cap\overrightarrow{P}`y\in 1 :.$

$[*10·11·23·35] \quad \supset\vdash :. \mathrm{Hp} : (\exists R) . R\in P_\Delta`\kappa . \mu=\mathrm{D}`R : \supset : y\in\kappa . \supset_y . \mu\cap\overrightarrow{P}`y\in 1 :.$

$[*37·6 . *33·12] \quad \supset\vdash :. \mathrm{Hp} . \mu\in\mathrm{D}``P_\Delta`\kappa . \supset : y\in\kappa . \supset_y . \mu\cap\overrightarrow{P}`y\in 1 \quad (1)$

$\vdash . *80·291 . *33·263 . \supset$

$\vdash : R\in P_\Delta`\kappa . \mu=\mathrm{D}`R . \supset . \mu\subset\mathrm{D}`(P\restriction\kappa) .$

$[*37·401] \quad \supset . \mu\subset P``\kappa :$

$[*10·11·23·35] \quad \supset\vdash : (\exists R) . R\in P_\Delta`\kappa . \mu=\mathrm{D}`R . \supset . \mu\subset P``\kappa :$

$[*37·6 . *33·12] \quad \supset\vdash : \mu\in\mathrm{D}``P_\Delta`\kappa . \supset . \mu\subset P``\kappa \quad (2)$

$\vdash . (1) . (2) . \supset\vdash . \mathrm{Prop}$

***81·212.** $\vdash :. y\in\kappa . \supset_y . \mu\cap\overrightarrow{P}`y\in 1 : \mu\subset P``\kappa : \supset . \mu\in\mathrm{D}``P_\Delta`\kappa . \mu\upharpoonleft P\restriction\kappa\in P_\Delta`\kappa$

Dem.

$\vdash . *35·442 . *37·402 . \supset$

$\vdash : R=\mu\upharpoonleft P\restriction\kappa . \supset . R\mathrel{\dot\subset} P . \text{Ɑ}`R=\kappa\cap\breve{P}``\mu . \mathrm{D}`R=\mu\cap P``\kappa \quad (1)$

$\vdash . *52·16 . \quad \supset\vdash :. \mathrm{Hp} . \supset : y\in\kappa . \supset_y . \exists ! \mu\cap\overrightarrow{P}`y .$

$[*37·46 . *32·241] \quad \supset_y . y\in\breve{P}``\mu :$

$[*22·1] \quad \supset : \kappa\subset\breve{P}``\mu \quad (2)$

$\vdash . (1) . (2) . *22·621 . \supset\vdash : \mathrm{Hp} . R=\mu\upharpoonleft P\restriction\kappa . \supset . R\mathrel{\dot\subset} P . \text{Ɑ}`R=\kappa . \mathrm{D}`R=\mu \quad (3)$

$\vdash . *32·18 . *35·102 . \supset\vdash :. \mathrm{Hp}(3) . \supset : y\in\kappa . \supset_y . \overrightarrow{R}`y=\mu\cap\overrightarrow{P}`y .$

$[\mathrm{Hp}] \quad \supset_y . \overrightarrow{R}`y\in 1 :$

$[*37·702] \quad \supset : \overrightarrow{R}``\kappa\subset 1 :$

$[(3) . *71·1] \quad \supset : R\in 1\to\mathrm{Cls} \quad (4)$

$\vdash . (3) . (4) . *80·14 . \supset\vdash : \mathrm{Hp} . \supset . \mu\upharpoonleft P\restriction\kappa\in P_\Delta`\kappa . \mathrm{D}`(\mu\upharpoonleft P\restriction\kappa)=\mu . \quad (5)$

$[*37·6] \quad \supset . \mu\in\mathrm{D}``P_\Delta`\kappa \quad (6)$

$\vdash . (5) . (6) . \supset\vdash . \mathrm{Prop}$

***81·22.** $\vdash : P\restriction\kappa\in\mathrm{Cls}\to 1 . \supset . \mathrm{D}``P_\Delta`\kappa=\hat{\mu}\{y\in\kappa . \supset_y . \mu\cap\overrightarrow{P}`y\in 1 : \mu\subset P``\kappa\}$
[*81·211·212]

***81·221.** $\vdash : P \restriction \kappa \in \mathrm{Cls} \to 1 \,.\supset .\, P_\Delta\text{‘}\kappa = \upharpoonleft(P \restriction \kappa)\text{“}\mathrm{D}\text{“}P_\Delta\text{‘}\kappa$

Dem.

$\vdash . *81\cdot14 . *37\cdot62 . \supset$

$\vdash :. \mathrm{Hp} . \supset : R \in P_\Delta\text{‘}\kappa \,.\supset_R .\, R = (\mathrm{D}\text{‘}R) \upharpoonleft P \restriction \kappa \,.\, \mathrm{D}\text{‘}R \in \mathrm{D}\text{“}P_\Delta\text{‘}\kappa \,.$

[*10·24] $\qquad \supset_R . (\exists \mu) . R = \mu \upharpoonleft P \restriction \kappa \,.\, \mu \in \mathrm{D}\text{“}P_\Delta\text{‘}\kappa \,.$

[*38·131] $\qquad \supset_R . R \in \upharpoonleft(P \restriction \kappa)\text{“}\mathrm{D}\text{“}P_\Delta\text{‘}\kappa \qquad (1)$

$\vdash . *81\cdot22\cdot212 . \supset \vdash :. \mathrm{Hp} . \supset : \mu \in \mathrm{D}\text{“}P_\Delta\text{‘}\kappa \,.\supset_\mu .\, \mu \upharpoonleft P \restriction \kappa \in P_\Delta\text{‘}\kappa :$

[*37·61] $\qquad \supset : \upharpoonleft(P \restriction \kappa)\text{“}\mathrm{D}\text{“}P_\Delta\text{‘}\kappa \subset P_\Delta\text{‘}\kappa \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***81·23.** $\vdash : P \restriction \kappa \in \mathrm{Cls} \to 1 \,.\, R \in P_\Delta\text{‘}\kappa \,.\, y \in \kappa \,.\supset .\, \mathrm{D}\text{‘}R - \overrightarrow{P}\text{‘}y = \mathrm{D}\text{‘}R - \iota\text{‘}R\text{‘}y$

Dem.

$\vdash . *22\cdot93 . \supset \vdash . \mathrm{D}\text{‘}R - \overrightarrow{P}\text{‘}y = \mathrm{D}\text{‘}R - (\mathrm{D}\text{‘}R \cap \overrightarrow{P}\text{‘}y) \qquad (1)$

$\vdash . *81\cdot15 . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{D}\text{‘}R - (\mathrm{D}\text{‘}R \cap \overrightarrow{P}\text{‘}y) = \mathrm{D}\text{‘}R - \iota\text{‘}R\text{‘}y \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***81·24.** $\vdash : P \restriction \kappa \in \mathrm{Cls} \to 1 \,.\, \mu \in \mathrm{D}\text{“}P_\Delta\text{‘}\kappa \,.\, y \in \kappa \,.\supset .\, \mu - \overrightarrow{P}\text{‘}y \in \mathrm{D}\text{“}P_\Delta\text{‘}(\kappa \dot{-} \iota\text{‘}y)$

Dem.

$\vdash . *80\cdot78 . \supset \vdash : R \in P_\Delta\text{‘}\kappa \,.\, y \in \kappa \,.\supset .\, R \dot{-} (R\text{‘}y) \downarrow y \in P_\Delta\text{‘}(\kappa - \iota\text{‘}y) \,.$

[*37·62.*33·12] $\qquad \supset . \mathrm{D}\text{‘}\{R \dot{-} (R\text{‘}y) \downarrow y\} \in \mathrm{D}\text{“}P_\Delta\text{‘}(\kappa - \iota\text{‘}y) \qquad (1)$

$\vdash . *81\cdot1 . *80\cdot14 . \supset$

$\vdash : P \restriction \kappa \in \mathrm{Cls} \to 1 \,.\, R \in P_\Delta\text{‘}\kappa \,.\, y \in \kappa \,.\supset .\, R \in 1 \to 1 \,.\, y \in \mathrm{C\!\!I}\text{‘}R \,.$

[*72·911.*71·31.*55·3] $\qquad \supset . \mathrm{D}\text{‘}\{R \dot{-} (R\text{‘}y) \downarrow y\} = \mathrm{D}\text{‘}R - \iota\text{‘}R\text{‘}y$

[*81·23] $\qquad = \mathrm{D}\text{‘}R - \overrightarrow{P}\text{‘}y \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash : \mathrm{Hp}(2) . \mathrm{D}\text{‘}R = \mu \,.\supset .\, \mu - \overrightarrow{P}\text{‘}y \in \mathrm{D}\text{“}P_\Delta\text{‘}(\kappa - \iota\text{‘}y) \qquad (3)$

$\vdash . (3) . *10\cdot11\cdot23\cdot35 . *37\cdot6 . *33\cdot12 . \supset \vdash . \mathrm{Prop}$

***81·25.** $\vdash : y \in \kappa \,.\, xPy \,.\, \mu \in \mathrm{D}\text{“}P_\Delta\text{‘}(\kappa - \iota\text{‘}y) \,.\supset .\, \mu \cup \iota\text{‘}x \in \mathrm{D}\text{“}P_\Delta\text{‘}\kappa$

Dem.

$\vdash . *80\cdot68 . \supset \vdash : y \in \kappa \,.\, xPy \,.\, R \in P_\Delta\text{‘}(\kappa - \iota\text{‘}y) \,.\supset .\, R \,\dot{\cup}\, x \downarrow y \in P_\Delta\text{‘}\kappa \,.$

[*37·62] $\qquad \supset . \mathrm{D}\text{‘}(R \,\dot{\cup}\, x \downarrow y) \in \mathrm{D}\text{“}P_\Delta\text{‘}\kappa \,.$

[*33·26.*55·15] $\qquad \supset . \mathrm{D}\text{‘}R \cup \iota\text{‘}x \in \mathrm{D}\text{“}P_\Delta\text{‘}\kappa \qquad (1)$

$\vdash . (1) . \supset \vdash : y \in \kappa \,.\, xPy \,.\, R \in P_\Delta\text{‘}(\kappa - \iota\text{‘}y) \,.\, \mu = \mathrm{D}\text{‘}R \,.\supset .\, \mu \cup \iota\text{‘}x \in \mathrm{D}\text{“}P_\Delta\text{‘}\kappa \qquad (2)$

$\vdash . (2) . *10\cdot11\cdot23\cdot35 . *37\cdot6 . \supset \vdash . \mathrm{Prop}$

***81·26.** $\vdash :. P \restriction \kappa \in \mathrm{Cls} \to 1 \,.\, y \in \kappa \,.\, \mu \cap \overrightarrow{P}\text{‘}y \in 1 \,.\supset :$

$\qquad \mu - \overrightarrow{P}\text{‘}y \in \mathrm{D}\text{“}P_\Delta\text{‘}(\kappa - \iota\text{‘}y) \,.\equiv .\, \mu \in \mathrm{D}\text{“}P_\Delta\text{‘}\kappa$

Dem.

$\vdash . *81\cdot24 . \supset \vdash :. \mathrm{Hp} . \supset : \mu \in \mathrm{D}\text{“}P_\Delta\text{‘}\kappa \,.\supset .\, \mu - \overrightarrow{P}\text{‘}y \in \mathrm{D}\text{“}P_\Delta\text{‘}(\kappa - \iota\text{‘}y) \qquad (1)$

$\vdash . *81\cdot25 . \supset \vdash :. \mathrm{Hp} . \supset : \mu \cap \overrightarrow{P}\text{‘}y = \iota\text{‘}x \,.\, \mu - \overrightarrow{P}\text{‘}y \in \mathrm{D}\text{“}P_\Delta\text{‘}(\kappa - \iota\text{‘}y) \,.\supset .$

$\qquad (\mu - \overrightarrow{P}\text{‘}y) \cup \iota\text{‘}x \in \mathrm{D}\text{“}P_\Delta\text{‘}\kappa \qquad (2)$

$\vdash . *22{\cdot}551 . \quad \supset \vdash : \mu \cap \overrightarrow{P}{}^{\backprime}y = \iota{}^{\backprime}x . \supset . (\mu - \overrightarrow{P}{}^{\backprime}y) \cup \iota{}^{\backprime}x = (\mu - \overrightarrow{P}{}^{\backprime}y) \cup (\mu \cap \overrightarrow{P}{}^{\backprime}y)$

$[*24{\cdot}41] \qquad = \mu \qquad (3)$

$\vdash . *52{\cdot}1 . \quad \supset \vdash : \mathrm{Hp} . \supset . (\exists x) . \mu \cap \overrightarrow{P}{}^{\backprime}y = \iota{}^{\backprime}x \qquad (4)$

$\vdash . (2) . (3) . (4) . \supset \vdash :. \mathrm{Hp} . \supset : \mu - \overrightarrow{P}{}^{\backprime}y \in \mathrm{D}{}^{\backprime\backprime}P_{\Delta}{}^{\backprime}(\kappa - \iota{}^{\backprime}y) . \supset . \mu \in \mathrm{D}{}^{\backprime\backprime}P_{\Delta}{}^{\backprime}\kappa \qquad (5)$

$\vdash . (1) . (5) . \quad \supset \vdash . \mathrm{Prop}$

$*81{\cdot}3$. $\vdash : P \upharpoonright \kappa \in \mathrm{Cls} \rightarrow 1 . \lambda = \overrightarrow{P}{}^{\backprime\backprime}\kappa . \supset . \mathrm{D}{}^{\backprime\backprime}P_{\Delta}{}^{\backprime}\kappa = \hat{\mu}\{\alpha \in \lambda . \supset_{\alpha} . \mu \cap \alpha \in 1 : \mu \subset s{}^{\backprime}\lambda\}$

Dem.

$\vdash . *37{\cdot}706 . \supset \vdash :. y \in \kappa . \supset_y . \mu \cap \overrightarrow{P}{}^{\backprime}y \in 1 : \equiv : \alpha \in \overrightarrow{P}{}^{\backprime\backprime}\kappa . \supset_{\alpha} . \mu \cap \alpha \in 1 \qquad (1)$

$\vdash . *40{\cdot}5 . \quad \supset \vdash : \mu \subset P{}^{\backprime\backprime}\kappa . \equiv . \mu \subset s{}^{\backprime}\overrightarrow{P}{}^{\backprime\backprime}\kappa \qquad (2)$

$\vdash . (1) . (2) . *81{\cdot}22 . \supset$

$\vdash : P \upharpoonright \kappa \in \mathrm{Cls} \rightarrow 1 . \supset . \mathrm{D}{}^{\backprime\backprime}P_{\Delta}{}^{\backprime}\kappa = \hat{\mu}\{\alpha \in \overrightarrow{P}{}^{\backprime\backprime}\kappa . \supset_{\alpha} . \mu \cap \alpha \in 1 : \mu \subset s{}^{\backprime}\overrightarrow{P}{}^{\backprime\backprime}\kappa\} \qquad (3)$

$\vdash . (3) . *13{\cdot}12 . \supset \vdash . \mathrm{Prop}$

$*81{\cdot}31$. $\vdash : P \upharpoonright \kappa, Q \upharpoonright \kappa \in \mathrm{Cls} \rightarrow 1 . \overrightarrow{P}{}^{\backprime\backprime}\kappa = \overrightarrow{Q}{}^{\backprime\backprime}\kappa . \supset . \mathrm{D}{}^{\backprime\backprime}P_{\Delta}{}^{\backprime}\kappa = \mathrm{D}{}^{\backprime\backprime}Q_{\Delta}{}^{\backprime}\kappa$

Dem.

$\vdash . *81{\cdot}3 . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{D}{}^{\backprime\backprime}P_{\Delta}{}^{\backprime}\kappa = \hat{\mu}\{\alpha \in \overrightarrow{Q}{}^{\backprime\backprime}\kappa . \supset_{\alpha} . \mu \cap \alpha \in 1 : \mu \subset s{}^{\backprime}\overrightarrow{Q}{}^{\backprime\backprime}\kappa\}$

$[*81{\cdot}3] \qquad = \mathrm{D}{}^{\backprime\backprime}Q_{\Delta}{}^{\backprime}\kappa : \supset \vdash . \mathrm{Prop}$

$*82$. SELECTIONS FROM RELATIVE PRODUCTS

Summary of $*82$.

The propositions contained in this number are not much used except in connection with the associative law for cardinal multiplication, but they have a certain intrinsic interest. We prove in this number that, with a suitable hypothesis, $(P \mid Q)_\Delta`\lambda$ results from $P_\Delta`Q``\lambda$ by multiplying each member by Q, *i.e.*

$*82{\cdot}272$. $\vdash : Q \upharpoonright \lambda \in 1 \to 1 \,.\, \lambda \in \mathrm{D}`(\breve{Q})_\epsilon \,.\supset.\, (P \mid Q)_\Delta`\lambda = |\, Q``P_\Delta`Q``\lambda$

Also under a suitable hypothesis the domains of $(P \mid Q)_\Delta`\lambda$ are the domains of $P_\Delta`Q``\lambda$, *i.e.*

$*82{\cdot}32$. $\vdash : Q \upharpoonright \lambda \in 1 \to 1 \,.\, \lambda \subset \text{Ɑ}`Q \,.\supset.\, \mathrm{D}``(P \mid Q)_\Delta`\lambda = \mathrm{D}``P_\Delta`Q``\lambda$

In the applications of propositions of the present number in $*85$, P and Q are replaced by ϵ and $\overrightarrow{Q}$. By $*62{\cdot}26$, $\epsilon \mid \overrightarrow{Q} = Q$; thus we obtain relations between $Q_\Delta`\lambda$ and $\epsilon_\Delta`\overrightarrow{Q}``\lambda$.

$*82{\cdot}2$. $\vdash : M \in P_\Delta`\kappa \,.\, N \in Q_\Delta`\lambda \,.\, Q``\lambda \subset \kappa \,.\supset.\, M \mid N \in (P \mid Q)_\Delta`\lambda$

Dem.

$\vdash . *80{\cdot}14 . \supset \vdash : \mathrm{Hp} . \supset . M, N \in 1 \to \mathrm{Cls} .$

$[*71{\cdot}25] \qquad \supset . M \mid N \in 1 \to \mathrm{Cls} \qquad (1)$

$\vdash . *80{\cdot}14 . \supset \vdash : \mathrm{Hp} . \supset . M \Subset P . N \Subset Q .$

$[*34{\cdot}34] \qquad \supset . M \mid N \Subset P \mid Q \qquad (2)$

$\vdash . *80{\cdot}14 . \supset \vdash : \mathrm{Hp} . \supset . \text{Ɑ}`M = \kappa .$

$[*37{\cdot}32] \qquad \supset . \text{Ɑ}`(M \mid N) = \breve{N}``\kappa \qquad (3)$

$\vdash . *80{\cdot}14 . \supset \vdash : \mathrm{Hp} . \supset . N \Subset Q . \text{Ɑ}`N = \lambda . \qquad (4)$

$[*37{\cdot}201{\cdot}25] \qquad \supset . N``\lambda \subset Q``\lambda . N``\lambda = \mathrm{D}`N .$

$[\mathrm{Hp}] \qquad \supset . \mathrm{D}`N \subset \kappa .$

$[*37{\cdot}271] \qquad \supset . \breve{N}``\kappa = \text{Ɑ}`N \qquad (5)$

$\vdash . (3) . (4) . (5) . \supset \vdash : \mathrm{Hp} . \supset . \text{Ɑ}`(M \mid N) = \lambda \qquad (6)$

$\vdash . (1) . (2) . (6) . *80{\cdot}14 . \supset \vdash . \mathrm{Prop}$

$*82{\cdot}21$. $\vdash : Q \upharpoonright \lambda \in 1 \to \mathrm{Cls} \,.\, \lambda \subset \text{Ɑ}`Q \,.\supset.\, Q_\Delta`\lambda = \iota`Q \upharpoonright \lambda$

Dem.

$\vdash . *80{\cdot}291{\cdot}14 . \supset \vdash :. \mathrm{Hp} . \supset : R \in Q_\Delta`\lambda . \supset . R \Subset Q \upharpoonright \lambda . \text{Ɑ}`R = \lambda .$

$[*72{\cdot}92] \qquad \supset . R = (Q \upharpoonright \lambda) \upharpoonright \text{Ɑ}`R . \text{Ɑ}`R = \lambda .$

$[*35{\cdot}31] \qquad \supset . R = Q \upharpoonright \lambda \qquad (1)$

$\vdash . *35{\cdot}441{\cdot}65 . \supset \vdash : \mathrm{Hp} . \supset . Q \upharpoonright \lambda \in 1 \to \mathrm{Cls} . Q \upharpoonright \lambda \Subset Q . \text{Ɑ}`(Q \upharpoonright \lambda) = \lambda .$

$[*80{\cdot}14] \qquad \supset . Q \upharpoonright \lambda \in Q_\Delta`\lambda \qquad (2)$

$\vdash . (1) . (2) . *51{\cdot}141 . \supset \vdash . \mathrm{Prop}$

***82·22.** $\vdash : Q \upharpoonright \lambda \in 1 \rightarrow \mathrm{Cls} \,.\, \lambda = \breve{Q}\text{“}\kappa \,.\, M \in P_\Delta\text{‘}\kappa \,.\, \supset .\, M \mid Q \in (P \mid Q)_\Delta\text{‘}\lambda$

Dem.

$$\begin{array}{lll}
\vdash .\, *80{\cdot}14 \,.\, *37{\cdot}32 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset .\, \text{Ɑ}\text{‘}(M \mid Q) = \breve{Q}\text{“}\kappa \,. & & \\
[\mathrm{Hp}] & \supset .\, \text{Ɑ}\text{‘}(M \mid Q) = \lambda & (1) \\
[*35{\cdot}452{\cdot}23] & \supset .\, M \mid Q = M \mid (Q \upharpoonright \lambda) \,. & \\
[*71{\cdot}25 . *80{\cdot}14] & \supset .\, M \mid Q \in 1 \rightarrow \mathrm{Cls} & (2) \\
\vdash .\, *34{\cdot}34 \,.\, *80{\cdot}14 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset .\, M \mid Q \subseteq P \mid Q & & (3) \\
\vdash .\, (1) \,.\, (2) \,.\, (3) \,.\, *80{\cdot}14 \,.\, \supset \vdash .\, \mathrm{Prop} & &
\end{array}$$

***82·221.** $\vdash : Q \upharpoonright \lambda \in 1 \rightarrow \mathrm{Cls} \,.\, \lambda \subset \text{Ɑ}\text{‘}Q \,.\, M \in P_\Delta\text{‘}Q\text{“}\lambda \,.\, \supset .\, M \mid Q \upharpoonright \lambda \in (P \mid Q)_\Delta\text{‘}\lambda$

Dem.

$$\begin{array}{lll}
\vdash .\, *71{\cdot}25 \,.\, *80{\cdot}14 \,. & \supset \vdash : \mathrm{Hp} \,.\, \supset .\, M \mid Q \upharpoonright \lambda \in 1 \rightarrow \mathrm{Cls} & (1) \\
\vdash .\, *34{\cdot}34 \,.\, *80{\cdot}14 \,. & \supset \vdash : \mathrm{Hp} \,.\, \supset .\, M \mid Q \upharpoonright \lambda \subseteq P \mid Q & (2) \\
\vdash .\, *37{\cdot}32 \,.\, *35{\cdot}64 \,.\, *80{\cdot}14 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset .\, \text{Ɑ}\text{‘}(M \mid Q \upharpoonright \lambda) = \lambda \cap \breve{Q}\text{“}Q\text{“}\lambda & & \\
[*37{\cdot}51 . *22{\cdot}621] & = \lambda & (3) \\
\vdash .\, (1) \,.\, (2) \,.\, (3) \,.\, \supset \vdash .\, \mathrm{Prop} & &
\end{array}$$

***82·23.** $\vdash : Q \upharpoonright \lambda \in 1 \rightarrow 1 \,.\, \kappa = Q\text{“}\lambda \,.\, R \in (P \mid Q)_\Delta\text{‘}\lambda \,.\, \supset .\, R \mid \breve{Q} \in P_\Delta\text{‘}\kappa$

Dem.

$$\begin{array}{lll}
\vdash .\, *80{\cdot}14 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset .\, \text{Ɑ}\text{‘}R = \lambda \,. & & (1) \\
[*35{\cdot}48] & \supset .\, R \mid \breve{Q} = R \mid (\lambda \upharpoonleft \breve{Q}) & \\
[*35{\cdot}51] & = R \mid \mathrm{Cnv}\text{‘}(Q \upharpoonright \lambda) \,. & (2) \\
[*71{\cdot}25] & \supset .\, R \mid \breve{Q} \in 1 \rightarrow \mathrm{Cls} & (3) \\
\vdash .\, *37{\cdot}32 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset .\, \text{Ɑ}\text{‘}(R \mid \breve{Q}) = Q\text{“}\text{Ɑ}\text{‘}R & & \\
[(1)] & = Q\text{“}\lambda & \\
[\mathrm{Hp}] & = \kappa & (4) \\
\vdash .\, *80{\cdot}291 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset .\, R \subseteq (P \mid Q) \upharpoonright \lambda \,. & & \\
[*35{\cdot}23] & \supset .\, R \subseteq P \mid (Q \upharpoonright \lambda) \,. & \\
[*34{\cdot}34] & \supset .\, R \mid \mathrm{Cnv}\text{‘}(Q \upharpoonright \lambda) \subseteq P \mid Q \upharpoonright \lambda \mid \mathrm{Cnv}\text{‘}(Q \upharpoonright \lambda) \,. & \\
[(2) . *72{\cdot}59] & \supset .\, R \mid \breve{Q} \subseteq P \upharpoonright \mathrm{D}\text{‘}(Q \upharpoonright \lambda) \,. & \\
[*35{\cdot}441] & \supset .\, R \mid \breve{Q} \subseteq P & (5) \\
\vdash .\, (3) \,.\, (4) \,.\, (5) \,.\, *80{\cdot}14 \,.\, \supset \vdash .\, \mathrm{Prop} & &
\end{array}$$

***82·231.** $\vdash : Q \upharpoonright \lambda \in 1 \rightarrow 1 \,.\, R \in (P \mid Q)_\Delta\text{‘}\lambda \,.\, \supset .\, R \mid \breve{Q} \in P_\Delta\text{‘}Q\text{“}\lambda \,.\, R = R \mid \breve{Q} \mid Q \upharpoonright \lambda$

Dem.

$$\begin{array}{lll}
\vdash .\, *80{\cdot}14 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset .\, \text{Ɑ}\text{‘}R = \lambda \,. & & (1) \\
[*74{\cdot}41] & \supset .\, R \mid \breve{Q} = R \mid \lambda \upharpoonleft \breve{Q} & \\
[*35{\cdot}51] & = R \mid \mathrm{Cnv}\text{‘}(Q \upharpoonright \lambda) \,. & \\
[*34{\cdot}27] & \supset .\, R \mid \breve{Q} \mid Q \upharpoonright \lambda = R \mid \mathrm{Cnv}\text{‘}(Q \upharpoonright \lambda) \mid Q \upharpoonright \lambda & \\
[*72{\cdot}591] & = R \upharpoonright \text{Ɑ}\text{‘}(Q \upharpoonright \lambda) & (2)
\end{array}$$

$\vdash . *80{\cdot}2 . \supset \vdash : \text{Hp} . \supset . \lambda \subset \text{Ɑ}`(P \mid Q) .$

$[*34{\cdot}36] \quad \supset . \lambda \subset \text{Ɑ}`Q .$

$[*35{\cdot}65] \quad \supset . \text{Ɑ}`Q \upharpoonright \lambda = \lambda .$

$[(1) . *74{\cdot}221] \quad \supset . R \upharpoonright \text{Ɑ}`(Q \upharpoonright \lambda) = R \qquad (3)$

$\vdash . (2) . (3) . \supset \vdash : \text{Hp} . \supset . R = R \mid \breve{Q} \mid Q \upharpoonright \lambda \qquad (4)$

$\vdash . (4) . *82{\cdot}23 . \supset \vdash . \text{Prop}$

$*82{\cdot}24$. $\vdash : Q \upharpoonright \lambda \in 1 \to 1 . \kappa \subset \text{D}`Q . \lambda = \breve{Q}``\kappa . R \in (P \mid Q)_\Delta`\lambda . \supset .$
$\kappa = Q``\lambda . R \mid \breve{Q} \in P_\Delta`\kappa . R = R \mid \breve{Q} \mid Q$

Dem.

$\vdash . *74{\cdot}16 . \supset \vdash : \text{Hp} . \supset . \kappa = Q``\lambda . \qquad (1)$

$[*82{\cdot}23] \quad \supset . R \mid \breve{Q} \in P_\Delta`\kappa . \qquad (2)$

$[*80{\cdot}14] \quad \supset . \text{Ɑ}`(R \mid \breve{Q}) = \kappa .$

$[\text{Hp}] \quad \supset . \breve{Q}``\text{Ɑ}`(R \mid \breve{Q}) = \lambda .$

$[*74{\cdot}4] \quad \supset . R \mid \breve{Q} \mid Q \upharpoonright \lambda = R \mid \breve{Q} \mid Q .$

$[*82{\cdot}231] \quad \supset . R = R \mid \breve{Q} \mid Q \qquad (3)$

$\vdash . (1) . (2) . (3) . \supset \vdash . \text{Prop}$

$*82{\cdot}241$. $\vdash : Q \upharpoonright \lambda \in 1 \to 1 . \lambda \in \text{D}`(\breve{Q})_\epsilon . R \in (P \mid Q)_\Delta`\lambda . \supset . R = R \mid \breve{Q} \mid Q$

Dem.

$\vdash . *74{\cdot}31 . \supset \vdash : \text{Hp} . \supset . \lambda = \breve{Q}``Q``\lambda$

$[*80{\cdot}14] \quad = \breve{Q}``Q``\text{Ɑ}`R$

$[*37{\cdot}32] \quad = \breve{Q}``\text{Ɑ}`(R \mid \breve{Q}) .$

$[*74{\cdot}4] \quad \supset . R \mid \breve{Q} \mid Q \upharpoonright \lambda = R \mid \breve{Q} \mid Q \qquad (1)$

$\vdash . (1) . *82{\cdot}231 . \supset \vdash . \text{Prop}$

$*82{\cdot}25$. $\vdash : Q \upharpoonright \lambda \in 1 \to 1 . \kappa \subset \text{D}`Q . \lambda = \breve{Q}``\kappa . R \in (P \mid Q)_\Delta`\lambda . \supset .$
$(\exists M) . M \in P_\Delta`\kappa . R = M \mid Q \quad [*82{\cdot}24 . *10{\cdot}24]$

$*82{\cdot}251$. $\vdash : Q \upharpoonright \lambda \in 1 \to 1 . R \in (P \mid Q)_\Delta`\lambda . \supset . (\exists M) . M \in P_\Delta`Q``\lambda . R = M \mid Q \upharpoonright \lambda$
$[*82{\cdot}231 . *10{\cdot}24]$

$*82{\cdot}26$. $\vdash :. Q \upharpoonright \lambda \in 1 \to 1 . \kappa \subset \text{D}`Q . \lambda = \breve{Q}``\kappa . \supset :$
$R \in (P \mid Q)_\Delta`\lambda . \equiv . (\exists M) . M \in P_\Delta`\kappa . R = M \mid Q \quad [*82{\cdot}22{\cdot}25]$

$*82{\cdot}261$. $\vdash :. Q \upharpoonright \lambda \in 1 \to 1 . \lambda \subset \text{Ɑ}`Q . \supset :$
$R \in (P \mid Q)_\Delta`\lambda . \equiv . (\exists M) . M \in P_\Delta`Q``\lambda . R = M \mid Q \upharpoonright \lambda$
$[*82{\cdot}221{\cdot}251]$

$*82{\cdot}27$. $\vdash : Q \upharpoonright \lambda \in 1 \to 1 . \kappa \subset \text{D}`Q . \lambda = \breve{Q}``\kappa . \supset . (P \mid Q)_\Delta`\lambda = \mid Q``P_\Delta`\kappa$
$[*82{\cdot}26 . *43{\cdot}121 . *37{\cdot}6]$

***82·271.** $\vdash : Q \upharpoonright \lambda \in 1 \to 1 \,.\, \lambda \subset \text{Ɑ}\text{‘}Q \,.\supset.\, (P \mid Q)_\Delta\text{‘}\lambda = \mid (Q \upharpoonright \lambda)\text{“}P_\Delta\text{‘}Q\text{“}\lambda$

[*82·261 . *43·121 . *37·6]

***82·272.** $\vdash : Q \upharpoonright \lambda \in 1 \to 1 \,.\, \lambda \in \mathrm{D}\text{‘}(\breve{Q})_\epsilon \,.\supset.\, (P \mid Q)_\Delta\text{‘}\lambda = \mid Q\text{“}P_\Delta\text{‘}Q\text{“}\lambda$

Dem.

$\vdash . *37{\cdot}23 \,.\supset \vdash : \mathrm{Hp} \,.\supset.\, (\exists \mu) \,.\, \lambda = \breve{Q}\text{“}\mu \,.$

$[*37{\cdot}261] \qquad \supset.\, (\exists \mu) \,.\, \lambda = \breve{Q}\text{“}(\mu \cap \mathrm{D}\text{‘}Q) \,.$

$[*22{\cdot}43] \qquad \supset.\, (\exists \kappa) \,.\, \lambda = \breve{Q}\text{“}\kappa \,.\, \kappa \subset \mathrm{D}\text{‘}Q \qquad (1)$

$\vdash . *82{\cdot}27 . *74{\cdot}16 \,.\supset$

$\vdash : Q \upharpoonright \lambda \in 1 \to 1 \,.\, \kappa \subset \mathrm{D}\text{‘}Q \,.\, \lambda = \breve{Q}\text{“}\kappa \,.\supset.\, (P \mid Q)_\Delta\text{‘}\lambda = \mid Q\text{“}P_\Delta\text{‘}Q\text{“}\lambda \qquad (2)$

$\vdash . (1) . (2) . *10{\cdot}11{\cdot}23{\cdot}35 \,.\supset \vdash . \mathrm{Prop}$

***82·28.** $\vdash :. \kappa \upharpoonleft Q \in 1 \to 1 \,.\, \lambda \subset \text{Ɑ}\text{‘}Q \,.\, \kappa = Q\text{“}\lambda \,.\supset :$

$R \in (P \mid Q)_\Delta\text{‘}\lambda \,.\equiv.\, (\exists M) \,.\, M \in P_\Delta\text{‘}\kappa \,.\, R = M \mid Q$

[*82·26 . *74·26]

***82·29.** $\vdash : \kappa \upharpoonleft Q \in 1 \to 1 \,.\, \lambda \subset \text{Ɑ}\text{‘}Q \,.\, \kappa = Q\text{“}\lambda \,.\supset.\, (P \mid Q)_\Delta\text{‘}\lambda = \mid Q\text{“}P_\Delta\text{‘}\kappa$

[*82·27 . *74·26]

***82·291.** $\vdash : \kappa \upharpoonleft Q \in 1 \to 1 \,.\, \kappa \in \mathrm{D}\text{‘}Q_\epsilon \,.\supset.\, (P \mid Q)_\Delta\text{‘}\breve{Q}\text{“}\kappa = \mid Q\text{“}P_\Delta\text{‘}\kappa$

[Proof as in *82·272]

***82·3.** $\vdash : M \in P_\Delta\text{‘}Q\text{“}\lambda \,.\supset.\, \mathrm{D}\text{‘}(M \mid Q \upharpoonright \lambda) = \mathrm{D}\text{‘}M$

Dem.

$\vdash . *80{\cdot}14 \,.\supset \vdash : \mathrm{Hp} \,.\supset.\, \text{Ɑ}\text{‘}M = Q\text{“}\lambda \,.$

$[*74{\cdot}42] \qquad \supset.\, \mathrm{D}\text{‘}(M \mid Q \upharpoonright \lambda) = \mathrm{D}\text{‘}M : \supset \vdash . \mathrm{Prop}$

***82·31.** $\vdash : R \in (P \mid Q)_\Delta\text{‘}\lambda \,.\supset.\, \mathrm{D}\text{‘}(R \mid \breve{Q}) = \mathrm{D}\text{‘}R$

Dem.

$\vdash . *80{\cdot}14{\cdot}2 \,.\supset \vdash : \mathrm{Hp} \,.\supset.\, \text{Ɑ}\text{‘}R = \lambda \,.\, \lambda \subset \text{Ɑ}\text{‘}(P \mid Q) \,.$

$[*34{\cdot}36] \qquad \supset.\, \text{Ɑ}\text{‘}R \subset \text{Ɑ}\text{‘}Q \,.$

$[*37{\cdot}321] \qquad \supset.\, \mathrm{D}\text{‘}(R \mid \breve{Q}) = \mathrm{D}\text{‘}R : \supset \vdash . \mathrm{Prop}$

***82·32.** $\vdash : Q \upharpoonright \lambda \in 1 \to 1 \,.\, \lambda \subset \text{Ɑ}\text{‘}Q \,.\supset.\, \mathrm{D}\text{“}(P \mid Q)_\Delta\text{‘}\lambda = \mathrm{D}\text{“}P_\Delta\text{‘}Q\text{“}\lambda$

Dem.

$\vdash . *82{\cdot}271 \,.\supset$

$\vdash :. \mathrm{Hp} \,.\supset : \mathrm{D}\text{“}(P \mid Q)_\Delta\text{‘}\lambda = \mathrm{D}\text{“} \mid (Q \upharpoonright \lambda)\text{“}P_\Delta\text{‘}Q\text{“}\lambda :$

$[*37{\cdot}67] \supset : \alpha \in \mathrm{D}\text{“}(P \mid Q)_\Delta\text{‘}\lambda \,.\equiv.\, (\exists M) \,.\, M \in P_\Delta\text{‘}Q\text{“}\lambda \,.\, \alpha = \mathrm{D}\text{‘}(M \mid Q \upharpoonright \lambda) \,.$

$[*82{\cdot}3] \qquad \supset.\, (\exists M) \,.\, M \in P_\Delta\text{‘}Q\text{“}\lambda \,.\, \alpha = \mathrm{D}\text{‘}M \,.$

$[*37{\cdot}6] \qquad \supset.\, \alpha \in \mathrm{D}\text{“}P_\Delta\text{‘}Q\text{“}\lambda \qquad (1)$

$\vdash . *82\cdot3\cdot221 . \supset \vdash :. \mathrm{Hp} . \supset : M \epsilon P_\Delta{}^\backprime Q^{\backprime\backprime}\lambda . \supset . \mathrm{D}^\backprime M = \mathrm{D}^\backprime(M \mid Q \upharpoonright \lambda) .$

$M \mid (Q \upharpoonright \lambda) \epsilon (P \mid Q)_\Delta{}^\backprime\lambda .$

[$*37\cdot62$] $\supset . \mathrm{D}^\backprime M \epsilon \mathrm{D}^{\backprime\backprime}(P \mid Q)_\Delta{}^\backprime\lambda :$

[$*37\cdot61$] $\supset : \mathrm{D}^{\backprime\backprime}P_\Delta{}^\backprime Q^{\backprime\backprime}\lambda \subset \mathrm{D}^{\backprime\backprime}(P \mid Q)_\Delta{}^\backprime\lambda$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

$*82\cdot33$. $\vdash : \kappa \rceil Q \epsilon 1 \to 1 . \kappa \epsilon \mathrm{D}^\backprime Q_\epsilon . \supset . \mathrm{D}^{\backprime\backprime}(P \mid Q)_\Delta{}^\backprime \breve{Q}^{\backprime\backprime}\kappa = \mathrm{D}^{\backprime\backprime}P_\Delta{}^\backprime\kappa$

Dem.

$\vdash . *37\cdot23\cdot26 . \supset \vdash : \kappa \epsilon \mathrm{D}^\backprime Q_\epsilon . \supset . (\exists\lambda) . \lambda \subset \text{Cl}^\backprime Q . \kappa = Q^{\backprime\backprime}\lambda$ (1)

$\vdash . *74\cdot26 . \supset$

$\vdash : \kappa \rceil Q \epsilon 1 \to 1 . \lambda \subset \text{Cl}^\backprime Q . \kappa = Q^{\backprime\backprime}\lambda . \supset . Q \upharpoonright \lambda \epsilon 1 \to 1 . \kappa \subset \mathrm{D}^\backprime Q . \lambda = \breve{Q}^{\backprime\backprime}\kappa .$ (2)

[$*82\cdot32$] $\supset . \mathrm{D}^{\backprime\backprime}(P \mid Q)_\Delta{}^\backprime\lambda = \mathrm{D}^{\backprime\backprime}P_\Delta{}^\backprime Q^{\backprime\backprime}\lambda .$

[(2).Hp (2)] $\supset . \mathrm{D}^{\backprime\backprime}(P \mid Q)_\Delta{}^\backprime \breve{Q}^{\backprime\backprime}\kappa = \mathrm{D}^{\backprime\backprime}P_\Delta{}^\backprime\kappa$ (3)

$\vdash . (3) . *10\cdot11\cdot23\cdot35 . \supset$

$\vdash :. \kappa \rceil Q \epsilon 1 \to 1 : (\exists\lambda) . \lambda \subset \text{Cl}^\backprime Q . \kappa = Q^{\backprime\backprime}\lambda : \supset . \mathrm{D}^{\backprime\backprime}(P \mid Q)_\Delta{}^\backprime \breve{Q}^{\backprime\backprime}\kappa = \mathrm{D}^{\backprime\backprime}P_\Delta{}^\backprime\kappa$ (4)

$\vdash . (1) . (4) . \supset \vdash . \mathrm{Prop}$

The following propositions ($*82\cdot4\cdot41\cdot411\cdot42$) are lemmas for $*82\cdot43$, which is used in the proof of $*114\cdot5$, in the theory of cardinal multiplication.

$*82\cdot4$. $\vdash : T \epsilon 1 \to \mathrm{Cls} . P^{\backprime\backprime}\lambda \subset \text{Cl}^\backprime T . \supset . T \mid {}^{\backprime\backprime}P_\Delta{}^\backprime\lambda \subset (T \mid P)_\Delta{}^\backprime\lambda$

Dem.

$\vdash . *80\cdot14 . *71\cdot25 .$ $\supset \vdash : \mathrm{Hp} . R \epsilon P_\Delta{}^\backprime\lambda . \supset . T \mid R \epsilon 1 \to \mathrm{Cls}$ (1)

$\vdash . *80\cdot14 . *34\cdot34 .$ $\supset \vdash : \mathrm{Hp} . R \epsilon P_\Delta{}^\backprime\lambda . \supset . T \mid R \mathrel{\subset\!\!\!\cdot} T \mid P$ (2)

$\vdash . *80\cdot33 .$ $\supset \vdash : \mathrm{Hp} . R \epsilon P_\Delta{}^\backprime\lambda . \supset . \mathrm{D}^\backprime R \subset \text{Cl}^\backprime T .$

[$*37\cdot322$] $\supset . \text{Cl}^\backprime(T \mid R) = \text{Cl}^\backprime R .$

[$*80\cdot14$] $\supset . \text{Cl}^\backprime(T \mid R) = \lambda$ (3)

$\vdash . (1) . (2) . (3) . *80\cdot14 . \supset \vdash :. \mathrm{Hp} . \supset : R \epsilon P_\Delta{}^\backprime\lambda . \supset . T \mid R \epsilon (T \mid P)_\Delta{}^\backprime\lambda :. \supset \vdash . \mathrm{Prop}$

$*82\cdot41$. $\vdash : T \epsilon \mathrm{Cls} \to 1 . M \epsilon (T \mid P)_\Delta{}^\backprime\lambda . \supset . \breve{T} \mid M \epsilon P_\Delta{}^\backprime\lambda . M = T \mid \breve{T} \mid M$

Dem.

$\vdash . *80\cdot14 . *71\cdot25 . \supset \vdash : \mathrm{Hp} . \supset . \breve{T} \mid M \epsilon 1 \to \mathrm{Cls}$ (1)

$\vdash . *80\cdot14 . *34\cdot34 . \supset \vdash : \mathrm{Hp} . \supset . \breve{T} \mid M \mathrel{\subset\!\!\!\cdot} \breve{T} \mid T \mid P .$

[$*71\cdot191 . *34\cdot2$] $\mathrel{\subset\!\!\!\cdot} P$ (2)

$\vdash . *80\cdot14 . *34\cdot36 . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{D}^\backprime M \subset \mathrm{D}^\backprime T .$

[$*37\cdot322$] $\supset . \text{Cl}^\backprime(\breve{T} \mid M) = \text{Cl}^\backprime M .$

[$*80\cdot14$] $\supset . \text{Cl}^\backprime(\breve{T} \mid M) = \lambda$ (3)

$\vdash . (1) . (2) . (3) . *80\cdot14 . \supset \vdash . \mathrm{Prop}$

$*82\cdot411$. $\vdash : T \epsilon \mathrm{Cls} \to 1 . \supset . (T \mid P)_\Delta{}^\backprime\lambda \subset T \mid {}^{\backprime\backprime}P_\Delta{}^\backprime\lambda$ [$*82\cdot41$]

***82·42.** $\vdash : T \in 1 \to 1 \,.\, P\text{“}\lambda \subset \text{Ɑ}\text{‘}T \,.\supset .\, (T \,|\, P)_\Delta\text{‘}\lambda = T \,|\text{“}P_\Delta\text{‘}\lambda$ [*82·4·411]

***82·43.** $\vdash : T, Q \upharpoonright \lambda \in 1 \to 1 \,.\, P\text{“}\lambda \subset \text{Ɑ}\text{‘}T \,.\, \lambda \subset \text{Ɑ}\text{‘}Q \,.\, \kappa = Q\text{“}\lambda \,.\supset .$
$$(T \,|\, P \upharpoonright \lambda \,|\, \breve{Q})_\Delta\text{‘}\kappa = (T \,\|\, \breve{Q})\text{“}P_\Delta\text{‘}\lambda$$

Dem.

$\vdash .\, *82·27 \dfrac{\lambda, \kappa}{\kappa, \lambda} .\supset \vdash : Q \in 1 \to 1 \,.\, \lambda \subset \text{D}\text{‘}Q \,.\, \kappa = \breve{Q}\text{“}\lambda \,.\supset .\, (P \,|\, Q)_\Delta\text{‘}\kappa = |\, Q\text{“}P_\Delta\text{‘}\lambda$ (1)

$\vdash .\, (1) \dfrac{\lambda \upharpoonleft \breve{Q}}{Q} . \quad \supset \vdash : Q \upharpoonright \lambda \in 1 \to 1 \,.\, \lambda \subset \text{D}\text{‘}(\lambda \upharpoonleft \breve{Q}) \,.\, \kappa = (Q \upharpoonright \lambda)\text{“}\lambda \,.\supset .$
$$(P \,|\, \lambda \upharpoonleft \breve{Q})_\Delta\text{‘}\kappa = |\, (\lambda \upharpoonleft \breve{Q})\text{“}P_\Delta\text{‘}\lambda \quad (2)$$

$\vdash .\, (2) \,.\, *35·61·354 \,.\, *37·412 \,.\, *43·481 \,.\, *80·14 \,.\supset$

$\vdash : Q \upharpoonright \lambda \in 1 \to 1 \,.\, \lambda \subset \text{Ɑ}\text{‘}Q \,.\, \kappa = Q\text{“}\lambda \,.\supset .\, (P \upharpoonright \lambda \,|\, \breve{Q})_\Delta\text{‘}\kappa = |\, \breve{Q}\text{“}P_\Delta\text{‘}\lambda$ (3)

$\vdash .\, (3) \dfrac{T \,|\, P}{P} . \quad \supset \vdash : Q \upharpoonright \lambda \in 1 \to 1 \,.\, \lambda \subset \text{Ɑ}\text{‘}Q \,.\, \kappa = Q\text{“}\lambda \,.\supset .$
$$(T \,|\, P \upharpoonright \lambda \,|\, \breve{Q})_\Delta\text{‘}\kappa = |\, \breve{Q}\text{“}(T \,|\, P)_\Delta\text{‘}\lambda \quad (4)$$

$\vdash .\, (4) \,.\, *82·42 \,.\supset \vdash : \text{Hp} \,.\supset .\, (T \,|\, P \upharpoonright \lambda \,|\, \breve{Q})_\Delta\text{‘}\kappa = |\, \breve{Q}\text{“}T \,|\text{“}P_\Delta\text{‘}\lambda$

[*43·202.*37·33] $\qquad = (T \,\|\, \breve{Q})\text{“}P_\Delta\text{‘}\lambda : \supset \vdash .\, \text{Prop}$

***82·45.** $\vdash : Q \upharpoonright \lambda \in 1 \to 1 \,.\, \lambda \subset \text{Ɑ}\text{‘}Q \,.\supset .\, (P \,|\, Q)_\Delta\text{‘}\lambda \,\text{sm}\, P_\Delta\text{‘}Q\text{“}\lambda$

Dem.

$\vdash .\, *80·14 \,.\, *37·15 \,.\supset \vdash : R \in P_\Delta\text{‘}Q\text{“}\lambda \,.\supset_R .\, \text{Ɑ}\text{‘}R = Q\text{“}\lambda \,.\, Q\text{“}\lambda \subset \text{D}\text{‘}Q \,.$

[*14·15] $\qquad \supset_R .\, \text{Ɑ}\text{‘}R \subset \text{D}\text{‘}Q :$

[*74·72] $\qquad \supset \vdash : \text{Hp} \,.\supset .\, |\, (Q \upharpoonright \lambda)\text{“}P_\Delta\text{‘}Q\text{“}\lambda \,\text{sm}\, P_\Delta\text{‘}Q\text{“}\lambda \,.$

[*82·271] $\qquad \supset .\, (P \,|\, Q)_\Delta\text{‘}\lambda \,\text{sm}\, P_\Delta\text{‘}Q\text{“}\lambda : \supset \vdash .\, \text{Prop}$

***82·5.** $\vdash : P \upharpoonright Q\text{“}\lambda \in \text{Cls} \to 1 \,.\, Q \upharpoonright \lambda \in 1 \to 1 \,.\, \lambda \subset \text{Ɑ}\text{‘}Q \,.\supset .$
$$(P \,|\, Q)_\Delta\text{‘}\lambda \,\text{sm}\, \text{D}\text{“}P_\Delta\text{‘}Q\text{“}\lambda$$ [*82·45 . *81·21]

***82·51.** $\vdash : P \upharpoonright \kappa \in \text{Cls} \to 1 \,.\, \kappa \upharpoonleft Q \in 1 \to 1 \,.\, \lambda \subset \text{Ɑ}\text{‘}Q \,.\, \kappa = Q\text{“}\lambda \,.\supset .$
$$(P \,|\, Q)_\Delta\text{‘}\lambda \,\text{sm}\, \text{D}\text{“}P_\Delta\text{‘}\kappa$$ [*82·5 . *74·251]

***82·52.** $\vdash : P \upharpoonright \kappa \in \text{Cls} \to 1 \,.\, \kappa \upharpoonleft Q \in 1 \to 1 \,.\, \kappa \in \text{D}\text{‘}Q_\epsilon \,.\supset .\, (P \,|\, Q)_\Delta\text{‘}\breve{Q}\text{“}\kappa \,\text{sm}\, \text{D}\text{“}P_\Delta\text{‘}\kappa$

Dem.

$\vdash .\, *37·23 . \quad \supset \vdash : \text{Hp} \,.\supset .\, (\exists \mu) \,.\, \kappa = Q\text{“}\mu$ (1)

$\vdash .\, *37·26 \,.\, *22·43 \,.\supset$
$$\vdash : \kappa = Q\text{“}\mu \,.\, \lambda = \mu \cap \text{Ɑ}\text{‘}Q \,.\supset .\, \kappa = Q\text{“}\lambda \,.\, \lambda \subset \text{Ɑ}\text{‘}Q \quad (2)$$

$\vdash .\, *74·161 . \quad \supset \vdash : \text{Hp} \,.\, \kappa = Q\text{“}\lambda \,.\, \lambda \subset \text{Ɑ}\text{‘}Q \,.\supset .\, \lambda = \breve{Q}\text{“}\kappa \,.$

[*82·51] $\qquad \supset .\, (P \,|\, Q)_\Delta\text{‘}\breve{Q}\text{“}\kappa \,\text{sm}\, \text{D}\text{“}P_\Delta\text{‘}\kappa :$

[*10·11·23·35] $\supset \vdash :.\, \text{Hp} : (\exists \lambda) \,.\, \kappa = Q\text{“}\lambda \,.\, \lambda \subset \text{Ɑ}\text{‘}Q : \supset .\, (P \,|\, Q)_\Delta\text{‘}\breve{Q}\text{“}\kappa \,\text{sm}\, \text{D}\text{“}P_\Delta\text{‘}\kappa$ (3)

$\vdash .\, (1) \,.\, (2) . \quad \supset \vdash : \text{Hp} \,.\supset .\, (\exists \lambda) \,.\, \kappa = Q\text{“}\lambda \,.\, \lambda \subset \text{Ɑ}\text{‘}Q$ (4)

$\vdash .\, (3) \,.\, (4) . \quad \supset \vdash .\, \text{Prop}$

***82·53.** $\vdash : P \restriction \kappa, R \restriction \kappa \in \mathrm{Cls} \to 1 \,.\, \kappa \,\rceil\, \breve{Q} \in 1 \to 1 \,.\, \kappa \in \mathrm{D}`Q_\epsilon \,.\, \overrightarrow{P}``\kappa = \overrightarrow{R}``\kappa \,.\, \supset .$

$(P \mid Q)_\Delta`\breve{Q}``\kappa \;\mathrm{sm}\; (R \mid Q)_\Delta`\breve{Q}``\kappa \,.$

$\mathrm{D}``(P \mid Q)_\Delta`\breve{Q}``\kappa = \mathrm{D}``(R \mid Q)_\Delta`\breve{Q}``\kappa =$

$\hat{\mu}\{\alpha \in \overrightarrow{P}``\kappa \,.\, \supset_\alpha .\, \mu \cap \alpha \in 1 : \mu \subset P``\kappa\}$

$= \mathrm{D}``P_\Delta`\kappa = \mathrm{D}``R_\Delta`\kappa$

Dem.

$\vdash .\, *82·52 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset .\, (P \mid Q)_\Delta`\breve{Q}``\kappa \;\mathrm{sm}\; \mathrm{D}``P_\Delta`\kappa \,.$

[*81·31] $\supset .\, (P \mid Q)_\Delta`\breve{Q}``\kappa \;\mathrm{sm}\; \mathrm{D}``R_\Delta`\kappa \,.$

[*82·52.*73·32] $\supset .\, (P \mid Q)_\Delta`\breve{Q}``\kappa \;\mathrm{sm}\; (R \mid Q)_\Delta`\breve{Q}``\kappa$ (1)

$\vdash .\, *82·33 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset .\, \mathrm{D}``(P \mid Q)_\Delta`\breve{Q}``\kappa = \mathrm{D}``P_\Delta`\kappa$ (2)

[*81·31] $= \mathrm{D}``R_\Delta`\kappa$ (3)

[*81·3.*40·5] $= \hat{\mu}\{\alpha \in \overrightarrow{P}``\kappa \,.\, \supset_\alpha .\, \mu \cap \alpha \in 1 : \mu \subset P``\kappa\}$ (4)

$\vdash .\, *82·33 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset .\, \mathrm{D}``(R \mid Q)_\Delta`\breve{Q}``\kappa = \mathrm{D}``R_\Delta`\kappa$ (5)

$\vdash .\, (1) \,.\, (2) \,.\, (3) \,.\, (4) \,.\, (5) \,.\, \supset \vdash .$ Prop

$*83$. SELECTIONS FROM CLASSES OF CLASSES

Summary of $*83$.

In this number, the general propositions which have been proved for $P_\Delta{}^\backprime\kappa$ are to be applied to the important special case where P is ϵ. In this case, we have selections from classes of classes: if $R \in \epsilon_\Delta{}^\backprime\kappa$, R picks out a *representative* $R^\backprime\alpha$ from each class α which is a member of κ; *i.e.* we have

$$\alpha \in \kappa \,.\supset_\alpha .\, R^\backprime\alpha \in \alpha.$$

The propositions of this number result from those of previous numbers either immediately, by the substitution of ϵ for P, or by the use of propositions of $*62$, notably $\overrightarrow{\epsilon}{}^\backprime\alpha = \alpha$ ($*62\cdot2$), and $\epsilon^{\backprime\backprime}\kappa = s^\backprime\kappa$ ($*62\cdot3$).

The propositions of the present number follow, in the main, the same course as those of $*80$, with ϵ substituted for P (except that the special forms of propositions before $*80\cdot2$ are not given). We have first a set of propositions resulting immediately from early propositions of $*80$. Of these the most used are:

$*83\cdot11$. $\vdash : \Lambda \in \kappa \,.\supset .\, \epsilon_\Delta{}^\backprime\kappa = \Lambda$

This leads to the proposition that an arithmetical product is null if one of its factors is null. (We cannot prove the converse universally without assuming the multiplicative axiom.)

$*83\cdot15$. $\vdash .\, \epsilon_\Delta{}^\backprime\Lambda = \iota^\backprime\Lambda$

Thus $\epsilon_\Delta{}^\backprime\Lambda$ is a unit class. This is the source of the proposition $\mu^0 = 1$, where μ is a cardinal (cf. note to $*83\cdot15$).

$*83\cdot2$. $\vdash :.\, R \in \epsilon_\Delta{}^\backprime\kappa \,.\supset :\, \alpha \in \kappa \,.\equiv .\, E!\, R^\backprime\alpha \,.\equiv .\, R^\backprime\alpha \in \alpha$

Here $R^\backprime\alpha$ is the "representative" of α.

$*83\cdot21$. $\vdash : R \in \epsilon_\Delta{}^\backprime\kappa \,.\supset .\, D^\backprime R \subset s^\backprime\kappa$

We have next a set of propositions ($*83\cdot4$—$\cdot44$) on selections from unit classes and classes of unit classes. We have

$*83\cdot41$. $\vdash .\, \epsilon_\Delta{}^\backprime\iota^\backprime\alpha \;\mathrm{sm}\; \alpha$

This leads to the proposition that a product of one factor is equal to that factor.

$*83\cdot43$. $\vdash : \kappa \subset 1 \,.\supset .\, \epsilon_\Delta{}^\backprime\kappa = \iota^\backprime(\breve{\iota} \upharpoonright \kappa) = \iota^\backprime(\epsilon \upharpoonright \kappa)$

This leads to

$*83\cdot44$. $\vdash : \kappa \subset 1 \,.\supset .\, \epsilon_\Delta{}^\backprime\kappa \in 1$

whence it follows that a product of factors, each of which is one, is one. This holds even if the number of factors is infinite or zero.

We have next a set of propositions (*83·5—·58) on changing the representative of a class, and on selections from a class of classes some of which are unit classes. These propositions are seldom referred to in the sequel.

We have next (*83·6—·74) a set of propositions on the domains of selections, *i.e.* on the class $\mathrm{D}''\epsilon_\Delta{}'\kappa$. We have

***83·66.** $\vdash : \exists ! \, \epsilon_\Delta{}'\kappa \,.\supset.\, s'\mathrm{D}''\epsilon_\Delta{}'\kappa = s'\kappa$

(The hypothesis here cannot be dispensed with unless we assume the multiplicative axiom.)

***83·7.** $\vdash .\, \mathrm{D}''\epsilon_\Delta{}'\iota'\alpha = \iota''\alpha$

***83·71.** $\vdash .\, \mathrm{D}''\epsilon_\Delta{}'\iota''\alpha = \iota'\alpha \,.\, \mathrm{D}'\alpha \rceil \breve{\iota} = \alpha$

We have next two propositions (*83·8·81) on the types of $\epsilon_\Delta{}'\kappa$ and $\mathrm{D}''\epsilon_\Delta{}'\kappa$. The type of $\mathrm{D}''\epsilon_\Delta{}'\kappa$ is the same as that of κ (*83·81).

The last set of propositions in this number (*83·9—·904) deals with the existence of selections. We have

***83·9.** $\vdash .\, \exists ! \, \epsilon_\Delta{}'\Lambda$

***83·901.** $\vdash : \exists ! \, \epsilon_\Delta{}'\iota'\alpha \,.\equiv.\, \exists ! \, \alpha$

***83·904.** $\vdash : \exists ! \, \epsilon_\Delta{}'(\kappa \cup \iota'\beta) \,.\equiv.\, \exists ! \, \epsilon_\Delta{}'\kappa \,.\, \exists ! \, \beta$

From these propositions we shall deduce by mathematical induction that whenever κ is a finite class, $\epsilon_\Delta{}'\kappa$ exists unless $\Lambda \in \kappa$ (cf. *120·62). Thus a product consisting of a finite number of factors (which may themselves be either finite or infinite) can only vanish if one of the factors vanishes.

***83·1.** $\vdash : \exists ! \, \epsilon_\Delta{}'\kappa \,.\supset.\, \Lambda \sim\in \kappa$

Dem.

$$\begin{aligned} &\vdash .\, *80\cdot2 \,.\supset \vdash : \mathrm{Hp} \,.\supset.\, \kappa \subset \mathrm{Cl}'\epsilon \,. \\ &[*62\cdot231] \qquad\qquad \supset.\, \Lambda \sim\in \kappa : \supset \vdash .\, \mathrm{Prop} \end{aligned}$$

***83·11.** $\vdash : \Lambda \in \kappa \,.\supset.\, \epsilon_\Delta{}'\kappa = \Lambda$ [*83·1 . Transp]

***83·12.** $\vdash .\, \epsilon_\Delta{}'\kappa = (\epsilon \upharpoonright \kappa)_\Delta{}'\kappa$ [*80·23]

***83·13.** $\vdash : \Lambda \sim\in \kappa \,.\, Q = \epsilon \upharpoonright \kappa \,.\supset.\, \epsilon_\Delta{}'\kappa = Q_\Delta{}'\mathrm{Cl}'Q$ [*80·24 . *62·231]

***83·14.** $\vdash : \exists ! \, \epsilon_\Delta{}'\kappa \,.\, Q = \epsilon \upharpoonright \kappa \,.\supset.\, \epsilon_\Delta{}'\kappa = Q_\Delta{}'\mathrm{Cl}'Q$ [*83·1·13]

***83·15.** $\vdash .\, \epsilon_\Delta{}'\Lambda = \iota'\dot{\Lambda}$ [*80·26]

In virtue of this proposition, the product of 0 cardinal numbers is 1—a proposition of which a particular case, namely $\mu^0 = 1$, is familiar. This arithmetical proposition results from the above as follows. We shall define the product of the numbers of members of κ as the number of members of $\epsilon_\Delta{}'\kappa$. Thus when $\kappa = \Lambda$, the number of members of $\epsilon_\Delta{}'\kappa$ is a product of 0 factors. Now by the above proposition, $\epsilon_\Delta{}'\Lambda$ has one member, namely $\dot{\Lambda}$. Hence a product of 0 factors is 1.

***83·16.** $\vdash : \exists ! \, \kappa \,.\supset.\, \dot{\Lambda} \sim\in \epsilon_\Delta{}'\kappa$ [*80·28]

*83·2. $\vdash :. R \epsilon \epsilon_\Delta{}'\kappa . \supset : \alpha \epsilon \kappa . \equiv . E ! R'\alpha . \equiv . R'\alpha \epsilon \alpha$ [*80·32 . *62·2]

*83·21. $\vdash : R \epsilon \epsilon_\Delta{}'\kappa . \supset . D'R \subset s'\kappa$ [*80·33 . *62·3]

*83·22. $\vdash : R \epsilon \epsilon_\Delta{}'\kappa . \supset . E !! R``\kappa . R``\kappa = D'R$ [*80·34]

*83·23. $\vdash : R \epsilon \epsilon_\Delta{}'\kappa . \supset . D'R = \hat{x} \{(\exists \alpha) . \alpha \epsilon \kappa . x = R'\alpha\}$ [*80·35]

*83·24. $\vdash : R \epsilon \epsilon_\Delta{}'\kappa . \alpha \epsilon \kappa . x \epsilon \alpha . \supset . [\{R \dot{-} (R'\alpha) \downarrow \alpha\} \cup x \downarrow \alpha] \epsilon \epsilon_\Delta{}'\kappa$ [*80·41]

*83·25. $\vdash : \exists ! \epsilon_\Delta{}'\kappa . \supset . \dot{s}'\epsilon_\Delta{}'\kappa = \epsilon \upharpoonright \kappa$ [*80·42]

*83·26. $\vdash : Q = \epsilon \upharpoonright \kappa . \exists ! Q_\Delta{}'\kappa . \supset . \dot{s}'Q_\Delta{}'\kappa = Q$ [*83·12·25]

*83·27. $\vdash :. R \subseteq \epsilon . R \epsilon 1 \to \mathrm{Cls} . \equiv : \alpha \epsilon \mathrm{G}'R . \supset_\alpha . R'\alpha \epsilon \alpha$ [*62·45 . *71·16]

*83·271. $\vdash :. R \epsilon \epsilon_\Delta{}'\mathrm{G}'R . \equiv : \alpha \epsilon \mathrm{G}'R . \supset_\alpha . R'\alpha \epsilon \alpha$ [*83·27 . *80·14]

*83·28. $\vdash :. R \epsilon \epsilon_\Delta{}'\kappa . \equiv : \alpha \epsilon \kappa . \supset_\alpha . R'\alpha \epsilon \alpha : \mathrm{G}'R = \kappa$
[*83·27 . *80·14 . *14·15]

*83·29. $\vdash :. R \epsilon \epsilon_\Delta{}'\kappa . \equiv : \alpha \epsilon \kappa . \equiv_\alpha . R'\alpha \epsilon \alpha : \mathrm{G}'R = \kappa$ [*83·2·28]

*83·3. $\vdash :. \kappa \cap \lambda = \Lambda . \supset : M \epsilon \epsilon_\Delta{}'(\kappa \cup \lambda) . \equiv .$
$(\exists R, S) . R \epsilon \epsilon_\Delta{}'\kappa . S \epsilon \epsilon_\Delta{}'\lambda . M = R \cup S$ [*80·66]

*83·31. $\vdash :. \kappa \cap \lambda = \Lambda . \supset : R \epsilon \epsilon_\Delta{}'\kappa . S \epsilon \epsilon_\Delta{}'\lambda . \equiv .$
$(\exists M) . M \epsilon \epsilon_\Delta{}'(\kappa \cup \lambda) . R = M \upharpoonright \kappa . S = M \upharpoonright \lambda$ [*80·67]

*83·4. $\vdash . \epsilon_\Delta{}'\iota'\alpha = \downarrow \alpha``\alpha$ [*80·45 . *62·2]

*83·41. $\vdash . \epsilon_\Delta{}'\iota'\alpha \,\mathrm{sm}\, \alpha$ [*83·4 . *73·611]

This proposition shows that a cardinal product of one factor is equal to that one factor. For the number of members of $\epsilon_\Delta{}'\iota'\alpha$ is the product of the numbers of members of members of $\iota'\alpha$, *i.e.* it is a product whose only factor is the number of members of α. By the above proposition, this product is equal to the number of members of α.

*83·42. $\vdash . \epsilon_\Delta{}'\iota``\alpha = \iota'(\alpha \rceil \breve{\iota}) = \iota'(\breve{\iota} \upharpoonright \iota``\alpha)$

Dem.

$\vdash . *83·12 . \quad \supset \vdash . \epsilon_\Delta{}'\iota``\alpha = (\epsilon \upharpoonright \iota``\alpha)_\Delta{}'\iota``\alpha$

[*62·56] $\quad = (\breve{\iota} \upharpoonright \iota``\alpha)_\Delta{}'\iota``\alpha$ (1)

$\vdash . *72·181 . *71·26 . \supset \vdash . \breve{\iota} \upharpoonright \iota``\alpha \epsilon 1 \to \mathrm{Cls}$ (2)

$\vdash . *37·15 . *33·21 . \supset \vdash . \iota``\alpha \subset \mathrm{G}'\breve{\iota} .$

[*35·65] $\quad \supset \vdash . \iota``\alpha = \mathrm{G}'(\breve{\iota} \upharpoonright \iota``\alpha)$ (3)

$\vdash . (2) . (3) . *82·21 . \supset \vdash . (\breve{\iota} \upharpoonright \iota``\alpha)_\Delta{}'\iota``\alpha = \iota'\{(\breve{\iota} \upharpoonright \iota``\alpha) \upharpoonright \iota``\alpha\}$

[*35·31] $\quad = \iota'(\breve{\iota} \upharpoonright \iota``\alpha)$ (4)

[*62·56] $\quad = \iota'(\alpha \rceil \breve{\iota})$ (5)

$\vdash . (1) . (4) . (5) . \supset \vdash . \mathrm{Prop}$

This proposition shows that a cardinal product whose factors are all 1 is 1. For $\iota``\alpha$ is a class whose members are all unit classes, and thus the number

of members of $\epsilon_\Delta`\iota``\alpha$ is the product of a number of 1's; and by the above proposition, $\epsilon_\Delta`\iota``\alpha$ is a unit class, its sole member being $\alpha \rceil \breve{\iota}$. This result is rendered more explicit by *83·43·44.

***83·43.** $\vdash : \kappa \subset 1 . \supset . \epsilon_\Delta`\kappa = \iota`(\breve{\iota} \upharpoonright \kappa) = \iota`(\epsilon \upharpoonright \kappa)$

Dem.

$$\vdash . *83{\cdot}42 . \supset \vdash : \kappa = \iota``\alpha . \supset . \epsilon_\Delta`\kappa = \iota`(\breve{\iota} \upharpoonright \kappa) \quad (1)$$

$$\vdash . (1) . *10{\cdot}11{\cdot}23 . \supset$$

$$\vdash : (\exists \alpha) . \kappa = \iota``\alpha . \supset . \epsilon_\Delta`\kappa = \iota`(\breve{\iota} \upharpoonright \kappa) :$$

$$[*52{\cdot}31] \quad \supset \vdash : \kappa \subset 1 . \supset . \epsilon_\Delta`\kappa = \iota`(\breve{\iota} \upharpoonright \kappa)$$

$$[*62{\cdot}55] \quad = \iota`(\epsilon \upharpoonright \kappa) : \supset \vdash . \text{Prop}$$

***83·44.** $\vdash : \kappa \subset 1 . \supset . \epsilon_\Delta`\kappa \, \epsilon \, 1$ [*83·43 . *52·22]

***83·5.** $\vdash : R \, \epsilon \, \epsilon_\Delta`\kappa . \alpha \sim\epsilon \, \kappa . x \, \epsilon \, \alpha . \supset . R \mathbin{\dot\cup} x \downarrow \alpha \, \epsilon \, \epsilon_\Delta`(\kappa \cup \iota`\alpha)$

Dem.

$$\vdash . *80{\cdot}43 . \supset \vdash : \text{Hp} . \supset . x \downarrow \alpha \, \epsilon \, \epsilon_\Delta`\iota`\alpha \quad (1)$$

$$\vdash . *51{\cdot}211 . \supset \vdash : \text{Hp} . \supset . \kappa \cap \iota`\alpha = \Lambda \quad (2)$$

$$\vdash . (1) . (2) . *80{\cdot}65 . \supset \vdash . \text{Prop}$$

It follows from this proposition that if κ is a class of classes for which there are selections, and if one member (not null) be added to κ, there are still selections from the resulting class of classes.

***83·51.** $\vdash : R \, \epsilon \, \epsilon_\Delta`\kappa . \alpha \, \epsilon \, \kappa . \supset . R \mathbin{\dot-} (R`\alpha) \downarrow \alpha \, \epsilon \, \epsilon_\Delta`(\kappa - \iota`\alpha)$ [*80·78]

***83·52.** $\vdash : R \, \epsilon \, \epsilon_\Delta`\kappa . \alpha \, \epsilon \, \kappa . x \, \epsilon \, \alpha . \supset . \{R \mathbin{\dot-} (R`\alpha) \downarrow \alpha\} \mathbin{\dot\cup} x \downarrow \alpha \, \epsilon \, \epsilon_\Delta`\kappa$ [*80·41]

***83·54.** $\vdash : \kappa \cap \lambda = \Lambda . \lambda \subset 1 . R \, \epsilon \, \epsilon_\Delta`\kappa . \supset . R \mathbin{\dot\cup} \breve{\iota} \upharpoonright \lambda \, \epsilon \, \epsilon_\Delta`(\kappa \cup \lambda)$

Dem.

$$\vdash . *80{\cdot}65 . \supset \vdash :. \text{Hp} . \supset : S \, \epsilon \, \epsilon_\Delta`\lambda . \supset . R \mathbin{\dot\cup} S \, \epsilon \, \epsilon_\Delta`(\kappa \cup \lambda) \quad (1)$$

$$\vdash . *83{\cdot}43 . \supset \vdash : \text{Hp} . \supset . \breve{\iota} \upharpoonright \lambda \, \epsilon \, \epsilon_\Delta`\lambda \quad (2)$$

$$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$$

***83·55.** $\vdash : \kappa \cap \lambda = \Lambda . \lambda \subset 1 . S \, \epsilon \, \epsilon_\Delta`(\kappa \cup \lambda) . \supset . S \mathbin{\dot-} \breve{\iota} \upharpoonright \lambda \, \epsilon \, \epsilon_\Delta`\kappa$

Dem.

$$\vdash . *80{\cdot}66 . \quad \supset \vdash : \text{Hp} . \supset . (\exists M, N) . M \, \epsilon \, \epsilon_\Delta`\kappa . N \, \epsilon \, \epsilon_\Delta`\lambda . S = M \mathbin{\dot\cup} N .$$

$$[*83{\cdot}43 . *51{\cdot}15] \quad \supset . (\exists M) . M \, \epsilon \, \epsilon_\Delta`\kappa . S = M \mathbin{\dot\cup} \breve{\iota} \upharpoonright \lambda \quad (1)$$

$$\vdash . *80{\cdot}14 . *35{\cdot}64 . \supset \vdash :. \text{Hp} . \supset : M \, \epsilon \, \epsilon_\Delta`\kappa . \supset . \text{C}`M \cap \text{C}`(\breve{\iota} \upharpoonright \lambda) = \Lambda .$$

$$[*33{\cdot}33] \quad \supset . M \mathbin{\dot\cap} \breve{\iota} \upharpoonright \lambda = \dot\Lambda .$$

$$[*25{\cdot}4] \quad \supset . (M \mathbin{\dot\cup} \breve{\iota} \upharpoonright \lambda) \mathbin{\dot-} \breve{\iota} \upharpoonright \lambda = M .$$

$$[*13{\cdot}12] \quad \supset : M \, \epsilon \, \epsilon_\Delta`\kappa . S = M \mathbin{\dot\cup} \breve{\iota} \upharpoonright \lambda . \supset . S \mathbin{\dot-} \breve{\iota} \upharpoonright \lambda \, \epsilon \, \epsilon_\Delta`\kappa \quad (2)$$

$$\vdash . (2) . *10{\cdot}11{\cdot}21{\cdot}23 . \supset$$

$$\vdash :. \text{Hp} . \supset : (\exists M) . M \, \epsilon \, \epsilon_\Delta`\kappa . S = M \mathbin{\dot\cup} \breve{\iota} \upharpoonright \lambda . \supset . S \mathbin{\dot-} \breve{\iota} \upharpoonright \lambda \, \epsilon \, \epsilon_\Delta`\kappa \quad (3)$$

$$\vdash . (1) . (3) . \supset \vdash . \text{Prop}$$

***83·56.** $\vdash : \kappa \cap \lambda = \Lambda . \lambda \subset 1 .$

$$\supset . \epsilon_\Delta{}^{\backprime}(\kappa \cup \lambda) = \hat{M}\{(\exists R) . R \in \epsilon_\Delta{}^{\backprime}\kappa . M = R \mathbin{\dot\cup} \breve{\iota} \restriction \lambda\}$$

Dem.

$\vdash . *80·66 . \supset \vdash :. \mathrm{Hp} . \supset :$

$M \in \epsilon_\Delta{}^{\backprime}(\kappa \cup \lambda) . \equiv . (\exists R, S) . R \in \epsilon_\Delta{}^{\backprime}\kappa . S \in \epsilon_\Delta{}^{\backprime}\lambda . M = R \mathbin{\dot\cup} S .$

$[*83·43] \quad \equiv . (\exists R) . R \in \epsilon_\Delta{}^{\backprime}\kappa . M = R \mathbin{\dot\cup} \breve{\iota} \restriction \lambda :. \supset \vdash . \mathrm{Prop}$

The following proposition is used in the theory of cardinal multiplication (*114·41).

***83·57.** $\vdash : \kappa \cap \lambda = \Lambda . \lambda \subset 1 . \supset . \epsilon_\Delta{}^{\backprime}(\kappa \cup \lambda) \,\mathrm{sm}\, \epsilon_\Delta{}^{\backprime}\kappa$

Dem.

$\vdash . *83·56 . *38·131 . \quad \supset \vdash : \mathrm{Hp} . \supset . \epsilon_\Delta{}^{\backprime}(\kappa \cup \lambda) = (\mathbin{\dot\cup} \breve{\iota} \restriction \lambda)^{\backprime\backprime}\epsilon_\Delta{}^{\backprime}\kappa \quad (1)$

$\vdash . *80·14 . *35·64 . \quad \supset \vdash : \mathrm{Hp} . R \in \epsilon_\Delta{}^{\backprime}\kappa . \supset . \text{Ꮧ}{}^{\backprime}R \cap \text{Ꮧ}{}^{\backprime}(\breve{\iota} \restriction \lambda) = \Lambda .$

$[*33·33] \quad \supset . R \mathbin{\dot\cap} \breve{\iota} \restriction \lambda = \dot\Lambda .$

$[*25·4] \quad \supset . R = (R \mathbin{\dot\cup} \breve{\iota} \restriction \lambda) \mathbin{\dot-} \breve{\iota} \restriction \lambda \quad (2)$

$\vdash . (2) . *23·481 . *13·172 . \supset$

$\vdash : \mathrm{Hp} . R, S \in \epsilon_\Delta{}^{\backprime}\kappa . R \mathbin{\dot\cup} \breve{\iota} \restriction \lambda = S \mathbin{\dot\cup} \breve{\iota} \restriction \lambda . \supset . R = S :$

$[\mathrm{Exp}. *11·11·3 . *38·11] \supset \vdash :. \mathrm{Hp} . \supset :$

$$R, S \in \epsilon_\Delta{}^{\backprime}\kappa . (\mathbin{\dot\cup} \breve{\iota} \restriction \lambda){}^{\backprime}R = (\mathbin{\dot\cup} \breve{\iota} \restriction \lambda){}^{\backprime}S . \supset_{R,S} . R = S :$$

$[*38·12 . *73·25] \quad \supset : (\mathbin{\dot\cup} \breve{\iota} \restriction \lambda)^{\backprime\backprime}\epsilon_\Delta{}^{\backprime}\kappa \,\mathrm{sm}\, \epsilon_\Delta{}^{\backprime}\kappa \quad (3)$

$\vdash . (1) . (3) . \supset \vdash . \mathrm{Prop}$

***83·58.** $\vdash . \epsilon_\Delta{}^{\backprime}\kappa \,\mathrm{sm}\, \epsilon_\Delta{}^{\backprime}(\kappa - 1)$

Dem.

$\vdash . *24·41·21 . *22·43 . \supset$

$\vdash . \kappa = (\kappa - 1) \cup (\kappa \cap 1) . (\kappa - 1) \cap (\kappa \cap 1) = \Lambda . \kappa \cap 1 \subset 1 \quad (1)$

$\vdash . (1) . *83·57 . \supset \vdash . \mathrm{Prop}$

This proposition shows that in a product any number of factors each equal to 1 may be omitted without altering the value of the product.

The following propositions, down to *83·74, are concerned with the domains of selective relations, *i.e.* with the selected classes.

***83·6.** $\vdash : R \in \epsilon_\Delta{}^{\backprime}\kappa . \alpha \in \kappa . \supset . \exists ! \alpha \cap \mathrm{D}{}^{\backprime}R$

Dem.

$\vdash . *83·2 . \supset \vdash : \mathrm{Hp} . \supset . R{}^{\backprime}\alpha \in \alpha .$

$[*33·43] \quad \supset . R{}^{\backprime}\alpha \in \alpha \cap \mathrm{D}{}^{\backprime}R .$

$[*10·24] \quad \supset . \exists ! \alpha \cap \mathrm{D}{}^{\backprime}R : \supset \vdash . \mathrm{Prop}$

***83·61.** $\vdash : R \in \epsilon_\Delta{}'\kappa \,.\, \alpha \in \kappa \,.\, \alpha \cap s'(\kappa - \iota'\alpha) = \Lambda \,.\, \supset . \, \alpha \cap \mathrm{D}'R = \iota' R'\alpha$

Dem.

$\vdash . *40{\cdot}27 . \supset \vdash :. \alpha \cap s'(\kappa - \iota'\alpha) = \Lambda \,.\, \equiv : \beta \in \kappa - \iota'\alpha \,.\, \supset_\beta . \, \alpha \cap \beta = \Lambda :$

[Transp.*51·15] $\equiv : \beta \in \kappa \,.\, \exists ! \alpha \cap \beta \,.\, \supset_\beta . \, \beta = \alpha$ (1)

$\vdash . *83{\cdot}23 . \supset \vdash :. \mathrm{Hp} . \supset : x \in \mathrm{D}'R \,.\, \equiv . (\exists \beta) . \beta \in \kappa \,.\, x = R'\beta \,.$

[*10·35.*14·15] $\supset : x \in \alpha \cap \mathrm{D}'R \,.\, \equiv . (\exists \beta) . \beta \in \kappa \,.\, x = R'\beta \,.\, R'\beta \in \alpha \,.$

[*83·2] $\equiv . (\exists \beta) . \beta \in \kappa \,.\, x = R'\beta \,.\, R'\beta \in \alpha \cap \beta \,.$

[(1).*4·71] $\equiv . (\exists \beta) . \beta \in \kappa \,.\, x = R'\beta \,.\, R'\beta \in \alpha \cap \beta \,.\, \alpha = \beta \,.$

[*13·195.*22·5] $\equiv . \alpha \in \kappa \,.\, x = R'\alpha \,.\, R'\alpha \in \alpha \,.$

[Hp.*4·73.*83·2] $\equiv . x = R'\alpha$ (2)

$\vdash . (2) . *51{\cdot}15 . \supset \vdash . \mathrm{Prop}$

***83·62.** $\vdash : \mu \in \mathrm{D}''\epsilon_\Delta{}'\kappa \,.\, \supset . \, \mu \subset s'\kappa$ [*83·21 . *37·63]

***83·63.** $\vdash : s'\kappa \cap s'\lambda = \Lambda \,.\, \mu \in \mathrm{D}''\epsilon_\Delta{}'(\kappa \cup \lambda) \,.\, \supset . \, \mu \cap s'\kappa \in \mathrm{D}''\epsilon_\Delta{}'\kappa \,.\, \mu \cap s'\lambda \in \mathrm{D}''\epsilon_\Delta{}'\lambda$

Dem.

$\vdash . *80{\cdot}62 . \supset \vdash : M \in \epsilon_\Delta{}'(\kappa \cup \lambda) \,.\, \supset . \, M \upharpoonright \kappa \in \epsilon_\Delta{}'\kappa \,.\, M \upharpoonright \lambda \in \epsilon_\Delta{}'\lambda \,.$ (1)

[*83·21] $\supset . \, \mathrm{D}'M \upharpoonright \kappa \subset s'\kappa \,.\, \mathrm{D}'M \upharpoonright \lambda \subset s'\lambda$ (2)

$\vdash . (2) . *24{\cdot}494 . \supset \vdash :. \mathrm{Hp} . \supset : M \in \epsilon_\Delta{}'(\kappa \cup \lambda) . \supset .$

$\mathrm{D}'M \upharpoonright \kappa = (\mathrm{D}'M \upharpoonright \kappa \cup \mathrm{D}'M \upharpoonright \lambda) - s'\lambda \,.\, \mathrm{D}'M \upharpoonright \lambda = (\mathrm{D}'M \upharpoonright \kappa \cup \mathrm{D}'M \upharpoonright \lambda) - s'\kappa \,.$

[*33·26.*35·412.*80·29] $\supset . \, \mathrm{D}'M \upharpoonright \kappa = \mathrm{D}'M - s'\lambda \,.\, \mathrm{D}'M \upharpoonright \lambda = \mathrm{D}'M - s'\kappa \,.$

[*24·491] $\supset . \, \mathrm{D}'M \upharpoonright \kappa = \mathrm{D}'M \cap s'\kappa \,.\, \mathrm{D}'M \upharpoonright \lambda = \mathrm{D}'M \cap s'\lambda$ (3)

$\vdash . (1) . (3) . *37{\cdot}6 . \supset \vdash :. \mathrm{Hp} . \supset :$

$M \in \epsilon_\Delta{}'(\kappa \cup \lambda) \,.\, \supset . \, \mathrm{D}'M \cap s'\kappa \in \mathrm{D}''\epsilon_\Delta{}'\kappa \,.\, \mathrm{D}'M \cap s'\lambda \in \mathrm{D}''\epsilon_\Delta{}'\lambda :$

[*37·63] $\supset : \mu \in \mathrm{D}''\epsilon_\Delta{}'(\kappa \cup \lambda) \,.\, \supset . \, \mu \cap s'\kappa \in \mathrm{D}''\epsilon_\Delta{}'\kappa \,.\, \mu \cap s'\lambda \in \mathrm{D}''\epsilon_\Delta{}'\lambda :. \supset \vdash . \mathrm{Prop}$

***83·64.** $\vdash :. \kappa \cap \lambda = \Lambda . \supset :$

$\mu \in \mathrm{D}''\epsilon_\Delta{}'(\kappa \cup \lambda) \,.\, \equiv . (\exists \rho, \sigma) . \rho \in \mathrm{D}''\epsilon_\Delta{}'\kappa \,.\, \sigma \in \mathrm{D}''\epsilon_\Delta{}'\lambda \,.\, \mu = \rho \cup \sigma$

Observe that the hypothesis required here is $\kappa \cap \lambda = \Lambda$, not $s'\kappa \cap s'\lambda = \Lambda$ as in *83·63.

Dem.

$\vdash . *80{\cdot}66 . \supset \vdash :. \mathrm{Hp} . \supset : M \in \epsilon_\Delta{}'(\kappa \cup \lambda) \,.\, \mu = \mathrm{D}'M \,.\, \equiv .$

$(\exists R, S) . R \in \epsilon_\Delta{}'\kappa \,.\, S \in \epsilon_\Delta{}'\lambda \,.\, M = R \cup S \,.\, \mu = \mathrm{D}'M \,.$

[*13·193.*33·26] $\equiv . (\exists R, S) . R \in \epsilon_\Delta{}'\kappa \,.\, S \in \epsilon_\Delta{}'\lambda \,.\, M = R \cup S \,.\, \mu = \mathrm{D}'R \cup \mathrm{D}'S$ (1)

$\vdash . (1) . *10{\cdot}11{\cdot}21{\cdot}281 . *37{\cdot}6 . \supset$

$\vdash :: \mathrm{Hp} . \supset :. \mu \in \mathrm{D}''\epsilon_\Delta{}'(\kappa \cup \lambda) . \equiv :$

$(\exists M, R, S) . R \in \epsilon_\Delta{}'\kappa \,.\, S \in \epsilon_\Delta{}'\lambda \,.\, M = R \cup S \,.\, \mu = \mathrm{D}'R \cup \mathrm{D}'S :$

[*10·35] $\equiv : (\exists R, S) : R \in \epsilon_\Delta{}'\kappa \,.\, S \in \epsilon_\Delta{}'\lambda \,.\, \mu = \mathrm{D}'R \cup \mathrm{D}'S : (\exists M) . M = R \cup S :$

[*21·2] $\equiv : (\exists R, S) . R \in \epsilon_\Delta{}'\kappa \,.\, S \in \epsilon_\Delta{}'\lambda \,.\, \mu = \mathrm{D}'R \cup \mathrm{D}'S :$

[*13·22] $\equiv : (\exists R, S, \rho, \sigma) . R \in \epsilon_\Delta{}'\kappa \,.\, \rho = \mathrm{D}'R \,.\, S \in \epsilon_\Delta{}'\lambda \,.\, \sigma = \mathrm{D}'S \,.\, \mu = \rho \cup \sigma :$

[*11·24·54] $\equiv : (\exists \rho, \sigma) : (\exists R) . R \in \epsilon_\Delta{}'\kappa \,.\, \rho = \mathrm{D}'R : (\exists S) . S \in \epsilon_\Delta{}'\lambda \,.\, \sigma = \mathrm{D}'S \,.$

$\mu = \rho \cup \sigma :$

[*37·6.*10·35] $\equiv : (\exists \rho, \sigma) . \rho \in \mathrm{D}''\epsilon_\Delta{}'\kappa \,.\, \sigma \in \mathrm{D}''\epsilon_\Delta{}'\lambda \,.\, \mu = \rho \cup \sigma :. \supset \vdash . \mathrm{Prop}$

The following proposition is used in connection with cardinal multiplication (*115·14).

***83·641.** $\vdash :. s`\kappa \cap s`\lambda = \Lambda . \supset :$

$$\mu \in D``\epsilon_\Delta`(\kappa \cup \lambda) . \equiv . (\exists \rho, \sigma) . \rho \in D``\epsilon_\Delta`\kappa . \sigma \in D``\epsilon_\Delta`\lambda . \mu = \rho \cup \sigma$$

Dem.

$\vdash . *53·25 . \supset \vdash :. \text{Hp} . \supset : \kappa \cap \lambda = \Lambda \cap \text{Cls} . \vee . \kappa \cap \lambda = \iota`\Lambda$ (1)

$\vdash . *83·64 . \supset \vdash :. \kappa \cap \lambda = \Lambda \cap \text{Cls} . \supset : \mu \in D``\epsilon_\Delta`(\kappa \cup \lambda) . \equiv .$

$(\exists \rho, \sigma) . \rho \in D``\epsilon_\Delta`\kappa . \sigma \in D``\epsilon_\Delta`\lambda . \mu = \rho \cup \sigma$ (2)

$\vdash . *51·16 . \supset \vdash :. \kappa \cap \lambda = \iota`\Lambda . \supset : \Lambda \in \kappa . \Lambda \in \lambda :$

[*83·11] $\supset : \epsilon_\Delta`\kappa = \Lambda . \epsilon_\Delta`\lambda = \Lambda . \epsilon_\Delta`(\kappa \cup \lambda) = \Lambda :$

[*37·29] $\supset : D``\epsilon_\Delta`\kappa = \Lambda . D``\epsilon_\Delta`\lambda = \Lambda . D``\epsilon_\Delta`(\kappa \cup \lambda) = \Lambda :$

[*24·15] $\supset : \mu \sim\in D``\epsilon_\Delta`(\kappa \cup \lambda) : (\rho) . \rho \sim\in D``\epsilon_\Delta`\kappa :$

[*11·55.Transp.*10·252] $\supset : \mu \sim\in D``\epsilon_\Delta`(\kappa \cup \lambda) :$

$\sim(\exists \rho, \sigma) . \rho \in D``\epsilon_\Delta`\kappa . \sigma \in D``\epsilon_\Delta`\lambda . \mu = \rho \cup \sigma :$

[*5·21] $\supset : \mu \in D``\epsilon_\Delta`(\kappa \cup \lambda) . \equiv .$

$(\exists \rho, \sigma) . \rho \in D``\epsilon_\Delta`\kappa . \sigma \in D``\epsilon_\Delta`\lambda . \mu = \rho \cup \sigma$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash . \text{Prop}$

***83·65.** $\vdash : s`\kappa \cap s`\lambda = \Lambda . \mu \in D``\epsilon_\Delta`(\kappa \cup \lambda) . \supset .$

$$\mu - s`\kappa \in D``\epsilon_\Delta`\lambda . \mu - s`\lambda \in D``\epsilon_\Delta`\kappa$$

Dem.

$\vdash . *83·62 . \qquad \supset \vdash : \text{Hp} . \supset . \mu \subset s`(\kappa \cup \lambda) .$

[*40·171] $\supset . \mu \subset s`\kappa \cup s`\lambda$ (1)

$\vdash . (1) . *24·491 . \supset \vdash : \text{Hp} . \supset . \mu - s`\kappa = \mu \cap s`\lambda . \mu - s`\lambda = \mu \cap s`\kappa$ (2)

$\vdash . (2) . *83·63 . \quad \supset \vdash . \text{Prop}$

***83·66.** $\vdash : \exists ! \epsilon_\Delta`\kappa . \supset . s`D``\epsilon_\Delta`\kappa = s`\kappa$

Dem.

$\vdash . *41·43 . \supset \vdash . s`D``\epsilon_\Delta`\kappa = D`\dot{s}`\epsilon_\Delta`\kappa$ (1)

$\vdash . *83·25 . \supset \vdash : \text{Hp} . \supset . D`\dot{s}`\epsilon_\Delta`\kappa = D`\epsilon \upharpoonright \kappa$

[*62·43] $= s`\kappa$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***83·7.** $\vdash . D``\epsilon_\Delta`\iota`\alpha = \iota``\alpha$ [*83·4 . *55·261]

***83·71.** $\vdash . D``\epsilon_\Delta`\iota``\alpha = \iota`\alpha . D`\alpha \rceil \breve{\iota} = \alpha$

Dem.

$\vdash . *83·42 . \supset \vdash . D``\epsilon_\Delta`\iota``\alpha = D``\iota`(\alpha \rceil \breve{\iota})$

[*53·31] $= \iota`D`(\alpha \rceil \breve{\iota})$ (1)

[*35·61] $= \iota`(\alpha \cap D`\breve{\iota})$

[*33·2] $= \iota`(\alpha \cap \text{Ɑ}`\iota)$

[*51·17.*24·26] $= \iota`\alpha$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***83·72.** $\vdash : \kappa \subset 1 . \supset . D``\epsilon_\Delta`\kappa = \iota`s`\kappa$

Dem.

$$\begin{aligned}
&\vdash . *83{\cdot}43 . \supset \vdash : \text{Hp} . \supset . D``\epsilon_\Delta`\kappa = D``\iota`(\epsilon \upharpoonright \kappa) \\
&[*53{\cdot}31] \qquad\qquad\qquad = \iota`D`(\epsilon \upharpoonright \kappa) \\
&[*62{\cdot}43] \qquad\qquad\qquad = \iota`s`\kappa : \supset \vdash . \text{Prop}
\end{aligned}$$

*83·73·731 are lemmas for *83·74.

***83·73.** $\vdash : \kappa \cap \lambda = \Lambda . \lambda \subset 1 . \supset .$

$$D``\epsilon_\Delta`(\kappa \cup \lambda) = \hat{\sigma}\{(\exists\rho) . \rho \in D``\epsilon_\Delta`\kappa . \sigma = \rho \cup s`\lambda\}$$

Dem.

$$\begin{aligned}
&\vdash . *83{\cdot}56 . *37{\cdot}6 . \supset \vdash :. \text{Hp} . \supset : \\
&\sigma \in D``\epsilon_\Delta`(\kappa \cup \lambda) . \equiv . (\exists R, S) . R \in \epsilon_\Delta`\kappa . S = R \,\dot{\cup}\, \breve{\iota} \upharpoonright \lambda . \sigma = D`S . \\
&[*13{\cdot}193] \qquad \equiv . (\exists R, S) . R \in \epsilon_\Delta`\kappa . S = R \,\dot{\cup}\, \breve{\iota} \upharpoonright \lambda . \sigma = D`(R \,\dot{\cup}\, \breve{\iota} \upharpoonright \lambda) . \\
&[*62{\cdot}43{\cdot}55] \qquad \equiv . (\exists R, S) . R \in \epsilon_\Delta`\kappa . S = R \,\dot{\cup}\, \breve{\iota} \upharpoonright \lambda . \sigma = D`R \cup s`\lambda . \\
&[*10{\cdot}35 . *21{\cdot}2] \qquad \equiv . (\exists R) . R \in \epsilon_\Delta`\kappa . \sigma = D`R \cup s`\lambda . \\
&[*37{\cdot}64] \qquad \equiv . (\exists\rho) . \rho \in D``\epsilon_\Delta`\kappa . \sigma = \rho \cup s`\lambda :. \supset \vdash . \text{Prop}
\end{aligned}$$

***83·731.** $\vdash :. \lambda \subset 1 . \supset : s`\kappa \cap s`\lambda = \Lambda . \supset . \kappa \cap \lambda = \Lambda$

Dem.

$$\begin{aligned}
&\vdash . *53{\cdot}25 . *51{\cdot}16 . \supset \vdash :. s`\kappa \cap s`\lambda = \Lambda . \supset : \kappa \cap \lambda = \Lambda . \vee . \Lambda \in \lambda \qquad (1) \\
&\vdash . *52{\cdot}16 . \qquad \supset \vdash :. \lambda \subset 1 . \supset : \alpha \in \lambda . \supset_\alpha . \exists ! \alpha : \\
&[*24{\cdot}63] \qquad\qquad \supset : \Lambda \sim\in \lambda \qquad (2) \\
&\vdash . (1) . (2) . \supset \vdash . \text{Prop}
\end{aligned}$$

***83·74.** $\vdash : s`\kappa \cap s`\lambda = \Lambda . \lambda \subset 1 . \supset . D``\epsilon_\Delta`(\kappa \cup \lambda) \operatorname{sm} D``\epsilon_\Delta`\kappa$

Dem.

$$\begin{aligned}
&\vdash . *83{\cdot}73{\cdot}731 . *38{\cdot}131 . \supset \vdash : \text{Hp} . \supset . D``\epsilon_\Delta`(\kappa \cup \lambda) = (\cup s`\lambda)``D``\epsilon_\Delta`\kappa \quad (1) \\
&\vdash . *83{\cdot}62 . *24{\cdot}13 . \supset \\
&\vdash :: \text{Hp} . \supset :. \mu, \nu \in D``\epsilon_\Delta`\kappa . \supset : \mu \cap s`\lambda = \Lambda . \nu \cap s`\lambda = \Lambda : \\
&[*24{\cdot}481] \qquad \supset : \mu \cup s`\lambda = \nu \cup s`\lambda . \equiv . \mu = \nu : \\
&[*38{\cdot}11] \qquad \supset : (\cup s`\lambda)`\mu = (\cup s`\lambda)`\nu . \equiv . \mu = \nu \qquad (2) \\
&\vdash . (2) . *73{\cdot}28 . \supset \vdash : \text{Hp} . \supset . (\cup s`\lambda)``D``\epsilon_\Delta`\kappa \operatorname{sm} D``\epsilon_\Delta`\kappa \qquad (3) \\
&\vdash . (1) . (3) . \supset \vdash . \text{Prop}
\end{aligned}$$

***83·8.** $\vdash . \epsilon_\Delta`\kappa \subset t_{10}`\kappa . \epsilon_\Delta`\kappa \in t`t_{10}`\kappa$

Dem.

$$\begin{aligned}
&\vdash . *80{\cdot}14 . *83{\cdot}21 . *35{\cdot}83 . \supset \vdash : R \in \epsilon_\Delta`\kappa . \supset . R \mathrel{\dot\subset} s`\kappa \uparrow \kappa . \\
&[*63{\cdot}105 . (*63{\cdot}03)] \qquad \supset . R \mathrel{\dot\subset} t_1`\kappa \uparrow t_0`\kappa . \\
&[*64{\cdot}201] \qquad \supset . R \in t`(t_1`\kappa \uparrow t_0`\kappa) . \\
&[(*64{\cdot}021)] \qquad \supset . R \in t_{10}`\kappa \qquad (1) \\
&\vdash . (1) . *63{\cdot}371 . \supset \vdash . \text{Prop}
\end{aligned}$$

***83·81.** $\vdash . D``\epsilon_\Delta`\kappa \subset t_0`\kappa . D``\epsilon_\Delta`\kappa \in t`\kappa$

Dem.

$\vdash . *83·62 . \supset \vdash : \mu \in D``\epsilon_\Delta`\kappa . \supset . \mu \subset s`\kappa .$

$[*63·105.(*63·03)] \quad \supset . \mu \subset t_1`\kappa .$

$[*63·51] \quad \supset . \mu \in t_0`\kappa \quad (1)$

$\vdash . (1) . *63·371 . \supset \vdash . \text{Prop}$

***83·9.** $\vdash . \exists ! \epsilon_\Delta`\Lambda$ [*83·15]

***83·901.** $\vdash : \exists ! \epsilon_\Delta`\iota`\alpha . \equiv . \exists ! \alpha$ [*80·46 . *62·2]

***83·902.** $\vdash : \exists ! \epsilon_\Delta`(\kappa \cup \lambda) . \equiv . \exists ! \epsilon_\Delta`\kappa . \exists ! \epsilon_\Delta`\lambda$ [*80·69]

***83·903.** $\vdash : \exists ! \epsilon_\Delta`(\iota`\alpha \cup \iota`\beta) . \equiv . \exists ! \alpha . \exists ! \beta$ [*83·901·902]

***83·904.** $\vdash : \exists ! \epsilon_\Delta`(\kappa \cup \iota`\beta) . \equiv . \exists ! \epsilon_\Delta`\kappa . \exists ! \beta$ [*83·901·902]

*83·9·904 lead to an inductive proof (to be given later) of $\exists ! \epsilon_\Delta`\kappa$ whenever κ is a finite class of classes none of which is Λ.

*84. CLASSES OF MUTUALLY EXCLUSIVE CLASSES

Summary of *84.

A class κ of mutually exclusive classes is one such that, if α and β are two different members of κ, α and β have no common members; *i.e.* it is a class composed of non-overlapping classes. Classes of mutually exclusive classes have many important properties. They are important in cardinal arithmetic, among other reasons, because if κ is a class of mutually exclusive classes, the cardinal number of $s\text{‘}\kappa$ is the sum of the cardinal numbers of the members of κ. Also if κ is a class of mutually exclusive classes, the number of selected classes of κ (*i.e.* $\mathrm{D}\text{“}\epsilon_\Delta\text{‘}\kappa$) is the same as the number of selective relations (*i.e.* $\epsilon_\Delta\text{‘}\kappa$).

"κ is a class of mutually exclusive classes" is written "$\kappa \in \mathrm{Cls}^2\,\mathrm{excl}$."

An important case is when no member of κ is null; in this case we write

$$\kappa \in \mathrm{Cls\ ex}^2\,\mathrm{excl}.$$

For a Cls^2 excl which is contained in a class of classes γ, we write

$$\mathrm{Cl\ excl}\text{‘}\gamma,$$

on the analogy of the notation $\mathrm{Cl}\text{‘}\gamma$.

The definitions are as follows:

***84·01.** $\mathrm{Cls}^2\,\mathrm{excl} = \hat\kappa\,(\alpha, \beta \in \kappa \,.\, \alpha \neq \beta \,.\, \supset_{\alpha,\beta} .\, \alpha \cap \beta = \Lambda)$ Df

***84·02.** $\mathrm{Cl\ excl}\text{‘}\gamma = \mathrm{Cls}^2\,\mathrm{excl} \cap \mathrm{Cl}\text{‘}\gamma$ Df

***84·03.** $\mathrm{Cls\ ex}^2\,\mathrm{excl} = \mathrm{Cls}^2\,\mathrm{excl} - \overleftarrow{\epsilon}\text{‘}\Lambda$ Df

The propositions of this number begin (*84·1—·14) with various equivalent forms for the definitions. Of these the most useful are:

***84·11.** $\vdash :. \kappa \in \mathrm{Cls}^2\,\mathrm{excl} \,.\equiv :\, \alpha, \beta \in \kappa \,.\, \exists !\, \alpha \cap \beta \,.\, \supset_{\alpha,\beta} .\, \alpha = \beta$

***84·13.** $\vdash : \kappa \in \mathrm{Cls\ ex}^2\,\mathrm{excl} \,.\equiv .\, \kappa \in \mathrm{Cls}^2\,\mathrm{excl} \,.\, \Lambda \sim\in \kappa$

***84·14.** $\vdash : \kappa \in \mathrm{Cls}^2\,\mathrm{excl} \,.\equiv .\, \epsilon \upharpoonright \kappa \in \mathrm{Cls} \to 1$

The last of these is specially important, because it renders the propositions of *81 applicable to $\epsilon_\Delta\text{‘}\kappa$ when $\kappa \in \mathrm{Cls}^2\,\mathrm{excl}$.

We have next (*84·2—·28) a set of propositions dealing with various special cases, such as Λ and 1. The most useful of these are

***84·23.** $\vdash .\, \iota\text{‘}\alpha \in \mathrm{Cls}^2\,\mathrm{excl}$

***84·241.** $\vdash .\, \iota\text{“}\alpha \in \mathrm{Cls\ ex}^2\,\mathrm{excl}$

***84·25.** $\vdash : \kappa \in \mathrm{Cls}^2\,\mathrm{excl} \,.\, \lambda \subset \kappa \,.\supset .\, \lambda \in \mathrm{Cls}^2\,\mathrm{excl}$

We next have a set of propositions (∗84·3—·37) which are immediate consequences of propositions in ∗81, by means of ∗84·14. The most useful of these is

∗84·3. $\vdash : \kappa \epsilon \mathrm{Cls}^2 \,\mathrm{excl} \,.\supset .\, \epsilon_\Delta{}^{\boldsymbol{\prime}}\kappa \subset 1 \rightarrow 1$

We next have a set of propositions (∗84·4—·43) dealing with the domains of selections from a Cls² excl. These are for the most part still immediate consequences of propositions in ∗81, in virtue of ∗84·14. The most useful are

∗84·41. $\vdash : \kappa \epsilon \mathrm{Cls}^2 \,\mathrm{excl} \,.\supset .\, \mathrm{D} \upharpoonright \epsilon_\Delta{}^{\boldsymbol{\prime}}\kappa \,\epsilon\, 1 \rightarrow 1 \,.\, \mathrm{D}^{\boldsymbol{\prime\prime}}\epsilon_\Delta{}^{\boldsymbol{\prime}}\kappa \,\mathrm{sm}\, \epsilon_\Delta{}^{\boldsymbol{\prime}}\kappa$

∗84·412. $\vdash : \kappa \epsilon \mathrm{Cls}^2 \,\mathrm{excl} \,.\supset .\, \mathrm{D}^{\boldsymbol{\prime\prime}}\epsilon_\Delta{}^{\boldsymbol{\prime}}\kappa = \hat{\mu}\{\alpha \epsilon \kappa \,.\supset_\alpha .\, \mu \cap \alpha \,\epsilon\, 1 : \mu \subset s^{\boldsymbol{\prime}}\kappa\}$

∗84·43. $\vdash :.\, \alpha, \beta \,\epsilon\, \mathrm{Cls}^2 \,\mathrm{excl} \,.\, s^{\boldsymbol{\prime}}\alpha = s^{\boldsymbol{\prime}}\beta \,.\supset : \alpha \subset \mathrm{D}^{\boldsymbol{\prime\prime}}\epsilon_\Delta{}^{\boldsymbol{\prime}}\beta \,.\equiv .\, \beta \subset \mathrm{D}^{\boldsymbol{\prime\prime}}\epsilon_\Delta{}^{\boldsymbol{\prime}}\alpha$

This proposition applies to such cases as the relations of rows and columns. Imagine any set of terms arranged in rows and columns so as to form a rectangle. Then each column is a selection from the rows, and each row is a selection from the columns. This is a particular case of the above proposition.

We next have a set of propositions on $\overrightarrow{R}^{\boldsymbol{\prime\prime}}\kappa$, $R^{\boldsymbol{\prime\prime\prime}}\kappa$, and $P_\Delta{}^{\boldsymbol{\prime\prime}}\kappa$ (∗84·5—·55). The most important of these are

∗84·51. $\vdash : R \upharpoonright \kappa \,\epsilon\, \mathrm{Cls} \rightarrow 1 \,.\supset .\, \overrightarrow{R}^{\boldsymbol{\prime\prime}}\kappa \,\epsilon\, \mathrm{Cls}^2 \,\mathrm{excl}$

∗84·53. $\vdash : R \,\epsilon\, \mathrm{Cls} \rightarrow 1 \,.\, \kappa \,\epsilon\, \mathrm{Cls}^2 \,\mathrm{excl} \,.\supset .\, R^{\boldsymbol{\prime\prime\prime}}\kappa \,\epsilon\, \mathrm{Cls}^2 \,\mathrm{excl}$

Finally we have a set of propositions (∗84·59—·62) showing circumstances under which $\kappa \cup \lambda$ is a Cls² excl. The only one of these which is used subsequently is

∗84·62. $\vdash :.\, \alpha \neq \beta \,.\supset : \iota^{\boldsymbol{\prime}}\alpha \cup \iota^{\boldsymbol{\prime}}\beta \,\epsilon\, \mathrm{Cls}^2 \,\mathrm{excl} \,.\equiv .\, \alpha \cap \beta = \Lambda$

∗84·01. $\mathrm{Cls}^2 \,\mathrm{excl} = \hat{\kappa}(\alpha, \beta \,\epsilon\, \kappa \,.\, \alpha \neq \beta \,.\supset_{\alpha,\beta} .\, \alpha \cap \beta = \Lambda)$ Df

∗84·02. $\mathrm{Cl\,excl}^{\boldsymbol{\prime}}\gamma = \mathrm{Cls}^2 \,\mathrm{excl} \cap \mathrm{Cl}^{\boldsymbol{\prime}}\gamma$ Df

∗84·03. $\mathrm{Cls\,ex}^2 \,\mathrm{excl} = \mathrm{Cls}^2 \,\mathrm{excl} - \overleftarrow{\epsilon}^{\boldsymbol{\prime}}\Lambda$ Df

∗84·1. $\vdash :.\, \kappa \,\epsilon\, \mathrm{Cls}^2 \,\mathrm{excl} \,.\equiv : \alpha, \beta \,\epsilon\, \kappa \,.\, \alpha \neq \beta \,.\supset_{\alpha,\beta} .\, \alpha \cap \beta = \Lambda$
[∗20·3 . (∗84·01)]

∗84·11. $\vdash :.\, \kappa \,\epsilon\, \mathrm{Cls}^2 \,\mathrm{excl} \,.\equiv : \alpha, \beta \,\epsilon\, \kappa \,.\, \exists ! \,\alpha \cap \beta \,.\supset_{\alpha,\beta} .\, \alpha = \beta$
[∗84·1 . Transp]

∗84·12. $\vdash :.\, \kappa \,\epsilon\, \mathrm{Cl\,excl}^{\boldsymbol{\prime}}\gamma \,.\equiv : \alpha, \beta \,\epsilon\, \kappa \,.\, \alpha \neq \beta \,.\supset_{\alpha,\beta} .\, \alpha \cap \beta = \Lambda : \kappa \subset \gamma : \equiv :$
$\kappa \,\epsilon\, \mathrm{Cls}^2 \,\mathrm{excl} \,.\, \kappa \subset \gamma$ [∗20·3 . (∗84·02) . ∗22·33 . ∗84·1]

∗84·121. $\vdash :.\, \kappa \,\epsilon\, \mathrm{Cl\,excl}^{\boldsymbol{\prime}}\gamma \,.\equiv : \alpha, \beta \,\epsilon\, \kappa \,.\, \exists ! \,\alpha \cap \beta \,.\supset_{\alpha,\beta} .\, \alpha = \beta : \kappa \subset \gamma$
[∗20·3 . (∗84·02) . ∗22·33 . ∗84·11]

***84·13.** $\vdash : \kappa \epsilon \mathrm{Cls\,ex^2\,excl} . \equiv . \kappa \epsilon \mathrm{Cls^2\,excl} . \Lambda \sim \epsilon \kappa$

Dem.

$\vdash . *22·33·35 . (*84·03) . \supset$

$\vdash : \kappa \epsilon \mathrm{Cls\,ex^2\,excl} . \equiv . \kappa \epsilon \mathrm{Cls^2\,excl} . \kappa \sim \epsilon \overleftarrow{\epsilon} {}^{\backprime}\Lambda .$

[*62·21] $\equiv . \kappa \epsilon \mathrm{Cls^2\,excl} . \Lambda \sim \epsilon \kappa : \supset \vdash .$ Prop

***84·131.** $\vdash :. \kappa \epsilon \mathrm{Cls\,ex^2\,excl} . \equiv : \alpha, \beta \epsilon \kappa . \alpha \neq \beta . \supset_{\alpha,\beta} . \alpha \cap \beta = \Lambda : \Lambda \sim \epsilon \kappa$
[*84·13·1]

***84·132.** $\vdash :. \kappa \epsilon \mathrm{Cls\,ex^2\,excl} . \equiv : \alpha, \beta \epsilon \kappa . \exists ! \alpha \cap \beta . \supset_{\alpha,\beta} . \alpha = \beta : \Lambda \sim \epsilon \kappa$
[*84·13·11]

***84·133.** $\vdash :. \kappa \epsilon \mathrm{Cls\,ex^2\,excl} . \equiv : \alpha, \beta \epsilon \kappa . \exists ! \alpha \cap \beta . \supset_{\alpha,\beta} . \alpha = \beta : \alpha \epsilon \kappa . \supset_\alpha . \exists ! \alpha$
[*84·132 . *24·63]

***84·134.** $\vdash :: \kappa \epsilon \mathrm{Cls\,ex^2\,excl} . \equiv :. \alpha, \beta \epsilon \kappa . \supset_{\alpha,\beta} : \exists ! \alpha . \exists ! \beta : \exists ! \alpha \cap \beta . \supset . \alpha = \beta$

Dem.

$\vdash . *11·59 . \quad \supset \vdash :. \alpha \epsilon \kappa . \supset_\alpha . \exists ! \alpha : \equiv : \alpha, \beta \epsilon \kappa . \supset_{\alpha,\beta} . \exists ! \alpha . \exists ! \beta$ (1)

$\vdash . *4·87 . *11·33 . \quad \supset \vdash :: \alpha, \beta \epsilon \kappa . \exists ! \alpha \cap \beta . \supset_{\alpha,\beta} . \alpha = \beta : \equiv :.$
$\alpha, \beta \epsilon \kappa . \supset_{\alpha,\beta} : \exists ! \alpha \cap \beta . \supset . \alpha = \beta$ (2)

$\vdash . (1) . (2) . *84·133 . \supset \vdash :: \kappa \epsilon \mathrm{Cls\,ex^2\,excl} . \equiv :.$
$\alpha, \beta \epsilon \kappa . \supset_{\alpha,\beta} . \exists ! \alpha . \exists ! \beta :. \alpha, \beta \epsilon \kappa . \supset_{\alpha,\beta} : \exists ! \alpha \cap \beta . \supset . \alpha = \beta :.$

[*11·391] $\equiv :. \alpha, \beta \epsilon \kappa . \supset_{\alpha,\beta} : \exists ! \alpha . \exists ! \beta : \exists ! \alpha \cap \beta . \supset . \alpha = \beta :: \supset \vdash .$ Prop

***84·135.** $\vdash :: \kappa \epsilon \mathrm{Cls\,ex^2\,excl} . \equiv :. \alpha, \beta \epsilon \kappa . \supset_{\alpha,\beta} : \exists ! \alpha \cap \beta . \equiv . \alpha = \beta$

Dem.

$\vdash . *84·133 . *22·5 . *13·191 . \supset$

$\vdash :: \kappa \epsilon \mathrm{Cls\,ex^2\,excl} . \quad \equiv :. \alpha, \beta \epsilon \kappa . \exists ! \alpha \cap \beta . \supset_{\alpha,\beta} . \alpha = \beta :$
$\alpha, \beta \epsilon \kappa . \alpha = \beta . \supset_{\alpha,\beta} . \exists ! \alpha \cap \beta :.$

[*11·31] $\equiv :. (\alpha, \beta) :. \alpha, \beta \epsilon \kappa . \exists ! \alpha \cap \beta . \supset . \alpha = \beta :$
$\alpha, \beta \epsilon \kappa . \alpha = \beta . \supset . \exists ! \alpha \cap \beta :.$

[*4·87.Comp.*11·33] $\equiv :. (\alpha, \beta) :. \alpha, \beta \epsilon \kappa . \supset : \exists ! \alpha \cap \beta . \equiv . \alpha = \beta :: \supset \vdash .$ Prop

***84·14.** $\vdash : \kappa \epsilon \mathrm{Cls^2\,excl} . \equiv . \epsilon \upharpoonright \kappa \epsilon \mathrm{Cls} \rightarrow 1$

Dem.

$\vdash . *10·23 . *84·11 . \supset \vdash :. \kappa \epsilon \mathrm{Cls^2\,excl} . \equiv : \alpha, \beta \epsilon \kappa . x \epsilon \alpha . x \epsilon \beta . \supset_{x,\alpha,\beta} . \alpha = \beta :$

[*35·101] $\equiv : x (\epsilon \upharpoonright \kappa) \alpha . x (\epsilon \upharpoonright \kappa) \beta . \supset_{x,\alpha,\beta} . \alpha = \beta :$

[*71·171] $\equiv : \epsilon \upharpoonright \kappa \epsilon \mathrm{Cls} \rightarrow 1 :. \supset \vdash .$ Prop

This proposition is important, since it enables us to apply the propositions of *81 to $\epsilon_\Delta {}^{\backprime} \kappa$ when $\kappa \epsilon \mathrm{Cls^2\,excl}$.

***84·2.** $\vdash . \Lambda \cap \mathrm{Cls} \epsilon \mathrm{Cls\,ex^2\,excl}$

Dem.

$\vdash . *24·105 . *11·57 . \supset \vdash . (\alpha, \beta) . \alpha, \beta \sim \epsilon \Lambda \cap \mathrm{Cls} .$

[*11·25·63] $\supset \vdash :. \alpha, \beta \epsilon \Lambda \cap \mathrm{Cls} . \supset_{\alpha,\beta} : \exists ! \alpha \cap \beta . \equiv . \alpha = \beta :.$

[*84·135] $\supset \vdash . \Lambda \cap \mathrm{Cls} \epsilon \mathrm{Cls\,ex^2\,excl}$

***84·21.** $\vdash . 1_{\text{Cls}} \subset \text{Cls}^2 \,\text{excl}$

Note. 1_{Cls} is the class of all unit classes whose members are classes; this results from *65·01. Thus "$\alpha \in 1_{\text{Cls}}$" is equivalent to "$\alpha$ consists of one class."

Dem.

$\vdash . *22{\cdot}33 . (*65{\cdot}01) . \supset \vdash :. \alpha \in 1_{\text{Cls}} . \equiv : \alpha \in 1 . \alpha \subset \text{Cls} :$

[*52·16] $\qquad \supset : \beta, \gamma \in \alpha . \supset_{\beta,\gamma} . \beta = \gamma :$

[*3·41] $\qquad \supset : \beta, \gamma \in \alpha . \exists ! \beta \cap \gamma . \supset_{\beta,\gamma} . \beta = \gamma :$

[*84·11] $\qquad \supset : \alpha \in \text{Cls}^2 \,\text{excl} :. \supset \vdash . \text{Prop}$

***84·22.** $\vdash . 1 \in \text{Cls ex}^2 \,\text{excl}$

Dem.

$\vdash . *52{\cdot}46 . \supset \vdash :. \alpha, \beta \in 1 . \supset : \exists ! \alpha \cap \beta . \equiv . \alpha = \beta \quad (1)$

$\vdash . (1) . *84{\cdot}135 . \supset \vdash . \text{Prop}$

***84·23.** $\vdash . \iota\text{`}\alpha \in \text{Cls}^2 \,\text{excl} \quad [*84{\cdot}21 . *52{\cdot}22]$

***84·24.** $\vdash \; \exists ! \alpha . \supset . \iota\text{`}\alpha \in \text{Cls ex}^2 \,\text{excl}$

Dem.

$\vdash . *13{\cdot}191 . \supset \vdash :. \text{Hp} . \supset : \beta = \alpha . \supset_\beta . \exists ! \beta :$

[*51·15] $\qquad \supset : \beta \in \iota\text{`}\alpha . \supset_\beta . \exists ! \beta :$

[*24·63] $\qquad \supset : \Lambda \sim\in \iota\text{`}\alpha \quad (1)$

$\vdash . (1) . *84{\cdot}23{\cdot}13 . \supset \vdash . \text{Prop}$

***84·241.** $\vdash . \iota\text{``}\alpha \in \text{Cls ex}^2 \,\text{excl}$

Dem.

$\vdash . *52{\cdot}3 . \supset \vdash :. \beta, \gamma \in \iota\text{``}\alpha . \supset_{\beta,\gamma} : \beta, \gamma \in 1 :$

[*52·46] $\qquad \supset_{\beta,\gamma} : \exists ! \beta \cap \gamma . \equiv . \beta = \gamma \quad (1)$

$\vdash . (1) . *84{\cdot}135 . \supset \vdash . \text{Prop}$

***84·242.** $\vdash : \kappa \subset 1 . \supset . \kappa \in \text{Cls ex}^2 \,\text{excl} \quad [*52{\cdot}46 . *84{\cdot}135]$

***84·25.** $\vdash : \kappa \in \text{Cls}^2 \,\text{excl} . \lambda \subset \kappa . \supset . \lambda \in \text{Cls}^2 \,\text{excl}$

Dem.

$\vdash . *22{\cdot}1 . *11{\cdot}59 . \supset \vdash :. \lambda \subset \kappa . \supset : \alpha, \beta \in \lambda . \supset_{\alpha,\beta} . \alpha, \beta \in \kappa :$

[*11·38] $\qquad \supset : \alpha, \beta \in \lambda . \alpha \neq \beta . \supset_{\alpha,\beta} . \alpha, \beta \in \kappa . \alpha \neq \beta \quad (1)$

$\vdash . *84{\cdot}1 . \quad \supset \vdash :. \kappa \in \text{Cls}^2 \,\text{excl} . \supset : \alpha, \beta \in \kappa . \alpha \neq \beta . \supset_{\alpha,\beta} . \alpha \cap \beta = \Lambda \quad (2)$

$\vdash . (1) . (2) . *11{\cdot}37 . \supset \vdash :. \text{Hp} . \supset : \alpha, \beta \in \lambda . \alpha \neq \beta . \supset_{\alpha,\beta} . \alpha \cap \beta = \Lambda :$

[*84·1] $\qquad \supset : \lambda \in \text{Cls}^2 \,\text{excl} :. \supset \vdash . \text{Prop}$

***84·26.** $\vdash : \kappa \in \text{Cls ex}^2 \,\text{excl} . \lambda \subset \kappa . \supset . \lambda \in \text{Cls ex}^2 \,\text{excl}$

Dem.

$\vdash . *84{\cdot}13{\cdot}25 . \quad \supset \vdash : \text{Hp} . \supset . \lambda \in \text{Cls}^2 \,\text{excl} \quad (1)$

$\vdash . *22{\cdot}1 . *10{\cdot}1 . \supset \vdash :. \text{Hp} . \supset : \Lambda \in \lambda . \supset . \Lambda \in \kappa :$

[Transp] $\qquad \supset : \Lambda \sim\in \kappa . \supset . \Lambda \sim\in \lambda \quad (2)$

$\vdash . *84{\cdot}13 . \quad \supset \vdash : \text{Hp} . \supset . \Lambda \sim\in \kappa \quad (3)$

$\vdash . (2) . (3) . \quad \supset \vdash : \text{Hp} . \supset . \Lambda \sim\in \lambda \quad (4)$

$\vdash . (1) . (4) . *84{\cdot}13 . \supset \vdash . \text{Prop}$

***84·28.** $\vdash : \kappa \in \mathrm{Cl\ excl}`\gamma \,.\, \lambda \subset \kappa \,.\, \gamma \subset \delta \,.\supset .\, \lambda \in \mathrm{Cl\ excl}`\delta$

Dem.

$$\vdash .\, {*}84{\cdot}12{\cdot}25 \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, \lambda \in \mathrm{Cls}^2\,\mathrm{excl} \qquad (1)$$

$$\vdash .\, {*}84{\cdot}12 \,.\quad \supset \vdash : \mathrm{Hp} \,.\supset .\, \kappa \subset \gamma \,.\, \lambda \subset \kappa \,.\, \gamma \subset \delta \,.$$

$$[{*}22{\cdot}44] \qquad \supset .\, \lambda \subset \delta \qquad (2)$$

$$\vdash .\, (1) \,.\, (2) \,.\, {*}84{\cdot}12 \,.\supset \vdash .\, \mathrm{Prop}$$

The following propositions are concerned with selections from a Cls^2 excl. In virtue of *84·14, the propositions of *81 which have the hypothesis $R \restriction \kappa \in \mathrm{Cls} \to 1$ become applicable when R is ϵ and κ is a Cls^2 excl. Thus $\epsilon_\Delta`\kappa$ has many important properties when κ is a Cls^2 excl which it does not have in the general case.

***84·3.** $\vdash : \kappa \in \mathrm{Cls}^2\,\mathrm{excl} \,.\supset .\, \epsilon_\Delta`\kappa \subset 1 \to 1$ [*84·14 . *81·1]

***84·31.** $\vdash : \kappa \in \mathrm{Cls}^2\,\mathrm{excl} \,.\, R \in \epsilon_\Delta`\kappa \,.\, x \in \mathrm{D}`R \,.\supset .\, \mathrm{E} \,!\, \breve{R}`x$ [*84·14 . *81·11]

***84·32.** $\vdash : \kappa \in \mathrm{Cls}^2\,\mathrm{excl} \,.\, R \in \epsilon_\Delta`\kappa \,.\, x \in \mathrm{D}`R \,.\supset .\, x \in \breve{R}`x \,.\, \breve{R}`x \in \kappa$
[*84·14 . *81·11 . *35·101]

***84·33.** $\vdash : \kappa \in \mathrm{Cls}^2\,\mathrm{excl} \,.\, R \in \epsilon_\Delta`\kappa \,.\, x \in \mathrm{D}`R \,.\supset .\, \breve{R}`x = (\imath\alpha)(\alpha \in \kappa \,.\, x \in \alpha) = (\kappa \upharpoonleft \epsilon)`x$
[*84·14 . *81·12]

***84·34.** $\vdash :. \kappa \in \mathrm{Cls}^2\,\mathrm{excl} \,.\, R \in \epsilon_\Delta`\kappa \,.\supset : xR\alpha \,.\equiv .\, x \in \alpha \,.\, x \in \mathrm{D}`R \,.\, \alpha \in \kappa$
[*81·13 . *84·14]

***84·341.** $\vdash : \kappa \in \mathrm{Cls}^2\,\mathrm{excl} \,.\, R \in \epsilon_\Delta`\kappa \,.\supset .\, R = \mathrm{D}`R \upharpoonleft \epsilon \restriction \kappa = \epsilon \mathbin{\dot\cap} \mathrm{D}`R \uparrow \kappa$
[*81·14 . *84·14]

***84·342.** $\vdash : \kappa \in \mathrm{Cls}^2\,\mathrm{excl} \,.\, R \in \epsilon_\Delta`\kappa \,.\, \alpha \in \kappa \,.\supset .\, \iota`R`\alpha = \alpha \cap \mathrm{D}`R$
[*81·15 . *84·14 . *62·2]

***84·35.** $\vdash :. \kappa \in \mathrm{Cls\ ex}^2\,\mathrm{excl} \,.\supset : R \in \epsilon_\Delta`\kappa \,.\equiv .\, R \in 1 \to 1 \,.\, R \mathrel{\dot\subset} \epsilon \restriction \kappa \,.\, \text{Ɑ}`R = \text{Ɑ}`\epsilon \restriction \kappa$

Dem.

$$\vdash .\, {*}84{\cdot}13 \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, \Lambda \sim\in \kappa \,.$$

$$[{*}62{\cdot}42] \qquad \supset .\, \text{Ɑ}`\epsilon \restriction \kappa = \kappa \qquad (1)$$

$$\vdash .\, (1) \,.\, {*}71{\cdot}103 \,.\, {*}80{\cdot}14 \,.\supset$$

$$\vdash :. \mathrm{Hp} \,.\supset : R \in 1 \to 1 \,.\, R \mathrel{\dot\subset} \epsilon \restriction \kappa \,.\, \text{Ɑ}`R = \text{Ɑ}`\epsilon \restriction \kappa \,.\supset .\, R \in \epsilon_\Delta`\kappa \qquad (2)$$

$$\vdash .\, (1) \,.\, {*}80{\cdot}14 \,.\supset \vdash :. \mathrm{Hp} \,.\supset : R \in \epsilon_\Delta`\kappa \,.\supset .\, \text{Ɑ}`R = \text{Ɑ}`\epsilon \restriction \kappa \qquad (3)$$

$$\vdash .\, (3) \,.\, {*}80{\cdot}291 \,.\, {*}84{\cdot}3 \,.\supset$$

$$\vdash :. \mathrm{Hp} \,.\supset : R \in \epsilon_\Delta`\kappa \,.\supset .\, R \in 1 \to 1 \,.\, R \mathrel{\dot\subset} \epsilon \restriction \kappa \,.\, \text{Ɑ}`R = \text{Ɑ}`\epsilon \restriction \kappa \qquad (4)$$

$$\vdash .\, (2) \,.\, (4) \,.\supset \vdash .\, \mathrm{Prop}$$

***84·37.** $\vdash : \kappa \in \mathrm{Cls}^2\,\mathrm{excl} \,.\, \exists \,!\, \epsilon_\Delta`\kappa \,.\supset .\, \kappa \in \mathrm{Cls\ ex}^2\,\mathrm{excl}$ [*83·1 . *84·13]

***84·4.** $\vdash :. \kappa \in \mathrm{Cls}^2\,\mathrm{excl} \,.\, R, S \in \epsilon_\Delta`\kappa \,.\supset : \mathrm{D}`R = \mathrm{D}`S \,.\equiv .\, R = S$ [*81·2 . *84·14]

***84·41.** $\vdash : \kappa \epsilon \mathrm{Cls}^2 \mathrm{excl} \,.\, \supset . \, \mathrm{D} \upharpoonright \epsilon_\Delta{}^\backprime\kappa \,\epsilon\, 1 \to 1 \,.\, \mathrm{D}^{\backprime\backprime}\epsilon_\Delta{}^\backprime\kappa \,\mathrm{sm}\, \epsilon_\Delta{}^\backprime\kappa$ [*81·21 . *84·14]

This is an important proposition, since it shows that, when κ is a Cls^2 excl, the number of classes that can be selected from κ is the product of the numbers of the various classes that are members of κ.

***84·411.** $\vdash :. \alpha \epsilon \kappa \,.\, \supset_\alpha . \, \mu \cap \alpha \epsilon 1 : \mu \subset s^\backprime\kappa : \supset . \, \mu \epsilon \mathrm{D}^{\backprime\backprime}\epsilon_\Delta{}^\backprime\kappa$ [*81·212 . *62·2·3]

***84·412.** $\vdash : \kappa \epsilon \mathrm{Cls}^2 \mathrm{excl} \,.\, \supset . \, \mathrm{D}^{\backprime\backprime}\epsilon_\Delta{}^\backprime\kappa = \hat{\mu}\{\alpha \epsilon \kappa \,.\, \supset_\alpha . \, \mu \cap \alpha \epsilon 1 : \mu \subset s^\backprime\kappa\}$
[*81·22 . *84·14 . *62·2·3]

This proposition gives what might be taken as the definition of the class of selected classes, namely

$$\hat{\mu}\{\alpha \epsilon \kappa \,.\, \supset_\alpha . \, \mu \cap \alpha \epsilon 1 : \mu \subset s^\backprime\kappa\}.$$

We might, starting with this as our definition, deal with the class of selected classes without first considering selective relations. The disadvantages of this method would be, first, that it requires that κ should be a Cls^2 excl if it is to give the results desired in arithmetic; secondly, that it is much more cumbrous technically than the method which proceeds by selective relations; thirdly, that it does not enable us to deal with selection from a class of classes as a particular case of selection from a relation (namely from $\epsilon \upharpoonright \kappa$), and therefore does not yield theorems of such generality as those obtained by the method adopted above.

***84·42.** $\vdash : \kappa \epsilon \mathrm{Cls}^2 \mathrm{excl} \,.\, \alpha \epsilon \kappa \,.\, \mu \epsilon \mathrm{D}^{\backprime\backprime}\epsilon_\Delta{}^\backprime\kappa \,.\, \supset . \, \mu - \alpha \,\epsilon\, \mathrm{D}^{\backprime\backprime}\epsilon_\Delta{}^\backprime(\kappa - \iota^\backprime\alpha)$
[*81·24 . *84·14 . *62·2]

***84·421.** $\vdash : \alpha \epsilon \kappa \,.\, x \epsilon \alpha \,.\, \mu \epsilon \mathrm{D}^{\backprime\backprime}\epsilon_\Delta{}^\backprime(\kappa - \iota^\backprime\alpha) \,.\, \supset . \, \mu \cup \iota^\backprime x \,\epsilon\, \mathrm{D}^{\backprime\backprime}\epsilon_\Delta{}^\backprime\kappa$ [*81·25]

***84·422.** $\vdash :. \kappa \epsilon \mathrm{Cls}^2 \mathrm{excl} \,.\, \alpha \epsilon \kappa \,.\, \mu \cap \alpha \epsilon 1 \,.\, \supset : \mu - \alpha \,\epsilon\, \mathrm{D}^{\backprime\backprime}\epsilon_\Delta{}^\backprime(\kappa - \iota^\backprime\alpha) \,.\, \equiv . \, \mu \epsilon \mathrm{D}^{\backprime\backprime}\epsilon_\Delta{}^\backprime\kappa$
[*81·26 . *84·14 . *62·2]

***84·43.** $\vdash :. \alpha, \beta \epsilon \mathrm{Cls}^2 \mathrm{excl} \,.\, s^\backprime\alpha = s^\backprime\beta \,.\, \supset : \alpha \subset \mathrm{D}^{\backprime\backprime}\epsilon_\Delta{}^\backprime\beta \,.\, \equiv . \, \beta \subset \mathrm{D}^{\backprime\backprime}\epsilon_\Delta{}^\backprime\alpha$

Dem.

$\vdash . *84·412 . \supset \vdash ::. \mathrm{Hp} . \supset ::$
$\alpha \subset \mathrm{D}^{\backprime\backprime}\epsilon_\Delta{}^\backprime\beta \,.\, \equiv :. \xi \epsilon \alpha \,.\, \supset_\xi : \eta \epsilon \beta \,.\, \supset_\eta . \, \xi \cap \eta \epsilon 1 : \xi \subset s^\backprime\beta :.$
$[*40·13.\mathrm{Hp}] \equiv :. \xi \epsilon \alpha \,.\, \supset_\xi : \eta \epsilon \beta \,.\, \supset_\eta . \, \xi \cap \eta \epsilon 1 :.$
$[*10·542·21] \equiv :. \eta \epsilon \beta \,.\, \supset_\eta : \xi \epsilon \alpha \,.\, \supset_\xi . \, \xi \cap \eta \epsilon 1 :.$
$[*40·13.\mathrm{Hp}] \equiv :. \eta \epsilon \beta \,.\, \supset_\eta : \xi \epsilon \alpha \,.\, \supset_\xi . \, \xi \cap \eta \epsilon 1 : \eta \subset s^\backprime\alpha :.$
$[*84·412] \quad \equiv :. \eta \epsilon \beta \,.\, \supset_\eta : \eta \epsilon \mathrm{D}^{\backprime\backprime}\epsilon_\Delta{}^\backprime\alpha ::. \supset \vdash . \mathrm{Prop}$

***84·5.** $\vdash : R \epsilon \mathrm{Cls} \to 1 \,.\, \supset . \, \overrightarrow{R}^{\backprime\backprime}\mathrm{Q}^\backprime R \,\epsilon\, \mathrm{Cls\,ex}^2\,\mathrm{excl}$

Dem.

$\vdash . *71·181 . \quad \supset \vdash :. \mathrm{Hp} . \supset : \exists ! \overrightarrow{R}^\backprime x \cap \overrightarrow{R}^\backprime y \,.\, \supset_{x,y} . \, x = y \,.$
$[*30·37] \quad \supset_{x,y} . \, \overrightarrow{R}^\backprime x = \overrightarrow{R}^\backprime y \quad (1)$
$\vdash . *33·41 . *11·59 . \supset \vdash : x, y \epsilon \mathrm{Q}^\backprime R \,.\, \supset_{x,y} . \, \exists ! \overrightarrow{R}^\backprime x \,.\, \exists ! \overrightarrow{R}^\backprime y \quad (2)$

$\vdash . (1) . (2) . \supset \vdash :: \text{Hp} . \supset :. x, y \in \text{Ɑ}`R . \supset_{x,y} :$

$\exists ! \overrightarrow{R}`x . \exists ! \overrightarrow{R}`y : \exists ! \overrightarrow{R}`x \cap \overrightarrow{R}`y . \supset . \overrightarrow{R}`x = \overrightarrow{R}`y :.$

[*37·63] $\supset :. \alpha, \beta \in \overrightarrow{R}``\text{Ɑ}`R . \supset_{\alpha,\beta} : \exists ! \alpha . \exists ! \beta : \exists ! \alpha \cap \beta . \supset . \alpha = \beta :.$

[*84·134] $\supset :. \overrightarrow{R}``\text{Ɑ}`R \in \text{Cls ex}^2 \text{ excl} :: \supset \vdash . \text{Prop}$

It might be supposed that the converse of the above would also hold. But this is not the case; for although $\overrightarrow{R}``\text{Ɑ}`R \in \text{Cls ex}^2 \text{ excl}$ secures that $\overrightarrow{R}`x$ and $\overrightarrow{R}`y$ cannot overlap when they are unequal, yet we may have $\overrightarrow{R}`x = \overrightarrow{R}`y$ without having $x = y$, so that if $\overrightarrow{R}`x = \alpha = \overrightarrow{R}`y$, we shall have $z \in \alpha . \supset . zRx . zRy$, whence, if $\exists ! \alpha . x \neq y$, it follows that R is not a $\text{Cls} \to 1$ even if $\overrightarrow{R}``\text{Ɑ}`R \in \text{Cls ex}^2 \text{ excl}$.

***84·51.** $\vdash : R \upharpoonright \kappa \in \text{Cls} \to 1 . \supset . \overrightarrow{R}``\kappa \in \text{Cls}^2 \text{ excl}$

Dem.

$\vdash . *71·171 . *35·101 . \supset$

$\vdash :. \text{Hp} . \supset : xRy . y \in \kappa . xRz . z \in \kappa . \supset_{x,y,z} . y = z .$

[*30·37] $\supset_{x,y,z} . \overrightarrow{R}`y = \overrightarrow{R}`z :$

[*32·18] $\supset : y, z \in \kappa . x \in \overrightarrow{R}`y \cap \overrightarrow{R}`z . \supset_{x,y,z} . \overrightarrow{R}`y = \overrightarrow{R}`z :$

[*10·23] $\supset : y, z \in \kappa . \exists ! \overrightarrow{R}`y \cap \overrightarrow{R}`z . \supset_{y,z} . \overrightarrow{R}`y = \overrightarrow{R}`z :$

[*37·63] $\supset : \alpha, \beta \in \overrightarrow{R}``\kappa . \exists ! \alpha \cap \beta . \supset_{\alpha,\beta} . \alpha = \beta :$

[*84·11] $\supset : \overrightarrow{R}``\kappa \in \text{Cls}^2 \text{ excl} :. \supset \vdash . \text{Prop}$

***84·52.** $\vdash : R \upharpoonright \kappa \in \text{Cls} \to 1 . \kappa \subset \text{Ɑ}`R . \supset . \overrightarrow{R}``\kappa \in \text{Cls ex}^2 \text{ excl}$

Dem.

$\vdash . *37·2 . \supset \vdash :. \text{Hp} . \supset : \alpha \in \overrightarrow{R}``\kappa . \supset . \alpha \in \overrightarrow{R}``\text{Ɑ}`R .$

[*37·77] $\supset . \exists ! \alpha$ (1)

$\vdash . (1) . *84·51·13 . *24·63 . \supset \vdash . \text{Prop}$

***84·521.** $\vdash : \overrightarrow{R} \upharpoonright \beta \in 1 \to 1 . \overrightarrow{R}``\beta \in \text{Cls}^2 \text{ excl} . \supset . R \upharpoonright \beta \in \text{Cls} \to 1$

Dem.

$\vdash . *71·55 . *84·11 . \supset$

$\vdash :. \overrightarrow{R} \upharpoonright \beta \in 1 \to 1 . \overrightarrow{R}``\beta \in \text{Cls}^2 \text{ excl} . \supset : y, z \in \beta . \overrightarrow{R}`y = \overrightarrow{R}`z . \supset_{y,z} . y = z :$

$y, z \in \beta . \exists ! \overrightarrow{R}`y \cap \overrightarrow{R}`z . \supset_{y,z} . \overrightarrow{R}`y = \overrightarrow{R}`z :$

[*11·37] $\supset : y, z \in \beta . \exists ! \overrightarrow{R}`y \cap \overrightarrow{R}`z . \supset_{y,z} . y = z :$

[*74·62.Transp] $\supset : R \upharpoonright \beta \in \text{Cls} \to 1 :. \supset \vdash . \text{Prop}$

The above proposition is a lemma for *84·522, which is used in an important proposition on relations of mutually exclusive relations (*163·17).

***84·522.** $\vdash :. \beta \subset \text{Q}^{\backprime}R . \supset : R \upharpoonright \beta \in \text{Cls} \to 1 . \equiv . \overrightarrow{R} \upharpoonright \beta \in 1 \to 1 . \overrightarrow{R}``\beta \in \text{Cls}^2 \text{excl}$

Dem.

$\vdash . *33\cdot31 . \supset \vdash :. \text{Hp} . \supset : y, z \in \beta . \supset . \exists ! \overrightarrow{R}`y . \exists ! \overrightarrow{R}`z :$

$[*22\cdot5] \qquad \supset : y, z \in \beta . \overrightarrow{R}`y = \overrightarrow{R}`z . \supset . \exists ! \overrightarrow{R}`y \cap \overrightarrow{R}`z :$

$[*74\cdot62] \qquad \supset : R \upharpoonright \beta \in \text{Cls} \to 1 . y, z \in \beta . \overrightarrow{R}`y = \overrightarrow{R}`z . \supset . y = z :$

$[*71\cdot55] \qquad \supset : R \upharpoonright \beta \in \text{Cls} \to 1 . \supset . \overrightarrow{R} \upharpoonright \beta \in 1 \to 1 \qquad (1)$

$\vdash . (1) . *84\cdot51 . \supset$

$\vdash :. \text{Hp} . \supset : R \upharpoonright \beta \in \text{Cls} \to 1 . \supset . \overrightarrow{R} \upharpoonright \beta \in 1 \to 1 . \overrightarrow{R}``\beta \in \text{Cls}^2 \text{excl} \qquad (2)$

$\vdash . (2) . *84\cdot521 . \supset \vdash . \text{Prop}$

***84·53.** $\vdash : R \in \text{Cls} \to 1 . \kappa \in \text{Cls}^2 \text{excl} . \supset . R```\kappa \in \text{Cls}^2 \text{excl}$

Dem.

$\vdash . *72\cdot421 . \supset$

$\vdash : R \in \text{Cls} \to 1 . \alpha, \beta \in \kappa . \exists ! R``\alpha \cap R``\beta . \supset . \exists ! \alpha \cap \beta \qquad (1)$

$\vdash . (1) . \text{Syll} . \supset \vdash :. R \in \text{Cls} \to 1 : \alpha, \beta \in \kappa . \exists ! \alpha \cap \beta . \supset_{\alpha, \beta} . \alpha = \beta : \supset :$

$\qquad \alpha, \beta \in \kappa . \exists ! R``\alpha \cap R``\beta . \supset_{\alpha, \beta} . \alpha = \beta .$

$[*30\cdot37 . *37\cdot11\cdot111] \qquad \supset_{\alpha, \beta} . R``\alpha = R``\beta :$

$[*37\cdot63 . (*37\cdot04)] \supset : \rho, \sigma \in R```\kappa . \exists ! \rho \cap \sigma . \supset_{\rho, \sigma} . \rho = \sigma \qquad (2)$

$\vdash . (2) . *84\cdot11 . \supset \vdash . \text{Prop}$

***84·54.** $\vdash : R \in 1 \to \text{Cls} . \kappa \in \text{Cls}^2 \text{excl} . \supset . \breve{R}```\kappa \in \text{Cls}^2 \text{excl} \qquad \left[*84\cdot53 \dfrac{\breve{R}}{R}\right]$

***84·55.** $\vdash . P_\Delta``\kappa \in \text{Cls}^2 \text{excl} \qquad [*80\cdot82]$

***84·59.** $\vdash : \kappa \cup \lambda \in \text{Cls}^2 \text{excl} . \equiv . \kappa, \lambda \in \text{Cls}^2 \text{excl} . s`(\kappa - \lambda) \cap s`\lambda = \Lambda$

Dem.

$\vdash . *84\cdot14 . \supset \vdash : \kappa \cup \lambda \in \text{Cls}^2 \text{excl} . \equiv . \epsilon \upharpoonright (\kappa \cup \lambda) \in \text{Cls} \to 1 .$

$[*74\cdot821] \qquad \equiv . \epsilon \upharpoonright \kappa, \epsilon \upharpoonright \lambda \in \text{Cls} \to 1 . \epsilon``(\kappa - \lambda) \cap \epsilon``\lambda = \Lambda .$

$[*84\cdot14 . *62\cdot3] \qquad \equiv . \kappa, \lambda \in \text{Cls}^2 \text{excl} . s`(\kappa - \lambda) \cap s`\lambda = \Lambda$

***84·6.** $\vdash :. \kappa \cap \lambda = \Lambda . \supset : \kappa \cup \lambda \in \text{Cls}^2 \text{excl} . \equiv . \kappa, \lambda \in \text{Cls}^2 \text{excl} . s`\kappa \cap s`\lambda = \Lambda$
$[*84\cdot59 . *24\cdot313]$

***84·61.** $\vdash :. \beta \sim\in \kappa . \supset : \kappa \cup \iota`\beta \in \text{Cls}^2 \text{excl} . \equiv . \kappa \in \text{Cls}^2 \text{excl} . \beta \cap s`\kappa = \Lambda$
$[*51\cdot211 . *53\cdot02 . *84\cdot23\cdot6]$

***84·62.** $\vdash :. \alpha \neq \beta . \supset : \iota`\alpha \cup \iota`\beta \in \text{Cls}^2 \text{excl} . \equiv . \alpha \cap \beta = \Lambda$
$[*84\cdot61 . *51\cdot15 . *53\cdot02 . *84\cdot23]$

$*85$. MISCELLANEOUS PROPOSITIONS

Summary of $*85$.

In this number certain important propositions are proved, and the other propositions of this number are mainly lemmas. The most important propositions are the following:

$*85\cdot1$ and $*85\cdot14$, which show that if $Q \upharpoonright \lambda$ is a $\mathrm{Cls} \to 1$, then the domains of $Q_\Delta{}'\lambda$ are the same as the domains of $\epsilon_\Delta{}'\overrightarrow{Q}{}''\lambda$, and $Q_\Delta{}'\lambda$ is similar to $\epsilon_\Delta{}'\overrightarrow{Q}{}''\lambda$, thus reducing the problem of selections from many-one relations to that of selections from classes of classes.

$*85\cdot27$ and $*85\cdot43$, which show that if $\kappa \in \mathrm{Cls}^2\,\mathrm{excl}$, $P_\Delta{}'s'\kappa$ consists of the relational sums of the domains of $\epsilon_\Delta{}'P_\Delta{}''\kappa$ and is similar to $\epsilon_\Delta{}'P_\Delta{}''\kappa$; *i.e.* the class of P-selections from $s'\kappa$ is similar to the class obtained as follows: take the members of κ one by one, and form the P-selections of each; we thus obtain a class of classes, each class being of the form $P_\Delta{}'\alpha$, where $\alpha \in \kappa$; we then make a selection from this class of classes; this selection is a member of $\epsilon_\Delta{}'P_\Delta{}''\kappa$; the number of such selections is the same as the number of $P_\Delta{}'s'\kappa$.

$*85\cdot28$ and $*85\cdot44$, which are special cases of $*85\cdot27$ and $*85\cdot43$, but more useful than these. $*85\cdot44$ is the source of the associative law in cardinal multiplication; it states that, if κ is a $\mathrm{Cls}^2\,\mathrm{excl}$, $\epsilon_\Delta{}'s'\kappa$ has the same number of members as $\epsilon_\Delta{}'\epsilon_\Delta{}''\kappa$. (On associative laws in general, see the notes to $*42\cdot1\cdot11$.) That is to say, if we form the class of selective relations ($\epsilon_\Delta{}'\alpha$) for every α which is a member of κ, and then form the class of selective relations for $\epsilon_\Delta{}''\kappa$, we get the same number of terms as if we proceeded to form the class of selective relations for $\epsilon_\Delta{}'s'\kappa$. The way in which this proposition yields the associative law of multiplication may be explained as follows. We shall define the product of the numbers of members of α as the number of $\epsilon_\Delta{}'\alpha$. Thus *e.g.* if the numbers of the members of α are $\mu_{\alpha1}$, $\mu_{\alpha2}$, $\mu_{\alpha3}$, the number of $\epsilon_\Delta{}'\alpha$ is $\mu_{\alpha1} \times \mu_{\alpha2} \times \mu_{\alpha3}$. Suppose the other members of κ are β and γ, and that β and γ again have three members each. Then the number of $\epsilon_\Delta{}'\epsilon_\Delta{}''\kappa$ is the product of the numbers of $\epsilon_\Delta{}'\alpha$, $\epsilon_\Delta{}'\beta$, $\epsilon_\Delta{}'\gamma$, *i.e.* it is the product of $\mu_{\alpha1} \times \mu_{\alpha2} \times \mu_{\alpha3}$, $\mu_{\beta1} \times \mu_{\beta2} \times \mu_{\beta3}$ and $\mu_{\gamma1} \times \mu_{\gamma2} \times \mu_{\gamma3}$.

But the numbers of the members of $s'\kappa$ are

$$\mu_{\alpha1},\ \mu_{\alpha2},\ \mu_{\alpha3},\ \mu_{\beta1},\ \mu_{\beta2},\ \mu_{\beta3},\ \mu_{\gamma1},\ \mu_{\gamma2},\ \mu_{\gamma3}.$$

Thus the number of $\epsilon_\Delta{}'s'\kappa$ is

$$\mu_{\alpha1} \times \mu_{\alpha2} \times \mu_{\alpha3} \times \mu_{\beta1} \times \mu_{\beta2} \times \mu_{\beta3} \times \mu_{\gamma1} \times \mu_{\gamma2} \times \mu_{\gamma3}.$$

Hence $*85{\cdot}44$ enables us to conclude that

$$(\mu_{\alpha 1} \times \mu_{\alpha 2} \times \mu_{\alpha 3}) \times (\mu_{\beta 1} \times \mu_{\beta 2} \times \mu_{\beta 3}) \times (\mu_{\gamma 1} \times \mu_{\gamma 2} \times \mu_{\gamma 3})$$
$$= \mu_{\alpha 1} \times \mu_{\alpha 2} \times \mu_{\alpha 3} \times \mu_{\beta 1} \times \mu_{\beta 2} \times \mu_{\beta 3} \times \mu_{\gamma 1} \times \mu_{\gamma 2} \times \mu_{\gamma 3},$$

which is a case of the associative law. In fact $*85{\cdot}44$ gives us this law in its general form, when the number of brackets, and of factors in each bracket, may be infinite or finite indifferently.

Another important pair of propositions is $*85{\cdot}53{\cdot}54$. These enable us to reduce the problem of selections for *any* relation to the problem of selections from a class of classes. The method is as follows: Given any term x, form the class of ordered couples of which x is relatum while the referent is a term having the relation P to x. Call this class of couples $P \downarrow x$. Form this class for every x which is a member of α; we thus obtain a class of classes, namely $P \downarrow$"α. Then the number of selections from this class of classes is the same as the number of P_Δ'α.

We have one other important pair of propositions in this number, namely $*85{\cdot}61{\cdot}63$. These show that what is called "Zermelo's axiom" is equivalent to what is called the "multiplicative axiom." Zermelo's axiom* is to the effect that if α is any class, ϵ_Δ'Cl ex'α is never null, *i.e.* $(\alpha) . \exists ! \epsilon_\Delta$'Cl ex'$\alpha$. The "multiplicative axiom" is to the effect that if $\kappa \in$ Cls ex² excl, there is at least one class formed by taking one representative from each member of κ, which is equivalent to

$$\kappa \in \text{Cls ex}^2 \text{ excl} . \supset_\kappa . \exists ! \epsilon_\Delta\text{'}\kappa.$$

In $*85{\cdot}63$, these two axioms are shown to be equivalent. From Zermelo's theorem† it follows that both are equivalent to the assumption that every class can be well-ordered. This will be proved later ($*258$).

The above-mentioned propositions, stated symbolically, are as follows:

$*85{\cdot}1.$ $\vdash : Q \upharpoonright \lambda \in \text{Cls} \to 1 . \supset . \text{D``}Q_\Delta\text{'}\lambda = \text{D``}\epsilon_\Delta\text{'}\overrightarrow{Q}\text{``}\lambda$

$*85{\cdot}14.$ $\vdash : Q \upharpoonright \lambda \in \text{Cls} \to 1 . \supset . Q_\Delta\text{'}\lambda \text{ sm } \epsilon_\Delta\text{'}\overrightarrow{Q}\text{``}\lambda$

$*85{\cdot}27.$ $\vdash : \kappa \in \text{Cls}^2 \text{ excl} . \supset . P_\Delta\text{'}s\text{'}\kappa = \dot{s}\text{``D``}\epsilon_\Delta\text{'}P_\Delta\text{``}\kappa$

$*85{\cdot}28.$ $\vdash : \kappa \in \text{Cls}^2 \text{ excl} . \supset . \epsilon_\Delta\text{'}s\text{'}\kappa = \dot{s}\text{``D``}\epsilon_\Delta\text{'}\epsilon_\Delta\text{``}\kappa$

$*85{\cdot}43.$ $\vdash : \kappa \in \text{Cls}^2 \text{ excl} . \supset . P_\Delta\text{'}s\text{'}\kappa \text{ sm } \epsilon_\Delta\text{'}P_\Delta\text{``}\kappa$

$*85{\cdot}44.$ $\vdash : \kappa \in \text{Cls}^2 \text{ excl} . \supset . \epsilon_\Delta\text{'}s\text{'}\kappa \text{ sm } \epsilon_\Delta\text{'}\epsilon_\Delta\text{``}\kappa$

The following propositions depend upon the definition

$*85{\cdot}5.$ $P \downarrow y = \downarrow y\text{``}\overrightarrow{P}\text{'}y$ Df

I.e. $P \downarrow y$ is the class of all couples whose relatum is y while the referent has the relation P to y. We then have

$*85{\cdot}53.$ $\vdash . P_\Delta\text{'}\alpha = \dot{s}\text{``D``}\epsilon_\Delta\text{'}P \downarrow\text{``}\alpha$

giving a construction for P_Δ'α by means of ϵ_Δ, and

* See *Math. Annalen*, Vol. LIX. † *loc. cit.*

***85·54.** $\vdash . P_\Delta\text{‘}\alpha \text{ sm } \epsilon_\Delta\text{‘}P \downarrow\text{“}\alpha$

which reduces the question of the existence of P-selections to that of the existence of ϵ-selections.

***85·61.** $\vdash . \epsilon \downarrow\text{“}\kappa \in \text{Cls}^2 \text{ excl} . \epsilon_\Delta\text{‘}\kappa = \dot{s}\text{“D“}\epsilon_\Delta\text{‘}\epsilon \downarrow\text{“}\kappa . \epsilon_\Delta\text{‘}\kappa \text{ sm } \epsilon_\Delta\text{‘}\epsilon \downarrow\text{“}\kappa$

This proposition gives a construction for any ϵ-selection in terms of an ϵ-selection from a Cls^2 excl, and reduces the question of the existence of the former to that of the existence of the latter. A particularly important case is when $\kappa = \text{Cl ex‘}\alpha$. This is considered in

***85·63.** $\vdash : \epsilon \downarrow\text{“Cl ex‘}\alpha \in \text{Cls ex}^2 \text{ excl} : \exists ! \epsilon_\Delta\text{‘Cl ex‘}\alpha . \equiv . \exists ! \epsilon_\Delta\text{‘}\epsilon \downarrow\text{“Cl ex‘}\alpha$

***85·1.** $\vdash : Q \upharpoonright \lambda \in \text{Cls} \to 1 . \supset . \text{D“}Q_\Delta\text{‘}\lambda = \text{D“}\epsilon_\Delta\text{‘}\overrightarrow{Q}\text{“}\lambda$

Dem.

$\vdash . *81\cdot3 . \supset \vdash : \text{Hp} . \supset . \text{D“}Q_\Delta\text{‘}\lambda = \hat{\mu}\{\alpha \in \overrightarrow{Q}\text{“}\lambda . \supset_\alpha . \mu \cap \alpha \in 1 : \mu \subset s\text{‘}\overrightarrow{Q}\text{“}\lambda\}$ (1)

$\vdash . *84\cdot51 . \supset \vdash : \text{Hp} . \supset . \overrightarrow{Q}\text{“}\lambda \in \text{Cls}^2 \text{ excl} .$

$[*84\cdot412] \quad \supset . \text{D“}\epsilon_\Delta\text{‘}\overrightarrow{Q}\text{“}\lambda = \hat{\mu}\{\alpha \in \overrightarrow{Q}\text{“}\lambda . \supset_\alpha . \mu \cap \alpha \in 1 : \mu \subset s\text{‘}\overrightarrow{Q}\text{“}\lambda\}$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***85·11.** $\vdash : \overrightarrow{Q} \upharpoonright \lambda \in 1 \to 1 . \supset . \text{D“}(P \mid \overrightarrow{Q})_\Delta\text{‘}\lambda = \text{D“}P_\Delta\text{‘}\overrightarrow{Q}\text{“}\lambda$

Dem.

$\vdash . *33\cdot431 . *32\cdot12 . \supset \vdash : \text{Hp} . \supset . \lambda \subset \text{Ꮯ‘}\overrightarrow{Q}$ (1)

$\vdash . (1) . *82\cdot32 . \quad \supset \vdash : \text{Hp} . \supset . \text{D“}(P \mid \overrightarrow{Q})_\Delta\text{‘}\lambda = \text{D“}P_\Delta\text{‘}\overrightarrow{Q}\text{“}\lambda : \supset \vdash . \text{Prop}$

***85·111.** $\vdash : M \in \epsilon_\Delta\text{‘}\overrightarrow{Q}\text{“}\lambda . \supset . \text{D‘}(M \mid \overrightarrow{Q} \upharpoonright \lambda) = \text{D‘}M$ [*82·3]

***85·112.** $\vdash : M \in \epsilon_\Delta\text{‘}\overrightarrow{Q}\text{“}\lambda . \supset . M \mid \overrightarrow{Q} \upharpoonright \lambda \in Q_\Delta\text{‘}\lambda$ $\left[*82\cdot221 \frac{\epsilon, \overrightarrow{Q}}{P, Q} . *62\cdot26\right]$

***85·12.** $\vdash : \overrightarrow{Q} \upharpoonright \lambda \in 1 \to 1 . \supset . \text{D“}Q_\Delta\text{‘}\lambda = \text{D“}\epsilon_\Delta\text{‘}\overrightarrow{Q}\text{“}\lambda$

Dem.

$\vdash . *62\cdot26 . \supset \vdash . \text{D“}Q_\Delta\text{‘}\lambda = \text{D“}(\epsilon \mid \overrightarrow{Q})_\Delta\text{‘}\lambda$ (1)

$\vdash . *82\cdot32 . \supset \vdash : \text{Hp} . \supset . \text{D“}(\epsilon \mid \overrightarrow{Q})_\Delta\text{‘}\lambda = \text{D“}\epsilon_\Delta\text{‘}\overrightarrow{Q}\text{“}\lambda$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

This proposition is used in connection with ordinal multiplication (*173·14).

***85·13.** $\vdash : \overrightarrow{Q} \upharpoonright \lambda \in 1 \to 1 . R \in Q_\Delta\text{‘}\lambda . \supset . R \mid \text{Cnv‘}\overrightarrow{Q} \in \epsilon_\Delta\text{‘}\overrightarrow{Q}\text{“}\lambda$

Dem.

$\vdash . *62\cdot26 . \supset \vdash : \text{Hp} . \supset . \overrightarrow{Q} \upharpoonright \lambda \in 1 \to 1 . R \in (\epsilon \mid \overrightarrow{Q})_\Delta\text{‘}\lambda .$

$[*82\cdot231] \quad \supset . R \mid \text{Cnv‘}\overrightarrow{Q} \in \epsilon_\Delta\text{‘}\overrightarrow{Q}\text{“}\lambda : \supset \vdash . \text{Prop}$

The above proposition is used in connection with "families" (*97·31).

***85·14.** $\vdash : Q \upharpoonright \lambda \in \text{Cls} \to 1 \,.\, \supset .\, Q_\Delta ‘\lambda \,\text{sm}\, \epsilon_\Delta ‘\overrightarrow{Q}‘‘\lambda$

Dem.

$$\vdash . *81{\cdot}21 . \supset \vdash : \text{Hp} . \supset . Q_\Delta ‘\lambda \,\text{sm}\, \text{D}‘‘Q_\Delta ‘\lambda .$$
$$[*85{\cdot}1] \qquad \supset . Q_\Delta ‘\lambda \,\text{sm}\, \text{D}‘‘\epsilon_\Delta ‘\overrightarrow{Q}‘‘\lambda \qquad (1)$$
$$\vdash . *84{\cdot}51 . \supset \vdash : \text{Hp} . \supset . \overrightarrow{Q}‘‘\lambda \in \text{Cls}^2 \,\text{excl} .$$
$$[*84{\cdot}41] \qquad \supset . \text{D}‘‘\epsilon_\Delta ‘\overrightarrow{Q}‘‘\lambda \,\text{sm}\, \epsilon_\Delta ‘\overrightarrow{Q}‘‘\lambda \qquad (2)$$
$$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$$

*85·21·22 are lemmas for *85·24, which, with *85·26, is required for *85·27.

***85·21.** $\vdash : \alpha \in \kappa \,.\, M \in P_\Delta ‘s‘\kappa \,.\, \supset .\, M \upharpoonright \alpha \in P_\Delta ‘\alpha \quad [*80{\cdot}6 . *40{\cdot}13]$

***85·22.** $\vdash : M \in P_\Delta ‘s‘\kappa \,.\, \supset .\, M \upharpoonright | \kappa \rceil \breve{P}_\Delta \in \epsilon_\Delta ‘P_\Delta ‘‘\kappa \,.\, \dot{s}‘\text{D}‘(M \upharpoonright | \kappa \rceil \breve{P}_\Delta) = M$

Here $M \upharpoonright | \kappa \rceil \breve{P}_\Delta \in \epsilon_\Delta ‘P_\Delta ‘‘\kappa$ can also be written $\{(M \upharpoonright) | (\kappa \rceil \breve{P}_\Delta)\} \in (\epsilon_\Delta ‘P_\Delta ‘‘\kappa)$. The brackets are omitted because no other meaning is possible.

Dem.

$$\vdash . *85{\cdot}21 . \qquad \supset \vdash :. \text{Hp} . \supset : \alpha \in \kappa . \supset_\alpha . \exists ! P_\Delta ‘\alpha :$$
$$[*80{\cdot}81] \qquad \supset : \alpha, \beta \in \kappa . P_\Delta ‘\alpha \,\dot{=}\, P_\Delta ‘\beta . \supset_{\alpha,\beta} . \alpha = \beta :$$
$$[*80{\cdot}12 . *71{\cdot}166{\cdot}55] \qquad \supset : P_\Delta \upharpoonright \kappa \in 1 \to 1 :$$
$$[*35{\cdot}52] \qquad \supset : \kappa \rceil \breve{P}_\Delta \in 1 \to 1 \qquad (1)$$
$$\vdash . (1) . *72{\cdot}14 . *71{\cdot}25 . \supset \vdash : \text{Hp} . \supset . M \upharpoonright | \kappa \rceil \breve{P}_\Delta \in 1 \to \text{Cls} \qquad (2)$$
$$\vdash . *34{\cdot}1 . *30{\cdot}4 . \qquad \supset \vdash : R \{M \upharpoonright | \kappa \rceil \breve{P}_\Delta\} \lambda . \equiv .$$
$$(\exists \alpha) . R = M \upharpoonright \alpha . \alpha \in \kappa . \lambda = P_\Delta ‘\alpha \qquad (3)$$
$$\vdash . (3) . *85{\cdot}21 . \qquad \supset \vdash : \text{Hp} . \supset . M \upharpoonright | \kappa \rceil \breve{P}_\Delta \subset \epsilon \qquad (4)$$
$$\vdash . *37{\cdot}322 . *33{\cdot}431 . \qquad \supset \vdash . \text{C\!\!I}‘\{M \upharpoonright | \kappa \rceil \breve{P}_\Delta\} = \text{C\!\!I}‘(\kappa \rceil \breve{P}_\Delta)$$
$$[*37{\cdot}4] \qquad = P_\Delta ‘‘\kappa \qquad (5)$$
$$\vdash . (2) . (4) . (5) . *80{\cdot}14 . \supset \vdash : \text{Hp} . \supset . \{M \upharpoonright | \kappa \rceil \breve{P}_\Delta\} \in \epsilon_\Delta ‘P_\Delta ‘‘\kappa \qquad (6)$$
$$\vdash . *37{\cdot}32 . *35{\cdot}62 . \qquad \supset \vdash . \text{D}‘(M \upharpoonright | \kappa \rceil \breve{P}_\Delta) = M \upharpoonright ‘‘\kappa .$$
$$[*41{\cdot}35] \qquad \supset \vdash . \dot{s}‘\text{D}‘(M \upharpoonright | \kappa \rceil \breve{P}_\Delta) = M \upharpoonright s‘\kappa \qquad (7)$$
$$\vdash . (7) . *80{\cdot}29 . \qquad \supset \vdash : \text{Hp} . \supset . \dot{s}‘\text{D}‘(M \upharpoonright | \kappa \rceil \breve{P}_\Delta) = M \qquad (8)$$
$$\vdash . (6) . (8) . \supset \vdash . \text{Prop}$$

***85·24.** $\vdash . P_\Delta ‘s‘\kappa \subset \dot{s}‘‘\text{D}‘‘\epsilon_\Delta ‘P_\Delta ‘‘\kappa$

Dem.

$$\vdash . *85{\cdot}22 . \supset$$
$$\vdash : M \in P_\Delta ‘s‘\kappa . \supset . (\exists X) . X \in \epsilon_\Delta ‘P_\Delta ‘‘\kappa . M = \dot{s}‘\text{D}‘X .$$
$$[*37{\cdot}67] \qquad \supset . M \in \dot{s}‘‘\text{D}‘‘\epsilon_\Delta ‘P_\Delta ‘‘\kappa : \supset \vdash . \text{Prop}$$

The following propositions are lemmas for ∗85·26.

∗85·241. $\vdash : X \in \epsilon_\Delta‘P_\Delta‘‘\kappa \,.\, \alpha \in \kappa \,.\, \supset .\, X‘P_\Delta‘\alpha \in P_\Delta‘\alpha$

Dem.

$\vdash .\, *83{\cdot}2 \,.\, \supset \vdash :.\, X \in \epsilon_\Delta‘P_\Delta‘‘\kappa \,.\, \supset : \lambda \in P_\Delta‘‘\kappa \,.\, \supset_\lambda .\, X‘\lambda \in \lambda :$

$[*37{\cdot}63] \qquad \supset : \alpha \in \kappa \,.\, \supset_\alpha .\, X‘P_\Delta‘\alpha \in P_\Delta‘\alpha :. \supset \vdash .\, \mathrm{Prop}$

∗85·243. $\vdash : \kappa \in \mathrm{Cls}^2\,\mathrm{excl} \,.\, X \in \epsilon_\Delta‘P_\Delta‘‘\kappa \,.\, \supset .\, \dot{s}‘\mathrm{D}‘X \in 1 \rightarrow \mathrm{Cls}$

Dem.

$\vdash .\, *83{\cdot}21 \,. \qquad \supset \vdash : \mathrm{Hp} \,.\, \supset .\, \mathrm{D}‘X \subset s‘P_\Delta‘‘\kappa \qquad (1)$

$\vdash .\, *40{\cdot}151 \,.\, *80{\cdot}11 \,. \qquad \supset \vdash .\, s‘P_\Delta‘‘\kappa \subset 1 \rightarrow \mathrm{Cls} \qquad (2)$

$\vdash .\, (1) \,.\, (2) \,. \qquad \supset \vdash : \mathrm{Hp} \,.\, \supset .\, \mathrm{D}‘X \subset 1 \rightarrow \mathrm{Cls} \qquad (3)$

$\vdash .\, *80{\cdot}35 \,.\, *11{\cdot}45{\cdot}55 \,.\, \supset \vdash :.\, \mathrm{Hp} \,.\, \supset : M, N \in \mathrm{D}‘X \,.\, \exists !\, \text{Ɑ}‘M \cap \text{Ɑ}‘N \,.\, \supset .$

$\qquad (\exists \alpha, \beta) \,.\, \alpha, \beta \in \kappa \,.\, M = X‘P_\Delta‘\alpha \,.\, N = X‘P_\Delta‘\beta \,.\, \exists !\, \text{Ɑ}‘M \cap \text{Ɑ}‘N \,.$

$[*85{\cdot}241 . *80{\cdot}14] \supset .\, (\exists \alpha, \beta) \,.\, \alpha, \beta \in \kappa \,.\, M = X‘P_\Delta‘\alpha \,.\, N = X‘P_\Delta‘\beta \,.$

$\qquad \exists !\, \text{Ɑ}‘M \cap \text{Ɑ}‘N \,.\, \alpha = \text{Ɑ}‘M \,.\, \beta = \text{Ɑ}‘N \,.$

$[*13{\cdot}22] \qquad \supset .\, \text{Ɑ}‘M, \text{Ɑ}‘N \in \kappa \,.\, M = X‘P_\Delta‘\text{Ɑ}‘M \,.\, N = X‘P_\Delta‘\text{Ɑ}‘N \,.$

$\qquad \exists !\, \text{Ɑ}‘M \cap \text{Ɑ}‘N \,.$

$[*84{\cdot}11] \qquad \supset .\, \text{Ɑ}‘M = \text{Ɑ}‘N \,.\, M = X‘P_\Delta‘\text{Ɑ}‘M \,.\, N = X‘P_\Delta‘\text{Ɑ}‘N \,.$

$[*30{\cdot}37] \qquad \supset .\, M = N \qquad (4)$

$\vdash .\, (3) \,.\, (4) \,.\, *72{\cdot}32 \,.\, \supset \vdash .\, \mathrm{Prop}$

∗85·244. $\vdash : X \in \epsilon_\Delta‘P_\Delta‘‘\kappa \,.\, \supset .\, \dot{s}‘\mathrm{D}‘X \mathrel{\dot\subset} P$

Dem.

$\vdash .\, *83{\cdot}21 \,.\, *40{\cdot}4 \,.\, \supset \vdash :.\, \mathrm{Hp} \,.\, \supset : R \in \mathrm{D}‘X \,.\, \supset_R .\, (\exists \alpha) \,.\, \alpha \in \kappa \,.\, R \in P_\Delta‘\alpha \,.$

$[*80{\cdot}14] \qquad \supset_R .\, R \mathrel{\dot\subset} P :$

$[*41{\cdot}151] \qquad \supset : \dot{s}‘\mathrm{D}‘X \mathrel{\dot\subset} P :. \supset \vdash .\, \mathrm{Prop}$

∗85·245. $\vdash : X \in \epsilon_\Delta‘P_\Delta‘‘\kappa \,.\, \supset .\, \text{Ɑ}‘\dot{s}‘\mathrm{D}‘X = s‘\kappa$

Dem.

$\vdash .\, *85{\cdot}241 \,.\, *80{\cdot}14 \,.\, \supset \vdash :.\, \mathrm{Hp} \,.\, \supset : \alpha \in \kappa \,.\, \supset_\alpha .\, \text{Ɑ}‘X‘P_\Delta‘\alpha = \alpha :$

$[*50{\cdot}17] \qquad \supset : \text{Ɑ}‘‘X‘‘P_\Delta‘‘\kappa = \kappa :$

$[*80{\cdot}34] \qquad \supset : \text{Ɑ}‘‘\mathrm{D}‘X = \kappa :$

$[*41{\cdot}44] \qquad \supset : \text{Ɑ}‘\dot{s}‘\mathrm{D}‘X = s‘\kappa :. \supset \vdash .\, \mathrm{Prop}$

∗85·25. $\vdash : \kappa \in \mathrm{Cls}^2\,\mathrm{excl} \,.\, X \in \epsilon_\Delta‘P_\Delta‘‘\kappa \,.\, \supset .\, \dot{s}‘\mathrm{D}‘X \in P_{-}‘s‘\kappa$

$[*85{\cdot}243{\cdot}244{\cdot}245 \,.\, *80{\cdot}14]$

∗85·26. $\vdash : \kappa \in \mathrm{Cls}^2\,\mathrm{excl} \,.\, \supset .\, \dot{s}‘‘\mathrm{D}‘‘\epsilon_\Delta‘P_\Delta‘‘\kappa \subset P_\Delta‘s‘\kappa$

Dem.

$\vdash .\, *85{\cdot}25 \,.\, \supset \vdash :.\, \mathrm{Hp} \,.\, \supset : X \in \epsilon_\Delta‘P_\Delta‘‘\kappa \,.\, \supset_X .\, \dot{s}‘\mathrm{D}‘X \in P_\Delta‘s‘\kappa :$

$[*37{\cdot}61{\cdot}33] \qquad \supset : \dot{s}‘‘\mathrm{D}‘‘\epsilon_\Delta‘P_\Delta‘‘\kappa \subset P_\Delta‘s‘\kappa :. \supset \vdash .\, \mathrm{Prop}$

∗85·27. $\vdash : \kappa \in \mathrm{Cls}^2\,\mathrm{excl} \,.\, \supset .\, P_\Delta‘s‘\kappa = \dot{s}‘‘\mathrm{D}‘‘\epsilon_\Delta‘P_\Delta‘‘\kappa \qquad [*85{\cdot}24{\cdot}26]$

∗85·28. $\vdash : \kappa \in \mathrm{Cls}^2\,\mathrm{excl} \,.\, \supset .\, \epsilon_\Delta‘s‘\kappa = \dot{s}‘‘\mathrm{D}‘‘\epsilon_\Delta‘\epsilon_\Delta‘‘\kappa \qquad \left[*85{\cdot}27 \dfrac{\epsilon}{P}\right]$

The following proposition is a lemma for *85·31.

***85·3.** $\vdash : M \in P_\Delta{}^\backprime\alpha \,.\, z \in \alpha \,.\, \supset . \, M^\backprime z \subset \dot{s}^\backprime \mathrm{D}^\backprime M \,.\, M^\backprime z \subset \dot{s}^\backprime \overrightarrow{P}{}^\backprime z$

The conditions of significance here and in *85·31·32·33·34 require $\mathrm{D}^\backprime P \subset \mathrm{Rel}$.

Dem.

$\vdash . *80{\cdot}32 . *33{\cdot}43 . \supset \vdash : \mathrm{Hp} . \supset . \, M^\backprime z \in \mathrm{D}^\backprime M \,.\, M^\backprime z \in \overrightarrow{P}{}^\backprime z \,.$

$[*41{\cdot}13] \qquad \supset . \, M^\backprime z \subset \dot{s}^\backprime \mathrm{D}^\backprime M \,.\, M^\backprime z \subset \dot{s}^\backprime \overrightarrow{P}{}^\backprime z : \supset \vdash . \mathrm{Prop}$

The following propositions, down to *85·42 inclusive, deal with circumstances under which we can infer $M = N$ from $\dot{s}^\backprime \mathrm{D}^\backprime M = \dot{s}^\backprime \mathrm{D}^\backprime N$. *85·32·33·34 are not subsequently used; the remainder are used in proving *85·43.

***85·31.** $\vdash :. \, z, w \in \alpha \,.\, z \neq w \,.\, \supset_{z,w} . \, \dot{s}^\backprime \overrightarrow{P}{}^\backprime z \,\dot{\cap}\, \dot{s}^\backprime \overrightarrow{P}{}^\backprime w = \dot{\Lambda} : \supset :$
$M, N \in P_\Delta{}^\backprime\alpha \,.\, \dot{s}^\backprime \mathrm{D}^\backprime M = \dot{s}^\backprime \mathrm{D}^\backprime N \,.\, \supset . \, M = N$

Dem.

$\vdash . *25{\cdot}54 . \supset \vdash : \mathrm{Hp} \,.\, z, w \in \alpha \,.\, \dot{\exists} ! \, \dot{s}^\backprime \overrightarrow{P}{}^\backprime z \,\dot{\cap}\, \dot{s}^\backprime \overrightarrow{P}{}^\backprime w \,.\, \supset_{z,w} . \, z = w :$

$[*11{\cdot}35] \quad \supset \vdash : \mathrm{Hp} \,.\, z, w \in \alpha \,.\, u \,(\dot{s}^\backprime \overrightarrow{P}{}^\backprime z)\, v \,.\, u \,(\dot{s}^\backprime \overrightarrow{P}{}^\backprime w)\, v \,.\, \supset_{z,w,u,v} . \, z = w \qquad (1)$

$\vdash . *85{\cdot}3 . \supset \vdash :: \mathrm{Hp} \,.\, z \in \alpha \,.\, M, N \in P_\Delta{}^\backprime\alpha \,.\, \dot{s}^\backprime \mathrm{D}^\backprime M = \dot{s}^\backprime \mathrm{D}^\backprime N \,.\, \supset :.$

$u \,(M^\backprime z)\, v \,.\, \supset : z \in \alpha \,.\, u \,(\dot{s}^\backprime \overrightarrow{P}{}^\backprime z)\, v \,.\, u \,(\dot{s}^\backprime \mathrm{D}^\backprime N)\, v :$

$[*80{\cdot}35] \qquad \supset : z \in \alpha \,.\, u \,(\dot{s}^\backprime \overrightarrow{P}{}^\backprime z)\, v : (\exists w) \,.\, w \in \alpha \,.\, u \,(N^\backprime w)\, v :$

$[*85{\cdot}3 . *10{\cdot}35] \supset : (\exists w) \,.\, z, w \in \alpha \,.\, u \,(\dot{s}^\backprime \overrightarrow{P}{}^\backprime z)\, v \,.\, u \,(\dot{s}^\backprime \overrightarrow{P}{}^\backprime w)\, v \,.\, u \,(N^\backprime w)\, v :$

$[(1) . *10{\cdot}28] \quad \supset : (\exists w) \,.\, z = w \,.\, u \,(N^\backprime w)\, v :$

$[*13{\cdot}195] \qquad \supset : u \,(N^\backprime z)\, v \qquad (2)$

$\vdash . (2) . \mathrm{Exp} . *10{\cdot}11{\cdot}21 . \supset \vdash :. \mathrm{Hp}\,(2) \,.\, \supset : z \in \alpha \,.\, \supset_z . \, M^\backprime z \subset N^\backprime z \qquad (3)$

Similarly $\qquad \vdash :. \mathrm{Hp}\,(2) \,.\, \supset : z \in \alpha \,.\, \supset_z . \, N^\backprime z \subset M^\backprime z \qquad (4)$

$\vdash . (3) . (4) . \qquad \supset \vdash :. \mathrm{Hp}\,(2) \,.\, \supset : z \in \alpha \,.\, \supset_z . \, M^\backprime z = N^\backprime z :$

$[*33{\cdot}45 . *80{\cdot}14] \qquad \supset : M = N :. \supset \vdash . \mathrm{Prop}$

***85·32.** $\vdash :. \, z, w \in \alpha \,.\, z \neq w \,.\, \supset_{z,w} . \, s^\backprime C^{\backprime\backprime} \overrightarrow{P}{}^\backprime z \cap s^\backprime C^{\backprime\backprime} \overrightarrow{P}{}^\backprime w = \Lambda : \supset :$
$M, N \in P_\Delta{}^\backprime\alpha \,.\, \dot{s}^\backprime \mathrm{D}^\backprime M = \dot{s}^\backprime \mathrm{D}^\backprime N \,.\, \supset . \, M = N$

Dem.

$\vdash . *41{\cdot}45 . \supset$

$\vdash :. \mathrm{Hp} \,.\, \supset : z, w \in \alpha \,.\, z \neq w \,.\, \supset_{z,w} . \, C^\backprime \dot{s}^\backprime \overrightarrow{P}{}^\backprime z \cap C^\backprime \dot{s}^\backprime \overrightarrow{P}{}^\backprime w = \Lambda \,.$

$[*33{\cdot}34] \qquad \supset_{z,w} . \, \dot{s}^\backprime \overrightarrow{P}{}^\backprime z \,\dot{\cap}\, \dot{s}^\backprime \overrightarrow{P}{}^\backprime w = \dot{\Lambda} \qquad (1)$

$\vdash . (1) . *85{\cdot}31 . \supset \vdash . \mathrm{Prop}$

***85·33.** $\vdash :. \, z, w \in \alpha \,.\, z \neq w \,.\, \supset_{z,w} . \, s^\backprime \mathrm{D}^{\backprime\backprime} \overrightarrow{P}{}^\backprime z \cap s^\backprime \mathrm{D}^{\backprime\backprime} \overrightarrow{P}{}^\backprime w = \Lambda : \supset :$
$M, N \in P_\Delta{}^\backprime\alpha \,.\, \dot{s}^\backprime \mathrm{D}^\backprime M = \dot{s}^\backprime \mathrm{D}^\backprime N \,.\, \supset . \, M = N$ [*41·43 . *33·32 . *85·31]

The proof proceeds exactly as in *85·32.

***85·34.** $\vdash :. \, z, w \in \alpha \,.\, z \neq w \,.\, \supset_{z,w} . \, s^\backprime \text{Ɑ}^{\backprime\backprime} \overrightarrow{P}{}^\backprime z \cap s^\backprime \text{Ɑ}^{\backprime\backprime} \overrightarrow{P}{}^\backprime w = \Lambda : \supset :$
$M, N \in P_\Delta{}^\backprime\alpha \,.\, \dot{s}^\backprime \mathrm{D}^\backprime M = \dot{s}^\backprime \mathrm{D}^\backprime N \,.\, \supset . \, M = N$ [*41·44 . *33·33 . *85·31]

The following propositions, *85·4·41·42, are lemmas for *85·43·44, which latter are of fundamental importance, since they are the source of the associative law in cardinal arithmetic.

***85·4.** $\vdash :. \lambda, \mu \epsilon \kappa . \lambda \neq \mu . \supset_{\lambda, \mu} . \dot{s}`\lambda \dot{\cap} \dot{s}`\mu = \dot{\Lambda} : \supset :$
$M, N \epsilon \epsilon_\Delta`\kappa . \dot{s}`\mathrm{D}`M = \dot{s}`\mathrm{D}`N . \supset . M = N$ $\left[*85·31 \frac{\epsilon}{P} . *62·2\right]$

***85·41.** $\vdash :. \kappa \epsilon \mathrm{Cls}^2 \mathrm{excl} . \supset : \alpha, \beta \epsilon \kappa . \alpha \neq \beta . \supset . \dot{s}`P_\Delta`\alpha \dot{\cap} \dot{s}`P_\Delta`\beta = \dot{\Lambda}$

Dem.

$\vdash . *80·14 . \supset \vdash : x (\dot{s}`P_\Delta`\alpha) y . x (\dot{s}`P_\Delta`\beta) y . \supset_{x,y} . y \epsilon \alpha . y \epsilon \beta .$
[*22·33.*10·24] $\supset_{x,y} . \exists ! \alpha \cap \beta :$
[Transp] $\supset \vdash : \alpha \cap \beta = \Lambda . \supset . \dot{s}`P_\Delta`\alpha \dot{\cap} \dot{s}`P_\Delta`\beta = \dot{\Lambda}$ (1)
$\vdash . (1) . *84·1 . \supset \vdash . \mathrm{Prop}$

***85·42.** $\vdash : \kappa \epsilon \mathrm{Cls}^2 \mathrm{excl} . M, N \epsilon \epsilon_\Delta`P_\Delta``\kappa . \dot{s}`\mathrm{D}`M = \dot{s}`\mathrm{D}`N . \supset . M = N$

Dem.

$\vdash . *30·37 . \mathrm{Transp} . \supset \vdash : P_\Delta`\alpha \neq P_\Delta`\beta . \supset_{\alpha, \beta} . \alpha \neq \beta :$
[Fact] $\supset \vdash : \kappa \epsilon \mathrm{Cls}^2 \mathrm{excl} . \alpha, \beta \epsilon \kappa . P_\Delta`\alpha \neq P_\Delta`\beta . \supset_{\alpha, \beta} .$
$\kappa \epsilon \mathrm{Cls}^2 \mathrm{excl} . \alpha, \beta \epsilon \kappa . \alpha \neq \beta .$
[*85·41] $\supset_{\alpha, \beta} . \dot{s}`P_\Delta`\alpha \dot{\cap} \dot{s}`P_\Delta`\beta = \dot{\Lambda} :$
[*37·63] $\supset \vdash : \kappa \epsilon \mathrm{Cls}^2 \mathrm{excl} . \lambda, \mu \epsilon P_\Delta``\kappa . \lambda \neq \mu . \supset_{\lambda, \mu} . \dot{s}`\lambda \dot{\cap} \dot{s}`\mu = \dot{\Lambda}$ (1)
$\vdash . (1) . *85·4 . \supset \vdash . \mathrm{Prop}$

***85·43.** $\vdash : \kappa \epsilon \mathrm{Cls}^2 \mathrm{excl} . \supset . P_\Delta`s`\kappa \mathrm{sm} \epsilon_\Delta`P_\Delta``\kappa$

Dem.

$\vdash . *34·41 . \supset \vdash . (M) . \dot{s}`\mathrm{D}`M = (\dot{s} \mid \mathrm{D})`M .$
[*13·12] $\supset \vdash :. M, N \epsilon \epsilon_\Delta`P_\Delta``\kappa . \dot{s}`\mathrm{D}`M = \dot{s}`\mathrm{D}`N . \supset_{M,N} . M = N : \supset :$
$M, N \epsilon \epsilon_\Delta`P_\Delta``\kappa . (\dot{s} \mid \mathrm{D})`M = (\dot{s} \mid \mathrm{D})`N . \supset_{M,N} . M = N$ (1)
$\vdash . (1) . *85·42 . \supset$
$\vdash :. \kappa \epsilon \mathrm{Cls}^2 \mathrm{excl} . \supset : M, N \epsilon \epsilon_\Delta`P_\Delta``\kappa . (\dot{s} \mid \mathrm{D})`M = (\dot{s} \mid \mathrm{D})`N . \supset_{M,N} . M = N :$
[*73·25] $\supset : (\dot{s} \mid \mathrm{D})``\epsilon_\Delta`P_\Delta``\kappa \mathrm{sm} \epsilon_\Delta`P_\Delta``\kappa :$
[*37·33] $\supset : \dot{s}``\mathrm{D}``\epsilon_\Delta`P_\Delta``\kappa \mathrm{sm} \epsilon_\Delta`P_\Delta``\kappa :$
[*85·27] $\supset : P_\Delta`s`\kappa \mathrm{sm} \epsilon_\Delta`P_\Delta``\kappa :. \supset \vdash . \mathrm{Prop}$

***85·44.** $\vdash : \kappa \epsilon \mathrm{Cls}^2 \mathrm{excl} . \supset . \epsilon_\Delta`s`\kappa \mathrm{sm} \epsilon_\Delta`\epsilon_\Delta``\kappa$ $\left[*85·43 \frac{\epsilon}{P}\right]$

The following proposition is used in connection with cardinal multiplication (*114·301).

***85·45.** $\vdash : \kappa \cap \lambda = \Lambda . \supset . \epsilon_\Delta`(\kappa \cup \lambda) \mathrm{sm} \epsilon_\Delta`(\iota`\epsilon_\Delta`\kappa \cup \iota`\epsilon_\Delta`\lambda)$

Dem.

$\vdash . *85·44 . \supset$
$\vdash : \iota`\kappa \cup \iota`\lambda \epsilon \mathrm{Cls}^2 \mathrm{excl} . \supset . \epsilon_\Delta`s`(\iota`\kappa \cup \iota`\lambda) \mathrm{sm} \epsilon_\Delta`\epsilon_\Delta``(\iota`\kappa \cup \iota`\lambda)$ (1)
$\vdash . *24·57 . \quad \supset \vdash :. \mathrm{Hp} . \supset : \kappa \neq \lambda . \vee . \kappa = \Lambda . \lambda = \Lambda :$
[*84·62·23] $\supset : \iota`\kappa \cup \iota`\lambda \epsilon \mathrm{Cls}^2 \mathrm{excl}$ (2)
$\vdash . *53·11·32 . \supset \vdash . s`(\iota`\kappa \cup \iota`\lambda) = \kappa \cup \lambda . \epsilon_\Delta``(\iota`\kappa \cup \iota`\lambda) = \iota`\epsilon_\Delta`\kappa \cup \iota`\epsilon_\Delta`\lambda$ (3)
$\vdash . (1) . (2) . (3) . \supset \vdash . \mathrm{Prop}$

The purpose of the following propositions, down to $*85{\cdot}55$, is to show how to get from a class of classes a class of selections having the same number of terms as $P_\Delta‘\kappa$. For this purpose we introduce a new notation, representing a rather important analysis of the couples contained in a given relation. A couple $x \downarrow y$ is contained in a relation P when xPy; thus if, keeping y fixed, we form the class of couples $\downarrow y“\overrightarrow{P}‘y$, all these couples are contained in P. We put

$*85{\cdot}5$. $P \overline{\downarrow} y = \downarrow y“\overrightarrow{P}‘y$ Df

Then $P\overline{\downarrow}“\text{ꓷ}‘P \in \text{Cls ex}^2\text{excl}$. Also $s‘P\overline{\downarrow}“\text{ꓷ}‘P$ is the class of all couples contained in P, and $\dot{s}‘s‘P\overline{\downarrow}“\text{ꓷ}‘P = P$. We shall now prove that $P_\Delta‘\alpha = \dot{s}“\text{D}“\epsilon_\Delta‘P\overline{\downarrow}“\alpha$, so that every member of $P_\Delta‘\alpha$ can be derived from a member of $\epsilon_\Delta‘P\overline{\downarrow}“\alpha$, and the problem of the existence of $P_\Delta‘\alpha$ is reduced to that of the existence of selections from a class of mutually exclusive existent classes.

$*85{\cdot}51$. $\vdash . P_\Delta‘\iota‘x = \downarrow x“\overrightarrow{P}‘x = P\overline{\downarrow}x$ $[*80{\cdot}45 . (*85{\cdot}5)]$

$*85{\cdot}52$. $\vdash . P_\Delta“\iota“\alpha = P\overline{\downarrow}“\alpha$ $[*37{\cdot}35 . *85{\cdot}51]$

$*85{\cdot}53$. $\vdash . P_\Delta‘\alpha = \dot{s}“\text{D}“\epsilon_\Delta‘P\overline{\downarrow}“\alpha$

Dem.

$\vdash . *84{\cdot}241 . *53{\cdot}22 . \supset \vdash . \iota“\alpha \in \text{Cls}^2\,\text{excl} . s‘\iota“\alpha = \alpha .$
$[*85{\cdot}27]$ $\supset \vdash . P_\Delta‘\alpha = \dot{s}“\text{D}“\epsilon_\Delta‘P_\Delta“\iota“\alpha$
$[*85{\cdot}52]$ $= \dot{s}“\text{D}“\epsilon_\Delta‘P\overline{\downarrow}“\alpha . \supset \vdash . \text{Prop}$

$*85{\cdot}54$. $\vdash . P_\Delta‘\alpha \,\text{sm}\, \epsilon_\Delta‘P\overline{\downarrow}“\alpha$

Dem.

$\vdash . *84{\cdot}241 . *53{\cdot}22 . \supset \vdash . \iota“\alpha \in \text{Cls}^2\,\text{excl} . s‘\iota“\alpha = \alpha .$
$[*85{\cdot}43]$ $\supset \vdash . P_\Delta‘\alpha \,\text{sm}\, \epsilon_\Delta‘P_\Delta“\iota“\alpha .$
$[*85{\cdot}52]$ $\supset \vdash . P_\Delta‘\alpha \,\text{sm}\, \epsilon_\Delta‘P\overline{\downarrow}“\alpha . \supset \vdash . \text{Prop}$

The following proposition is frequently useful.

$*85{\cdot}55$. $\vdash . P_\Delta‘\alpha \,\text{sm}\, \text{D}“\epsilon_\Delta‘P\overline{\downarrow}“\alpha . P\overline{\downarrow}“\alpha \in \text{Cls}^2\,\text{excl}$

Dem.

$\vdash . *85{\cdot}51 . *80{\cdot}14 . \supset \vdash : R \in P\overline{\downarrow}x . \supset . \text{ꓷ}‘R = \iota‘x : R \in P\overline{\downarrow}y . \supset . \text{ꓷ}‘R = \iota‘y :$
$[*3{\cdot}47]$ $\supset \vdash : R \in P\overline{\downarrow}x \cap P\overline{\downarrow}y . \supset . \text{ꓷ}‘R = \iota‘x . \text{ꓷ}‘R = \iota‘y .$
$[*13{\cdot}171 . *51{\cdot}23]$ $\supset . x = y .$
$[*30{\cdot}37]$ $\supset . P\overline{\downarrow}x = P\overline{\downarrow}y :$
$[*10{\cdot}11{\cdot}23]$ $\supset \vdash : \exists ! P\overline{\downarrow}x \cap P\overline{\downarrow}y . \supset . P\overline{\downarrow}x = P\overline{\downarrow}y :$
$[*3{\cdot}42 . *11{\cdot}11]$ $\supset \vdash : x, y \in \alpha . \exists ! P\overline{\downarrow}x \cap P\overline{\downarrow}y . \supset_{x,y} . P\overline{\downarrow}x = P\overline{\downarrow}y :$
$[*37{\cdot}63]$ $\supset \vdash : \lambda, \mu \in P\overline{\downarrow}“\alpha . \exists ! \lambda \cap \mu . \supset_{\lambda,\mu} . \lambda = \mu :$
$[*84{\cdot}11]$ $\supset \vdash . P\overline{\downarrow}“\alpha \in \text{Cls}^2\,\text{excl} .$ (1)
$[*84{\cdot}41]$ $\supset \vdash . \text{D}“\epsilon_\Delta‘P\overline{\downarrow}“\alpha \,\text{sm}\, \epsilon_\Delta‘P\overline{\downarrow}“\alpha .$
$[*85{\cdot}54]$ $\supset \vdash . P_\Delta‘\alpha \,\text{sm}\, \text{D}“\epsilon_\Delta‘P\overline{\downarrow}“\alpha$ (2)
$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***85·56.** $\vdash : P \upharpoonright \alpha \in \mathrm{Cls} \to 1 \,.\supset .\, \epsilon_\Delta{}^{\epsilon}\overrightarrow{P}{}^{\epsilon\epsilon}\alpha \,\mathrm{sm}\, \epsilon_\Delta{}^{\epsilon}P \,\overline{\downarrow}{}^{\epsilon\epsilon}\alpha$ [*85·14·54]

***85·6.** $\vdash .\, \epsilon_\Delta{}^{\epsilon\epsilon}\iota^{\epsilon\epsilon}\kappa = \hat{\mu}\{(\exists\beta) . \beta \in \kappa . \mu = \downarrow \beta^{\epsilon\epsilon}\beta\} = \epsilon \,\overline{\downarrow}{}^{\epsilon\epsilon}\kappa$

Dem.

$\vdash . *37{\cdot}67 . \supset \vdash .\, \epsilon_\Delta{}^{\epsilon\epsilon}\iota^{\epsilon\epsilon}\kappa = \hat{\mu}\{(\exists\beta) . \beta \in \kappa . \mu = \epsilon_\Delta{}^{\epsilon}\iota^{\epsilon}\beta\}$

$[*83{\cdot}4] \qquad = \hat{\mu}\{(\exists\beta) . \beta \in \kappa . \mu = \downarrow \beta^{\epsilon\epsilon}\beta\}$ (1)

$\vdash . (1) . *85{\cdot}52 . \supset \vdash .$ Prop

The following proposition is frequently employed.

***85·601.** $\vdash .\, \epsilon \,\overline{\downarrow}\, \alpha = \downarrow \alpha^{\epsilon\epsilon}\alpha \,.\, \epsilon \,\overline{\downarrow}\, \alpha \,\mathrm{sm}\, \alpha \,.\, \epsilon \,\overline{\downarrow}{}^{\epsilon\epsilon}\kappa \,\mathrm{sm}\, \kappa \,.\, \epsilon \,\overline{\downarrow} \in 1 \to 1 \,.\, \mathrm{E}\,!\, \epsilon \,\overline{\downarrow}{}^{\epsilon}\alpha$

Dem.

$\vdash . *85{\cdot}51 . *62{\cdot}2 . \quad \supset \vdash .\, \epsilon \,\overline{\downarrow}\, \alpha = \downarrow \alpha^{\epsilon\epsilon}\alpha$ (1)

$[*73{\cdot}611] \quad \supset \vdash .\, \epsilon \,\overline{\downarrow}\, \alpha \,\mathrm{sm}\, \alpha$ (2)

$\vdash . *38{\cdot}12 . \quad \supset \vdash .\, \mathrm{E}\,!\, \epsilon \,\overline{\downarrow}{}^{\epsilon}\alpha$ (3)

$[*71{\cdot}166] \quad \supset \vdash .\, \epsilon \,\overline{\downarrow} \in 1 \to \mathrm{Cls}$ (4)

$\vdash . (2) . *73{\cdot}47 . \quad \supset \vdash : \alpha = \Lambda \,.\, \epsilon \,\overline{\downarrow}\, \alpha = \epsilon \,\overline{\downarrow}\, \beta \,.\supset .\, \epsilon \,\overline{\downarrow}\, \beta = \Lambda \,.$

$[*73{\cdot}47 . (2)] \quad \supset .\, \beta = \Lambda$ (5)

$\vdash . (1) . *38{\cdot}131 . \quad \supset \vdash : x \in \alpha \,.\, \epsilon \,\overline{\downarrow}\, \alpha = \epsilon \,\overline{\downarrow}\, \beta \,.\supset .\, x \downarrow \alpha \in \downarrow \beta^{\epsilon\epsilon}\beta \,.$

$[*38{\cdot}131] \quad \supset .\, (\exists y) . x \downarrow \alpha = y \downarrow \beta \,.$

$[*55{\cdot}202] \quad \supset .\, \alpha = \beta$ (6)

$\vdash . (6) . *10{\cdot}11{\cdot}23{\cdot}35 . \supset \vdash : \exists\,!\,\alpha \,.\, \epsilon \,\overline{\downarrow}\, \alpha = \epsilon \,\overline{\downarrow}\, \beta \,.\supset .\, \alpha = \beta$ (7)

$\vdash . (5) . (7) . \quad \supset \vdash : \epsilon \,\overline{\downarrow}\, \alpha = \epsilon \,\overline{\downarrow}\, \beta \,.\supset .\, \alpha = \beta$ (8)

$\vdash . (4) . (8) . *71{\cdot}54 . \quad \supset \vdash .\, \epsilon \,\overline{\downarrow} \in 1 \to 1$ (9)

$\vdash . (9) . (3) . *73{\cdot}26 . \quad \supset \vdash .\, \epsilon \,\overline{\downarrow}{}^{\epsilon\epsilon}\kappa \,\mathrm{sm}\, \kappa$ (10)

$\vdash . (1) . (2) . (3) . (9) . (10) . \supset \vdash .$ Prop

***85·61.** $\vdash .\, \epsilon \,\overline{\downarrow}{}^{\epsilon\epsilon}\kappa \in \mathrm{Cls}^2 \,\mathrm{excl} \,.\, \epsilon_\Delta{}^{\epsilon}\kappa = \breve{s}^{\epsilon\epsilon}\mathrm{D}^{\epsilon\epsilon}\epsilon_\Delta{}^{\epsilon}\epsilon \,\overline{\downarrow}{}^{\epsilon\epsilon}\kappa \,.\, \epsilon_\Delta{}^{\epsilon}\kappa \,\mathrm{sm}\, \epsilon_\Delta{}^{\epsilon}\epsilon \,\overline{\downarrow}{}^{\epsilon\epsilon}\kappa$

$\left[*85{\cdot}53{\cdot}54{\cdot}55 \dfrac{\epsilon}{P}\right]$

***85·62.** $\vdash : \exists\,!\,\epsilon_\Delta{}^{\epsilon}\kappa \,.\equiv .\, \exists\,!\,\epsilon_\Delta{}^{\epsilon}\epsilon \,\overline{\downarrow}{}^{\epsilon\epsilon}\kappa$ [*85·61 . *73·36]

***85·63.** $\vdash : \epsilon \,\overline{\downarrow}{}^{\epsilon\epsilon}\mathrm{Cl\,ex}{}^{\epsilon}\alpha \in \mathrm{Cls\,ex}^2\,\mathrm{excl} : \exists\,!\,\epsilon_\Delta{}^{\epsilon}\mathrm{Cl\,ex}{}^{\epsilon}\alpha \,.\equiv .\, \exists\,!\,\epsilon_\Delta{}^{\epsilon}\epsilon \,\overline{\downarrow}{}^{\epsilon\epsilon}\mathrm{Cl\,ex}{}^{\epsilon}\alpha$

Dem.

$\vdash . *85{\cdot}6 . *60{\cdot}21 . \supset$

$\vdash : \lambda \in \epsilon \,\overline{\downarrow}{}^{\epsilon\epsilon}\mathrm{Cl\,ex}{}^{\epsilon}\alpha \,.\equiv .\, (\exists\beta) . \beta \subset \alpha \,.\, \exists\,!\,\beta \,.\, \lambda = \downarrow \beta^{\epsilon\epsilon}\beta$ (1)

$\vdash . *73{\cdot}611{\cdot}36 . \supset \vdash : \exists\,!\,\beta \,.\, \lambda = \downarrow \beta^{\epsilon\epsilon}\beta \,.\supset .\, \exists\,!\,\lambda :$

$[*3{\cdot}42] \quad \supset \vdash : \beta \subset \alpha \,.\, \exists\,!\,\beta \,.\, \lambda = \downarrow \beta^{\epsilon\epsilon}\beta \,.\supset .\, \exists\,!\,\lambda :$

$[*10{\cdot}11{\cdot}23] \quad \supset \vdash : (\exists\beta) . \beta \subset \alpha \,.\, \exists\,!\,\beta \,.\, \lambda = \downarrow \beta^{\epsilon\epsilon}\beta \,.\supset .\, \exists\,!\,\lambda$ (2)

$\vdash . (1) . (2) . \quad \supset \vdash : \lambda \in (\epsilon \,\overline{\downarrow}{}^{\epsilon\epsilon}\mathrm{Cl\,ex}{}^{\epsilon}\alpha) \,.\supset .\, \exists\,!\,\lambda :$

$[*10{\cdot}11 . *24{\cdot}63] \supset \vdash .\, \Lambda \sim\in (\epsilon \,\overline{\downarrow}{}^{\epsilon\epsilon}\mathrm{Cl\,ex}{}^{\epsilon}\alpha)$ (3)

$\vdash . (3) . *85{\cdot}61 . *84{\cdot}13 . \supset \vdash .\, \epsilon \,\overline{\downarrow}{}^{\epsilon\epsilon}\mathrm{Cl\,ex}{}^{\epsilon}\alpha \in \mathrm{Cls\,ex}^2\,\mathrm{excl}$ (4)

$\vdash . (4) . *85{\cdot}62 . \supset \vdash .$ Prop

Note. $(\alpha) . \exists\,!\,\epsilon_\Delta{}^{\epsilon}\mathrm{Cl\,ex}{}^{\epsilon}\alpha$ is "Zermelo's axiom." The above proposition shows that this is true if

$$\kappa \in \mathrm{Cls\,ex}^2\,\mathrm{excl} \,.\supset_\kappa .\, \exists\,!\,\epsilon_\Delta{}^{\epsilon}\kappa,$$

which again is true if

$$\kappa \epsilon \text{Cls ex}^2 \text{excl} . \supset : (\exists \mu) : \alpha \epsilon \kappa . \supset_\alpha . \mu \cap \alpha \epsilon 1$$

in virtue of *84·412. The last of these is the "multiplicative axiom," which is thus shown to imply "Zermelo's axiom."

The following propositions lead up to *85·72, which is used in the theory of double similarity (*111·3).

***85·7.** $\vdash :. \beta \epsilon \lambda . \supset_\beta . R'\beta \subset \beta : M \epsilon \epsilon_\Delta{}'R``\lambda : \supset .$
$$M | R \upharpoonright \lambda \epsilon \epsilon_\Delta{}'\lambda . \text{D}'(M | R \upharpoonright \lambda) = \text{D}'M$$

Dem.

$\vdash . *14{\cdot}21 . \supset \vdash :. \text{Hp} . \supset : \beta \epsilon \lambda . \supset_\beta . \text{E} ! R'\beta :$

[*74·11] $\supset : R \upharpoonright \lambda \epsilon 1 \to \text{Cls} . \lambda \subset \text{Ɑ}'R$ (1)

[*80·14.*71·25] $\supset : M | R \upharpoonright \lambda \epsilon 1 \to \text{Cls}$ (2)

$\vdash . (1) . *71{\cdot}7 . *35{\cdot}7 . \supset \vdash :. \text{Hp} . \supset : x (M | R \upharpoonright \lambda) \beta . \supset . \beta \epsilon \lambda . xM(R'\beta) .$

[*80·14.Hp] $\supset . \beta \epsilon \lambda . x \epsilon R'\beta .$

[Hp] $\supset . x \epsilon \beta$ (3)

$\vdash . *80{\cdot}14 . *74{\cdot}44 . \supset$

$\vdash : \text{Hp} . \supset . \text{D}'(M | R \upharpoonright \lambda) = \text{D}'M . \text{Ɑ}'(M | R \upharpoonright \lambda) = \lambda \cap \text{Ɑ}'R$

[(1)] $= \lambda$ (4)

$\vdash . (2) . (3) . (4) . *80{\cdot}14 . \supset \vdash . \text{Prop}$

***85·701.** $\vdash :. \beta \epsilon \lambda . \supset_\beta . R'\beta \subset \beta : \supset . \text{D}``\epsilon_\Delta{}'R``\lambda \subset \text{D}``\epsilon_\Delta{}'\lambda$ [*85·7]

***85·702.** $\vdash :. \beta \epsilon \lambda . \supset_\beta . R'\text{Cl}'\beta \epsilon \text{Cl}'\beta : \supset . \text{D}``\epsilon_\Delta{}'R``\text{Cl}``\lambda \subset \text{D}``\epsilon_\Delta{}'\lambda$

$$\left[*85{\cdot}701 \frac{R | \text{Cl}}{R} \right]$$

***85·71.** $\vdash : R \epsilon \epsilon_\Delta{}'\text{Cl}``\lambda . \supset . \text{D}``\epsilon_\Delta{}'\text{D}'R \subset \text{D}``\epsilon_\Delta{}'\lambda$ [*85·702 . *83·2]

This proposition asserts that if we can select one sub-class out of each member of λ (where λ is a class of classes), then selections from the sub-classes so obtained are selections from λ.

***85·72.** $\vdash :. (S``\beta) \rceil S \epsilon 1 \to 1 : \beta \epsilon \lambda . \supset_\beta . R'\beta \subset S'\beta : \supset .$
$$\text{D}``\epsilon_\Delta{}'R``\lambda \subset \text{D}``\epsilon_\Delta{}'S``\lambda$$

Dem.

$\vdash . *14{\cdot}21 . *33{\cdot}43 . \supset \vdash :. \text{Hp} . \supset : \beta \epsilon \lambda . \supset . \beta \epsilon \text{Ɑ}'S$ (1)

$\vdash . *85{\cdot}701 \dfrac{R | \breve{S}, S``\lambda}{R, \lambda} . \supset$

$\vdash :. \gamma \epsilon S``\lambda . \supset_\gamma . (R | \breve{S})'\gamma \subset \dot{\gamma} : \supset . \text{D}``\epsilon_\Delta{}'R``\breve{S}``S``\lambda \subset \text{D}``\epsilon_\Delta{}'S``\lambda$ (2)

$\vdash . *37{\cdot}63 . *14{\cdot}21 . \supset$

$\vdash :: \text{Hp} . \supset :. \gamma \epsilon S``\lambda . \supset_\gamma . (R | \breve{S})'\gamma \subset \gamma : \equiv : \beta \epsilon \lambda . \supset_\beta . (R | \breve{S})'S'\beta \subset S'\beta :$

[*74·53.(1)] $\equiv : \beta \epsilon \lambda . \supset_\beta . R'\beta \subset S'\beta$ (3)

$\vdash . *74{\cdot}171 . \supset \vdash : \text{Hp} . \supset . \breve{S}``S``\lambda = \lambda$ (4)

$\vdash . (2) . (3) . (4) . \supset \vdash . \text{Prop}$

The following proposition is a lemma employed in the theory of double similarity (*111·313).

***85·81.** $\vdash :. \lambda \in \mathrm{Cl}'\mathrm{s}^2\,\mathrm{excl} : \beta \in \lambda \,.\, \supset_\beta .\, s'\text{Ɑ}``T`\beta \subset \beta : R \in \epsilon_\Delta`T``\lambda : \supset :$

$\beta \in \lambda \,.\, \supset_\beta .\, (\dot{s}`\mathrm{D}`R) \upharpoonright \beta = R`T`\beta$

Dem.

$\vdash .\, *14{\cdot}21 \,.\quad \supset \vdash :. \mathrm{Hp} \,.\, \supset : \beta \in \lambda \,.\, \supset .\, \mathrm{E}!\, T`\beta :$ (1)

$[*83{\cdot}2 . *37{\cdot}6] \quad \supset : \beta \in \lambda \,.\, \supset .\, R`T`\beta \in T`\beta \,.$ (2)

$[*35{\cdot}452 . \mathrm{Hp}] \quad \supset .\, R`T`\beta = (R`T`\beta) \upharpoonright \beta$ (3)

$\vdash .\, (1) \,.\, *83{\cdot}22 \,.\, \supset \vdash :. \mathrm{Hp} \,.\, \supset : \beta \in \lambda \,.\, \supset .\, \mathrm{E}!\, R`T`\beta \,.$

$[*33{\cdot}43 . *41{\cdot}13] \quad \supset .\, R`T`\beta \mathrel{\dot\subset} \dot{s}`\mathrm{D}`R \,.$

$[*35{\cdot}461] \quad \supset .\, (R`T`\beta) \upharpoonright \beta \mathrel{\dot\subset} (\dot{s}`\mathrm{D}`R) \upharpoonright \beta \,.$

$[(3)] \quad \supset .\, R`T`\beta \mathrel{\dot\subset} (\dot{s}`\mathrm{D}`R) \upharpoonright \beta$ (4)

$\vdash .\, (1) \,.\, *37{\cdot}6 \,.\, *83{\cdot}23 \,.\, \supset \vdash :. \mathrm{Hp} \,.\, \supset : \mathrm{D}`R = \hat{M}\{(\exists \gamma) \,.\, \gamma \in \lambda \,.\, M = R`T`\gamma\} :$

$[*41{\cdot}11 . *13{\cdot}195] \quad \supset : x\,(\dot{s}`\mathrm{D}`R)\,y \,.\, \equiv .\, (\exists \gamma) \,.\, \gamma \in \lambda \,.\, x\,(R`T`\gamma)\,y :$

$[*35{\cdot}101] \quad \supset : x\,\{(\dot{s}`\mathrm{D}`R) \upharpoonright \beta\}\,y \,.\, \equiv .\, (\exists \gamma) \,.\, \gamma \in \lambda \,.\, x\,(R`T`\gamma)\,y \,.\, y \in \beta$ (5)

$\vdash .\, (2) \,.\, *33{\cdot}14 \,.\, \supset \vdash :. \mathrm{Hp} \,.\, \gamma \in \lambda \,.\, \supset : x\,(R`T`\gamma)\,y \,.\, \supset .\, y \in \text{Ɑ}`R`T`\gamma \,.\, R`T`\gamma \in T`\gamma \,.$

$[*40{\cdot}4] \quad \supset .\, y \in s`\text{Ɑ}``T`\gamma \,.$

$[\mathrm{Hp}] \quad \supset .\, y \in \gamma$ (6)

$\vdash .\, (5) \,.\, (6) \,.\quad \supset \vdash :: \mathrm{Hp} \,.\, \supset :. \beta \in \lambda \,.\, \supset :$

$x\,\{(\dot{s}`\mathrm{D}`R) \upharpoonright \beta\}\,y \,.\, \equiv .\, (\exists \gamma) \,.\, \beta, \gamma \in \lambda \,.\, x\,(R`T`\gamma)\,y \,.\, y \in \beta \,.\, y \in \gamma \,.$

$[*84{\cdot}11 . \mathrm{Hp}] \quad \supset .\, (\exists \gamma) \,.\, \beta, \gamma \in \lambda \,.\, x\,(R`T`\gamma)\,y \,.\, \beta = \gamma \,.$

$[*13{\cdot}195] \quad \supset .\, x\,(R`T`\beta)\,y$ (7)

$\vdash .\, (4) \,.\, (7) \,.\, \supset \vdash .\, \mathrm{Prop}$

*88. CONDITIONS FOR THE EXISTENCE OF SELECTIONS

Summary of *88.

The existence of selections cannot, so far as is known at present, be proved in general. That is, we cannot prove any of the following:

$$(P, \kappa) : \kappa \subset \text{Ɑ}`P \,.\, \supset \,.\, \exists \,!\, P_\Delta`\kappa$$

$$(P, \kappa) : P \in \mathrm{Cls} \rightarrow 1 \,.\, \kappa \subset \text{Ɑ}`P \,.\, \supset \,.\, \exists \,!\, P_\Delta`\kappa$$

$$(P) \,.\, \exists \,!\, P_\Delta`\text{Ɑ}`P$$

$$(\kappa) : \Lambda \sim\in \kappa \,.\, \supset \,.\, \exists \,!\, \epsilon_\Delta`\kappa$$

$$(\kappa) : \kappa \in \mathrm{Cls\,ex^2\,excl} \,.\, \supset \,.\, \exists \,!\, \epsilon_\Delta`\kappa$$

$$(\alpha) \,.\, \exists \,!\, \epsilon_\Delta`\mathrm{Cl\,ex}`\alpha$$

$$(\kappa) :. \kappa \in \mathrm{Cls\,ex^2\,excl} \,.\, \supset : (\exists\mu) : \alpha \in \kappa \,.\, \supset_\alpha \,.\, \mu \cap \alpha \in 1$$

These various propositions can be shown to be all equivalent *inter se*; and in virtue of Zermelo's theorem (cf. *258), they are equivalent to the proposition "every class can be well-ordered." In the present number we have to prove the above equivalences, as well as certain propositions giving the existence of selections in various particular cases.

The most apparently obvious of the above propositions is the last, namely: "If κ is a class of mutually exclusive classes, no one of which is null, there is at least one class μ which takes one and only one member from each member of κ." This we shall define as the "multiplicative axiom."

We will call P a *multipliable* relation (denoted by "Rel Mult") if $P_\Delta`\text{Ɑ}`P$ exists, or, what is equivalent, if $\kappa \subset \text{Ɑ}`P \,.\, \supset_\kappa \,.\, \exists \,!\, P_\Delta`\kappa$. Thus we put

$$\mathrm{Rel\,Mult} = \hat{P}\{\exists \,!\, P_\Delta`\text{Ɑ}`P\} \quad \mathrm{Df.}$$

We will call κ a *multipliable* class of classes if $\epsilon_\Delta`\kappa$ exists, *i.e.* we put

$$\mathrm{Cls^2\,Mult} = \hat{\kappa}\{\exists \,!\, \epsilon_\Delta`\kappa\} \quad \mathrm{Df.}$$

The multiplicative axiom will be denoted by "Mult ax." Thus we put

$$\mathrm{Mult\,ax} \,.\, = :. \kappa \in \mathrm{Cls\,ex^2\,excl} \,.\, \supset_\kappa : (\exists\mu) : \alpha \in \kappa \,.\, \supset_\alpha \,.\, \mu \cap \alpha \in 1 \quad \mathrm{Df.}$$

In the present number, we shall first give various equivalent forms of the assumption that P is a multipliable relation (*88·1—·15); we shall then do the same for multipliable classes of classes (*88·2—·26); next we shall give various equivalent forms of the multiplicative axiom (*88·3—·39). (Some important equivalent forms cannot be given at this stage, as they depend upon definitions not yet given, such as the definitions of cardinal multiplication and of well-ordered series. Cf. *114·26 and *258·37.) Finally we shall give propositions showing that various special classes of classes are multipliable. Most of these propositions will not be used in the sequel, but they illustrate

the nature of the difficulties involved in proving that a class of classes is multipliable, and some of them show that mere size does not prevent a class from being multipliable. For example, $*88{\cdot}48$ shows that, given any class of classes κ, if each member α is replaced by $\iota``\alpha \cup \iota`\alpha$, the result is a multipliable class of classes; but the only effect of this change is to increase the number of members of each member of our class of classes by one.

The chief propositions in this number which are afterwards referred to are the following:

$*88{\cdot}22.$ $\vdash : \kappa \in \mathrm{Cls}^2\,\mathrm{Mult} \,.\, \lambda \subset \kappa \,.\supset.\, \lambda \in \mathrm{Cls}^2\,\mathrm{Mult}$

$*88{\cdot}32.$ $\vdash :.\, \mathrm{Mult\,ax} \,.\equiv:\, \kappa \in \mathrm{Cls\,ex}^2\,\mathrm{excl} \,.\supset_\kappa .\, \exists\,!\,\epsilon_\Delta`\kappa$

$*88{\cdot}33.$ $\vdash : \mathrm{Mult\,ax} \,.\equiv.\, (\alpha) \,.\, \exists\,!\,\epsilon_\Delta`\mathrm{Cl\,ex}`\alpha$

$*88{\cdot}361.$ $\vdash :.\, \mathrm{Mult\,ax} \,.\equiv:\, \kappa \subset \text{Ɑ}`R \,.\equiv_{R,\kappa} .\, \exists\,!\,R_\Delta`\kappa$

$*88{\cdot}37.$ $\vdash :.\, \mathrm{Mult\,ax} \,.\equiv:\, \Lambda \sim\in \kappa \,.\supset_\kappa .\, \exists\,!\,\epsilon_\Delta`\kappa$

The above is usually the most convenient form of the multiplicative axiom.

$*88{\cdot}372.$ $\vdash :.\, \mathrm{Mult\,ax} \,.\equiv:\, \Lambda \in \kappa \,.\equiv_\kappa .\, \epsilon_\Delta`\kappa = \Lambda$

This proposition is used in $*114$, to prove that the multiplicative axiom is equivalent to the proposition that a cardinal product vanishes when, and only when, one of its factors vanishes.

$*88{\cdot}01.$ $\mathrm{Rel\,Mult} = \hat{P}\,\{\exists\,!\,P_\Delta`\text{Ɑ}`P\}$ Df

$*88{\cdot}02.$ $\mathrm{Cls}^2\,\mathrm{Mult} = \hat{\kappa}\,\{\exists\,!\,\epsilon_\Delta`\kappa\}$ Df

$*88{\cdot}03.$ $\mathrm{Mult\,ax} \,.=:.\, \kappa \in \mathrm{Cls\,ex}^2\,\mathrm{excl} \,.\supset_\kappa : (\exists\mu) : \alpha \in \kappa \,.\supset_\alpha .\, \mu \cap \alpha \in 1$ Df

$*88{\cdot}1.$ $\vdash : P \in \mathrm{Rel\,Mult} \,.\equiv.\, \exists\,!\,P_\Delta`\text{Ɑ}`P$ [$*20{\cdot}3 \,.\, (*88{\cdot}01)$]

$*88{\cdot}11.$ $\vdash : P \in \mathrm{Rel\,Mult} \,.\, \lambda \subset \text{Ɑ}`P \,.\supset.\, \exists\,!\,P_\Delta`\lambda$

Dem.

$\vdash .\, *80{\cdot}6 \,.\quad \supset \vdash : R \in P_\Delta`\text{Ɑ}`P \,.\, \lambda \subset \text{Ɑ}`P \,.\supset.\, R \upharpoonright \lambda \in P_\Delta`\lambda \,.$

$[*10{\cdot}24] \qquad \supset .\, \exists\,!\,P_\Delta`\lambda :$

$[*10{\cdot}11{\cdot}23{\cdot}35] \supset \vdash : \exists\,!\,P_\Delta`\text{Ɑ}`P \,.\, \lambda \subset \text{Ɑ}`P \,.\supset.\, \exists\,!\,P_\Delta`\lambda$ (1)

$\vdash .\, (1) \,.\, *88{\cdot}1 \,.\supset \vdash .\, \mathrm{Prop}$

$*88{\cdot}12.$ $\vdash :.\, P \in \mathrm{Rel\,Mult} \,.\equiv:\, \lambda \subset \text{Ɑ}`P \,.\supset_\lambda .\, \exists\,!\,P_\Delta`\lambda$

Dem.

$\vdash .\, *88{\cdot}11 \,.\, \mathrm{Exp} \,.\, *10{\cdot}11{\cdot}21 \,.\supset$

$\vdash :.\, P \in \mathrm{Rel\,Mult} \,.\supset:\, \lambda \subset \text{Ɑ}`P \,.\supset_\lambda .\, \exists\,!\,P_\Delta`\lambda$ (1)

$\vdash .\, *10{\cdot}1 \,.\, *22{\cdot}42 \,.\supset$

$\vdash :.\, \lambda \subset \text{Ɑ}`P \,.\supset_\lambda .\, \exists\,!\,P_\Delta`\lambda :\supset.\, \exists\,!\,P_\Delta`\text{Ɑ}`P \,.$

$[*88{\cdot}1] \qquad \supset .\, P \in \mathrm{Rel\,Mult}$ (2)

$\vdash .\, (1) \,.\, (2) \,.\supset \vdash .\, \mathrm{Prop}$

$*88{\cdot}13.$ $\vdash : P \in \mathrm{Rel\,Mult} \,.\equiv.\, \exists\,!\,\epsilon_\Delta`P\!\downarrow``\text{Ɑ}`P$ [$*85{\cdot}54 \,.\, *73{\cdot}36 \,.\, *88{\cdot}1$]

***88·14.** $\vdash :. \kappa \subset \mathrm{Cl}'P . \supset : P \upharpoonright \kappa \in \mathrm{Rel\,Mult} . \equiv . \exists ! P_\Delta{}'\kappa$

Dem.

$\vdash . *80·23 . \supset \vdash : \exists ! P_\Delta{}'\kappa . \equiv . \exists ! (P \upharpoonright \kappa)_\Delta{}'\kappa$ (1)

$\vdash . *35·65 . \supset \vdash : \kappa \subset \mathrm{Cl}'P . \supset . \mathrm{Cl}'(P \upharpoonright \kappa) = \kappa$ (2)

$\vdash . (1) . (2) . \supset \vdash :. \mathrm{Hp} . \supset : \exists ! P_\Delta{}'\kappa . \equiv . \exists ! (P \upharpoonright \kappa)_\Delta{}'\mathrm{Cl}'(P \upharpoonright \kappa) .$

[*88·1] $\equiv . P \upharpoonright \kappa \in \mathrm{Rel\,Mult} :. \supset \vdash . \mathrm{Prop}$

***88·15.** $\vdash :. \mathrm{Cl}'P = \mathrm{V} . \supset : P \upharpoonright \kappa \in \mathrm{Rel\,Mult} . \equiv . \exists ! P_\Delta{}'\kappa$ [*88·14 . *24·11]

***88·2.** $\vdash : \kappa \in \mathrm{Cls}^2\,\mathrm{Mult} . \equiv . \exists ! \epsilon_\Delta{}'\kappa$ [*20·3 . (*88·02)]

***88·21.** $\vdash : P \in \mathrm{Rel\,Mult} . \equiv . P \downharpoonright {}``\mathrm{Cl}'P \in \mathrm{Cls}^2\,\mathrm{Mult}$ [*88·13·2]

***88·22.** $\vdash : \kappa \in \mathrm{Cls}^2\,\mathrm{Mult} . \lambda \subset \kappa . \supset . \lambda \in \mathrm{Cls}^2\,\mathrm{Mult}$

Dem.

$\vdash . *80·6 . \supset \vdash : R \in \epsilon_\Delta{}'\kappa . \lambda \subset \kappa . \supset . R \upharpoonright \lambda \in \epsilon_\Delta{}'\lambda .$

[*10·24] $\supset . \exists ! \epsilon_\Delta{}'\lambda :$

[*10·11·23·35] $\supset \vdash : \exists ! \epsilon_\Delta{}'\kappa . \lambda \subset \kappa . \supset . \exists ! \epsilon_\Delta{}'\lambda$ (1)

$\vdash . (1) . *88·2 . \supset \vdash . \mathrm{Prop}$

***88·23.** $\vdash : \kappa \in \mathrm{Cls}^2\,\mathrm{Mult} . \supset . \mathrm{Cl}'\kappa \subset \mathrm{Cls}^2\,\mathrm{Mult}$ [*88·22 . *60·2]

***88·24.** $\vdash :. P \in \mathrm{Cls} \rightarrow 1 . \supset : P \in \mathrm{Rel\,Mult} . \equiv . \overrightarrow{P}{}``\mathrm{Cl}'P \in \mathrm{Cls}^2\,\mathrm{Mult}$

Dem.

$\vdash . *85·14 . *73·36 . \supset \vdash :. \mathrm{Hp} . \supset : \exists ! P_\Delta{}'\mathrm{Cl}'P . \equiv . \exists ! \epsilon_\Delta{}'\overrightarrow{P}{}``\mathrm{Cl}'P$ (1)

$\vdash . (1) . *88·1·2 . \supset \vdash . \mathrm{Prop}$

***88·25.** $\vdash :. P \upharpoonright \kappa \in \mathrm{Cls} \rightarrow 1 . \kappa \subset \mathrm{Cl}'P . \supset : P \upharpoonright \kappa \in \mathrm{Rel\,Mult} . \equiv . \overrightarrow{P}{}``\kappa \in \mathrm{Cls}^2\,\mathrm{Mult}$

Dem.

$\vdash . *85·14 . *73·36 . \supset$

$\vdash :. \mathrm{Hp} . \supset : \exists ! P_\Delta{}'\kappa . \equiv . \exists ! \epsilon_\Delta{}'\overrightarrow{P}{}``\kappa :$

[*88·14·2] $\supset : P \upharpoonright \kappa \in \mathrm{Rel\,Mult} . \equiv . \overrightarrow{P}{}``\kappa \in \mathrm{Cls}^2\,\mathrm{Mult} :. \supset \vdash . \mathrm{Prop}$

***88·26.** $\vdash :: \kappa \in \mathrm{Cls}^2\,\mathrm{excl} . \supset :. \kappa \in \mathrm{Cls}^2\,\mathrm{Mult} . \equiv : (\exists \mu) : \alpha \in \kappa . \supset_\alpha . \mu \cap \alpha \in 1$

Dem.

$\vdash . *88·2 . *37·45 . \supset \vdash : \kappa \in \mathrm{Cls}^2\,\mathrm{Mult} . \equiv . \exists ! \mathrm{D}``\epsilon_\Delta{}'\kappa$ (1)

$\vdash . (1) . *84·412 . \supset$

$\vdash :: \mathrm{Hp} . \supset :. \kappa \in \mathrm{Cls}^2\,\mathrm{Mult} . \equiv : (\exists \mu) : \alpha \in \kappa . \supset_\alpha . \mu \cap \alpha \in 1 : \mu \subset s'\kappa :$ (2)

[*10·5] $\supset : (\exists \mu) : \alpha \in \kappa . \supset_\alpha . \mu \cap \alpha \in 1$ (3)

$\vdash . *40·13 . *22·621 . \supset \vdash : \alpha \in \kappa . \supset_\alpha . s'\kappa \cap \alpha = \alpha .$

[*22·481] $\supset_\alpha . \mu \cap s'\kappa \cap \alpha = \mu \cap \alpha :$

[*2·77 . *10·27] $\supset \vdash :. \alpha \in \kappa . \supset_\alpha . \mu \cap \alpha \in 1 : \supset : \alpha \in \kappa . \supset_\alpha . \mu \cap s'\kappa \cap \alpha \in 1$ (4)

$\vdash . (4) . *22·43 . \supset \vdash :. \alpha \in \kappa . \supset_\alpha . \mu \cap \alpha \in 1 : \supset :$

$\alpha \in \kappa . \supset_\alpha . \mu \cap s'\kappa \cap \alpha \in 1 : \mu \cap s'\kappa \subset s'\kappa :$

[*10·24] $\supset : (\exists \nu) : \alpha \in \kappa . \supset_\alpha . \nu \cap \alpha \in 1 : \nu \subset s'\kappa$ (5)

$\vdash . (5) . *10{\cdot}11{\cdot}23 . \supset$

$\vdash :. (\exists \mu) : \alpha \in \kappa . \supset_\alpha . \mu \cap \alpha \in 1 : \supset : (\exists \nu) : \alpha \in \kappa . \supset_\alpha . \nu \cap \alpha \in 1 : \nu \subset s{}'\kappa$ (6)

$\vdash . (6) . (2) . \supset \vdash :: \text{Hp} . \supset :. (\exists \mu) : \alpha \in \kappa . \supset_\alpha . \mu \cap \alpha \in 1 : \supset . \kappa \in \text{Cls}^2\,\text{Mult}$ (7)

$\vdash . (3) . (7) . \supset \vdash . \text{Prop}$

$*88{\cdot}3$. $\vdash :: \text{Mult ax} . \equiv :. \kappa \in \text{Cls ex}^2\,\text{excl} . \supset_\kappa : (\exists \mu) : \alpha \in \kappa . \supset_\alpha . \mu \cap \alpha \in 1$
$[*4{\cdot}2 . (*88{\cdot}03)]$

$*88{\cdot}31$. $\vdash : \text{Mult ax} . \equiv . \text{Cls ex}^2\,\text{excl} \subset \text{Cls}^2\,\text{Mult}$

Dem.

$\vdash . *88{\cdot}26 . *5{\cdot}74 . \supset \vdash :: \kappa \in \text{Cls ex}^2\,\text{excl} . \supset_\kappa . \kappa \in \text{Cls}^2\,\text{Mult} : \equiv :.$

$\kappa \in \text{Cls ex}^2\,\text{excl} . \supset_\kappa : (\exists \mu) : \alpha \in \kappa . \supset_\alpha . \mu \cap \alpha \in 1 :.$

$[*88{\cdot}3] \quad \equiv :. \text{Mult ax} :: \supset \vdash . \text{Prop}$

$*88{\cdot}32$. $\vdash :. \text{Mult ax} . \equiv : \kappa \in \text{Cls ex}^2\,\text{excl} . \supset_\kappa . \exists ! \epsilon_\Delta{}'\kappa \quad [*88{\cdot}31{\cdot}2]$

$*88{\cdot}33$. $\vdash : \text{Mult ax} . \equiv . (\alpha) . \exists ! \epsilon_\Delta{}'\text{Cl ex}{}'\alpha$

Note that $(\alpha) . \exists ! \epsilon_\Delta{}'\text{Cl ex}{}'\alpha$ is Zermelo's axiom.

Dem.

$\vdash . *88{\cdot}32 . *85{\cdot}63 . \supset \vdash : \text{Mult ax} . \supset . \exists ! \epsilon_\Delta{}'\epsilon \downarrow {}''\text{Cl ex}{}'\alpha .$

$[*85{\cdot}63] \quad \supset . \exists ! \epsilon_\Delta{}'\text{Cl ex}{}'\alpha$ (1)

$\vdash . *60{\cdot}57 . \quad \supset \vdash . \kappa \subset \text{Cl}{}'s{}'\kappa .$

$[*60{\cdot}24] \quad \supset \vdash . \kappa - \iota{}'\Lambda \subset \text{Cl ex}{}'s{}'\kappa .$

$[*84{\cdot}13] \quad \supset \vdash : \kappa \in \text{Cls ex}^2\,\text{excl} . \supset . \kappa \subset \text{Cl ex}{}'s{}'\kappa$ (2)

$\vdash . (2) . *80{\cdot}6 . \supset \vdash : \kappa \in \text{Cls ex}^2\,\text{excl} . R \in \epsilon_\Delta{}'\text{Cl ex}{}'s{}'\kappa . \supset . R \upharpoonright \kappa \in \epsilon_\Delta{}'\kappa$ (3)

$\vdash . (3) . *10{\cdot}11{\cdot}28{\cdot}35 . \supset \vdash : \kappa \in \text{Cls ex}^2\,\text{excl} . \exists ! \epsilon_\Delta{}'\text{Cl ex}{}'s{}'\kappa . \supset_\kappa . \exists ! \epsilon_\Delta{}'\kappa :$

$[*10{\cdot}1] \supset \vdash :. (\alpha) . \exists ! \epsilon_\Delta{}'\text{Cl ex}{}'\alpha . \supset : \kappa \in \text{Cls ex}^2\,\text{excl} . \supset_\kappa . \exists ! \epsilon_\Delta{}'\kappa :$

$[*88{\cdot}32] \quad \supset : \text{Mult ax}$ (4)

$\vdash . (1) . (4) . \supset \vdash . \text{Prop}$

$*88{\cdot}34$. $\vdash : \text{Mult ax} . \equiv . \text{Cls} \rightarrow 1 \subset \text{Rel Mult}$

Dem.

$\vdash . *84{\cdot}5 . *88{\cdot}32 . \supset \vdash :. \text{Mult ax} . \supset : R \in \text{Cls} \rightarrow 1 . \supset . \exists ! \epsilon_\Delta{}'\overrightarrow{R}{}''\text{Ɑ}{}'R .$

$[*85{\cdot}14 . *73{\cdot}36] \quad \supset . \exists ! R_\Delta{}'\text{Ɑ}{}'R .$

$[*88{\cdot}1] \quad \supset . R \in \text{Rel Mult}$ (1)

$\vdash . *84{\cdot}14 . \quad \supset \vdash :. \text{Cls} \rightarrow 1 \subset \text{Rel Mult} . \supset :$

$\kappa \in \text{Cls ex}^2\,\text{excl} . \supset . \epsilon \upharpoonright \kappa \in \text{Rel Mult} .$

$[*88{\cdot}1] \quad \supset . \exists ! (\epsilon \upharpoonright \kappa)_\Delta{}'\text{Ɑ}{}'\epsilon \upharpoonright \kappa .$

$[*84{\cdot}13 . *62{\cdot}42] \quad \supset . \exists ! (\epsilon \upharpoonright \kappa)_\Delta{}'\kappa .$

$[*80{\cdot}23] \quad \supset . \exists ! \epsilon_\Delta{}'\kappa$ (2)

$\vdash . (2) . *10{\cdot}11{\cdot}21 . *88{\cdot}32 . \supset \vdash : \text{Cls} \rightarrow 1 \subset \text{Rel Mult} . \supset . \text{Mult ax}$ (3)

$\vdash . (1) . (3) . \supset \vdash . \text{Prop}$

***88·35.** $\vdash : \text{Mult ax} . \equiv . (R) . R \in \text{Rel Mult}$

Dem.

$\vdash . *37{\cdot}45 . *55{\cdot}121 . (*85{\cdot}5) . \supset \vdash : \exists ! P \downharpoonleft x . \equiv . \exists ! \overrightarrow{P}{}^{\backprime}x .$

$[*33{\cdot}41] \qquad \equiv . x \in \text{Ɑ}{}^{\backprime}P \qquad (1)$

$\vdash . (1) . *10{\cdot}11 . *37{\cdot}63 . \supset \vdash : \alpha \in P \downharpoonleft {}^{\backprime\backprime}\text{Ɑ}{}^{\backprime}P . \supset_\alpha . \exists ! \alpha :$

$[*24{\cdot}63] \qquad \supset \vdash . \Lambda \sim\in P \downharpoonleft {}^{\backprime\backprime}\text{Ɑ}{}^{\backprime}P \qquad (2)$

$\vdash . (2) . *84{\cdot}13 . *85{\cdot}55 . \supset \vdash . P \downharpoonleft {}^{\backprime\backprime}\text{Ɑ}{}^{\backprime}P \in \text{Cls ex}^2 \text{ excl} .$

$[*88{\cdot}32] \qquad \supset \vdash : \text{Mult ax} . \supset . \exists ! \epsilon_\Delta{}^{\backprime}P \downharpoonleft {}^{\backprime\backprime}\text{Ɑ}{}^{\backprime}P .$

$[*85{\cdot}54 . *73{\cdot}36] \qquad \supset . \exists ! P_\Delta{}^{\backprime}\text{Ɑ}{}^{\backprime}P .$

$[*88{\cdot}1] \qquad \supset . P \in \text{Rel Mult} \qquad (3)$

$\vdash . *10{\cdot}1 . *88{\cdot}1 . \supset \vdash : (R) . R \in \text{Rel Mult} . \supset . \exists ! (\epsilon \upharpoonright \text{Cl ex}{}^{\backprime}\alpha)_\Delta{}^{\backprime}\text{Ɑ}{}^{\backprime}(\epsilon \upharpoonright \text{Cl ex}{}^{\backprime}\alpha) .$

$[*62{\cdot}42] \qquad \supset . \exists ! (\epsilon \upharpoonright \text{Cl ex}{}^{\backprime}\alpha)_\Delta{}^{\backprime}\text{Cl ex}{}^{\backprime}\alpha .$

$[*80{\cdot}23] \qquad \supset . \exists ! \epsilon_\Delta{}^{\backprime}\text{Cl ex}{}^{\backprime}\alpha \qquad (4)$

$\vdash . (4) . *10{\cdot}11{\cdot}21 . *88{\cdot}33 . \supset \vdash : (R) . R \in \text{Rel Mult} . \supset . \text{Mult ax} \qquad (5)$

$\vdash . (3) . (5) . \supset \vdash . \text{Prop}$

***88·36.** $\vdash :. \text{Mult ax} . \equiv : \kappa \subset \text{Ɑ}{}^{\backprime}R . \supset_{R,\kappa} . \exists ! R_\Delta{}^{\backprime}\kappa \quad [*88{\cdot}35{\cdot}12]$

***88·361.** $\vdash :. \text{Mult ax} . \equiv : \kappa \subset \text{Ɑ}{}^{\backprime}R . \equiv_{R,\kappa} . \exists ! R_\Delta{}^{\backprime}\kappa \quad [*88{\cdot}36 . *80{\cdot}2]$

***88·37.** $\vdash :. \text{Mult ax} . \equiv : \Lambda \sim\in \kappa . \supset_\kappa . \exists ! \epsilon_\Delta{}^{\backprime}\kappa$

Dem.

$\vdash . *88{\cdot}36 . *62{\cdot}231 . \supset \vdash :. \text{Mult ax} . \supset : \Lambda \sim\in \kappa . \supset_\kappa . \exists ! \epsilon_\Delta{}^{\backprime}\kappa \qquad (1)$

$\vdash . *84{\cdot}13 . *88{\cdot}32 . \supset \vdash :. \Lambda \sim\in \kappa . \supset_\kappa . \exists ! \epsilon_\Delta{}^{\backprime}\kappa : \supset . \text{Mult ax} \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***88·371.** $\vdash :. \text{Mult ax} . \equiv : \Lambda \sim\in \kappa . \equiv_\kappa . \exists ! \epsilon_\Delta{}^{\backprime}\kappa \quad [*88{\cdot}37 . *83{\cdot}1]$

***88·372.** $\vdash :. \text{Mult ax} . \equiv : \Lambda \in \kappa . \equiv_\kappa . \epsilon_\Delta{}^{\backprime}\kappa = \Lambda \quad [*88{\cdot}371 . \text{Transp}]$

This proposition shows that the multiplicative axiom is equivalent to the assumption that a cardinal product is zero when, and only when, one of its factors is zero.

***88·373.** $\vdash : \text{Mult ax} . \equiv . \text{Cl}{}^{\backprime}(\text{Cls} - \iota{}^{\backprime}\Lambda) \subset \text{Cls}^2 \text{ Mult}$

Dem.

$\vdash . *24{\cdot}63 . *53{\cdot}5 . \supset \vdash :. \Lambda \sim\in \kappa . \equiv : \alpha \in \kappa . \supset_\alpha . \alpha \in \text{Cls} - \iota{}^{\backprime}\Lambda :$

$[*22{\cdot}1] \qquad \equiv : \kappa \subset \text{Cls} - \iota{}^{\backprime}\Lambda :$

$[*60{\cdot}2] \qquad \equiv : \kappa \in \text{Cl}{}^{\backprime}(\text{Cls} - \iota{}^{\backprime}\Lambda) \qquad (1)$

$\vdash . (1) . *88{\cdot}37 . \supset \vdash :. \text{Mult ax} . \equiv : \kappa \in \text{Cl}{}^{\backprime}(\text{Cls} - \iota{}^{\backprime}\Lambda) . \supset_\kappa . \exists ! \epsilon_\Delta{}^{\backprime}\kappa :$

$[*88{\cdot}2] \qquad \equiv : \text{Cl}{}^{\backprime}(\text{Cls} - \iota{}^{\backprime}\Lambda) \subset \text{Cls}^2 \text{ Mult} :. \supset \vdash . \text{Prop}$

***88·38.** $\vdash : \text{Mult ax} . \equiv . \text{Cls} - \iota{}^{\backprime}\Lambda \in \text{Cls}^2 \text{ Mult} \quad [*88{\cdot}23{\cdot}373]$

***88·39.** $\vdash : \text{Mult ax} . \equiv . (\exists R) . R \in 1 \to \text{Cls} . R \subseteq \epsilon . \text{D}{}^{\backprime}R = \text{V} . \text{Ɑ}{}^{\backprime}R = \text{Cls} - \iota{}^{\backprime}\Lambda$

Dem.

$\vdash . *88{\cdot}38{\cdot}2 . *80{\cdot}14 . \supset$

$\vdash : \text{Mult ax} . \equiv . (\exists R) . R \in 1 \to \text{Cls} . R \subseteq \epsilon . \text{Ɑ}{}^{\backprime}R = \text{Cls} - \iota{}^{\backprime}\Lambda \qquad (1)$

$\vdash . *51{\cdot}161 . *53{\cdot}5 . \supset \vdash : \text{Ɑ}{}^{\backprime}R = \text{Cls} - \iota{}^{\backprime}\Lambda . \supset . \iota{}^{\backprime}x \in \text{Ɑ}{}^{\backprime}R \qquad (2)$

$\vdash . *23{\cdot}621 . \supset \vdash : R \Subset \epsilon . \supset . R = R \mathbin{\dot\cap} \epsilon$ (3)

$\vdash . (2) . (3) . \supset \vdash : R \Subset \epsilon . \text{Ɑ}\text{‘}R = \text{Cls} - \iota\text{‘}\Lambda . \supset . \iota\text{‘}x \epsilon \text{Ɑ}\text{‘}(R \mathbin{\dot\cap} \epsilon) .$

$[*33{\cdot}131] \qquad \supset . (\exists y) . yR\iota\text{‘}x . y \epsilon \iota\text{‘}x .$

$[*51{\cdot}15] \qquad \supset . (\exists y) . yR\iota\text{‘}x . y = x .$

$[*13{\cdot}195] \qquad \supset . xR\iota\text{‘}x .$

$[*33{\cdot}14] \qquad \supset . x \epsilon \text{D}\text{‘}R$ (4)

$\vdash . (4) . *10{\cdot}11{\cdot}21 . *24{\cdot}14 . \supset \vdash : R \Subset \epsilon . \text{Ɑ}\text{‘}R = \text{Cls} - \iota\text{‘}\Lambda . \supset . \text{D}\text{‘}R = \text{V}$ (5)

$\vdash . (1) . (5) . \supset \vdash . \text{Prop}$

The following propositions are concerned with certain cases in which a construction exists by which the existence of selections can be proved.

***88·4.** $\vdash . \kappa \rceil \breve{\text{Cl}} \epsilon \epsilon_\Delta\text{‘}\text{Cl}\text{“}\kappa$

Dem.

$\vdash . *72{\cdot}19 . *71{\cdot}27 . \supset \vdash . \kappa \rceil \breve{\text{Cl}} \epsilon 1 \rightarrow \text{Cls}$ (1)

$\vdash . *35{\cdot}52{\cdot}101 . \quad \supset \vdash : \alpha (\kappa \rceil \breve{\text{Cl}}) \lambda . \equiv . \alpha \epsilon \kappa . \lambda = \text{Cl}\text{‘}\alpha .$

$[*60{\cdot}34] \qquad \supset . \alpha \epsilon \lambda$ (2)

$\vdash . (2) . *11{\cdot}11 . \quad \supset \vdash . \kappa \rceil \breve{\text{Cl}} \Subset \epsilon$ (3)

$\vdash . *35{\cdot}52 . \quad \supset \vdash . \text{Ɑ}\text{‘}(\kappa \rceil \breve{\text{Cl}}) = \text{D}\text{‘}(\text{Cl} \upharpoonright \kappa)$

$[*37{\cdot}401] \qquad = \text{Cl}\text{“}\kappa$ (4)

$\vdash . (1) . (3) . (4) . *80{\cdot}14 . \supset \vdash . \text{Prop}$

***88·41.** $\vdash . \text{Cl}\text{“}\kappa \epsilon \text{Cls}^2 \text{Mult} \quad [*88{\cdot}4{\cdot}2]$

***88·411.** $\vdash . \kappa \epsilon \text{D}\text{“}\epsilon_\Delta\text{‘}\text{Cl}\text{“}\kappa$

Dem.

$\vdash . *35{\cdot}52 . \quad \supset \vdash . \text{D}\text{‘}(\kappa \rceil \breve{\text{Cl}}) = \text{Ɑ}\text{‘}(\text{Cl} \upharpoonright \kappa)$

$[*35{\cdot}65 . *33{\cdot}431] \qquad = \kappa$ (1)

$\vdash . (1) . *88{\cdot}4 . \supset \vdash . (\exists R) . R \epsilon \epsilon_\Delta\text{‘}\text{Cl}\text{“}\kappa . \text{D}\text{‘}R = \kappa .$

$[*37{\cdot}6] \qquad \supset \vdash . \kappa \epsilon \text{D}\text{“}\epsilon_\Delta\text{‘}\text{Cl}\text{“}\kappa . \supset \vdash . \text{Prop}$

***88·42.** $\vdash : \kappa \epsilon \text{Cls}^2 \text{Mult} . \exists ! \alpha . \equiv . \kappa \cup \iota\text{‘}\alpha \epsilon \text{Cls}^2 \text{Mult} \quad [*83{\cdot}904 . *88{\cdot}2]$

In virtue of this proposition, as will be proved later, every finite class of existent classes is a Cls^2 Mult. For we have $\dot{\Lambda} \epsilon \epsilon_\Delta\text{‘}\Lambda$; and, by the above, a Cls^2 Mult remains a Cls^2 Mult when one existent class is added as an additional member; hence the result follows by induction.

***88·43.** $\vdash : s\text{‘}\kappa \epsilon \text{Cls}^2 \text{Mult} . \supset . \epsilon_\Delta\text{“}\kappa \epsilon \text{Cls}^2 \text{Mult}$

Dem.

$\vdash . *88{\cdot}2 . \supset \vdash : \text{Hp} . \supset . \exists ! \epsilon_\Delta\text{‘}s\text{‘}\kappa .$

$[*85{\cdot}24] \qquad \supset . \exists ! \dot{s}\text{“}\text{D}\text{“}\epsilon_\Delta\text{‘}\epsilon_\Delta\text{“}\kappa .$

$[*37{\cdot}45] \qquad \supset . \exists ! \epsilon_\Delta\text{‘}\epsilon_\Delta\text{“}\kappa .$

$[*88{\cdot}2] \qquad \supset . \epsilon_\Delta\text{“}\kappa \epsilon \text{Cls}^2 \text{Mult} : \supset \vdash . \text{Prop}$

***88·431.** $\vdash :. \kappa \epsilon \text{Cls}^2 \text{excl} . \supset : \epsilon_\Delta\text{“}\kappa \epsilon \text{Cls}^2 \text{Mult} . \equiv . s\text{‘}\kappa \epsilon \text{Cls}^2 \text{Mult}$
$[*88{\cdot}2 . *85{\cdot}28 . *37{\cdot}45]$

***88·44.** $\vdash : \text{Cl ex}`s`\kappa \in \text{Cls}^2\,\text{Mult} . \supset . \kappa - \iota`\Lambda \in \text{Cls}^2\,\text{Mult}$ [*60·57 . *88·22]

***88·441.** $\vdash : \Lambda \sim\in \kappa . \text{Cl ex}`s`\kappa \in \text{Cls}^2\,\text{Mult} . \supset . \kappa \in \text{Cls}^2\,\text{Mult}$ [*88·44]

***88·45.** $\vdash : \text{D}`R \cap \text{Ɑ}`R = \Lambda . P = \hat{x}\hat{\alpha}\{x \in \text{Ɑ}`R . \alpha = \overrightarrow{R}`x \cup \iota`x\} . \supset . P \in \epsilon_\Delta`\text{Ɑ}`P$

Dem.

$\vdash . *21·3 . \supset \vdash :. \text{Hp} . \supset : xP\alpha . \equiv_{x,\alpha} . x \in \text{Ɑ}`R . \alpha = \overrightarrow{R}`x \cup \iota`x .$ (1)

[*51·16] $\supset_{x,\alpha} . x \in \alpha$ (2)

$\vdash . (1) . *33·15 . *51·2 . \supset$

$\vdash :. \text{Hp} . \supset : xP\alpha . \supset_x . \alpha = \overrightarrow{R}`x \cup \iota`x . \overrightarrow{R}`x \subset \text{D}`R . \iota`x \subset \text{Ɑ}`R .$

[*24·494] $\supset_x . \alpha - \text{D}`R = \iota`x$ (3)

$\vdash . (3) . *11·59 . \supset \vdash :. \text{Hp} . \supset : xP\alpha . yP\alpha . \supset_{x,y} . \alpha - \text{D}`R = \iota`x . \alpha - \text{D}`R = \iota`y .$

[*20·23.*51·23] $\supset_{x,y} . x = y :$

[*71·17] $\supset : P \in 1 \rightarrow \text{Cls}$ (4)

$\vdash . (2) . (4) . *80·14 . \supset \vdash . \text{Prop}$

***88·46.** $\vdash : \text{D}`R \cap \text{Ɑ}`R = \Lambda . \lambda = \hat{\alpha}\{(\exists x) . x \in \text{Ɑ}`R . \alpha = \overrightarrow{R}`x \cup \iota`x\} . \supset .$
$\lambda \in \text{Cls}^2\,\text{Mult}$

Dem.

$\vdash . *21·3 . *10·281 . *33·131 . \supset \vdash :. P = \hat{x}\hat{\alpha}\{x \in \text{Ɑ}`R . \alpha = \overrightarrow{R}`x \cup \iota`x\} . \supset :$

$\alpha \in \text{Ɑ}`P . \equiv_\alpha . (\exists x) . x \in \text{Ɑ}`R . \alpha = \overrightarrow{R}`x \cup \iota`x$ (1)

$\vdash . (1) . *88·45 . \supset \vdash : \text{Hp} . \supset . \hat{x}\hat{\alpha}\{x \in \text{Ɑ}`R . \alpha = \overrightarrow{R}`x \cup \iota`x\} \in \epsilon_\Delta`\lambda .$

[*10·24] $\supset . \exists ! \epsilon_\Delta`\lambda .$

[*88·2] $\supset . \lambda \in \text{Cls}^2\,\text{Mult} : \supset \vdash . \text{Prop}$

***88·47.** $\vdash : P = \hat{\alpha}\hat{\beta}\{\alpha \in \kappa . \beta = \iota``\alpha \cup \iota`\alpha\} . \supset . P \in \epsilon_\Delta`\text{Ɑ}`P$

Dem.

$\vdash . *21·3 . \qquad \supset \vdash :. \text{Hp} . \supset : \alpha P\beta . \equiv_{\alpha,\beta} . \alpha \in \kappa . \beta = \iota``\alpha \cup \iota`\alpha .$ (1)

[*51·16] $\supset_{\alpha,\beta} . \alpha \in \beta$ (2)

$\vdash . (1) . *11·59 . \supset \vdash :. \text{Hp} . \supset : \alpha P\beta . \gamma P\beta . \supset_{\alpha,\beta,\gamma} . \beta = \iota``\alpha \cup \iota`\alpha . \beta = \iota``\gamma \cup \iota`\gamma .$

[*40·171.*53·22·02] $\supset_{\alpha,\beta,\gamma} . s`\beta = \alpha . s`\beta = \gamma .$

[*20·23] $\supset_{\alpha,\beta,\gamma} . \alpha = \gamma$

[*71·17] $\supset : P \in 1 \rightarrow \text{Cls}$ (3)

$\vdash . (2) . (3) . *80·14 . \supset \vdash . \text{Prop}$

***88·48.** $\vdash . \hat{\beta}\{(\exists\alpha) . \alpha \in \kappa . \beta = \iota``\alpha \cup \iota`\alpha\} \in \text{Cls}^2\,\text{Mult}$ [*88·47]

The proof proceeds as in *88·46.

***88·5.** $\vdash . \Lambda \cap \text{Cls} \in \text{Cls}^2\,\text{Mult}$ [*83·9 . *88·2]

***88·51.** $\vdash : \exists ! \alpha . \supset . \iota`\alpha \in \text{Cls}^2\,\text{Mult}$ [*83·901 . *88·2]

***88·52.** $\vdash . \iota``\alpha \in \text{Cls}^2\,\text{Mult}$ [*83·42]

***88·53.** $\vdash : \kappa \subset 1 . \supset . \kappa \in \text{Cls}^2\,\text{Mult}$ [*83·44]

SECTION E

INDUCTIVE RELATIONS

Summary of Section E.

The subjects to be treated in this section are certain general ideas of which a particular instance is afforded by mathematical induction. Mathematical induction is, in fact, the application to the number-series of a conception which is applicable to all relations, and is often very important. The conception in question is that which we shall call the *ancestral relation* with respect to a given relation. If R is the given relation, we denote the corresponding ancestral relation by "R_*"; the name is chosen because, if R is the relation of parent and child, R_* will be the relation of ancestor and descendant—where, for convenience of language, we include x among his own ancestors if x is a parent or a child of anything.

It would commonly be said that a has to z the relation of ancestor to descendant if there are a certain number of intermediate people $b, c, d, \ldots$ such that in the series $a, b, c, d, \ldots z$ each term has to the next the relation of parent and child. But this is not an adequate definition, because the dots in

$$\text{"}a, b, c, d, \ldots z\text{"}$$

represent an unanalysed idea. We may then try to amend this definition by saying that there is a finite class α of intermediate terms such that one member (b) of α is a child of a, one (y) is a parent of z, every member of α except b is a child of one (and only one) member of α, and every member of α except y is a parent of one (and only one) member of α. This definition is open to several objections. In the first place, it is very complicated; in the second place, there will, in regard to a general relation, be difficulty in securing the uniqueness of the member of α which is to be a parent (or a child) of a given member of α; in the third place (and this is the really fatal objection) the proposed definition states that α is to be a *finite* class, and we shall find that finitude, in the relevant sense, is only defined by means of the very conception of the ancestral relation which we are here engaged in defining. In fact, if N denotes the relation of ν to $\nu+1$, where ν is a cardinal number, then a finite cardinal (in the sense we require) is one to which 0 has the relation N_*, *i.e.* one of which 0 is an ancestor with respect to the relation

$$\hat{\nu}\hat{\mu}\,(\mu = \nu + 1).$$

Hence we must not use the notion of finitude in defining the ancestral relation. In fact, the ancestral relation is defined as follows.

Let us call μ a *hereditary class with respect to* R if $\breve{R}``\mu \subset \mu$, *i.e.* if successors of μ's (with respect to R) are μ's. Thus, for example, if μ is the class of persons named Smith, μ is hereditary with respect to the relation of father to son. If μ is the Peerage, μ is hereditary with respect to the relation of father to surviving eldest son. If μ is numbers greater than 100, μ is hereditary with respect to the relation of ν to $\nu+1$; and so on. If now a is an ancestor of z, and μ is a hereditary class to which a belongs, then z also belongs to this class. Conversely, if z belongs to every hereditary class to which a belongs, then (in the sense in which a is one of his own ancestors if a is anybody's parent or child) a must be an ancestor of z. For to have a for one's ancestor is a hereditary property which belongs to a, and therefore, by hypothesis, to z. Hence a is an ancestor of z when, and only when, a belongs to the field of the relation in question and z belongs to every hereditary class to which a belongs. This property may be used to define the ancestral relation; *i.e.* since we have

$$aR_*z \,.\equiv: a \,\epsilon\, C`R : \breve{R}``\mu \subset \mu \,.\, a \,\epsilon\, \mu \,.\supset_\mu .\, z \,\epsilon\, \mu$$

we put

$$R_* = \hat{a}\hat{z}\,\{a \,\epsilon\, C`R : \breve{R}``\mu \subset \mu \,.\, a \,\epsilon\, \mu \,.\supset_\mu .\, z \,\epsilon\, \mu\} \quad \text{Df.}$$

We then have

$$\vdash : a \,\epsilon\, C`R \,.\supset .\, \overleftarrow{R_*}`a = \hat{z}\,\{\breve{R}``\mu \subset \mu \,.\, a \,\epsilon\, \mu \,.\supset_\mu .\, z \,\epsilon\, \mu\}.$$

Here $\overleftarrow{R_*}`a$ may be called "the descendants of a." It is the class of terms of which a is an ancestor.

To make plain the relation of the above to mathematical induction, put 0 for a, and $\hat{\alpha}\hat{\beta}\,(\beta = \alpha+1)$ for R. Then, since $1 = 0+1$, we have $0 \,\epsilon\, C`R$. Again

$$\breve{R}``\mu \subset \mu \,.\equiv: \alpha \,\epsilon\, \mu \,.\supset_\alpha .\, \alpha+1 \,\epsilon\, \mu.$$

Thus we find

$$\overleftarrow{R_*}`0 = \hat{\beta}\,\{\alpha \,\epsilon\, \mu \,.\supset_\alpha .\, \alpha+1 \,\epsilon\, \mu : 0 \,\epsilon\, \mu :\supset_\mu .\, \beta \,\epsilon\, \mu\}.$$

Thus if β is a descendant of 0, β belongs to every class to which 0 belongs and to which $\alpha+1$ belongs whenever α belongs. Hence mathematical induction, starting from 0, will prove properties of β. In elementary mathematics it is customary to speak as if this held of *all* integers, *i.e.* as if $\overleftarrow{R_*}`0$ (as above defined) included all integers; but in fact only *finite* integers (in one of the two senses which the word *finite* may have) belong to the class $\overleftarrow{R_*}`0$, and they belong to it *by definition*, being defined as the class

$$\hat{\beta}\,\{\alpha \,\epsilon\, \mu \,.\supset_\alpha .\, \alpha+1 \,\epsilon\, \mu : 0 \,\epsilon\, \mu :\supset_\mu .\, \beta \,\epsilon\, \mu\},$$

i.e. as $\overleftarrow{R_*}`0$ in the above sense. To infinite numbers, inductive proofs of this kind starting from 0 cannot be applied.

The study of R_* will occupy $*90$. The relation R_* holds between x and y if $x(I \upharpoonright C'R)y$ or xRy or xR^2y or etc. The study of this "etc." occupies $*91$, "on the powers of a relation." We may, for many technical purposes, regard $I \upharpoonright C'R$ as the 0th power of R; the other powers are R, R^2, etc. If S is a power of R, so is $S|R$. Now $S|R$ is $|R'S$, according to the definition in $*38$. Thus if we have

$$R \epsilon \mu : S \epsilon \mu . \supset_S . S | R \epsilon \mu : \supset_\mu . P \epsilon \mu,$$

P must be a power of R, because the class of powers of R is a value of μ which satisfies the hypothesis

$$R \epsilon \mu : S \epsilon \mu . \supset_S . S | R \epsilon \mu.$$

Conversely, if P is a power of R, then P is reached by repetitions of the process of turning S into $S|R$, starting this process with R. Hence if P is a power of R, we shall have

$$R \epsilon \mu : S \epsilon \mu . \supset_S . S | R \epsilon \mu : \supset_\mu . P \epsilon \mu.$$

Consequently, if we denote the class of powers of R by Pot'R, we have

$$P \epsilon \mathrm{Pot}'R . \equiv :. R \epsilon \mu : S \epsilon \mu . \supset_S . S | R \epsilon \mu : \supset_\mu . P \epsilon \mu.$$

We might use this as the definition of Pot'R; but we can get a somewhat simpler form. For the above is shown, without much difficulty, to be equivalent to

$$P \epsilon \mathrm{Pot}'R . \equiv . P (|R)_* R,$$

that is, P belongs to the ancestry of R with respect to $|R$, in other words, P is reached from R by proceeding along the series

$$R, \ |R'R, \ |R'|R'R, \text{ etc.}$$

which is the same as the series

$$R, R^2, R^3, \text{ etc.}$$

The relation $(|R)_*$ is important on its own account. We put

$$R_{ts} = (|R)_* \quad \mathrm{Df},$$

and then we put

$$\mathrm{Pot}'R = \overrightarrow{R_{ts}}'R \quad \mathrm{Df}.$$

We often want to include $I \upharpoonright C'R$ among the powers of R; the class consisting of Pot'R together with $I \upharpoonright C'R$ we call Potid'R. The definition is

$$\mathrm{Potid}'R = \overrightarrow{R_{ts}}'(I \upharpoonright C'R),$$

whence we easily prove

$$\mathrm{Potid}'R = \mathrm{Pot}'R \cup \iota'(I \upharpoonright C'R).$$

The relation of being related by some power of R (other than $I \upharpoonright C'R$) is a very important one. We denote it by R_{po}, and put

$$R_{po} = \dot{s}'\mathrm{Pot}'R \quad \mathrm{Df}.$$

Thus when $xR_{po}y$, we have one of xRy, xR^2y, xR^3y, etc. It is easy to prove that

$$R_* = R_{po} \,\dot{\cup}\, I \upharpoonright C'R.$$

In a series in which every term (except the first, if there is a first) has an immediate predecessor, and every term (except the last, if there is a last) has an immediate successor, if R is the relation of a term to its immediate successor, R_{po} is the relation of any earlier term to any later one.

The next number (*92) concerns itself with some special properties of the powers of one-many, many-one and one-one relations.

The next number (*93) analyses the field of a relation into successive *generations*; *e.g.* if the relation is that of parent and child, the first generation will consist of Adam and Eve, the second of their children, the third of their grandchildren, and so on, taking always the longest route from Adam and Eve when there have been intermarriages between generations. That is, taking any relation P, the first generation is $\text{D}\text{‘}P - \text{Ɑ}\text{‘}P$, the second is $\text{Ɑ}\text{‘}P - \text{Ɑ}\text{‘}(P^2)$ the third is $\text{Ɑ}\text{‘}(P^2) - \text{Ɑ}\text{‘}(P^3)$, and so on. Generally, if T is a power of P (including $I \upharpoonright C\text{‘}P$), the corresponding generation is

$$\text{Ɑ}\text{‘}T - \text{Ɑ}\text{‘}(T \mid P),$$

i.e.

$$\text{Ɑ}\text{‘}T - \breve{P}\text{“}\text{Ɑ}\text{‘}T.$$

In order to express this more conveniently, we introduce a new symbol $\min_P$, which is required also on other grounds, especially in series. "$\min_P$" may be read "minimum with respect to P." We regard "xPy" as "x precedes y"; then in a class α, the "minima of α" will be those members of α which belong to $C\text{‘}P$ and are not preceded by any other members of α, *i.e.* $\alpha \cap C\text{‘}P - \breve{P}\text{“}\alpha$. We put therefore

$$x \min_P \alpha \,.\equiv.\, x \in \alpha \cap C\text{‘}P - \breve{P}\text{“}\alpha,$$

i.e.

$$\min_P = \hat{x}\hat{\alpha}\,(x \in \alpha \cap C\text{‘}P - \breve{P}\text{“}\alpha) \quad \text{Df.}$$

Hence we have

$$\overrightarrow{\min_P}\text{‘}\alpha = \alpha \cap C\text{‘}P - \breve{P}\text{“}\alpha,$$

i.e. $\overrightarrow{\min_P}\text{‘}\alpha$ consists of those members of $\alpha \cap C\text{‘}P$ which are not preceded by any other members of α. (If α has a single first term, this term is $\min_P\text{‘}\alpha$.) Thus we have, when T is a power of P,

$$\overrightarrow{\min_P}\text{‘}\text{Ɑ}\text{‘}T = \text{Ɑ}\text{‘}T - \breve{P}\text{“}\text{Ɑ}\text{‘}T.$$

Thus $\overrightarrow{\min_P}\text{‘}\text{Ɑ}\text{‘}T$, where T is any power of P (including $I \upharpoonright C\text{‘}P$), is the generation of P corresponding to T; thus the whole class of generations is $\overrightarrow{\min_P}\text{“}\text{Ɑ}\text{“}\text{Potid}\text{‘}P$. Hence we put

$$\text{gen}\text{‘}P = \overrightarrow{\min_P}\text{“}\text{Ɑ}\text{“}\text{Potid}\text{‘}P \quad \text{Df,}$$

where "gen" stands for "generation."

The notation "$\min_P$" will not be much used until we come to series, but then it will be constantly used. At present, we shall only give such properties of $\min_P$ as are necessary for our immediate purposes, but in Part V (on series) we shall devote a number (*205) to its properties.

In this number we also introduce the notation "xBP" for "$x \in \text{D}\text{‘}P - \text{Ɑ}\text{‘}P$." "$xBP$" may be read "$x$ begins P." If there is a single beginning of P, this is $B\text{‘}P$; otherwise the class of beginnings is $\overrightarrow{B}\text{‘}P$, which $= \text{D}\text{‘}P - \text{Ɑ}\text{‘}P$. Thus if P is the relation of father and son, $B\text{‘}P =$ Adam; if P is the relation of parent and child, $\overrightarrow{B}\text{‘}P =$ Adam and Eve. $B\text{‘}\breve{P}$ will be the end of P, if there is one; generally, $\overrightarrow{B}\text{‘}\breve{P}$ will be the class of ends, *i.e.* $\text{Ɑ}\text{‘}P - \text{D}\text{‘}P$. The first generation of P is $\overrightarrow{B}\text{‘}P$. If $P \in 1 \rightarrow \text{Cls}$, any generation of P is $\breve{T}\text{“}\overrightarrow{B}\text{‘}P$, where T is the corresponding power of P.

The field of a relation consists, in general, not only of the generations of P, but also of another part, the part in which, however far we go backwards, we never reach a beginning. This part is $p\text{‘}\text{Ɑ}\text{“}\text{Pot}\text{‘}P$. The two parts $s\text{‘}\text{gen}\text{‘}P$ and $p\text{‘}\text{Ɑ}\text{“}\text{Pot}\text{‘}P$ are mutually exclusive, and together exhaust $C\text{‘}P$.

The two next numbers, ∗94 and ∗95, are hardly ever relevant in subsequent propositions, and may therefore be omitted by any reader who is not interested in their subject-matter. ∗94 deals with powers of relative products. It is only used in the following number (∗95), on "equi-factor relations." The matter to be dealt with in this number (∗95) may be explained as follows. In dealing with correlations and similar topics, we often wish to consider the series of relations

$$R,\ P \mid R \mid Q,\ P^2 \mid R \mid Q^2,\ P^3 \mid R \mid Q^3, \text{ etc.}$$

Now we have not yet at our command a definition of P^ν, where ν is any finite number; thus we cannot define a general term of this series as $P^\nu \mid R \mid Q^\nu$. We need therefore a different method of definition. We have

$$P \mid R \mid Q = (P \parallel Q)\text{‘}R, \quad P^2 \mid R \mid Q^2 = (P \parallel Q)^2\text{‘}R,$$

and so on. Thus if T is any power of $(P \mid) \mid (\mid Q)$, a general term of our series is $T\text{‘}R$. For convenience of notation, we put

$$P{*}Q = \text{sg}\text{‘}(P \parallel Q)_* \quad \text{Dft.}$$

Then our series consists of $(P{*}Q)\text{‘}R$. The sum of all relations of this class is considered in this number.

The principal propositions proved in ∗94 and ∗95 are two which have the same hypothesis as the Schröder-Bernstein theorem, namely

$$R, S \in 1 \rightarrow 1 \,.\, \text{Ɑ}\text{‘}S \subset \text{D}\text{‘}R \,.\, \text{Ɑ}\text{‘}R \subset \text{D}\text{‘}S.$$

These two propositions state that, with the above hypothesis,

$$s\text{‘}\text{gen}\text{‘}(R \mid S) \text{ sm } s\text{‘}\text{gen}\text{‘}(S \mid R)$$

and $$p\text{‘}\text{Ɑ}\text{“}\text{Pot}\text{‘}(R \mid S) \text{ sm } p\text{‘}\text{Ɑ}\text{“}\text{Pot}\text{‘}(S \mid R).$$

The two combined reconstitute the Schröder-Bernstein theorem, since

$$s\text{‘}\text{gen}\text{‘}(R \mid S) \cup p\text{‘}\text{Ɑ}\text{“}\text{Pot}\text{‘}(R \mid S) = \text{D}\text{‘}R$$

and $$s\text{‘}\text{gen}\text{‘}(S \mid R) \cup p\text{‘}\text{Ɑ}\text{“}\text{Pot}\text{‘}(S \mid R) = \text{D}\text{‘}S.$$

Thus they present, so to speak, an itemized account of the equality proved by the Schröder-Bernstein theorem.

*96, on the posterity of a term, is concerned with the properties of $\overleftarrow{R_*}\text{‘}x$, chiefly when $R \in \mathrm{Cls} \to 1$. In this case, in general, $\overleftarrow{R_*}\text{‘}x$ consists of two parts, first an open series and then a cyclic series. Either of these may vanish, or may reduce to a single term. If we call the two parts β and γ, the whole of β precedes the whole of γ, and $\beta \upharpoonleft R$, $\gamma \upharpoonleft R \in 1 \to 1$. Thus if either β or γ vanishes, $\overleftarrow{R_*}\text{‘}x \upharpoonleft R \in 1 \to 1$. If γ vanishes, the series never returns into itself, that is, $\overleftarrow{R_*}\text{‘}x \upharpoonleft R_{\mathrm{po}} \mathrel{\subset\!\!\!\cdot} J$. If γ exists, there is a definite power of R, say T, such that $y \in \gamma \,.\supset_y .\, yTy$. If β and γ both exist, there is one term, namely the successor of the last term of β, which has just two immediate predecessors, one in β and one in γ; every other term of $\overleftarrow{R_{\mathrm{po}}}\text{‘}x$ has only one immediate predecessor in $\overleftarrow{R_*}\text{‘}x$. Thus $\overleftarrow{R_*}\text{‘}x$ is shaped like a Q, with x at the tip of the tail.

*97 deals with the analysis of the field of a relation into families. Taking any member x of $C\text{‘}R$, the family of x with respect to R is $\overrightarrow{R_*}\text{‘}x \cup \overleftarrow{R_*}\text{‘}x$, which we write $\overleftrightarrow{R_*}\text{‘}x$. Thus the class of families is $\overleftrightarrow{R_*}\text{“}C\text{‘}R$. Those families which contain a member of $\overrightarrow{B}\text{‘}R$ are $\overleftarrow{R_*}\text{“}\overrightarrow{B}\text{‘}R$. If we regard $\breve{R}_*\text{“}\overrightarrow{B}\text{‘}R$ as arranged in a rectangle, in which the generations are the successive rows, then $\overleftarrow{R_*}\text{“}\overrightarrow{B}\text{‘}R$ will be the columns. Thus the relation of $\mathrm{gen}\text{‘}R$ to $\overleftarrow{R_*}\text{“}\overrightarrow{B}\text{‘}R$ may be regarded as a generalized form of the relation of rows and columns. Under a suitable hypothesis, each row is a selection from the columns, and each column a selection from the rows. This is expressed in the following proposition:

$$\vdash : R \in 1 \to 1 \,.\, \overrightarrow{B}\text{‘}\breve{R} \in \mathrm{gen}\text{‘}R \cup \iota\text{‘}\Lambda \,.\supset .$$
$$\overleftarrow{R_*}\text{“}\overrightarrow{B}\text{‘}R \subset \mathrm{D}\text{“}\epsilon_\Delta\text{‘}(\mathrm{gen}\text{‘}R - \iota\text{‘}\Lambda) \,.\, \mathrm{gen}\text{‘}R - \iota\text{‘}\Lambda \subset \mathrm{D}\text{“}\epsilon_\Delta\text{‘}\overleftarrow{R_*}\text{“}\overrightarrow{B}\text{‘}R$$

whence we derive existence-theorems for selections in the cases concerned.

The importance of the ideas dealt with in the present section is very great. These ideas dominate the treatment of finite and infinite, the theory of progressions and $\aleph_0$, and the transition from series generated by one-one or many-one relations of consecutive terms to series generated by transitive relations of *before* and *after*. Wherever, in short, mathematical induction is used the ideas treated in this section are required. The portions of our subsequent work in which this section is most referred to are the two sections on finite and infinite cardinals and ordinals (Part III, Section C and Part V, Section E). In the general theory of cardinals, *i.e.* in Part III, Sections A and B, before the distinction of finite and infinite has been introduced, the present section will be seldom if ever referred to*.

* The present section is based on the work of Frege, who first defined the ancestral relation. See his *Begriffsschrift* (Halle, 1879) Part III., pp. 55—87. Cf. also his *Grundgesetze der Arithmetik*, Vol. I. (Jena, 1893), §§ 45, 46 (pp. 59, 60). In this work the ancestral relation is used to prove the properties of finite cardinals and $\aleph_0$.

$*90$. ON THE ANCESTRAL RELATION

Summary of $*90$.

If R is any relation, "xR_*y" is to mean "x is an ancestor of y with respect to R," where a term counts as its own ancestor provided it belongs to the field of R. The definition of R_* is as follows:

$*90·01$. $R_* = \hat{x}\hat{y}\{x \in C\text{‘}R : \breve{R}\text{“}\mu \subset \mu \,.\, x \in \mu \,.\, \supset_\mu \,.\, y \in \mu\}$ Df

That is, xR_*y is to hold when x belongs to the field of R, and y belongs to every hereditary class to which x belongs; a hereditary class being a class μ such that $\breve{R}\text{“}\mu \subset \mu$, *i.e.* such that all successors of μ's are μ's.

$*90·02$. $\breve{R}_* = \mathrm{Cnv}\text{‘}R_*$ Df

This definition serves merely to decide the ambiguity between $(\breve{R})_*$ and $\mathrm{Cnv}\text{‘}R_*$, either of which might be meant of $\breve{R}_*$. It will be shown, however, that the two are equal ($*90·132$).

The most important propositions of this number are the following:

$*90·112$. $\vdash :. xR_*y : \phi z \,.\, zRw \,.\, \supset_{z,w} \,.\, \phi w : \phi x : \supset \,.\, \phi y$

I.e. if xR_*y and if $\phi\hat{z}$ is a hereditary property belonging to x, then it belongs to y.

$*90·12$. $\vdash : x \in C\text{‘}R \,.\, \equiv \,.\, xR_*x$

I.e. R_* is reflexive throughout the field of R, but not elsewhere.

$*90·14$. $\vdash \,.\, \mathrm{D}\text{‘}R_* = \text{Ɑ}\text{‘}R_* = C\text{‘}R_* = C\text{‘}R$

$*90·15$. $\vdash \,.\, I \upharpoonright C\text{‘}R \Subset R_*$

$*90·151$. $\vdash \,.\, R \Subset R_*$

$*90·16$. $\vdash \,.\, R_* \,|\, R \Subset R_*$

$*90·163$. $\vdash \,.\, \breve{R}\text{“}\overleftarrow{R_*}\text{‘}x \subset \overleftarrow{R_*}\text{‘}x$

I.e. $\overleftarrow{R_*}\text{‘}x$ is a hereditary class.

$*90·17$. $\vdash \,.\, R_*^2 = R_*$

$*90·21$. $\vdash : \alpha \subset C\text{‘}R \,.\, \equiv \,.\, \alpha \subset \breve{R}_*\text{“}\alpha \,.\, \equiv \,.\, \alpha \subset R_*\text{“}\alpha$

$*90·22$. $\vdash : \breve{R}\text{“}\alpha \subset \alpha \,.\, \equiv \,.\, \breve{R}_*\text{“}\alpha \subset \alpha$

I.e. the classes that are hereditary with respect to R are the same as those that are hereditary with respect to R_*.

$*90·31$. $\vdash \,.\, R_* = I \upharpoonright C\text{‘}R \cup R_* \,|\, R$

$*90·32$. $\vdash \,.\, R \,|\, R_* = R \cup R \,|\, R_* \,|\, R = R_* \,|\, R$

$*90·33$. $\vdash \,.\, R_*\text{“}\alpha = (\alpha \cap C\text{‘}R) \cup R_*\text{“}R\text{“}\alpha = (\alpha \cap C\text{‘}R) \cup R\text{“}R_*\text{“}\alpha$

$*90·4$. $\vdash \,.\, (R_*)_* = R_*$

***90·01.** $R_* = \hat{x}\hat{y}\{x \in C'R : \breve{R}``\mu \subset \mu \,.\, x \in \mu \,.\, \supset_\mu \,.\, y \in \mu\}$ Df

***90·02.** $\breve{R}_* = \text{Cnv}'R_*$ Df

***90·1.** $\vdash :. \, xR_*y \,.\, \equiv : x \in C'R : \breve{R}``\mu \subset \mu \,.\, x \in \mu \,.\, \supset_\mu \,.\, y \in \mu$ [*21·3 . (*90·01)]

***90·101.** $\vdash : \breve{R}``\mu \subset \mu \,.\, \equiv \,.\, R`` - \mu \subset - \mu$

Dem.

$\vdash \,.\, *37\cdot171 \,.\, \supset \vdash :. \, \breve{R}``\mu \subset \mu \,.\, \equiv : x \in \mu \,.\, xRy \,.\, \supset_{x,y} \,.\, y \in \mu :$

[Transp] $\equiv : y \in -\mu \,.\, xRy \,.\, \supset_{x,y} \,.\, x \in -\mu :$

[*37·17] $\equiv : R`` - \mu \subset - \mu :. \supset \vdash \,.\, \text{Prop}$

*90·102 is a lemma for *90·11.

***90·102.** $\vdash :. \, \breve{R}``\mu \subset \mu \,.\, x \in \mu \,.\, \supset_\mu \,.\, y \in \mu : \equiv : R``\mu \subset \mu \,.\, y \in \mu \,.\, \supset_\mu \,.\, x \in \mu$

Dem.

$\vdash \,.\, *90\cdot101 \,.\, \supset$

$\vdash :. \, \breve{R}``\mu \subset \mu \,.\, x \in \mu \,.\, \supset \,.\, y \in \mu : \equiv : R`` - \mu \subset - \mu \,.\, x \in \mu \,.\, \supset \,.\, y \in \mu :$

[Transp] $\equiv : R`` - \mu \subset - \mu \,.\, y \in - \mu \,.\, \supset \,.\, x \in - \mu$ (1)

$\vdash \,.\, (1) \,.\, *10\cdot11\cdot271 \,.\, \supset$

$\vdash : \breve{R}``\mu \subset \mu \,.\, x \in \mu \,.\, \supset_\mu \,.\, y \in \mu : \equiv : R`` - \mu \subset - \mu \,.\, y \in - \mu \,.\, \supset_\mu \,.\, x \in - \mu :$

[*22·94] $\equiv : R``\mu \subset \mu \,.\, y \in \mu \,.\, \supset_\mu \,.\, x \in \mu :. \supset \vdash \,.\, \text{Prop}$

***90·11.** $\vdash :. \, xR_*y \,.\, \equiv : x \in C'R : R``\mu \subset \mu \,.\, y \in \mu \,.\, \supset_\mu \,.\, x \in \mu$ [*90·1·102]

***90·111.** $\vdash :: xR_*y \,.\, \equiv :. \, x \in C'R :. \, z \in \mu \,.\, zRw \,.\, \supset_{z,w} \,.\, w \in \mu : x \in \mu : \supset_\mu \,.\, y \in \mu$ [*90·1 . *37 171]

***90·112.** $\vdash :. \, xR_*y : \phi z \,.\, zRw \,.\, \supset_{z,w} \,.\, \phi w : \phi x : \supset \,.\, \phi y$

Dem.

$\vdash \,.\, *90\cdot111 \dfrac{\hat{z}(\phi z)}{\mu} \,.\, \supset$

$\vdash :: xR_*y \,.\, \supset :. \, z \in \hat{z}(\phi z) \,.\, zRw \,.\, \supset_{z,w} \,.\, w \in \hat{z}(\phi z) : x \in \hat{z}(\phi z) : \supset : y \in \hat{z}(\phi z) :.$

[*20·3] $\supset :. \, \phi z \,.\, zRw \,.\, \supset_{z,w} \,.\, \phi w : \phi x : \supset \,.\, \phi y$ (1)

$\vdash \,.\, (1) \,.\, \text{Imp} \,.\, \supset \vdash \,.\, \text{Prop}$

***90·12.** $\vdash : x \in C'R \,.\, \equiv \,.\, xR_*x$

Dem.

$\vdash \,.\, *90\cdot1 \,.\, \supset \vdash : xR_*x \,.\, \supset \,.\, x \in C'R$ (1)

$\vdash \,.\, *3\cdot27 \,.\, *10\cdot11 \,.\, \supset \vdash : \breve{R}``\mu \subset \mu \,.\, x \in \mu \,.\, \supset_\mu \,.\, x \in \mu :$

[*3·21] $\supset \vdash :. \, x \in C'R \,.\, \supset : x \in C'R : \breve{R}``\mu \subset \mu \,.\, x \in \mu \,.\, \supset_\mu \,.\, x \in \mu :$

[*90·1] $\supset : xR_*x$ (2)

$\vdash \,.\, (1) \,.\, (2) \,.\, \supset \vdash \,.\, \text{Prop}$

***90·13.** $\vdash : xR_*y \,.\, \supset \,.\, x, y \in C`R \,.\, xR_*x \,.\, yR_*y$

Dem.

$\vdash \,.\, *37\cdot16 \,.\, *33\cdot161 \,.\, \supset \vdash \,.\, \breve{R}``C`R \subset C`R$ (1)

$\vdash \,.\, *90\cdot1 \,.\, \qquad \supset \vdash : xR_*y \,.\, \supset \,.\, x \in C`R$ (2)

$\vdash \,.\, *90\cdot1 \dfrac{C`R}{\mu} \,.\, \qquad \supset \vdash :. \, xR_*y \,.\, \supset : \breve{R}``C`R \subset C`R \,.\, x \in C`R \,.\, \supset \,.\, y \in C`R :$

[(1).(2)] $\qquad \supset : y \in C`R$ (3)

$\vdash \,.\, (2) \,.\, (3) \,.\, *90\cdot12 \,.\, \supset \vdash : xR_*y \,.\, \supset \,.\, xR_*x \,.\, yR_*y$ (4)

$\vdash \,.\, (2) \,.\, (3) \,.\, (4) \,.\, \supset \vdash \,.\,$ Prop

The following proposition is a lemma for *90·132.

***90·131.** $\vdash :. \, xR_*y \,.\, \equiv : y \in C`R : R``\mu \subset \mu \,.\, y \in \mu \,.\, \supset_\mu \,.\, x \in \mu$

Dem.

$\vdash \,.\, *90\cdot11\cdot13 \,.\, \supset$

$\vdash :. \, xR_*y \,.\, \supset : y \in C`R : R``\mu \subset \mu \,.\, y \in \mu \,.\, \supset_\mu \,.\, x \in \mu$ (1)

$\vdash \,.\, *37\cdot15 \,.\, *33\cdot161 \,.\, \supset \vdash \,.\, R``C`R \subset C`R$ (2)

$\vdash \,.\, *10\cdot1 \,.\, \supset \vdash :. \, y \in C`R : R``\mu \subset \mu \,.\, y \in \mu \,.\, \supset_\mu \,.\, x \in \mu : \supset :$

$y \in C`R : R``C`R \subset C`R \,.\, y \in C`R \,.\, \supset \,.\, x \in C`R :$

[*5·33] $\qquad \supset : R``C`R \subset C`R \,.\, \supset \,.\, x \in C`R :$

[(2)] $\qquad \supset : x \in C`R$ (3)

$\vdash \,.\, (3) \,.\, *5\cdot3 \,.\, \supset \vdash :. \, y \in C`R : R``\mu \subset \mu \,.\, y \in \mu \,.\, \supset_\mu \,.\, x \in \mu : \supset :$

$x \in C`R : R``\mu \subset \mu \,.\, y \in \mu \,.\, \supset_\mu \,.\, x \in \mu :$

[*90·11] $\qquad \supset : xR_*y$ (4)

$\vdash \,.\, (1) \,.\, (4) \,.\, \supset \vdash \,.\,$ Prop

***90·132.** $\vdash \,.\, (\breve{R})_* = \breve{R}_*$

Dem.

$\vdash \,.\, *31\cdot33 \,.\, *33\cdot22 \,.\, *90\cdot1 \,.\, \supset$

$\vdash :. \, y(\breve{R})_* x \,.\, \equiv : y \in C`R : R``\mu \subset \mu \,.\, y \in \mu \,.\, \supset_\mu \,.\, x \in \mu :$

[*90·131] $\qquad \equiv : xR_*y :$

[*31·11] $\qquad \equiv : y\breve{R}_*x :. \supset \vdash \,.\,$ Prop

In accordance with our general convention as regards suffixes, and with the definition *90·02, $\breve{R}_*$ means $\text{Cnv}`R_*$, not $(\breve{R})_*$.

***90·14.** $\vdash \,.\, \text{D}`R_* = \text{Ɑ}`R_* = C`R_* = C`R$

Dem.

$\vdash \,.\, *90\cdot12 \,.\, *33\cdot14\cdot17 \,.\, \supset \vdash : x \in C`R \,.\, \supset \,.\, x \in \text{D}`R_* \,.\, x \in \text{Ɑ}`R_* \,.\, x \in C`R_*$ (1)

$\vdash \,.\, *33\cdot13 \,.\, \qquad \supset \vdash : x \in \text{D}`R_* \,.\, \equiv \,.\, (\exists y) \,.\, xR_*y \,.$

[*90·13] $\qquad \supset \,.\, x \in C`R$ (2)

Similarly $\qquad \vdash : x \in \text{Ɑ}`R_* \,.\, \supset \,.\, x \in C`R$ (3)

$\vdash \,.\, (2) \,.\, (3) \,.\, *33\cdot16 \,.\, \quad \supset \vdash : x \in C`R_* \,.\, \supset \,.\, x \in C`R$ (4)

$\vdash \,.\, (1) \,.\, (2) \,.\, (3) \,.\, (4) \,.\, \supset \vdash \,.\,$ Prop

✱90·141. $\vdash : \dot{\exists} \,! R_* . \equiv . \dot{\exists}\,! R$ [✱90·14 . ✱33·24]

✱90·15. $\vdash . I \upharpoonright C`R \Subset R_*$

Dem.

$\vdash . ✱50·1 . ✱35·101 . \supset \vdash : x (I \upharpoonright C`R) y . \equiv . x = y . y \in C`R .$
[✱90·12] $\equiv . x = y . y R_* y .$
[✱13·13] $\supset . x R_* y : \supset \vdash .$ Prop

Note that $I \upharpoonright C`R$ may be conveniently regarded as the 0th power of R. By ✱50·64·65, when multiplied by R it gives R; also it is contained in $R \,|\, \breve{R}$, $R^2 \,|\, \breve{R}^2$, etc. I has properties, as regards relational multiplication, analogous to those of 1 in ordinary multiplication; thus to regard $I \upharpoonright C`R$ as the 0th power of R is analogous to regarding 1 as the 0th power of n, where n is a number.

✱90·151. $\vdash . R \Subset R_*$

Dem.

$\vdash . ✱11·1 . \supset \vdash :: z \in \mu . zRw . \supset_{z,w} . w \in \mu : \supset :. x \in \mu . xRy . \supset . y \in \mu :.$
[Exp.Comm] $\supset :. xRy . \supset : x \in \mu . \supset . y \in \mu$ (1)
$\vdash . (1) .$ Comm . Imp $. \supset$
$\vdash :: xRy . \supset :. z \in \mu . zRw . \supset_{z,w} . w \in \mu : x \in \mu : \supset . y \in \mu$ (2)
$\vdash . (2) . ✱10·11·21 . \supset$
$\vdash :: xRy . \supset :. z \in \mu . zRw . \supset_{z,w} . w \in \mu : x \in \mu : \supset_\mu . y \in \mu :.$
[✱90·111.✱33·17] $\supset :. x R_* y :: \supset \vdash .$ Prop

✱90·16. $\vdash . R_* \,|\, R \Subset R_*$

Dem.

$\vdash . ✱11·1 . \supset \vdash :. z \in \mu . zRw . \supset_{z,w} . w \in \mu : \supset : y \in \mu . yRv . \supset . v \in \mu$ (1)
$\vdash . ✱90·111 . ✱10·1 .$ Fact $. \supset$
$\vdash :: x R_* y . yRv . \supset :. z \in \mu . zRw . \supset_{z,w} . w \in \mu : x \in \mu : \supset . y \in \mu . yRv$ (2)
$\vdash . (1) . (2) . \supset$
$\vdash :: x R_* y . yRv . \supset :. z \in \mu . zRw . \supset_{z,w} . w \in \mu : x \in \mu : \supset . v \in \mu$ (3)
$\vdash . (3) . ✱10·11·21 . ✱90·111 . \supset$
$\vdash : x R_* y . yRv . \supset . x R_* v$ (4)
$\vdash . (4) . ✱10·11·23 . ✱34·1 . \supset \vdash .$ Prop

✱90·161. $\vdash : S \Subset R_* . \supset . S \,|\, R \Subset R_*$

Dem.

$\vdash . ✱34·34 . \supset \vdash :$ Hp $. \supset . S \,|\, R \Subset R_* \,|\, R$ (1)
$\vdash . (1) . ✱90·16 . \supset \vdash .$ Prop

✱90·162. $\vdash . R^2 \Subset R_*$ [✱90·151·161]

✱90·163. $\vdash . \breve{R}``\overleftarrow{R_*}`x \subset \overleftarrow{R_*}`x$ [✱37·301 . ✱32·19 . ✱90·16]

This proposition is important, since it proves that $\overleftarrow{R_*}`x$ is a hereditary class.

***90·164.** $\vdash . \breve{R}``\breve{R}_*``\alpha \subset \breve{R}_*``\alpha$ [*37·33·201 . *90·16]

This proposition shows that $\breve{R}_*``\alpha$ is a hereditary class.

***90·17.** $\vdash . R_*^2 = R_*$

Note that R_*^2 means $(R_*)^2$, not $(R^2)_*$.

Dem.

$\vdash . *90·13 . \quad \supset \vdash : xR_*y . \supset . xR_*y . yR_*y .$

$[*34·5 . *10·24] \quad \supset . xR_*^2 y \quad (1)$

$\vdash . *90·163·1 \dfrac{\overleftarrow{R}`x}{\mu} . \supset \vdash :. yR_*z . \supset : y \,\epsilon\, \overleftarrow{R}_*`x . \supset . z \,\epsilon\, \overleftarrow{R}_*`x :$

$[*32·181] \quad \supset : xR_*y . \supset . xR_*z \quad (2)$

$\vdash . (2) . \mathrm{Imp} . \quad \supset \vdash : xR_*y . yR_*z . \supset . xR_*z :$

$[*11·11 . *34·55] \quad \supset \vdash : R_*^2 \mathrel{\dot\subset} R_* \quad (3)$

$\vdash . (1) . (3) . \supset \vdash . \mathrm{Prop}$

***90·171.** $\vdash . \breve{R}_*``\breve{R}_*``\alpha = \breve{R}_*``\alpha$ [*90·17 . *37·33]

***90·172.** $\vdash . R \,|\, R_* \mathrel{\dot\subset} R_*$

Dem.

$\vdash . *90·151 . \supset \vdash . R \,|\, R_* \mathrel{\dot\subset} R_*^2 \quad (1)$

$\vdash . (1) . *90·17 . \supset \vdash . \mathrm{Prop}$

***90·18.** $\vdash : P \mathrel{\dot\subset} Q . \supset . P_* \mathrel{\dot\subset} Q_*$

Dem.

$\vdash . *33·265 . \supset \vdash :. \mathrm{Hp} . \supset : x \,\epsilon\, C`P . \supset . x \,\epsilon\, C`Q \quad (1)$

$\vdash . *37·201 . \supset \vdash :: \mathrm{Hp} . \supset :. \breve{P}``\mu \subset \breve{Q}``\mu :.$

$[*22·44] \quad \supset :. \breve{Q}``\mu \subset \mu . \supset . \breve{P}``\mu \subset \mu :.$

$[\mathrm{Fact}] \quad \supset :. \breve{Q}``\mu \subset \mu . x \,\epsilon\, \mu . \supset . \breve{P}``\mu \subset \mu . x \,\epsilon\, \mu :.$

$[\mathrm{Syll}] \quad \supset :. \breve{P}``\mu \subset \mu . x \,\epsilon\, \mu . \supset . y \,\epsilon\, \mu : \supset : \breve{Q}``\mu \subset \mu . x \,\epsilon\, \mu . \supset . y \,\epsilon\, \mu \quad (2)$

$\vdash . (2) . *10·11·21·27 . \supset$

$\vdash :: \mathrm{Hp} . \supset :. \breve{P}``\mu \subset \mu . x \,\epsilon\, \mu . \supset_\mu . y \,\epsilon\, \mu : \supset : \breve{Q}``\mu \subset \mu . x \,\epsilon\, \mu . \supset_\mu . y \,\epsilon\, \mu \quad (3)$

$\vdash . (1) . (3) . *90·1 . \supset \vdash :. \mathrm{Hp} . \supset : xP_*y . \supset . xQ_*y :. \supset \vdash . \mathrm{Prop}$

***90·21.** $\vdash : \alpha \subset C`R . \equiv . \alpha \subset \breve{R}_*``\alpha . \equiv . \alpha \subset R_*``\alpha$

Dem.

$\vdash . *4·7 . \supset \vdash :. \alpha \subset C`R . \supset : x \,\epsilon\, \alpha . \supset . x \,\epsilon\, \alpha . x \,\epsilon\, C`R .$

$[*90·12] \quad \supset . x \,\epsilon\, \alpha . xR_*x .$

$[*10·24 . *37·1·105] \quad \supset . x \,\epsilon\, \breve{R}_*``\alpha . x \,\epsilon\, R_*``\alpha \quad (1)$

$\vdash . *37·16 . \supset \vdash : \alpha \subset \breve{R}_*``\alpha . \supset . \alpha \subset \text{Ɑ}`R_* .$

$[*90·14] \quad \supset . \alpha \subset C`R \quad (2)$

$\vdash . *37·15 . *90·14 . \supset \vdash : \alpha \subset R_*``\alpha . \supset . \alpha \subset C`R \quad (3)$

$\vdash . (1) . (2) . (3) . \supset \vdash . \mathrm{Prop}$

***90·22.** $\vdash : \breve{R}\text{“}\alpha \subset \alpha \,.\equiv .\, \breve{R}_*\text{“}\alpha \subset \alpha$

Dem.

$\vdash .\, *90\cdot1 \,.\supset \vdash :.\, xR_*y \,.\supset_{x,y} : \breve{R}\text{“}\alpha \subset \alpha \,.\, x \epsilon \alpha \,.\supset .\, y \epsilon \alpha :.$

$[\text{Comm}] \quad \supset \vdash :.\, \breve{R}\text{“}\alpha \subset \alpha \,.\supset : xR_*y \,.\, x \epsilon \alpha \,.\supset_{x,y} .\, y \epsilon \alpha :$

$[*37\cdot171] \qquad \supset : \breve{R}_*\text{“}\alpha \subset \alpha \qquad (1)$

$\vdash .\, *90\cdot151 \,.\, *37\cdot201 \,.\supset \vdash .\, \breve{R}\text{“}\alpha \subset \breve{R}_*\text{“}\alpha \,.$

$[*22\cdot44] \qquad \supset \vdash : \breve{R}_*\text{“}\alpha \subset \alpha \,.\supset .\, \breve{R}\text{“}\alpha \subset \alpha \qquad (2)$

$\vdash .\, (1) \,.\, (2) \,.\supset \vdash .\, \text{Prop}$

***90·23.** $\vdash : \alpha \subset C\text{‘}R \,.\, \breve{R}\text{“}\alpha \subset \alpha \,.\equiv .\, \alpha = \breve{R}_*\text{“}\alpha \quad [*90\cdot21\cdot22]$

*90·23 is useful in the theory of sections of a series (*211). A section of the series generated by R is defined as a class α satisfying

$$\alpha \subset C\text{‘}R \,.\, R\text{“}\alpha \subset \alpha.$$

***90·24.** $\vdash : \breve{R}\text{“}\mu \subset \mu \,.\, \alpha \subset \mu \,.\supset .\, \breve{R}_*\text{“}\alpha \subset \mu$

Dem.

$\vdash .\, *37\cdot2 \,. \quad \supset \vdash : \text{Hp} \,.\supset .\, \breve{R}_*\text{“}\alpha \subset \breve{R}_*\text{“}\mu \qquad (1)$

$\vdash .\, *90\cdot22 \,. \quad \supset \vdash : \text{Hp} \,.\supset .\, \breve{R}_*\text{“}\mu \subset \mu \qquad (2)$

$\vdash .\, (1) \,.\, (2) \,.\supset \vdash .\, \text{Prop}$

This proposition shows that if μ is a hereditary class which contains α, then μ contains all the descendants of α's.

***90·25.** $\vdash : \alpha \subset C\text{‘}R \,.\, \breve{R}_*\text{“}\alpha \subset \mu \,.\supset .\, \alpha \subset \mu$

Dem.

$\vdash .\, *90\cdot21 \,.\supset \vdash : \text{Hp} \,.\supset .\, \alpha \subset \breve{R}_*\text{“}\alpha \,.$

$[\text{Hp}] \qquad \supset .\, \alpha \subset \mu : \supset \vdash .\, \text{Prop}$

***90·26.** $\vdash :.\, \alpha \subset C\text{‘}R \,.\, \breve{R}\text{“}\mu \subset \mu \,.\supset : \alpha \subset \mu \,.\equiv .\, \breve{R}_*\text{“}\alpha \subset \mu$

Dem.

$\vdash .\, *90\cdot24 \,. \quad \supset \vdash :.\, \text{Hp} \,.\supset : \alpha \subset \mu \,.\supset .\, \breve{R}_*\text{“}\alpha \subset \mu \qquad (1)$

$\vdash .\, *90\cdot25 \,. \quad \supset \vdash :.\, \text{Hp} \,.\supset : \breve{R}_*\text{“}\alpha \subset \mu \,.\supset .\, \alpha \subset \mu \qquad (2)$

$\vdash .\, (1) \,.\, (2) \,.\supset \vdash .\, \text{Prop}$

***90·27.** $\vdash :.\, \alpha \subset C\text{‘}R \,.\supset : \alpha \cup \breve{R}\text{“}\mu \subset \mu \,.\equiv .\, \breve{R}_*\text{“}\alpha \cup \breve{R}\text{“}\mu \subset \mu$

Dem.

$\vdash .\, *90\cdot26 \,.\, \text{Exp} \,.\, *5\cdot32 \,.\supset$

$\vdash :.\, \alpha \subset C\text{‘}R \,.\supset : \breve{R}\text{“}\mu \subset \mu \,.\, \alpha \subset \mu \,.\equiv .\, \breve{R}\text{“}\mu \subset \mu \,.\, \breve{R}_*\text{“}\alpha \subset \mu :$

$[*22\cdot59] \qquad \supset : \alpha \cup \breve{R}\text{“}\mu \subset \mu \,.\equiv .\, \breve{R}_*\text{“}\alpha \cup \breve{R}\text{“}\mu \subset \mu :.\supset \vdash .\, \text{Prop}$

***90·31.** $\vdash .\, R_* = I \upharpoonright C\text{‘}R \mathbin{\dot\cup} R_* \mid R$

Dem.

$\vdash .\, *90\cdot15\cdot16 \,.\supset \vdash .\, I \upharpoonright C\text{‘}R \mathbin{\dot\cup} R_* \mid R \mathbin{\dot\subset} R_* \qquad (1)$

$$
\begin{array}{lll}
[\text{Fact}] & \supset \vdash : x(I \upharpoonright C`R \cup R_* \mid R)z \,.\, zRw \,.\supset .\, xR_*z \,.\, zRw \,. & \\
[*10\cdot24.*34\cdot1] & \supset .\, x(R_* \mid R)w \,. & \\
[*23\cdot58] & \supset .\, x(I \upharpoonright C`R \cup R_* \mid R)w & (2) \\
\vdash .\, *90\cdot13 \,.\, *50\cdot3 \,.\supset \vdash : xR_*y \,.\supset .\, xIx \,.\, x \in C`R \,. & & \\
[*35\cdot101] & \supset .\, x(I \upharpoonright C`R)x \,. & \\
[*23\cdot58] & \supset .\, x(I \upharpoonright C`R \cup R_* \mid R)x \,. & \\
[*4\cdot7] & \supset .\, xR_*y \,.\, x(I \upharpoonright C`R \cup R_* \mid R)x & (3)
\end{array}
$$

$$\vdash .\, (2) \,.\, (3) \,.\, *90\cdot112 \frac{x(I \upharpoonright C`R \cup R_* \mid R)z}{\phi z} \,.\supset$$

$$\vdash : xR_*y \,.\supset .\, x(I \upharpoonright C`R \cup R_* \mid R)y \qquad (4)$$

$\vdash .\, (1) \,.\, (4) \,.\supset \vdash .$ Prop

In the last line of the above proof, the process is as follows. Writing ϕz for $x(I \upharpoonright C`R \cup R_* \mid R)z$, (2) becomes $\phi z \,.\, zRw \,.\supset .\, \phi w$, while (3) becomes $xR_*y \,.\supset .\, xR_*y \,.\, \phi x$. Hence, by (2) and (3),

$$xR_*y \,.\supset : xR_*y : \phi z \,.\, zRw \,.\supset_{z,w} .\, \phi w : \phi x.$$

Hence, by $*90\cdot112$, $xR_*y \,.\supset .\, \phi y$, which is the proposition to be proved.

$*90\cdot311$. $\vdash .\, R_* = I \upharpoonright C`R \cup R \mid R_*$

Dem.

$$\vdash .\, *90\cdot31 \frac{\breve{R}}{R} \,.\, *90\cdot132 \,.\supset$$

$$
\begin{array}{lll}
\vdash .\, \breve{R}_* = I \upharpoonright C`\breve{R} \cup \breve{R}_* \mid \breve{R} & & \\
[*33\cdot22.*34\cdot2] & = I \upharpoonright C`R \cup \mathrm{Cnv}`(R \mid R_*) & \\
[*50\cdot5\cdot51] & = \mathrm{Cnv}`(I \upharpoonright C`R) \cup \mathrm{Cnv}`(R \mid R_*) & \\
[*31\cdot15] & = \mathrm{Cnv}`(I \upharpoonright C`R \cup R \mid R_*) & (1) \\
\vdash .\, (1) \,.\, *31\cdot32 \,.\supset \vdash .\, \text{Prop} & &
\end{array}
$$

$*90\cdot32$. $\vdash .\, R \mid R_* = R \cup R \mid R_* \mid R = R_* \mid R$ (2)

Dem.

$$
\begin{array}{lll}
\vdash .\, *90\cdot31 \,.\supset \vdash .\, R \mid R_* = R \mid I \upharpoonright C`R \cup R \mid R_* \mid R & & \\
[*50\cdot64] & = R \cup R \mid R_* \mid R & (1) \\
[*50\cdot65] & = (I \upharpoonright C`R) \mid R \cup R \mid R_* \mid R & \\
[*90\cdot311.*34\cdot26] & = R_* \mid R & (2) \\
\vdash .\, (1) \,.\, (2) \,.\supset \vdash .\, \text{Prop} & &
\end{array}
$$

$*90\cdot33$. $\vdash .\, R_*``\alpha = (\alpha \cap C`R) \cup R_*``R``\alpha = (\alpha \cap C`R) \cup R``R_*``\alpha$

Dem.

$$
\begin{array}{lll}
\vdash .\, *90\cdot31 \,.\, *37\cdot221 \,.\supset & & \\
\vdash .\, R_*``\alpha = (I \upharpoonright C`R)``\alpha \cup (R_* \mid R)``\alpha & & \\
[*37\cdot412\cdot33] = I``(C`R \cap \alpha) \cup R_*``R``\alpha & & \\
[*50\cdot16] & = (C`R \cap \alpha) \cup R_*``R``\alpha & (1)
\end{array}
$$

Similarly, by $*90\cdot311$,

$$\vdash .\, R_*``\alpha = (C`R \cap \alpha) \cup R``R_*``\alpha \qquad (2)$$

$\vdash .\, (1) \,.\, (2) \,.\supset \vdash .$ Prop

***90·331.** $\vdash . \breve{R}_*\text{“}\alpha = (\alpha \cap C\text{‘}R) \cup \breve{R}_*\text{“}\breve{R}\text{“}\alpha = (\alpha \cap C\text{‘}R) \cup \breve{R}\text{“}\breve{R}_*\text{“}\alpha$
[Proof as in *90·33]

***90·34.** $\vdash : \alpha \subset C\text{‘}R . \supset . R_*\text{“}\alpha = \alpha \cup R_*\text{“}R\text{“}\alpha = \alpha \cup R\text{“}R_*\text{“}\alpha$
[*90·33 . *22·621]

***90·341.** $\vdash : \alpha \subset C\text{‘}R . \supset . \breve{R}_*\text{“}\alpha = \alpha \cup \breve{R}_*\text{“}\breve{R}\text{“}\alpha = \alpha \cup \breve{R}\text{“}\breve{R}_*\text{“}\alpha$
[*90·331 . *22·621]

***90·35.** $\vdash :. xR \,|\, R_*z . \supset : \breve{R}\text{“}\mu \subset \mu . \overleftarrow{R}\text{‘}x \subset \mu . \supset_\mu . z \in \mu$

Dem.

$\vdash . *32{\cdot}181 . \supset \vdash :. xRy . \supset : y \in \overleftarrow{R}\text{‘}x :$

$[*22{\cdot}46] \quad \supset : \overleftarrow{R}\text{‘}x \subset \mu . \supset . y \in \mu :$

$[\text{Fact}] \quad \supset : \breve{R}\text{“}\mu \subset \mu . \overleftarrow{R}\text{‘}x \subset \mu . \supset . \breve{R}\text{“}\mu \subset \mu . y \in \mu \quad (1)$

$\vdash . *90{\cdot}1 . \quad \supset \vdash :. yR_*z . \supset : \breve{R}\text{“}\mu \subset \mu . y \in \mu . \supset . z \in \mu \quad (2)$

$\vdash . (1) . (2) . \supset \vdash :. xRy . yR_*z . \supset : \breve{R}\text{“}\mu \subset \mu . \overleftarrow{R}\text{‘}x \subset \mu . \supset . z \in \mu :.$

$[*10{\cdot}11{\cdot}23 . *34{\cdot}1] \supset \vdash :. xR \,|\, R_*z . \supset : \breve{R}\text{“}\mu \subset \mu . \overleftarrow{R}\text{‘}x \subset \mu . \supset . z \in \mu \quad (3)$

$\vdash . (3) . *10{\cdot}11{\cdot}21 . \supset \vdash . \text{Prop}$

***90·351.** $\vdash :. \breve{R}\text{“}\mu \subset \mu . \overleftarrow{R}\text{‘}x \subset \mu . \supset_\mu . z \in \mu : \supset . xR \,|\, R_*z$

Dem.

$\vdash . *90{\cdot}172 . \text{Fact} . \supset \vdash : xR \,|\, R_*z . zRw . \supset . xR_*z . zRw .$

$[*34{\cdot}1] \quad \supset . xR_* \,|\, Rw .$

$[*90{\cdot}32] \quad \supset . xR \,|\, R_*w \quad (1)$

$\vdash . (1) . *37{\cdot}171 . \supset \vdash . \breve{R}\text{“}\hat{z}(xR \,|\, R_*z) \subset \hat{z}(xR \,|\, R_*z) \quad (2)$

$\vdash . *90{\cdot}32 . \quad \supset \vdash : xRy . \supset . xR \,|\, R_*y :$

$[*32{\cdot}181 . *20{\cdot}3] \quad \supset \vdash : y \in \overleftarrow{R}\text{‘}x . \supset . y \in \hat{z}(xR \,|\, R_*z) :$

$[*10{\cdot}11 . *22{\cdot}1] \quad \supset \vdash . \overleftarrow{R}\text{‘}x \subset \hat{z}(xR \,|\, R_*z) \quad (3)$

$\vdash . (2) . (3) . *10{\cdot}1 . \supset$

$\vdash :. \breve{R}\text{“}\mu \subset \mu . \overleftarrow{R}\text{‘}x \subset \mu . \supset_\mu . z \in \mu : \supset . z \in \hat{z}(xR \,|\, R_*z) .$

$[*20{\cdot}3] \quad \supset . xR \,|\, R_*z :. \supset \vdash . \text{Prop}$

***90·36.** $\vdash :. xR \,|\, R_*z . \equiv : \breve{R}\text{“}\mu \subset \mu . \overleftarrow{R}\text{‘}x \subset \mu . \supset_\mu . z \in \mu$ [*90·35·351]

***90·4.** $\vdash . (R_*)_* = R_*$

Dem.

$\vdash . *90{\cdot}151{\cdot}18 . \supset \vdash . R_* \in (R_*)_* \quad (1)$

$\vdash . *90{\cdot}112 \dfrac{R_*, \; xR_*z}{R, \; \phi z} . \supset$

$\vdash :. x(R_*)_*y : xR_*z . zR_*w . \supset_{z,w} . xR_*w : xR_*x : \supset . xR_*y \quad (2)$

$\vdash . *90 \cdot 13 . \supset \vdash : x (R_*)_* y . \supset . x \epsilon C ‘ R_* .$

$[*90 \cdot 14] \qquad \supset . x \epsilon C ‘ R .$

$[*90 \cdot 12] \qquad \supset . x R_* x \qquad (3)$

$\vdash . *90 \cdot 17 . \supset \vdash : x R_* z . z R_* w . \supset_{z,w} . x R_* w \qquad (4)$

$\vdash . (2) . (3) . (4) . \supset \vdash : x (R_*)_* y . \supset . x R_* y \qquad (5)$

$\vdash . (1) . (5) . \supset \vdash .$ Prop

$*90 \cdot 41$. $\vdash . C ‘ P_* \upharpoonright \alpha = \alpha \cap C ‘ P$

Dem.

$\vdash . *37 \cdot 41 . \qquad \supset \vdash . C ‘ P_* \upharpoonright \alpha = \alpha \cap (P_* ‘‘ \alpha \cup \breve{P}_* ‘‘ \alpha) \qquad (1)$

$\vdash . (1) . *37 \cdot 15 \cdot 16 . *90 \cdot 14 . \supset \vdash . C ‘ P_* \upharpoonright \alpha \subset \alpha \cap C ‘ P \qquad (2)$

$\vdash . *90 \cdot 33 \cdot 331 . \qquad \supset \vdash . \alpha \cap C ‘ P \subset P_* ‘‘ \alpha \cup \breve{P}_* ‘‘ \alpha \qquad (3)$

$\vdash . (3) . (1) . \qquad \supset \vdash . \alpha \cap C ‘ P \subset C ‘ P_* \upharpoonright \alpha \qquad (4)$

$\vdash . (2) . (4) . \qquad \supset \vdash .$ Prop

$*90 \cdot 42$. $\vdash . (Q_* \upharpoonright \alpha)_* = Q_* \upharpoonright \alpha$

Dem.

$\vdash . *90 \cdot 18 . \supset \vdash . (Q_* \upharpoonright \alpha)_* \in (Q_*)_*$

$[*90 \cdot 4] \qquad \in Q_* \qquad (1)$

$\vdash . *90 \cdot 13 . \supset \vdash : x (Q_* \upharpoonright \alpha)_* y . \supset . x, y \epsilon C ‘ Q_* \upharpoonright \alpha .$

$[*90 \cdot 41] \qquad \supset . x, y \epsilon \alpha \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . (Q_* \upharpoonright \alpha)_* \in Q_* \upharpoonright \alpha \qquad (3)$

$\vdash . (3) . *90 \cdot 151 . \supset \vdash .$ Prop

$*91$. ON POWERS OF A RELATION

Summary of $*91$.

In the present number, we consider the class of relations

$$R,\ R^2,\ R^3,\ \ldots.$$

Each of these has to its predecessor the relation $|R$; we have

$$R^2 = |R\text{`}R,\ R^3 = |R\text{`}R^2, \text{ etc.}$$

Thus every term of the series has to R the relation $(|R)_*$; hence the powers of R may be defined as those relations which have to R the relation $(|R)_*$. The series of powers starting with $I \upharpoonright C\text{`}R$ instead of with $R$ is similarly composed of those relations which have to $I \upharpoonright C\text{`}R$ the relation $(|R)_*$. (This class consists of the previous class together with $I \upharpoonright C\text{`}R$.) To say that the relation R_* holds between x and y turns out to be equivalent to saying that one of the relations

$$I \upharpoonright C\text{`}R,\ R,\ R^2,\ R^3,\ \ldots$$

holds between x and y; and to say that the relation $R|R_*$ holds between x and y turns out to be equivalent to saying that one of the relations

$$R,\ R^2,\ R^3,\ \ldots$$

holds between x and y. Thus we might have begun by defining powers of R, and proceeded to define R_* as their sum.

For notational convenience we put

$$R_{\text{ts}} = (|R)_* \quad \text{Df.}$$

Then the definition of powers of R excluding $I \upharpoonright C\text{`}R$ is

$$\text{Pot}\text{`}R = \overrightarrow{R_{\text{ts}}}\text{`}R \quad \text{Df,}$$

and the definition of powers of R including $I \upharpoonright C\text{`}R$ is

$$\text{Potid}\text{`}R = \overrightarrow{R_{\text{ts}}}\text{`}(I \upharpoonright C\text{`}R) \quad \text{Df.}$$

(Here the letters "id" are added to suggest that identity is to be added to $\text{Pot}\text{`}R$.)

We put also

$$R_{\text{po}} = \dot{s}\text{`Pot`}R \quad \text{Df.}$$

Many of the propositions in this number are very often used. Among the more important propositions are the following:

$*91{\cdot}17$. $\vdash :. P \in \text{Potid}\text{`}R : \phi S \,.\supset_S .\, \phi(S|R) : \phi(I \upharpoonright C\text{`}R) : \supset .\, \phi P$

$*91{\cdot}171$. $\vdash :. P \in \text{Pot}\text{`}R : \phi S \,.\supset_S .\, \phi(S|R) : \phi R : \supset .\, \phi P$

$*91{\cdot}373$. $\vdash :. P \in \text{Pot}\text{`}R \,.\supset_P .\, \phi P : \equiv : \phi R : S \in \text{Pot}\text{`}R \,.\, \phi S \,.\supset_S .\, \phi(S|R)$

These are formulae of induction. The first two state that if the property ϕ is hereditary with respect to $|R$, then if ϕ belongs to $I \upharpoonright C'R$ it belongs to any member of Potid$'R$, while if ϕ belongs to R it belongs to any member of Pot$'R$. The third gives a form of induction which is sometimes more powerful than the second. It states that if ϕ is hereditary provided its argument is a power of R, and if ϕR, then every power of R satisfies ϕ, and vice versa.

***91·23.** $\vdash . \text{Potid}'R = \iota'(I \upharpoonright C'R) \cup \text{Pot}'R$

***91·24.** $\vdash . \text{Pot}'R = |R``\text{Potid}'R$

These two propositions are very useful as giving relations of Pot$'R$ and Potid$'R$.

***91·27.** $\vdash : P \in \text{Potid}'R . \supset . C'P \subset C'R$

***91·271.** $\vdash : P \in \text{Pot}'R . \supset . \text{D}'P \subset \text{D}'R . \text{Q}'P \subset \text{Q}'R$

We do not have in general $P \in \text{Pot}'R . \supset . \text{D}'P = \text{D}'R . \text{Q}'P = \text{Q}'R$. If R is the sort of relation which generates a series (*i.e.* is either itself serial, or such that R_{po} is serial), the above would characterize a series without a first or last term. To illustrate the matter, consider a series of four terms, x, y, z, w, and let R be the relation of immediately preceding in this series. Thus R holds between x and y, y and z, z and w. Then R^2 holds between x and z, y and w; thus z, which belongs to D$'R$, does not belong to D$'R^2$. R^3 holds only between x and w; thus neither y nor z belongs to D$'R^3$. All powers of R beyond the third are null. On the other hand, if we take a cyclic relation, such as that of left-hand neighbour at a dinner-table, we shall always have $\text{D}'P = \text{D}'R . \text{Q}'P = \text{Q}'R$, whatever power of R P may be.

***91·282.** $\vdash : P \in \text{Pot}'R . \supset . P | R \in \text{Pot}'R$

This proposition shows that Pot$'R$ is a hereditary class with respect to $|R$.

***91·34.** $\vdash : P, Q \in \text{Potid}'R . \supset . P | Q = Q | P$

This proposition states that the relative product is commutative when each factor is $I \upharpoonright C'R$ or a power of R.

We come next to propositions concerning R_{po}. We have

***91·502.** $\vdash . R \subseteq R_{\text{po}}$

***91·504.** $\vdash . \text{D}'R_{\text{po}} = \text{D}'R . \text{Q}'R_{\text{po}} = \text{Q}'R . C'R_{\text{po}} = C'R$

***91·511.** $\vdash . R_{\text{po}} | R \subseteq R_{\text{po}}$

***91·52.** $\vdash . R_{\text{po}} = R_* | R = R | R_*$

***91·54.** $\vdash . R_* = I \upharpoonright C'R \cup R_{\text{po}}$

*91·52·54 are fundamental in the theory of inductive relations.

***91·542.** $\vdash : xR_*y . x \neq y . \equiv . xR_{\text{po}}y . x \neq y$

This proposition is particularly useful when (as often happens) we have $R_{\text{po}} \subseteq J$. In that case, it gives $R_{\text{po}} = R_* \cap J$.

***91·55.** $\vdash . R_* = \dot{s}\text{‘Potid‘}R$

***91·56.** $\vdash . R_{\text{po}}^2 \subset R_{\text{po}}$

Thus R_{po} is always transitive, which is one of the three characteristics of serial relations (cf. *204). We shall find that R_{po} is often serial when R is not so.

***91·574.** $\vdash . R_* \,|\, R_{\text{po}} = R_{\text{po}} \,|\, R_* = R_{\text{po}} = R \,|\, R_* = R_* \,|\, R$

***91·602.** $\vdash . (R_{\text{po}})_* = R_*$

***91·01.** $R_{\text{st}} = (R\,|)_*$ Df

***91·02.** $R_{\text{ts}} = (|\,R)_*$ Df

***91·03.** $\text{Pot‘}R = \overrightarrow{R_{\text{ts}}}\text{‘}R$ Df

***91·04.** $\text{Potid‘}R = \overrightarrow{R_{\text{ts}}}\text{‘}(I \restriction C\text{‘}R)$ Df

***91·05.** $R_{\text{po}} = \dot{s}\text{‘Pot‘}R$ Df

The first two of the above definitions are introduced merely for notational convenience. The other three represent ideas of great importance. The last is especially useful when a series is given as the field of a one-one relation between consecutive terms—as, *e.g.*, when the series of natural numbers is given as the field of the relation of n to $n+1$. Then R_{po} is the relation of any earlier term to any later term—*e.g.*, in the above case of the natural numbers, the relation of a less integer to a greater.

***91·1.** $\vdash :: PR_{\text{st}}Q . \equiv :. S \in \mu . \supset_S . R\,|\,S \in \mu : Q \in \mu : \supset_\mu . P \in \mu$

Dem.

$\vdash . *4\cdot2 . (*91\cdot01) . \supset$

$\vdash :: PR_{\text{st}}Q . \equiv :. P(R\,|)_* Q :.$

$[*90\cdot11] \quad \equiv :. P \in C\text{‘}(R\,|) : (R\,|)\text{“}\mu \subset \mu . Q \in \mu . \supset_\mu . P \in \mu :.$

$[*43\cdot3 . *33\cdot161] \quad \equiv :. (R\,|)\text{“}\mu \subset \mu . Q \in \mu . \supset_\mu . P \in \mu :.$

$[*37\cdot61] \quad \equiv :. S \in \mu . \supset_S . R\,|\text{‘}S \in \mu : Q \in \mu : \supset_\mu . P \in \mu :.$

$[*43\cdot11] \quad \equiv :. S \in \mu . \supset_S . R\,|\,S \in \mu : Q \in \mu : \supset_\mu . P \in \mu :: \supset \vdash . \text{Prop}$

***91·11.** $\vdash :: PR_{\text{ts}}Q . \equiv :. S \in \mu . \supset_S . S\,|\,R \in \mu : Q \in \mu : \supset_\mu . P \in \mu$

***91·12.** $\vdash : P \in \text{Pot‘}R . \equiv . PR_{\text{ts}}R$ [*32·18 . (*91·03)]

***91·13.** $\vdash :: P \in \text{Pot‘}R . \equiv :. S \in \mu . \supset_S . S\,|\,R \in \mu : R \in \mu : \supset_\mu . P \in \mu$
[*91·11·12]

***91·14.** $\vdash : P \in \text{Potid‘}R . \equiv . PR_{\text{ts}}(I \restriction C\text{‘}R)$ [*32·18 . (*91·04)]

***91·15.** $\vdash :: P \in \text{Potid‘}R . \equiv :. S \in \mu . \supset_S . S\,|\,R \in \mu : I \restriction C\text{‘}R \in \mu : \supset_\mu . P \in \mu$
[*91·11·14]

***91·16.** $\vdash :: xR_{\text{po}}y . \equiv :. (\exists P) :. S \in \mu . \supset_S . S\,|\,R \in \mu : R \in \mu : \supset_\mu . P \in \mu :. xPy$
[*41·11 . (*91·05) . *91·13]

***91·17.** $\vdash :. P \epsilon \text{Potid}`R : \phi S . \supset_S . \phi(S \mid R) : \phi(I \upharpoonright C`R) : \supset . \phi P$

$\left[*91\cdot15 \dfrac{\hat{S}(\phi S)}{\mu}\right]$

***91·171.** $\vdash :. P \epsilon \text{Pot}`R : \phi S . \supset_S . \phi(S \mid R) : \phi R : \supset . \phi P$

$\left[*91\cdot13 \dfrac{\hat{S}(\phi S)}{\mu}\right]$

These propositions are of great importance, because they enable us to prove that a property ϕ belongs to every power of R if it belongs to R (or $I \upharpoonright C`R$) and also belongs to $S \mid R$ whenever it belongs to S.

***91·2.** $\vdash : QR_{ts}P . \supset . (Q \mid R) R_{ts} P$

Dem.

$\vdash . *43\cdot101 . (*91\cdot02) . \supset \vdash : \text{Hp} . \supset . (Q \mid R)(\mid R) Q . Q (\mid R)_* P .$
$[*90\cdot172] \qquad \supset . (Q \mid R)(\mid R)_* P .$
$[\text{Id}.(*91\cdot02)] \qquad \supset . (Q \mid R) R_{ts} P : \supset \vdash . \text{Prop}$

***91·201.** $\vdash : QR_{st}P . \supset . (R \mid Q) R_{st} P$ [Proof as in *91·2]

***91·204.** $\vdash : P \{R_{ts} \mid (\mid R)\} Q . \equiv . PR_{ts}(Q \mid R)$

Dem.

$\vdash . *34\cdot1 . \supset \vdash : P \{R_{ts} \mid (\mid R)\} Q . \equiv . (\exists T) . PR_{ts}T . T (\mid R) Q .$
$[*43\cdot101] \qquad \equiv . (\exists T) . PR_{ts}T . T = Q \mid R .$
$[*13\cdot195] \qquad \equiv . PR_{ts}(Q \mid R) : \supset \vdash . \text{Prop}$

***91·205.** $\vdash : P \{R_{st} \mid (R \mid)\} Q . \equiv . PR_{st}(R \mid Q)$

***91·21.** $\vdash . R_{ts} = I \cup R_{ts} \mid (\mid R)$

Dem.

$\vdash . *90\cdot31 . (*91\cdot02) . \supset \vdash . R_{ts} = I \upharpoonright C`(\mid R) \cup R_{ts} \mid (\mid R)$
$[*43\cdot311] \qquad = I \cup R_{ts} \mid (\mid R) . \supset \vdash . \text{Prop}$

***91·211.** $\vdash . R_{st} = I \cup R_{st} \mid (R \mid)$

***91·212.** $\vdash :. PR_{ts}Q . \equiv : P = Q . \vee . PR_{ts}(Q \mid R)$

Dem.

$\vdash . *91\cdot21 . *50\cdot1 . \supset \vdash :. PR_{ts}Q . \equiv : P = Q . \vee . P \{R_{ts} \mid (\mid R)\} Q :$
$[*91\cdot204] \qquad \equiv : P = Q . \vee . PR_{ts}(Q \mid R) :. \supset \vdash :. \text{Prop}$

***91·213.** $\vdash :. PR_{st}Q . \equiv : P = Q . \vee . PR_{st}(R \mid Q)$

***91·22.** $\vdash . \overrightarrow{R_{ts}}`Q = \iota`Q \cup \overrightarrow{R_{ts}}`(Q \mid R)$ [*91·212 . *32·18 . *51·15]

***91·221.** $\vdash . \overrightarrow{R_{st}}`Q = \iota`Q \cup \overrightarrow{R_{st}}`(R \mid Q)$

***91·23.** $\vdash . \text{Potid}`R = \iota`(I \upharpoonright C`R) \cup \text{Pot}`R$

Dem.

$\vdash . *91\cdot22 . (*91\cdot04) . \supset \vdash . \text{Potid}`R = \iota`(I \upharpoonright C`R) \cup \overrightarrow{R_{ts}}`\{(I \upharpoonright C`R) \mid R\}$
$[*50\cdot65.(*91\cdot03)] \qquad = \iota`(I \upharpoonright C`R) \cup \text{Pot}`R . \supset \vdash . \text{Prop}$

***91·231.** $\vdash . \overrightarrow{R_{ts}}`I = \iota`I \cup \text{Pot}`R$ [*91·22 . (*91·03) . *50·4]

***91·24.** $\vdash . \text{Pot}`R = |R``\text{Potid}`R$

Dem.

$\vdash . *91·12 . \supset \vdash : P \epsilon \text{Pot}`R . \equiv . PR_{ts}R .$

[*50·65] $\equiv . PR_{ts}(I \upharpoonright C`R | R) .$

[*91·204] $\equiv . P \{R_{ts} | (|R)\} (I \upharpoonright C`R) .$

[*90·32.(*91·02)] $\equiv . P \{(|R) | R_{ts}\} (I \upharpoonright C`R) .$

[*37·3] $\equiv . P \epsilon |R``\overrightarrow{R_{ts}}`(I \upharpoonright C`R) .$

[*4·2.(*91·04)] $\equiv . P \epsilon |R``\text{Potid}`R : \supset \vdash . \text{Prop}$

***91·241.** $\vdash : TR_{ts}P . \supset . (Q|T) R_{ts} (Q|P)$

Dem.

$\vdash . *91·212 . \supset \vdash . (Q|P) R_{ts} (Q|P)$ (1)

$\vdash . *91·2 . \quad \supset \vdash : (Q|S) R_{ts} (Q|P) . \supset . (Q|S|R) R_{ts} (Q|P)$ (2)

$\vdash . (1) . (2) . *91·11 \dfrac{\hat{S} \{(Q|S) R_{ts} (Q|P)\}}{\mu} . \supset \vdash . \text{Prop}$

The last line of the above proof is obtained as follows: writing μ for $\hat{S} \{(Q|S) R_{ts} (Q|P)\}$, (1) becomes

$P \epsilon \mu$ (1),

while (2) becomes $S \epsilon \mu . \supset . S|R \epsilon \mu$ (2).

But by *91·11, writing T for the P of *91·11, and P for the Q, we have

$$TR_{ts}P . \supset :. S \epsilon \mu . \supset_S . S|R \epsilon \mu : P \epsilon \mu : \supset . T \epsilon \mu.$$

Hence, by (1) and (2), $TR_{ts}P . \supset . T \epsilon \mu$, *i.e.*

$$TR_{ts}P . \supset . (Q|T) R_{ts} (Q|P),$$

which is the proposition to be proved.

***91·242.** $\vdash : SR_{ts} (Q|P) . \supset . S \epsilon Q|``\overrightarrow{R_{ts}}`P$

Dem.

$\vdash . *91·22 . *43·11 . \supset \vdash . Q|P \epsilon Q|``\overrightarrow{R_{ts}}`P$ (1)

$\vdash . *37·1 . *43·1 . \supset$

$\vdash : S \epsilon Q|``\overrightarrow{R_{ts}}`P . \equiv . (\exists T) . T \epsilon \overrightarrow{R_{ts}}`P . S = Q|T .$

[*91·2] $\supset . (\exists T) . T|R \epsilon \overrightarrow{R_{ts}}`P . S|R = Q|T|R .$

[*37·1.*43·1] $\supset . S|R \epsilon Q|``\overrightarrow{R_{ts}}`P$ (2)

$\vdash . (1) . (2) . *91·11 \dfrac{Q|``\overrightarrow{R_{ts}}`P}{\mu} . \supset \vdash . \text{Prop}$

***91·25.** $\vdash . \overrightarrow{R_{ts}}`(Q|P) = Q|``\overrightarrow{R_{ts}}`P$

Dem.

$\vdash . *91·242 . \supset \vdash . \overrightarrow{R_{ts}}`(Q|P) \subset Q|``\overrightarrow{R_{ts}}`P$ (1)

$\vdash . *91\cdot241 . \supset \vdash : T \epsilon \overrightarrow{R_{ts}}\text{‘}P . S = Q \mid T . \supset . S \epsilon \overrightarrow{R_{ts}}\text{‘}(Q \mid P) :$

$[*10\cdot11\cdot23] \quad \supset \vdash : (\exists T) . T \epsilon \overrightarrow{R_{ts}}\text{‘}P . S = Q \mid T . \supset . S \epsilon \overrightarrow{R_{ts}}\text{‘}(Q \mid P) :$

$[*37\cdot1 . *43\cdot1] \supset \vdash : S \epsilon Q \mid \text{“}\overrightarrow{R_{ts}}\text{‘}P . \supset . S \epsilon \overrightarrow{R_{ts}}\text{‘}(Q \mid P)$ (2)

$\vdash . (1) . (2) . \quad \supset \vdash . \text{Prop}$

$*91\cdot251$. $\vdash . \overrightarrow{R_{st}}\text{‘}(Q \mid P) = \mid P\text{“}\overrightarrow{R_{st}}\text{‘}Q$ [Proof as in $*91\cdot25$]

$*91\cdot26$. $\vdash . \overrightarrow{R_{ts}}\text{‘}Q = Q \mid \text{“}\overrightarrow{R_{ts}}\text{‘}I \qquad \left[*91\cdot25 \frac{I}{P}\right]$

$*91\cdot261$. $\vdash . \overrightarrow{R_{st}}\text{‘}Q = \mid Q\text{“}\overrightarrow{R_{st}}\text{‘}I \qquad \left[*91\cdot251 \frac{I, Q}{Q, P}\right]$

$*91\cdot262$. $\vdash : \text{Ɑ}\text{‘}Q \subset C\text{‘}R . \supset . \overrightarrow{R_{ts}}\text{‘}Q = Q \mid \text{“Potid‘}R$

$\left[*91\cdot25 \frac{I \upharpoonright C\text{‘}R}{P} . *50\cdot62 . (*91\cdot04)\right]$

$*91\cdot263$. $\vdash . \overrightarrow{R_{ts}}\text{‘}(Q \mid R) = Q \mid \text{“Pot‘}R \qquad \left[*91\cdot25 \frac{R}{P} . (*91\cdot03)\right]$

$*91\cdot264$. $\vdash . \text{Pot‘}R = \iota\text{‘}R \cup R \mid \text{“Pot‘}R \qquad \left[*91\cdot22\cdot263 \frac{R}{Q}\right]$

$*91\cdot27$. $\vdash : P \epsilon \text{Potid‘}R . \supset . C\text{‘}P \subset C\text{‘}R$

Dem.

$\vdash . *50\cdot5\cdot52 . \supset \vdash . C\text{‘}(I \upharpoonright C\text{‘}R) = C\text{‘}R .$

$[*22\cdot42] \quad \supset \vdash . C\text{‘}(I \upharpoonright C\text{‘}R) \subset C\text{‘}R$ (1)

$\vdash . *34\cdot38 . \quad \supset \vdash : C\text{‘}S \subset C\text{‘}R . \supset . C\text{‘}(S \mid R) \subset C\text{‘}R$ (2)

$\vdash . (1) . (2) . *91\cdot17 \frac{C\text{‘}S \subset C\text{‘}R}{\phi S} . \supset \vdash . \text{Prop}$

$*91\cdot271$. $\vdash : P \epsilon \text{Pot‘}R . \supset . \text{D‘}P \subset \text{D‘}R . \text{Ɑ‘}P \subset \text{Ɑ‘}R$

Dem.

$\vdash . *22\cdot42 . \supset \vdash . \text{D‘}R \subset \text{D‘}R . \text{Ɑ‘}R \subset \text{Ɑ‘}R$ (1)

$\vdash . *34\cdot36 . \supset \vdash : \text{D‘}S \subset \text{D‘}R . \supset . \text{D‘}(S \mid R) \subset \text{D‘}R . \text{Ɑ‘}(S \mid R) \subset \text{Ɑ‘}R$ (2)

$\vdash . (1) . (2) . *91\cdot171 \frac{\text{D‘}S \subset \text{D‘}R . \text{Ɑ‘}S \subset \text{Ɑ‘}R}{\phi S} . \supset \vdash . \text{Prop}$

$*91\cdot28$. $\vdash : P \epsilon \text{Potid‘}R . \supset . P \mid R \epsilon \text{Pot‘}R$ [$*91\cdot24$]

$*91\cdot281$. $\vdash : \text{Pot‘}R \subset \text{Potid‘}R . \mid R\text{“Potid‘}R \subset \text{Potid‘}R$ [$*91\cdot23\cdot24$]

$*91\cdot282$. $\vdash : P \epsilon \text{Pot‘}R . \supset . P \mid R \epsilon \text{Pot‘}R$ [$*91\cdot28\cdot281$]

$*91\cdot283$. $\vdash : \mid R\text{“Pot‘}R \subset \text{Pot‘}R$ [$*91\cdot282$]

The following propositions show that the relative product of two powers of R is commutative, *i.e.* (cf. $*91\cdot34$)

$$P, Q \epsilon \text{Potid‘}R . \supset . P \mid Q = Q \mid P.$$

We also have (cf. $*91\cdot341$)

$$P, Q \epsilon \text{Potid‘}R . \supset . P \mid Q \epsilon \text{Potid‘}R.$$

It is these propositions (as will appear in the sequel) which are the source of the commutative law for the addition of finite ordinals. Ordinals in general are not commutative, just as relative products in general are not commutative; but owing to the fact that relative products whose factors are powers of a given relation are commutative, *finite* ordinals are commutative.

∗91·3. $\vdash : P \epsilon \text{Potid`}R \,.\supset .\, R \,|\, P = P \,|\, R$

Dem.

$\vdash .\, {*}50{\cdot}64{\cdot}65 \,.\supset \vdash .\, R \,|\, I \upharpoonright C`R = I \upharpoonright C`R \,|\, R$ (1)

$\vdash .\, {*}34{\cdot}21 \,.\quad \supset \vdash .\, R \,|\, (S \,|\, R) = (R \,|\, S) \,|\, R$ (2)

$\vdash .\, {*}34{\cdot}27 \,.\quad \supset \vdash : R \,|\, S = S \,|\, R \,.\supset .\, (R \,|\, S) \,|\, R = (S \,|\, R) \,|\, R \,.$

[(2)] $\qquad \supset .\, R \,|\, (S \,|\, R) = (S \,|\, R) \,|\, R$ (3)

$\vdash .\, {*}91{\cdot}17 \dfrac{R \,|\, S = S \,|\, R}{\phi S} \,.\supset$

$\vdash :.\, P \epsilon \text{Potid`}R : R \,|\, S = S \,|\, R \,.\supset_S .\, R \,|\, (S \,|\, R) = (S \,|\, R) \,|\, R : R \,|\, I \upharpoonright C`R = I \upharpoonright C`R \,|\, R :$

$\qquad \supset .\, R \,|\, P = P \,|\, R$ (4)

$\vdash .\, (1) \,.\, (3) \,.\, (4) \,.\supset \vdash .\, \text{Prop}$

∗91·301. $\vdash : P \epsilon \overrightarrow{R_{\text{st}}}`(I \upharpoonright C`R) \,.\supset .\, R \,|\, P = P \,|\, R$ [Proof as in ∗91·3]

∗91·302. $\vdash .\, |\, R``\text{Potid`}R = R \,|``\text{Potid`}R$

Dem.

$\vdash .\, {*}91{\cdot}3 \,.\, {*}13{\cdot}182 \,.\supset \vdash :.\, P \epsilon \text{Potid`}R \,.\supset : S = R \,|\, P \,.\equiv .\, S = P \,|\, R :$

[∗43·1·101] $\qquad \supset : S \,(R \,|)\, P \,.\equiv .\, S \,(|\, R)\, P$ (1)

$\vdash .\, (1) \,.\, {*}5{\cdot}32 \,.\supset \vdash : P \epsilon \text{Potid`}R \,.\, S \,(R \,|)\, P \,.\equiv .\, P \epsilon \text{Potid`}R \,.\, S \,(|\, R)\, P :$

[∗10·11·281] $\supset \vdash : (\exists P) \,.\, P \epsilon \text{Potid`}R \,.\, S \,(R \,|)\, P \,.\equiv .$

$\qquad (\exists P) \,.\, P \epsilon \text{Potid`}R \,.\, S \,(|\, R)\, P :$

[∗37·1] $\qquad \supset \vdash : S \epsilon R \,|``\text{Potid`}R \,.\equiv .\, S \epsilon \,|\, R``\text{Potid`}R : \supset \vdash .\, \text{Prop}$

∗91·303. $\vdash .\, |\, R``\overrightarrow{R_{\text{st}}}`(I \upharpoonright C`R) = R \,|``\overrightarrow{R_{\text{st}}}`(I \upharpoonright C`R)$ [Proof as in ∗91·302]

∗91·304. $\vdash .\, |\, R``\text{Pot`}R = R \,|``\text{Pot`}R$ [Proof as in ∗91·302]

∗91·31. $\vdash .\, \text{Pot`}R = R \,|``\text{Potid`}R$ [∗91·24·301]

∗91·33. $\vdash .\, \text{Potid`}R = \overrightarrow{R_{\text{st}}}`(I \upharpoonright C`R)$

Dem.

$\vdash .\, {*}91{\cdot}23 \,.\supset \vdash .\, I \upharpoonright C`R \,\epsilon\, \text{Potid`}R$ (1)

$\vdash .\, {*}91{\cdot}3 \,.\quad \supset \vdash : P \epsilon \text{Potid`}R \,.\supset .\, R \,|\, P = P \,|\, R \,.$

[∗91·281] $\qquad \supset .\, R \,|\, P \epsilon \text{Potid`}R$ (2)

$\vdash .\, (1) \,.\, (2) \,.\, {*}91{\cdot}1 \dfrac{\text{Potid`}R}{\mu} \,.\supset \vdash : P R_{\text{st}} (I \upharpoonright C`R) \,.\supset .\, P \epsilon \text{Potid`}R$ (3)

$\vdash .\, {*}91{\cdot}301 \,.\supset \vdash : P R_{\text{st}} (I \upharpoonright C`R) \,.\supset .\, P \,|\, R = R \,|\, P \,.$

[∗91·201] $\qquad \supset .\, (P \,|\, R)\, R_{\text{st}} (I \upharpoonright C`R)$ (4)

$\vdash .\, {*}91{\cdot}213 \,.\supset \vdash .\, (I \upharpoonright C`R)\, R_{\text{st}} (I \upharpoonright C`R)$ (5)

$\vdash .\, (4) \,.\, (5) \,.\, {*}91{\cdot}17 \,.\supset \vdash : P \epsilon \text{Potid`}R \,.\supset .\, P R_{\text{st}} (I \upharpoonright C`R)$ (6)

$\vdash .\, (3) \,.\, (6) \,.\supset \vdash .\, \text{Prop}$

***91·331.** $\vdash . \text{Pot}\text{`}R = \overrightarrow{R_{st}}\text{`}R$

Dem.

$\vdash . *91·24·33 . \supset \vdash . \text{Pot}\text{`}R = |R\text{``}\overrightarrow{R_{st}}\text{`}(I \upharpoonright C\text{`}R)$

$[*91·251.*50·65] \quad = \overrightarrow{R_{st}}\text{`}R . \supset \vdash . \text{Prop}$

***91·34.** $\vdash : P, Q \in \text{Potid}\text{`}R . \supset . P | Q = Q | P$

Dem.

$*50·62 . *91·27 . \supset \vdash : P \in \text{Potid}\text{`}R . \supset . P | (I \upharpoonright C\text{`}R) = P$

$[*50·63.*91·27] \quad = (I \upharpoonright C\text{`}R) | P \quad (1)$

$\vdash . *34·27 . \quad \supset \vdash : P \in \text{Potid}\text{`}R . P | S = S | P . \supset . P | S | R = S | P | R$

$[*91·3] \quad = S | R | P \quad (2)$

$\vdash . (1) . (2) . *91·17 \dfrac{P | S = S | P}{\phi S} . \supset \vdash . \text{Prop}$

This is the commutative law for the relative product of two powers of R.

***91·341.** $\vdash : P, Q \in \text{Potid}\text{`}R . \supset . P | Q \in \text{Potid}\text{`}R$

Dem.

$\vdash . *50·62 . *91·27 . \supset \vdash : P \in \text{Potid}\text{`}R . \supset . P | (I \upharpoonright C\text{`}R) = P .$

$[*13·12] \quad \supset . P | (I \upharpoonright C\text{`}R) \in \text{Potid}\text{`}R \quad (1)$

$\vdash . *91·281 . \quad \supset \vdash : P | S \in \text{Potid}\text{`}R . \supset . P | S | R \in \text{Potid}\text{`}R \quad (2)$

$\vdash . (1) . (2) . *91·17 \dfrac{P | S \in \text{Potid}\text{`}R}{\phi S} . \supset \vdash . \text{Prop}$

***91·342.** $\vdash : P \in \text{Potid}\text{`}R . Q \in \text{Pot}\text{`}R . \supset . P | Q \in \text{Pot}\text{`}R$

Dem.

$\vdash . *91·28 . \quad \supset \vdash : P \in \text{Potid}\text{`}R . \supset . P | R \in \text{Pot}\text{`}R \quad (1)$

$\vdash . *91·282 . \supset \vdash : P | Q \in \text{Pot}\text{`}R . \supset . P | Q | R \in \text{Pot}\text{`}R \quad (2)$

$\vdash . (1) . (2) . *91·171 . \supset \vdash . \text{Prop}$

***91·343.** $\vdash : P, Q \in \text{Pot}\text{`}R . \supset . P | Q \in \text{Pot}\text{`}R$ [*91·342·23]

***91·35.** $\vdash . I \upharpoonright C\text{`}R \in \text{Potid}\text{`}R$ [*91·23]

***91·351.** $\vdash . R \in \text{Pot}\text{`}R$ [*91·264]

***91·352.** $\vdash . R^2 \in \text{Pot}\text{`}R$ [*91·282·351]

***91·36.** $\vdash : P \in \text{Pot}\text{`}R . \supset . P | R, R | P \in \text{Pot}\text{`}R$ [*91·343·351]

***91·37.** $\vdash :. \text{Potid}\text{`}R \subset \mu . \equiv : I \upharpoonright C\text{`}R \in \mu : S \in \text{Potid}\text{`}R . S \in \mu . \supset_S . S | R \in \mu$

Dem.

$\vdash . *91·281·35 . \supset$

$\vdash :. I \upharpoonright C\text{`}R \in \mu : S \in \text{Potid}\text{`}R . S \in \mu . \supset_S . S | R \in \mu : \equiv :$

$I \upharpoonright C\text{`}R \in \text{Potid}\text{`}R . I \upharpoonright C\text{`}R \in \mu : S \in \text{Potid}\text{`}R . S \in \mu . \supset_S . S | R \in \text{Potid}\text{`}R . S | R \in \mu :$

$[*91·17] \supset : P \in \text{Potid}\text{`}R . \supset . P \in \mu \quad (1)$

$\vdash . *91·35 . \quad \supset \vdash : \text{Potid}\text{`}R \subset \mu . \supset . I \upharpoonright C\text{`}R \in \mu \quad (2)$

$\vdash . *91·281 . \supset \vdash :. \text{Potid}\text{`}R \subset \mu . \supset : S \in \text{Potid}\text{`}R . \supset_S . S | R \in \mu :$

$[*3·41] \quad \supset : S \in \text{Potid}\text{`}R . S \in \mu . \supset_S . S | R \in \mu \quad (3)$

$\vdash . (1) . (2) . (3) . \supset \vdash . \text{Prop}$

***91·371.** $\vdash :. P \in \text{Potid}`R . \supset_P . \phi P : \equiv :$
$\phi(I \upharpoonright C`R) : S \in \text{Potid}`R . \phi S . \supset_S . \phi(S|R)$ [*91·37]

***91·372.** $\vdash :. \text{Pot}`R \subset \mu . \equiv : R \in \mu : S \in \text{Pot}`R . S \in \mu . \supset_S . S|R \in \mu$
[Proof as in *91·37]

***91·373.** $\vdash :. P \in \text{Pot}`R . \supset_P . \phi P : \equiv : \phi R : S \in \text{Pot}`R . \phi S . \supset_S . \phi(S|R)$
[*91·372]

***91·41.** $\vdash . \overrightarrow{R_{ts}}`(P|R) = P|``\text{Pot}`R$ $\left[*91\cdot25 \dfrac{P, R}{Q, P} . (*91\cdot03) \right]$

***91·411.** $\vdash . \overrightarrow{R_{st}}`(R|P) = |P``\text{Pot}`R$ $\left[*91\cdot251 \dfrac{R}{Q} . *91\cdot331 \right]$

***91·42.** $\vdash . \overrightarrow{R_{ts}}`P = \iota`P \cup P|``\text{Pot}`R$ [*91·22·41]

***91·421.** $\vdash . \overrightarrow{R_{st}}`P = \iota`P \cup |P``\text{Pot}`R$ [*91·221·411]

***91·43.** $\vdash : P \in \text{Pot}`R . QR_{ts}P . \supset . Q \in \text{Pot}`R$

Dem.

$\vdash . *91\cdot42 . \supset \vdash :. \text{Hp} . \supset : Q = P . \vee . Q \in P|``\text{Pot}`R :$
[*37·1.*43·1] $\supset : Q = P . \vee . (\exists T) . T \in \text{Pot}`R . Q = P|T :$
[*13·12.*91·343] $\supset : Q \in \text{Pot}`R :. \supset \vdash . \text{Prop}$

***91·431.** $\vdash : P \in \text{Potid}`R . QR_{ts}P . \supset . Q \in \text{Potid}`R$ [Proof as in *91·43]

***91·44.** $\vdash :. P, Q \in \text{Potid}`R . \supset : QR_{ts}P . \vee . PR_{ts}Q$

Dem.

$\vdash . *91\cdot14 . \supset \vdash : P \in \text{Potid}`R . \supset . PR_{ts}(I \upharpoonright C`R)$ (1)
$\vdash . *91\cdot2 . \supset \vdash : QR_{ts}P . \supset . (Q|R) R_{ts} P$ (2)
$\vdash . *91\cdot212 . \supset \vdash :. PR_{ts}Q . \supset : P = Q . \vee . PR_{ts}(Q|R)$ (3)
$\vdash . *91\cdot212 . \supset \vdash : P = Q . \supset . QR_{ts}P .$
[*91·2] $\supset . (Q|R) R_{ts} P$ (4)
$\vdash . (3) . (4) . \supset \vdash :. PR_{ts}Q . \supset : (Q|R) R_{ts} P . \vee . PR_{ts}(Q|R)$ (5)
$\vdash . (2) . (5) . \supset \vdash :. QR_{ts}P . \vee . PR_{ts}Q : \supset : (Q|R) R_{ts} P . \vee . PR_{ts}(Q|R)$ (6)
$\vdash . (1) . (6) . *91\cdot17 . \supset \vdash . \text{Prop}$

***91·45.** $\vdash :. P, Q \in \text{Potid}`R . \supset : (\exists T) : T \in \text{Potid}`R : Q = P|T . \vee . P = Q|T$

Dem.

$\vdash . *91\cdot262\cdot27 . \supset \vdash :. \text{Hp} . \supset : \overrightarrow{R_{ts}}`P = P|``\text{Potid}`R . \overrightarrow{R_{ts}}`Q = Q|``\text{Potid}`R :$
[*37·1.*43·1] $\supset : QR_{ts}P . \equiv . (\exists T) . T \in \text{Potid}`R . Q = P|T :$
$PR_{ts}Q . \equiv . (\exists T) . T \in \text{Potid}`R . P = Q|T$ (1)
$\vdash . (1) . *91\cdot44 . *10\cdot42 . \supset \vdash . \text{Prop}$

***91·46.** $\vdash :. P, Q \in \text{Potid}`R . \supset : (\exists T) : T \in \text{Potid}`R : Q = T|P . \vee . P = T|Q$
[*91·45·34]

The remainder of this number is concerned with R_{po} and its relations to R_*.

***91·502.** $\vdash . R \subseteq R_{po}$ [*91·351 . (*91·05) . *41·13]

***91·503.** $\vdash . R^2 \subseteq R_{po}$ [*91·352 . (*91·05) . *41·13]

$*91{\cdot}504$. $\vdash . \mathrm{D}`R_{po} = \mathrm{D}`R . \text{ɑ}`R_{po} = \text{ɑ}`R . C`R_{po} = C`R$

Dem.

$\vdash . *91{\cdot}502 . \quad \supset \vdash . \mathrm{D}`R \subset \mathrm{D}`R_{po}$ (1)

$\vdash . *91{\cdot}271 . *40{\cdot}43 . \supset \vdash . s`\mathrm{D}``\mathrm{Pot}`R \subset \mathrm{D}`R .$

$[*41{\cdot}43] \quad \supset \vdash . \mathrm{D}`R_{po} \subset \mathrm{D}`R$ (2)

$\vdash . (1) . (2) . \quad \supset \vdash . \mathrm{D}`R = \mathrm{D}`R_{po}$ (3)

Similarly $\quad \vdash . \text{ɑ}`R = \text{ɑ}`R_{po} . C`R = C`R_{po}$ (4)

$\vdash . (3) . (4) . \supset \vdash . \mathrm{Prop}$

The following propositions are concerned mainly with the relations of R_{po} and R_*. These relations are embodied in the propositions

$$R_{po} = R_* \mid R = R \mid R_* \quad (*91{\cdot}52)$$
$$R_* = I \upharpoonright C`R \cup R_{po} \quad (*91{\cdot}54)$$

and
$$R_* = \dot{s}`\mathrm{Potid}`R \quad (*91{\cdot}55)$$

$*91{\cdot}51$. $\vdash . R_{po} \mid R = R \mid R_{po}$

Dem.

$\vdash . *43{\cdot}421 . (*91{\cdot}05) . \supset \vdash . R_{po} \mid R = \dot{s}`{\mid}R``\mathrm{Pot}`R$

$[*91{\cdot}304] \quad = \dot{s}`R\mid``\mathrm{Pot}`R$

$[*43{\cdot}42 . (*91{\cdot}05)] \quad = R \mid R_{po} . \supset \vdash . \mathrm{Prop}$

$*91{\cdot}511$. $\vdash . R_{po} \mid R \subseteq R_{po} \quad [*43{\cdot}421 . *91{\cdot}283 . *41{\cdot}161]$

$*91{\cdot}512$. $\vdash . R_{po} \subseteq R_* \mid R$

Dem.

$\vdash . *90{\cdot}32 . \supset \vdash . R \subseteq R_* \mid R$ (1)

$\vdash . *90{\cdot}16 . \supset \vdash : S \subseteq R_* \mid R . \supset . S \subseteq R_* .$

$[*34{\cdot}34] \quad \supset . S \mid R \subseteq R_* \mid R$ (2)

$\vdash . (1) . (2) . *91{\cdot}171 \dfrac{S \subseteq R_* \mid R}{\phi S} . \supset \vdash : P \in \mathrm{Pot}`R . \supset . P \subseteq R_* \mid R :$

$[*41{\cdot}151 . (*91{\cdot}05)] \quad \supset \vdash . R_{po} \subseteq R_* \mid R . \supset \vdash . \mathrm{Prop}$

$*91{\cdot}513$. $\vdash . R_* \subseteq \dot{s}`\mathrm{Potid}`R$

Dem.

$\vdash . *90{\cdot}112 \dfrac{x\,(\dot{s}`\mathrm{Potid}`R)\,z}{\phi z} . \supset$

$\vdash :. xR_*y : x\,(\dot{s}`\mathrm{Potid}`R)\,z . zRw . \supset_{z,w} . x\,(\dot{s}`\mathrm{Potid}`R)\,w :$

$x\,(\dot{s}`\mathrm{Potid}`R)\,x : \supset . x\,(\dot{s}`\mathrm{Potid}`R)\,y$ (1)

$\vdash . *43{\cdot}421 . \quad \supset \vdash . (\dot{s}`\mathrm{Potid}`R) \mid R = \dot{s}`{\mid}R``\mathrm{Potid}`R$

$[*91{\cdot}281 . *41{\cdot}161] \quad \subseteq \dot{s}`\mathrm{Potid}`R .$

$[*34{\cdot}1 . *10{\cdot}23] \supset \vdash : x\,(\dot{s}`\mathrm{Potid}`R)\,z . zRw . \supset_{z,w} . x\,(\dot{s}`\mathrm{Potid}`R)\,w$ (2)

$\vdash . *90{\cdot}13 . \quad \supset \vdash : xR_*y . \supset . x \in C`R .$

$[*50{\cdot}3 . *35{\cdot}101] \quad \supset . x\,(I \upharpoonright C`R)\,x .$

$[*91{\cdot}35 . *41{\cdot}13] \quad \supset . x\,(\dot{s}`\mathrm{Potid}`R)\,x$ (3)

$\vdash . (2) . (3) . *4{\cdot}71{\cdot}73 . \supset \vdash : \mathrm{Hp}(1) . \equiv . xR_*y$ (4)

$\vdash . (1) . (4) . \supset \vdash : xR_*y . \supset . x\,(\dot{s}`\mathrm{Potid}`R)\,y : \supset \vdash . \mathrm{Prop}$

***91·514.** $\vdash . R_* | R \subset R_{po}$

Dem.

$$\begin{aligned} &\vdash . *91\cdot513 . \supset \vdash . R_* | R \subset (\dot{s}\text{`Potid`}R) | R \\ &[*43\cdot421] \qquad \subset \dot{s}\text{`}|R\text{``Potid`}R \\ &[*91\cdot24] \qquad \subset \dot{s}\text{`Pot`}R \\ &[(*91\cdot05)] \qquad \subset R_{po} . \supset \vdash . \text{Prop} \end{aligned}$$

***91·52.** $\vdash . R_{po} = R_* | R = R | R_*$ [*91·512·514 . *90·32]

***91·521.** $\vdash : P \in \text{Potid`}R . \equiv . \breve{P} \in \text{Potid`}\breve{R}$

Dem.

$$\vdash . *91\cdot15 \frac{\text{Cnv``}\mu}{\mu} . \supset \vdash :: \breve{P} \in \text{Potid`}\breve{R} . \supset :.$$

$$I \upharpoonright C\text{`}R \in \text{Cnv``}\mu : S \in \text{Cnv``}\mu . \supset_S . S | \breve{R} \in \text{Cnv``}\mu : \supset . \breve{P} \in \text{Cnv``}\mu \quad (1)$$

$$\vdash . *72\cdot513\cdot11 . \quad \supset \vdash : \breve{P} \in \text{Cnv``}\mu . \equiv . P \in \mu \quad (2)$$

$$\vdash . (2) . *50\cdot5\cdot51 . \supset \vdash : I \upharpoonright C\text{`}R \in \text{Cnv``}\mu . \equiv . I \upharpoonright C\text{`}R \in \mu \quad (3)$$

$$\begin{aligned} &\vdash . *31\cdot51 . \quad \supset \vdash :. S \in \text{Cnv``}\mu . \supset_S . S | \breve{R} \in \text{Cnv``}\mu : \equiv : \\ &\qquad \breve{S} \in \text{Cnv``}\mu . \supset_S . \breve{S} | \breve{R} \in \text{Cnv``}\mu : \\ &[(2).*34\cdot2] \quad \equiv : S \in \mu . \supset_S . R | S \in \mu \end{aligned} \quad (4)$$

$\vdash . (1) . (2) . (3) . (4) . \supset$

$$\vdash :: \breve{P} \in \text{Potid`}\breve{R} . \supset :. I \upharpoonright C\text{`}R \in \mu : S \in \mu . \supset_S . R | S \in \mu : \supset . P \in \mu \quad (5)$$

$\vdash . (5) . *10\cdot11\cdot21 . *91\cdot1\cdot33 . \supset$

$$\vdash : \breve{P} \in \text{Potid`}\breve{R} . \supset . P \in \text{Potid`}R \quad (6)$$

$$\vdash . (6) \frac{\breve{P}, \breve{R}}{P, R} . *31\cdot33 . \supset \vdash : P \in \text{Potid`}R . \supset . \breve{P} \in \text{Potid`}\breve{R} \quad (7)$$

$\vdash . (6) . (7) . \supset \vdash . \text{Prop}$

***91·522.** $\vdash : P \in \text{Pot`}R . \equiv . \breve{P} \in \text{Pot`}\breve{R}$ [Proof as in *91·521]

***91·53.** $\vdash . \breve{R}_{po} = (\breve{R})_{po}$

Dem.

$$\begin{aligned} &\vdash . *91\cdot52 . \supset \vdash . \breve{R}_{po} = \breve{R} | \breve{R}_* \\ &[*90\cdot132] \qquad = \breve{R} | (\breve{R})_* \\ &[*91\cdot52] \qquad = (\breve{R})_{po} . \supset \vdash . \text{Prop} \end{aligned}$$

***91·54.** $\vdash . R_* = I \upharpoonright C\text{`}R \cup R_{po}$ [*90·31 . *91·52]

***91·541.** $\vdash . R_* \dot{\cap} J = R_{po} \dot{\cap} J$ [*25·401 . (*50·02) . *35·441 . *91·54]

***91·542.** $\vdash : xR_*y . x \neq y . \equiv . xR_{po}y . x \neq y$ [*91·541 . *50·11]

***91·543.** $\vdash . R_*\text{``}\beta = (\beta \cap C\text{`}R) \cup R_{po}\text{``}\beta$

Dem.

$$\begin{aligned} &\vdash . *91\cdot54 . *37\cdot221 . \supset \vdash . R_*\text{``}\beta = (I \upharpoonright C\text{`}R)\text{``}\beta \cup R_{po}\text{``}\beta \\ &[*50\cdot59] \qquad = (\beta \cap C\text{`}R) \cup R_{po}\text{``}\beta . \supset \vdash . \text{Prop} \end{aligned}$$

***91·544.** $\vdash . \breve{R}_*``\beta = (\beta \cap C`R) \cup \breve{R}_{po}``\beta$

***91·545.** $\vdash : \beta \subset C`R . \supset . R_*``\beta = \beta \cup R_{po}``\beta$ [*91·543 . *22·621]

***91·546.** $\vdash : \beta \subset C`R . \supset . \breve{R}_*``\beta = \beta \cup \breve{R}_{po}``\beta$

***91·55.** $\vdash . R_* = \dot{s}`\text{Potid}`R$

Dem.

$\vdash . *91·23 . \supset \vdash . \dot{s}`\text{Potid}`R = \dot{s}`\{\iota`(I \upharpoonright C`R) \cup \text{Pot}`R\}$
[*53·17.(*91·05)] $= I \upharpoonright C`R \,\dot{\cup}\, R_{po}$
[*91·54] $= R_* . \supset \vdash . \text{Prop}$

***91·56.** $\vdash . R_{po}^2 \mathrel{\dot{\subset}} R_{po}$

Dem.

$\vdash . *91·52 . \supset \vdash . R_{po}^2 = R_* \,|\, R \,|\, R_* \,|\, R$
[*90·16] $\dot{\subset} R_* \,|\, R_* \,|\, R$
[*90·17] $\dot{\subset} R_* \,|\, R$
[*91·52] $\dot{\subset} R_{po} . \supset \vdash . \text{Prop}$

***91·561.** $\vdash : S \mathrel{\dot{\subset}} R_{po} . T \mathrel{\dot{\subset}} R_{po} . \supset . S \,|\, T \mathrel{\dot{\subset}} R_{po}$ [*34·34 . *91·56]

***91·562.** $\vdash : S \mathrel{\dot{\subset}} R_{po} . \supset . S \,|\, R \mathrel{\dot{\subset}} R_{po} . R \,|\, S \mathrel{\dot{\subset}} R_{po}$ [*91·561·502]

***91·57.** $\vdash . R_{po} = R \,\dot{\cup}\, R_{po} \,|\, R = R \,\dot{\cup}\, R \,|\, R_{po}$ [*90·32 . *91·52]

***91·571.** $\vdash . R_{po} \,|\, R = R \,|\, R_{po}$ [*91·52]

***91·572.** $\vdash . R_{po} \mathbin{\dot{-}} (R_{po} \,|\, R) \mathrel{\dot{\subset}} R$ [*91·57 . *22·9·43]

***91·573.** $\vdash . R_{po} \mathbin{\dot{-}} (R \,|\, R_{po}) \mathrel{\dot{\subset}} R$ [*91·571·572]

***91·574.** $\vdash . R_* \,|\, R_{po} = R_{po} \,|\, R_* = R_{po} = R \,|\, R_* = R_* \,|\, R$

Dem.

$\vdash . *91·52 . \supset \vdash . R_* \,|\, R_{po} = R_* \,|\, R_* \,|\, R$
[*90·17] $= R_* \,|\, R$ (1)
$\vdash . *91·52 . \supset \vdash . R_{po} \,|\, R_* = R \,|\, R_* \,|\, R_*$
[*90·17] $= R \,|\, R_*$ (2)
$\vdash . (1) . (2) . *91·52 . \supset \vdash . \text{Prop}$

***91·575.** $\vdash . R_{po}^2 = R \,|\, R_{po} = R_{po} \,|\, R = R^2 \,|\, R_* = R_* \,|\, R^2 = R \,|\, R_* \,|\, R$

Dem.

$\vdash . *91·574·52 . \supset \vdash . R_{po}^2 = R \,|\, R_{po} = R_{po} \,|\, R$ (1)
$\vdash . (1) . *91·52 . \supset \vdash . \text{Prop}$

***91·58.** $\vdash : P \in \text{Potid}`R . \supset . P \mathrel{\dot{\subset}} R_*$ [*91·55 . *41·13]

***91·581.** $\vdash : P \in \text{Pot}`R . \supset . P \mathrel{\dot{\subset}} R_{po}$ [*41·13 . (*91·05)]

***91·59.** $\vdash : R \mathrel{\dot{\subset}} S . \supset . R_{po} \mathrel{\dot{\subset}} S_{po}$

Dem.

$\vdash . *90·18 . \supset \vdash : \text{Hp} . \supset . R_* \mathrel{\dot{\subset}} S_* .$
[*34·34] $\supset . R_* \,|\, R \mathrel{\dot{\subset}} S_* \,|\, S .$
[*91·52] $\supset . R_{po} \mathrel{\dot{\subset}} S_{po} : \supset \vdash . \text{Prop}$

***91·6.** $\vdash : Q \epsilon \text{Pot}‘R . \supset . \text{Pot}‘Q \subset \text{Pot}‘R . Q_{po} \mathrel{\subset\!\!\!\cdot} R_{po}$

Dem.

$\vdash . *91·171 \dfrac{Q, S \epsilon \text{Pot}‘R}{R, \quad \phi S} . \supset$

$\vdash :. P \epsilon \text{Pot}‘Q : S \epsilon \text{Pot}‘R . \supset_S . S | Q \epsilon \text{Pot}‘R : Q \epsilon \text{Pot}‘R : \supset . P \epsilon \text{Pot}‘R$ (1)

$\vdash . *91·343 . \quad \supset \vdash :. Q \epsilon \text{Pot}‘R . \supset : S \epsilon \text{Pot}‘R . \supset_S . S | Q \epsilon \text{Pot}‘R$ (2)

$\vdash . (1) . (2) . \quad \supset \vdash : P \epsilon \text{Pot}‘Q . Q \epsilon \text{Pot}‘R . \supset . P \epsilon \text{Pot}‘R :$

$[\text{Exp}.*10·11·21] \supset \vdash : Q \epsilon \text{Pot}‘R . \supset . \text{Pot}‘Q \subset \text{Pot}‘R .$ (3)

$[*41·161] \qquad \supset . Q_{po} \mathrel{\subset\!\!\!\cdot} R_{po}$ (4)

$\vdash . (3) . (4) . \supset \vdash . \text{Prop}$

***91·601.** $\vdash . (R_{po})_{po} = R_{po}$

Dem.

$\vdash . *91·502 . \supset \vdash . R_{po} \mathrel{\subset\!\!\!\cdot} (R_{po})_{po}$ (1)

$\vdash . *91·171 \dfrac{R_{po}, S \mathrel{\subset\!\!\!\cdot} R_{po}}{R, \quad \phi S} . \supset$

$\vdash :. P \epsilon \text{Pot}‘R_{po} : S \mathrel{\subset\!\!\!\cdot} R_{po} . \supset_S . S | R_{po} \mathrel{\subset\!\!\!\cdot} R_{po} : R_{po} \mathrel{\subset\!\!\!\cdot} R_{po} : \supset . P \mathrel{\subset\!\!\!\cdot} R_{po}$ (2)

$\vdash . *34·34 . *91·56 . \quad \supset \vdash : S \mathrel{\subset\!\!\!\cdot} R_{po} . \supset_S . S | R_{po} \mathrel{\subset\!\!\!\cdot} R_{po}$ (3)

$\vdash . (2) . (3) . *23·42 . \supset \vdash : P \epsilon \text{Pot}‘R_{po} . \supset . P \mathrel{\subset\!\!\!\cdot} R_{po} :$

$[*41·151] \qquad \supset \vdash . (R_{po})_{po} \mathrel{\subset\!\!\!\cdot} R_{po}$ (4)

$\vdash . (1) . (4) . \supset \vdash . \text{Prop}$

***91·602.** $\vdash . (R_{po})_* = R_*$

Dem.

$\vdash . *91·54 . \supset \vdash . (R_{po})_* = I \upharpoonright C‘R_{po} \cup (R_{po})_{po}$

$[*91·504·601] \qquad = I \upharpoonright C‘R \cup R_{po}$

$[*91·54] \qquad = R_* . \supset \vdash . \text{Prop}$

***91·603.** $\vdash . (R_*)_{po} = R_*$

Dem.

$\vdash . *91·52 . \supset \vdash . (R_*)_{po} = (R_*)_* | R_*$

$[*90·4] \qquad = R_* | R_*$

$[*90·17] \qquad = R_* . \supset \vdash . \text{Prop}$

***91·62.** $\vdash :. x R_{po} y . \equiv : \breve{R}“\mu \subset \mu . \overleftarrow{R}‘x \subset \mu . \supset_\mu . y \epsilon \mu \quad [*91·52 . *90·36]$

This formula should be compared with $*90·1$, in which an analogous formula is given for R_*. It will be observed that here we do not require to add $x \epsilon C‘R$, for if $\overleftarrow{R}‘x = \Lambda$, the above formula leads to $x R_{po} y . \supset . y \epsilon \Lambda$, *i.e.* to $\sim(x R_{po} y)$. Hence $x R_{po} y . \supset . \exists ! \overleftarrow{R}‘x$, *i.e.* $x R_{po} y . \supset . x \epsilon \text{D}‘R$. It will be observed that $x R_{po} y$ holds whenever y belongs to every hereditary class which contains the immediate successors of x, whereas $x R_* y$ holds whenever y belongs to every hereditary class to which x itself belongs.

***91·7.** $\vdash . R_{po}“\text{Ꮯ}‘R = \text{D}‘R . \breve{R}_{po}“\text{D}‘R = \text{Ꮯ}‘R \quad [*91·504 . *37·25]$

***91·71.** $\vdash : R``\mu \subset \mu . \equiv . R_{po}``\mu \subset \mu . \equiv . R_*``\mu \subset \mu$

Dem.

$\vdash . *90·22·132 . \supset \vdash : R``\mu \subset \mu . \equiv . R_*``\mu \subset \mu .$ (1)

[*91·602] $\equiv . (R_{po})_*``\mu \subset \mu .$

$\left[(1)\frac{R_{po}}{R}\right]$ $\equiv . R_{po}``\mu \subset \mu$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***91·711.** $\vdash : R``\mu \subset \mu . \supset . R_{po}``\mu = R``\mu$

Dem.

$\vdash . *91·71·52 . *37·2 . \supset \vdash : \mathrm{Hp} . \supset . R_{po}``\mu \subset R``\mu$ (1)

$\vdash . *91·502 . \supset \vdash . R``\mu \subset R_{po}``\mu$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

The above proposition is used in the theory of minimum points in a series (*205·68).

***91·72.** $\vdash . R``(\alpha \cup R_{po}``\alpha) = R_{po}``\alpha$

Dem.

$\vdash . *37·22·33 . \supset \vdash . R``(\alpha \cup R_{po}``\alpha) = R``\alpha \cup (R \mid R_{po})``\alpha$

[*37·221] $= (R \cup R \mid R_{po})``\alpha$

[*91·57] $= R_{po}``\alpha . \supset \vdash . \mathrm{Prop}$

***91·721.** $\vdash . \breve{R}``(\alpha \cup \breve{R}_{po}``\alpha) = \breve{R}_{po}``\alpha$ $\left[*91·72\frac{\breve{R}}{R} . *91·53\right]$

***91·73.** $\vdash :. P, Q \in \mathrm{Potid}`R . P \neq Q . \supset : (\exists T) : T \in \mathrm{Pot}`R : Q = P \mid T . \vee . P = Q \mid T$

Dem.

$\vdash . *91·45 . \supset$

$\vdash :. \mathrm{Hp} . \supset : (\exists T) : T \in \mathrm{Potid}`R : Q = P \mid T . P \mid T \neq P . \vee . P = Q \mid T . Q \mid T \neq Q$ (1)

$\vdash . *91·27 . *50·62 . \supset \vdash : P \in \mathrm{Potid}`R . \supset . P \mid I \upharpoonright C`R = P :$

[Transp] $\supset \vdash : P, T \in \mathrm{Potid}`R . P \mid T \neq P . \supset . T \neq I \upharpoonright C`R$ (2)

$\vdash . (1) . (2) . \supset$

$\vdash :. \mathrm{Hp} . \supset : (\exists T) : T \in \mathrm{Potid}`R . T \neq I \upharpoonright C`R : Q = P \mid T . \vee . P = Q \mid T$ (3)

$\vdash . *91·23 . \supset \vdash : T \in \mathrm{Potid}`R . T \neq I \upharpoonright C`R . \supset . T \in \mathrm{Pot}`R$ (4)

$\vdash . (3) . (4) . \supset \vdash . \mathrm{Prop}$

***91·731.** $\vdash :. P, Q \in \mathrm{Potid}`R . P \neq Q . \supset : (\exists T) : T \in \mathrm{Pot}`R : Q = T \mid P . \vee . P = T \mid Q$
[*91·73·34]

By means of *91·73 or *91·731, the powers of R can often be arranged in a series, the rule of arrangement being that P comes earlier than Q if $Q = P \mid T$, and later in the converse case. But we shall only get an open series from this arrangement if $P \in \mathrm{Potid}`R . T \in \mathrm{Pot}`R . \supset_{P,T} . P \mid T \neq P$; otherwise the powers from a certain point onwards form a cyclic series.

***91·732.** $\vdash :. P, Q \epsilon \text{Potid}`R . P \neq Q . \supset :$

$$(\exists S) : S \epsilon \text{Potid}`R : Q = S \,|\, R \,|\, P . \text{v} . P = S \,|\, R \,|\, Q$$

Dem.

$\vdash . *91·731·24 . \supset$

$\vdash :. \text{Hp} . \supset : (\exists S, T) : S \epsilon \text{Potid}`R . T = S \,|\, R : Q = T \,|\, P . \text{v} . P = T \,|\, Q :$

$[*13·195] \supset : (\exists S) : S \epsilon \text{Potid}`R : Q = S \,|\, R \,|\, P . \text{v} . P = S \,|\, R \,|\, Q :. \supset \vdash . \text{Prop}$

***91·74.** $\vdash . \breve{R}``\overleftarrow{R_*}`x = \overleftarrow{R_{po}}`x . R``\overrightarrow{R_*}`x = \overrightarrow{R_{po}}`x$ [*91·52 . *37·302]

***91·75.** $\vdash . R_* \cup \breve{R}_* = R_* \cup \breve{R}_{po} = R_{po} \cup \breve{R}_* = R_{po} \cup I \upharpoonright C`R \cup \breve{R}_{po}$

Dem.

$\vdash . *50·5·51 . \quad \supset \vdash . \text{Cnv}`(I \upharpoonright C`R) = I \upharpoonright C`R .$

$[*91·54] \quad \supset \vdash . \breve{R}_* = I \upharpoonright C`R \cup \breve{R}_{po} . \quad (1)$

$[*91·54 . *23·56] \supset \vdash . R_* \cup \breve{R}_* = R_{po} \cup I \upharpoonright C`R \cup \breve{R}_{po} \quad (2)$

$[*91·54] \quad = R_* \cup \breve{R}_{po} \quad (3)$

$[(1)] \quad = R_{po} \cup \breve{R}_* \quad (4)$

$\vdash . (2) . (3) . (4) . \supset \vdash . \text{Prop}$

*92. POWERS OF ONE-MANY AND MANY-ONE RELATIONS

Summary of *92.

If $R \epsilon \text{Cls} \rightarrow 1$, it follows that, starting from a given term x, there is only one series of terms $x_1, x_2, x_3, \ldots$ such that

$$xRx_1 . x_1Rx_2 . x_2Rx_3 . \ldots .$$

Thus for example the relation of son to father is a $\text{Cls} \rightarrow 1$; and starting from a given man, the series of ancestors in the direct male line (which is the above series $x_1, x_2, x_3, \ldots$) is unique and determinate. A result of this property of many-one relations is that if, starting from a term y, we go backwards a certain number of steps to a term x, and then forward a greater number of steps to a term z, we must pass through y in going from x to z; while if the number of steps from x to z is less than that from x to y, z must lie on the road from x to y. These facts are expressed by the proposition:

$$R \epsilon \text{Cls} \rightarrow 1 . \supset . \breve{R}_* \,|\, R_* \subseteq R_* \cup \breve{R}_* .$$

In the present number, we have to establish various propositions of this kind.

We prove in this number various propositions which are used in the discussion of "families" in *96 and *97, and some which are used in the theory of finite and infinite. But on the whole the propositions of this number are not much used. The most important of them are the following:

***92·11.** $\vdash : R \epsilon 1 \rightarrow \text{Cls} . \supset . R_{\text{po}} \,|\, \breve{R} \subseteq R_* . R_{\text{po}} \,|\, \breve{R} = R_* \restriction \text{D}‘R$

with a similar proposition (*92·111) for $\text{Cls} \rightarrow 1$.

***92·132.** $\vdash : R \epsilon 1 \rightarrow \text{Cls} . Q, T \epsilon \text{Potid}‘R . \supset . Q \,|\, T \,|\, \breve{Q} \subseteq T$

with a similar proposition (*92·133) for $\text{Cls} \rightarrow 1$.

***92·14.** $\vdash : \text{G}‘R \subset \text{D}‘R . Q \epsilon \text{Pot}‘R . \supset . \text{D}‘Q = \text{D}‘R$

On this proposition, compare the remarks on *91·271 in the introduction to *91. If R is a serial relation, $\text{G}‘R \subset \text{D}‘R$ is the condition that the series may have no last term.

***92·31.** $\vdash : R \epsilon 1 \rightarrow \text{Cls} . \supset . R_* \,|\, \breve{R}_* = R_* \cup \breve{R}_*$

***92·311.** $\vdash : R \epsilon \text{Cls} \rightarrow 1 . \supset . \breve{R}_* \,|\, R_* = R_* \cup \breve{R}_*$

***92·1.** $\vdash : R \epsilon 1 \rightarrow \text{Cls} . \supset . \text{Potid}‘R \subset 1 \rightarrow \text{Cls}$

Dem.

$\vdash . *72·17 . *71·26 . \supset \vdash . I \restriction C‘R \epsilon 1 \rightarrow \text{Cls}$ (1)

$\vdash . *71·25 . \quad \supset \vdash :. \text{Hp} . \supset : S \epsilon 1 \rightarrow \text{Cls} . \supset . S \,|\, R \epsilon 1 \rightarrow \text{Cls}$ (2)

$\vdash . (1) . (2) . *91·17 . \supset \vdash . \text{Prop}$

***92·101.** $\vdash : R \epsilon \text{Cls} \to 1 . \supset . \text{Potid}'R \subset \text{Cls} \to 1$ [Proof as in *92·1]

***92·102.** $\vdash : R \epsilon 1 \to 1 . \supset . \text{Potid}'R \subset 1 \to 1$ [Proof as in *92·1]

***92·11.** $\vdash : R \epsilon 1 \to \text{Cls} . \supset . R_{po} | \breve{R} \in R_* . R_{po} | \breve{R} = R_* \upharpoonright D'R$

Dem.

$\vdash . *91\cdot52 . \quad \supset \vdash . R_{po} | \breve{R} = R_* | R | \breve{R}$ (1)

$\vdash . *71\cdot19 . \quad \supset \vdash : \text{Hp} . \supset . R | \breve{R} = I \upharpoonright D'R$ (2)

$\vdash . (1) . (2) . *50\cdot6 . \supset \vdash : \text{Hp} . \supset . R_{po} | \breve{R} = R_* \upharpoonright D'R$ (3)

$\vdash . (3) . *35\cdot441 . \quad \supset \vdash . \text{Prop}$

***92·111.** $\vdash : R \epsilon \text{Cls} \to 1 . \supset . \breve{R} | R_{po} \in R_* . \breve{R} | R_{po} = (\text{Ɑ}'R) \upharpoonleft R_*$
[Proof as in *92·11]

***92·112.** $\vdash : R \epsilon 1 \to \text{Cls} . \supset . R | R_{po} | \breve{R} = R_{po} \upharpoonright D'R$ [*92·11 . *91·52]

***92·113.** $\vdash : R \epsilon \text{Cls} \to 1 . \supset . \breve{R} | R_{po} | R = (\text{Ɑ}'R) \upharpoonleft R_{po}$ [*92·111 . *91·52]

***92·12.** $\vdash : R \epsilon 1 \to \text{Cls} . \text{Ɑ}'R \subset D'R . \supset . R_{po} | \breve{R} = R_*$ [*92·11 . *35·66]

***92·121.** $\vdash : R \epsilon \text{Cls} \to 1 . D'R \subset \text{Ɑ}'R . \supset . \breve{R} | R_{po} = R_*$ [*92·111 . *35·63]

***92·13.** $\vdash : R \epsilon 1 \to \text{Cls} . Q, T \epsilon \text{Potid}'R . \supset . T | Q | \breve{Q} = T \upharpoonright D'Q$

Dem.

$\vdash . *92\cdot1 . \supset \vdash : \text{Hp} . \supset . Q \epsilon 1 \to \text{Cls} .$

$[*71\cdot19] \quad \supset . Q | \breve{Q} = I \upharpoonright D'Q .$

$[*50\cdot6] \quad \supset . T | Q | \breve{Q} = T \upharpoonright D'Q : \supset \vdash . \text{Prop}$

***92·131.** $\vdash : R \epsilon \text{Cls} \to 1 . Q, T \epsilon \text{Potid}'R . \supset . \breve{Q} | Q | T = (\text{Ɑ}'Q) \upharpoonleft T$

In this number, when proofs have been given for $R \epsilon 1 \to \text{Cls}$, we shall omit the proofs of corresponding propositions for $R \epsilon \text{Cls} \to 1$, as these are always exactly analogous to the proofs for $R \epsilon 1 \to \text{Cls}$.

***92·132.** $\vdash : R \epsilon 1 \to \text{Cls} . Q, T \epsilon \text{Potid}'R . \supset . Q | T | \breve{Q} \in T$ [*92·13 . *91·34]

***92·133.** $\vdash : R \epsilon \text{Cls} \to 1 . Q, T \epsilon \text{Potid}'R . \supset . \breve{Q} | T | Q \in T$

***92·14.** $\vdash : \text{Ɑ}'R \subset D'R . Q \epsilon \text{Pot}'R . \supset . D'Q = D'R$

Dem.

$\vdash . *91\cdot271 . \supset \vdash :. \text{Hp} . \supset : \text{Ɑ}'Q \subset D'R :$

$[*37\cdot321] \quad \supset : D'(Q | R) = D'Q :$

$[*13\cdot182] \quad \supset : D'Q = D'R . \supset . D'(Q | R) = D'R$ (1)

$\vdash . *13\cdot15 . \quad \supset \vdash . D'R = D'R$ (2)

$\vdash . (1) . (2) . *91\cdot171 \dfrac{D'S = D'R}{\phi S} . \supset \vdash . \text{Prop}$

***92·141.** $\vdash : D'R \subset \text{Ɑ}'R . Q \epsilon \text{Pot}'R . \supset . \text{Ɑ}'Q = \text{Ɑ}'R$

***92·142.** $\vdash : \text{Ɑ}‘R \subset \text{D}‘R \,.\, Q \in \text{Potid}‘R \,.\, \supset \,.\, \text{D}‘Q = \text{D}‘R$

Dem.

$\vdash \,.\, *50·5·52 \,.\, \supset \vdash : Q = I \upharpoonright C‘R \,.\, \supset \,.\, \text{D}‘Q = C‘R$ (1)

$\vdash \,.\, *33·181 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, C‘R = \text{D}‘R$ (2)

$\vdash \,.\, (1) \,.\, (2) \,.\, \supset \vdash : \text{Hp} \,.\, Q = I \upharpoonright C‘R \,.\, \supset \,.\, \text{D}‘Q = \text{D}‘R$ (3)

$\vdash \,.\, *91·23 \,.\, \supset \vdash :. \text{Hp} \,.\, \supset : Q = I \upharpoonright C‘R \,.\, \text{v} \,.\, Q \in \text{Pot}‘R$ (4)

$\vdash \,.\, (3) \,.\, (4) \,.\, *92·14 \,.\, \supset \vdash \,.\, \text{Prop}$

***92·143.** $\vdash : \text{D}‘R \subset \text{Ɑ}‘R \,.\, Q \in \text{Potid}‘R \,.\, \supset \,.\, \text{Ɑ}‘Q = \text{Ɑ}‘R$

***92·144.** $\vdash : \text{Ɑ}‘R \subset \text{D}‘R \,.\, Q \in \text{Potid}‘R \,.\, \supset \,.\, \text{Ɑ}‘Q \subset \text{D}‘R \,.\, \text{Ɑ}‘Q \subset \text{D}‘Q$

Dem.

$\vdash \,.\, *91·271 \,.\, \supset \vdash : \text{Hp} \,.\, Q \in \text{Pot}‘R \,.\, \supset \,.\, \text{Ɑ}‘Q \subset \text{D}‘R$ (1)

$\vdash \,.\, *50·5·52 \,.\, \supset \vdash : Q = I \upharpoonright C‘R \,.\, \supset \,.\, \text{Ɑ}‘Q = C‘R$ (2)

$\vdash \,.\, *33·181 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, C‘R = \text{D}‘R$ (3)

$\vdash \,.\, (2) \,.\, (3) \,.\, *23·42 \,.\, \supset \vdash : \text{Hp} \,.\, Q = I \upharpoonright C‘R \,.\, \supset \,.\, \text{Ɑ}‘Q \subset \text{D}‘R$ (4)

$\vdash \,.\, *91·23 \,.\, \supset \vdash :. \text{Hp} \,.\, \supset : Q = I \upharpoonright C‘R \,.\, \text{v} \,.\, Q \in \text{Pot}‘R$ (5)

$\vdash \,.\, (1) \,.\, (4) \,.\, (5) \,.\, *92·142 \,.\, \supset \vdash \,.\, \text{Prop}$

***92·145.** $\vdash : \text{D}‘R \subset \text{Ɑ}‘R \,.\, Q \in \text{Potid}‘R \,.\, \supset \,.\, \text{D}‘Q \subset \text{Ɑ}‘R \,.\, \text{D}‘Q \subset \text{Ɑ}‘Q$

***92·146.** $\vdash : \text{Ɑ}‘R \subset \text{D}‘R \,.\, Q, T \in \text{Potid}‘R \,.\, \supset \,.\, T \upharpoonright \text{D}‘Q = T$

Dem.

$\vdash \,.\, *92·142·144 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, \text{D}‘Q = \text{D}‘R \,.\, \text{Ɑ}‘T \subset \text{D}‘R \,.$

$[*13·13] \quad \supset \,.\, \text{Ɑ}‘T \subset \text{D}‘Q \,.$

$[*35·66] \quad \supset \,.\, T \upharpoonright \text{D}‘Q = T : \supset \vdash \,.\, \text{Prop}$

***92·147.** $\vdash : \text{D}‘R \subset \text{Ɑ}‘R \,.\, Q, T \in \text{Potid}‘R \,.\, \supset \,.\, (\text{Ɑ}‘Q) \upharpoonleft T = T$

***92·15.** $\vdash : R \in 1 \to \text{Cls} \,.\, \text{Ɑ}‘R \subset \text{D}‘R \,.\, Q, T \in \text{Potid}‘R \,.\, \supset \,.\, T \,|\, Q \,|\, \breve{Q} = T$
$[*92·13·146]$

***92·151.** $\vdash : R \in \text{Cls} \to 1 \,.\, \text{D}‘R \subset \text{Ɑ}‘R \,.\, Q, T \in \text{Potid}‘R \,.\, \supset \,.\, \breve{Q} \,|\, Q \,|\, T = T$

***92·152.** $\vdash : R \in 1 \to \text{Cls} \,.\, \text{Ɑ}‘R \subset \text{D}‘R \,.\, Q, T \in \text{Potid}‘R \,.\, \supset \,.\, Q \,|\, T \,|\, \breve{Q} = T$
$[*92·15 \,.\, *91·34]$

***92·153.** $\vdash : R \in \text{Cls} \to 1 \,.\, \text{D}‘R \subset \text{Ɑ}‘R \,.\, Q, T \in \text{Potid}‘R \,.\, \supset \,.\, \breve{Q} \,|\, T \,|\, Q = T$

***92·16.** $\vdash :. R \in 1 \to \text{Cls} \,.\, P, Q \in \text{Potid}‘R \,.\, \supset :$

$$(\exists T) : T \in \text{Potid}‘R : P \,|\, \breve{Q} = T \upharpoonright \text{D}‘Q \,.\, \text{v} \,.\, P \,|\, \breve{Q} = \text{Cnv}‘(T \upharpoonright \text{D}‘P)$$

Dem.

$\vdash \,.\, *91·46 \,.\, \supset \vdash :. \text{Hp} \,.\, \supset : (\exists T) : T \in \text{Potid}‘R : Q = T \,|\, P \,.\, \text{v} \,.\, P = T \,|\, Q$ (1)

$\vdash \,.\, *92·13 \,.\, \supset \vdash : \text{Hp} \,.\, T \in \text{Potid}‘R \,.\, P = T \,|\, Q \,.\, \supset \,.\, P \,|\, \breve{Q} = T \upharpoonright \text{D}‘Q$ (2)

$\vdash \,.\, *92·13 \,.\, \supset \vdash : \text{Hp} \,.\, T \in \text{Potid}‘R \,.\, Q = T \,|\, P \,.\, \supset \,.\, Q \,|\, \breve{P} = T \upharpoonright \text{D}‘P \,.$

$[*34·2] \quad \supset \,.\, P \,|\, \breve{Q} = \text{Cnv}‘(T \upharpoonright \text{D}‘P)$ (3)

$\vdash \,.\, (1) \,.\, (2) \,.\, (3) \,.\, \supset \vdash \,.\, \text{Prop}$

***92·161.** $\vdash :. R \in \mathrm{Cls} \to 1 \,.\, P, Q \in \mathrm{Potid}`R \,.\, \supset :$

$$(\exists T) : T \in \mathrm{Potid}`\breve{R} : Q \,|\, P = (\mathrm{G}`Q) \upharpoonleft T \,.\, \mathrm{v} \,.\, \breve{Q} \,|\, P = \mathrm{Cnv}`\{(\mathrm{G}`P) \upharpoonleft T\}$$

***92·17.** $\vdash : R \in 1 \to \mathrm{Cls} \,.\, P, Q \in \mathrm{Potid}`R \,.\, \supset .\, (\exists T) \,.\, T \in \mathrm{Potid}`R \,.\, P \,|\, \breve{Q} \subset T \cup \breve{T}$

Dem.

$\vdash . *35·441 \,.\, \supset \vdash : P \,|\, \breve{Q} = T \upharpoonright \mathrm{D}`Q \,.\, \supset .\, P \,|\, \breve{Q} \subset T .$

$[*23·58] \qquad \supset .\, P \,|\, \breve{Q} \subset T \cup \breve{T} \qquad (1)$

$\vdash . *35·52·44 \,.\, \supset \vdash : P \,|\, \breve{Q} = \mathrm{Cnv}`(T \upharpoonright \mathrm{D}`P) \,.\, \supset .\, P \,|\, \breve{Q} \subset \breve{T} .$

$[*23·58] \qquad \supset .\, P \,|\, \breve{Q} \subset T \cup \breve{T} \qquad (2)$

$\vdash . (1) . (2) . *92·16 \,.\, \supset \vdash .$ Prop

***92·171.** $\vdash : R \in \mathrm{Cls} \to 1 \,.\, P, Q \in \mathrm{Potid}`R \,.\, \supset .\, (\exists T) \,.\, T \in \mathrm{Potid}`R \,.\, \breve{Q} \,|\, P \subset T \cup \breve{T}$

***92·18.** $\vdash : R \in 1 \to \mathrm{Cls} \,.\, \mathrm{G}`R \subset \mathrm{D}`R \,.\, P, Q \in \mathrm{Potid}`R \,.\, \supset .$

$$P \,|\, \breve{Q} \in \mathrm{Potid}`R \cup \mathrm{Potid}`\breve{R}$$

Dem.

$\vdash . *92·16·146 \,.\, \supset$

$\vdash :. \mathrm{Hp} \,.\, \supset : (\exists T) : T \in \mathrm{Potid}`R : P \,|\, \breve{Q} = T \,.\, \mathrm{v} \,.\, P \,|\, \breve{Q} = \breve{T} :$

$[*10·42] \supset : (\exists T) \,.\, T \in \mathrm{Potid}`R \,.\, P \,|\, \breve{Q} = T \,.\, \mathrm{v} \,.\, (\exists T) \,.\, T \in \mathrm{Potid}`R \,.\, P \,|\, \breve{Q} = \breve{T} :$

$[*91·521] \supset : (\exists T) \,.\, T \in \mathrm{Potid}`R \,.\, P \,|\, \breve{Q} = T \,.\, \mathrm{v} \,.\, (\exists T) \,.\, T \in \mathrm{Potid}`\breve{R} \,.\, P \,|\, \breve{Q} = T :$

$[*13·195] \supset : P \,|\, \breve{Q} \in \mathrm{Potid}`R \,.\, \mathrm{v} \,.\, P \,|\, \breve{Q} \in \mathrm{Potid}`\breve{R} :. \supset \vdash .$ Prop

***92·181.** $\vdash : R \in \mathrm{Cls} \to 1 \,.\, \mathrm{D}`R \subset \mathrm{G}`R \,.\, P, Q \in \mathrm{Potid}`R \,.\, \supset .$

$$\breve{Q} \,|\, P \in \mathrm{Potid}`R \cup \mathrm{Potid}`\breve{R}$$

***92·19.** $\vdash : R \in 1 \to \mathrm{Cls} \,.\, \mathrm{G}`R \subset \mathrm{D}`R \,.\, P, Q \in \mathrm{Potid}`R \,.\, \supset :$

$$P \,|\, \breve{Q} \in \mathrm{Potid}`R \,.\, \mathrm{v} \,.\, Q \,|\, \breve{P} \in \mathrm{Potid}`R$$

Dem.

$\vdash . *92·18 \,. \qquad \supset \vdash :. \mathrm{Hp} \,.\, \supset : P \,|\, \breve{Q} \in \mathrm{Potid}`R \,.\, \mathrm{v} \,.\, P \,|\, \breve{Q} \in \mathrm{Potid}`\breve{R} \qquad (1)$

$\vdash . *91·521 \,.\, *34·2 \,.\, \supset \vdash : P \,|\, \breve{Q} \in \mathrm{Potid}`\breve{R} \,.\, \equiv .\, Q \,|\, \breve{P} \in \mathrm{Potid}`R \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash .$ Prop

***92·191.** $\vdash : R \in \mathrm{Cls} \to 1 \,.\, \mathrm{D}`R \subset \mathrm{G}`R \,.\, P, Q \in \mathrm{Potid}`R \,.\, \supset :$

$$\breve{P} \,|\, Q \in \mathrm{Potid}`R \,.\, \mathrm{v} \,.\, \breve{Q} \,|\, P \in \mathrm{Potid}`R$$

***92·3.** $\vdash : R \in 1 \to \mathrm{Cls} \,.\, P, Q \in \mathrm{Potid}`R \,.\, \supset .\, P \,|\, \breve{Q} \subset R_* \cup \breve{R}_*$

Dem.

$\vdash . *91·58 \,.\, \supset \vdash : T \in \mathrm{Potid}`R \,.\, \supset .\, T \cup \breve{T} \subset R_* \cup \breve{R}_* :$

$[*23·44] \quad \supset \vdash : T \in \mathrm{Potid}`R \,.\, P \,|\, \breve{Q} \subset T \cup \breve{T} \,.\, \supset .\, P \,|\, \breve{Q} \subset R_* \cup \breve{R}_* :$

$[*10·11·23] \supset \vdash : (\exists T) \,.\, T \in \mathrm{Potid}`R \,.\, P \,|\, \breve{Q} \subset T \cup \breve{T} \,.\, \supset .\, P \,|\, \breve{Q} \subset R_* \cup \breve{R}_* \qquad (1)$

$\vdash . (1) . *92·17 \,.\, \supset \vdash .$ Prop

***92·301.** $\vdash : R \in \text{Cls} \to 1 \,.\, P, Q \in \text{Potid}`R \,.\, \supset .\, \breve{P} \,|\, Q \subseteq R_* \cup \breve{R}_*$

***92·31.** $\vdash : R \in 1 \to \text{Cls} \,.\, \supset .\, R_* \,|\, \breve{R}_* = R_* \cup \breve{R}_*$

Dem.

$\vdash .\, *90·14 \,.\, *50·64 \,.\, \supset \vdash .\, R_* = R_* \,|\, I \upharpoonright C`R$ (1)

$\vdash .\, *90·15·132 \,.\, *33·22 \,.\, \supset \vdash .\, I \upharpoonright C`R \subseteq \breve{R}_* \,.$

$[*34·34] \quad \supset \vdash .\, R_* \,|\, I \upharpoonright C`R \subseteq R_* \,|\, \breve{R}_* \,.$

$[(1)] \quad \supset \vdash .\, R_* \subseteq R_* \,|\, \breve{R}_*$ (2)

Similarly $\quad \vdash .\, \breve{R}_* \subseteq R_* \,|\, \breve{R}_*$ (3)

$\vdash .\, *91·55 \,.\, *90·132 \,.\, \supset \vdash .\, R_* \,|\, \breve{R}_* = \dot{s}`\text{Potid}`R \,|\, \dot{s}`\text{Potid}`\breve{R}$

$[*41·51] \quad = \dot{s}`\hat{T}\{(\exists P, Q) \,.\, P \in \text{Potid}`R \,.\, Q \in \text{Potid}`\breve{R} \,.\, T = P \,|\, Q\}$

$[*91·521] \quad = \dot{s}`\hat{T}\{(\exists P, Q) \,.\, P, Q \in \text{Potid}`R \,.\, T = P \,|\, \breve{Q}\}$ (4)

$\vdash .\, *92·3 \,.\, \supset \vdash :.\, \text{Hp} \,.\, \supset : (\exists P, Q) \,.\, P, Q \in \text{Potid}`R \,.\, T = P \,|\, \breve{Q} \,.\, \supset_T .\, T \subseteq R_* \cup \breve{R}_* :$

$[*41·151] \quad \supset : \dot{s}`\hat{T}\{(\exists P, Q) \,.\, P, Q \in \text{Potid}`R \,.\, T = P \,|\, \breve{Q}\} \subseteq R_* \cup \breve{R}_*$ (5)

$\vdash .\, (4) \,.\, (5) \,.\, \supset \vdash : \text{Hp} \,.\, \supset .\, R_* \,|\, \breve{R}_* \subseteq R_* \cup \breve{R}_*$ (6)

$\vdash .\, (2) \,.\, (3) \,.\, (6) \,.\, \supset \vdash .$ Prop

***92·311.** $\vdash : R \in \text{Cls} \to 1 \,.\, \supset .\, \breve{R}_* \,|\, R_* = R_* \cup \breve{R}_*$

***92·312.** $\vdash : R \in 1 \to 1 \,.\, \supset .\, R_* \,|\, \breve{R}_* = \breve{R}_* \,|\, R_* = R_* \cup \breve{R}_*$ [*92·31·311]

***92·32.** $\vdash : R \in 1 \to 1 \,.\, \supset .\, (R_* \cup \breve{R}_*) \,|\, (R \cup \breve{R}) \subseteq R_* \cup \breve{R}_*$

Dem.

$\vdash .\, *34·25·26 \,.\, \supset \vdash .\, (R_* \cup \breve{R}_*) | (R \cup \breve{R}) = R_* | R \cup R_* | \breve{R} \cup \breve{R}_* | R \cup \breve{R}_* | \breve{R}$ (1)

$\vdash .\, *90·16·132 \,.\, \supset \vdash .\, R_* \,|\, R \subseteq R_* \,.\, \breve{R}_* \,|\, \breve{R} \subseteq \breve{R}_*$ (2)

$\vdash .\, *90·151 \,.\, \supset \vdash .\, R_* \,|\, \breve{R} \subseteq R_* \,|\, \breve{R}_* \,.\, \breve{R}_* \,|\, R \subseteq \breve{R}_* \,|\, R_*$ (3)

$\vdash .\, (3) \,.\, *92·312 \,.\, \supset \vdash : \text{Hp} \,.\, \supset .\, R_* \,|\, \breve{R} \subseteq R_* \cup \breve{R}_* \,.\, \breve{R}_* \,|\, R \subseteq R_* \cup \breve{R}_*$ (4)

$\vdash .\, (1) \,.\, (2) \,.\, (4) \,.\, \supset \vdash .$ Prop

***92·33.** $\vdash : R \in 1 \to 1 \,.\, \supset .\, (R \cup \breve{R})_* = R_* \cup \breve{R}_*$

Dem.

$\vdash .\, *90·18 \,.\, \supset \vdash .\, R_* \subseteq (R \cup \breve{R})_* \,.\, \breve{R}_* \subseteq (R \cup \breve{R})_* \,.$

$[*23·59] \quad \supset \vdash .\, R_* \cup \breve{R}_* \subseteq (R \cup \breve{R})_*$ (1)

$\vdash .\, *33·272 \,.\, \supset \vdash .\, I \upharpoonright C`(R \cup \breve{R}) = I \upharpoonright C`R \,.$

$[*90·15 \,.\, *23·58] \quad \supset \vdash .\, I \upharpoonright C`(R \cup \breve{R}) \subseteq R_* \cup \breve{R}_*$ (2)

$\vdash .\, *92·32 \,.\, *34·34 \,.\, \supset \vdash :.\, \text{Hp} \,.\, \supset : S \subseteq R_* \cup \breve{R}_* \,.\, \supset .\, S \,|\, (R \cup \breve{R}) \subseteq R_* \cup \breve{R}_*$ (3)

$\vdash . (2) . (3) . *91{\cdot}17 \dfrac{R \cup \breve{R}, S \subset R_* \cup \breve{R}_*}{R, \qquad \phi S} . \supset$

$\vdash :. \text{Hp} . \supset : P \in \text{Potid}`(R \cup \breve{R}) . \supset_P . P \subset R_* \cup \breve{R}_* :$

[$*41{\cdot}151$] $\supset : \dot{s}`\text{Potid}`(R \cup \breve{R}) \subset R_* \cup \breve{R}_* :$

[$*91{\cdot}55$] $\supset : (R \cup \breve{R})_* \subset R_* \cup \breve{R}_*$ (4)

$\vdash . (1) . (4) . \supset \vdash .$ Prop

$*92{\cdot}34$. $\vdash : R \in 1 \to 1 . \supset . (R \cup \breve{R})_{po} = R_* \cup \breve{R}_*$

Dem.

$\vdash . *92{\cdot}33 . *91{\cdot}52 . \supset$

$\vdash : \text{Hp} . \supset .$

$(R \cup \breve{R})_{po} = (R_* \cup \breve{R}_*) | (R \cup \breve{R})$

[$*34{\cdot}25{\cdot}26$] $= R_* | R \cup \breve{R}_* | R \cup R_* | \breve{R} \cup \breve{R}_* | \breve{R}$

[$*91{\cdot}52{\cdot}54{\cdot}57$] $= R_{po} \cup (I \upharpoonright C`R \cup \breve{R} \cup \breve{R}_* | \breve{R}) | R \cup (I \upharpoonright C`R \cup R \cup R_* | R) | \breve{R} \cup \breve{R}_{po}$

[$*50{\cdot}65 . *71{\cdot}192 . *72{\cdot}59{\cdot}591$]

$= R_{po} \cup R \cup I \upharpoonright \text{Cl}`R \cup \breve{R}_* \upharpoonright \text{Cl}`R \cup \breve{R} \cup I \upharpoonright \text{D}`R \cup R_* \upharpoonright \text{D}`R \cup \breve{R}_{po}$

[$*35{\cdot}412 . *91{\cdot}502$]

$= R_{po} \cup I \upharpoonright C`R \cup \breve{R}_* \upharpoonright \text{Cl}`R \cup R_* \upharpoonright \text{D}`R \cup \breve{R}_{po}$

[$*91{\cdot}75$] $= R_* \cup \breve{R}_* \cup \breve{R}_* \upharpoonright \text{Cl}`R \cup R_* \upharpoonright \text{D}`R$

[$*35{\cdot}441$] $= R_* \cup \breve{R}_* : \supset \vdash .$ Prop

*93. INDUCTIVE ANALYSIS OF THE FIELD OF A RELATION

Summary of *93.

For this number, we introduce three new notations, of which the first two will be used constantly, especially in the theory of series, while the third will be seldom used except in the present section. The two which are constantly used are

$$xBP, \text{ meaning } x \in \mathrm{D}\text{‘}P - \text{ᗡ}\text{‘}P$$

and
$$x \min_P \alpha, \text{ meaning } x \in \alpha \cap C\text{‘}P - \breve{P}\text{“}\alpha,$$

i.e. x is a member of α and of $C\text{‘}P$, and no member of α precedes x in $C\text{‘}P$.

The letter B may be regarded as standing for "begins." Thus if we take any member y of $C\text{‘}P$, and proceed backwards and forwards as far as possible by P-steps, we obtain a series which may be called the "family" of y: this series, if it has a first term, has one which is a member of $\mathrm{D}\text{‘}P - \text{ᗡ}\text{‘}P$; thus the members of $\mathrm{D}\text{‘}P - \text{ᗡ}\text{‘}P$ are the beginners of families. For example, if P is the relation of a peer to his heir, "xBP" will mean "x is a peer who is not the heir of a peer"; thus x is the first of his family. If P is the relation of parent and child, "xBP" will be satisfied only by Adam and Eve; and so for other relations.

The definition of B is

$$B = \hat{x}\hat{P}\,(x \in \mathrm{D}\text{‘}P - \text{ᗡ}\text{‘}P) \quad \text{Df.}$$

Hence $\overrightarrow{B}\text{‘}P = \mathrm{D}\text{‘}P - \text{ᗡ}\text{‘}P$. If P is the generating relation of a series which has a first term, that first term is $B\text{‘}P$; if there is a last term it is $B\text{‘}\breve{P}$.

If α is any class, we may call a term x a *minimum* of α with respect to P if it is a member of α and of $C\text{‘}P$, but does not follow any member of α, *i.e.* is not a member of $\breve{P}\text{“}\alpha$. We denote this relation of x to α by "$\min_P$"; thus we have

$$x \min_P \alpha \,.\equiv.\, x \in \alpha \cap C\text{‘}P - \breve{P}\text{“}\alpha,$$

and the definition of $\min_P$ is

$$\min_P = \hat{x}\hat{\alpha}\,(x \in \alpha \cap C\text{‘}P - \breve{P}\text{“}\alpha) \quad \text{Df.}$$

We shall also, when convenient, write "$\min(P)$" in place of "$\min_P$."

We have
$$\overrightarrow{\min_P}\text{‘}\alpha = \alpha \cap C\text{‘}P - \breve{P}\text{“}\alpha.$$

If P is serial, $\overrightarrow{\min_P}\text{‘}\alpha$ reduces to a single term if it is not null; thus if a class α has a first term, this term is $\min_P\text{‘}\alpha$. We also put

$$\max_P = \min(\breve{P}) \quad \text{Df,}$$

and then $\max_P\text{‘}\alpha$, if it exists, is the last term of α in the P-series. Thus if α

is the class of peers, and P is the relation of father to son, $\overrightarrow{\min}_P{}^{\backprime}\alpha$ consists of those peers who are the first of their line, while $\overrightarrow{\max}_P{}^{\backprime}\alpha$ consists of those peers who are the last of their line. If α is a class of numbers, and P is the relation of less to greater, $\min_P{}^{\backprime}\alpha$ is the smallest member of α (if it exists), and $\max_P{}^{\backprime}\alpha$ is the largest (if it exists).

B and "$\max_P$" and "$\min_P$" will be used constantly in connection with series, where the two latter will be considered in detail, but the present number is more specially concerned with a less general idea, namely that of *generations.* Take, *e.g.*, the relation of parent and child; let us call it P. Then the first generation consists of those who are parents but not children, *i.e.* $\overrightarrow{B}{}^{\backprime}P$; the second consists of those who are children but not grandchildren, *i.e.* $\text{Ɑ}{}^{\backprime}P - \text{Ɑ}{}^{\backprime}P^2$, *i.e.* $\text{Ɑ}{}^{\backprime}P - \breve{P}{}^{\backprime\backprime}\text{Ɑ}{}^{\backprime}P$, *i.e.* $\overrightarrow{\min}_P{}^{\backprime}\text{Ɑ}{}^{\backprime}P$; the third consists of those who are grandchildren but not great-grandchildren, *i.e.* $\text{Ɑ}{}^{\backprime}P^2 - \text{Ɑ}{}^{\backprime}P^3$, *i.e.* $\text{Ɑ}{}^{\backprime}P^2 - \breve{P}{}^{\backprime\backprime}\text{Ɑ}{}^{\backprime}P^2$, *i.e.* $\overrightarrow{\min}_P{}^{\backprime}\text{Ɑ}{}^{\backprime}P^2$; and so on. Also we have

$$\overrightarrow{B}{}^{\backprime}P = \overrightarrow{\min}_P{}^{\backprime}\text{Ɑ}{}^{\backprime}(I \upharpoonright C{}^{\backprime}P);$$

hence the generations of P are $\overrightarrow{\min}_P{}^{\backprime\backprime}\text{Ɑ}{}^{\backprime\backprime}\text{Potid}{}^{\backprime}P$. Thus we put

$$\text{gen}{}^{\backprime}P = \overrightarrow{\min}_P{}^{\backprime\backprime}\text{Ɑ}{}^{\backprime\backprime}\text{Potid}{}^{\backprime}P \quad \text{Df},$$

where "gen" stands for "generation."

When P is a one-many relation, such as that of father and son, every generation is of the form $\breve{T}{}^{\backprime\backprime}\overrightarrow{B}{}^{\backprime}P$, where T is a power of P (including $I \upharpoonright C{}^{\backprime}P$). When P is not a one-many relation, this is not in general the case.

The generations of P do not in general exhaust the field of P. For x will only belong to a generation of P if x can be reached by successive P-steps starting from a member of $\overrightarrow{B}{}^{\backprime}P$. If some of the families constituting the field of P have no beginning, the members of these families will not belong to any generation of P. Such terms together constitute the class

$$p{}^{\backprime}\text{Ɑ}{}^{\backprime\backprime}\text{Pot}{}^{\backprime}P,$$

or

$$p{}^{\backprime}\text{Ɑ}{}^{\backprime\backprime}\text{Potid}{}^{\backprime}P,$$

which is the same class.

Thus the field of P may be divided into two mutually exclusive portions, $s{}^{\backprime}\text{gen}{}^{\backprime}P$ and $p{}^{\backprime}\text{Ɑ}{}^{\backprime\backprime}\text{Pot}{}^{\backprime}P$.

The present number begins with some elementary properties of B and $\min_P$ and $\max_P$. We then ($*93{\cdot}2$—$\cdot 275$) consider such properties of generations as do not demand any hypothesis as to P. We prove

$*93{\cdot}25$. $\vdash . \text{gen}{}^{\backprime}P \in \text{Cls}^2 \text{excl}$

$*93{\cdot}261$. $\vdash . p{}^{\backprime}\text{Ɑ}{}^{\backprime\backprime}\text{Pot}{}^{\backprime}P = p{}^{\backprime}\text{Ɑ}{}^{\backprime\backprime}\text{Potid}{}^{\backprime}P . p{}^{\backprime}\text{Ɑ}{}^{\backprime\backprime}\text{Pot}{}^{\backprime}P \subset \text{Ɑ}{}^{\backprime}P$

and we prove (*93·274·275) that $s\text{‘}\mathrm{gen}\text{‘}P$ and $p\text{‘}\mathrm{Ɑ}\text{“}\mathrm{Pot}\text{‘}P$ are mutually exclusive, and together constitute $C\text{‘}P$. We then proceed to a set of propositions (*93·3—·41) demanding that P should be one-many or many-one or one-one. We prove

***93·32.** $\vdash :. P \epsilon 1 \rightarrow \mathrm{Cls} . \supset : \alpha \epsilon \mathrm{gen}\text{‘}P . \equiv . (\exists T) . T \epsilon \mathrm{Potid}\text{‘}P . \alpha = \breve{T}\text{“}\overrightarrow{B}\text{‘}P$

***93·36.** $\vdash : P \epsilon 1 \rightarrow \mathrm{Cls} . \supset . s\text{‘}\mathrm{gen}\text{‘}P = \breve{P}_*\text{“}\overrightarrow{B}\text{‘}P$

***93·381.** $\vdash :. P \epsilon \mathrm{Cls} \rightarrow 1 . \supset : x \epsilon p\text{‘}\mathrm{Ɑ}\text{“}\mathrm{Pot}\text{‘}\breve{P} . \equiv . \overleftarrow{P_*}\text{‘}x \subset \mathrm{D}\text{‘}P . x \epsilon C\text{‘}P$

and various other properties of $\mathrm{gen}\text{‘}P$ and $p\text{‘}\mathrm{Ɑ}\text{“}\mathrm{Pot}\text{‘}P$ when $P \epsilon 1 \rightarrow \mathrm{Cls}$.

The propositions of this number are used throughout the rest of this section; they are also used in the cardinal theory of finite and infinite. The early propositions, down to *93·12 inclusive, are also used in the theory of series.

***93·01.** $B = \hat{x}\hat{P}\,(x \epsilon \mathrm{D}\text{‘}P - \mathrm{Ɑ}\text{‘}P)$ Df

***93·02.** $\mathrm{min}_P = \mathrm{min}\,(P) = \hat{x}\hat{\alpha}\,(x \epsilon \alpha \cap C\text{‘}P - \breve{P}\text{“}\alpha)$ Df

***93·021.** $\mathrm{max}_P = \mathrm{max}\,(P) = \mathrm{min}\,(\breve{P})$ Df

***93·03.** $\mathrm{gen}\text{‘}P = \overrightarrow{\mathrm{min}_P}\text{“}\mathrm{Ɑ}\text{“}\mathrm{Potid}\text{‘}P$ Df

***93·1.** $\vdash : xBP . \equiv . x \epsilon \mathrm{D}\text{‘}P - \mathrm{Ɑ}\text{‘}P$ [*21·3 . (*93·01)]

***93·101.** $\vdash . \overrightarrow{B}\text{‘}P = \mathrm{D}\text{‘}P - \mathrm{Ɑ}\text{‘}P$ [*93·1 . *32·18]

***93·102.** $\vdash : x = B\text{‘}P . \equiv . x = \breve{\iota}\text{‘}(\mathrm{D}\text{‘}P - \mathrm{Ɑ}\text{‘}P) . \equiv . \mathrm{D}\text{‘}P - \mathrm{Ɑ}\text{‘}P \epsilon 1 . x \epsilon \mathrm{D}\text{‘}P - \mathrm{Ɑ}\text{‘}P$
[*93·101 . *53·4]

***93·103.** $\vdash . \overrightarrow{B}\text{‘}P = C\text{‘}P - \mathrm{Ɑ}\text{‘}P$

Dem.

$\vdash . *22·9 . *33·16 . \supset \vdash . C\text{‘}P - \mathrm{Ɑ}\text{‘}P = \mathrm{D}\text{‘}P - \mathrm{Ɑ}\text{‘}P$ (1)

$\vdash . (1) . *93·101 . \supset \vdash . \mathrm{Prop}$

***93·104.** $\vdash : xBR . \supset . \overrightarrow{R_*}\text{‘}x = \iota\text{‘}x . \overrightarrow{R_{\mathrm{po}}}\text{‘}x = \Lambda$

Dem.

$\vdash . *93·1 . \quad \supset \vdash : \mathrm{Hp} . \supset . x \epsilon C\text{‘}R .$

[*90·12] $\quad \supset . x \epsilon \overrightarrow{R_*}\text{‘}x$ (1)

$\vdash . *91·504 . \quad \supset \vdash : \exists ! \overrightarrow{R_{\mathrm{po}}}\text{‘}x . \supset . x \epsilon \mathrm{Ɑ}\text{‘}R :$

[Transp.*93·1] $\supset \vdash : xBR . \supset . \overrightarrow{R_{\mathrm{po}}}\text{‘}x = \Lambda$ (2)

$\vdash . *91·542 . \quad \supset \vdash : yR_*x . y \neq x . \supset . yR_{\mathrm{po}}x :$

[*32·18] $\quad \supset \vdash :. yR_*x . \supset : y = x . \vee . y \epsilon \overrightarrow{R_{\mathrm{po}}}\text{‘}x$ (3)

$\vdash . (2) . (3) . \quad \supset \vdash :. \mathrm{Hp} . \supset : yR_*x . \supset . y = x$ (4)

$\vdash . (1) . (4) . \quad \supset \vdash : \mathrm{Hp} . \supset . \overrightarrow{R_*}\text{‘}x = \iota\text{‘}x$ (5)

$\vdash . (2) . (5) . \supset \vdash . \mathrm{Prop}$

***93·11.** $\vdash : x \min_P \alpha . \equiv . x \in \alpha \cap C\text{‘}P - \breve{P}\text{‘‘}\alpha$ [(*93·02)]

***93·111.** $\vdash . \overrightarrow{\min}_P\text{‘}\alpha = \alpha \cap C\text{‘}P - \breve{P}\text{‘‘}\alpha$ [*93·11 . *32·18]

***93·112.** $\vdash . \overrightarrow{B}\text{‘}P = \overrightarrow{\min}_P\text{‘}\mathrm{D}\text{‘}P = \overrightarrow{\min}_P\text{‘}C\text{‘}P$

Dem.

$$\begin{aligned}
&\vdash . *93\cdot111 . \supset \vdash . \overrightarrow{\min}_P\text{‘}\mathrm{D}\text{‘}P = \mathrm{D}\text{‘}P - \breve{P}\text{‘‘}\mathrm{D}\text{‘}P \\
&[*37\cdot25] \qquad = \mathrm{D}\text{‘}P - \text{Ɑ}\text{‘}P \\
&[*93\cdot101] \qquad = \overrightarrow{B}\text{‘}P \qquad (1) \\
&\text{Similarly} \quad \vdash . \overrightarrow{\min}_P\text{‘}C\text{‘}P = \overrightarrow{B}\text{‘}P \qquad (2) \\
&\vdash . (1) . (2) . \supset \vdash . \text{Prop}
\end{aligned}$$

***93·113.** $\vdash . \overrightarrow{\min}_P\text{‘}\alpha \subset \alpha \cap C\text{‘}P$ [*93·111]

***93·114.** $\vdash . \max_P = \min(\breve{P})$ [(*93·021)]

***93·115.** $\vdash : x \max_P \alpha . \equiv . x \in \alpha \cap C\text{‘}P - P\text{‘‘}\alpha$ [*93·11·114]

***93·116.** $\vdash . \overrightarrow{\max}_P\text{‘}\alpha = \alpha \cap C\text{‘}P - P\text{‘‘}\alpha$ [*93·115 . *32·18]

***93·117.** $\vdash . \overrightarrow{B}\text{‘}\breve{P} = \overrightarrow{\max}_P\text{‘}\text{Ɑ}\text{‘}P = \overrightarrow{\max}_P\text{‘}C\text{‘}P$ [*93·112·114]

***93·118.** $\vdash . \overrightarrow{\max}_P\text{‘}\alpha \subset \alpha \cap C\text{‘}P$ [*93·116]

***93·12.** $\vdash . \overrightarrow{B}\text{‘}\breve{P} = \text{Ɑ}\text{‘}P - \mathrm{D}\text{‘}P = C\text{‘}P - \mathrm{D}\text{‘}P$ [*93·101·103 . *33·2·21·22]

***93·13.** $\vdash . \overrightarrow{\min}_P\text{‘}\text{Ɑ}\text{‘}(I \upharpoonright C\text{‘}P) = \overrightarrow{B}\text{‘}P$ [*50·5·52 . *93·112]

***93·131.** $\vdash . \overrightarrow{\min}_P\text{‘}\text{Ɑ}\text{‘}P = \text{Ɑ}\text{‘}P - \text{Ɑ}\text{‘}P^2$

Dem.

$$\begin{aligned}
&\vdash . *93\cdot111 . \supset \vdash . \overrightarrow{\min}_P\text{‘}\text{Ɑ}\text{‘}P = \text{Ɑ}\text{‘}P - \breve{P}\text{‘‘}\text{Ɑ}\text{‘}P \\
&[*37\cdot36] \qquad = \text{Ɑ}\text{‘}P - \text{Ɑ}\text{‘}P^2 . \supset \vdash . \text{Prop}
\end{aligned}$$

***93·132.** $\vdash . \overrightarrow{\min}_P\text{‘}\text{Ɑ}\text{‘}T = C\text{‘}P \cap \text{Ɑ}\text{‘}T - \text{Ɑ}\text{‘}(T \mid P)$

Dem.

$$\begin{aligned}
&\vdash . *93\cdot111 . \supset \vdash . \overrightarrow{\min}_P\text{‘}\text{Ɑ}\text{‘}T = C\text{‘}P \cap \text{Ɑ}\text{‘}T - \breve{P}\text{‘‘}\text{Ɑ}\text{‘}T \\
&[*37\cdot32] \qquad = C\text{‘}P \cap \text{Ɑ}\text{‘}T - \text{Ɑ}\text{‘}(T \mid P) . \supset \vdash . \text{Prop}
\end{aligned}$$

***93·2.** $\vdash : \alpha \in \text{gen}\text{‘}P . \equiv . (\exists T) . T \in \text{Potid}\text{‘}P . \alpha = \overrightarrow{\min}_P\text{‘}\text{Ɑ}\text{‘}T$
[*37·67 . (*93·03)]

***93·21.** $\vdash : \alpha \in \text{gen}\text{‘}P . \equiv . (\exists T) . T \in \text{Potid}\text{‘}P . \alpha = \text{Ɑ}\text{‘}T - \text{Ɑ}\text{‘}(T \mid P)$
[*93·2·132 . *91·27]

***93·22.** $\vdash . \overrightarrow{B}\text{‘}P \in \text{gen}\text{‘}P$ [*93·2·13 . *91·35]

***93·221.** $\vdash . \text{Ɑ}\text{‘}P - \text{Ɑ}\text{‘}P^2 \in \text{gen}\text{‘}P$ [*93·2·131 . *91·351·23]

***93·23.** $\vdash . \mathrm{gen}\text{‘}P = \iota\text{‘}\overrightarrow{B}\text{‘}P \cup \overrightarrow{\mathrm{min}}_P\text{“}\text{Ɑ}\text{“}\mathrm{Pot}\text{‘}P$

Dem.

$\vdash . *91·23 . *37·22 . \supset$

$\vdash . \mathrm{gen}\text{‘}P = \overrightarrow{\mathrm{min}}_P\text{“}\text{Ɑ}\text{“}\iota\text{‘}(I \upharpoonright C\text{‘}P) \cup \overrightarrow{\mathrm{min}}_P\text{“}\text{Ɑ}\text{“}\mathrm{Pot}\text{‘}P$

$[*53·31] \quad = \iota\text{‘}\overrightarrow{\mathrm{min}}_P\text{‘}\text{Ɑ}\text{‘}(I \upharpoonright C\text{‘}P) \cup \overrightarrow{\mathrm{min}}_P\text{“}\text{Ɑ}\text{“}\mathrm{Pot}\text{‘}P$

$[*93·13] \quad = \iota\text{‘}\overrightarrow{B}\text{‘}P \cup \overrightarrow{\mathrm{min}}_P\text{“}\text{Ɑ}\text{“}\mathrm{Pot}\text{‘}P$

***93·231.** $\vdash :. S, T \in \mathrm{Potid}\text{‘}P . S \neq T . \supset : \text{Ɑ}\text{‘}S \subset \breve{P}\text{“}\text{Ɑ}\text{‘}T . \vee . \text{Ɑ}\text{‘}T \subset \breve{P}\text{“}\text{Ɑ}\text{‘}S$

Dem.

$\vdash . *91·732 . \supset$

$\vdash :. \mathrm{Hp} . \supset : (\exists M) : S = M \mid P \mid T . \vee . T = M \mid P \mid S :$

$[*91·3] \quad \supset : (\exists M) : S = M \mid T \mid P . \vee . T = M \mid S \mid P \qquad (1)$

$\vdash . *34·36 . \quad \supset \vdash : S = M \mid T \mid P . \supset . \text{Ɑ}\text{‘}S \subset \text{Ɑ}\text{‘}(T \mid P) .$

$[*37·32] \qquad \supset . \text{Ɑ}\text{‘}S \subset \breve{P}\text{“}\text{Ɑ}\text{‘}T \qquad (2)$

Similarly $\quad \vdash : T = M \mid S \mid P . \supset . \text{Ɑ}\text{‘}T \subset \breve{P}\text{“}\text{Ɑ}\text{‘}S \qquad (3)$

$\vdash . (1) . (2) . (3) . \supset \vdash . \mathrm{Prop}$

***93·24.** $\vdash : S, T \in \mathrm{Potid}\text{‘}P . S \neq T . \supset . \overrightarrow{\mathrm{min}}_P\text{‘}\text{Ɑ}\text{‘}S \cap \overrightarrow{\mathrm{min}}_P\text{‘}\text{Ɑ}\text{‘}T = \Lambda$

Dem.

$\vdash . *24·3 . \quad \supset \vdash : \text{Ɑ}\text{‘}S \subset \breve{P}\text{“}\text{Ɑ}\text{‘}T . \supset . \text{Ɑ}\text{‘}S - \breve{P}\text{“}\text{Ɑ}\text{‘}T = \Lambda .$

$[*24·34] \qquad \supset . \text{Ɑ}\text{‘}S \cap \text{Ɑ}\text{‘}T - \breve{P}\text{“}\text{Ɑ}\text{‘}T = \Lambda .$

$[*24·34] \qquad \supset . (\text{Ɑ}\text{‘}S - \breve{P}\text{“}\text{Ɑ}\text{‘}S) \cap (\text{Ɑ}\text{‘}T - \breve{P}\text{“}\text{Ɑ}\text{‘}T) = \Lambda .$

$[*93·111] \qquad \supset . \overrightarrow{\mathrm{min}}_P\text{‘}\text{Ɑ}\text{‘}S \cap \overrightarrow{\mathrm{min}}_P\text{‘}\text{Ɑ}\text{‘}T = \Lambda \qquad (1)$

$\vdash . (1)\dfrac{T, S}{S, T} . \supset \vdash : \text{Ɑ}\text{‘}T \subset \breve{P}\text{“}\text{Ɑ}\text{‘}S . \supset . \overrightarrow{\mathrm{min}}_P\text{‘}\text{Ɑ}\text{‘}S \cap \overrightarrow{\mathrm{min}}_P\text{‘}\text{Ɑ}\text{‘}T = \Lambda \qquad (2)$

$\vdash . (1) . (2) . *93·231 . \supset \vdash . \mathrm{Prop}$

***93·25.** $\vdash . \mathrm{gen}\text{‘}P \in \mathrm{Cls}^2\,\mathrm{excl}$

Dem.

$\vdash . *30·37 . \mathrm{Transp} . \supset$

$\vdash :. S, T \in \mathrm{Potid}\text{‘}P . \alpha = \overrightarrow{\mathrm{min}}_P\text{‘}\text{Ɑ}\text{‘}S . \beta = \overrightarrow{\mathrm{min}}_P\text{‘}\text{Ɑ}\text{‘}T . \alpha \neq \beta . \supset :$

$\qquad S, T \in \mathrm{Potid}\text{‘}P . S \neq T :$

$[*93·24] \supset : \alpha \cap \beta = \Lambda \qquad (1)$

$\vdash . (1) . *11·11·35·54 . \supset$

$\vdash :. (\exists S) . S \in \mathrm{Potid}\text{‘}P . \alpha = \overrightarrow{\mathrm{min}}_P\text{‘}\text{Ɑ}\text{‘}S : (\exists T) . T \in \mathrm{Potid}\text{‘}P . \beta = \overrightarrow{\mathrm{min}}_P\text{‘}\text{Ɑ}\text{‘}T :$

$\qquad \alpha \neq \beta : \supset . \alpha \cap \beta = \Lambda \qquad (2)$

$\vdash . (2) . *93·2 . \supset \vdash : \alpha, \beta \in \mathrm{gen}\text{‘}P . \alpha \neq \beta . \supset . \alpha \cap \beta = \Lambda \qquad (3)$

$\vdash . (3) . *84·1 . \supset \vdash . \mathrm{Prop}$

***93·26.** $\vdash : S, T \in \text{Potid}'P \,.\, T \in |S''\text{Pot}'P \,.\supset .\, \overrightarrow{\min_P}'\text{ᗡ}'S \cap \overrightarrow{\min_P}'\text{ᗡ}'T = \Lambda$

Dem.

$\vdash .\, *91{\cdot}24 \,.\supset \vdash : \text{Hp} \,.\supset .\, T \in |S''|P''\text{Potid}'P \,.$

$[*43{\cdot}111 . *37{\cdot}67] \quad \supset .\, (\exists M) \,.\, T = M|P|S \,.$

$[*91{\cdot}3] \quad \supset .\, (\exists M) \,.\, T = M|S|P \,.$

$[*34{\cdot}36 . *37{\cdot}32] \quad \supset .\, \text{ᗡ}'T \subset \breve{P}''\text{ᗡ}'S \,.$

$[*24{\cdot}3] \quad \supset .\, \text{ᗡ}'T - \breve{P}''\text{ᗡ}'S = \Lambda \,.$

$[*24{\cdot}34] \quad \supset .\, (\text{ᗡ}'S - \breve{P}''\text{ᗡ}'S) \cap (\text{ᗡ}'T - \breve{P}''\text{ᗡ}'T) = \Lambda \,.$

$[*93{\cdot}111 . *91{\cdot}27] \quad \supset .\, \overrightarrow{\min_P}'\text{ᗡ}'S \cap \overrightarrow{\min_P}'\text{ᗡ}'T = \Lambda : \supset \vdash .\, \text{Prop}$

***93·261.** $\vdash .\, p'\text{ᗡ}''\text{Pot}'P = p'\text{ᗡ}''\text{Potid}'P \,.\, p'\text{ᗡ}''\text{Pot}'P \subset \text{ᗡ}'P$

Dem.

$\vdash .\, *91{\cdot}23 \,. \quad \supset \vdash .\, \text{ᗡ}''\text{Potid}'P = \text{ᗡ}''\text{Pot}'P \cup \iota'\text{ᗡ}'(I \upharpoonright C'P)$

$[*50{\cdot}5{\cdot}52] \quad = \text{ᗡ}''\text{Pot}'P \cup \iota'C'P \quad (1)$

$\vdash .\, (1) \,.\, *53{\cdot}14 \,. \quad \supset \vdash .\, p'\text{ᗡ}''\text{Potid}'P = p'\text{ᗡ}''\text{Pot}'P \cap C'P \quad (2)$

$\vdash .\, *40{\cdot}12 \,.\, *91{\cdot}351 \,.\supset \vdash .\, p'\text{ᗡ}''\text{Pot}'P \subset \text{ᗡ}'P \quad (3)$

$\vdash .\, (2) \,.\, (3) \,.\, *22{\cdot}621 \,.\supset \vdash .\, \text{Prop}$

***93·27.** $\vdash :.\, x \in C'P \,.\supset : x \sim\in s'\text{gen}'P \,.\equiv .\, x \in p'\text{ᗡ}''\text{Pot}'P$

Dem.

$\vdash .\, *40{\cdot}11 \,.\, *10{\cdot}51 \,.\supset$

$\vdash :.\, x \sim\in s'\text{gen}'P \,.\equiv : \alpha \in \text{gen}'P \,.\supset_\alpha .\, x \sim\in \alpha :$

$[*93{\cdot}21] \quad \equiv : T \in \text{Potid}'P \,.\supset_T .\, x \sim\in \text{ᗡ}'T - \text{ᗡ}'(T|P) :$

$[*4{\cdot}53 . *5{\cdot}6] \quad \equiv : T \in \text{Potid}'P \,.\, x \in \text{ᗡ}'T \,.\supset_T .\, x \in \text{ᗡ}'(T|P) \quad (1)$

$\vdash .\, *50{\cdot}5{\cdot}52 \,.\supset \vdash : x \in C'P \,.\supset .\, x \in \text{ᗡ}'(I \upharpoonright C'P) \quad (2)$

$\vdash .\, (1) \,.\, (2) \,.\supset \vdash :: x \in C'P \,.\supset :.\, x \sim\in s'\text{gen}'P \,.\equiv :$

$x \in \text{ᗡ}'(I \upharpoonright C'P) : T \in \text{Potid}'P \,.\, x \in \text{ᗡ}'T \,.\supset_T .\, x \in \text{ᗡ}'(T|P) :$

$[*91{\cdot}371] \equiv : T \in \text{Potid}'P \,.\supset_T .\, x \in \text{ᗡ}'T :$

$[*40{\cdot}41] \quad \equiv : x \in p'\text{ᗡ}''\text{Potid}'P :$

$[*93{\cdot}261] \equiv : x \in p'\text{ᗡ}''\text{Pot}'P :: \supset \vdash .\, \text{Prop}$

***93·271.** $\vdash .\, C'P - s'\text{gen}'P = p'\text{ᗡ}''\text{Pot}'P$

Dem.

$\vdash .\, *5{\cdot}32 \,.\, *93{\cdot}27 \,.\supset \vdash : x \in C'P - s'\text{gen}'P \,.\equiv .\, x \in C'P \,.\, x \in p'\text{ᗡ}''\text{Pot}'P \,.$

$[*93{\cdot}261 . *4{\cdot}71] \quad \equiv .\, x \in p'\text{ᗡ}''\text{Pot}'P : \supset \vdash .\, \text{Prop}$

***93·272.** $\vdash .\, s'\text{gen}'P \subset C'P$

Dem.

$\vdash .\, *93{\cdot}2{\cdot}113 \,.\supset \vdash : \alpha \in \text{gen}'P \,.\supset .\, (\exists T) \,.\, T \in \text{Potid}'P \,.\, \alpha \subset \text{ᗡ}'T \,.$

$[*91{\cdot}27] \quad \supset .\, \alpha \subset C'P \quad (1)$

$\vdash .\, (1) \,.\, *40{\cdot}151 \,.\supset \vdash .\, \text{Prop}$

***93·273.** $\vdash . C`P - p`\text{Ɑ}``\mathrm{Pot}`P = s`\mathrm{gen}`P$ [*93·271·272 . *24·492]

***93·274.** $\vdash . C`P = s`\mathrm{gen}`P \cup p`\text{Ɑ}``\mathrm{Pot}`P$ [*24·411 . *93·271·272]

***93·275.** $\vdash . s`\mathrm{gen}`P \cap p`\text{Ɑ}``\mathrm{Pot}`P = \Lambda$ [*93·271 . *24·21]

***93·3.** $\vdash : P \in 1 \to \mathrm{Cls} . T \in \mathrm{Potid}`P . \supset . \overrightarrow{\min_P}`\text{Ɑ}`T = \breve{T}``\overrightarrow{B}`P$

Dem.

$\vdash . *71\cdot38 . *93\cdot101 . \supset \vdash : \mathrm{Hp} . \supset . \breve{T}``\overrightarrow{B}`P = \breve{T}``\mathrm{D}`P - \breve{T}``\text{Ɑ}`P$

$[*37\cdot25] \qquad = \breve{T}``\mathrm{D}`P - \breve{T}``\breve{P}``\mathrm{D}`P$

$[*37\cdot33 . *91\cdot3] \qquad = \breve{T}``\mathrm{D}`P - \breve{P}``\breve{T}``\mathrm{D}`P$

$[*93\cdot111 . *91\cdot27] \qquad = \overrightarrow{\min_P}`\breve{T}``\mathrm{D}`P$ (1)

$\vdash . *91\cdot271 . *37\cdot271 . \supset \vdash : T \in \mathrm{Pot}`P . \supset . \breve{T}``\mathrm{D}`P = \text{Ɑ}`T$ (2)

$\vdash . *50\cdot5\cdot51\cdot59 . \quad \supset \vdash : T = I \upharpoonright C`P . \supset . \breve{T}``\mathrm{D}`P = \mathrm{D}`P .$

$[*93\cdot112] \qquad \supset . \overrightarrow{\min_P}`\breve{T}``\mathrm{D}`P = \overrightarrow{B}`P$

$[*93\cdot13] \qquad = \overrightarrow{\min_P}`\text{Ɑ}`T$ (3)

$\vdash . (2) . (3) . *91\cdot23 . \quad \supset \vdash : T \in \mathrm{Potid}`P . \supset . \overrightarrow{\min_P}`\breve{T}``\mathrm{D}`P = \overrightarrow{\min_P}`\text{Ɑ}`T$ (4)

$\vdash . (1) . (4) . \supset \vdash . \mathrm{Prop}$

***93·31.** $\vdash : P \in 1 \to \mathrm{Cls} . \supset . \breve{P}``\overrightarrow{\min_P}`\text{Ɑ}`T = \overrightarrow{\min_P}`\text{Ɑ}`(T \mid P)$

Dem.

$\vdash . *71\cdot38 . *93\cdot111 . *37\cdot265 . \supset$

$\vdash : \mathrm{Hp} . \supset . \breve{P}``\overrightarrow{\min_P}`\text{Ɑ}`T = \breve{P}``\text{Ɑ}`T - \breve{P}``\breve{P}``\text{Ɑ}`T$

$[*37\cdot32] \qquad = \text{Ɑ}`(T \mid P) - \breve{P}``\text{Ɑ}`(T \mid P)$

$[*93\cdot111 . *34\cdot36] \qquad = \overrightarrow{\min_P}`\text{Ɑ}`(T \mid P) : \supset \vdash . \mathrm{Prop}$

***93·32.** $\vdash :. P \in 1 \to \mathrm{Cls} . \supset : \alpha \in \mathrm{gen}`P . \equiv . (\exists T) . T \in \mathrm{Potid}`P . \alpha = \breve{T}``\overrightarrow{B}`P$
[*93·2·3]

***93·33.** $\vdash : P \in 1 \to \mathrm{Cls} . \alpha \in \mathrm{gen}`P . \supset . \breve{P}``\alpha \in \mathrm{gen}`P$
[*93·2·31 . *91·28·281]

***93·34.** $\vdash : P \in 1 \to \mathrm{Cls} . \supset . \breve{P}``\overrightarrow{B}`P \in \mathrm{gen}`P$ [*93·22·33]

***93·35.** $\vdash : P \in 1 \to \mathrm{Cls} . \alpha \in \mathrm{gen}`P . T \in \mathrm{Potid}`P . \supset . \breve{T}``\alpha \in \mathrm{gen}`P$

Dem.

$\vdash . *91\cdot341 . *37\cdot33 . *34\cdot2 . \supset$

$\vdash : S, T \in \mathrm{Potid}`P . \alpha = \breve{S}``\overrightarrow{B}`P . \supset . S \mid T \in \mathrm{Potid}`P . \breve{T}``\alpha = \{\mathrm{Cnv}`(S \mid T)\}``\overrightarrow{B}`P$ (1)

$\vdash . (1) . *93\cdot32 . \supset \vdash : \mathrm{Hp}(1) . P \in 1 \to \mathrm{Cls} . \supset . \breve{T}``\alpha \in \mathrm{gen}`P$ (2)

$\vdash . (2) . *10\cdot11\cdot23\cdot35 . *93\cdot32 . \supset \vdash . \mathrm{Prop}$

***93·36.** $\vdash : P \in 1 \to \mathrm{Cls} \,.\supset .\, s`\mathrm{gen}`P = \breve{P}_*``\overrightarrow{B}`P$

Dem.

$\vdash .\, *93{\cdot}32 \,.\supset \vdash :: \mathrm{Hp} \,.\supset :.$

$y \in s`\mathrm{gen}`P \,.\equiv : (\exists T) \,.\, T \in \mathrm{Potid}`P \,.\, y \in \breve{T}``\overrightarrow{B}`P :$

$[*37{\cdot}105] \quad \equiv : (\exists T, x) \,.\, T \in \mathrm{Potid}`P \,.\, x \in \overrightarrow{B}`P \,.\, xTy :$

$[*11{\cdot}55] \quad \equiv : (\exists x) : x \in \overrightarrow{B}`P : (\exists T) \,.\, T \in \mathrm{Potid}`P \,.\, xTy :$

$[*41{\cdot}11] \quad \equiv : (\exists x) \,.\, x \in \overrightarrow{B}`P \,.\, x(\dot{s}`\mathrm{Potid}`P)\, y :$

$[*91{\cdot}55] \quad \equiv : (\exists x) \,.\, x \in \overrightarrow{B}`P \,.\, xP_*y :$

$[*37{\cdot}105] \quad \equiv : y \in \breve{P}_*``\overrightarrow{B}`P :: \supset \vdash .\, \mathrm{Prop}$

***93·37.** $\vdash : P \in 1 \to \mathrm{Cls} \,.\supset .\, C`P = \breve{P}_*``\overrightarrow{B}`P \cup p`\text{Ɑ}``\mathrm{Pot}`P \quad [*93{\cdot}274{\cdot}36]$

***93·38.** $\vdash :.\, P \in 1 \to \mathrm{Cls} \,.\supset : x \in p`\text{Ɑ}``\mathrm{Pot}`P \,.\equiv .\, \overrightarrow{P_*}`x \subset \text{Ɑ}`P \,.\, x \in C`P$

Dem.

$\vdash .\, *93{\cdot}271{\cdot}36 \,.\supset$

$\vdash :: \mathrm{Hp} \,.\supset :.\, x \in p`\text{Ɑ}``\mathrm{Pot}`P \,.\equiv : x \in C`P \,.\, x \sim\in \breve{P}_*``\overrightarrow{B}`P :$

$[*37{\cdot}105 . *10{\cdot}51] \quad \equiv : x \in C`P : yP_*x \,.\supset_y .\, y \sim\in \overrightarrow{B}`P :$

$[*93{\cdot}101 . *22{\cdot}84{\cdot}8] \quad \equiv : x \in C`P : yP_*x \,.\supset_y .\, y \in \text{Ɑ}`P \cup -\mathrm{D}`P :$

$[*90{\cdot}13 . *33{\cdot}16] \quad \equiv : x \in C`P : yP_*x \,.\supset_y .$

$\qquad y \in (\text{Ɑ}`P \cup -\mathrm{D}`P) \cap (\text{Ɑ}`P \cup \mathrm{D}`P) :$

$[*22{\cdot}69 . *24{\cdot}21] \quad \equiv : x \in C`P : yP_*x \,.\supset_y .\, y \in \text{Ɑ}`P :: \supset \vdash .\, \mathrm{Prop}$

***93·381.** $\vdash :.\, P \in \mathrm{Cls} \to 1 \,.\supset : x \in p`\text{Ɑ}``\mathrm{Pot}`\breve{P} \,.\equiv .\, \overleftarrow{P_*}`x \subset \mathrm{D}`P \,.\, x \in C`P$

***93·382.** $\vdash :.\, P \in 1 \to 1 \,.\supset : x \in p`\text{Ɑ}``\mathrm{Pot}`P \cap p`\text{Ɑ}``\mathrm{Pot}`\breve{P} \,.\equiv .$
$\overrightarrow{P_*}`x \cup \overleftarrow{P_*}`x \subset \mathrm{D}`P \cap \text{Ɑ}`P \,.\, x \in C`P \quad [*93{\cdot}38{\cdot}381{\cdot}261 \,.\, *90{\cdot}31{\cdot}311]$

***93·4.** $\vdash : P \in 1 \to \mathrm{Cls} \,.\, \text{Ɑ}`P \subset \mathrm{D}`P \,.\, \exists !\, \overrightarrow{B}`P \,.\, T \in \mathrm{Potid}`P \,.\supset .\, \exists !\, \overrightarrow{\min_P}`\text{Ɑ}`T$

Dem.

$\vdash .\, *93{\cdot}13 \,. \qquad \supset \vdash : \mathrm{Hp} \,.\supset .\, \exists !\, \overrightarrow{\min_P}`\text{Ɑ}`(I \upharpoonright C`P) \qquad (1)$

$\vdash .\, *93{\cdot}113 \,.\, *33{\cdot}181 \,.\supset \vdash :.\, \mathrm{Hp} \,.\supset : \overrightarrow{\min_P}`\text{Ɑ}`T \subset \mathrm{D}`P :$

$[*37{\cdot}431] \qquad \supset : \exists !\, \overrightarrow{\min_P}`\text{Ɑ}`T \,.\supset .\, \exists !\, \breve{P}``\overrightarrow{\min_P}`\text{Ɑ}`T .$

$[*93{\cdot}31] \qquad \supset .\, \exists !\, \overrightarrow{\min_P}`\text{Ɑ}`(T \mid P) \qquad (2)$

$\vdash .\, (1) \,.\, (2) \,.\, *91{\cdot}17 \,.\supset \vdash .\, \mathrm{Prop}$

***93·41.** $\vdash : P \in 1 \to \mathrm{Cls} \,.\, \text{Ɑ}`P \subset \mathrm{D}`P \,.\, \exists !\, \overrightarrow{B}`P \,.\supset .\, \mathrm{gen}`P \in \mathrm{Cls\ ex^2\ excl}$
$[*93{\cdot}2{\cdot}4{\cdot}25 \,.\, *84{\cdot}13 \,.\, *24{\cdot}63]$

***93·412.** $\vdash . \breve{P}``p`\text{Ɑ}``\mathrm{Pot}`P \subset p`\text{Ɑ}``\mathrm{Pot}`P$

Dem.

$\vdash . *93\cdot261 . \supset \vdash . \breve{P}``p`\text{Ɑ}``\mathrm{Pot}`P = \breve{P}``p`\text{Ɑ}``\mathrm{Potid}`P$

$[*40\cdot37] \qquad \subset p`\breve{P}```\text{Ɑ}``\mathrm{Potid}`P$

$[*43\cdot411] \qquad \subset p`\text{Ɑ}``|P``\mathrm{Potid}`P$

$[*91\cdot24] \qquad \subset p`\text{Ɑ}``\mathrm{Pot}`P . \supset \vdash . \mathrm{Prop}$

***93·42.** $\vdash : P \epsilon 1 \rightarrow \mathrm{Cls} . \supset . \breve{P}``p`\text{Ɑ}``\mathrm{Pot}`P = p`\text{Ɑ}``\mathrm{Pot}`P$

Dem.

$\vdash . *93\cdot261 . \supset \vdash . \breve{P}``p`\text{Ɑ}``\mathrm{Pot}`P = \breve{P}``p`\text{Ɑ}``\mathrm{Potid}`P \qquad (1)$

$\vdash . (1) . *72\cdot34 . *91\cdot35 . *10\cdot24 . \supset$

$\vdash : \mathrm{Hp} . \supset . \breve{P}``p`\text{Ɑ}``\mathrm{Pot}`P = p`\breve{P}```\text{Ɑ}``\mathrm{Potid}`P$

$[*43\cdot411] \qquad = p`\text{Ɑ}``|P``\mathrm{Potid}`P$

$[*91\cdot24] \qquad = p`\text{Ɑ}``\mathrm{Pot}`P : \supset \vdash . \mathrm{Prop}$

***93·431.** $\vdash . p`\text{Ɑ}``\mathrm{Pot}`P = p`\text{Ɑ}``|P``\mathrm{Pot}`P$

Dem.

$\vdash . *91\cdot264\cdot304 . \qquad \supset \vdash . \mathrm{Pot}`P = \iota`P \cup |P``\mathrm{Pot}`P .$

$[*53\cdot14] \qquad \supset \vdash . p`\text{Ɑ}``\mathrm{Pot}`P = \text{Ɑ}`P \cap p`\text{Ɑ}``|P``\mathrm{Pot}`P .$

$[*91\cdot271\cdot283 . *40\cdot151\cdot23] \supset \vdash . p`\text{Ɑ}``\mathrm{Pot}`P = p`\text{Ɑ}``|P``\mathrm{Pot}`P . \supset \vdash . \mathrm{Prop}$

The following propositions, not being needed in subsequent propositions, are here inserted without proof, merely for the sake of their intrinsic interest.

***93·5.** $\vdash : T \epsilon \mathrm{Potid}`P . \supset . \overrightarrow{P_{ts}}`T = \overrightarrow{P_{st}}`T = T|``\mathrm{Potid}`P = |T``\mathrm{Potid}`P$

***93·51.** $\vdash : T \epsilon \mathrm{Pot}`P . \supset . \mathrm{Pot}`T \subset \overrightarrow{P_{ts}}`T \subset \mathrm{Pot}`P$

***93·52.** $\vdash : T \epsilon \mathrm{Pot}`P . \supset . p`\text{Ɑ}``\mathrm{Pot}`T = p`\text{Ɑ}``\overrightarrow{P_{ts}}`T = p`\text{Ɑ}``\mathrm{Pot}`P$

***93·53.** $\vdash : S, T \epsilon \mathrm{Pot}`P . xSx . \supset . (\exists y) . y (S | T) x$

***93·54.** $\vdash : S \epsilon \mathrm{Pot}`P . xSx . \supset . x \epsilon p`\text{Ɑ}``\mathrm{Pot}`P$

***93·55.** $\vdash . C`(P_{po} \dot{\cap} I) \subset p`\text{Ɑ}``\mathrm{Pot}`P$

***93·56.** $\vdash : \dot{\exists} ! (P_{po} \dot{\cap} I) . \supset . \exists ! p`\text{Ɑ}``\mathrm{Pot}`P$

*94. ON POWERS OF RELATIVE PRODUCTS

Summary of *94.

In this number we shall be chiefly concerned with propositions connecting powers of $R \mid S$ with powers of $S \mid R$. If P is a power of $R \mid S$, $S \mid P \mid R$ will be a power of $S \mid R$. If P is a power of $R \mid S$, it is a product of the form

$$(R \mid S) \mid (R \mid S) \mid \ldots \mid (R \mid S).$$

If we transfer the initial R to the end, we get a power of $S \mid R$. Thus there is a power of $S \mid R$, say T, such that

$$P \mid R = R \mid T.$$

If $R \in 1 \rightarrow \text{Cls} \,.\, \text{Ɑ}‘(R \mid S) \subset \text{D}‘R$, we find

$$R \mid (S \mid R) \mid (S \mid R) \mid \ldots \mid (S \mid R) \mid \breve{R} = (R \mid S) \mid (R \mid S) \ldots (R \mid S)$$

by rearranging and observing that $R \mid \breve{R} = I \upharpoonright \text{D}‘R$. Thus

$$R \in 1 \rightarrow \text{Cls} \,.\, \text{Ɑ}‘(R \mid S) \subset \text{D}‘R \,.\, P \in \text{Pot}‘R \mid S \,.\supset .\, (\exists T) \,.\, T \in \text{Pot}‘S \mid R \,.\, P = R \mid T \mid \breve{R}.$$

Expressions of the form $R \mid T \mid \breve{R}$ are constantly needed. They will be specially dealt with in *150, and will occur constantly in the sequel.

The above connections of $\text{Pot}‘(R \mid S)$ and $\text{Pot}‘(S \mid R)$ are embodied in the following propositions:

***94·14.** $\vdash . \mid R“\text{Pot}‘(R \mid S) = R \mid “\text{Pot}‘(S \mid R)$

***94·21.** $\vdash . \text{Pot}‘(S \mid R) = (S \parallel R)“\{\text{Pot}‘(R \mid S) \cup \iota‘I\}$

***94·31.** $\vdash : R \in 1 \rightarrow \text{Cls} \,.\, \text{Ɑ}‘(R \mid S) \subset \text{D}‘R \,.\supset .\, \text{Pot}‘(R \mid S) = (R \parallel \breve{R})“\text{Pot}‘(S \mid R)$.

From *94·4 to *94·54, the propositions are all concerned with $p‘\text{Ɑ}“(R \mid S)$ and $p‘\text{Ɑ}“(S \mid R)$. We prove

***94·5.** $\vdash . p‘\text{Ɑ}“\text{Pot}‘(S \mid R) = p‘\text{Ɑ}“R \mid “\text{Pot}‘(S \mid R)$

***94·51.** $\vdash : R \in 1 \rightarrow \text{Cls} \,.\supset .\, p‘\text{Ɑ}“\text{Pot}‘(S \mid R) = \breve{R}“p‘\text{Ɑ}“\text{Pot}‘(R \mid S)$

Finally we prove (*94·53·54) that if either R is one-one and $\text{Ɑ}‘(R \mid S) \subset \text{D}‘R$, or S is one-one and $\text{Ɑ}‘(S \mid R) \subset \text{D}‘S$, then $p‘\text{Ɑ}“\text{Pot}‘(R \mid S)$ is similar to $p‘\text{Ɑ}“\text{Pot}‘(S \mid R)$.

The only proposition of this number which is ever subsequently referred to is the last, *94·64, which, owing to the fact that the Schröder-Bernstein theorem has been already proved (*73·88), is only used in *95·23. But *95·23 itself is never referred to again. The reader may therefore omit the reading of the propositions of this number (as also of *95) without detriment to the understanding of what follows; he should, however, read the summaries.

The chief importance of the propositions in the present number is when R and S fulfil the hypothesis of the Schröder-Bernstein theorem, *i.e.*

$$R, S \epsilon 1 \to 1 \,.\, \text{Ɑ}`R \subset \text{D}`S \,.\, \text{Ɑ}`S \subset \text{D}`R.$$

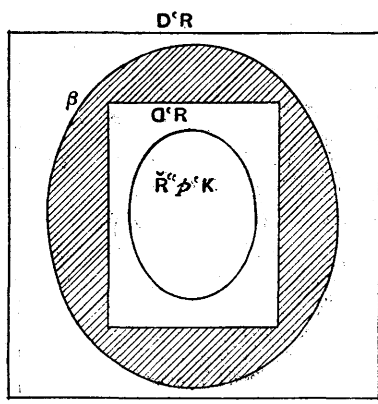
In this case, $R \mid S$ gives what we may call a "reflexion" of $\text{D}`R$ into part of itself; this part may be again reflected by $R \mid S$ into a part of itself, and so on. The terms in $\text{D}`R$ which are eliminated sooner or later by this process of reflexion constitute $s`\text{gen}`(R \mid S)$, since any one reflexion eliminates terms which constitute one generation of $R \mid S$. The terms not eliminated by any number of reflexions constitute $p`\text{Ɑ}``\text{Pot}`(R \mid S)$. These two sets of terms together constitute $\text{D}`(R \mid S)$, *i.e.* $\text{D}`R$. In this number and $*95$ we shall prove that, with the Schröder-Bernstein hypothesis,

$$s`\text{gen}`(R \mid S) \text{ sm } s`\text{gen}`(S \mid R) \,.\, p`\text{Ɑ}``\text{Pot}`(R \mid S) \text{ sm } p`\text{Ɑ}``\text{Pot}`(S \mid R).$$

These two propositions together yield a proof of the Schröder-Bernstein theorem, in virtue of $*93{\cdot}274{\cdot}275$. This proof is essentially the same as Bernstein's published originally by Borel*.

The nature of the two proofs of the Schröder-Bernstein theorem, namely Zermelo's (that given in $*73$) and Bernstein's (that to be given in this number and $*95$) will be best apprehended by means of figures.

In Zermelo's proof, we first prove that if R is one-one, and β is a class contained in $\text{D}`R$ and containing $\text{Ɑ}`R$, then β is similar both to $\text{D}`R$ and to $\text{Ɑ}`R$. In the figure, the points of the outer rectangle form $\text{D}`R$, those of the



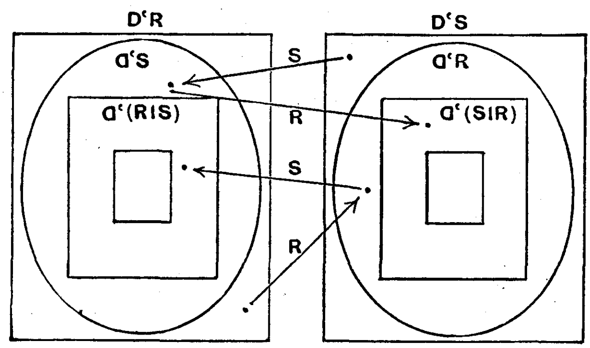
inner rectangle form $\text{Ɑ}`R$, and those of the outer oval form $\beta$. Thus the shaded portion of the figure is $\beta - \text{Ɑ}`R$. We now define a class of classes κ by the following characteristics: α is a member of κ if (1) α is contained in $\text{D}`R$, (2) α contains the whole of the shaded area, (3) $\breve{R}``\alpha \subset \alpha$, *i.e.* if x is a member of α, so is any term to which x has the relation R. Our proposition is obtained by considering $p`\kappa$, *i.e.* the area common to all the members of κ. We prove

* *Leçons sur la théorie des fonctions* (Paris, 1898), Note I (pp. 102—7).

(*73·81) that $p`\kappa \in \kappa$, and (*73·811) that $\breve{R}``p`\kappa$ does not contain any of the shaded area. In the figure, $\breve{R}``p`\kappa$ is the smaller oval. We then prove (*73·83) that $p`\kappa$ consists entirely of the shaded portion and the smaller oval. Hence β (the larger oval) consists of two mutually exclusive parts, namely $p`\kappa$ and $\text{Ɑ}`R - \breve{R}``p`\kappa$, the latter being that part of the inner rectangle which lies outside the inner oval. Assuming now that $R$ is one-one, $p`\kappa$ is similar to $\breve{R}``p`\kappa$; hence, adding $\text{Ɑ}`R - \breve{R}``p`\kappa$, it follows that $\beta$ is similar to $\text{Ɑ}`R$, and therefore to $\text{D}`R$.

In order to obtain hence the Schröder-Bernstein theorem, it is only necessary to replace R by $R \mid S$ and β by $\text{Ɑ}`S$, and to assume further that $S$ is a one-one whose domain contains $\text{Ɑ}`R$. Then $\text{D}`R = \text{D}`(R \mid S)$, and we obtain (*73·87) $\text{Ɑ}`S \,\text{sm}\, \text{D}`R$, and therefore $\text{D}`S \,\text{sm}\, \text{D}`R$, which was to be proved.

In Bernstein's proof, we have the two relations R and S from the beginning. In the left-hand part of the figure, the outer rectangle is $\text{D}`R$, which $= \text{D}`(R \mid S)$,



the oval is $\text{Ɑ}`S$, and the second rectangle is $\text{Ɑ}`(R \mid S)$. Thus the points of the outer but not the second rectangle form the first generation of $R \mid S$. Within $\text{Ɑ}`(R \mid S)$ we can form a third rectangle, which will be $\breve{S}``\breve{R}``\text{Ɑ}`(R \mid S)$, *i.e.* $\text{Ɑ}`(R \mid S)^2$. The points belonging to the second rectangle but not to the third form the second generation of $R \mid S$. We can proceed in this way to continually smaller rectangles. The points which sooner or later are left outside some rectangle form $s`\text{gen}`(R \mid S)$; those which are common to all the rectangles form $p`\text{Ɑ}``\text{Pot}`(R \mid S)$. A similar analysis, exhibited in the right-hand part of the figure, may be applied to $\text{D}`S$, which is thus divided into $s`\text{gen}`(S \mid R)$ and $p`\text{Ɑ}``\text{Pot}`(S \mid R)$. We prove in this number (*94·53) that, with a hypothesis which is part of the hypothesis of the Schröder-Bernstein theorem, $p`\text{Ɑ}``\text{Pot}`(R \mid S) \,\text{sm}\, p`\text{Ɑ}``\text{Pot}`(S \mid R)$; in the next number (*95·71) we prove that with the hypothesis of the Schröder-Bernstein theorem, $s`\text{gen}`(R \mid S) \,\text{sm}\, s`\text{gen}`(S \mid R)$. Hence by addition, $\text{D}`R \,\text{sm}\, \text{D}`S$.

***94·12.** $\vdash : P \epsilon \text{Pot}'(R \mid S) . \supset . (\exists T) . T \epsilon \text{Pot}'(S \mid R) . P \mid R = R \mid T$

Dem.

$\vdash . *34·21 . \quad \supset \vdash . (R \mid S) \mid R = R \mid (S \mid R)$ (1)

$\vdash . *91·36 . *34·27 . \supset \vdash : T \epsilon \text{Pot}'(S \mid R) . P \mid R = R \mid T . \supset .$

$T \mid S \mid R \epsilon \text{Pot}'(S \mid R) . P \mid R \mid S \mid R = R \mid T \mid S \mid R .$

[*10·24] $\supset . (\exists T') . T' \epsilon \text{Pot}'(S \mid R) . P \mid R \mid S \mid R = R \mid T'$ (2)

$\vdash . (2) . *10·11·23 . \supset \vdash : (\exists T) . T \epsilon \text{Pot}'(S \mid R) . P \mid R = R \mid T . \supset .$

$(\exists T') . T' \epsilon \text{Pot}'(S \mid R) . P \mid R \mid S \mid R = R \mid T'$ (3)

$\vdash . (1) . (3) . *91·171 . \supset \vdash . \text{Prop}$

***94·13.** $\vdash : T \epsilon \text{Pot}'(S \mid R) . \supset . (\exists P) . P \epsilon \text{Pot}'(R \mid S) . P \mid R = R \mid T$

[Proof as in *94·12]

***94·14.** $\vdash . \mid R``\text{Pot}'(R \mid S) = R \mid ``\text{Pot}'(S \mid R)$

Dem.

$\vdash . *94·12 . *43·111·1 . *37·1 . \supset \vdash : P \epsilon \text{Pot}'(R \mid S) . \supset . \mid R'P \epsilon R \mid ``\text{Pot}'(S \mid R) :$

[*37·61] $\supset \vdash . \mid R``\text{Pot}'(R \mid S) \subset R \mid ``\text{Pot}'(S \mid R)$ (1)

$\vdash . *94·13 . *43·11·101 . *37·1 . \supset$

$\vdash : T \epsilon \text{Pot}'(S \mid R) . \supset . R \mid 'T \epsilon \mid R``\text{Pot}'(R \mid S) :$

[*37·61] $\supset \vdash . R \mid ``\text{Pot}'(S \mid R) \subset \mid R``\text{Pot}'(R \mid S)$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***94·2.** $\vdash : P \epsilon \text{Pot}'(R \mid S) \cup \iota'I . \supset . S \mid P \mid R \epsilon \text{Pot}'(S \mid R)$

Dem.

$\vdash . *34·21 . \supset \vdash . S \mid (R \mid S) \mid R = (S \mid R)^2 .$

[*91·352] $\supset \vdash . S \mid (R \mid S) \mid R \epsilon \text{Pot}'(S \mid R)$ (1)

$\vdash . *34·21 . *91·282 . \supset$

$\vdash : S \mid P \mid R \epsilon \text{Pot}'(S \mid R) . \supset . S \mid (P \mid R \mid S) \mid R = (S \mid P \mid R) \mid S \mid R .$

$(S \mid P \mid R) \mid S \mid R \epsilon \text{Pot}'(S \mid R)$ (2)

$\vdash . (1) . (2) . *91·171 \dfrac{S \mid P \mid R \epsilon \text{Pot}'(S \mid R)}{\phi S} . \supset$

$\vdash : P \epsilon \text{Pot}'(R \mid S) . \supset . S \mid P \mid R \epsilon \text{Pot}'(S \mid R)$ (3)

$\vdash . *50·4 . *91·351 . \supset \vdash . S \mid I \mid R \epsilon \text{Pot}'(S \mid R)$ (4)

$\vdash . (3) . (4) . \supset \vdash . \text{Prop}$

***94·201.** $\vdash : T \epsilon \text{Pot}'(S \mid R) . \supset . (\exists P) . P \epsilon \text{Pot}'(R \mid S) \cup \iota'I . T = S \mid P \mid R$

Dem.

$\vdash . *50·4 . *51·16 . \quad \supset \vdash . S \mid R = S \mid I \mid R . I \epsilon \text{Pot}'(R \mid S) \cup \iota'I .$

[*10·24] $\supset \vdash . (\exists P) . P \epsilon \text{Pot}'(R \mid S) \cup \iota'I . S \mid R = S \mid P \mid R$ (1)

$\vdash . *91·282 . *34·21 . \supset \vdash : P \epsilon \text{Pot}'(R \mid S) . T = S \mid P \mid R . \supset .$

$P \mid R \mid S \epsilon \text{Pot}'(R \mid S) . T \mid S \mid R = S \mid (P \mid R \mid S) \mid R .$

[*10·24] $\supset . (\exists Q) . Q \epsilon \text{Pot}'(R \mid S) . T \mid S \mid R = S \mid Q \mid R$ (2)

$\vdash . *50{\cdot}4 . *34{\cdot}21 . \supset$

$\vdash : P = I . T = S | P | R . \supset . T | S | R = S | (R | S) | R .$

$[*91{\cdot}351] \qquad \supset . (\exists Q) . Q \in \mathrm{Pot}`(R | S) . T | S | R = S | Q | R \qquad (3)$

$\vdash . (2) . (3) . *10{\cdot}11{\cdot}23 . \supset \vdash : (\exists P) . P \in \mathrm{Pot}`(R | S) \cup \iota`I . T = S | P | R . \supset .$

$\qquad (\exists Q) . Q \in \mathrm{Pot}`(R | S) . T | S | R = S | Q | R .$

$[*22{\cdot}58] \qquad \supset . (\exists Q) . Q \in \mathrm{Pot}`(R | S) \cup \iota`I . T | S | R = S | Q | R \qquad (4)$

$\vdash . (1) . (4) . *91{\cdot}171 \dfrac{S | R, (\exists P) . P \in \mathrm{Pot}`(R | S) \cup \iota`I . T = S | P | R}{P, \qquad \phi T} . \supset \vdash . \mathrm{Prop}$

$*94{\cdot}21.$ $\vdash . \mathrm{Pot}`(S | R) = (S \parallel R)``\{\mathrm{Pot}`(R | S) \cup \iota`I\}$

Dem.

$\vdash . *94{\cdot}2 . *43{\cdot}112 . *37{\cdot}61 . \supset \vdash . (S \parallel R)``\{\mathrm{Pot}`(R | S) \cup \iota`I\} \subset \mathrm{Pot}`(S | R) \quad (1)$

$\vdash . *94{\cdot}201 . *43{\cdot}102 . *37{\cdot}1 . \supset \vdash . \mathrm{Pot}`(S | R) \subset (S \parallel R)``\{\mathrm{Pot}`(R | S) \cup \iota`I\} \quad (2)$

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

$*94{\cdot}22.$ $\vdash :. \text{Ɑ}`R \subset \mathrm{D}`S . \vee . \mathrm{D}`S \subset \text{Ɑ}`R : \supset .$

$\qquad \mathrm{Pot}`(S | R) = (S \parallel R)``\mathrm{Potid}`(R | S)$

Dem.

$\vdash . *94{\cdot}21 . *43{\cdot}112 . *50{\cdot}4 . *53{\cdot}31 . \supset$

$\vdash . \mathrm{Pot}`(S | R) = (S \parallel R)``\mathrm{Pot}`(R | S) \cup \iota`(S | R) \qquad (1)$

$\vdash . *37{\cdot}321 . \supset \vdash : \text{Ɑ}`R \subset \mathrm{D}`S . \supset . \mathrm{D}`R = \mathrm{D}`(R | S) .$

$[*33{\cdot}161] \qquad \supset . \mathrm{D}`R \subset C`(R | S) .$

$[*50{\cdot}63] \qquad \supset . I \upharpoonright C`(R | S) | R = R .$

$[*34{\cdot}28] \qquad \supset . S | I \upharpoonright C`(R | S) | R = S | R .$

$[*43{\cdot}112] \qquad \supset . (S \parallel R)`I \upharpoonright C`(R | S) = S | R \qquad (2)$

Similarly $\quad \vdash : \mathrm{D}`S \subset \text{Ɑ}`R . \supset . (S \parallel R)`I \upharpoonright C`(R | S) = S | R \qquad (3)$

$\vdash . (1) . (2) . (3) . \supset$

$\vdash : \mathrm{Hp} . \supset . \mathrm{Pot}`(S | R) = (S \parallel R)``\mathrm{Pot}`(R | S) \cup \iota`(S \parallel R)`I \upharpoonright C`(R | S)$

$[*91{\cdot}23] \qquad = (S \parallel R)``\mathrm{Potid}`(R | S) : \supset \vdash . \mathrm{Prop}$

$*94{\cdot}3.$ $\vdash :. R \in 1 \rightarrow \mathrm{Cls} . \text{Ɑ}`(R | S) \subset \mathrm{D}`R . \supset :$

$\qquad P \in \mathrm{Pot}`(R | S) . \equiv . (\exists T) . T \in \mathrm{Pot}`(S | R) . P = R | T | \breve{R}$

Dem.

$\vdash . *94{\cdot}12 . \supset \vdash : P \in \mathrm{Pot}`(R | S) . \supset . (\exists T) . T \in \mathrm{Pot}`(S | R) . P | R | \breve{R} = R | T | \breve{R} \quad (1)$

$\vdash . *91{\cdot}271 . \supset \vdash :. \mathrm{Hp} . \supset : P \in \mathrm{Pot}`(R | S) . \supset . \text{Ɑ}`P \subset \mathrm{D}`R .$

$[*72{\cdot}6] \qquad \supset . P | R | \breve{R} = P \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash :. \mathrm{Hp} . \supset : P \in \mathrm{Pot}`(R | S) . \supset .$

$\qquad (\exists T) . T \in \mathrm{Pot}`(S | R) . P = R | T | \breve{R} \qquad (3)$

$\vdash . *94{\cdot}13 . \supset \vdash : T \in \mathrm{Pot}`(S | R) . \supset .$

$\qquad (\exists P) . P \in \mathrm{Pot}`(R | S) . P | R | \breve{R} = R | T | \breve{R} \qquad (4)$

$\vdash . (2) . (4) . \supset \vdash :. \text{Hp} . \supset : T \epsilon \text{Pot}`(S \mid R) . \supset .$

$(\exists P) . P \epsilon \text{Pot}`(R \mid S) . P = R \mid T \mid \breve{R} .$

[∗13·195] $\supset . R \mid T \mid \breve{R} \epsilon \text{Pot}`(R \mid S) :$

[∗13·12] $\supset : T \epsilon \text{Pot}`(S \mid R) . P = R \mid T \mid \breve{R} . \supset . P \epsilon \text{Pot}`(R \mid S)$ (5)

$\vdash . (5) . ∗10·11·21·23 . \supset$

$\vdash :. \text{Hp} . \supset : (\exists T) . T \epsilon \text{Pot}`(S \mid R) . P = R \mid T \mid \breve{R} . \supset . P \epsilon \text{Pot}`(R \mid S)$ (6)

$\vdash . (3) . (6) . \supset \vdash . \text{Prop}$

∗94·31. $\vdash : R \epsilon 1 \rightarrow \text{Cls} . \text{Ɑ}`(R \mid S) \subset \text{D}`R . \supset . \text{Pot}`(R \mid S) = (R \parallel \breve{R})``\text{Pot}`(S \mid R)$ [∗94·3]

The following series of propositions lead up to the proof that when $R \epsilon 1 \rightarrow 1 . \text{Ɑ}`(R \mid S) \subset \text{D}`R$, or $S \epsilon 1 \rightarrow 1 . \text{Ɑ}`(S \mid R) \subset \text{D}`S$, we have

$$p`\text{Ɑ}``\text{Pot}`(R \mid S) \,\text{sm}\, p`\text{Ɑ}``\text{Pot}`(S \mid R).$$

∗94·4. $\vdash . p`\text{Ɑ}``\text{Pot}`(R \mid S) = p`\text{Ɑ}``\mid S``\mid R``\text{Pot}`(R \mid S)$

$= p`\breve{S}```\text{Ɑ}``\mid R``\text{Pot}`(R \mid S)$

$= p`\breve{S}```\breve{R}```\text{Ɑ}``\text{Pot}`(R \mid S)$

Dem.

$\vdash . ∗93·431 . \supset \vdash . p`\text{Ɑ}``\text{Pot}`(R \mid S) = p`\text{Ɑ}``\mid (R \mid S)``\text{Pot}`(R \mid S)$

[∗43·201.∗37·33] $= p`\text{Ɑ}``\mid S``\mid R``\text{Pot}`(R \mid S)$ (1)

[∗43·411] $= p`\breve{S}```\text{Ɑ}``\mid R``\text{Pot}`(R \mid S)$ (2)

[∗43·411] $= p`\breve{S}```\breve{R}```\text{Ɑ}``\text{Pot}`(R \mid S)$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash . \text{Prop}$

∗94·401. $\vdash . p`\text{Ɑ}``\text{Pot}`(R \mid S) = p`\text{Ɑ}``R \mid ``S \mid ``\text{Pot}`(R \mid S)$

Dem.

$\vdash . ∗93·431 . ∗91·304 . \supset$

$\vdash . p`\text{Ɑ}``\text{Pot}`(R \mid S) = p`\text{Ɑ}``(R \mid S) \mid ``\text{Pot}`(R \mid S)$

[∗43·2.∗37·33] $= p`\text{Ɑ}``R \mid ``S \mid ``\text{Pot}`(R \mid S) . \supset \vdash . \text{Prop}$

∗94·402. $\vdash . p`\text{Ɑ}``R \mid ``\lambda \subset p`\text{Ɑ}``\lambda$

Dem.

$\vdash . ∗43·11 . ∗34·36 . \supset \vdash . (P) . \text{Ɑ}`R \mid `P \subset \text{Ɑ}`P$ (1)

$\vdash . (1) . ∗40·451 . \quad \supset \vdash . \text{Prop}$

∗94·41. $\vdash : S \epsilon 1 \rightarrow \text{Cls} . \text{Ɑ}`(S \mid R) \subset \text{D}`S . \supset .$

$S``p`\text{Ɑ}``\text{Pot}`(R \mid S) = p`\text{Ɑ}``\mid R``\text{Pot}`(R \mid S)$

Dem.

$\vdash . ∗40·12 . ∗91·351 . \supset \vdash . p`\text{Ɑ}``\mid R``\text{Pot}`(R \mid S) \subset \text{Ɑ}`\mid R`(R \mid S)$

[∗43·111] $\subset \text{Ɑ}`(R \mid S \mid R)$

[∗34·36] $\subset \text{Ɑ}`(S \mid R)$ (1)

$\vdash . (1) . \supset \vdash : \text{Hp} . \supset . p\text{‘}\text{ᗡ}\text{“}|R\text{“}\text{Pot}\text{‘}(R|S) \subset \text{D}\text{‘}S .$

$[*72{\cdot}502] \qquad \supset . p\text{‘}\text{ᗡ}\text{“}|R\text{“}\text{Pot}\text{‘}(R|S) = S\text{“}\breve{S}\text{“}p\text{‘}\text{ᗡ}\text{“}|R\text{“}\text{Pot}\text{‘}(R|S)$

$[*72{\cdot}34] \qquad = S\text{“}p\text{‘}\breve{S}\text{‘‘‘}\text{ᗡ}\text{“}|R\text{“}\text{Pot}\text{‘}(R|S)$

$[*94{\cdot}4] \qquad = S\text{“}p\text{‘}\text{ᗡ}\text{“}\text{Pot}\text{‘}(R|S) : \supset \vdash . \text{Prop}$

***94·42.** $\vdash : R \in 1 \rightarrow \text{Cls} . \supset . \breve{R}\text{“}p\text{‘}\text{ᗡ}\text{“}\text{Pot}\text{‘}(R|S) = p\text{‘}\text{ᗡ}\text{“}|R\text{“}\text{Pot}\text{‘}(R|S)$

Dem.

$\vdash . *72{\cdot}34 . \supset \vdash : \text{Hp} . \supset . \breve{R}\text{“}p\text{‘}\text{ᗡ}\text{“}\text{Pot}\text{‘}(R|S) = p\text{‘}\breve{R}\text{‘‘‘}\text{ᗡ}\text{“}\text{Pot}\text{‘}(R|S)$

$[*43{\cdot}411] \qquad = p\text{‘}\text{ᗡ}\text{“}|R\text{“}\text{Pot}\text{‘}(R|S) : \supset \vdash . \text{Prop}$

***94·43.** $\vdash : R, S \in 1 \rightarrow \text{Cls} . \text{ᗡ}\text{‘}(S|R) \subset \text{D}\text{‘}S . \supset .$
$S\text{“}p\text{‘}\text{ᗡ}\text{“}\text{Pot}\text{‘}(R|S) = \breve{R}\text{“}p\text{‘}\text{ᗡ}\text{“}\text{Pot}\text{‘}(R|S)$ [*94·41·42]

***94·441.** $\vdash : S \in 1 \rightarrow \text{Cls} . \text{ᗡ}\text{‘}(S|R) \subset \text{D}\text{‘}S . \supset .$
$S\text{“}p\text{‘}\text{ᗡ}\text{“}\text{Pot}\text{‘}(R|S) = p\text{‘}\text{ᗡ}\text{“}R|\text{“}\text{Pot}\text{‘}(S|R)$ [*94·14·41]

***94·442.** $\vdash : R \in 1 \rightarrow \text{Cls} . \supset . \breve{R}\text{“}p\text{‘}\text{ᗡ}\text{“}\text{Pot}\text{‘}(R|S) = p\text{‘}\text{ᗡ}\text{“}R|\text{“}\text{Pot}\text{‘}(S|R)$
[*94·14·42]

***94·5.** $\vdash . p\text{‘}\text{ᗡ}\text{“}\text{Pot}\text{‘}(S|R) = p\text{‘}\text{ᗡ}\text{“}R|\text{“}\text{Pot}\text{‘}(S|R)$

Dem.

$\vdash . *94{\cdot}402 . \supset \vdash . p\text{‘}\text{ᗡ}\text{“}R|\text{“}\text{Pot}\text{‘}(S|R) \subset p\text{‘}\text{ᗡ}\text{“}\text{Pot}\text{‘}(S|R) \qquad (1)$

$\vdash . *94{\cdot}402 . \supset \vdash . p\text{‘}\text{ᗡ}\text{“}S|\text{“}R|\text{“}\text{Pot}\text{‘}(S|R) \subset p\text{‘}\text{ᗡ}\text{“}R|\text{“}\text{Pot}\text{‘}(S|R) .$

$[*94{\cdot}401] \quad \supset \vdash . p\text{‘}\text{ᗡ}\text{“}\text{Pot}\text{‘}(S|R) \subset p\text{‘}\text{ᗡ}\text{“}R|\text{“}\text{Pot}\text{‘}(S|R) \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***94·51.** $\vdash : R \in 1 \rightarrow \text{Cls} . \supset . p\text{‘}\text{ᗡ}\text{“}\text{Pot}\text{‘}(S|R) = \breve{R}\text{“}p\text{‘}\text{ᗡ}\text{“}\text{Pot}\text{‘}(R|S)$
[*94·5·442]

***94·52.** $\vdash : S \in 1 \rightarrow \text{Cls} . \text{ᗡ}\text{‘}(S|R) \subset \text{D}\text{‘}S . \supset .$
$p\text{‘}\text{ᗡ}\text{“}\text{Pot}\text{‘}(S|R) = S\text{“}p\text{‘}\text{ᗡ}\text{“}\text{Pot}\text{‘}(R|S)$ [*94·5·441]

***94·53.** $\vdash : R \in 1 \rightarrow 1 . \text{ᗡ}\text{‘}(R|S) \subset \text{D}\text{‘}R . \supset .$
$p\text{‘}\text{ᗡ}\text{“}\text{Pot}\text{‘}(R|S) \text{ sm } p\text{‘}\text{ᗡ}\text{“}\text{Pot}\text{‘}(S|R)$

Dem.

$\vdash . *93{\cdot}261 . \supset \vdash . p\text{‘}\text{ᗡ}\text{“}\text{Pot}\text{‘}(R|S) \subset \text{ᗡ}\text{‘}(R|S) \qquad (1)$

$\vdash . (1) . \quad \supset \vdash : \text{Hp} . \supset . p\text{‘}\text{ᗡ}\text{“}\text{Pot}\text{‘}(R|S) \subset \text{D}\text{‘}R \qquad (2)$

$\vdash . (2) . *94{\cdot}51 . *73{\cdot}21 . \supset \vdash . \text{Prop}$

***94·54.** $\vdash : S \in 1 \rightarrow 1 . \text{ᗡ}\text{‘}(S|R) \subset \text{D}\text{‘}S . \supset . p\text{‘}\text{ᗡ}\text{“}\text{Pot}\text{‘}(R|S) \text{ sm } p\text{‘}\text{ᗡ}\text{“}\text{Pot}\text{‘}(S|R)$

$\left[*94{\cdot}53 \dfrac{S, R}{R, S}\right]$

[Or, *94·52 . *93·261 . *73·22]

$*94{\cdot}6$. $\vdash :. R \,|\, S = S \,|\, R . \supset : M \in \mathrm{Pot}`R . N \in \mathrm{Pot}`S . \supset . M \,|\, N = N \,|\, M$

Dem.

$\vdash . *34{\cdot}27{\cdot}28 . \supset \vdash : \mathrm{Hp} . M \,|\, S = S \,|\, M . \supset . M \,|\, R \,|\, S = S \,|\, M \,|\, R$ (1)

$\vdash . (1) . *91{\cdot}171 \dfrac{M \,|\, S = S \,|\, M}{\phi M} . \supset$

$\vdash :. \mathrm{Hp} . M \in \mathrm{Pot}`R . \supset : M \,|\, S = S \,|\, M :$ (2)

$\left[(2) \dfrac{S, M, N}{R, S, M}\right] \supset : N \in \mathrm{Pot}`S . \supset . M \,|\, N = N \,|\, M :. \supset \vdash . \mathrm{Prop}$

$*94{\cdot}61$. $\vdash :. R \,|\, S = S \,|\, R . \supset : M \in \mathrm{Pot}`R . \supset . M \,|\, S_{\mathrm{po}} = S_{\mathrm{po}} \,|\, M :$
$N \in \mathrm{Pot}`S . \supset . N \,|\, R_{\mathrm{po}} = R_{\mathrm{po}} \,|\, N$

Dem.

$\vdash . *43{\cdot}42 . \quad \supset \vdash . M \,|\, S_{\mathrm{po}} = \dot{s}`M \,|\, ``\mathrm{Pot}`S$ (1)

$\vdash . (1) . *94{\cdot}6 . \supset \vdash : \mathrm{Hp} . M \in \mathrm{Pot}`R . \supset . M \,|\, S_{\mathrm{po}} = \dot{s}`|\, M``\mathrm{Pot}`S$

$[*43{\cdot}421] \qquad = S_{\mathrm{po}} \,|\, M$ (2)

$\vdash . (2) \dfrac{S, R}{R, S} . \quad \supset \vdash : \mathrm{Hp} . N \in \mathrm{Pot}`S . \supset . N \,|\, R_{\mathrm{po}} = R_{\mathrm{po}} \,|\, N$ (3)

$\vdash . (2) . (3) . \quad \supset \vdash . \mathrm{Prop}$

$*94{\cdot}62$. $\vdash : R \,|\, S = S \,|\, R . \supset . R_{\mathrm{po}} \,|\, S_{\mathrm{po}} = S_{\mathrm{po}} \,|\, R_{\mathrm{po}}$

Dem.

$\vdash . *43{\cdot}42 . *94{\cdot}61 . \supset \vdash : \mathrm{Hp} . \supset . R_{\mathrm{po}} \,|\, S_{\mathrm{po}} = \dot{s}`|\, R_{\mathrm{po}}``\mathrm{Pot}`S$

$[*43{\cdot}421] \qquad = S_{\mathrm{po}} \,|\, R_{\mathrm{po}} : \supset \vdash . \mathrm{Prop}$

$*94{\cdot}63$. $\vdash : R \,|\, S = S \,|\, R . \supset . (R \,|\, S)_{\mathrm{po}} \subseteq R_{\mathrm{po}} \,|\, S_{\mathrm{po}}$

Dem.

$\vdash . *91{\cdot}502 . \qquad \supset \vdash . R \,|\, S \subseteq R_{\mathrm{po}} \,|\, S_{\mathrm{po}}$ (1)

$\vdash . *94{\cdot}61 . \qquad \supset \vdash : \mathrm{Hp} . M \subseteq R_{\mathrm{po}} \,|\, S_{\mathrm{po}} . \supset . M \,|\, R \,|\, S \subseteq R_{\mathrm{po}} \,|\, R \,|\, S_{\mathrm{po}} \,|\, S$

$[*91{\cdot}511] \qquad \subseteq R_{\mathrm{po}} \,|\, S_{\mathrm{po}}$ (2)

$\vdash . (1) . (2) . *91{\cdot}171 . \supset \vdash :. \mathrm{Hp} . \supset : M \in \mathrm{Pot}`(R \,|\, S) . \supset . M \subseteq R_{\mathrm{po}} \,|\, S_{\mathrm{po}} :$

$[*41{\cdot}151] \qquad \supset : (R \,|\, S)_{\mathrm{po}} \subseteq R_{\mathrm{po}} \,|\, S_{\mathrm{po}} :. \supset \vdash . \mathrm{Prop}$

$*94{\cdot}64$. $\vdash : R \,|\, S = S \,|\, R . \supset . (R \,|\, S)_* \subseteq R_* \,|\, S_*$

Dem.

$\vdash . *34{\cdot}36 . \qquad \supset \vdash . \mathrm{D}`(R \,|\, S) \subset \mathrm{D}`R . \mathrm{Cl}`(S \,|\, R) \subset \mathrm{Cl}`R .$

$[*33{\cdot}16] \qquad \supset \vdash : \mathrm{Hp} . \supset . C`(R \,|\, S) \subset C`R$ (1)

Similarly $\qquad \vdash : \mathrm{Hp} . \supset . C`(R \,|\, S) \subset C`S$ (2)

$\vdash . (1) . (2) . *50{\cdot}6 . *35{\cdot}31 . \supset \vdash : \mathrm{Hp} . \supset . I \upharpoonright C`(R \,|\, S) \subseteq I \upharpoonright C`R \,|\, I \upharpoonright C`S$ (3)

$\vdash . (3) . *94{\cdot}63 . *91{\cdot}54 . \quad \supset \vdash . \mathrm{Prop}$

$*95$. ON THE EQUI-FACTOR RELATION

Summary of $*95$.

The purpose of this number may be explained as follows. Consider the series of relations

$$R,\ P \mid R \mid Q,\ P^2 \mid R \mid Q^2,\ P^3 \mid R \mid Q^3, \ldots;$$

it is required to find a means of defining this series without the use of numbers. If we used numbers, and had the definition given later ($*301$) of P^ν, where ν is any finite integer, the general term of the series would be $P^\nu \mid R \mid Q^\nu$. But we have not yet defined numbers, and we therefore desire some means, not involving numbers, of expressing what is intended when we say that, in a given term of the series, the same power of P and of Q is to be involved. This we do as follows. Using the definition of $P \| Q$ in $*43$, we have

$$P \mid R \mid Q = (P \| Q)`R \,.\, P^2 \mid R \mid Q^2 = (P \| Q)^2`R \,.\, P^3 \mid R \mid Q^3 = (P \| Q)^3`R \ldots.$$

Thus the general term of our series is got by taking any power S of $(P \| Q)$, and forming $S`R$. The whole of the terms of the series are therefore constituted by the terms which have to $R$ the relation $(P \| Q)_*$; *i.e.* they are $\{\text{sg}`(P \| Q)_*\}`R$. For convenience of notation we put*

$$(P*Q) = \text{sg}`\{(P \| Q)_*\} \quad \text{Dft} \ [*95]$$

Thus the class of relations we wish to consider is $(P*Q)`R$.

To illustrate the nature of $(P*Q)`R$, suppose $R$ is the relation "first cousin," while $P$ is the relation of child to parent and $Q$ is the relation of parent to child. Then $P \mid R \mid Q$ is the relation "second cousin," $P^2 \mid R \mid Q^2$ is the relation "third cousin," and so on. Thus $(P*Q)`R$ is the class of all relations of cousinship which do not involve a difference of generation; and "$x \, \{\dot{s}`(P*Q)`R\} \, y$" will mean "$x$ is a cousin of y in the same generation."

Most of the propositions in this number are inserted because they are required in the proof of $*95{\cdot}52$, which states that, under suitable circumstances, $\dot{s}`(P*Q)`R \in 1 \rightarrow 1$. This proposition itself is proved mainly because it is required in the proof of $*95{\cdot}63$, which states that, if P, Q are one-one's each of which has its converse domain contained in its domain, and if the first generation of P is similar to the first generation of Q, then the sum of the generations of P is similar to the sum of the generations of Q. This leads immediately to a proposition ($*95{\cdot}71$) which is half of the Schröder-Bernstein theorem (the other half being $*94{\cdot}53$ or $*94{\cdot}54$), namely: "If

* This notation is used in the present number only. In $*257$, we shall introduce a different and wholly unconnected meaning for $(P*Q)$. A temporary definition is indicated by the letters "Dft" followed by a reference in square brackets to the number or numbers in which the definition is used.

R and S are one-one's each of which has its converse domain contained in the domain of the other, then the sum of the generations of $R \mid S$ is similar to the sum of the generations of $S \mid R$."

∗95·01. $(P \ast Q) = \text{sg}`\{(P \parallel Q)_\ast\}$ Dft [∗95]

∗95·1. $\vdash :: M \epsilon (P \ast Q)`R \,.\equiv :. R \epsilon \mu : N \epsilon \mu \,.\supset_N .\, P|N|Q \epsilon \mu : \supset_\mu .\, M \epsilon \mu$

Dem.

$\vdash .\, \ast 32\cdot 18 \,.\, (\ast 95\cdot 01) \,.\supset$

$\vdash :: M \epsilon (P \ast Q)`R \,.\equiv :. M (P \parallel Q)_\ast R :.$

[∗90·11] $\equiv :. M \epsilon C`(P \parallel Q) :. N \epsilon \mu \,.\, T (P \parallel Q) N \,.\supset_{N,T} .\, T \epsilon \mu :$
$R \epsilon \mu : \supset_\mu .\, M \epsilon \mu :.$

[∗43·302·102] $\equiv :. N \epsilon \mu \,.\, T = P|N|Q \,.\supset_{T,N} .\, T \epsilon \mu : R \epsilon \mu : \supset_\mu .\, M \epsilon \mu :.$

[∗13·191] $\equiv :. N \epsilon \mu \,.\supset_N .\, P|N|Q \epsilon \mu : R \epsilon \mu : \supset_\mu .\, M \epsilon \mu :: \supset \vdash .$ Prop

∗95·11. $\vdash :. \phi R : \phi N \,.\supset_N .\, \phi (P|N|Q) : \supset : M \epsilon (P \ast Q)`R \,.\supset_M .\, \phi M$

Dem.

$\vdash .\, \ast 95\cdot 1 \dfrac{\hat{N}(\phi N)}{\mu} \,.\supset$

$\vdash :: M \epsilon (P \ast Q)`R \,.\supset :. \phi R : \phi N \,.\supset_N .\, \phi (P|N|Q) : \supset .\, \phi M$ (1)

$\vdash .\, (1) \,.\,$ Comm $.\, \ast 10\cdot 11\cdot 21 \,.\supset \vdash .$ Prop

∗95·12. $\vdash :. M \epsilon (P \ast Q)`R \,.\supset_M .\, \phi (P|M|Q) : \supset : N \epsilon (P \ast Q)`R - \iota`R \,.\supset_N .\, \phi N$

Dem.

$\vdash .\, \ast 43\cdot 112 \,.\supset$

$\vdash :. \text{Hp} \,.\equiv : M \epsilon (P \ast Q)`R \,.\supset_M .\, \phi \{(P \parallel Q)`M\} :$

[∗37·63] $\equiv : N \epsilon (P \parallel Q)``(P \ast Q)`R \,.\supset_N .\, \phi N$ (1)

$\vdash .\, \ast 90\cdot 311 \dfrac{P \parallel Q}{R} \,.\supset$

$\vdash : N \epsilon (P \ast Q)`R - \iota`R \,.\supset .\, N \epsilon (P \parallel Q)``(P \ast Q)`R$ (2)

$\vdash .\, (1) \,.\, (2) \,.\supset \vdash .$ Prop

∗95·13. $\vdash .\, R \epsilon (P \ast Q)`R$ [∗95·1]

∗95·131. $\vdash .\, P|R|Q \epsilon (P \ast Q)`R$

Dem.

$\vdash .\, \ast 90\cdot 151 \dfrac{P \parallel Q}{R} \,.\supset \vdash : S (P \parallel Q) R \,.\supset .\, S (P \parallel Q)_\ast R$ (1)

$\vdash .\, (1) \,.\, \ast 43\cdot 102 \,.\, (\ast 95\cdot 01) \,.\supset \vdash .$ Prop

∗95·132. $\vdash : M \epsilon (P \ast Q)`R \,.\supset .\, P|M|Q \epsilon (P \ast Q)`R$

$\left[\ast 90\cdot 172 \dfrac{P \parallel Q}{R} \,.\, \ast 43\cdot 102\right]$

***95·14.** $\vdash :. \phi R : N \epsilon (P*Q)\text{'}R . \phi N . \supset_N . \phi(P|N|Q) : \supset : M \epsilon (P*Q)\text{'}R . \supset_M . \phi M$

Dem.

$\vdash . *95\cdot 13\cdot 132 . \supset \vdash :. \text{Hp} . \supset :$

$\phi R . R \epsilon (P*Q)\text{'}R : N \epsilon (P*Q)\text{'}R . \phi N . \supset_N . P|N|Q \epsilon (P*Q)\text{'}R . \phi (P|N|Q) :$

$[*95\cdot 11] \supset : M \epsilon (P*Q)\text{'}R . \supset_M . M \epsilon (P*Q)\text{'}R . \phi M :. \supset \vdash . \text{Prop}$

The use of *95·11 in the last line of the above proof proceeds by substituting $M \epsilon (P*Q)\text{'}R . \phi M$ for ϕM.

***95·21.** $\vdash : M \epsilon (P*Q)\text{'}R . \supset . (\exists S, T) . S \epsilon \text{Pot}\text{'}P \cup \iota\text{'}I . T \epsilon \text{Pot}\text{'}Q \cup \iota\text{'}I . M = S|R|T$

Dem.

$\vdash . *50\cdot 4 . \supset \vdash . R = I|R|I .$

$[*51\cdot 16] \supset \vdash . (\exists S, T) . S \epsilon \text{Pot}\text{'}P \cup \iota\text{'}I . T \epsilon \text{Pot}\text{'}Q \cup \iota\text{'}I . R = S|R|T \quad (1)$

$\vdash . *91\cdot 36\cdot 351 . *50\cdot 4 . *34\cdot 27\cdot 28 . \supset$

$\vdash : S \epsilon \text{Pot}\text{'}P \cup \iota\text{'}I . T \epsilon \text{Pot}\text{'}Q \cup \iota\text{'}I . M = S|R|T . \supset .$

$P|S \epsilon \text{Pot}\text{'}P \cup \iota\text{'}I . T|Q \epsilon \text{Pot}\text{'}Q \cup \iota\text{'}I . P|M|Q = (P|S)|R|(T|Q) .$

$[*11\cdot 36] \supset . (\exists S', T') . S' \epsilon \text{Pot}\text{'}P \cup \iota\text{'}I . T' \epsilon \text{Pot}\text{'}Q \cup \iota\text{'}I . P|M|Q = S'|R|T' \quad (2)$

$\vdash . (2) . *11\cdot 11\cdot 35 . \supset$

$\vdash : (\exists S, T) . S \epsilon \text{Pot}\text{'}P \cup \iota\text{'}I . T \epsilon \text{Pot}\text{'}Q \cup \iota\text{'}I . M = S|R|T . \supset .$

$(\exists S, T) . S \epsilon \text{Pot}\text{'}P \cup \iota\text{'}I . T \epsilon \text{Pot}\text{'}Q \cup \iota\text{'}I . P|M|Q = S|R|T \quad (3)$

$\vdash . (1) . (3) . *95\cdot 11 . \supset \vdash . \text{Prop}$

***95·211.** $\vdash : \text{G}\text{'}R \subset C\text{'}Q . M \epsilon (P*Q)\text{'}R . \supset .$

$(\exists S, T) . S \epsilon \text{Pot}\text{'}P \cup \iota\text{'}I . T \epsilon \text{Potid}\text{'}Q . M = S|R|T$

Dem.

$\vdash . *50\cdot 62\cdot 4 . \supset \vdash :. \text{Hp} . \supset : S|R|I \upharpoonright C\text{'}Q = S|R|I :$

$[*51\cdot 239 . *91\cdot 23] \quad \supset : (\exists S, T) . S \epsilon \text{Pot}\text{'}P \cup \iota\text{'}I . T \epsilon \text{Potid}\text{'}Q . M = S|R|T . \equiv .$

$(\exists S, T) . S \epsilon \text{Pot}\text{'}P \cup \iota\text{'}I . T \epsilon \text{Pot}\text{'}Q \cup \iota\text{'}I . M = S|R|T :$

$[*95\cdot 21] \quad \supset : (\exists S, T) . S \epsilon \text{Pot}\text{'}P \cup \iota\text{'}I . T \epsilon \text{Potid}\text{'}Q . M = S|R|T :.$

$\supset \vdash . \text{Prop}$

***95·212.** $\vdash : \text{D}\text{'}R \subset C\text{'}P . M \epsilon (P*Q)\text{'}R . \supset .$

$(\exists S, T) . S \epsilon \text{Potid}\text{'}P . T \epsilon \text{Pot}\text{'}Q \cup \iota\text{'}I . M = S|R|T$

[Proof as in *95·211]

***95·22.** $\vdash : \text{D}\text{'}R \subset C\text{'}P . \text{G}\text{'}R \subset C\text{'}Q . M \epsilon (P*Q)\text{'}R . \supset .$

$(\exists S, T) . S \epsilon \text{Potid}\text{'}P . T \epsilon \text{Potid}\text{'}Q . M = S|R|T$

[Proof as in *95·211]

***95·221.** $\vdash : T \epsilon \text{Pot}\text{'}Q . \supset . (\exists S) . S \epsilon \text{Pot}\text{'}P . S|R|T \epsilon (P*Q)\text{'}R$

Dem.

$\vdash . *95\cdot 131 . *91\cdot 351 . \supset \vdash . (\exists S) . S \epsilon \text{Pot}\text{'}P . S|R|Q \epsilon (P*Q)\text{'}R \quad (1)$

$\vdash . *95\cdot 132 . \supset$

$\vdash : S \epsilon \text{Pot}\text{'}P . T \epsilon \text{Pot}\text{'}Q . S|R|T \epsilon (P*Q)\text{'}R . \supset . P|S|R|T|Q \epsilon (P*Q)\text{'}R .$

$[*91\cdot 36] \quad \supset . (\exists S') . S' \epsilon \text{Pot}\text{'}P . S'|R|T|Q \epsilon (P*Q)\text{'}R \quad (2)$

$\vdash . (1) . (2) . *91\cdot 373 . \supset \vdash . \text{Prop}$

***95·222.** $\vdash : S \in \text{Pot}`P . \supset . (\exists T) . T \in \text{Pot}`Q . S | R | T \in (P * Q)`R$
[Proof as in *95·221]

***95·23.** $\vdash : M \in (P * Q)`R . \supset . M (P_{st} \mid Q_{ts}) R$

Dem.

$\vdash . *32·18 . (*95·01) . \supset \vdash : \text{Hp} . \supset . M \{(P \| Q)_*\} R .$
[*43·202] $\supset . M \{(P |) (| Q)\}_* R .$
[*43·202.*94·64] $\supset . M \{(P |)_* | (| Q)_*\} R .$
[(*91·01·02)] $\supset . M (P_{st} \mid Q_{ts}) R . \supset \vdash . \text{Prop}$

***95·24.** $\vdash : M \in (P * Q)`R . \supset . M (Q_{ts} \mid P_{st}) R$ [Proof as in *95·23]

***95·3.** $\vdash :. \dot{\exists} ! R . \text{Cl}`Q \subset \text{D}`Q . \text{Cl}`R \subset \text{D}`Q . \supset : T \in \text{Potid}`Q . \supset . \dot{\exists} ! R | T$

Dem.

$\vdash . *50·62 . \quad \supset \vdash : \text{Hp} . \supset . R | I \upharpoonright C`Q = R .$
[*13·12] $\supset . \dot{\exists} ! R | (I \upharpoonright C`Q)$ (1)
$\vdash . *91·27 . *33·181 . \supset \vdash :. \text{Hp} . T \in \text{Potid}`Q . \supset : \text{Cl}`T \subset \text{D}`Q :$
[*34·35] $\supset : \dot{\exists} ! T . \supset . \dot{\exists} ! (T | Q)$ (2)
$\vdash . (1) . (2) . *91·371 . \supset \vdash . \text{Prop}$

***95·301.** $\vdash :. \dot{\exists} ! R . \text{D}`P \subset \text{Cl}`P . \text{D}`R \subset \text{Cl}`P . \supset : S \in \text{Potid}`P . \supset . \dot{\exists} ! S | R$
[Proof as in *95·3]

***95·302.** $\vdash :. \text{Cl}`Q \subset \text{D}`Q . \text{Cl}`R \subset \text{D}`Q . \supset : T \in \text{Potid}`Q . \supset . \text{Cl}`(R | T) \subset \text{D}`Q$

Dem.

$\vdash . *91·271 . *34·36 . \supset \vdash : T \in \text{Potid}`Q . \supset . \text{Cl}`(R | T) \subset \text{Cl}`Q$ (1)
$\vdash . (1) . *22·44 . \supset \vdash . \text{Prop}$

***95·303.** $\vdash :. \text{D}`R \subset \text{Cl}`P . \text{D}`P \subset \text{Cl}`P . \supset : S \in \text{Potid}`P . \supset . \text{D}`(S | R) \subset \text{Cl}`P$
[Proof as in *95·302]

***95·304.** $\vdash :. \text{Cl}`Q \subset \text{D}`Q . \text{Cl}`R \subset \text{D}`Q . \text{D}`P \subset \text{Cl}`P . \text{D}`R \subset \text{Cl}`P . \supset :$
$S \in \text{Potid}`P . T \in \text{Potid}`Q . \supset . \text{D}`(S | R | T) \subset \text{Cl}`P . \text{Cl}`(S | R | T) \subset \text{D}`Q$
[*95·302·303 . *34·36]

***95·305.** $\vdash :. \text{Hp } *95·304 . \supset : M \in (P * Q)`R . \supset . \text{D}`M \subset \text{Cl}`P . \text{Cl}`M \subset \text{D}`Q$
[*95·304·22]

***95·31.** $\vdash :. \text{Hp } *95·304 . \dot{\exists} ! R . \supset : S \in \text{Potid}`P . T \in \text{Potid}`Q . \supset . \dot{\exists} ! S | R | T$

Dem.

$\vdash . *92·142·143 . \supset \vdash :. \text{Hp} . \supset : S \in \text{Potid}`P . T \in \text{Potid}`Q . \supset .$
$\text{D}`R \subset \text{Cl}`S . \text{Cl}`R \subset \text{D}`T .$
[*34·361] $\supset . \dot{\exists} ! S | R | T :. \supset \vdash . \text{Prop}$

***95·32.** $\vdash :. \text{Hp } *95·31 . \supset : M \in (P * Q)`R . \supset . \dot{\exists} ! M$ [*95·31·22]

***95·33.** $\vdash : \text{Cl}`R \subset \overrightarrow{B}`Q . \supset . \text{Cl}`(S | R | T) \subset \breve{T}``\overrightarrow{B}`Q$

Dem.

$\vdash . *34·36 . \supset \vdash : \text{Hp} . \supset . \text{Cl}`(S | R) \subset \overrightarrow{B}`Q .$
[*37·32·2] $\supset . \text{Cl}`(S | R | T) \subset \breve{T}``\overrightarrow{B}`Q : \supset \vdash . \text{Prop}$

***95·34.** $\vdash : \text{Ɑ}\text{‘}R \subset \overrightarrow{B}\text{‘}Q \,.\, M \in (P{*}Q)\text{‘}R \,.\supset .\, (\exists T) \,.\, T \in \text{Potid}\text{‘}Q \,.\, \text{Ɑ}\text{‘}M \subset \breve{T}\text{“}\overrightarrow{B}\text{‘}Q$
[*95·33·211]

***95·35.** $\vdash : Q \in 1 \to \text{Cls} \,.\, \text{Ɑ}\text{‘}R \subset \overrightarrow{B}\text{‘}Q \,.\, M \in (P{*}Q)\text{‘}R \,.\supset .\, (\exists \alpha) \,.\, \alpha \in \text{gen}\text{‘}Q \,.\, \text{Ɑ}\text{‘}M \subset \alpha$
[*95·34 . *93·32]

***95·351.** $\vdash :.\, Q \in 1 \to \text{Cls} \,.\, \text{Ɑ}\text{‘}R \subset \overrightarrow{B}\text{‘}Q \,.\supset :$
$T, T' \in \text{Potid}\text{‘}Q \,.\, \exists !\, \text{Ɑ}\text{‘}(S \,|\, R \,|\, T) \cap \text{Ɑ}\text{‘}(S' \,|\, R \,|\, T') \,.\supset .\, T = T'$

Dem.

$\vdash .\, {*}95{\cdot}33 \,.\supset \vdash :.\, \text{Hp} \,.\supset :$
$T, T' \in \text{Potid}\text{‘}Q \,.\, \exists !\, \text{Ɑ}\text{‘}(S \,|\, R \,|\, T) \cap \text{Ɑ}\text{‘}(S' \,|\, R \,|\, T') \,.\supset .\, \exists !\, \breve{T}\text{“}\overrightarrow{B}\text{‘}Q \cap \breve{T}'\text{“}\overrightarrow{B}\text{‘}Q \,.$
[*93·3] $\supset .\, \exists !\, \overrightarrow{\min}_Q\text{‘}\text{Ɑ}\text{‘}T \cap \overrightarrow{\min}_Q\text{‘}\text{Ɑ}\text{‘}T' \,.$
[*93·24.Transp] $\supset .\, T = T' :. \supset \vdash .\, \text{Prop}$

***95·352.** $\vdash :.\, P \in \text{Cls} \to 1 \,.\, \text{D}\text{‘}R \subset \overrightarrow{B}\text{‘}\breve{P} \,.\supset :$
$S, S' \in \text{Potid}\text{‘}P \,.\, \exists !\, \text{D}\text{‘}(S \,|\, R \,|\, T) \cap \text{D}\text{‘}(S' \,|\, R \,|\, T') \,.\supset .\, S = S'$
[Proof as in *95·351]

***95·36.** $\vdash :.\, Q \in 1 \to \text{Cls} \,.\, \text{Ɑ}\text{‘}R \subset \overrightarrow{B}\text{‘}Q \,.\, \dot{\exists} !\, R \,.\, \text{D}\text{‘}R \subset \text{Ɑ}\text{‘}P \,.$
$\text{D}\text{‘}P \subset \text{Ɑ}\text{‘}P \,.\, \text{Ɑ}\text{‘}Q \subset \text{D}\text{‘}Q \,.\supset :$
$S, S' \in \text{Potid}\text{‘}P \,.\, T, T' \in \text{Potid}\text{‘}Q \,.\, S \,|\, R \,|\, T = S' \,|\, R \,|\, T' \,.\supset .\, T = T'$

Dem.

$\vdash .\, {*}95{\cdot}31 \,.\, {*}93{\cdot}101 \,.\supset \vdash :.\, \text{Hp} \,.\supset :$
$S, S' \in \text{Potid}\text{‘}P \,.\, T, T' \in \text{Potid}\text{‘}Q \,.\, S \,|\, R \,|\, T = S' \,|\, R \,|\, T' \,.\supset .$
$\dot{\exists} !\, S \,|\, R \,|\, T \,.\, S \,|\, R \,|\, T = S' \,|\, R \,|\, T' \,.$
[*22·5.*33·24] $\supset .\, \exists !\, \text{Ɑ}\text{‘}(S \,|\, R \,|\, T) \cap \text{Ɑ}\text{‘}(S' \,|\, R \,|\, T') \,.$
[*95·351] $\supset .\, T = T' :. \supset \vdash .\, \text{Prop}$

***95·361.** $\vdash :.\, P \in \text{Cls} \to 1 \,.\, \text{D}\text{‘}R \subset \overrightarrow{B}\text{‘}\breve{P} \,.\, \dot{\exists} !\, R \,.\, \text{D}\text{‘}P \subset \text{Ɑ}\text{‘}P \,.$
$\text{Ɑ}\text{‘}R \subset \text{D}\text{‘}Q \,.\, \text{Ɑ}\text{‘}Q \subset \text{D}\text{‘}Q \,.\supset :$
$S, S' \in \text{Potid}\text{‘}P \,.\, T, T' \in \text{Potid}\text{‘}Q \,.\, S \,|\, R \,|\, T = S' \,|\, R \,|\, T' \,.\supset .\, S = S'$
[Proof as in *95·36]

***95·37.** $\vdash :.\, P \in \text{Cls} \to 1 \,.\, Q \in 1 \to \text{Cls} \,.\, \text{D}\text{‘}R \subset \overrightarrow{B}\text{‘}\breve{P} \,.\, \text{Ɑ}\text{‘}R \subset \overrightarrow{B}\text{‘}Q \,.\, \dot{\exists} !\, R$
$\text{D}\text{‘}P \subset \text{Ɑ}\text{‘}P \,.\, \text{Ɑ}\text{‘}Q \subset \text{D}\text{‘}Q \,.\supset :$
$S, S' \in \text{Potid}\text{‘}P \,.\, T, T' \in \text{Potid}\text{‘}Q \,.\, S \,|\, R \,|\, T = S' \,|\, R \,|\, T' \,.\supset .\, S = S' \,.\, T = T'$
[*95·36·361]

***95·38.** $\vdash :.\, \exists !\, \overrightarrow{B}\text{‘}Q \cap \text{Ɑ}\text{‘}R \,.\supset : T \in \text{Pot}\text{‘}Q \,.\supset .\, R \,|\, T \neq R$

Dem.

$\vdash .\, {*}91{\cdot}271 \,.\supset \vdash : T \in \text{Pot}\text{‘}Q \,.\supset .\, \text{Ɑ}\text{‘}(R \,|\, T) \subset \text{Ɑ}\text{‘}Q \,.$
[*93·101] $\supset .\, \text{Ɑ}\text{‘}(R \,|\, T) \cap \overrightarrow{B}\text{‘}Q = \Lambda$ (1)
$\vdash .\, {*}24{\cdot}54 \,.\supset \vdash : \text{Hp} \,.$ $\supset .\, \sim \{\text{Ɑ}\text{‘}R \cap \overrightarrow{B}\text{‘}Q = \Lambda\}$ (2)
$\vdash .\, (1) \,.\, (2) \,.\, {*}13{\cdot}14 \,.\supset \vdash .\, \text{Prop}$

***95·381.** $\vdash :. \exists ! \overrightarrow{B}\text{‘}\breve{P} \cap \text{D}\text{‘}R . \supset : S \in \text{Pot}\text{‘}P . \supset . S \,|\, R \neq R$

[Proof as in *95·38]

***95·382.** $\vdash :. \exists ! \overrightarrow{B}\text{‘}\breve{P} \cap \text{D}\text{‘}R . \vee . \exists ! \overrightarrow{B}\text{‘}Q \cap \text{Ɑ}\text{‘}R : \supset :$

$$S \in \text{Pot}\text{‘}P . T \in \text{Pot}\text{‘}Q . \supset . S \,|\, R \,|\, T \neq R$$

Dem.

$\vdash . *91{\cdot}271 . *93{\cdot}101 . \supset \vdash : T \in \text{Pot}\text{‘}Q . \supset . \text{Ɑ}\text{‘}(S \,|\, R \,|\, T) \cap \overrightarrow{B}\text{‘}Q = \Lambda \quad (1)$

$\vdash . *24{\cdot}54 . \supset \vdash : \exists ! \overrightarrow{B}\text{‘}Q \cap \text{Ɑ}\text{‘}R . \supset . \sim \{\text{Ɑ}\text{‘}R \cap \overrightarrow{B}\text{‘}Q = \Lambda\} \quad (2)$

$\vdash . (1) . (2) . *13{\cdot}14 . \supset \vdash :. \exists ! \overrightarrow{B}\text{‘}Q \cap \text{Ɑ}\text{‘}R . \supset : T \in \text{Pot}\text{‘}Q . \supset . S \,|\, R \,|\, T \neq R \quad (3)$

$\vdash . *91{\cdot}271 . *93{\cdot}12 . \supset \vdash : S \in \text{Pot}\text{‘}P . \supset . \text{D}\text{‘}(S \,|\, R \,|\, T) \cap \overrightarrow{B}\text{‘}\breve{P} = \Lambda \quad (4)$

$\vdash . *24{\cdot}54 . \supset \vdash : \exists ! \overrightarrow{B}\text{‘}\breve{P} \cap \text{D}\text{‘}R . \supset . \sim \{\text{D}\text{‘}R \cap \overrightarrow{B}\text{‘}\breve{P} = \Lambda\} \quad (5)$

$\vdash . (4) . (5) . *13{\cdot}14 . \supset \vdash :. \exists ! \overrightarrow{B}\text{‘}\breve{P} \cap \text{D}\text{‘}R . \supset : S \in \text{Pot}\text{‘}P . \supset . S \,|\, R \,|\, T \neq R \quad (6)$

$\vdash . (3) . (6) . \supset \vdash . \text{Prop}$

***95·383.** $\vdash :. \dot{\exists} ! R : \text{D}\text{‘}R \subset \overrightarrow{B}\text{‘}\breve{P} . \vee . \text{Ɑ}\text{‘}R \subset \overrightarrow{B}\text{‘}Q : \supset :$

$S \in \text{Pot}\text{‘}P . T \in \text{Pot}\text{‘}Q . \supset . S \,|\, R \,|\, T \neq R$ [*95·382 . *33·24 . *22·621]

***95·4.** $\vdash : M \in (P * Q)\text{‘}R . S \in \text{Pot}\text{‘}P . T \in \text{Pot}\text{‘}Q . S \,|\, R \,|\, T \in (P * Q)\text{‘}R . \supset .$

$$S \,|\, M \,|\, T \in (P * Q)\text{‘}R$$

Dem.

$\vdash . \text{Simp} . \supset \vdash : \text{Hp} . \supset . S \,|\, R \,|\, T \in (P * Q)\text{‘}R \quad (1)$

$\vdash . *91{\cdot}34 . *95{\cdot}132 . \supset$

$\vdash : \text{Hp} . S \,|\, M \,|\, T \in (P * Q)\text{‘}R . \supset . S \,|\, P \,|\, M \,|\, Q \,|\, T = P \,|\, S \,|\, M \,|\, T \,|\, Q .$

$P \,|\, S \,|\, M \,|\, T \,|\, Q \in (P * Q)\text{‘}R .$

[*13·13] $\supset . S \,|\, (P \,|\, M \,|\, Q) \,|\, T \in (P * Q)\text{‘}R \quad (2)$

$\vdash . (1) . (2) . *95{\cdot}14 . \supset \vdash . \text{Prop}$

***95·41.** $\vdash :. P \in \text{Cls} \to 1 . Q \in 1 \to \text{Cls} . \text{D}\text{‘}P \subset \text{Ɑ}\text{‘}P . \text{Ɑ}\text{‘}Q \subset \text{D}\text{‘}Q . \supset :$

$S, S' \in \text{Potid}\text{‘}P . T, T' \in \text{Potid}\text{‘}Q . \supset . \breve{S} \,|\, S \,|\, S' \,|\, N \,|\, T' \,|\, T \,|\, \breve{T} = S' \,|\, N \,|\, T'$

[*92·15·151]

***95·411.** $\vdash :. \text{Hp}\, *95{\cdot}41 . \text{D}\text{‘}R \subset C\text{‘}P . \text{Ɑ}\text{‘}R \subset C\text{‘}Q . \supset :$

$S \in \text{Potid}\text{‘}P . T \in \text{Potid}\text{‘}Q . M \in (P * Q)\text{‘}R . \supset . M = \breve{S} \,|\, S \,|\, M \,|\, T \,|\, \breve{T}$

[*95·41·22]

***95·42.** $\vdash :. \text{Hp}\, *95{\cdot}411 . \supset : M \in (P * Q)\text{‘}R - \iota\text{‘}R . \supset . \breve{P} \,|\, M \,|\, \breve{Q} \in (P * Q)\text{‘}R$

Dem.

$\vdash . *95{\cdot}411 . *91{\cdot}351{\cdot}281 . \supset$

$\vdash :. \text{Hp} . \supset : M \in (P * Q)\text{‘}R . \supset . \breve{P} \,|\, (P \,|\, M \,|\, Q) \,|\, \breve{Q} \in (P * Q)\text{‘}R \quad (1)$

$\vdash . (1) . *95{\cdot}12 . \supset \vdash . \text{Prop}$

***95·43.** $\vdash :. \text{Hp} *95{\cdot}411 . \text{Hp} *95{\cdot}382 . \supset : S \in \text{Potid}`P . T \in \text{Potid}`Q .$
$P | S | R | T | Q \in (P*Q)`R . \supset . S | R | T \in (P*Q)`R$

Dem.

$\vdash . *95{\cdot}42{\cdot}382 . *91{\cdot}28{\cdot}3 . \supset \vdash :. \text{Hp} . \supset : S \in \text{Potid}`P . T \in \text{Potid}`Q .$
$P | S | R | T | Q \in (P*Q)`R . \supset . \breve{P} | P | S | R | T | Q | \breve{Q} \in (P*Q)`R$ (1)

$\vdash . *95{\cdot}41 . \supset \vdash :. \text{Hp} . \supset : S \in \text{Potid}`P . T \in \text{Potid}`Q . \supset .$
$\breve{P} | P | S | R | T | Q | \breve{Q} = S | R | T$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***95·431.** $\vdash : \text{Hp} *95{\cdot}43 . S \in \text{Potid}`P . T \in \text{Potid}`Q . M \in (P*Q)`R .$
$P | S | M | T | Q \in (P*Q)`R . \supset . S | M | T \in (P*Q)`R$

Dem.

$\vdash . *95{\cdot}22 . \supset \vdash : \text{Hp} . \supset . (\exists S', T') . S' \in \text{Potid}`P . T' \in \text{Potid}`Q . M = S' | R | T' .$
$P | S | M | T | Q \in (P*Q)`R .$

[*91·341] $\supset . (\exists S', T') . S' \in \text{Potid}`P . T' \in \text{Potid}`Q . M = S' | R | T' .$
$S | S' \in \text{Potid}`P . T' | T \in \text{Potid}`Q . P | S | S' | R | T' | T | Q \in (P*Q)`R .$

[*95·43] $\supset . (\exists S', T') . S' \in \text{Potid}`P . T' \in \text{Potid}`Q . M = S' | R | T' .$
$S | S' | R | T' | T \in (P*Q)`R .$

[*13·195] $\supset . S | M | T \in (P*Q)`R : \supset \vdash . \text{Prop}$

***95·44.** $\vdash :. \text{Hp} *95{\cdot}43 . S \in \text{Potid}`P . T \in \text{Potid}`Q . \supset :$
$M \in (P*Q)`R . S | M | T \in (P*Q)`R . \supset . S | R | T \in (P*Q)`R$

Dem.

$\vdash . \text{Id} . \supset \vdash :: \phi M . \equiv_M : S | M | T \in (P*Q)`R . \supset . S | R | T \in (P*Q)`R :. \supset . \phi R$ (1)

$\vdash . *95{\cdot}431 . *91{\cdot}3 . \supset$

$\vdash ::. \text{Hp} . \supset :: S | P | M | Q | T \in (P*Q)`R . \supset :. S | M | T \in (P*Q)`R :.$

[*2·27] $\supset :. S | M | T \in (P*Q)`R . \supset . S | R | T \in (P*Q)`R : \supset .$
$S | R | T \in (P*Q)`R$ (2)

$\vdash . (2) . \text{Comm} . \supset$

$\vdash :: \text{Hp} . \supset :. S | M | T \in (P*Q)`R . \supset . S | R | T \in (P*Q)`R : \supset :$
$S | (P | M | Q) | T \in (P*Q)`R . \supset . S | R | T \in (P*Q)`R$ (3)

$\vdash . (3) . \supset \vdash :. \text{Hp} . \text{Hp}(1) . \supset : \phi M . \supset . \phi (P | M | Q)$ (4)

$\vdash . (1) . (4) . *95{\cdot}14 . \supset \vdash : \text{Hp} . \text{Hp}(1) . M \in (P*Q)`R . \supset . \phi M : \supset \vdash . \text{Prop}$

***95·45.** $\vdash :. \text{Hp} *95{\cdot}43 . S, S' \in \text{Potid}`P . T, T' \in \text{Potid}`Q .$
$S | S' | R | T' | T \in (P*Q)`R . \supset : S | R | T \in (P*Q)`R . \equiv . S' | R | T' \in (P*Q)`R$

Dem.

$\vdash . *95{\cdot}44 . \supset \vdash : \text{Hp} . S' | R | T' \in (P*Q)`R . \supset . S | R | T \in (P*Q)`R$ (1)

$\vdash . *91{\cdot}34 . \supset \vdash :. \text{Hp} . \supset : S' | S | R | T | T' \in (P*Q)`R :$

[*95·44] $\supset : S | R | T \in (P*Q)`R . \supset . S' | R | T' \in (P*Q)`R$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***95·46.** $\vdash :. \text{Hp}\,*95{\cdot}41 \,.\, \dot{\exists}\,!\,R \,.\, \text{D}{`}R \subset \overrightarrow{B}{`}\breve{P} \,.\, \text{Ɑ}{`}R \subset \overrightarrow{B}{`}Q \,.\, \supset :$

$$T \in \text{Pot}{`}Q \,.\, \supset \,.\, R \,|\, T \sim\in (P*Q){`}R$$

Dem.

$\vdash \,.\, *95{\cdot}38 \,.\, \supset \vdash :. \text{Hp} \,.\, T \in \text{Pot}{`}Q \,.\, \supset : R \,|\, T \neq R :$

$[*95{\cdot}42] \qquad \supset : R \,|\, T \in (P*Q){`}R \,.\, \supset \,.\, \breve{P} \,|\, R \,|\, T \,|\, \breve{Q} \in (P*Q){`}R \,.$

$[*95{\cdot}32] \qquad \supset \,.\, \dot{\exists}\,!\,\breve{P} \,|\, R \,|\, T \,|\, \breve{Q} \,.$

$[*34{\cdot}31] \qquad \supset \,.\, \dot{\exists}\,!\,\breve{P} \,|\, R \,.$

$[*34{\cdot}3] \qquad \supset \,.\, \exists\,!\,\text{D}{`}P \cap \text{D}{`}R \quad (1)$

$\vdash \,.\, *93{\cdot}12 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, \text{D}{`}P \cap \text{D}{`}R = \Lambda \quad (2)$

$\vdash \,.\, (2) \,.\, (1) \,.\, \text{Transp} \,.\, \supset \vdash : \text{Hp} \,.\, T \in \text{Pot}{`}Q \,.\, \supset \,.\, R \,|\, T \sim\in (P*Q){`}R : \supset \vdash \,.\, \text{Prop}$

***95·47.** $\vdash : \text{Hp}\,*95{\cdot}46 \,.\, S \in \text{Potid}{`}P \,.\, T, T' \in \text{Potid}{`}Q \,.$

$$S \,|\, R \,|\, T, S \,|\, R \,|\, T' \in (P*Q){`}R \,.\, \supset \,.\, T = T'$$

Dem.

$\vdash \,.\, *91{\cdot}46 \,.\, \supset \vdash :. \text{Hp} \,.\, \supset : (\exists U) : U \in \text{Potid}{`}Q : T = U \,|\, T' \,.\, \text{v} \,.\, T' = U \,|\, T \quad (1)$

$\vdash \,.\, *50{\cdot}62 \,.\, *91{\cdot}35 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, S = S \,|\, I \restriction C{`}P \,.\, I \restriction C{`}P \in \text{Potid}{`}P \quad (2)$

$\vdash \,.\, *95{\cdot}45 \,.\, *33{\cdot}24 \,.\, *22{\cdot}621 \,.\, (2) \,.\, \supset$

$\vdash : \text{Hp} \,.\, U \in \text{Potid}{`}Q \,.\, T = U \,|\, T' \,.\, \supset \,.\, I \restriction C{`}P \,|\, R \,|\, U \in (P*Q){`}R \,.\, U \in \text{Potid}{`}Q \,.$

$[*50{\cdot}63] \qquad \supset \,.\, R \,|\, U \in (P*Q){`}R \,.\, U \in \text{Potid}{`}Q \,.$

$[*95{\cdot}46.\text{Transp}] \qquad \supset \,.\, U \sim\in \text{Pot}{`}Q \,.\, U \in \text{Potid}{`}Q \,.$

$[*91{\cdot}23] \qquad \supset \,.\, U = I \restriction C{`}Q \,.$

$[*91{\cdot}27.*50{\cdot}63] \qquad \supset \,.\, U \,|\, T' = T' \,.$

$[*13{\cdot}12] \qquad \supset \,.\, T = T' \quad (3)$

Similarly $\quad \vdash : \text{Hp} \,.\, U \in \text{Potid}{`}Q \,.\, T' = U \,|\, T \,.\, \supset \,.\, T = T' \quad (4)$

$\vdash \,.\, (1) \,.\, (3) \,.\, (4) \,.\, \supset \vdash \,.\, \text{Prop}$

***95·471.** $\vdash : \text{Hp}\,*95{\cdot}46 \,.\, S, S' \in \text{Potid}{`}P \,.\, T \in \text{Potid}{`}Q \,.$

$$S \,|\, R \,|\, T, S' \,|\, R \,|\, T \in (P*Q){`}R \,.\, \supset \,.\, S = S'$$

[Proof as in *95·47]

***95·51.** $\vdash : \text{Hp}\,*95{\cdot}46 \,.\, M, M' \in (P*Q){`}R \,.\, \exists\,!\,\text{Ɑ}{`}M \cap \text{Ɑ}{`}M' \,.\, \supset \,.\, M = M'$

Dem.

$\vdash \,.\, *95{\cdot}22 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, (\exists S, S', T, T') \,.\, S, S' \in \text{Potid}{`}P \,.\, T, T' \in \text{Potid}{`}Q \,.$

$$M = S \,|\, R \,|\, T \,.\, M' = S' \,|\, R \,|\, T' \,.$$

$$S \,|\, R \,|\, T, S' \,|\, R \,|\, T' \in (P*Q){`}R \,.$$

$$\exists\,!\,\text{Ɑ}{`}(S \,|\, R \,|\, T) \cap \text{Ɑ}{`}(S' \,|\, R \,|\, T') \,.$$

$[*95{\cdot}351] \supset \,.\, (\exists S, S', T) \,.\, S, S' \in \text{Potid}{`}P \,.\, T \in \text{Potid}{`}Q \,.\, M = S|R|T \,.\, M' = S'|R|T \,.$

$$S \,|\, R \,|\, T, S' \,|\, R \,|\, T \in (P*Q){`}R \,.$$

$[*95{\cdot}471] \supset \,.\, (\exists S, T) \,.\, S \in \text{Potid}{`}P \,.\, T \in \text{Potid}{`}Q \,.\, M = S \,|\, R \,|\, T \,.\, M' = S \,|\, R \,|\, T \,.$

$[*13{\cdot}172] \supset \,.\, M = M' : \supset \vdash \,.\, \text{Prop}$

***95·511.** $\vdash : \mathrm{Hp}\ *95{\cdot}46 \,.\, M, M' \in (P*Q){}^{\backprime}R \,.\, \exists ! \, \mathrm{D}{}^{\backprime}M \cap \mathrm{D}{}^{\backprime}M' \,.\supset .\, M = M'$

[Proof as in *95·51]

***95·52.** $\vdash : P, Q, R \in 1 \to 1 \,.\, \mathrm{D}{}^{\backprime}P \subset \text{Ɑ}{}^{\backprime}P \,.\, \text{Ɑ}{}^{\backprime}Q \subset \mathrm{D}{}^{\backprime}Q \,.\, \mathrm{D}{}^{\backprime}R \subset \overrightarrow{B}{}^{\backprime}\breve{P} \,.\, \text{Ɑ}{}^{\backprime}R \subset \overrightarrow{B}{}^{\backprime}Q \,.\supset .$

$\dot{s}{}^{\backprime}(P*Q){}^{\backprime}R \in 1 \to 1$

Dem.

$\vdash . *95{\cdot}21 \,.\, *34{\cdot}32 \,.\supset \vdash : R = \dot{\Lambda} \,.\supset .\, (P*Q){}^{\backprime}R \subset \iota{}^{\backprime}\dot{\Lambda} \,.$

$[*53{\cdot}04] \qquad \supset .\, \dot{s}{}^{\backprime}(P*Q){}^{\backprime}R = \dot{\Lambda} \,.$

$[*72{\cdot}1] \qquad \supset .\, \dot{s}{}^{\backprime}(P*Q){}^{\backprime}R \in 1 \to 1 \qquad (1)$

$\vdash . *92{\cdot}102 \,.\, *95{\cdot}21 \,.\, *71{\cdot}252 \,.\supset \vdash : \mathrm{Hp} \,.\, M \in (P*Q){}^{\backprime}R \,.\supset .\, M \in 1 \to 1 \qquad (2)$

$\vdash . *41{\cdot}11 \,.\supset \vdash : x \,\{\dot{s}{}^{\backprime}(P*Q){}^{\backprime}R\}\, y \,.\, x \,\{\dot{s}{}^{\backprime}(P*Q){}^{\backprime}R\}\, z \,.\supset .$

$(\exists M, M') \,.\, M, M' \in (P*Q){}^{\backprime}R \,.\, xMy \,.\, xM'z \,.$

$[*33{\cdot}14] \quad \supset .\, (\exists M, M') \,.\, M, M' \in (P*Q){}^{\backprime}R \,.\, xMy \,.\, xM'z \,.\, \exists ! \, \mathrm{D}{}^{\backprime}M \cap \mathrm{D}{}^{\backprime}M' \qquad (3)$

$\vdash . (3) \,.\, *95{\cdot}511 \,.\supset$

$\vdash : \mathrm{Hp} \,.\, \dot{\exists} ! \, R \,.\, \mathrm{Hp}\,(3) \,.\supset .\, (\exists M) \,.\, M \in (P*Q){}^{\backprime}R \,.\, xMy \,.\, xMz \,.$

$[(2)] \qquad \supset .\, y = z \qquad (4)$

Similarly

$\vdash : \mathrm{Hp} \,.\, \dot{\exists} ! \, R \,.\, x \,\{\dot{s}{}^{\backprime}(P*Q){}^{\backprime}R\}\, z \,.\, y \,\{\dot{s}{}^{\backprime}(P*Q){}^{\backprime}R\}\, z \,.\supset .\, x = y \qquad (5)$

$\vdash . (4) . (5) \,.\, *71{\cdot}172 \,.\supset \vdash : \mathrm{Hp} \,.\, \dot{\exists} ! \, R \,.\supset .\, \dot{s}{}^{\backprime}(P*Q){}^{\backprime}R \in 1 \to 1 \qquad (6)$

$\vdash . (1) . (6) \,.\supset \vdash . \mathrm{Prop}$

***95·6.** $\vdash : \mathrm{D}{}^{\backprime}R \subset \text{Ɑ}{}^{\backprime}P \,.\, \mathrm{D}{}^{\backprime}P \subset \text{Ɑ}{}^{\backprime}P \,.\, \text{Ɑ}{}^{\backprime}R = \overrightarrow{B}{}^{\backprime}Q \,.\, Q \in 1 \to \mathrm{Cls} \,.\supset .$

$\text{Ɑ}{}^{\backprime\backprime}(P*Q){}^{\backprime}R = \mathrm{gen}{}^{\backprime}Q$

Dem.

$\vdash . *92{\cdot}143 \,.\supset \vdash : \mathrm{Hp} \,.\, S \in \mathrm{Potid}{}^{\backprime}P \,.\supset .\, \text{Ɑ}{}^{\backprime}S = \text{Ɑ}{}^{\backprime}P \,.$

$[\mathrm{Hp}] \qquad \supset .\, \mathrm{D}{}^{\backprime}R \subset \text{Ɑ}{}^{\backprime}S \,.$

$[*37{\cdot}322] \qquad \supset .\, \text{Ɑ}{}^{\backprime}(S \mid R) = \text{Ɑ}{}^{\backprime}R \,.$

$[*37{\cdot}32] \qquad \supset .\, \text{Ɑ}{}^{\backprime}(S \mid R \mid T) = \breve{T}{}^{\backprime\backprime}\text{Ɑ}{}^{\backprime}R \qquad (1)$

$\vdash . (1) \,.\supset \vdash : \mathrm{Hp} \,.\, S \in \mathrm{Potid}{}^{\backprime}P \,.\, T \in \mathrm{Potid}{}^{\backprime}Q \,.\supset .\, \text{Ɑ}{}^{\backprime}(S \mid R \mid T) = \breve{T}{}^{\backprime\backprime}\overrightarrow{B}{}^{\backprime}Q \,. \qquad (2)$

$[*93{\cdot}32] \qquad \supset .\, \text{Ɑ}{}^{\backprime}(S \mid R \mid T) \in \mathrm{gen}{}^{\backprime}Q \qquad (3)$

$\vdash . (3) \,.\, *95{\cdot}22 \,. \qquad \supset \vdash : \mathrm{Hp} \,.\supset .\, \text{Ɑ}{}^{\backprime\backprime}(P*Q){}^{\backprime}R \subset \mathrm{gen}{}^{\backprime}Q \qquad (4)$

$\vdash . (2) \,.\, *95{\cdot}221 \,.\, *93{\cdot}32 \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, \mathrm{gen}{}^{\backprime}Q \subset \text{Ɑ}{}^{\backprime\backprime}(P*Q){}^{\backprime}R \qquad (5)$

$\vdash . (4) . (5) \,.\supset \vdash . \mathrm{Prop}$

***95·601.** $\vdash : \text{Ɑ}{}^{\backprime}R \subset \mathrm{D}{}^{\backprime}Q \,.\, \text{Ɑ}{}^{\backprime}Q \subset \mathrm{D}{}^{\backprime}Q \,.\, \mathrm{D}{}^{\backprime}R = \overrightarrow{B}{}^{\backprime}\breve{P} \,.\, P \in \mathrm{Cls} \to 1 \,.\supset .$

$\mathrm{D}{}^{\backprime\backprime}(P*Q){}^{\backprime}R = \mathrm{gen}{}^{\backprime}\breve{P}$

[Proof as in *95·6]

***95·61.** $\vdash : P, Q, R \in 1 \to 1 \,.\, \mathrm{D}{}^{\backprime}P \subset \text{Ɑ}{}^{\backprime}P \,.\, \text{Ɑ}{}^{\backprime}Q \subset \mathrm{D}{}^{\backprime}Q \,.\, \mathrm{D}{}^{\backprime}R = \overrightarrow{B}{}^{\backprime}\breve{P} \,.\, \text{Ɑ}{}^{\backprime}R = \overrightarrow{B}{}^{\backprime}Q \,.\supset .$

$\dot{s}{}^{\backprime}(P*Q){}^{\backprime}R \in 1 \to 1 \,.\, \mathrm{D}{}^{\backprime}\dot{s}{}^{\backprime}(P*Q){}^{\backprime}R = s{}^{\backprime}\mathrm{gen}{}^{\backprime}\breve{P} \,.\, \text{Ɑ}{}^{\backprime}\dot{s}{}^{\backprime}(P*Q){}^{\backprime}R = s{}^{\backprime}\mathrm{gen}{}^{\backprime}Q$

$[*95{\cdot}52{\cdot}6{\cdot}601 \,.\, *41{\cdot}43{\cdot}44]$

***95·62.** $\vdash : \text{Hp} *95\cdot61 . \supset . s\text{'gen'}\breve{P} \text{ sm } s\text{'gen'}Q$ [*95·61 . *73·2]

***95·63.** $\vdash : P, Q \in 1 \to 1 . \text{G'}P \subset \text{D'}P . \text{G'}Q \subset \text{D'}Q . \overrightarrow{B}\text{'}P \text{ sm } \overrightarrow{B}\text{'}Q . \supset .$
$$s\text{'gen'}P \text{ sm } s\text{'gen'}Q$$

Dem.

$$\vdash . *95\cdot62 \frac{\breve{P}}{P} . \supset \vdash : P, Q, R \in 1 \to 1 . \text{G'}P \subset \text{D'}P . \text{G'}Q \subset \text{D'}Q .$$
$$\text{D'}R = \overrightarrow{B}\text{'}P . \text{G'}R = \overrightarrow{B}\text{'}Q . \supset . s\text{'gen'}P \text{ sm } s\text{'gen'}Q \quad (1)$$
$$\vdash . (1) . *10\cdot11\cdot23\cdot35 . *73\cdot1 . \supset \vdash . \text{Prop}$$

***95·64.** $\vdash : P, Q \in 1 \to 1 . \text{G'}P \subset \text{D'}P . \text{G'}Q \subset \text{D'}Q . \overrightarrow{B}\text{'}P \text{ sm } \overrightarrow{B}\text{'}Q .$
$$p\text{'G''Pot'}P = \Lambda . p\text{'G''Pot'}Q = \Lambda . \supset . \text{D'}P \text{ sm } \text{D'}Q$$
[*95·63 . *93·274 . *33·181]

***95·65.** $\vdash : P, Q \in 1 \to 1 . \text{G'}P \subset \text{D'}P . \text{G'}Q \subset \text{D'}Q . \overrightarrow{B}\text{'}P \text{ sm } \overrightarrow{B}\text{'}Q .$
$$C\text{'}P = \breve{P}_*\text{''}\overrightarrow{B}\text{'}P . C\text{'}Q = \breve{Q}_*\text{''}\overrightarrow{B}\text{'}Q . \supset . C\text{'}P \text{ sm } C\text{'}Q$$
[*95·63 . *93·36]

The following example may illustrate the scope of *95·65. Let R, S be the generating relations of two well-ordered series, neither of which has a last term. Put $P = R \dot{-} R^2 . Q = S \dot{-} S^2$. Then P is the relation of immediately preceding in the R-series, and Q is the relation of immediately preceding in the S-series. We shall have

$$P, Q \in 1 \to 1 . \text{G'}P \subset \text{D'}P . \text{G'}Q \subset \text{D'}Q.$$

Also, except in certain exceptional cases, $\overrightarrow{B}\text{'}P$, $\overrightarrow{B}\text{'}Q$ are the first derivatives of the two series (including the first terms of the two series).

$$\text{"}C\text{'}P = \breve{P}_*\text{''}\overrightarrow{B}\text{'}P\text{"}$$

states that, starting from any term of the series and going backwards, a finite number of steps will bring us to a member of the first derivative, which is true. Hence, by *95·65, neglecting certain exceptional cases, we arrive at the result that if the first derivatives of two well-ordered series have the same cardinal number of terms, then the series themselves have the same cardinal number of terms. This proposition can of course be proved otherwise; the above is merely mentioned as an illustration of the results of *95·65.

***95·7.** $\vdash : R, S \in 1 \to 1 . \text{G'}R \subset \text{D'}S . \text{G'}S \subset \text{D'}R . \supset . \overrightarrow{B}\text{'}(R \mid S) \text{ sm } \overrightarrow{B}\text{'}(S \mid R)$

Dem.

$\vdash . *93\cdot101 . *24\cdot412 . *37\cdot16\cdot321 . \supset$

$$\vdash : \text{Hp} . \supset . \overrightarrow{B}\text{'}(R \mid S) = (\text{D'}R - \text{G'}S) \cup (\text{G'}S - \breve{S}\text{''G'}R) .$$
$$\overrightarrow{B}\text{'}(S \mid R) = (\text{D'}S - \text{G'}R) \cup (\text{G'}R - \breve{R}\text{''G'}S) \quad (1)$$
$$\vdash . *71\cdot38 . *37\cdot32 . \supset \vdash : \text{Hp} . \supset . \breve{R}\text{''}(\text{D'}R - \text{G'}S) = \text{G'}R - \breve{R}\text{''G'}S \quad (2)$$
$$\vdash . *71\cdot381 . *37\cdot32 . \supset \vdash : \text{Hp} . \supset . S\text{''}(\text{G'}S - \breve{S}\text{''G'}R) = \text{D'}S - S\text{''}\breve{S}\text{''G'}R$$
[*72·502]
$$= \text{D'}S - \text{G'}R \quad (3)$$

$\vdash . (2) . (3) . *73{\cdot}21{\cdot}22 . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{D}{`}R \mathbin{\dot{-}} \mathrm{Cl}{`}S \,\mathrm{sm}\, \mathrm{Cl}{`}R - \breve{R}{``}\mathrm{Cl}{`}S .$

$\mathrm{Cl}{`}S - \breve{S}{``}\mathrm{Cl}{`}R \,\mathrm{sm}\, \mathrm{D}{`}S - \mathrm{Cl}{`}R \qquad (4)$

$\vdash . *24{\cdot}21 . \qquad \supset \vdash : \mathrm{Hp} . \supset . (\mathrm{D}{`}R - \mathrm{Cl}{`}S) \cap (\mathrm{Cl}{`}S - \breve{S}{``}\mathrm{Cl}{`}R) = \Lambda .$

$(\mathrm{Cl}{`}R - \breve{R}{``}\mathrm{Cl}{`}S) \cap (\mathrm{D}{`}S - \mathrm{Cl}{`}R) = \Lambda \qquad (5)$

$\vdash . (1) . (4) . (5) . *73{\cdot}71 . \supset \vdash . \mathrm{Prop}$

***95·71.** $\vdash : R, S \in 1 \to 1 . \mathrm{Cl}{`}R \subset \mathrm{D}{`}S . \mathrm{Cl}{`}S \subset \mathrm{D}{`}R . \supset . s{`}\mathrm{gen}{`}(R \mid S) \,\mathrm{sm}\, s{`}\mathrm{gen}{`}(S \mid R)$

Dem.

$\vdash . *34{\cdot}36 . *37{\cdot}321 . \quad \supset \vdash : \mathrm{Hp} . \supset . \mathrm{Cl}{`}(R \mid S) \subset \mathrm{D}{`}(R \mid S) . \mathrm{Cl}{`}(S \mid R) \subset \mathrm{D}{`}(S \mid R) \quad (1)$

$\vdash . *71{\cdot}252 . \qquad \supset \vdash : \mathrm{Hp} . \supset . R \mid S, S \mid R \in 1 \to 1 \qquad (2)$

$\vdash . (1) . (2) . *95{\cdot}7{\cdot}63 . \supset \vdash . \mathrm{Prop}$

This proposition and *94·53 or *94·54 together reconstitute the Schröder-Bernstein theorem (*73·88). For, in virtue of *93·274·275 and *73·71, they together give

$$R, S \in 1 \to 1 . \mathrm{Cl}{`}R \subset \mathrm{D}{`}S . \mathrm{Cl}{`}S \subset \mathrm{D}{`}R . \supset . C{`}(R \mid S) \,\mathrm{sm}\, C{`}(S \mid R),$$

and with this hypothesis

$$C{`}(R \mid S) = \mathrm{D}{`}R . C{`}(S \mid R) = \mathrm{D}{`}S.$$

$*96$. ON THE POSTERITY OF A TERM

Summary of $*96$.

By the "posterity" of a term with respect to a relation R we mean the class $\overleftarrow{R_*}`x$. In the present number, we shall be chiefly concerned with the relation $(\overleftarrow{R_*}`x) \upharpoonright R$, *i.e.* the relation R confined to the posterity of x. We shall also be concerned with $(\overleftarrow{R_*}`x) \upharpoonright R_*$ and $(\overleftarrow{R_*}`x) \upharpoonright R_{\text{po}}$, which, as is proved in $*96 \cdot 13$, are respectively

$$\{(\overleftarrow{R_*}`x) \upharpoonright R\}_* \text{ and } \{(\overleftarrow{R_*}`x) \upharpoonright R\}_{\text{po}}.$$

The most interesting case is when $R \in \text{Cls} \to 1$. In this case, $\overleftarrow{R_*}`x$ is in general shaped like a $Q$, with $x$ at the tip of the tail; that is, $\overleftarrow{R_*}`x$ may be divided into two parts, the first an open series, the second a closed series. If y is the junction of the two, we shall have

$$xR_*z \,.\, zR_{\text{po}}y \,.\, \supset \,.\, \sim (zR_{\text{po}}z),$$

$$yR_*z \,.\, \supset \,.\, zR_{\text{po}}z;$$

in fact, $\qquad (\exists P) : P \in \text{Pot}`R : yR_*z \,.\, \supset_z \,.\, zPz.$

We have also, when $R \in \text{Cls} \to 1$,

$$y, z \in \overleftarrow{R_*}`x \,.\, \supset : yR_*z \,.\, \vee \,.\, zR_*y.$$

It thus appears that $\overleftarrow{R_*}`x$ is divided into two parts, the first consisting of those terms $z$ for which $\sim (zR_{\text{po}}z)$, the second of those for which $zR_{\text{po}}z$. The first wholly precedes the second; the first exists if $\sim (xR_{\text{po}}x)$, the second if $\dot{\exists} ! \{(\overleftarrow{R_*}`x) \upharpoonright R_{\text{po}} \dot{\cap} I\}$. Every term in $\overleftarrow{R_{\text{po}}}`x$ has one and only one immediate predecessor, except the term (if it exists) at the junction of the tail and circle of the Q; this term has just two immediate predecessors, one in the tail and one in the circle. But if either the tail or the circle is null, then every term in $\overleftarrow{R_{\text{po}}}`x$ has only one immediate predecessor, and therefore

$$(\overleftarrow{R_*}`x) \upharpoonright R \in 1 \to 1.$$

Put $\qquad I_R`x = \overleftarrow{R_*}`x \cap \hat{z}(zR_{\text{po}}z) \quad \text{Dft}$

$$J_R`x = \overleftarrow{R_*}`x \cap \hat{z}\{\sim (zR_{\text{po}}z)\} \quad \text{Dft}$$

(these definitions being only to apply within $*96$). Then $J_R`x$ is the open part of the series $\overleftarrow{R_*}`x$, and $I_R`x$ is the circular part. The open part wholly precedes the circular part, provided $R \in \text{Cls} \to 1$; *i.e.*

$$R \in \text{Cls} \to 1 \,.\, \supset \,.\, J_R`x \subset p`\overrightarrow{R_{\text{po}}}``I_R`x.$$

If J_R'x and I_R'x both exist, J_R'x has a last term, say y. The successor of this term, $\breve{R}$'y, is the only term in $\overleftarrow{R_*}$'x which has two immediate predecessors in $\overleftarrow{R_*}$'x, namely y and $\breve{\iota}$'$(I_R$'$x \cap \overrightarrow{R}$'$\breve{R}$'$y)$.

The most important applications of the propositions of the present number are in the theory of finite and infinite, both cardinal and ordinal. When R is many-one, then if I_R'x exists, or, more generally, if J_R'x has a last term, $\overleftarrow{R_*}$'x is a finite class, *i.e.* what we shall call a "Cls induct" (cf. $*120$). That is, we have

$$\vdash : R \epsilon \mathrm{Cls} \rightarrow 1 \,.\, \mathrm{E} \,!\, \max_R\text{'}J_R\text{'}x \,.\, \supset \,.\, \overleftarrow{R_*}\text{'}x \epsilon \mathrm{Cls\ induct}.$$

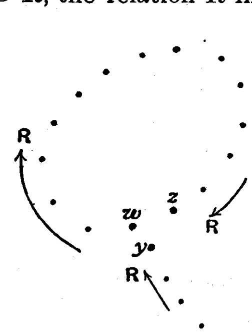
If J_R'x exists, but has no last term, $\overleftarrow{R_*}$'x is a *progression* (cf. $*122$) when its terms are arranged in the order generated by R. That is, giving to $\aleph_0$ and ω the meanings given by Cantor (cf. $*123$ and $*263$), and using "Prog" for the class of one-one relations which generate progressions, we have

$$\vdash : R \epsilon \mathrm{Cls} \rightarrow 1 \,.\, \sim \mathrm{E} \,!\, \max_R\text{'}J_R\text{'}x \,.\, \exists \,!\, J_R\text{'}x \,.\, \supset \,.$$
$$\overleftarrow{R_*}\text{'}x \epsilon \aleph_0 \,.\, (\overleftarrow{R_*}\text{'}x) \upharpoonright R \epsilon \mathrm{Prog} \,.\, (\overleftarrow{R_*}\text{'}x) \upharpoonright R_{\mathrm{po}} \epsilon \omega.$$

Another very important proposition in the proof of which the present number is useful is $*121{\cdot}47$, which proves that if R is either one-many or many-one, and a and z are any two terms whatever, then $\overleftarrow{R_*}$'$a \cap \overrightarrow{R_*}$'$z$ (which we call the "interval" from a to z) is always a finite class. The proof that progressions are well-ordered series depends upon the propositions of this number, since it uses $*122{\cdot}23$, which depends upon $*96{\cdot}52$.

The present number begins with a series of propositions (ending with $*96{\cdot}16$) on $\alpha \upharpoonright R_{\mathrm{po}}$ and $\alpha \upharpoonright R_*$, both in general and when $\alpha = \overleftarrow{R_*}$'$x$. We then proceed to a few propositions ($*96{\cdot}2$—$\cdot 25$) on $(\overleftarrow{R_*}$'$x) \upharpoonright R$ when $R \epsilon 1 \rightarrow \mathrm{Cls}$; with the exception of $*96{\cdot}24$, these propositions are all used in the cardinal theory of finite and infinite. They are, however, less important than the subsequent propositions, which are concerned with $\overleftarrow{R_*}$'x when $R \epsilon \mathrm{Cls} \rightarrow 1$.

If R is a many-one relation, and x is a member of D'R, the relation R in general arranges $\overleftarrow{R_*}$'x (*i.e.* the posterity of x) in a figure such as is here given. The relation R holds between each dot and the next, starting from x, and travelling round the circle in the sense indicated by the arrow. The dots from x to y constitute J_R'x, and the dots in the circle constitute I_R'x. y is the last term of J_R'x, *i.e.* $\max_R$'J_R'x; w is $\breve{R}$'y, and z is $\breve{\iota}$'$(\overrightarrow{R}$'$w \cap I_R$'$x)$, or, what comes to the same thing, $\{(I_R$'$x) \upharpoonright R\}$'w. w is the only term which has more than one immediate predecessor in $\overleftarrow{R_*}$'x; w always



exists if neither $J_R\text{‘}x$ nor $I_R\text{‘}x$ is null, and conversely, if w exists, neither $J_R\text{‘}x$ nor $I_R\text{‘}x$ is null. The proof of these propositions is long; the following are useful stages in the proof.

If xRx, the whole posterity of x is x itself (∗96·33); if xRy and yRx, x and y constitute the whole posterity of x (∗96·331), and so on. The successors of members of $I_R\text{‘}x$ belong to $I_R\text{‘}x$ (∗96·341), and the predecessors of members of $J_R\text{‘}x$, if they belong to $\overleftarrow{R_*}\text{‘}x$, belong to $J_R\text{‘}x$ (∗96·351). (It should be observed that, since R is only assumed to be many-one, not one-one, every member of $\overleftarrow{R_*}\text{‘}x$ may have any number of predecessors which do not belong to $\overleftarrow{R_*}\text{‘}x$.) We have a series of propositions, beginning with ∗96·4, which deal with the hypothesis $yRw \,.\, zRw$. We prove (∗96·42) that if $yRw \,.\, zRw$ and $yR_{\text{po}}z$, then $zR_{\text{po}}z$, *i.e.* z belongs to $I_R\text{‘}x$. We prove (∗96·431) that $J_R\text{‘}x$ wholly precedes $I_R\text{‘}x$; that $(J_R\text{‘}x) \upharpoonleft R$ and $(I_R\text{‘}x) \upharpoonleft R$ are both one-one (∗96·45), so that if $yRw \,.\, zRw \,.\, y \neq z$, one of y and z must belong to $J_R\text{‘}x$ and the other to $I_R\text{‘}x$ (∗96·441). Hence it follows (∗96·453) that if either $xR_{\text{po}}x$ (in which case $J_R\text{‘}x = \Lambda$) or $(\overleftarrow{R_*}\text{‘}x) \upharpoonleft R_{\text{po}} \subset J$ (in which case $I_R\text{‘}x = \Lambda$), then $(\overleftarrow{R_*}\text{‘}x) \upharpoonleft R$ is a one-one relation. (This proposition is used twice in the cardinal theory of finite and infinite, namely in ∗121·43 and ∗122·17.) Hence we arrive at the proposition (∗96·47) that if two different members y and z of $\overleftarrow{R_*}\text{‘}x$ both immediately precede a term w, then one of y and z (say y) is the last term of $J_R\text{‘}x$, w is its immediate successor and z is the immediate predecessor of w in $I_R\text{‘}x$, *i.e.* we have

$$y = \max_R\text{‘}J_R\text{‘}x \,.\, w = \breve{R}\text{‘}\max_R\text{‘}J_R\text{‘}x \,.\, z = \{(I_R\text{‘}x) \upharpoonleft R\}\text{‘}\breve{R}\text{‘}\max_R\text{‘}J_R\text{‘}x.$$

Thus y, z, w are unique if they exist. We prove next (∗96·475) that y, z, w exist when, and only when, neither $I_R\text{‘}x$ nor $J_R\text{‘}x$ is null.

It follows from the above propositions that if R is one-one, either $I_R\text{‘}x$ or $J_R\text{‘}x$ must be null (∗96·491), *i.e.* the posterity of a term is either an open series or a cycle, and cannot have the Q-shape.

∗96·01. $I_R\text{‘}x = \overleftarrow{R_*}\text{‘}x \cap \hat{z}(zR_{\text{po}}z)$ Dft [∗96]

∗96·02. $J_R\text{‘}x = \overleftarrow{R_*}\text{‘}x - I_R\text{‘}x$ Dft [∗96]

∗96·1. $\vdash : z \in I_R\text{‘}x \,.\equiv.\, xR_*z \,.\, zR_{\text{po}}z$ [∗20·3 . ∗32·181 . (∗96·01)]

∗96·101. $\vdash : z \in J_R\text{‘}x \,.\equiv.\, xR_*z \,.\, \sim(zR_{\text{po}}z)$ [∗96·1 . ∗22·93 . (∗96·02)]

∗96·102. $\vdash .\, \overleftarrow{R_*}\text{‘}x = J_R\text{‘}x \cup I_R\text{‘}x \,.\, J_R\text{‘}x \cap I_R\text{‘}x = \Lambda$ [∗24·41·21 . (∗96·01·02)]

∗96·103. $\vdash .\, (J_R\text{‘}x) \upharpoonleft R_{\text{po}} \subset J$

Dem.

$\vdash .\, ∗96·101 \,.\supset \vdash :.\, y\,\{(J_R\text{‘}x) \upharpoonleft R_{\text{po}}\}\,z \,.\equiv:\, xR_*y \,.\, \sim(yR_{\text{po}}y) \,.\, yR_{\text{po}}z :$

[∗13·14] $\supset : y \neq z :.\supset \vdash .$ Prop

***96·104.** $\vdash : I_R{}^{\backprime}x = \Lambda . \equiv . (\overleftarrow{R_*}{}^{\backprime}x) \upharpoonleft R_{po} \subset J . \equiv . J_R{}^{\backprime}x \doteqdot \overleftarrow{R_*}{}^{\backprime}x$

Dem.

$\vdash . *96·1 . \supset \vdash :. I_R{}^{\backprime}x = \Lambda . \equiv : xR_*y . \supset_y . \sim(yR_{po}y) :$

$[*13·196] \quad \equiv : xR_*y . yR_{po}z . \supset_{y,z} . y \neq z :$

$[*35·1] \quad \equiv : (\overleftarrow{R_*}{}^{\backprime}x) \upharpoonleft R_{po} \subset J \quad (1)$

$\vdash . (1) . *96·102 . \supset \vdash . \text{Prop}$

***96·11.** $\vdash . (\alpha \upharpoonleft R)_{po} \subset \alpha \upharpoonleft R_{po}$

Dem.

$\vdash . *91·502 . *35·46 . \supset \vdash . \alpha \upharpoonleft R \subset \alpha \upharpoonleft R_{po} \quad (1)$

$\vdash . *35·1 . \supset$

$\vdash :. P \subset \alpha \upharpoonleft R_{po} . \supset : xPy . y(\alpha \upharpoonleft R) z . \supset . x \epsilon \alpha . xR_{po}y . yRz .$

$[*91·511.*35·1] \quad \supset . x(\alpha \upharpoonleft R_{po}) z .$

$[*34·1] \quad \supset : P \mid (\alpha \upharpoonleft R) \subset \alpha \upharpoonleft R_{po} \quad (2)$

$\vdash . (1) . (2) . *91·171 . \supset \vdash : P \epsilon \text{Pot}{}^{\backprime}(\alpha \upharpoonleft R) . \supset . P \subset \alpha \upharpoonleft R_{po} :$

$[*41·151] \quad \supset \vdash . (\alpha \upharpoonleft R)_{po} \subset \alpha \upharpoonleft R_{po} . \supset \vdash . \text{Prop}$

***96·111.** $\vdash : \breve{R}{}^{\backprime\backprime}\alpha \subset \alpha . \supset . (\alpha \upharpoonleft R)_{po} = \alpha \upharpoonleft R_{po}$

Dem.

$\vdash . *91·502 . \supset \vdash . \alpha \upharpoonleft R \subset (\alpha \upharpoonleft R)_{po} \quad (1)$

$\vdash . *90·22 . *91·54 . \supset \vdash :. \text{Hp} . \supset : P \epsilon \text{Pot}{}^{\backprime}R . x \epsilon \alpha . xPy . \supset . y \epsilon \alpha :$

$[*35·1.\text{Fact}] \supset : P \epsilon \text{Pot}{}^{\backprime}R . x(\alpha \upharpoonleft P) y . yRz . \supset . y(\alpha \upharpoonleft R) z :$

$[*91·511] \quad \supset : P \epsilon \text{Pot}{}^{\backprime}R . \alpha \upharpoonleft P \subset (\alpha \upharpoonleft R)_{po} . \supset . (\alpha \upharpoonleft P) \mid R \subset (\alpha \upharpoonleft R)_{po} \quad (2)$

$\vdash . (1) . (2) . *91·373 . \supset \vdash :. \text{Hp} . \supset : P \epsilon \text{Pot}{}^{\backprime}R . \supset . \alpha \upharpoonleft P \subset (\alpha \upharpoonleft R)_{po} :$

$[*41·52] \quad \supset : \alpha \upharpoonleft R_{po} \subset (\alpha \upharpoonleft R)_{po} :$

$[*96·11] \quad \supset : \alpha \upharpoonleft R_{po} = (\alpha \upharpoonleft R)_{po} :. \supset \vdash . \text{Prop}$

***96·112.** $\vdash : \alpha \subset \text{D}{}^{\backprime}R . \breve{R}{}^{\backprime\backprime}\alpha \subset \alpha . \supset . (\alpha \upharpoonleft R)_* = \alpha \upharpoonleft R_*$

Dem.

$\vdash . *35·62 . *37·4 . \supset \vdash : \text{Hp} . \supset . C{}^{\backprime}(\alpha \upharpoonleft R) = \alpha \cup \breve{R}{}^{\backprime\backprime}\alpha$

$[*22·62] \quad = \alpha .$

$[*50·5] \quad \supset . I \upharpoonright C{}^{\backprime}(\alpha \upharpoonleft R) = \alpha \upharpoonleft I \quad (1)$

$\vdash . *50·53 . \quad \supset \vdash . \alpha \upharpoonleft I \upharpoonright C{}^{\backprime}R = (\alpha \cap C{}^{\backprime}R) \upharpoonleft I \quad (2)$

$\vdash . (2) . *22·621 . \quad \supset \vdash : \text{Hp} . \supset . \alpha \upharpoonleft I \upharpoonright C{}^{\backprime}R = \alpha \upharpoonleft I \quad (3)$

$\vdash . *91·54 . \quad \supset \vdash : (\alpha \upharpoonleft R)_* = (\alpha \upharpoonleft R)_{po} \cup I \upharpoonright C{}^{\backprime}(\alpha \upharpoonleft R) \quad (4)$

$\vdash . *91·54 . *35·42 . \supset \vdash : \alpha \upharpoonleft R_* = \alpha \upharpoonleft R_{po} \cup \alpha \upharpoonleft I \upharpoonright C{}^{\backprime}R \quad (5)$

$\vdash . (1) . (3) . (4) . (5) . *96·111 . \supset \vdash . \text{Prop}$

***96·121.** $\vdash : R{}^{\backprime\backprime}\alpha \subset \alpha . \supset . (R \upharpoonright \alpha)_{po} = R_{po} \upharpoonright \alpha$ [Proof as in *96·111]

***96·122.** $\vdash : \alpha \subset \text{G}{}^{\backprime}R . R{}^{\backprime\backprime}\alpha \subset \alpha . \supset . (R \upharpoonright \alpha)_* = R_* \upharpoonright \alpha$ [Proof as in *96·112]

***96·13.** $\vdash . (\overleftarrow{R_*}{}^{\backprime}x) \upharpoonleft R_{po} = \{(\overleftarrow{R_*}{}^{\backprime}x) \upharpoonleft R\}_{po}$ [*96·111 . *90·163]

***96·131.** $\vdash : x \in \mathrm{D}‘R \,.\supset .\, (\overleftarrow{R_*}‘x) \upharpoonleft R_* = \{(\overleftarrow{R_*}‘x) \upharpoonleft R\}_*$ [*96·112 . *90·163]

***96·14.** $\vdash : x \in C‘R \,.\supset .\, \overleftarrow{R_*}‘x = \iota‘x \cup \overleftarrow{R_{\mathrm{po}}}‘x$ [*91·54 . *32·33]

***96·141.** $\vdash .\, C‘(\alpha \upharpoonleft R_*) = \breve{R}_*‘‘\alpha$

Dem.

$\vdash .\, *35·61 .\, *37·4 .\, *90·14 .\supset \vdash .\, C‘(\alpha \upharpoonleft R_*) = (\alpha \cap C‘R) \cup \breve{R}_*‘‘\alpha$

[*90·331] $= \breve{R}_*‘‘\alpha \,.\supset \vdash .\, \mathrm{Prop}$

***96·142.** $\vdash .\, C‘(\alpha \upharpoonleft R_{\mathrm{po}}) = (\alpha \cap \mathrm{D}‘R) \cup \breve{R}_{\mathrm{po}}‘‘\alpha$ [*35·61 . *37·4 . *91·504]

***96·143.** $\vdash .\, C‘(\alpha \upharpoonleft R_{\mathrm{po}}) = \breve{R}_*‘‘(\alpha \cap \mathrm{D}‘R)$

Dem.

$\vdash .\, *37·261 .\, *91·504 .\supset \vdash .\, \breve{R}_{\mathrm{po}}‘‘\alpha = \breve{R}_{\mathrm{po}}‘‘(\alpha \cap \mathrm{D}‘R)$ (1)

$\vdash .\, (1) .\, *91·546 .\, *96·142 .\supset \vdash .\, \mathrm{Prop}$

***96·144.** $\vdash : \alpha \cap \text{Ɑ}‘R \subset \breve{R}_*‘‘(\alpha \cap \mathrm{D}‘R) \,.\supset .\, C‘(\alpha \upharpoonleft R_{\mathrm{po}}) = \breve{R}_*‘‘\alpha$

Dem.

$\vdash .\, *22·62 .\supset \vdash : \mathrm{Hp} .\supset .\, \breve{R}_*‘‘(\alpha \cap \mathrm{D}‘R) = (\alpha \cap \text{Ɑ}‘R) \cup \breve{R}_*‘‘(\alpha \cap \mathrm{D}‘R)$

[*91·546] $= (\alpha \cap \text{Ɑ}‘R) \cup (\alpha \cap \mathrm{D}‘R) \cup \breve{R}_{\mathrm{po}}‘‘(\alpha \cap \mathrm{D}‘R)$

[*37·261.*91·504] $= (\alpha \cap C‘R) \cup \breve{R}_{\mathrm{po}}‘‘\alpha$

[*91·544] $= \breve{R}_*‘‘\alpha$ (1)

$\vdash .\, (1) .\, *96·143 .\supset \vdash .\, \mathrm{Prop}$

***96·15.** $\vdash .\, \mathrm{D}‘\{(\overleftarrow{R_*}‘x) \upharpoonleft R\} = \overleftarrow{R_*}‘x \cap \mathrm{D}‘R \,.\, \text{Ɑ}‘\{(\overleftarrow{R_*}‘x) \upharpoonleft R\} = \overleftarrow{R_{\mathrm{po}}}‘x$

Dem.

$\vdash .\, *35·61 .\supset \vdash .\, \mathrm{D}‘\{(\overleftarrow{R_*}‘x) \upharpoonleft R\} = \overleftarrow{R_*}‘x \cap \mathrm{D}‘R$ (1)

$\vdash .\, *37·4 .\ \supset \vdash .\, \text{Ɑ}‘\{(\overleftarrow{R_*}‘x) \upharpoonleft R\} = \breve{R}‘‘\overleftarrow{R_*}‘x$

[*91·74] $= \overleftarrow{R_{\mathrm{po}}}‘x$ (2)

$\vdash .\, (1) .\, (2) .\supset \vdash .\, \mathrm{Prop}$

***96·151.** $\vdash : x \in \mathrm{D}‘R \,.\supset .\, C‘\{(\overleftarrow{R_*}‘x) \upharpoonleft R\} = \overleftarrow{R_*}‘x$

Dem.

$\vdash .\, *96·14 .\supset \vdash : \mathrm{Hp} .\supset .\, \overleftarrow{R_*}‘x \cap \mathrm{D}‘R = \iota‘x \cup (\overleftarrow{R_{\mathrm{po}}}‘x \cap \mathrm{D}‘R) .$

[*22·63] $\supset .\, (\overleftarrow{R_*}‘x \cap \mathrm{D}‘R) \cup \overleftarrow{R_{\mathrm{po}}}‘x = \iota‘x \cup \overleftarrow{R_{\mathrm{po}}}‘x$

[*96·14] $= \overleftarrow{R_*}‘x$ (1)

$\vdash .\, (1) .\, *96·15 .\supset \vdash .\, \mathrm{Prop}$

***96·152.** $\vdash .\, \breve{R}_*‘‘\overleftarrow{R_*}‘x = \overleftarrow{R_*}‘x$ [*90·17]

***96·153.** $\vdash .\, \breve{R}_*‘‘\overleftarrow{R_{\mathrm{po}}}‘x = \breve{R}_{\mathrm{po}}‘‘\overleftarrow{R_*}‘x = \overleftarrow{R_{\mathrm{po}}}‘x$ [*91·574]

***96·154.** $\vdash .\, C‘\{(\overleftarrow{R_*}‘x) \upharpoonleft R_*\} = \overleftarrow{R_*}‘x$ [*96·141·152]

***96·155.** $\vdash . \text{D}‘\{(\overleftarrow{R_*}‘x) \rceil R_{po}\} = \overleftarrow{R_*}‘x \cap \text{D}‘R \,.\, \text{Ɑ}‘\{(\overleftarrow{R_*}‘x) \rceil R_{po}\} = \overleftarrow{R_{po}}‘x$

Dem.

$\vdash . *35·61 . *91·504 . \supset \vdash . \text{D}‘\{(\overleftarrow{R_*}‘x) \rceil R_{po}\} = \overleftarrow{R_*}‘x \cap \text{D}‘R$ (1)

$\vdash . *37·4 . \qquad \supset \vdash . \text{Ɑ}‘\{(\overleftarrow{R_*}‘x) \rceil R_{po}\} = \breve{R}_{po}“\overleftarrow{R_*}‘x$

$[*96·153] \qquad = \overleftarrow{R_{po}}‘x$ (2)

$\vdash . (1) . (2) . \supset \vdash .$ Prop

***96·156.** $\vdash . C‘\{(\overleftarrow{R_*}‘x) \rceil R_{po}\} = (\iota‘x \cap \text{D}‘R) \cup \overleftarrow{R_{po}}‘x$

Dem.

$\vdash . *96·155 . \supset$

$\vdash . C‘\{(\overleftarrow{R_*}‘x) \rceil R_{po}\} = (\overleftarrow{R_*}‘x \cap \text{D}‘R) \cup \overleftarrow{R_{po}}‘x$

$[*91·54] \qquad = (\iota‘x \cap C‘R \cap \text{D}‘R) \cup (\overleftarrow{R_{po}}‘x \cap \text{D}‘R) \cup \overleftarrow{R_{po}}‘x$

$[*22·62 . *33·161] \qquad = (\iota‘x \cap \text{D}‘R) \cup \overleftarrow{R_{po}}‘x . \supset \vdash .$ Prop

***96·157.** $\vdash : x \in \text{D}‘R . \supset . C‘\{(\overleftarrow{R_*}‘x) \rceil R_{po}\} = \overleftarrow{R_*}‘x \quad [*96·156·14]$

***96·158.** $\vdash : x \sim\in \text{D}‘R . \supset . (\overleftarrow{R_*}‘x) \rceil R_{po} = \dot{\Lambda}$

Dem.

$\vdash . *91·504 . \supset \vdash : \text{Hp} . \supset . x \sim\in \text{D}‘R_{po} .$

$[*33·4] \qquad \supset . \overleftarrow{R_{po}}‘x = \Lambda$ (1)

$\vdash . (1) . *96·155 . \supset \vdash .$ Prop

***96·159.** $\vdash : \dot{\exists} ! (\overleftarrow{R_*}‘x) \rceil R_{po} . \supset . C‘\{(\overleftarrow{R_*}‘x) \rceil R_{po}\} = \overleftarrow{R_*}‘x \quad [*96·157·158]$

***96·16.** $\vdash . (\overleftarrow{R_*}‘x) \rceil R = R \sqsubset \overleftarrow{R_*}‘x$

Dem.

$\vdash . *35·1 . \supset \vdash : y \{(\overleftarrow{R_*}‘x) \rceil R\} z . \equiv . y \in \overleftarrow{R_*}‘x . yRz .$

$[*90·16 . *4·71] \qquad \equiv . y \in \overleftarrow{R_*}‘x . yRz . z \in \overleftarrow{R_*}‘x .$

$[*36·13] \qquad \equiv . y (R \sqsubset \overleftarrow{R_*}‘x) z : \supset \vdash .$ Prop

***96·2.** $\vdash : R \in 1 \rightarrow \text{Cls} . \supset . (\overleftarrow{R_*}‘x) \rceil R = R \upharpoonright \overleftarrow{R_{po}}‘x$

Dem.

$\vdash . *72·55 . \supset \vdash : \text{Hp} . \supset . (\overleftarrow{R_*}‘x) \rceil R = R \upharpoonright \breve{R}“\overleftarrow{R_*}‘x$

$[*91·74] \qquad = R \upharpoonright \overleftarrow{R_{po}}‘x : \supset \vdash .$ Prop

***96·21.** $\vdash : R \in 1 \rightarrow \text{Cls} . xBR . \supset . (\overleftarrow{R_*}‘x) \rceil R = R \upharpoonright \overleftarrow{R_*}‘x$

Dem.

$\vdash . *96·14 . \qquad \supset \vdash : \text{Hp} . \supset . R \upharpoonright \overleftarrow{R_*}‘x = R \upharpoonright \iota‘x \cup R \upharpoonright \overleftarrow{R_{po}}‘x$ (1)

$\vdash . *35·64 . *93·1 . \supset \vdash : \text{Hp} . \supset . \text{Ɑ}‘(R \upharpoonright \iota‘x) = \Lambda .$

$[*33·241] \qquad \supset . R \upharpoonright \iota‘x = \dot{\Lambda}$ (2)

$\vdash . (1) . (2) . \qquad \supset \vdash : \text{Hp} . \supset . R \upharpoonright \overleftarrow{R_*}‘x = R \upharpoonright \overleftarrow{R_{po}}‘x$

$[*96·2] \qquad = (\overleftarrow{R_*}‘x) \rceil R : \supset \vdash .$ Prop

***96·22.** $\vdash : R \in 1 \to \mathrm{Cls} \,.\, \sim(xRx) \,.\, \supset \,.\, (\overleftarrow{R}_{*}\text{‘}x) \upharpoonright R \subset J$

Dem.

$\vdash \,.\, *31\cdot11 \,.\quad \supset \vdash : xQy \,.\, yRy \,.\, \supset \,.\, xQy \,.\, yRy \,.\, y\breve{Q}x \,.$

[*10·24.*34·1] $\quad \supset \,.\, xQ \,|\, R \,|\, \breve{Q}x \qquad (1)$

$\vdash \,.\, (1) \,.\, *92\cdot132 \,.\, \supset \vdash : R \in 1 \to \mathrm{Cls} \,.\, \supset : Q \in \mathrm{Potid}\text{‘}R \,.\, xQy \,.\, yRy \,.\, \supset \,.\, xRx :$

[*10·11·21·23·35.*91·55] $\quad \supset : xR_{*}y \,.\, yRy \,.\, \supset \,.\, xRx :$

[Transp] $\quad \supset : \sim(xRx) \,.\, xR_{*}y \,.\, \supset \,.\, \sim(yRy) :$

[*13·196] $\quad \supset : \sim(xRx) \,.\, xR_{*}y \,.\, yRz \,.\, \supset \,.\, y \neq z :$

[*32·181.*35·1] $\quad \supset : \sim(xRx) \,.\, \supset \,.\, (\overleftarrow{R_{*}}\text{‘}x) \upharpoonright R \subset J :. \supset \vdash \,.\, \mathrm{Prop}$

***96·23.** $\vdash : R \in 1 \to \mathrm{Cls} \,.\, xBR \,.\, \supset \,.\, I_R\text{‘}x = \Lambda \,.\, (\overleftarrow{R_{*}}\text{‘}x) \upharpoonright R_{\mathrm{po}} \subset J$

Dem.

$\vdash \,.\, *31\cdot11 \,.\, \supset \vdash : xQy \,.\, yTy \,.\, \supset \,.\, xQy \,.\, yTy \,.\, y\breve{Q}x \,.$

[*34·1] $\quad \supset \,.\, xQ \,|\, T \,|\, \breve{Q}x \qquad (1)$

$\vdash \,.\, (1) \,.\, *92\cdot132 \,.\, \supset$

$\vdash :. R \in 1 \to \mathrm{Cls} \,.\, \supset : Q, T \in \mathrm{Potid}\text{‘}R \,.\, xQy \,.\, yTy \,.\, \supset \,.\, xTx :$

[*91·271] $\quad \supset : Q \in \mathrm{Potid}\text{‘}R \,.\, T \in \mathrm{Pot}\text{‘}R \,.\, xQy \,.\, yTy \,.\, \supset \,.\, x \in \mathrm{G}\text{‘}R$

[*11·11·3·35·54.*91·55.(*91·05)] $\supset : y \in \overleftarrow{R_{*}}\text{‘}x \,.\, yR_{\mathrm{po}}y \,.\, \supset \,.\, x \in \mathrm{G}\text{‘}R :$

[Transp.*93·1] $\quad \supset : xBR \,.\, \supset \,.\, \sim(y \in \overleftarrow{R_{*}}\text{‘}x \,.\, yR_{\mathrm{po}}y) :$

[*96·1.*10·11·21] $\quad \supset : xBR \,.\, \supset \,.\, I_R\text{‘}x = \Lambda \qquad (2)$

$\vdash \,.\, (2) \,.\, *96\cdot104 \,.\, \supset \vdash \,.\, \mathrm{Prop}$

***96·24.** $\vdash : R \in 1 \to \mathrm{Cls} \,.\, C\text{‘}R = \breve{R}_{*}\text{‘‘}\overrightarrow{B}\text{‘}R \,.\, \supset \,.\, R_{\mathrm{po}} \subset J$

Dem.

$\vdash \,.\, *37\cdot105 \,.\, \supset \vdash :. \mathrm{Hp} \,.\, \supset : y \in C\text{‘}R \,.\, \supset \,.\, (\exists x) \,.\, x \in \overrightarrow{B}\text{‘}R \,.\, xR_{*}y :$

[*91·504] $\quad \supset : yR_{\mathrm{po}}z \,.\, \supset \,.\, (\exists x) \,.\, x \in \overrightarrow{B}\text{‘}R \,.\, xR_{*}y :$

[*4·7.*32·18·181] $\quad \supset : yR_{\mathrm{po}}z \,.\, \supset \,.\, (\exists x) \,.\, xBR \,.\, y \in \overleftarrow{R_{*}}\text{‘}x \,.\, yR_{\mathrm{po}}z \,.$

[*96·23] $\quad \supset \,.\, yJz :. \supset \vdash \,.\, \mathrm{Prop}$

***96·25.** $\vdash :. R \in 1 \to \mathrm{Cls} \,.\, xBR \,.\, xR_{*}y : yR_{*}z \,.\, \mathsf{v} \,.\, zR_{*}y : \supset \,.\, xR_{*}z$

Dem.

$\vdash \,.\, *90\cdot17 \,.\, \supset \vdash : xR_{*}y \,.\, yR_{*}z \,.\, \supset \,.\, xR_{*}z \qquad (1)$

$\vdash \,.\, *92\cdot31 \,.\, *91\cdot75 \,.\, \supset$

$\vdash :. \mathrm{Hp} \,.\, \supset : xR_{*}y \,.\, zR_{*}y \,.\, \supset : xR_{*}z \,.\, \mathsf{v} \,.\, zR_{\mathrm{po}}x \qquad (2)$

$\vdash \,.\, *91\cdot504 \,.\, *93\cdot1 \,.\, \supset \vdash : xBR \,.\, \supset \,.\, \sim(zR_{\mathrm{po}}x) \qquad (3)$

$\vdash \,.\, (2) \,.\, (3) \,.\, \supset \vdash :. \mathrm{Hp} \,.\, \supset : xR_{*}y \,.\, zR_{*}y \,.\, \supset \,.\, xR_{*}z \qquad (4)$

$\vdash \,.\, (1) \,.\, (4) \,.\, \supset \vdash \,.\, \mathrm{Prop}$

The following propositions lead up to $*96{\cdot}32$, *i.e.*

$$\vdash : R \in 1 \to 1 \,.\, xR_*y \,.\, \supset .\, \overrightarrow{R_*}\text{‘}x \cup \overleftarrow{R_*}\text{‘}x = \overrightarrow{R_*}\text{‘}y \cup \overleftarrow{R_*}\text{‘}y,$$

which is a proposition used in the following number ($*97$).

$*96{\cdot}3{\cdot}301{\cdot}302{\cdot}303$ are also frequently used elsewhere.

$*96{\cdot}3$. $\vdash : xR_*y \,.\, \supset .\, \overleftarrow{R_*}\text{‘}y \subset \overleftarrow{R_*}\text{‘}x$ [$*90{\cdot}17$]

$*96{\cdot}301$. $\vdash : xR_*y \,.\, \supset .\, \overrightarrow{R_*}\text{‘}x \subset \overrightarrow{R_*}\text{‘}y$ [$*90{\cdot}17$]

$*96{\cdot}302$. $\vdash :. R \in \mathrm{Cls} \to 1 \,.\, xR_*y \,.\, xR_*z \,.\, \supset : yR_*z \,.\, \vee \,.\, zR_*y$ [$*92{\cdot}311$]

$*96{\cdot}303$. $\vdash :. R \in \mathrm{Cls} \to 1 \,.\, xR_*y \,.\, xR_*z \,.\, y \neq z \,.\, \supset : yR_{\mathrm{po}}z \,.\, \vee \,.\, zR_{\mathrm{po}}y$
[$*96{\cdot}302 \,.\, *91{\cdot}542$]

$*96{\cdot}31$. $\vdash : R \in \mathrm{Cls} \to 1 \,.\, xR_*y \,.\, \supset .\, \overleftarrow{R_*}\text{‘}x \subset \overrightarrow{R_*}\text{‘}y \cup \overleftarrow{R_*}\text{‘}y$ [$*96{\cdot}302$]

$*96{\cdot}311$. $\vdash : R \in 1 \to \mathrm{Cls} \,.\, xR_*y \,.\, \supset .\, \overrightarrow{R_*}\text{‘}y \subset \overrightarrow{R_*}\text{‘}x \cup \overleftarrow{R_*}\text{‘}x$ [$*92{\cdot}31$]

$*96{\cdot}32$. $\vdash : R \in 1 \to 1 \,.\, xR_*y \,.\, \supset .\, \overrightarrow{R_*}\text{‘}x \cup \overleftarrow{R_*}\text{‘}x = \overrightarrow{R_*}\text{‘}y \cup \overleftarrow{R_*}\text{‘}y$

Dem.

$\vdash .\, *96{\cdot}301{\cdot}31 \,.\, \supset \vdash : R \in \mathrm{Cls} \to 1 \,.\, xR_*y \,.\, \supset .\, \overrightarrow{R_*}\text{‘}x \cup \overleftarrow{R_*}\text{‘}x \subset \overrightarrow{R_*}\text{‘}y \cup \overleftarrow{R_*}\text{‘}y$ (1)

$\vdash .\, *96{\cdot}3{\cdot}311 \,.\, \supset \vdash : R \in 1 \to \mathrm{Cls} \,.\, xR_*y \,.\, \supset .\, \overrightarrow{R_*}\text{‘}y \cup \overleftarrow{R_*}\text{‘}y \subset \overrightarrow{R_*}\text{‘}x \cup \overleftarrow{R_*}\text{‘}x$ (2)

$\vdash .\, (1) \,.\, (2) \,.\, \supset \vdash .\, \mathrm{Prop}$

$*96{\cdot}33$. $\vdash : R \in \mathrm{Cls} \to 1 \,.\, xRx \,.\, \supset .\, \overleftarrow{R_*}\text{‘}x = \iota\text{‘}x$

Dem.

$\vdash .\, *71{\cdot}171 \,.\, \supset \vdash :. \mathrm{Hp} \,.\, \supset : z = x \,.\, zRw \,.\, \supset_{z,w} .\, w = x$ (1)

$\vdash .\, (1) \,.\, *13{\cdot}15 \,.\, *90{\cdot}112 \dfrac{z = x}{\phi z} \,.\, \supset \vdash : xR_*y \,.\, \supset .\, y = x$ (2)

$\vdash .\, *90{\cdot}12 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset .\, xR_*x$ (3)

$\vdash .\, (2) \,.\, (3) \,.\, \supset \vdash :. \mathrm{Hp} \,.\, \supset : xR_*y \,.\, \equiv .\, y = x :. \supset \vdash .\, \mathrm{Prop}$

$*96{\cdot}331$. $\vdash : R \in \mathrm{Cls} \to 1 \,.\, xRy \,.\, yRx \,.\, \supset .\, \overleftarrow{R_*}\text{‘}x = \iota\text{‘}x \cup \iota\text{‘}y$

Dem.

$\vdash .\, *90{\cdot}151{\cdot}162 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset .\, \iota\text{‘}x \cup \iota\text{‘}y \subset \overleftarrow{R_*}\text{‘}x$ (1)

$\vdash .\, *71{\cdot}171 \,.\, \supset \vdash :. \mathrm{Hp} \,.\, \supset : z = x \,.\, zRw \,.\, \supset_{z,w} .\, w = y \,.$

[$*51{\cdot}232$] $\supset_{z,w} .\, w \in \iota\text{‘}x \cup \iota\text{‘}y$ (2)

$\vdash .\, *71{\cdot}171 \,.\, \supset \vdash :. \mathrm{Hp} \,.\, \supset : z = y \,.\, zRw \,.\, \supset_{z,w} .\, w = x \,.$

[$*51{\cdot}232$] $\supset_{z,w} .\, w \in \iota\text{‘}x \cup \iota\text{‘}y$ (3)

$\vdash .\, (2) \,.\, (3) \,.\, \supset \vdash :. \mathrm{Hp} \,.\, \supset : z \in \iota\text{‘}x \cup \iota\text{‘}y \,.\, zRw \,.\, \supset_{z,w} .\, w \in \iota\text{‘}x \cup \iota\text{‘}y$ (4)

$\vdash .\, *51{\cdot}16 \,.\, \supset \vdash .\, x \in \iota\text{‘}x \cup \iota\text{‘}y$ (5)

$\vdash .\, (4) \,.\, (5) \,.\, *90{\cdot}112 \,.\, \supset \vdash :. \mathrm{Hp} \,.\, \supset : xR_*z \,.\, \supset .\, z \in \iota\text{‘}x \cup \iota\text{‘}y$ (6)

$\vdash .\, (1) \,.\, (6) \,.\, \supset \vdash .\, \mathrm{Prop}$

This process of proof can obviously be extended to any finite cycle of terms.

***96·34.** $\vdash : R \epsilon \text{Cls} \to 1 . \supset . \breve{R}_{\text{po}}``\hat{z}(zR_{\text{po}}z) \subset \hat{z}(zR_{\text{po}}z)$

Dem.

$\vdash . *31\cdot11 . *34\cdot1 . \supset \vdash : zR_{\text{po}}z . zRw . \supset . w\breve{R} \,|\, R_{\text{po}} \,|\, Rw$ (1)

$\vdash . (1) . *92\cdot113 . \supset \vdash :. \text{Hp} . \supset : zR_{\text{po}}z . zRw . \supset . wR_{\text{po}}w :$

[*20·3] $\supset : z \epsilon \hat{z}(zR_{\text{po}}z) . zRw . \supset . w \epsilon \hat{z}(zR_{\text{po}}z) :$

[*37·171] $\supset : \breve{R}``\hat{z}(zR_{\text{po}}z) \subset \hat{z}(zR_{\text{po}}z) :$

[*91·71·53] $\supset : \breve{R}_{\text{po}}``\hat{z}(zR_{\text{po}}z) \subset \hat{z}(zR_{\text{po}}z) :. \supset \vdash . \text{Prop}$

***96·341.** $\vdash : R \epsilon \text{Cls} \to 1 . \supset . \breve{R}_{\text{po}}``I_R`x \subset I_R`x$

Dem.

$\vdash . *37\cdot21 . (*96\cdot01) . \supset \vdash . \breve{R}_{\text{po}}``I_R`x \subset \breve{R}_{\text{po}}``\overleftarrow{R_*}`x \cap \breve{R}_{\text{po}}``\hat{z}(zR_{\text{po}}z)$

[*90·163.*91·602] $\subset \overleftarrow{R_*}`x \cap \breve{R}_{\text{po}}``\hat{z}(zR_{\text{po}}z)$ (1)

$\vdash . (1) . *96\cdot34 . \supset \vdash : \text{Hp} . \supset . \breve{R}_{\text{po}}``I_R`x \subset \overleftarrow{R_*}`x \cap \hat{z}(zR_{\text{po}}z)$

[(*96·01)] $\subset I_R`x : \supset \vdash . \text{Prop}$

***96·342.** $\vdash : R \epsilon \text{Cls} \to 1 . \supset . \breve{R}_*``I_R`x \subset I_R`x$ [*96·341 . *91·71]

***96·35.** $\vdash :. R \epsilon \text{Cls} \to 1 . \supset : \sim(wR_{\text{po}}w) . zR_{\text{po}}w . \supset . \sim(zR_{\text{po}}z)$
[*96·34 . Transp]

***96·351.** $\vdash : R \epsilon \text{Cls} \to 1 . \supset . R_{\text{po}}``J_R`x \cap \overleftarrow{R_*}`x \subset J_R`x$

Dem.

$\vdash . *96\cdot35 . \text{Fact} . *96\cdot101 . \supset$

$\vdash :. \text{Hp} . \supset : w \epsilon J_R`x . zR_{\text{po}}w . z \epsilon \overleftarrow{R_*}`x . \supset . z \epsilon J_R`x :. \supset \vdash . \text{Prop}$

***96·352.** $\vdash : R \epsilon \text{Cls} \to 1 . \supset . R_*``J_R`x \cap \overleftarrow{R_*}`x \subset J_R`x$ [*91·543 . *96·351]

The following propositions are lemmas for *96·45·47.

***96·4.** $\vdash : R \epsilon \text{Cls} \to 1 . S, T \epsilon \text{Pot}`R . ySy . yTz . \supset . zSz$

Dem.

$\vdash . *31\cdot11 . \supset \vdash : \text{Hp} . \supset . z\breve{T} \,|\, S \,|\, Tz .$

[*92·133] $\supset . zSz : \supset \vdash . \text{Prop}$

***96·401.** $\vdash : R \epsilon \text{Cls} \to 1 . S, T \epsilon \text{Pot}`R . ySy . yTz . yRw . zRw . \supset . wSw . wTw$

Dem.

$\vdash . *31\cdot11 . \supset \vdash : \text{Hp} . \supset . w\breve{R}z . z\breve{T}y . ySy . yTz . zRw .$

[*34·1·2] $\supset . w \{\text{Cnv}`(T \,|\, R) \,|\, S \,|\, (T \,|\, R)\} w$ (1)

$\vdash . *91\cdot282 . \supset \vdash : \text{Hp} . \supset . T \,|\, R \,\dot{\epsilon}\, \text{Pot}`R$ (2)

$\vdash . (1) . (2) . *92\cdot133 . \supset \vdash : \text{Hp} . \supset . wSw$ (3)

$\vdash . *31\cdot11 . \supset \vdash : \text{Hp} . \supset . w\breve{R}y . yTz . zRw .$

[*34·1] $\supset . w\breve{R} \,|\, T \,|\, Rw .$

[*91·351.*92·133] $\supset . wTw$ (4)

$\vdash . (3) . (4) . \supset \vdash . \text{Prop}$

∗96·402. $\vdash : R \in \text{Cls} \to 1 \,.\, T \in \text{Pot}`R \,.\, yRy \,.\, yTz \,.\, yRw \,.\, zRw \,.\supset .\, y = w \,.\, y = z$

Dem.

$\vdash .\, ∗71{\cdot}171 \,. \quad \supset \vdash : \text{Hp} \,.\supset .\, y = w \quad (1)$

$\vdash .\, ∗96{\cdot}4 \,.\, ∗91{\cdot}351 \,.\supset \vdash : \text{Hp} \,.\supset .\, zRz \,.$

$[∗71{\cdot}171] \quad \supset .\, z = w \quad (2)$

$\vdash .\, (1) \,.\, (2) \,.\supset \vdash .\, \text{Prop}$

∗96·403. $\vdash : R \in \text{Cls} \to 1 \,.\, S, T \in \text{Pot}`R \,.\, yS \,|\, Ry \,.\, yTz \,.\, yRw \,.\, zRw \,.\supset .$

$wSy \,.\, wSz \,.\, y = z$

Dem.

$\vdash .\, ∗31{\cdot}11 \,. \quad \supset \vdash : \text{Hp} \,.\supset .\, w\breve{R} \,|\, S \,|\, Ry \,.$

$[∗92{\cdot}133] \quad \supset .\, wSy \quad (1)$

$\vdash .\, ∗96{\cdot}4 \,.\, ∗91{\cdot}343 \,. \quad \supset \vdash : \text{Hp} \,.\supset .\, zS \,|\, Rz \,.$

$[∗31{\cdot}11] \quad \supset .\, w\breve{R} \,|\, S \,|\, Rz \,.$

$[∗92{\cdot}133] \quad \supset .\, wSz \quad (2)$

$\vdash .\, (1) \,.\, (2) \,.\, ∗92{\cdot}101 \,.\, ∗71{\cdot}171 \,.\supset \vdash : \text{Hp} \,.\supset .\, y = z \quad (3)$

$\vdash .\, (1) \,.\, (2) \,.\, (3) \,.\supset \vdash .\, \text{Prop}$

∗96·41. $\vdash : R \in \text{Cls} \to 1 \,.\, S, T \in \text{Pot}`R \,.\, ySy \,.\, yTz \,.\, yRw \,.\, zRw \,.\supset .\, y = z$

Dem.

$\vdash .\, ∗91{\cdot}264{\cdot}304 \,.\supset \vdash .\, \text{Pot}`R = \iota`R \cup |R``\text{Pot}`R \,.$

$[∗51{\cdot}236] \quad \supset \vdash :.\, S \in \text{Pot}`R \,.\equiv : S = R \,.\vee .\, (\exists S') \,.\, S' \in \text{Pot}`R \,.\, S = S' \,|\, R \quad (1)$

$\vdash .\, ∗96{\cdot}402 \,.\supset$

$\vdash :.\, S = R \,.\supset : R \in \text{Cls} \to 1 \,.\, T \in \text{Pot}`R \,.\, ySy \,.\, yTz \,.\, yRw \,.\, zRw \,.\supset .\, y = z \quad (2)$

$\vdash .\, ∗96{\cdot}403 \,.\supset$

$\vdash :.\, (\exists S') \,.\, S' \in \text{Pot}`R \,.\, S = S' \,|\, R \,.\supset :$

$R \in \text{Cls} \to 1 \,.\, T \in \text{Pot}`R \,.\, ySy \,.\, yTz \,.\, yRw \,.\, zRw \,.\supset .\, y = z \quad (3)$

$\vdash .\, (1) \,.\, (2) \,.\, (3) \,.\supset \vdash :.\, S \in \text{Pot}`R \,.\supset :$

$R \in \text{Cls} \to 1 \,.\, T \in \text{Pot}`R \,.\, ySy \,.\, yTz \,.\, yRw \,.\, zRw \,.\supset .\, y = z :. \supset \vdash .\, \text{Prop}$

∗96·42. $\vdash : R \in \text{Cls} \to 1 \,.\, yRw \,.\, zRw \,.\, yR_{\text{po}}z \,.\supset .\, zR_{\text{po}}z$

Dem.

$\vdash .\, ∗31{\cdot}11 \,.\supset \vdash : \text{Hp} \,.\supset .\, w\breve{R}y \,.\, yR_{\text{po}}z \,.$

$[∗92{\cdot}111] \quad \supset .\, wR_*z \,.$

$[\text{Hp}.∗34{\cdot}1] \quad \supset .\, zR \,|\, R_*z \,.$

$[∗91{\cdot}52] \quad \supset .\, zR_{\text{po}}z : \supset \vdash .\, \text{Prop}$

∗96·421. $\vdash :.\, R \in \text{Cls} \to 1 \,.\, y, z \in \overleftarrow{R_*}`x \,.\, yRw \,.\, zRw \,.\, y \neq z \,.\supset : yR_{\text{po}}y \,.\vee .\, zR_{\text{po}}z$

Dem.

$\vdash .\, ∗96{\cdot}303 \,.\supset \vdash :.\, \text{Hp} \,.\supset : yR_{\text{po}}z \,.\vee .\, zR_{\text{po}}y \quad (1)$

$\vdash .\, ∗96{\cdot}42 \,. \quad \supset \vdash : \text{Hp} \,.\, yR_{\text{po}}z \,.\supset .\, zR_{\text{po}}z \quad (2)$

$\vdash .\, ∗96{\cdot}42 \,. \quad \supset \vdash : \text{Hp} \,.\, zR_{\text{po}}y \,.\supset .\, yR_{\text{po}}y \quad (3)$

$\vdash .\, (1) \,.\, (2) \,.\, (3) \,.\supset \vdash .\, \text{Prop}$

***96·431.** $\vdash : R \in \text{Cls} \rightarrow 1 \,.\, y \in J_R{}^{\backprime}x \,.\, z \in I_R{}^{\backprime}x \,.\, \supset .\, y R_{\text{po}} z$

Dem.

$\vdash .\, *96{\cdot}102 \,.\, \supset \vdash :.\, \text{Hp} \,.\, \supset : y \neq z :$

$[*96{\cdot}303] \quad \supset : y R_{\text{po}} z \,.\, \mathbf{v} \,.\, z R_{\text{po}} y$ (1)

$\vdash .\, *96{\cdot}341 \,.\, \supset \vdash :.\, \text{Hp} \,.\, \supset : z R_{\text{po}} y \,.\, \supset .\, y \in I_R{}^{\backprime}x :$

$[\text{Transp}.*96{\cdot}102] \quad \supset : y \in J_R{}^{\backprime}x \,.\, \supset .\, \sim(z R_{\text{po}} y)$ (2)

$\vdash .\, (2) \,.\, \quad \supset \vdash : \text{Hp} \,.\, \supset .\, \sim(z R_{\text{po}} y)$ (3)

$\vdash .\, (1) \,.\, (3) \,.\, \supset \vdash .\, \text{Prop}$

***96·432.** $\vdash : R \in \text{Cls} \rightarrow 1 \,.\, y, z \in I_R{}^{\backprime}x \,.\, yRw \,.\, zRw \,.\, \supset .\, y = z$

Dem.

$\vdash .\, *96{\cdot}1 \,.\, \quad \supset \vdash : \text{Hp} \,.\, \supset .\, (\exists S, T) \,.\, S, T \in \text{Pot}{}^{\backprime}R \,.\, ySy \,.\, zTz$ (1)

$\vdash .\, *96{\cdot}303 \,.\, \supset \vdash :.\, \text{Hp} \,.\, \supset : y = z : \mathbf{v} : (\exists U) : U \in \text{Pot}{}^{\backprime}R : yUz \,.\, \mathbf{v} \,.\, zUy$ (2)

$\vdash .\, (1) \,.\, (2) \,.\, \supset \vdash :.\, \text{Hp} \,.\, \supset :$

$y = z : \mathbf{v} : (\exists S, T, U) : S, T, U \in \text{Pot}{}^{\backprime}R \,.\, ySy \,.\, zTz : yUz \,.\, \mathbf{v} \,.\, zUy$ (3)

$\vdash .\, *96{\cdot}41 \,.\, \quad \supset \vdash :.\, \text{Hp} \,.\, \supset : (\exists S, U) \,.\, S, U \in \text{Pot}{}^{\backprime}R \,.\, ySy \,.\, yUz \,.\, \supset .\, y = z$ (4)

$\vdash .\, *96{\cdot}41 \,.\, \quad \supset \vdash :.\, \text{Hp} \,.\, \supset : (\exists T, U) \,.\, T, U \in \text{Pot}{}^{\backprime}R \,.\, zTz \,.\, zUy \,.\, \supset .\, y = z$ (5)

$\vdash .\, (4) \,.\, (5) \,.\, \supset \vdash :: \text{Hp} \,.\, \supset :.$

$(\exists S, T, U) : S, T, U \in \text{Pot}{}^{\backprime}R \,.\, ySy \,.\, zTz : yUz \,.\, \mathbf{v} \,.\, zUy : \supset .\, y = z$ (6)

$\vdash .\, (3) \,.\, (6) \,.\, \supset \vdash .\, \text{Prop}$

***96·44.** $\vdash :.\, R \in \text{Cls} \rightarrow 1 \,.\, y, z \in \overleftarrow{R_*}{}^{\backprime}x \,.\, yRw \,.\, zRw \,.\, y \neq z \,.\, \supset : y \in I_R{}^{\backprime}x \,.\, \mathbf{v} \,.\, z \in I^{R}{}^{\backprime}x$
[*96·421·1]

***96·441.** $\vdash :.\, R \in \text{Cls} \rightarrow 1 \,.\, y, z \in \overleftarrow{R_*}{}^{\backprime}x \,.\, yRw \,.\, zRw \,.\, y \neq z \,.\, \supset :$

$w \in I_R{}^{\backprime}x : y \in J_R{}^{\backprime}x \,.\, z \in I_R{}^{\backprime}x \,.\, \mathbf{v} \,.\, y \in I_R{}^{\backprime}x \,.\, z \in J_R{}^{\backprime}x$

Dem.

$\vdash .\, *96{\cdot}432 \,.\, \text{Transp} \,.\, (*96{\cdot}02) \,.\, \supset$

$\vdash :.\, \text{Hp} \,.\, \supset : z \in I_R{}^{\backprime}x \,.\, \supset .\, y \in J_R{}^{\backprime}x : y \in I_R{}^{\backprime}x \,.\, \supset .\, z \in J_R{}^{\backprime}x$ (1)

$\vdash .\, (1) \,.\, *96{\cdot}44 \,.\, \supset \vdash :.\, \text{Hp} \,.\, \supset : y \in J_R{}^{\backprime}x \,.\, z \in I_R{}^{\backprime}x \,.\, \mathbf{v} \,.\, y \in I_R{}^{\backprime}x \,.\, z \in J_R{}^{\backprime}x$ (2)

$\vdash .\, *91{\cdot}502 \,.\, *96{\cdot}341 \,.\, \supset$

$\vdash :.\, \text{Hp} \,.\, \supset : z \in I_R{}^{\backprime}x \,.\, \supset .\, w \in I_R{}^{\backprime}x : y \in I_R{}^{\backprime}x \,.\, \supset .\, w \in I_R{}^{\backprime}x :$

$[*96{\cdot}44] \supset : w \in I_R{}^{\backprime}x$ (3)

$\vdash .\, (2) \,.\, (3) \,.\, \supset \vdash .\, \text{Prop}$

***96·442.** $\vdash : R \in \text{Cls} \rightarrow 1 \,.\, y, z \in J_R{}^{\backprime}x \,.\, yRw \,.\, zRw \,.\, \supset .\, y = z$
[*96·44 . Transp]

The following proposition (*96·45) is important.

***96·45.** $\vdash : R \in \text{Cls} \rightarrow 1 \,.\, \supset .\, (J_R{}^{\backprime}x) \upharpoonright R, (I_R{}^{\backprime}x) \upharpoonright R \in 1 \rightarrow 1$
[*96·442·432]

***96·451.** $\vdash :.\, R \in \text{Cls} \rightarrow 1 : J_R{}^{\backprime}x = \Lambda \,.\, \mathbf{v} \,.\, I_R{}^{\backprime}x = \Lambda : \supset .\, (\overleftarrow{R_*}{}^{\backprime}x) \upharpoonright R \in 1 \rightarrow 1$
[*96·45·102]

***96·452.** $\vdash :. R \in \text{Cls} \to 1 . \supset : \exists ! J_R\text{‘}x . \equiv . x \in J_R\text{‘}x$

Dem.

$\vdash . *10\cdot24 . \qquad \supset \vdash : x \in J_R\text{‘}x . \supset . \exists ! J_R\text{‘}x \qquad (1)$

$\vdash . *96\cdot342 . \qquad \supset \vdash : \text{Hp} . x \in I_R\text{‘}x . \supset . \overleftarrow{R_*}\text{‘}x \subset I_R\text{‘}x .$

$[*96\cdot102] \qquad \supset . J_R\text{‘}x = \Lambda \qquad (2)$

$\vdash . *96\cdot101 . \qquad \supset \vdash : \exists ! J_R\text{‘}x . \supset . \exists ! \overleftarrow{R_*}\text{‘}x .$

$[*90\cdot13] \qquad \supset . xR_*x \qquad (3)$

$\vdash . (3) . (2) . \text{Transp} . \supset \vdash : \text{Hp} . \exists ! J_R\text{‘}x . \supset . x \in \overleftarrow{R_*}\text{‘}x - I_R\text{‘}x .$

$[(*96\cdot02)] \qquad \supset . x \in J_R\text{‘}x \qquad (4)$

$\vdash . (1) . (4) . \supset \vdash . \text{Prop}$

***96·453.** $\vdash :. R \in \text{Cls} \to 1 : xR_{\text{po}}x . \text{v} . (\overleftarrow{R_*}\text{‘}x) \upharpoonright R_{\text{po}} \mathrel{\dot\subset} J : \supset . (\overleftarrow{R_*}\text{‘}x) \upharpoonright R \in 1 \to 1$

Dem.

$\vdash . *96\cdot452 . \text{Transp} . \supset \vdash : R \in \text{Cls} \to 1 . xR_{\text{po}}x . \supset . J_R\text{‘}x = \Lambda \qquad (1)$

$\vdash . *96\cdot104 . \qquad \supset \vdash : R \in \text{Cls} \to 1 . (\overleftarrow{R_*}\text{‘}x) \upharpoonright R_{\text{po}} \mathrel{\dot\subset} J . \supset . I_R\text{‘}x = \Lambda \qquad (2)$

$\vdash . (1) . (2) . *96\cdot451 . \supset \vdash . \text{Prop}$

***96·46.** $\vdash : R \in \text{Cls} \to 1 . y, y' \in J_R\text{‘}x . \breve{R}\text{‘}y, \breve{R}\text{‘}y' \in I_R\text{‘}x . \supset . y = y'$

Dem.

$\vdash . *92\cdot111 . \supset$

$\vdash : R \in \text{Cls} \to 1 . y \in J_R\text{‘}x . \breve{R}\text{‘}y \in I_R\text{‘}x . yR_{\text{po}}y' . \supset . \breve{R}\text{‘}y \in I_R\text{‘}x . \breve{R}\text{‘}yR_*y' .$

$[*96\cdot342] \qquad \supset . y' \in I_R\text{‘}x \qquad (1)$

$\vdash . (1) . \text{Transp} . \supset \vdash : R \in \text{Cls} \to 1 . y, y' \in J_R\text{‘}x . \breve{R}\text{‘}y \in I_R\text{‘}x . \supset . \sim(yR_{\text{po}}y') \qquad (2)$

$\vdash . (2)\dfrac{y', y}{y, y'} . \supset \vdash : R \in \text{Cls} \to 1 . y, y' \in J_R\text{‘}x . \breve{R}\text{‘}y' \in I_R\text{‘}x . \supset . \sim(y'R_{\text{po}}y) \qquad (3)$

$\vdash . (2) . (3) . \supset \vdash : \text{Hp} . \supset . \sim(yR_{\text{po}}y') . \sim(y'R_{\text{po}}y) .$

$[*96\cdot303.\text{Transp}] \qquad \supset . y = y' : \supset \vdash . \text{Prop}$

***96·461.** $\vdash : R \in \text{Cls} \to 1 . y \in J_R\text{‘}x . \breve{R}\text{‘}y \in I_R\text{‘}x . \supset . y = \max_R\text{‘}J_R\text{‘}x$

Dem.

$\vdash . *14\cdot21 . \supset \vdash :. \text{Hp} . \supset : \text{E} ! \breve{R}\text{‘}y :$

$[*30\cdot13] \qquad \supset : \breve{R}\text{‘}y \sim\in J_R\text{‘}x . \equiv . \sim(\breve{R}\text{‘}y \in J_R\text{‘}x) .$

$[*71\cdot371.\text{Transp}] \qquad \equiv . y \sim\in R\text{‘‘}J_R\text{‘}x \qquad (1)$

$\vdash . (1) . *93\cdot115 . *96\cdot102 . \supset \vdash : \text{Hp} . \supset . y \max_R (J_R\text{‘}x) \qquad (2)$

$\vdash . *96\cdot431 . \supset \vdash :. \text{Hp} . y' \in J_R\text{‘}x . \supset : y'R_{\text{po}}\breve{R}\text{‘}y :$

$[*91\cdot504] \qquad \supset : y' \in \text{D}\text{‘}R :$

$[*71\cdot164] \qquad \supset : \text{E} ! \breve{R}\text{‘}y' :$

$[*30\cdot13] \qquad \supset : \breve{R}\text{‘}y' \sim\in J_R\text{‘}x . \equiv . \sim(\breve{R}\text{‘}y' \in J_R\text{‘}x) .$

$[*71\cdot371.\text{Transp}] \qquad \equiv . y' \sim\in R\text{‘‘}J_R\text{‘}x \qquad (3)$

$\vdash . (3) . *93\cdot115 . \supset \vdash :. \text{Hp} . \supset : y' \max_R (J_R{}^{\backprime}x) . \supset . y' \in J_R{}^{\backprime}x . \breve{R}{}^{\backprime}y' \sim\in J_R{}^{\backprime}x .$

[$*96\cdot102$] $\supset . y' \in J_R{}^{\backprime}x . \breve{R}{}^{\backprime}y' \in I_R{}^{\backprime}x .$

[$*96\cdot46$] $\supset . y = y'$ (4)

$\vdash . (2) . (4) . *30\cdot31 . \supset \vdash . \text{Prop}$

$*96\cdot462$. $\vdash : R \in \text{Cls} \to 1 . y \in J_R{}^{\backprime}x . z \in I_R{}^{\backprime}x . yRw . zRw . \supset .$

$$y = \max_R{}^{\backprime}J_R{}^{\backprime}x . w = \breve{R}{}^{\backprime}\max_R{}^{\backprime}J_R{}^{\backprime}x . z = \{(J_R{}^{\backprime}x) \upharpoonright R\}{}^{\backprime}\breve{R}{}^{\backprime}\max_R{}^{\backprime}J_R{}^{\backprime}x$$

Dem.

$\vdash . *96\cdot441\cdot102 . *71\cdot361 . \supset$

$\vdash : \text{Hp} . \supset . w \in I_R{}^{\backprime}x . w = \breve{R}{}^{\backprime}y .$

[$*96\cdot461$] $\supset . y = \max_R{}^{\backprime}J_R{}^{\backprime}x . w = \breve{R}{}^{\backprime}\max_R{}^{\backprime}J_R{}^{\backprime}x$ (1)

$\vdash . *96\cdot45 . \supset \vdash : \text{Hp} . \supset . z = \{(I_R{}^{\backprime}x) \upharpoonright R\}{}^{\backprime}w$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

The above proposition, since it exhibits y, z, w as functions of x and R, shows that there is at most one w in $\overleftarrow{R_*}{}^{\backprime}x$ having more than one immediate predecessor, and that this one has exactly one immediate predecessor in $J_R{}^{\backprime}x$ and one in $I_R{}^{\backprime}x$. (These results require $*96\cdot441$, in addition to $*96\cdot462$.) Thus we arrive at the following proposition:

$*96\cdot47$. $\vdash :. R \in \text{Cls} \to 1 . y, z \in \overleftarrow{R_*}{}^{\backprime}x . yRw . zRw . y \neq z . \supset : w = \breve{R}{}^{\backprime}\max_R{}^{\backprime}J_R{}^{\backprime}x :$

$$y = \max_R{}^{\backprime}J_R{}^{\backprime}x . z = \{(I_R{}^{\backprime}x) \upharpoonright R\}{}^{\backprime}\breve{R}{}^{\backprime}\max_R{}^{\backprime}J_R{}^{\backprime}x . \lor .$$

$$z = \max_R{}^{\backprime}J_R{}^{\backprime}x . y = \{(I_R{}^{\backprime}x) \upharpoonright R\}{}^{\backprime}\breve{R}{}^{\backprime}\max_R{}^{\backprime}J_R{}^{\backprime}x$$

[$*96\cdot441\cdot462$]

We still have to prove

$R \in \text{Cls} \to 1 . \exists ! J_R{}^{\backprime}x . \exists ! I_R{}^{\backprime}x . \supset . (\exists y, z, w) . y, z \in \overleftarrow{R_*}{}^{\backprime}x . yRw . zRw . y \neq z,$

or, what comes to the same thing because of $*96\cdot441$,

$R \in \text{Cls} \to 1 . \exists ! J_R{}^{\backprime}x . \exists ! I_R{}^{\backprime}x . \supset . (\exists y, z, w) . y \in J_R{}^{\backprime}x . z \in I_R{}^{\backprime}x . yRw . zRw.$

This is effected in the following propositions.

$*96\cdot472$. $\vdash : R \in \text{Cls} \to 1 . \exists ! J_R{}^{\backprime}x . \exists ! I_R{}^{\backprime}x . \supset . (\exists y) . y \in J_R{}^{\backprime}x . \breve{R}{}^{\backprime}y \in I_R{}^{\backprime}x$

Dem.

$\vdash . *90\cdot1 . \supset \vdash :. x \in J_R{}^{\backprime}x . \breve{R}{}^{\backprime\backprime}J_R{}^{\backprime}x \subset J_R{}^{\backprime}x . \supset : xR_*y . \supset . y \in J_R{}^{\backprime}x :$

[$*96\cdot104$] $\supset : I_R{}^{\backprime}x = \Lambda$ (1)

$\vdash . (1) . \text{Transp} . *96\cdot452 . \supset \vdash : \text{Hp} . \supset . \exists ! \breve{R}{}^{\backprime\backprime}J_R{}^{\backprime}x - J_R{}^{\backprime}x .$

[$*71\cdot401$] $\supset . (\exists y, z) . y \in J_R{}^{\backprime}x . z = \breve{R}{}^{\backprime}y . z \sim\in J_R{}^{\backprime}x .$

[$*13\cdot195$] $\supset . (\exists y) . y \in J_R{}^{\backprime}x . \breve{R}{}^{\backprime}y \sim\in J_R{}^{\backprime}x .$

[$*96\cdot102$] $\supset . (\exists y) . y \in J_R{}^{\backprime}x . \breve{R}{}^{\backprime}y \in I_R{}^{\backprime}x : \supset \vdash . \text{Prop}$

∗96·473. $\vdash : R \epsilon \mathrm{Cls} \to 1 \,.\, \exists ! J_R`x \,.\, \exists ! I_R`x \,.\, \supset .\, \mathrm{E} ! \max_R`J_R`x \,.\, \mathrm{E} ! \breve{R}`\max_R`J_R`x$ [∗96·461·472]

∗96·474. $\vdash : R \epsilon \mathrm{Cls} \to 1 \,.\, w = \breve{R}`\max_R`J_R`x \,.\, \supset .$

$$\mathrm{E} ! \{(I_R`x) \rceil R\}`w \,.\, \mathrm{E} ! \max_R`J_R`x \,.\, \{(J_R`x) \rceil R\}`w = \max_R`J_R`x$$

Dem.

$\vdash .\, ∗71·361 \,.\quad \supset \vdash : \mathrm{Hp} \,.\, \supset .\, (\max_R`J_R`x)\, Rw \,.$ (1)

[∗14·21] $\supset .\, \mathrm{E} ! \max_R`J\ `x \,.$ (2)

[∗93·11] $\supset .\, \max_R`J_R`x \,\epsilon\, J_R`x \,.$

[(1).∗96·45] $\supset .\, \{(J_R`x) \rceil R\}`w = \max_R`J_R`x$ (3)

$\vdash .\, (2) \,.\, ∗93·11 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset .\, \max_R`J_R`x \sim\epsilon R``J_R`x \,.$

[∗71·371.∗30·13] $\supset .\, \breve{R}`\max_R`J_R`x \sim\epsilon J_R`x \,.$

[Hp.∗96·102] $\supset .\, w \,\epsilon\, I_R`x \,.$

[∗96·1.∗91·52] $\supset .\, wR_{\mathrm{po}}w \,.\, wR_* \,|\, Rw \,.$

[∗34·1] $\supset .\, (\exists z) \,.\, wR_{\mathrm{po}}w \,.\, wR_*z \,.\, zRw \,.$

[∗96·342] $\supset .\, (\exists z) \,.\, z \,\epsilon\, I_R`x \,.\, zRw \,.$

[∗96·45] $\supset .\, \mathrm{E} ! \{(I_R`x) \rceil R\}`w$ (4)

$\vdash .\, (2) \,.\, (3) \,.\, (4) \,.\, \supset \vdash .\, \mathrm{Prop}$

∗96·475. $\vdash :.\, R \epsilon \mathrm{Cls} \to 1 \,.\, \supset : \mathrm{E} ! \breve{R}`\max_R`J_R`x \,.\, \equiv .\, \exists ! J_R`x \,.\, \exists ! I_R`x$ [∗96·473·474]

This proposition and ∗96·45·47 embody the main results of this number.

∗96·48. $\vdash :.\, R \epsilon \mathrm{Cls} \to 1 \,.\, S = (\overleftarrow{R_*}`x) \rceil R \,.\, w \,\epsilon\, \overleftarrow{R_{\mathrm{po}}}`x \,.\, \supset :$

$$\sim(w = \breve{R}`\max_R`J_R`x) \,.\, \equiv .\, \overrightarrow{S}`w \,\epsilon\, 1 : w = \breve{R}`\max_R`J_R`x \,.\, \equiv .\, \overrightarrow{S}`w \,\epsilon\, 2$$

Dem.

$\vdash .\, ∗96·15 \,.\, ∗33·41 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset .\, \exists ! \overrightarrow{S}`w$ (1)

$\vdash .\, ∗96·47 \,.\, \supset \vdash :.\, \mathrm{Hp} \,.\, \supset : (\exists y, z) \,.\, ySw \,.\, zSw \,.\, y \neq z \,.\, \supset .\, w = \breve{R}`\max_R`J_R`x :$

[(1).∗52·41] $\supset : \overrightarrow{S}`w \sim\epsilon 1 \,.\, \supset .\, w = \breve{R}`\max_R`J_R`x$ (2)

$\vdash .\, ∗96·474·102 \,.\quad \supset \vdash :.\, \mathrm{Hp} \,.\, \supset : w = \breve{R}`\max_R`J_R`x \,.\, \supset .\, \overrightarrow{S}`w \sim\epsilon 1$ (3)

$\vdash .\, (2) \,.\, (3) \,.\, \mathrm{Transp} \,.\quad \supset \vdash :.\, \mathrm{Hp} \,.\, \supset : \sim(w = \breve{R}`\max_R`J_R`x) \,.\, \equiv .\, \overrightarrow{S}`w \,\epsilon\, 1$ (4)

$\vdash .\, (2) \,.\, ∗52·4 \,.\, ∗54·101 \,.\, \supset \vdash :.\, \mathrm{Hp} \,.\, \supset : \overrightarrow{S}`w \,\epsilon\, 2 \,.\, \supset .\, w = \breve{R}`\max_R`J_R`x$ (5)

$\vdash .\, ∗96·474·102 \,.\quad \supset \vdash :.\, \mathrm{Hp} \,.\, \supset : w = \breve{R}`\max_R`J_R`x \,.\, \supset .$

$\mathrm{E} ! \{(J_R`x) \rceil R\}`w \,.\, \mathrm{E} ! \{(I_R`x) \rceil R\}`w \,.\, \iota`\{(J_R`x) \rceil R\}`w \cup \iota`\{(I_R`x) \rceil R\}`w = \overrightarrow{S}`w \,.$

[∗96·102.∗54·101] $\supset .\, \overrightarrow{S}`w \,\epsilon\, 2$ (6)

$\vdash .\, (5) \,.\, (6) \,.\, \supset \vdash :.\, \mathrm{Hp} \,.\, \supset : w = \breve{R}`\max_R`J_R`x \,.\, \equiv .\, \overrightarrow{S}`w \,\epsilon\, 2$ (7)

$\vdash .\, (4) \,.\, (7) \,.\, \supset \vdash .\, \mathrm{Prop}$

In the above proposition we write "$\sim(w = \breve{R}`\max_R`J_R`x)$" rather than "$w \neq \breve{R}`\max_R`J_R`x$," because the latter implies the existence of $\breve{R}`\max_R`J_R`x$.

***96·49.** $\vdash :: R \in \text{Cls} \to 1 . \supset :. (\overleftarrow{R_*}{}^{\backprime}x) \upharpoonright R \in 1 \to 1 . \equiv : I_R{}^{\backprime}x = \Lambda . \vee . J_R{}^{\backprime}x = \Lambda$

Dem.

$\vdash . *96{\cdot}48 . \text{Transp} . \supset \vdash :. \text{Hp} . S = (\overleftarrow{R_*}{}^{\backprime}x) \upharpoonright R . \supset :$

$w \in \overleftarrow{R_{\text{po}}}{}^{\backprime}x . w = \breve{R}{}^{\backprime}\text{max}_R{}^{\backprime}J_R{}^{\backprime}x . \equiv . w \in \overleftarrow{R_{\text{po}}}{}^{\backprime}x . \overrightarrow{S}{}^{\backprime}w \sim \in 1 :$

$[*96{\cdot}15 . *91{\cdot}52] \supset : w = \breve{R}{}^{\backprime}\text{max}_R{}^{\backprime}J_R{}^{\backprime}x . \equiv . w \in \text{Cl}{}^{\backprime}S . \overrightarrow{S}{}^{\backprime}w \sim \in 1 :$

$[*14{\cdot}204] \qquad \supset : \text{E} ! \breve{R}{}^{\backprime}\text{max}_R{}^{\backprime}J_R{}^{\backprime}x . \equiv . (\exists w) . w \in \text{Cl}{}^{\backprime}S . \overrightarrow{S}{}^{\backprime}w \sim \in 1 :$

$[*96{\cdot}475 . *71{\cdot}1] \supset : \exists ! J_R{}^{\backprime}x . \exists ! I_R{}^{\backprime}x . \equiv . S \sim \in 1 \to \text{Cls} .$

$[*71{\cdot}261{\cdot}103] \qquad \equiv . S \sim \in 1 \to 1 \qquad (1)$

$\vdash . (1) . \text{Transp} . \supset \vdash . \text{Prop}$

***96·491.** $\vdash :. R \in 1 \to 1 . \supset : I_R{}^{\backprime}x = \Lambda . \vee . J_R{}^{\backprime}x = \Lambda$

Dem.

$\vdash . *96{\cdot}49 . \qquad \supset \vdash :. \text{Hp} . x \in \text{D}{}^{\backprime}R . \supset : I_R{}^{\backprime}x = \Lambda . \vee . J_R{}^{\backprime}x = \Lambda \qquad (1)$

$\vdash . *91{\cdot}54{\cdot}504 . \supset \vdash : \text{Hp} . x \sim \in \text{D}{}^{\backprime}R . \supset . \overleftarrow{R_*}{}^{\backprime}x = \iota{}^{\backprime}x \cap C{}^{\backprime}R . \sim (x R_{\text{po}} x) .$

$[*96{\cdot}1] \qquad \supset . I_R{}^{\backprime}x = \Lambda \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***96·492.** $\vdash :. R \in 1 \to 1 . x \in \text{D}{}^{\backprime}R . \supset :$

$\sim (x R_{\text{po}} x) . \equiv . I_R{}^{\backprime}x = \Lambda : x R_{\text{po}} x . \equiv . J_R{}^{\backprime}x = \Lambda$

Dem.

$\vdash . *96{\cdot}1{\cdot}101 . \supset$

$\vdash : I_R{}^{\backprime}x = \Lambda . \supset . \sim (x R_{\text{po}} x) : x \in \text{D}{}^{\backprime}R . \sim (x R_{\text{po}} x) . \supset . \exists ! J_R{}^{\backprime}x \qquad (1)$

$\vdash . (1) . *96{\cdot}491 . \supset \vdash :. \text{Hp} . \supset : \sim (x R_{\text{po}} x) . \equiv . I_R{}^{\backprime}x = \Lambda \qquad (2)$

Similarly $\qquad \vdash :. \text{Hp} . \supset : x R_{\text{po}} x . \equiv . J_R{}^{\backprime}x = \Lambda \qquad (3)$

$\vdash . (2) . (3) . \supset \vdash . \text{Prop}$

The above proposition is used in *122·52.

The following propositions, not being needed in the sequel, are merely stated:

$\vdash : R \in \text{Cls} \to 1 . \exists ! J_R{}^{\backprime}x . \exists ! I_R{}^{\backprime}x . \supset . I_R{}^{\backprime}x \cap \breve{R}{}^{\backprime\backprime}J_R{}^{\backprime}x \in 1 . J_R{}^{\backprime}x \cap R{}^{\backprime\backprime}I_R{}^{\backprime}x \in 1$

$\vdash : R \in \text{Cls} \to 1 . J_R{}^{\backprime}x = \Lambda . \supset . (\overleftarrow{R_*}{}^{\backprime}x) \upharpoonright I \in \text{Pot}{}^{\backprime}\{(\overleftarrow{R_*}{}^{\backprime}x) \upharpoonright R\}$

***96·5.** $\vdash : R \in 1 \to 1 . x \in \text{D}{}^{\backprime}R . \supset . \overrightarrow{R_{\text{po}}}{}^{\backprime}\breve{R}{}^{\backprime}x = \overrightarrow{R_*}{}^{\backprime}x = \overrightarrow{R_{\text{po}}}{}^{\backprime}x \cup \iota{}^{\backprime}x$

Dem.

$\vdash . *71{\cdot}7 . \supset \vdash :. \text{Hp} . \supset : y \in \overrightarrow{R_{\text{po}}}{}^{\backprime}\breve{R}{}^{\backprime}x . \equiv . y R_{\text{po}} | \breve{R} x .$

$[*92{\cdot}11] \qquad \equiv . y R_* x . x \in \text{D}{}^{\backprime}R .$

$[\text{Hp} . *4{\cdot}71] \qquad \equiv . y R_* x :$

$\left[*32{\cdot}18 . *96{\cdot}14 \frac{\breve{R}}{R}\right] \supset : \overrightarrow{R_{\text{po}}}{}^{\backprime}\breve{R}{}^{\backprime}x = \overrightarrow{R_*}{}^{\backprime}x = \overrightarrow{R_{\text{po}}}{}^{\backprime}x \cup \iota{}^{\backprime}x :. \supset \vdash . \text{Prop}$

***96·501.** $\vdash : R \in 1 \to 1 . x \in \text{Cl}{}^{\backprime}R . \supset . \overleftarrow{R_{\text{po}}}{}^{\backprime}R{}^{\backprime}x = \overleftarrow{R_*}{}^{\backprime}x = \overleftarrow{R_{\text{po}}}{}^{\backprime}x \cup \iota{}^{\backprime}x$

***96·502.** $\vdash : R \epsilon 1 \to \text{Cls} . xRy . \supset . \overrightarrow{R_*}`y = \overrightarrow{R_*}`x \cup \iota`y$

Dem.

$\vdash . *90\cdot31 . \supset \vdash :: \text{Hp} . \supset :. zR_*y . \equiv : zR_*(R`y) . \vee . z = y :: \supset \vdash . \text{Prop}$

***96·51.** $\vdash : R \epsilon 1 \to 1 . \alpha \subset \breve{R}_*``\overrightarrow{B}`R . \alpha \subset \breve{R}_{po}``\alpha . \supset . \alpha = \Lambda$

Dem.

$\vdash . *37\cdot105 . \supset \vdash :. \text{Hp} . \supset : y \epsilon \alpha . \supset_y . (\exists x) . x \epsilon \alpha . xR_{po}y .$

[*32·18] $\supset_y . \exists ! \alpha \cap \overrightarrow{R_{po}}`y :$

[*14·18·21] $\supset : \breve{R}`x \epsilon \alpha . \supset . \exists ! \alpha \cap \overrightarrow{R_{po}}`\breve{R}`x .$

[*96·5] $\supset . \exists ! \alpha \cap \overrightarrow{R_*}`x :$

[Transp] $\supset : \alpha \cap \overrightarrow{R_*}`x = \Lambda . xRy . \supset . y \sim \epsilon \alpha .$

[*51·211] $\supset . \alpha \cap (\overrightarrow{R_*}`x \cup \iota`y) = \Lambda .$

[*96·502] $\supset . \alpha \cap \overrightarrow{R_*}`y = \Lambda$ (1)

$\vdash . *91\cdot504 . \supset \vdash :. \alpha \subset \breve{R}_{po}``\alpha . \supset : \alpha \subset \text{Cl}`R :$

[*93·104] $\supset : x \epsilon \overrightarrow{B}`R . \supset . \alpha \cap \overrightarrow{R_*}`x = \Lambda$ (2)

$\vdash . (1) . (2) . *90\cdot112 . \supset \vdash :. \text{Hp} . \supset : x \epsilon \overrightarrow{B}`R . xR_*y . \supset . \alpha \cap \overrightarrow{R_*}`y = \Lambda .$

[*90·13] $\supset . y \sim \epsilon \alpha :$

[*37·105] $\supset : \breve{R}_*``\overrightarrow{B}`R \cap \alpha = \Lambda$ (3)

$\vdash . *22\cdot621 . \quad \supset \vdash : \text{Hp} . \supset . \alpha = \breve{R}_*``\overrightarrow{B}`R \cap \alpha$ (4)

$\vdash . (3) . (4) . \supset \vdash . \text{Prop}$

***96·52.** $\vdash : R \epsilon 1 \to 1 . \alpha \subset \breve{R}_*``\overrightarrow{B}`R . \exists ! \alpha . \supset . \exists ! \overrightarrow{\min}(R_{po})`\alpha$

Dem.

$\vdash . *96\cdot51 . \text{Transp} . \supset \vdash : \text{Hp} . \supset . \exists ! \alpha - \breve{R}_{o}``\alpha$ (1)

$\vdash . (1) . *93\cdot111 . \quad \supset \vdash . \text{Prop}$

This proposition is used in *122·23.

$*97$. ANALYSIS OF THE FIELD OF A RELATION INTO FAMILIES

Summary of $*97$.

In this number, we consider not only the posterity of a term, but the ancestry and posterity together, *i.e.* $\overrightarrow{R_*}\text{‘}x \cup \overleftarrow{R_*}\text{‘}x$. We put

$$\overleftrightarrow{R}\text{‘}x = \overrightarrow{R}\text{‘}x \cup (\iota\text{‘}x \cap C\text{‘}R) \cup \overleftarrow{R}\text{‘}x \quad \text{Df.}$$

Thus the whole family of a term, *i.e.* its ancestry and posterity together, is $\overleftrightarrow{R_*}\text{‘}x$. The most important case here is when $R \in 1 \to 1$; in this case families are mutually exclusive, *i.e.* we have

$$\vdash : R \in 1 \to 1 \,.\supset .\, \overleftrightarrow{R_*}\text{“}C\text{‘}R \in \text{Cls ex}^2\text{ excl.}$$

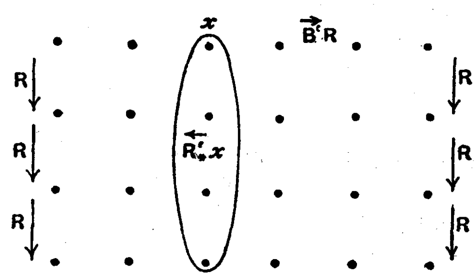
In case $R \in 1 \to 1$ and y belongs to a family which has a beginning, *i.e.* in case $\exists ! \overleftrightarrow{R_*}\text{‘}y \cap \overrightarrow{B}\text{‘}R$, the whole family of y consists of the posterity of the beginning, *i.e.* we have

$$\vdash : R \in 1 \to 1 \,.\, xBR \,.\, xR_*y \,.\supset .\, \overleftrightarrow{R_*}\text{‘}y = \overleftarrow{R_*}\text{‘}x,$$

whence

$*97{\cdot}21$. $\vdash : R \in 1 \to 1 \,.\supset .\, \overleftrightarrow{R_*}\text{“}s\text{‘gen‘}R = \overleftarrow{R_*}\text{“}\overrightarrow{B}\text{‘}R$

When $R \in 1 \to 1$, the relation of gen‘R to $\overleftarrow{R_*}\text{“}\overrightarrow{B}\text{‘}R$ may be pictured as the relation of rows to columns. *E.g.* let the field of R consist of the dots



in the accompanying rectangle, and let each dot have the relation R to the dot below it. Then the top row is $\overrightarrow{B}\text{‘}R$, the second row is $\text{Ɑ‘}R - \text{Ɑ‘}R^2$, the third is $\text{Ɑ‘}R^2 - \text{Ɑ‘}R^3$, and so on; thus the rows are the generations of R. Again, if x is any dot in the top row, the column beginning with x is $\overleftarrow{R_*}\text{‘}x$, and if y is any member of this column, the column is $\overleftrightarrow{R_*}\text{‘}y$. Thus the columns are the families of R. It will be seen that in the case represented by the above figure, every family consists of a selection from the generations, and every generation consists of a selection from the families, *i.e.*

$$\overleftarrow{R_*}\text{“}\overrightarrow{B}\text{‘}R \subset \text{D“}\epsilon_\Delta\text{‘gen‘}R \,.\, \text{gen‘}R \subset \text{D“}\epsilon_\Delta\text{‘}\overleftarrow{R_*}\text{“}\overrightarrow{B}\text{‘}R.$$

The circumstances under which this occurs will be considered in the present number (∗97·3—·47). The results are summed up in ∗97·47.

The remaining propositions (∗97·5—·58) are concerned with *circular* families of one-one relations. If $R \epsilon 1 \to 1$, $\overleftrightarrow{R}_*‘x$ is a circular family if $xR_{\mathrm{po}}x$. In that case, we have $xR_{\mathrm{po}}y \,.\supset.\, yR_{\mathrm{po}}x$; moreover there is a definite power of R, say P, such that every member of the family of x has the relation P to itself (∗97·54). (The same will hold, of course, of all powers of P.) The families of a $1 \to 1$ are all either circular or open, *i.e.* we have (∗97·55) either $y \epsilon \overleftrightarrow{R}_*‘x \,.\supset_y.\, yR_{\mathrm{po}}y$, or $y \epsilon \overleftrightarrow{R}_*‘x \,.\supset_y.\, \sim(yR_{\mathrm{po}}y)$. The Q-shaped families considered in ∗96 are not possible for a $1 \to 1$, since in such families the term at the junction of the tail and the circle has two predecessors. The family of any member of $s‘\mathrm{gen}‘R$ must be open (∗97·57). The family of a member of $p‘\text{Ɑ}‘‘\mathrm{Pot}‘R$ need not be closed, but cannot have a beginning; if open, it forms a series of type $^*\omega$ or $^*\omega + \omega$, according as it has or has not an end*. Finite open families are contained in $s‘\mathrm{gen}‘R \cap s‘\mathrm{gen}‘\breve{R}$; families of type ω are contained in $s‘\mathrm{gen}‘R \cap p‘\text{Ɑ}‘‘\mathrm{Pot}‘\breve{R}$; those of type $^*\omega$, in $s‘\mathrm{gen}‘\breve{R} \cap p‘\text{Ɑ}‘‘\mathrm{Pot}‘R$; those of type $^*\omega + \omega$ and circular families are contained in $p‘\text{Ɑ}‘‘\mathrm{Pot}‘R \cap p‘\text{Ɑ}‘‘\mathrm{Pot}‘\breve{R}$. Those of type $^*\omega + \omega$ are distinguished from circular families by the fact that in the former we do not have $xR_{\mathrm{po}}x$, while in the latter we do have this.

In addition to the propositions already mentioned, the most useful propositions of the present number are the following:

∗97·13. $\vdash .\, \overleftrightarrow{R}_*‘x = \overrightarrow{R}_*‘x \cup \overleftarrow{R}_*‘x$

∗97·17. $\vdash .\, \overleftrightarrow{R}_*‘x = \overleftrightarrow{R}_{\mathrm{po}}‘x = \overrightarrow{R}_*‘x \cup \overleftarrow{R}_{\mathrm{po}}‘x = \overrightarrow{R}_{\mathrm{po}}‘x \cup \overleftarrow{R}_*‘x$

∗97·5. $\vdash : R \epsilon \mathrm{Cls} \to 1 \,.\, xR_{\mathrm{po}}x \,.\, xR_{\mathrm{po}}y \,.\supset.\, yR_{\mathrm{po}}x$

∗97·501. $\vdash : R \epsilon 1 \to \mathrm{Cls} \,.\, xR_{\mathrm{po}}x \,.\, yR_{\mathrm{po}}x \,.\supset.\, xR_{\mathrm{po}}y$

∗97·01. $\overleftrightarrow{R}‘x = \overrightarrow{R}‘x \cup (\iota‘x \cap C‘R) \cup \overleftarrow{R}‘x$ Df

Observe that "$\iota‘x \cap C‘R$" means that x is to be included if it is a member of $C‘R$, but not otherwise; for $\iota‘x \cap C‘R = \iota‘x$ if $x \epsilon C‘R$, and otherwise $\iota‘x \cap C‘R = \Lambda$.

* Here the type "$^*\omega$" is the type of converses of relations of type ω, *i.e.* the type of the negative integers in order of magnitude, ending with -1, ω being the type of the positive integers in order of magnitude, and therefore $^*\omega+\omega$ being the type of negative and positive integers in order of magnitude.

***97·1.** $\vdash :. y \in \overleftrightarrow{R}\text{‘}x . \equiv : yRx . \vee . y = x . x \in C\text{‘}R . \vee . xRy$

Dem.

$\vdash . *32·18·181 . *51·15 . (*97·01) . \supset$

$\vdash :. y \in \overleftrightarrow{R}\text{‘}x . \equiv : yRx . \vee . y = x . y \in C\text{‘}R . \vee . xRy :$

$[*13·193] \quad \equiv : yRx . \vee . y = x . x \in C\text{‘}R . \vee . xRy :. \supset \vdash . \text{Prop}$

***97·101.** $\vdash : y \in \overleftrightarrow{R}\text{‘}x . \equiv . x \in \overleftrightarrow{R}\text{‘}y$

Dem.

$\vdash . *32·18·181 . *51·15 . (*97·01) . \supset$

$\vdash :. x \in \overleftrightarrow{R}\text{‘}y . \equiv : xRy . \vee . x = y . x \in C\text{‘}R . \vee . yRx :$

$[*97·1] \quad \equiv : y \in \overleftrightarrow{R}\text{‘}x :. \supset \vdash . \text{Prop}$

***97·11.** $\vdash . s\text{‘}\overleftrightarrow{R}\text{“}C\text{‘}R = C\text{‘}R$

Dem.

$\vdash . *97·1 . *40·11 . \supset$

$\vdash :. y \in s\text{‘}\overleftrightarrow{R}\text{“}C\text{‘}R . \equiv : (\exists x) . yRx . \vee . (\exists x) . y = x . x \in C\text{‘}R . \vee . (\exists x) . xRy :$

$[*33·13·131 . *13·195] \equiv : y \in \mathrm{D}\text{‘}R . \vee . y \in C\text{‘}R . \vee . y \in \mathrm{G}\text{‘}R :$

$[*33·16] \quad \equiv : y \in C\text{‘}R :. \supset \vdash . \text{Prop}$

***97·111.** $\vdash : x \in C\text{‘}R . \equiv . x \in \overleftrightarrow{R}\text{‘}x . \equiv . \exists ! \overleftrightarrow{R}\text{‘}x$

Dem.

$\vdash . *97·1 . \supset \vdash :. x \in \overleftrightarrow{R}\text{‘}x . \equiv : xRx . \vee . x \in C\text{‘}R :$

$[*33·17] \quad \equiv : x \in C\text{‘}R \qquad (1)$

$\vdash . *97·1 . \supset \vdash :. \exists ! \overleftrightarrow{R}\text{‘}x . \equiv : (\exists y) : yRx . \vee . xRy : \vee : (\exists y) . x \in C\text{‘}R . y = x :$

$[*33·132 . *13·19] \quad \equiv : x \in C\text{‘}R \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***97·12.** $\vdash . \Lambda \sim\in \overleftrightarrow{R}_*\text{“}C\text{‘}R$

Dem.

$\vdash . *97·111 . *37·63 . \supset \vdash : \alpha \in \overleftrightarrow{R}_*\text{“}C\text{‘}R . \supset_\alpha . \exists ! \alpha \qquad (1)$

$\vdash . (1) . *24·63 . \supset \vdash . \text{Prop}$

***97·13.** $\vdash . \overleftrightarrow{R}_*\text{‘}x = \overrightarrow{R}_*\text{‘}x \cup \overleftarrow{R}_*\text{‘}x$

Note. $\overleftrightarrow{R}_*$ is to mean $(\overleftrightarrow{R_*})$, not $(\overleftrightarrow{R})_*$. The latter is unmeaning, since $\overleftrightarrow{R}$ is never a homogeneous relation, and therefore its square and higher powers are unmeaning.

Dem.

$\vdash . *90·12 . \supset \vdash : y = x . y \in C\text{‘}R . \supset . yR_*x :$

$[*51·15] \quad \supset \vdash . \iota\text{‘}x \cap C\text{‘}R \subset \overleftarrow{R}_*\text{‘}x .$

$[*90·14] \quad \supset \vdash . \iota\text{‘}x \cap C\text{‘}R_* \subset \overleftarrow{R}_*\text{‘}x \qquad (1)$

$\vdash . (1) . (*97·01) . \supset \vdash . \text{Prop}$

***97·14.** $\vdash : R \in 1 \to 1 \,.\, xR_*y \,.\, \supset . \overleftrightarrow{R_*}\text{‘}x = \overleftrightarrow{R_*}\text{‘}y$ [*96·32 . *97·13]

***97·15.** $\vdash : R \in 1 \to 1 \,.\, x \in \overleftrightarrow{R_*}\text{‘}y \,.\, \supset . \overleftrightarrow{R_*}\text{‘}x = \overleftrightarrow{R_*}\text{‘}y$

Dem.

$$\vdash . *97\cdot13 . \supset \vdash :. \text{Hp} . \supset : xR_*y . \vee . yR_*x \quad (1)$$
$$\vdash . (1) . *97\cdot14 . \supset \vdash . \text{Prop}$$

***97·16.** $\vdash : R \in 1 \to 1 \,.\, \supset . \overleftrightarrow{R_*}\text{‘‘}C\text{‘}R \in \text{Cls ex}^2\text{ excl}$

Dem.

$$\vdash . *97\cdot15 . \supset \vdash :. \text{Hp} . \supset : x \in \overleftrightarrow{R_*}\text{‘}y . x \in \overleftrightarrow{R_*}\text{‘}z . \supset_x . \overleftrightarrow{R_*}\text{‘}x = \overleftrightarrow{R_*}\text{‘}y . \overleftrightarrow{R_*}\text{‘}x = \overleftrightarrow{R_*}\text{‘}z .$$
$$[*13\cdot171] \qquad \supset_x . \overleftrightarrow{R_*}\text{‘}y = \overleftrightarrow{R_*}\text{‘}z :$$
$$[*10\cdot23] \qquad \supset : \exists ! \overleftrightarrow{R_*}\text{‘}y \cap \overleftrightarrow{R_*}\text{‘}z . \supset . \overleftrightarrow{R_*}\text{‘}y = \overleftrightarrow{R_*}\text{‘}z \quad (1)$$
$$\vdash . (1) . *11\cdot11\cdot3 . *37\cdot63 . \supset$$
$$\vdash :. \text{Hp} . \supset : \alpha, \beta \in \overleftrightarrow{R_*}\text{‘‘}C\text{‘}R . \exists ! \alpha \cap \beta . \supset_{\alpha,\beta} . \alpha = \beta \quad (2)$$
$$\vdash . (2) . *97\cdot12 . *84\cdot132 . \supset \vdash . \text{Prop}$$

***97·17.** $\vdash . \overleftrightarrow{R_*}\text{‘}x = \overleftrightarrow{R_{\text{po}}}\text{‘}x = \overrightarrow{R_*}\text{‘}x \cup \overleftarrow{R_{\text{po}}}\text{‘}x = \overrightarrow{R_{\text{po}}}\text{‘}x \cup \overleftarrow{R_*}\text{‘}x$

Dem.

$$\vdash . *97\cdot13 . *91\cdot54 . \supset \vdash . \overleftrightarrow{R_*}\text{‘}x = \overrightarrow{R_{\text{po}}}\text{‘}x \cup (\iota\text{‘}x \cap C\text{‘}R) \cup \overleftarrow{R_{\text{po}}}\text{‘}x \quad (1)$$
$$[*91\cdot504.(*97\cdot01)] \qquad = \overleftrightarrow{R_{\text{po}}}\text{‘}x \quad (2)$$
$$\vdash . (1) . *91\cdot54 . \quad \supset \vdash . \overleftrightarrow{R_*}\text{‘}x = \overrightarrow{R_{\text{po}}}\text{‘}x \cup \overleftarrow{R_*}\text{‘}x = \overrightarrow{R_*}\text{‘}x \cup \overleftarrow{R_{\text{po}}}\text{‘}x \quad (3)$$
$$\vdash . (2) . (3) . \supset \vdash . \text{Prop}$$

***97·18.** $\vdash . C\text{‘}(R \upharpoonright \overleftrightarrow{R}\text{‘}x) = \overleftrightarrow{R}\text{‘}x$

Dem.

$$\vdash . *37\cdot41 . \supset \vdash . C\text{‘}(R \upharpoonright \overleftrightarrow{R}\text{‘}x) \subset \overleftrightarrow{R}\text{‘}x \quad (1)$$
$$\vdash . *97\cdot1 . *36\cdot13 . \supset$$
$$\vdash :. x \in C\text{‘}R . y \in \overrightarrow{R}\text{‘}x \cup \overleftarrow{R}\text{‘}x . \supset : x (R \upharpoonright \overleftrightarrow{R}\text{‘}x) y . \vee . y (R \upharpoonright \overleftrightarrow{R}\text{‘}x) x :$$
$$[*33\cdot17] \qquad \supset : x, y \in C\text{‘}(R \upharpoonright \overleftrightarrow{R}\text{‘}x) \quad (2)$$
$$\vdash . (2) . *97\cdot1 . \quad \supset \vdash : x \in C\text{‘}R . \supset . \overleftrightarrow{R}\text{‘}x \subset C\text{‘}(R \upharpoonright \overleftrightarrow{R}\text{‘}x) \quad (3)$$
$$\vdash . *97\cdot111 . \text{Transp} . \supset \vdash : x \sim\in C\text{‘}R . \supset . \overleftrightarrow{R}\text{‘}x \subset C\text{‘}(R \upharpoonright \overleftrightarrow{R}\text{‘}x) \quad (4)$$
$$\vdash . (1) . (3) . (4) . \supset \vdash . \text{Prop}$$

***97·2.** $\vdash : xBR \,.\, \supset . \overleftrightarrow{R_*}\text{‘}x = \overleftarrow{R_*}\text{‘}x$

Dem.

$$\vdash . *93\cdot104 . *97\cdot13 . \supset \vdash : \text{Hp} . \supset . \overleftrightarrow{R_*}\text{‘}x = \iota\text{‘}x \cup \overleftarrow{R_*}\text{‘}x \quad (1)$$
$$\vdash . *93\cdot101 . *90\cdot12 . \supset \vdash : \text{Hp} . \supset . x \in \overleftarrow{R_*}\text{‘}x \quad (2)$$
$$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$$

***97·21.** $\vdash : R \epsilon 1 \to 1 . \supset . \overleftrightarrow{R_*}``s`\mathrm{gen}`R = \overleftarrow{R_*}``\overrightarrow{B}`R$

Dem.

$\vdash . *97\cdot14\cdot2 . \supset \vdash :. \mathrm{Hp} . \supset : xBR . xR_*y . \supset . \overleftarrow{R_*}`x = \overleftrightarrow{R_*}`y .$

$[*37\cdot62] \qquad \supset . \overleftrightarrow{R_*}`y \,\epsilon\, \overleftarrow{R_*}``\overrightarrow{B}`R :$

$[*93\cdot36] \qquad \supset : y \,\epsilon\, s`\mathrm{gen}`R . \supset . \overleftrightarrow{R_*}`y \,\epsilon\, \overleftarrow{R_*}``\overrightarrow{B}`R :$

$[*37\cdot61] \qquad \supset : \overleftrightarrow{R_*}``s`\mathrm{gen}`R \subset \overleftarrow{R_*}``\overrightarrow{B}`R \qquad (1)$

$\vdash . *97\cdot2 . *93\cdot22 . \supset \vdash . \overleftarrow{R_*}``\overrightarrow{B}`R \subset \overleftrightarrow{R_*}``s`\mathrm{gen}`R \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***97·22.** $\vdash : R \epsilon 1 \to 1 . \supset . \overleftarrow{R_*}``\overrightarrow{B}`R \cup \overleftrightarrow{R_*}``p`\mathrm{Cl}``\mathrm{Pot}`R = \overleftrightarrow{R_*}``C`R$
$[*97\cdot21 . *93\cdot37]$

***97·23.** $\vdash :: \overleftrightarrow{R}``C`R \,\epsilon\, 0 \cup 1 . \equiv :. x, y \,\epsilon\, C`R . \supset_{x,y} : x = y . \vee . xRy . \vee . yRx$

Dem.

$\vdash . *52\cdot4 . (*54\cdot01) . \supset$

$\vdash ::. \overleftrightarrow{R}``C`R \,\epsilon\, 0 \cup 1 . \equiv :: \alpha, \beta \,\epsilon\, \overleftrightarrow{R}``C`R . \supset_{\alpha,\beta} . \alpha = \beta ::$

$[*37\cdot63] \qquad \equiv :: x, y \,\epsilon\, C`R . \supset_{x,y} . \overleftrightarrow{R}`x = \overleftrightarrow{R}`y ::$

$[*97\cdot1] \qquad \equiv :: x, y \,\epsilon\, C`R . \supset_{x,y} :. zRx . \vee . x \,\epsilon\, C`R . z = x . \vee . xRz . \equiv_z :$
$zRy . \vee . y \,\epsilon\, C`R . z = y . \vee . yRz :.$

$[*4\cdot71] \qquad \equiv :: x, y \,\epsilon\, C`R . \supset_{x,y} :. zRx . \vee . z = x . \vee . xRz : \equiv_z :$
$zRy . \vee . z = y . \vee . yRz :. \qquad (1)$

$[*10\cdot1] \qquad \supset :: x, y \,\epsilon\, C`R . \supset_{x,y} :. xRx . \vee . x = x . \vee . xRx : \equiv :$
$xRy . \vee . x = y . \vee . yRx :.$

$[*13\cdot15] \qquad \supset_{x,y} :. xRy . \vee . x = y . \vee . yRx \qquad (2)$

$\vdash . *10\cdot1 . \supset \vdash :: x, y \,\epsilon\, C`R : z \,\epsilon\, C`R :. x, y \,\epsilon\, C`R . \supset_{x,y} : xRy . \vee . x = y . \vee . yRx :. \supset :.$
$xRz . \vee . x = z . \vee . zRx : yRz . \vee . y = z . \vee . zRy :.$

$[*5\cdot1] \qquad \supset :. xRz . \vee . x = z . \vee . zRx : \equiv : yRz . \vee . y = z . \vee . zRy \qquad (3)$

$\vdash . *33\cdot132 . \mathrm{Transp} . *13\cdot14 . \supset$

$\vdash :: x, y \,\epsilon\, C`R : z \sim\epsilon\, C`R : \supset :. \sim(xRz . \vee . zRx) . x \neq z : \sim(yRz . \vee . zRy) . y \neq z :.$

$[*5\cdot21] \qquad \supset :. xRz . \vee . x = z . \vee . zRx : \equiv :$
$yRz . \vee . y = z . \vee . zRy \qquad (4)$

$\vdash . (3) . (4) . \supset \vdash ::. x, y \,\epsilon\, C`R . \supset_{x,y} : xRy . \vee . x = y . \vee . yRx :. \supset ::$
$x, y \,\epsilon\, C`R . \supset_{x,y} :. xRz . \vee . x = z . \vee . zRx : \equiv_z : yRz . \vee . y = z . \vee . zRy \qquad (5)$

$\vdash . (1) . (2) . (5) . \supset \vdash . \mathrm{Prop}$

***97·231.** $\vdash :. \overleftrightarrow{R}``C`R \,\epsilon\, 0 \cup 1 . \equiv : x \,\epsilon\, C`R . \supset_x . C`R = \overrightarrow{R}`x \cup \iota`x \cup \overleftarrow{R}`x$

Dem.

$\vdash . *97\cdot23 . *32\cdot18\cdot181 . *51\cdot15 . \supset$

$\vdash :. \overleftrightarrow{R}``C`R \,\epsilon\, 0 \cup 1 . \equiv : x \,\epsilon\, C`R . \supset . C`R \subset \overrightarrow{R}`x \cup \iota`x \cup \overleftarrow{R}`x \qquad (1)$

$\vdash . *33\cdot152 . *51\cdot2 . \supset \vdash : x \,\epsilon\, C`R . \supset . \overrightarrow{R}`x \cup \iota`x \cup \overleftarrow{R}`x \subset C`R \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***97·24.** $\vdash :. \overleftrightarrow{R_*}\text{“}C\text{‘}R \in 0 \cup 1 . \equiv : x \in C\text{‘}R . \supset_x . C\text{‘}R = \overrightarrow{R_*}\text{‘}x \cup \overleftarrow{R_*}\text{‘}x$

Dem.

$\vdash . *97\cdot231 . *90\cdot14 . \supset$

$\vdash :. \overleftrightarrow{R_*}\text{“}C\text{‘}R \in 0 \cup 1 . \equiv : x \in C\text{‘}R . \supset_x . C\text{‘}R = \overrightarrow{R_*}\text{‘}x \cup \iota\text{‘}x \cup \overleftarrow{R_*}\text{‘}x$ (1)

$\vdash . *90\cdot12 . \quad \supset \vdash : x \in C\text{‘}R . \supset . \iota\text{‘}x \subset \overrightarrow{R_*}\text{‘}x$ (2)

$\vdash . (1) . (2) . *22\cdot62 . \supset \vdash . \text{Prop}$

***97·241.** $\vdash :: \overleftrightarrow{R_*}\text{“}C\text{‘}R \in 0 \cup 1 . \equiv :. x, y \in C\text{‘}R . \supset_{x,y} : xR_*y . \vee . yR_*x$

Dem.

$\vdash . *97\cdot24 . *32\cdot18\cdot181 . \supset$

$\vdash ::. \overleftrightarrow{R_*}\text{“}C\text{‘}R \in 0 \cup 1 . \equiv :: x \in C\text{‘}R . \supset_x :. y \in C\text{‘}R . \equiv_y : xR_*y . \vee . yR_*x$ (1)

$\vdash . *90\cdot13 . \supset \vdash :. xR_*y . \vee . yR_*x : \supset . y \in C\text{‘}R$ (2)

$\vdash . (1) . (2) . *4\cdot73 . \supset \vdash . \text{Prop}$

***97·242.** $\vdash :: \overleftrightarrow{R_*}\text{“}C\text{‘}R \in 0 \cup 1 . \equiv :. x, y \in C\text{‘}R . \supset_{x,y} : x = y . \vee . xR_{\text{po}}y . \vee . yR_{\text{po}}x :.$
$\equiv :. \overleftrightarrow{R_{\text{po}}}\text{“}C\text{‘}R \in 0 \cup 1$

[*91·542 . *97·23 . *91·504]

The remaining propositions of this number (except *97·5 ff.) are concerned with proving that, under certain hypotheses,

$$\overleftarrow{R_*}\text{“}\overrightarrow{B}\text{‘}R \subset \text{D}\text{“}\epsilon_\Delta\text{‘gen‘}R, \quad \textit{i.e.}\ \overleftrightarrow{R_*}\text{“}s\text{‘gen‘}R \subset \text{D}\text{“}\epsilon_\Delta\text{‘gen‘}R,$$

and $$\text{gen‘}R - \iota\text{‘}\Lambda \subset \text{D}\text{“}\epsilon_\Delta\text{‘}\overleftarrow{R_*}\text{“}\overrightarrow{B}\text{‘}R.$$

These propositions have the merit of proving the existence of selections in the cases to which they apply.

***97·3.** $\vdash . \overleftarrow{R_*} \restriction \overrightarrow{B}\text{‘}R \in 1 \to 1$

Dem.

$\vdash . *90\cdot12 . \supset$

$\vdash :. x, y \in \overrightarrow{B}\text{‘}R . \overleftarrow{R_*}\text{‘}x = \overleftarrow{R_*}\text{‘}y . \supset : y \in \overleftarrow{R_*}\text{‘}x :$

[*91·54] $\quad \supset : y = x . \vee . xR_{\text{po}}y$ (1)

$\vdash . *91\cdot504 . \quad \supset \vdash : xR_{\text{po}}y . \supset . y \in \text{ɑ}\text{‘}R :$

[Transp.*93·101] $\supset \vdash : y \in \overrightarrow{B}\text{‘}R . \supset . \sim(xR_{\text{po}}y)$ (2)

$\vdash . (1) . (2) . \quad \supset \vdash : x, y \in \overrightarrow{B}\text{‘}R . \overleftarrow{R_*}\text{‘}x = \overleftarrow{R_*}\text{‘}y . \supset . x = y$ (3)

$\vdash . (3) . *71\cdot55 . *72\cdot12 . \supset \vdash . \text{Prop}$

***97·301.** $\vdash . I \restriction \overrightarrow{B}\text{‘}R \in (\breve{R}_*)_\Delta\text{‘}\overrightarrow{B}\text{‘}R$

Dem.

$\vdash . *72\cdot17 . \quad \supset \vdash . I \restriction \overrightarrow{B}\text{‘}R \in 1 \to \text{Cls}$ (1)

$\vdash . *90\cdot15 . \quad \supset \vdash . I \restriction \overrightarrow{B}\text{‘}R \mathrel{\mathsf{C}} \breve{R}_*$ (2)

$\vdash . *50\cdot5\cdot52 . \supset \vdash . \text{ɑ}\text{‘}I \restriction \overrightarrow{B}\text{‘}R = \overrightarrow{B}\text{‘}R$ (3)

$\vdash . (1) . (2) . (3) . *80\cdot14 . \supset \vdash . \text{Prop}$

***97·31.** $\vdash . (\overrightarrow{B}\text{‘}R) \uparrow \mathrm{Cnv}\text{‘}\overleftarrow{R_*} \in \epsilon_\Delta\text{‘}\overleftarrow{R_*}\text{“}\overrightarrow{B}\text{‘}R . \mathrm{D}\text{‘}\{(\overrightarrow{B}\text{‘}R) \uparrow \mathrm{Cnv}\text{‘}\overleftarrow{R_*}\} = \overrightarrow{B}\text{‘}R$

Dem.

$\vdash . *97\cdot3 . *85\cdot13 \dfrac{\breve{R}_*}{Q} . \supset$

$\vdash : S \in (\breve{R}_*)_\Delta\text{‘}\overrightarrow{B}\text{‘}R . \supset . S \mid \mathrm{Cnv}\text{‘}\overleftarrow{R_*} \in \epsilon_\Delta\text{‘}\overleftarrow{R_*}\text{“}\overrightarrow{B}\text{‘}R$ (1)

$\vdash . (1) . *97\cdot301 . \quad \supset \vdash . I \upharpoonright \overrightarrow{B}\text{‘}R \mid \mathrm{Cnv}\text{‘}\overleftarrow{R_*} \in \epsilon_\Delta\text{‘}\overleftarrow{R_*}\text{“}\overrightarrow{B}\text{‘}R .$

[*50·61] $\quad \supset \vdash . (\overrightarrow{B}\text{‘}R) \uparrow \mathrm{Cnv}\text{‘}\overleftarrow{R_*} \in \epsilon_\Delta\text{‘}\overleftarrow{R_*}\text{“}\overrightarrow{B}\text{‘}R$ (2)

$\vdash . *35\cdot62 . *33\cdot431 . \supset \vdash . \mathrm{D}\text{‘}\{(\overrightarrow{B}\text{‘}R) \uparrow \mathrm{Cnv}\text{‘}\overleftarrow{R_*}\} = \overrightarrow{B}\text{‘}R$ (3)

$\vdash . (2) . (3) . \supset \vdash . \mathrm{Prop}$

***97·32.** $\vdash . \overrightarrow{B}\text{‘}R \in \mathrm{D}\text{“}\epsilon_\Delta\text{‘}\overleftarrow{R_*}\text{“}\overrightarrow{B}\text{‘}R \quad [*97\cdot31]$

***97·33.** $\vdash : R \in 1 \to 1 . \alpha \subset s\text{‘}\overleftrightarrow{R_*}\text{“}\beta . \beta \subset s\text{‘}\overleftrightarrow{R_*}\text{“}\alpha . \supset . \overleftrightarrow{R_*}\text{“}\alpha = \overleftrightarrow{R_*}\text{“}\beta$

Dem.

$\vdash . *97\cdot15 . \mathrm{Fact} . \supset \vdash :. \mathrm{Hp} . \supset : y \in \beta . x \in \overleftrightarrow{R_*}\text{‘}y . \supset . \overleftrightarrow{R_*}\text{‘}y = \overleftrightarrow{R_*}\text{‘}x . y \in \beta .$

[*37·62] $\quad \supset . \overleftrightarrow{R_*}\text{‘}x \in \overleftrightarrow{R_*}\text{“}\beta$ (1)

$\vdash . (1) . *10\cdot11\cdot21\cdot23 . *40\cdot4 . \supset \vdash :. \mathrm{Hp} . \supset : x \in s\text{‘}\overleftrightarrow{R_*}\text{“}\beta . \supset_x . \overleftrightarrow{R_*}\text{‘}x \in \overleftrightarrow{R_*}\text{“}\beta :$

[Hp.Syll] $\quad \supset : x \in \alpha . \supset_x . \overleftrightarrow{R_*}\text{‘}x \in \overleftrightarrow{R_*}\text{“}\beta :$

[*37·61] $\quad \supset : \overleftrightarrow{R_*}\text{“}\alpha \subset \overleftrightarrow{R_*}\text{“}\beta$ (2)

$\vdash . *40\cdot4 . \supset \vdash :. \mathrm{Hp} . \supset : y \in \beta . \supset . (\exists x) . x \in \alpha . y \in \overleftrightarrow{R_*}\text{‘}x .$

[*97·15] $\quad \supset . (\exists x) . x \in \alpha . \overleftrightarrow{R_*}\text{‘}x = \overleftrightarrow{R_*}\text{‘}y .$

[*37·62] $\quad \supset . \overleftrightarrow{R_*}\text{‘}y \in \overleftrightarrow{R_*}\text{“}\alpha$ (3)

$\vdash . (3) . *37\cdot61 . \supset \vdash : \mathrm{Hp} . \supset . \overleftrightarrow{R_*}\text{“}\beta \subset \overleftrightarrow{R_*}\text{“}\alpha$ (4)

$\vdash . (2) . (4) . \supset \vdash . \mathrm{Prop}$

***97·34.** $\vdash : R \in 1 \to 1 . \beta \in \mathrm{D}\text{“}\epsilon_\Delta\text{‘}\overleftrightarrow{R_*}\text{“}\alpha . \supset . \overleftrightarrow{R_*}\text{“}\alpha = \overleftrightarrow{R_*}\text{“}\beta$

Dem.

$\vdash . *83\cdot6\cdot62 . \quad \supset \vdash :. \mathrm{Hp} . \supset : x \in \alpha . \supset_x . \exists ! \beta \cap \overleftrightarrow{R_*}\text{‘}x : \beta \subset s\text{‘}\overleftrightarrow{R_*}\text{“}\alpha$ (1)

$\vdash . *40\cdot4 . *97\cdot101 . \supset \vdash :. x \in \alpha . \supset_x . \exists ! \beta \cap \overleftrightarrow{R_*}\text{‘}x . \equiv . \alpha \subset s\text{‘}\overleftrightarrow{R_*}\text{“}\beta$ (2)

$\vdash . (1) . (2) . *97\cdot33 . \supset \vdash . \mathrm{Prop}$

***97·341.** $\vdash : R \in 1 \to 1 . \beta \in \mathrm{D}\text{“}\epsilon_\Delta\text{‘}\overleftarrow{R_*}\text{“}\overrightarrow{B}\text{‘}R . \supset . \overleftrightarrow{R_*}\text{“}\beta = \overleftarrow{R_*}\text{“}\overrightarrow{B}\text{‘}R$

$\left[*97\cdot34 \dfrac{\overrightarrow{B}\text{‘}R}{\alpha} . *97\cdot2\right]$

***97·35.** $\vdash : R \in \mathrm{Cls} \to 1 . T \in \mathrm{Potid}\text{‘}R . \overrightarrow{B}\text{‘}R \subset \mathrm{D}\text{‘}T . \supset .$

$\mathrm{Cnv}\text{‘}\{(\overleftarrow{R_*} \upharpoonright \overrightarrow{B}\text{‘}R) \mid T\} \in \epsilon_\Delta\text{‘}\overleftarrow{R_*}\text{“}\overrightarrow{B}\text{‘}R . \mathrm{Cl}\text{‘}\{(\overleftarrow{R_*} \upharpoonright \overrightarrow{B}\text{‘}R) \mid T\} = \breve{T}\text{“}\overrightarrow{B}\text{‘}R$

Dem.

$\vdash . *97\cdot3 . *92\cdot101 . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{Cnv}\text{‘}\{(\overleftarrow{R_*} \upharpoonright \overrightarrow{B}\text{‘}R) \mid T\} \in 1 \to \mathrm{Cls}$ (1)

$\vdash . *35\cdot101 . *30\cdot4 . \supset$

$\vdash : \alpha \{(\overleftarrow{R_*} \upharpoonright \overrightarrow{B}\text{‘}R) \mid T\} y . \equiv . (\exists x) . x \in \overrightarrow{B}\text{‘}R . \alpha = \overleftarrow{R_*}\text{‘}x . xTy$ (2)

$\vdash . *91\cdot58 . \quad \supset \vdash :. \text{Hp} . \supset : xTy . \supset . y \in \overleftarrow{R_*}\text{‘}x :$

$[*13\cdot12] \quad \supset : \alpha = \overleftarrow{R_*}\text{‘}x . xTy . \supset . y \in \alpha$ (3)

$\vdash . (2) . (3) . \quad \supset \vdash :. \text{Hp} . \supset : \alpha \{(\overleftarrow{R_*} \upharpoonright \overrightarrow{B}\text{‘}R) \mid T\} y . \supset_{\alpha, y} . y \in \alpha :$

$[*23\cdot1 . *31\cdot131] \quad \supset : \text{Cnv}\text{‘}\{(\overleftarrow{R_*} \upharpoonright \overrightarrow{B}\text{‘}R) \mid T\} \subseteq \epsilon$ (4)

$\vdash . *37\cdot321 . *35\cdot65 . \supset \vdash : \text{Hp} . \supset . \text{D}\text{‘}\{(\overleftarrow{R_*} \upharpoonright \overrightarrow{B}\text{‘}R) \mid T\} = \overleftarrow{R_*}\text{“}\overrightarrow{B}\text{‘}R$ (5)

$\vdash . (1) . (4) . (5) . *80\cdot14 . \supset \vdash : \text{Hp} . \supset . \text{Cnv}\text{‘}\{(\overleftarrow{R_*} \upharpoonright \overrightarrow{B}\text{‘}R) \mid T\} \in \epsilon_\Delta\text{‘}\overleftarrow{R_*}\text{“}\overrightarrow{B}\text{‘}R$ (6)

$\vdash . *35\cdot65 . \quad \supset \vdash . \text{Ɑ}\text{‘}(\overleftarrow{R_*} \upharpoonright \overrightarrow{B}\text{‘}R) = \overrightarrow{B}\text{‘}R .$

$[*37\cdot32] \quad \supset \vdash . \text{Ɑ}\text{‘}\{(\overleftarrow{R_*} \upharpoonright \overrightarrow{B}\text{‘}R) \mid T\} = \breve{T}\text{“}\overrightarrow{B}\text{‘}R$ (7)

$\vdash . (6) . (7) . \supset \vdash . \text{Prop}$

***97·36.** $\vdash : R \in \text{Cls} \rightarrow 1 . T \in \text{Potid}\text{‘}R . \overrightarrow{B}\text{‘}R \subset \text{D}\text{‘}T . \supset . \breve{T}\text{“}\overrightarrow{B}\text{‘}R \in \text{D}\text{“}\epsilon_\Delta\text{‘}\overleftarrow{R_*}\text{“}\overrightarrow{B}\text{‘}R$
[*97·35]

***97·37.** $\vdash : R \in 1 \rightarrow 1 . \text{Ɑ}\text{‘}R \subset \text{D}\text{‘}R . \supset . \text{gen}\text{‘}R \subset \text{D}\text{“}\epsilon_\Delta\text{‘}\overleftarrow{R_*}\text{“}\overrightarrow{B}\text{‘}R$

Dem.

$\vdash . *92\cdot14 . \supset \vdash :. \text{Hp} . \supset : T \in \text{Potid}\text{‘}R . \supset . \overrightarrow{B}\text{‘}R \subset \text{D}\text{‘}T$ (1)

$\vdash . *93\cdot32 . \supset \vdash :. \text{Hp} . \supset : \alpha \in \text{gen}\text{‘}R . \equiv . (\exists T) . T \in \text{Potid}\text{‘}R . \alpha = \breve{T}\text{“}\overrightarrow{B}\text{‘}R$ (2)

$\vdash . (1) . (2) . *97\cdot36 . \supset \vdash . \text{Prop}$

***97·38.** $\vdash : R \in 1 \rightarrow 1 . \text{Ɑ}\text{‘}R \subset \text{D}\text{‘}R . \supset . \overleftarrow{R_*}\text{“}\overrightarrow{B}\text{‘}R \subset \text{D}\text{“}\epsilon_\Delta\text{‘}\text{gen}\text{‘}R$

Dem.

$\vdash . *93\cdot36 . *40\cdot52 . \supset \vdash : \text{Hp} . \supset . s\text{‘}\overleftarrow{R_*}\text{“}\overrightarrow{B}\text{‘}R = s\text{‘}\text{gen}\text{‘}R$ (1)

$\vdash . (1) . *84\cdot43 . *97\cdot37 . *93\cdot25 . *97\cdot16\cdot21 . \supset \vdash . \text{Prop}$

***97·4.** $\vdash : S \in \text{Pot}\text{‘}R . \supset . \breve{S}\text{“}\overrightarrow{B}\text{‘}\breve{R} = \Lambda$

Dem.

$\vdash . *91\cdot31 . \supset \vdash : \text{Hp} . \supset . (\exists T) . T \in \text{Potid}\text{‘}R . S = R \mid T .$

$[*37\cdot341] \quad \supset . (\exists T) . T \in \text{Potid}\text{‘}R . \breve{S}\text{“}\overrightarrow{B}\text{‘}\breve{R} = \breve{T}\text{“}\breve{R}\text{“}\overrightarrow{B}\text{‘}\breve{R}$

$[*37\cdot261\cdot29 . *93\cdot101] \quad = \Lambda .$

$[*10\cdot35] \quad \supset . \breve{S}\text{“}\overrightarrow{B}\text{‘}\breve{R} = \Lambda : \supset \vdash . \text{Prop}$

***97·401.** $\vdash :. x \in \text{D}\text{‘}R : S \in \text{Pot}\text{‘}R . xSy . \supset_{S,y} . y \in \text{D}\text{‘}R : \supset : S \in \text{Pot}\text{‘}R . \supset_S . x \in \text{D}\text{‘}S$

Dem.

$\vdash . *33\cdot13 . \supset \vdash :. \text{Hp} . \supset : S \in \text{Pot}\text{‘}R . xSy . \supset_{S,y} . (\exists z) . yRz . xSy .$

$[*34\cdot1 . *33\cdot13] \quad \supset_{S,y} . x \in \text{D}\text{‘}(S \mid R) :$

$[*10\cdot28 . *33\cdot13] \quad \supset : S \in \text{Pot}\text{‘}R . x \in \text{D}\text{‘}S . \supset_S . x \in \text{D}\text{‘}(S \mid R)$ (1)

$\vdash . (1) . *91\cdot373 . \supset \vdash . \text{Prop}$

$*97{\cdot}402.\ \vdash :.\ R \in \text{Cls}\to 1\ .\ x \in \text{D}\text{‘}R : (\exists S)\ .\ S \in \text{Pot}\text{‘}R\ .\ x \sim\in \text{D}\text{‘}S : \supset .$

$(\exists S)\ .\ S \in \text{Pot}\text{‘}R\ .\ \breve{S}\text{‘}x \in \overrightarrow{B}\text{‘}\breve{R}$

Dem.

$\vdash .\ *97{\cdot}401\ .\ \text{Transp}\ .\ \supset \vdash : \text{Hp}\ .\ \supset .\ (\exists S, y)\ .\ S \in \text{Pot}\text{‘}R\ .\ xSy\ .\ y \sim\in \text{D}\text{‘}R\ .$

$[*91{\cdot}271.*33{\cdot}14.*93{\cdot}101]\quad \supset .\ (\exists S, y)\ .\ S \in \text{Pot}\text{‘}R\ .\ xSy\ .\ y \in \overrightarrow{B}\text{‘}\breve{R}\ .$

$[*71{\cdot}321]\quad \supset .\ (\exists S)\ .\ S \in \text{Pot}\text{‘}R\ .\ \breve{S}\text{‘}x \in \overrightarrow{B}\text{‘}\breve{R} : \supset \vdash .\ \text{Prop}$

$*97{\cdot}403.\ \vdash : R \in \text{Cls}\to 1\ .\ x \in \overrightarrow{B}\text{‘}R\ .\ T \in \text{Pot}\text{‘}R\ .\ \overrightarrow{B}\text{‘}\breve{R} = \breve{T}\text{‘‘}\overrightarrow{B}\text{‘}R\ .\ \supset .$

$(\exists S)\ .\ S \in \text{Pot}\text{‘}R\ .\ x \sim\in \text{D}\text{‘}S$

Dem. $\vdash .\ *92{\cdot}131\ .\ \supset \vdash :.\ \text{Hp}\ .\ \supset : xTy\ .\ xTz\ .\ zRw\ .\ \supset .\ yRw\ .$

$[*33{\cdot}14]\quad \supset .\ y \sim\in \overrightarrow{B}\text{‘}\breve{R}\qquad (1)$

$\vdash .\ (1)\ .\ *11{\cdot}11{\cdot}3{\cdot}35\ .\ \supset$

$\vdash :: \text{Hp}\ .\ \supset :.\ xTy : (\exists z, w)\ .\ xTz\ .\ zRw : \supset .\ y \sim\in \overrightarrow{B}\text{‘}\breve{R} :.$

$[*34{\cdot}1.*33{\cdot}13] \supset :.\ xTy\ .\ x \in \text{D}\text{‘}(T \mid R)\ .\ \supset .\ y \sim\in \overrightarrow{B}\text{‘}\breve{R} :.$

$[\text{Transp}]\quad \supset :.\ xTy\ .\ y \in \overrightarrow{B}\text{‘}\breve{R}\ .\ \supset .\ x \sim\in \text{D}\text{‘}(T \mid R)\qquad (2)$

$\vdash .\ *10{\cdot}24\ .\ \supset \vdash : \text{Hp}\ .\ x \sim\in \text{D}\text{‘}T\ .\ \supset .\ (\exists S)\ .\ S \in \text{Pot}\text{‘}R\ .\ x \sim\in \text{D}\text{‘}S\qquad (3)$

$\vdash .\ *37{\cdot}105\ .\ \supset \vdash : \text{Hp}\ .\ xTy\ .\ \supset .\ y \in \overrightarrow{B}\text{‘}\breve{R}\ .$

$[(2)]\quad \supset .\ x \sim\in \text{D}\text{‘}(T \mid R)\qquad (4)$

$\vdash .\ (4)\ .\ *10{\cdot}11{\cdot}23{\cdot}35\ .\ *33{\cdot}13\ .\ \supset$

$\vdash : \text{Hp}\ .\ x \in \text{D}\text{‘}T\ .\ \supset .\ x \sim\in \text{D}\text{‘}(T \mid R)\ .$

$[*91{\cdot}282]\quad \supset .\ (\exists S)\ .\ S \in \text{Pot}\text{‘}R\ .\ x \sim\in \text{D}\text{‘}S\qquad (5)$

$\vdash .\ (3)\ .\ (5)\ .\ \supset \vdash .\ \text{Prop}$

$*97{\cdot}41.\ \vdash : R \in \text{Cls}\to 1\ .\ x \in \overrightarrow{B}\text{‘}R\ .\ T \in \text{Pot}\text{‘}R\ .\ \overrightarrow{B}\text{‘}\breve{R} = \breve{T}\text{‘‘}\overrightarrow{B}\text{‘}R\ .\ \supset .$

$(\exists S)\ .\ S \in \text{Pot}\text{‘}R\ .\ \breve{S}\text{‘}x \in \overrightarrow{B}\text{‘}\breve{R}$

$[*97{\cdot}402{\cdot}403]$

$*97{\cdot}42.\ \vdash : R \in 1\to 1\ .\ x \in \overrightarrow{B}\text{‘}R\ .\ S, T \in \text{Pot}\text{‘}R\ .\ \overrightarrow{B}\text{‘}\breve{R} = \breve{T}\text{‘‘}\overrightarrow{B}\text{‘}R\ .\ \breve{S}\text{‘}x \in \overrightarrow{B}\text{‘}\breve{R}\ .\ \supset .\ S = T$

Dem. $\vdash .\ *37{\cdot}6\ .\quad \supset \vdash : \text{Hp}\ .\ \supset .\ (\exists y)\ .\ y \in \overrightarrow{B}\text{‘}R\ .\ \breve{S}\text{‘}x = \breve{T}\text{‘}y\qquad (1)$

$\vdash .\ *37{\cdot}62\ .\ (1)\ .\ \supset \vdash : \text{Hp}\ .\ \supset .\ \breve{S}\text{‘}x \in \breve{S}\text{‘‘}\overrightarrow{B}\text{‘}R \cap \breve{T}\text{‘‘}\overrightarrow{B}\text{‘}R\ .$

$[*93{\cdot}3]\quad \supset .\ \breve{S}\text{‘}x \in \text{min}_R\text{‘}\overrightarrow{\text{Ɑ}}\text{‘}S \cap \text{min}_R\text{‘}\overrightarrow{\text{Ɑ}}\text{‘}T\ .$

$[*93{\cdot}24.\text{Transp}]\quad \supset .\ S = T : \supset \vdash .\ \text{Prop}$

$*97{\cdot}43.\ \vdash : R \in 1\to 1\ .\ T \in \text{Pot}\text{‘}R\ .\ \overrightarrow{B}\text{‘}\breve{R} = \breve{T}\text{‘‘}\overrightarrow{B}\text{‘}R\ .\ \supset .\ \overrightarrow{B}\text{‘}R \subset \text{D}\text{‘}T$

Dem. $\vdash .\ *97{\cdot}42\ .\ \supset$

$\vdash :.\ \text{Hp}\ .\ x \in \overrightarrow{B}\text{‘}R\ .\ \supset : S \in \text{Pot}\text{‘}R\ .\ \breve{S}\text{‘}x \in \overrightarrow{B}\text{‘}\breve{R}\ .\ \supset .\ \breve{T}\text{‘}x \in \overrightarrow{B}\text{‘}\breve{R} :$

$[*10{\cdot}11{\cdot}21{\cdot}23]\quad \supset : (\exists S)\ .\ S \in \text{Pot}\text{‘}R\ .\ \breve{S}\text{‘}x \in \overrightarrow{B}\text{‘}\breve{R}\ .\ \supset .\ \breve{T}\text{‘}x \in \overrightarrow{B}\text{‘}\breve{R} :$

$[*97{\cdot}41]\quad \supset : \breve{T}\text{‘}x \in \overrightarrow{B}\text{‘}\breve{R} :$

$[*14{\cdot}21]\quad \supset : \text{E}\,!\,\breve{T}\text{‘}x :$

$[*33{\cdot}44]\quad \supset : x \in \text{D}\text{‘}T :.\ \supset \vdash .\ \text{Prop}$

***97·44.** $\vdash : R \in 1 \to 1 \,.\, S, T \in \text{Pot}\text{‘}R \,.\, \vec{B}\text{‘}\breve{R} = \breve{T}\text{“}\vec{B}\text{‘}R \,.\, \exists ! \breve{S}\text{“}\vec{B}\text{‘}R \,.\supset .\, \vec{B}\text{‘}R \subset \text{D}\text{‘}S$

Dem.

$\vdash . *91{\cdot}45 . \supset \vdash :. \text{Hp} . \supset : (\exists U) : U \in \text{Potid}\text{‘}R : S = U \,|\, T \,.\vee.\, T = U \,|\, S \qquad (1)$

$\vdash . *97{\cdot}4 . \quad \supset \vdash :. \text{Hp} . \supset : U \in \text{Pot}\text{‘}R \,.\, S = U \,|\, T \,.\supset .\, \breve{S}\text{“}\vec{B}\text{‘}R = \Lambda :$

$[*91{\cdot}23] \qquad \supset : U \in \text{Potid}\text{‘}R \,.\, S = U \,|\, T \,.\, \exists ! \breve{S}\text{“}\vec{B}\text{‘}R \,.\supset .\, U = I \upharpoonright C\text{‘}R \,.$

$[*50{\cdot}63 . *91{\cdot}271] \qquad \supset . S = T .$

$[*97{\cdot}43] \qquad \supset . \vec{B}\text{‘}R \subset \text{D}\text{‘}S \qquad (2)$

$\vdash . *91{\cdot}34 . \supset \vdash :. \text{Hp} . \supset : U \in \text{Potid}\text{‘}R \,.\, T = U \,|\, S \,.\supset .\, T = S \,|\, U .$

$[*34{\cdot}36] \qquad \supset . \text{D}\text{‘}T \subset \text{D}\text{‘}S .$

$[*97{\cdot}43] \qquad \supset . \vec{B}\text{‘}R \subset \text{D}\text{‘}S \qquad (3)$

$\vdash . (1) . (2) . (3) . \supset \vdash . \text{Prop}$

***97·45.** $\vdash : R \in 1 \to 1 \,.\, \vec{B}\text{‘}\breve{R} \in \text{gen}\text{‘}R \,.\supset .\, \text{gen}\text{‘}R - \iota\text{‘}\Lambda \subset \text{D}\text{“}\epsilon_\Delta\text{‘}\overleftarrow{R}_*\text{“}\vec{B}\text{‘}R$

Dem.

$\vdash . *97{\cdot}44 . *10{\cdot}11{\cdot}23{\cdot}35 . *93{\cdot}32 . \supset$

$\vdash : R \in 1 \to 1 \,.\, \vec{B}\text{‘}\breve{R} \in \text{gen}\text{‘}R \,.\, S \in \text{Pot}\text{‘}R \,.\, \exists ! \breve{S}\text{“}\vec{B}\text{‘}R \,.\supset .\, \vec{B}\text{‘}R \subset \text{D}\text{‘}S .$

$[*97{\cdot}36] \qquad \supset . \breve{S}\text{“}\vec{B}\text{‘}R \in \text{D}\text{“}\epsilon_\Delta\text{‘}\overleftarrow{R}_*\text{“}\vec{B}\text{‘}R \qquad (1)$

$\vdash . (1) . *13{\cdot}12 . \supset$

$\vdash : R \in 1 \to 1 \,.\, \vec{B}\text{‘}\breve{R} \in \text{gen}\text{‘}R \,.\, S \in \text{Pot}\text{‘}R \,.\, \alpha = \breve{S}\text{“}\vec{B}\text{‘}R \,.\, \exists ! \alpha .$

$\qquad \supset . \alpha \in \text{D}\text{“}\epsilon_\Delta\text{‘}\overleftarrow{R}_*\text{“}\vec{B}\text{‘}R \qquad (2)$

$\vdash . (2) . *10{\cdot}11{\cdot}23{\cdot}35 . *93{\cdot}32 . \supset$

$\vdash : R \in 1 \to 1 \,.\, \vec{B}\text{‘}\breve{R} \in \text{gen}\text{‘}R \,.\, \alpha \in \text{gen}\text{‘}R \,.\, \exists ! \alpha \,.\supset .\, \alpha \in \text{D}\text{“}\epsilon_\Delta\text{‘}\overleftarrow{R}_*\text{“}\vec{B}\text{‘}R \qquad (3)$

$\vdash . (3) . *53{\cdot}52 . \supset \vdash . \text{Prop}$

***97·46.** $\vdash : R \in 1 \to 1 \,.\, \vec{B}\text{‘}\breve{R} \in \text{gen}\text{‘}R \,.\supset .\, \overleftarrow{R}_*\text{“}\vec{B}\text{‘}R \subset \text{D}\text{“}\epsilon_\Delta\text{‘}(\text{gen}\text{‘}R - \iota\text{‘}\Lambda)$

Dem.

$\vdash . *93{\cdot}36 . *40{\cdot}52 . \supset \vdash : \text{Hp} . \supset . s\text{‘}\overleftarrow{R}_*\text{“}\vec{B}\text{‘}R = s\text{‘}\text{gen}\text{‘}R$

$[*53{\cdot}18] \qquad = s\text{‘}(\text{gen}\text{‘}R - \iota\text{‘}\Lambda) \qquad (1)$

$\vdash . (1) . *84{\cdot}43 . *97{\cdot}45{\cdot}16{\cdot}21 . *93{\cdot}25 . \supset \vdash . \text{Prop}$

***97·47.** $\vdash : R \in 1 \to 1 \,.\, \vec{B}\text{‘}\breve{R} \in \text{gen}\text{‘}R \cup \iota\text{‘}\Lambda \,.\supset .$

$\qquad \text{gen}\text{‘}R - \iota\text{‘}\Lambda \subset \text{D}\text{“}\epsilon_\Delta\text{‘}\overleftarrow{R}_*\text{“}\vec{B}\text{‘}R \,.\, \overleftarrow{R}_*\text{“}\vec{B}\text{‘}R \subset \text{D}\text{“}\epsilon_\Delta\text{‘}(\text{gen}\text{‘}R - \iota\text{‘}\Lambda)$

Dem.

$\vdash . *93{\cdot}32 . \supset \vdash : \vec{B}\text{‘}R = \Lambda \,.\supset .\, \text{gen}\text{‘}R = \iota\text{‘}\Lambda \qquad (1)$

$\vdash . (1) . *37{\cdot}29 . \supset \vdash : \vec{B}\text{‘}R = \Lambda \,.\supset .\, \text{gen}\text{‘}R - \iota\text{‘}\Lambda = \Lambda \,.\, \overleftarrow{R}_*\text{“}\vec{B}\text{‘}R = \Lambda .$

$[*24{\cdot}12] \qquad \supset . \text{gen}\text{‘}R - \iota\text{‘}\Lambda \subset \text{D}\text{“}\epsilon_\Delta\text{‘}\overleftarrow{R}_*\text{“}\vec{B}\text{‘}R .$

$\qquad \overleftarrow{R}_*\text{“}\vec{B}\text{‘}R \subset \text{D}\text{“}\epsilon_\Delta\text{‘}(\text{gen}\text{‘}R - \iota\text{‘}\Lambda) \qquad (2)$

$\vdash . *24\cdot3 . \text{Fact} . \supset$

$\vdash : R \epsilon 1 \to 1 . \exists ! \overrightarrow{B}\text{‘}R . \overrightarrow{B}\text{‘}\breve{R} = \Lambda . \supset . R \epsilon 1 \to 1 . \exists ! \overrightarrow{B}\text{‘}R . \text{C}\text{‘}R \subset \text{D}\text{‘}R .$ (3)

[$*93\cdot41$] $\supset . \Lambda \sim\epsilon \text{gen}\text{‘}R .$

[$*51\cdot222$] $\supset . \text{gen}\text{‘}R - \iota\text{‘}\Lambda = \text{gen}\text{‘}R .$

[(3).$*97\cdot37\cdot38$] $\supset . \text{gen}\text{‘}R - \iota\text{‘}\Lambda \subset \text{D}\text{“}\epsilon_\Delta\text{‘}\overleftarrow{R_*}\text{“}\overrightarrow{B}\text{‘}R .$

$\overleftarrow{R_*}\text{“}\overrightarrow{B}\text{‘}R \subset \text{D}\text{“}\epsilon_\Delta\text{‘}(\text{gen}\text{‘}R - \iota\text{‘}\Lambda)$ (4)

$\vdash . (2) . (4) . \supset \vdash : R \epsilon 1 \to 1 . \overrightarrow{B}\text{‘}\breve{R} = \Lambda . \supset . \text{gen}\text{‘}R - \iota\text{‘}\Lambda \subset \text{D}\text{“}\epsilon_\Delta\text{‘}\overleftarrow{R_*}\text{“}\overrightarrow{B}\text{‘}R .$

$\overleftarrow{R_*}\text{“}\overrightarrow{B}\text{‘}R \subset \text{D}\text{“}\epsilon_\Delta\text{‘}(\text{gen}\text{‘}R - \iota\text{‘}\Lambda)$ (5)

$\vdash . *97\cdot45\cdot46 . \supset$

$\vdash : R \epsilon 1 \to 1 . \overrightarrow{B}\text{‘}\breve{R} \epsilon \text{gen}\text{‘}R . \supset . \text{gen}\text{‘}R - \iota\text{‘}\Lambda \subset \text{D}\text{“}\epsilon_\Delta\text{‘}\overleftarrow{R_*}\text{“}\overrightarrow{B}\text{‘}R .$

$\overleftarrow{R_*}\text{“}\overrightarrow{B}\text{‘}R \subset \text{D}\text{“}\epsilon_\Delta\text{‘}(\text{gen}\text{‘}R - \iota\text{‘}\Lambda)$ (6)

$\vdash . (5) . (6) . \supset \vdash . \text{Prop}$

$*97\cdot5$. $\vdash : R \epsilon \text{Cls} \to 1 . xR_{po}x . xR_{po}y . \supset . yR_{po}x$

Dem.

$\vdash . *92\cdot111 . \quad \supset \vdash :. R \epsilon \text{Cls} \to 1 . xR_{po}x . xRy . \supset : yR_*x :$

[$*91\cdot54$] $\supset : y = x . \vee . yR_{po}x :$

[Hp] $\supset : yR_{po}x$ (1)

$\vdash . *10\cdot1 . *34\cdot1 . \supset \vdash :. R \epsilon \text{Cls} \to 1 . xR_{po}x . P \epsilon \text{Pot}\text{‘}R :$

$xPy . \supset_y . yR_{po}x : xP \mid Rz : \supset : (\exists y) . yR_{po}x . yRz :$

[$*92\cdot111$] $\supset : zR_*x :$

[$*91\cdot54$] $\supset : z = x . \vee . zR_{po}x :$

[Hp] $\supset : zR_{po}x$ (2)

$\vdash . (1) . (2) . *91\cdot171 . \supset \vdash : R \epsilon \text{Cls} \to 1 . xR_{po}x . P \epsilon \text{Pot}\text{‘}R . xPy . \supset . yR_{po}x :$

[($*91\cdot05$)] $\supset \vdash : R \epsilon \text{Cls} \to 1 . xR_{po}x . xR_{po}y . \supset . yR_{po}x : \supset \vdash . \text{Prop}$

$*97\cdot501$. $\vdash : R \epsilon 1 \to \text{Cls} . xR_{po}x . yR_{po}x . \supset . xR_{po}y$ [Proof as in $*97\cdot5$]

$*97\cdot51$. $\vdash : R \epsilon 1 \to 1 . xR_{po}x . \supset . \overleftrightarrow{R_*}\text{‘}x = \overrightarrow{R_*}\text{‘}x = \overleftarrow{R_*}\text{‘}x = \overleftarrow{R_*}\text{‘}x \cap \overrightarrow{R_*}\text{‘}x$
[$*97\cdot5\cdot501\cdot17$]

$*97\cdot52$. $\vdash : R \epsilon 1 \to 1 . xR_{po}x . xR_{po}y . \supset . \overleftrightarrow{R_*}\text{‘}x = \overleftarrow{R_*}\text{‘}x \cap \overrightarrow{R_*}\text{‘}y$ [$*97\cdot5\cdot501\cdot51\cdot14$]

$*97\cdot53$. $\vdash : R \epsilon 1 \to 1 . P \epsilon \text{Pot}\text{‘}R . xPx . y \epsilon \overleftrightarrow{R_*}\text{‘}x . \supset . yPy$ [$*92\cdot132\cdot133$]

$*97\cdot54$. $\vdash : R \epsilon 1 \to 1 . xR_{po}x . \supset . (\exists P) . P \epsilon \text{Pot}\text{‘}R . P \upharpoonright \overleftrightarrow{R_*}\text{‘}x = I \upharpoonright \overleftrightarrow{R_*}\text{‘}x$
[$*97\cdot53$]

$*97\cdot55$. $\vdash :: R \epsilon 1 \to 1 . \supset :. y \epsilon \overleftrightarrow{R_*}\text{‘}x . \supset_y . yR_{po}y : \vee : y \epsilon \overleftrightarrow{R_*}\text{‘}x . \supset_y . \sim(yR_{po}y)$

Dem.

$\vdash . *97\cdot53 . \quad \supset \vdash :. \text{Hp} . xR_{po}x . \supset : y \epsilon \overleftrightarrow{R_*}\text{‘}x . \supset_y . yR_{po}y$ (1)

$\vdash . (1) \dfrac{y, x}{x, y} . \text{Transp} . \supset \vdash : \text{Hp} . \sim(xR_{po}x) . x \epsilon \overleftrightarrow{R_*}\text{‘}y . \supset . \sim(yR_{po}y)$ (2)

$\vdash . (2) . *97\cdot101 . \quad \supset \vdash :. \text{Hp} . \sim(xR_{po}x) . \supset : y \epsilon \overleftrightarrow{R_*}\text{‘}x . \supset_y . \sim(yR_{po}y)$ (3)

$\vdash . (1) . (3) . \supset \vdash . \text{Prop}$

***97·56.** $\vdash :. R \epsilon 1 \rightarrow 1 . x \epsilon \overrightarrow{B}`R . \supset : y \epsilon \overleftrightarrow{R_*}`x . \supset_y . \sim (y R_{po} y)$
[*96·23·1 . *97·55]

***97·57.** $\vdash :. R \epsilon 1 \rightarrow 1 . x \epsilon s`\text{gen}`R . \supset : y \epsilon \overleftrightarrow{R_*}`x . \supset_y . \sim (y R_{po} y)$
[*97·21·56]

***97·58.** $\vdash :. R \epsilon 1 \rightarrow \text{Cls} . \supset : x \epsilon s`\text{gen}`R . \supset . \overleftrightarrow{R_*}`x \subset s`\text{gen}`R :$
$x \epsilon p`\text{Ꮖ}``\text{Pot}`R . \supset . \overleftrightarrow{R_*}`x \subset p`\text{Ꮖ}``\text{Pot}`R$

Dem.

$\vdash . *93·412 . \quad \supset \vdash . \breve{R}``p`\text{Ꮖ}``\text{Pot}`R \subset p`\text{Ꮖ}``\text{Pot}`R$ (1)

$[*90·101 . *93·273 . *37·265] \supset \vdash . R``s`\text{gen}`R \subset s`\text{gen}`R$ (2)

$\vdash . *93·33 . *40·13·38·43 . \quad \supset \vdash : R \epsilon 1 \rightarrow \text{Cls} . \supset . \breve{R}``s`\text{gen}`R \subset s`\text{gen}`R .$ (3)

$[*90·101 . *93·271 . *37·265] \quad \supset . R``p`\text{Ꮖ}``\text{Pot}`R \subset p`\text{Ꮖ}``\text{Pot}`R$ (4)

$\vdash . (1) . (2) . (3) . (4) . *90·22 . *40·5·52 . \supset$

$\vdash : R \epsilon 1 \rightarrow \text{Cls} . \supset . s`\overleftrightarrow{R_*}``s`\text{gen}`R \subset s`\text{gen}`R .$
$s`\overleftrightarrow{R_*}``p`\text{Ꮖ}``\text{Pot}`R \subset p`\text{Ꮖ}``\text{Pot}`R : \supset \vdash . \text{Prop}$

It follows from this proposition that every family is either wholly contained in the generations of R or wholly contained in $p`\text{Ꮖ}``\text{Pot}`R$, which may be called the *residue* of the field of R.

APPENDIX A

$*8$. THE THEORY OF DEDUCTION FOR PROPOSITIONS CONTAINING APPARENT VARIABLES*

ALL propositions, of whatever order, are derived from a matrix composed of elementary propositions combined by means of the stroke. Given such a matrix, any constituent may be left constant or turned into an apparent variable; the latter may be done in two ways, by taking "all values" or "some values." Thus, if p and q are elementary propositions, giving rise to $p \mid q$, we may replace p by ϕx or q by ψy or both, where ϕx, ψy are propositional functions whose values are elementary propositions. We thus arrive, to begin with, at four new propositions:

$$(x) \,.\, (\phi x \mid q), \quad (\exists x) \,.\, (\phi x \mid q), \quad (y) \,.\, (p \mid \psi y), \quad (\exists y) \,.\, (p \mid \psi y).$$

By means of definitions, we can separate out the constant and the variable part in these expressions; we put

$*8{\cdot}01$. $\{(x) \,.\, \phi x\} \mid q \,. = .\, (\exists x) \,.\, (\phi x \mid q)$ Df

$*8{\cdot}011$. $\{(\exists x) \,.\, \phi x\} \mid q \,. = .\, (x) \,.\, (\phi x \mid q)$ Df

$*8{\cdot}012$. $p \mid \{(y) \,.\, \psi y\} \,. = .\, (\exists y) \,.\, (p \mid \psi y)$ Df

$*8{\cdot}013$. $p \mid \{(\exists y) \,.\, \psi y\} \,. = .\, (y) \,.\, (p \mid \psi y)$ Df

These definitions define the meaning of the stroke when it occurs between two propositions of which one is elementary while the other is of the first order.

When the stroke occurs between two propositions which are both of the first order, we shall adopt the rule that the above definitions are to be applied first to the one on the left, treating the one on the right as if it were elementary, and are then to be applied to the one on the right. Thus

$$\begin{aligned} \{(x) \,.\, \phi x\} \mid \{(y) \,.\, \psi y\} \,. &= : (\exists x) : \phi x \mid \{(y) \,.\, \psi y\} : \\ &= : (\exists x) : (\exists y) \,.\, (\phi x \mid \psi y). \end{aligned}$$

The same rule can be applied to n propositions; they are to be eliminated from left to right. If a proposition occurs more than once, its occurrences must be eliminated successively as if they were different propositions. These rules are only required for the sake of definiteness, as different orders of elimination give equivalent results. This is only true because we are dealing with various functions each containing one variable, and no variable occurs on both sides of the stroke; it would not be true if we were dealing with functions of several variables. We have *e.g.*

$$(\exists x) : (y) \,.\, (\phi x \mid \psi y) : \equiv : (y) : (\exists x) \,.\, (\phi x \mid \psi y).$$

* This chapter is to replace $*9$ of the text.

But we do not have in general

$$(\exists x):(y).\chi(x,y):\equiv:(y):(\exists x).\chi(x,y);$$

here the right-hand side is more likely to be true than the left-hand side. For the present, however, we are not concerned with variable functions of two variables.

It should be observed that this possibility of changing the order of the variables is a merit of the stroke. We have

$$(\exists x):(y).\phi x|\psi y:\equiv:(y):(\exists x).\phi x|\psi y:\equiv:(\exists x).\sim\phi x.\vee.(y).\sim\psi y.$$

That is, these equivalent propositions are true when, and only when, either ϕ is sometimes false or ψ is always false. But if we take *e.g.*

$$\phi x \vee \psi y . \sim\phi x \vee \sim\psi y$$

we shall not get the same result. For

$$(\exists x):(y).\phi x \vee \psi y . \sim\phi x \vee \sim\psi y:\supset:(y).\psi y.\vee.(y).\sim\psi y,$$

whereas $(y):(\exists x).\phi x \vee \psi y . \sim\phi x \vee \sim\psi y$ does not imply this.

Written in stroke notation, after some reduction, the above matrix is

$$\{\phi x|(\psi y|\psi y)\}|\{\psi y|(\phi x|\phi x)\}.$$

Here both x and y occur on both sides of the principal matrix. Thus in order to be able to change the order of "$(\exists x)$" and "(y)," it is sufficient (though not *always* necessary) that the matrix should contain some part of the form $\phi x|\psi y$, and that x and y should not occur in any other part of the matrix. (This part may of course be the whole matrix.) We assume the legitimacy of this interchange by a primitive proposition, and in practice arrange to have all the $\exists$-prefixes as far to the right as possible, because this facilitates proofs.

Our primitive propositions are the following:

***8·1.** $\vdash.(\exists x,y).\phi a|(\phi x|\phi y)$ Pp

On applying the definitions, this is seen to be

$$\vdash:\phi a.\supset.(\exists x).\phi x.$$

***8·11.** $\vdash.(\exists x).\phi x|(\phi a|\phi b)$ Pp

On applying the definitions, this becomes

$$\vdash:(x).\phi x.\supset.\phi a.\phi b.$$

We have $\qquad \phi a|(\phi a|\phi b).\vee.\phi b|(\phi a|\phi b)$

and by *8·1 $\qquad \vdash:\phi a|(\phi a|\phi b).\supset.(\exists x).\phi x|(\phi a|\phi b):$

$\qquad \phi b|(\phi a|\phi b).\supset.(\exists x).\phi x|(\phi a|\phi b),$

but we cannot deduce $(\exists x).\phi x|(\phi a|\phi b)$ without *8·11 or an equivalent.

***8·12.** From "$(x).\phi x$" and "$(x).\phi x\supset\psi x$" we can infer "$(x).\psi x$," even when ϕ and ψ are not elementary. Pp

***8·13.** If all occurrences of x are separated from all occurrences of y by a certain stroke, we can change the order of x and y in the prefix, *i.e.* we can replace "$(y):(\exists x).\phi x|\psi y$" by "$(\exists x):(y).\phi x|\psi y$" and *vice versa*. Pp

The above primitive propositions are to be assumed, not only for one or two variables, but for any number. Thus *e.g.* *8·1 allows us to assert

$$\vdash : \phi(a_1, a_2, \ldots a_n) \,.\supset.\, (\exists x_1, x_2, \ldots x_n) \,.\, \phi(x_1, x_2, \ldots x_n).$$

***8·2.** $\vdash : (x) \,.\, \phi x \,.\supset.\, \phi a \quad \left[*8{\cdot}11 \frac{a}{b}\right]$

In what follows, the method of proof is invariably the same. We first apply the definitions until the whole asserted proposition is brought into the form of a matrix with a prefix. If necessary, we apply *8·13 to change the order of the variables in the prefix. When the proposition to be proved has been brought into this form, we deduce it by means of *8·1·11, using *8·12 in the deduction if necessary. It will be observed that *8·1 is $\vdash : \phi a \,.\supset.\, (\exists x) \,.\, \phi x$. Hence, by *8·12, whenever we know ϕa, we can assert $(\exists x) \,.\, \phi x$; *8·1 is often used in this way.

***8·21.** $\vdash :. (x) \,.\, \phi x \supset \psi x \,.\supset : (\exists x) \,.\, \phi x \,.\supset.\, (\exists x) \,.\, \psi x$

Dem.

Applying the definitions, and using *8·13, the proposition to be proved becomes

$$(y, y') : (\exists x, z, w, z', w') \,.\, \{\phi x \mid (\psi x \mid \psi x)\} \mid [\{\phi y \mid (\psi z \mid \psi w)\} \mid \{\phi y' \mid (\psi z' \mid \psi w')\}].$$

Putting $z = w = z' = w' = x$, the above becomes

$$(y, y') : (\exists x) \,.\, \{\phi x \mid (\psi x \mid \psi x)\} \mid [\{\phi y \mid (\psi x \mid \psi x)\} \mid \{\phi y' \mid (\psi x \mid \psi x)\}].$$

By *8·1, the proposition to be proved is true if this is true. But this is true by *8·11, putting y, y' for a, b and $\phi y \mid (\psi x \mid \psi x)$ for ϕa. Hence the proposition is true.

***8·22.** $\vdash : \phi a \vee \phi b \,.\supset.\, (\exists x) \,.\, \phi x$

Dem.

$\vdash \,.\, *8{\cdot}11 \,.\supset \vdash \,.\, (\exists z) \,.\, (\sim \phi z) \mid (\sim \phi a \mid \sim \phi b)$		(1)
Transp.	$\supset \vdash : (\sim \phi z) \mid (\sim \phi a \mid \sim \phi b) \,.\supset.\, (\phi a \vee \phi b) \mid (\phi z \mid \phi z)$	(2)
$\vdash \,.\, (1) \,.\, (2) \,.\, *8{\cdot}21 \,.\supset \vdash \,.\, (\exists z) \,.\, (\phi a \vee \phi b) \mid (\phi z \mid \phi z)$		(3)
$\vdash \,.\, (3) \,.\, *8{\cdot}1{\cdot}21 \,.$	$\supset \vdash \,.\, (\exists z, w) \,.\, (\phi a \vee \phi b) \mid (\phi z \mid \phi w) \,.$	
[(*8·012·013)]	$\supset \vdash : \phi a \vee \phi b \,.\supset.\, (\exists x) \,.\, \phi x : \supset \vdash .$ Prop	

These propositions, as well as all the others in *8, apply to any number of variables, since the primitive propositions do so.

***8·23.** $\vdash : (\exists x) \,.\, \phi x \vee \phi c \,.\supset.\, (\exists x) \,.\, \phi x$

Dem.

Applying the definitions, this proposition is

$$(x) : (\exists y, z) \,.\, (\phi x \vee \phi c) \mid (\phi y \mid \phi z),$$

i.e.

$$(x) : \phi x \vee \phi c \,.\supset.\, (\exists x) \,.\, \phi x,$$

which follows from *8·22.

The following propositions are concerned with forms of the syllogism.

***8·24.** $\vdash :: p \supset q . \supset :. q . \supset . (\exists x) . \phi x : \supset : p . \supset . (\exists x) . \phi x$

Dem.

Applying the definitions, we obtain a matrix

$(p \supset q) \mid [\{(q \mid (\phi x \mid \phi y)) \mid (p \mid (\phi z \mid \phi w) \mid p \mid (\phi u \mid \phi v))\} \mid$ {the same with accented letters}]

with a prefix

$$(x, y, x', y') : (\exists z, w, u, v, z', w', u', v').$$

By *8·1, this will be true if it is true for chosen values of $z, w, u, v, z', w', u', v'$. Put $z = u = x . w = v = y . z' = u' = x' . w' = v' = y'$. Then what has to be proved becomes

$p \supset q . \supset :. q . \supset . \phi x . \phi y : \supset : p . \supset . \phi x . \phi y :. q . \supset . \phi x' . \phi y' : \supset : p . \supset . \phi x' . \phi y'$,

which is true by Syll. Hence the proposition follows.

***8·241.** $\vdash :: (x) . \phi x . \supset . p : \supset :. p \supset q . \supset : (x) . \phi x . \supset . q$

Putting $f(y, z) . = . \{p \mid (q \mid q)\} \mid [\{\phi y \mid (q \mid q)\} \mid \{\phi z \mid (q \mid q)\}]$,

the matrix of the proposition to be proved is

$$\{\phi x \mid (p \mid p)\} \mid \{f(y, z) \mid f(y', z')\}$$

and the prefix is $(x) : (\exists y, z, y', z')$. Putting $y = z = y' = z' = x$, the matrix reduces to $\phi x \supset p . \supset : p \supset q . \supset . \phi x \supset q$, which is true by Syll. Hence the proposition is true by *8·1.

***8·25.** $\vdash :: p . \supset . (\exists x) . \phi x : \supset :. (\exists x) . \phi x . \supset . (\exists x) . \psi x : \supset : p . \supset . (\exists x) . \psi x$

Dem.

Put $f(x, y, z, u, v, m, n) . = . \{\phi x \mid (\psi y \mid \psi z)\} \mid [\{p \mid (\psi u \mid \psi v)\} \mid \{p \mid (\psi m \mid \psi n)\}]$. Then the proposition to be proved, on applying the definitions, is found to have a matrix

$$\{p \mid (\phi a \mid \phi b)\} \mid \{f(x, y, z, u, v, m, n) \mid f(x', y', z', u', v', m', n')\}$$

with the prefix

$$(a, b, y, z, y', z') : (\exists x, u, v, m, n, x', u', v', m', n').$$

Put $x = a . x' = b . u = v = y . m = n = z . u' = v' = y' . m' = n' = z'$.

Then the matrix reduces to

$$p . \supset . \phi a . \phi b : \supset :. \phi a . \supset . \psi y . \psi z : \supset : p . \supset . \psi y . \psi z :.$$
$$\phi b . \supset . \psi y' . \psi z' : \supset : p . \supset . \psi y' . \psi z',$$

which is true by Syll. Hence our proposition results by repeated applications of *8·1·13.

Analogous proofs apply to other forms of the syllogism.

***8·26.** $\vdash : \phi a \vee \phi b \vee \phi c . \supset . (\exists x) . \phi x \vee \phi c$

Dem.

$\vdash : \phi a \vee \phi b \vee \phi c . \supset . (\phi a \vee \phi c) \vee (\phi b \vee \phi c)$ (1)

$\vdash . *8·22 . \supset \vdash : (\phi a \vee \phi c) \vee (\phi b \vee \phi c) . \supset . (\exists x) . \phi x \vee \phi c$ (2)

$\vdash . (1) . (2) . *8·24 . \supset \vdash . \text{Prop}$

***8·261.** $\vdash : \phi a \vee \phi b \vee \phi c \,.\supset.\, (\exists x) \,.\, \phi x$
[*8·25·26·23]

It is obvious that we can prove in like manner

$$\phi a \vee \phi b \vee \phi c \vee \phi d \,.\supset.\, (\exists x) \,.\, \phi x$$

and so on.

***8·27.** $\vdash :: q \,.\supset.\, (\exists x) \,.\, \phi x : \supset :. p \supset q \,.\supset : p \,.\supset.\, (\exists x) \,.\, \phi x$

Dem.

Put $\quad f(x, y, u, v) \,.=.\, \{p \mid (\phi x \mid \phi y)\} \mid \{p \mid (\phi u \mid \phi v)\}.$

Then the matrix is

$$\{q \mid (\phi a \mid \phi b)\} \mid [\{(p \supset q) \mid f(x, y, u, v)\} \mid \{(p \supset q) \mid f(x', y', u', v')\}]$$

and the prefix is $\quad (a, b) : (\exists x, y, u, v, x', y', u', v').$

Putting $x = u = x' = u' = a \,.\, y = v = y' = v' = b$, the matrix becomes

$$q \,.\supset.\, \phi a \,.\, \phi b : \supset :. p \supset q \,.\supset : p \,.\supset.\, \phi a \,.\, \phi b,$$

which is true. Hence the proposition.

***8·271.** $\vdash :: q \,.\supset.\, (\exists x, y) \,.\, \phi(x, y) : \supset :. p \supset q \,.\supset : p \,.\supset.\, (\exists x, y) \,.\, \phi(x, y)$
[Proof as in *8·27]

It is obvious that we can prove similarly the analogous proposition with $\phi(x_1, x_2, \ldots x_n)$ in place of $\phi(x, y)$.

***8·272.** $\vdash ::. p \,.\supset : q \,.\supset.\, (\exists x) \,.\, \phi x :. \supset :: r \supset p \,.\supset :. r \,.\supset : q \,.\supset.\, (\exists x) \,.\, \phi x$

Dem.

$q \,.\supset.\, (\exists x) \,.\, \phi x$ is $(\exists x, y) \,.\, q \mid (\phi x \mid \phi y)$. Hence the proposition results from *8·271 by the substitution of p for q, r for p, and $q \mid (\phi x \mid \phi y)$ for $\phi(x, y)$.

***8·28.** $\vdash :: p \,.\supset.\, (\exists x) \,.\, \phi x : \supset :. q \,.\supset.\, (\exists x) \,.\, \phi x : \supset : p \vee q \,.\supset.\, (\exists x) \,.\, \phi x$

Dem.

Put $\quad f(x, y, z, w) \,.=.\, \{(p \vee q) \mid (\phi x \mid \phi y)\} \mid \{(p \vee q) \mid (\phi z \mid \phi w)\}.$

Then the matrix is

$$\{p \mid (\phi a \mid \phi b)\} \mid [\{(q \mid (\phi c \mid \phi d)) \mid f(x, y, z, w)\} \mid \{(q \mid (\phi c' \mid \phi d')) \mid f(x', y', z', w')\}]$$

and the prefix is

$$(a, b, c, d, c', d') : (\exists x, y, z, w, x', y', z', w').$$

The matrix is

$$p \,.\supset.\, \phi a \,.\, \phi b : \supset :. q \,.\supset.\, \phi c \,.\, \phi d : \supset . f(x, y, z, w) :.$$
$$q \,.\supset.\, \phi c' \,.\, \phi d' \,.\supset . f(x', y', z', w'),$$

while $\quad f(x, y, z, w) \,.\equiv : p \vee q \,.\supset.\, \phi x \,.\, \phi y \,.\, \phi z \,.\, \phi w.$

Call the matrix $\quad F(x, y, z, w, x', y', z', w').$

Then $\quad \vdash : p \,.\supset.\, F(a, b, a, b, a, b, a, b),$

$\quad \vdash : \sim p \,.\supset.\, F(c, d, c, d, c', d', c', d').$

Hence $\quad \vdash : F(a, b, a, b, a, b, a, b) \,.\vee.\, F(c, d, c, d, c', d', c', d').$

Hence, by the extension of $*8{\cdot}261$ to eight variables,

$$\vdash . (\exists x, y, z, w, x', y', z', w') . F(x, y, z, w, x', y', z', w'),$$

which was to be proved.

$*8{\cdot}29$. $\vdash :. (x) . \phi x \supset \psi x . \supset : (x) . \phi x . \supset . (x) . \psi x$

Dem.

Applying the definitions, our proposition is found to have a matrix

$$(\phi x \supset \psi x) \mid [\{\phi y \mid (\psi u \mid \psi v)\} \mid \{\phi y' \mid (\psi u' \mid \psi v')\}]$$

with a prefix (after using $*8{\cdot}13$)

$$(u, v, u', v') : (\exists x, y, y').$$

The matrix is equivalent to

$$\phi x \supset \psi x . \supset : \phi y . \supset . \psi u . \psi v : \phi y' . \supset . \psi u' . \psi v'.$$

Calling this $M(x, y, y')$, we have to prove

$$(\exists x, y, y') . M(x, y, y').$$

If $\psi u . \psi v . \psi u' . \psi v'$, $M(x, y, y')$ is always true. (1)

If $\sim \psi u$, put $x = y = y' = u$. Then if ϕu is true, $\phi u \supset \psi u$ is false and $M(u, u, u)$ is true. But if ϕu is false, $\phi u . \supset . \psi u . \psi v$ and $\phi u . \supset . \psi u' . \psi v'$ are true, so that $M(u, u, u)$ is true. Hence

$$\sim \psi u . \supset . M(u, u, u) . \supset . (\exists x, y, y') . M(x, y, y'). \quad (2)$$

Similarly if $\qquad \sim \psi v \vee \sim \psi u' \vee \sim \psi v'.$ (3)

(1), (2), and (3) exhaust possible cases. Hence the result by $*8{\cdot}28$.

We are now in a position to prove that all the propositions of $*1$—$*5$ remain true when one or more of the propositions $p, q, r, \ldots$ are first-order propositions instead of being elementary propositions. For this purpose, we take, not the one primitive proposition which Nicod has shown to be sufficient, but the two which he has shown to be equivalent to it, namely:

$$p \supset p \text{ and } p \supset q . \supset . s \mid q \supset p \mid s.$$

We show that these are true when one, or two, or three, of the propositions p, q, s are first-order propositions. From this, the rest follows. The first of these primitive propositions, $p \supset p$, gives rise to two cases, according as we substitute $(x) . \phi x$ or $(\exists x) . \phi x$ for p; the second primitive proposition gives rise to 26 cases. These have to be considered one by one.

$*8{\cdot}3$. $\vdash : (x) . \phi x . \supset . (x) . \phi x$

Applying the definitions, this is $(\exists x) : (y, z) . \phi x \mid (\phi y \mid \phi z)$, which follows from $*8{\cdot}11$ by $*8{\cdot}13$.

$*8{\cdot}31$. $\vdash : (\exists x) . \phi x . \supset . (\exists x) . \phi x$

Applying the definitions, this is $(x) : (\exists y, z) . \phi x \mid (\phi y \mid \phi z)$. This is $*8{\cdot}1$.

This completes the proof of $p \supset p$.

∗8·32. $\vdash :. (x) . \phi x . \supset . q : \supset : s \mid q . \supset . \{(x) . \phi x\} \mid s$

Putting $p . = . (x) . \phi x$, the proposition to be proved is

$$(p \mid \sim q) \mid \sim \{(s \mid q) \mid \sim (p \mid s)\}.$$

By the definitions,

$$p \mid \sim q . = . (\exists a) . \phi a \mid (q \mid q), \qquad (1)$$

$$p \mid s . = . (\exists x) . \phi x \mid s,$$

$$\sim (p \mid s) . = . (x, y) . (\phi x \mid s) \mid (\phi y \mid s),$$

$$(s \mid q) \mid \sim (p \mid s) . = . (\exists x, y) . (s \mid q) \mid \{(\phi x \mid s) \mid (\phi y \mid s)\}.$$

Put $\qquad f(x, y) . = . (s \mid q) \mid \{(\phi x \mid s) \mid (\phi y \mid s)\}.$

Then $\qquad \sim \{(s \mid q) \mid \sim (p \mid s)\} . = . (x, y, x', y') . f(x, y) \mid f(x', y'). \qquad (2)$

By (1) and (2), the proposition to be proved is

$$(a) : (\exists x, y, x', y') . \{\phi a \mid (q \mid q)\} \mid \{f(x, y) \mid f(x', y')\}.$$

Putting $x = y = x' = y' = a$, the matrix of this proposition reduces to

$$\phi a \supset q . \supset . s \mid q \supset \phi a \mid s,$$

which is our primitive proposition with ϕa substituted for p, and is therefore true. Hence the proposition follows by ∗8·1.

In what follows, the reduction of the proposition to be proved to a matrix and prefix, by means of the definitions, proceeds always by the same method, and the steps will usually be omitted.

∗8·321. $\vdash :. (\exists x) . \phi x . \supset . q : \supset : s \mid q . \supset . \{(\exists x) . \phi x\} \mid s$

We obtain the same matrix as in ∗8·32, but the opposite prefix, *i.e.* the prefix is

$$(x, y, x', y') : (\exists a).$$

The matrix is equivalent to

$$\phi a \supset q . \supset : q \supset \sim s . \supset . \phi x \supset \sim s . \phi y \supset \sim s . \phi x' \supset \sim s . \phi y' \supset \sim s.$$

Calling this fa, we have to prove $(\exists a) . fa$, for any x, y, x', y'. We have

$$\phi a . \sim q . \supset . fa.$$

Also $\phi a . q . \supset :. fa . \equiv : \sim s . \supset . \phi x \supset \sim s . \phi y \supset \sim s . \phi x' \supset \sim s . \phi y' \supset \sim s :.$
$\qquad \supset :. fa.$

Hence $\qquad \phi a . \supset . fa.$

Hence by ∗8·1·24 $\qquad \phi x . \supset . (\exists a) . fa,$

and similarly for ϕy, $\phi x'$, $\phi y'$. Hence by ∗8·261

$$\phi x \vee \phi y \vee \phi x' \vee \phi y' . \supset . (\exists a) . fa.$$

Also $\qquad \sim \phi x . \sim \phi y . \sim \phi x' . \sim \phi y' . \supset . fa .$

$\qquad$ [∗8·1·24] $\qquad \supset . (\exists a) . fa.$

Hence by ∗8·28

$$\phi x \vee \phi y \vee \phi x' \vee \phi y' \vee \sim \phi x . \sim \phi y . \sim \phi x' . \sim \phi y' : \supset . (\exists a) . fa.$$

Hence, by ∗8·12, $(\exists a) . fa$, which was to be proved.

***8·322.** $\vdash :. p . \supset . (x) . \psi x : \supset : s \mid \{(x) . \psi x\} . \supset . p \mid s$

Dem.

Put $$fy . = . (s \mid \psi y) \mid \{(p \mid s) \mid (p \mid s)\}.$$

Then the proposition to be proved is

$$(y, y') : (\exists b, c) . \{p \mid (\psi b \mid \psi c)\} \mid (fy \mid fy').$$

The matrix here is equivalent to

$$p . \supset . \psi b . \psi c : \supset : s \mid \psi y . \supset . p \mid s : s \mid \psi y' . \supset . p \mid s.$$

Putting $b = y . c = y'$, this follows at once from the primitive proposition, which gives

$$p \supset \psi y . \supset : s \mid \psi y . \supset . p \mid s,$$
$$p \supset \psi y' . \supset : s \mid \psi y' . \supset . p \mid s.$$

Hence the proposition.

***8·323.** $\vdash :. p . \supset . (\exists x) . \psi x : \supset : s \mid \{(\exists x) . \psi x\} . \supset . p \mid s$

We have the same matrix as in *8·322, but the opposite prefix, *i.e.*

$$(b, c) : (\exists y, y').$$

Putting $y = b . y' = c$, the matrix is satisfied, as in *8·322.

***8·324.** $\vdash :. p \supset q . \supset : \{(x) . \chi x\} \mid q . \supset . p \mid \{(x) . \chi x\}$

Dem.

Put $f(x, y, z) . = . (\chi x \mid q) \mid \{(p \mid \chi y) \mid (p \mid \chi z)\}$. Then the matrix is

$$\{p \mid (q \mid q)\} \mid \{f(x, y, z) \mid f(x', y', z')\}$$

and the prefix is $(x, x') : (\exists y, z, y', z')$. Putting

$$y = z = x . y' = z' = x',$$

the matrix is equivalent to

$$p \supset q . \supset : \chi x \mid q . \supset . p \mid \chi x : \chi x' \mid q . \supset . p \mid \chi x',$$

which follows from our primitive proposition by Comp.

***8·325.** $\vdash :. p \supset q . \supset : \{(\exists x) . \chi x\} \mid q . \supset . p \mid \{(\exists x) . \chi x\}$

Dem.

The matrix is the same as in *8·324, but the prefix is the opposite, *i.e.*

$$(y, z, y', z') : (\exists x, x').$$

Calling the matrix $M(x, x')$, we have, if $\theta w . \equiv_w . \sim \chi w$,

$$M(x, x') . \equiv :: p \supset q . \supset :. q \supset \theta x . \supset : p . \supset . \theta y . \theta z :. q \supset \theta x' . \supset : p . \supset . \theta y' . \theta z'.$$

Hence $$\theta y . \theta z . \theta y' . \theta z' . \supset . M(x, x') . \supset . (\exists x, x') . M(x, x') \quad (1)$$

But $\sim \theta x . \sim \theta x' . \supset . M(x, x')$. Hence

$$\sim \theta x . \supset . M(x, x) . \supset . (\exists x, x') . M(x, x') \quad (2)$$

Similarly with $\theta y, \theta x', \theta y'$. Hence the result follows as in *8·321.

This ends the cases in which only one of p, q, r in

$$p \supset q . \supset : s \mid q . \supset . p \mid s$$

is of the first order instead of being elementary. We have now to deal with the cases in which two, but not three, are of the first order.

***8·33.** $\vdash :. (x) . \phi x . \supset . (x) . \psi x : \supset : s \mid \{(x) . \psi x\} . \supset . \{(x) . \phi x\} \mid s$

Putting $f(x, y, z) . = . (s \mid \psi x) \mid \{(\phi y \mid s) \mid (\phi z \mid s)\}$, the matrix is

$$\{\phi a \mid (\psi b \mid \psi c)\} \mid \{f(x, y, z) \mid f(x', y', z')\}$$

and the prefix is $(a, x, x') : (\exists b, c, y, z, y', z')$. The matrix is satisfied by

$$b = x . c = x' . y = z = y' = z' = a,$$

in which case it is equivalent to

$$\phi a . \supset . \psi x . \psi x' : \supset :. \psi x \supset \sim s . \supset . \phi a \supset \sim s : \psi x' \supset \sim s . \supset . \phi a \supset \sim s.$$

Hence Prop.

We have the same matrix in the three following propositions, only with different prefixes.

***8·331.** $\vdash :. (x) . \phi x . \supset . (\exists x) . \psi x : \supset : s \mid \{(\exists x) . \psi x\} . \supset . \{(x) . \phi x\} \mid s$

Here the prefix to the matrix is $(a, b, c) : (\exists x, y, z, x', y', z')$. The matrix is satisfied by $x = b . x' = c . y = z = y' = z' = a$. Hence Prop.

***8·332.** $\vdash :. (\exists x) . \phi x . \supset . (x) . \psi x : \supset : s \mid \{(x) . \psi x\} . \supset . \{(\exists x) . \phi x\} \mid s$

The prefix here is $(x, y, z, x', y', z') : (\exists a, b, c)$. Writing r for $\sim s$, matrix becomes

$$\phi a . \supset . \psi b . \psi c : \supset :. \psi x \supset r . \supset . \phi y \vee \phi z \supset r : \psi x' \supset r . \supset . \phi y' \vee \phi z' \supset r.$$

(Here only a, b, c can be chosen arbitrarily.) This is true if $\phi y, \phi z, \phi y', \phi z'$ are all false. Suppose ϕy is true. Put $a = y$. Then if ψb or ψc is false, $\phi a . \supset . \psi b . \psi c$ is false, and the matrix is true. Therefore if ψx is false, put $b = c = x$; if $\psi x'$ is false, put $b = c = x'$. If ψx and $\psi x'$ are both true, putting $a = y . b = c = x$, the matrix becomes equivalent to

$$r . \supset . \phi y \vee \phi z \supset r : r . \supset . \phi y' \vee \phi z' \supset r,$$

which is true. Hence if ϕy is true, the matrix can be made true. Similarly for z, y', z'. This exhausts possible cases. Hence Prop, by *8·28.

***8·333.** $\vdash :. (\exists x) . \phi x . \supset . (\exists x) . \psi x : \supset : s \mid \{(\exists x) . \psi x\} . \supset . \{(\exists x) . \phi x\} \mid s$

Dem.

The matrix is as before, and the prefix (after using *8·13) is

$$(b, c, y, z, y', z') : (\exists a, x, x').$$

Call the matrix $M(a, x, x')$. Then

$\vdash : \psi b . \supset . M(a, b, b) . \supset . (\exists a, x, x') . M(a, x, x')$ (1)

$\vdash : \psi c . \supset . M(a, c, c) . \supset . (\exists a, x, x') . M(a, x, x')$ (2)

$\vdash : \sim \psi b . \sim \psi c . \phi y . \supset . M(y, b, c) . \supset . (\exists a, x, x') . M(a, x, x')$ (3)

$(1) . (2) . (3) . \supset \vdash : \phi y . \supset . (\exists a, x, x') . M(a, x, x')$ [using *8·28] (4)

Similarly for $\phi y', \phi z, \phi z'$. Hence by *8·28

$\vdash : \phi y \vee \phi y' \vee \phi z \vee \phi z' . \supset . (\exists a, x, x') . M(a, x, x')$ (5)

But $\vdash :. \sim \phi y . \sim \phi y' . \sim \phi z . \sim \phi z' . \supset : \phi y \vee \phi z \supset r . \phi y' \vee \phi z' \supset r :$

$\supset : M(a, x, x')$

[*8·1] $\supset : (\exists a, x, x') . M(a, x, x')$ (6)

$\vdash . (5) . (6) . *8·28 . \supset \vdash . (\exists a, x, x') . M(a, x, x') . \supset \vdash . \text{Prop}$

This ends the cases in which p and q but not s contain apparent variables. We take next the four cases in which p and s, but not q, contain apparent variables.

***8·34.** $\vdash :. (x) . \phi x . \supset . q : \supset : \{(x) . \chi x\} \mid q . \supset . \{(x) . \phi x\} \mid \{(x) . \chi x\}$

Putting $f(x, y, z, u, v) . = . (\chi x \mid q) \mid \{(\phi y \mid \chi z) \mid (\phi u \mid \chi v)\}$, the matrix is

$$(\phi a \mid \sim q) \mid \{f(x, y, z, u, v) \mid f(x', y', z', u', v')\}.$$

(This is also the matrix of the three following propositions.)

The prefix is $(a, x, x') : (\exists y, z, u, v, y', z', u', v')$.

The matrix is equivalent to

$$\phi a \supset q . \supset . f(x, y, z, u, v) . f(x', y', z', u', v')$$

and $\qquad f(x, y, z, u, v) . \equiv : \chi x \mid q . \supset . \phi y \mid \chi z . \phi u \mid \chi v :$

$$\equiv : q \supset \sim \chi x . \supset . \phi y \supset \sim \chi z . \phi u \supset \sim \chi v.$$

Putting $y = u = y' = u' = a . z = v = x . z' = v' = x'$, the matrix is satisfied. Hence Prop.

***8·341.** $\vdash :. (x) . \phi x . \supset . q : \supset : \{(\exists x) . \chi x\} \mid q . \supset . \{(x) . \phi x\} \mid \{(\exists x) . \chi x\}$

Matrix as in *8·34. Prefix $(a, z, v, z', v') : (\exists x, y, u, x', y', u')$.

Matrix is equivalent to

$$\phi a \supset q . \supset :. q \supset \sim \chi x . \supset . \phi y \supset \sim \chi z . \phi u \supset \sim \chi v :$$
$$q \supset \sim \chi x' . \supset . \phi y' \supset \sim \chi z' . \phi u' \supset \sim \chi v'.$$

If ϕa is false, this becomes true by putting $y = u = y' = u' = a$. If ϕa is true, the matrix is true if q is false. Suppose q true. Then the matrix is equivalent to

$$\sim \chi x . \supset . \phi y \supset \sim \chi z . \phi u \supset \sim \chi v : \sim \chi x' . \supset . \phi y' \supset \sim \chi z' . \phi u' \supset \sim \chi v'.$$

This is true if χz, χv, $\chi z'$, $\chi v'$ are false. If one of them, say χz, is true, put $x = x' = z$, and the matrix is true. This exhausts possible cases. Hence Prop, by *8·28.

***8·342.** $\vdash :. (\exists x) . \phi x . \supset . q : \supset : \{(x) . \chi x\} \mid q . \supset . \{(\exists x) . \phi x\} \mid \{(x) . \chi x\}$

Matrix as before. Prefix (after using *8·13) $(x, y, u, x', y', u') : (\exists a, z, v, z', v')$.

Call the matrix $M(a, z, v, z', v')$. Then

$$\vdash : \sim \chi x . \quad \supset . M(a, x, x, x, x) \tag{1}$$
$$\vdash : \sim \chi x' . \quad \supset . M(a, x', x', x', x') \tag{2}$$
$$\vdash : q . \chi x . \chi x' . \supset . \sim(q \supset \sim \chi x) . \sim(q \supset \sim \chi x') .$$
$$\supset . M(a, z, v, z', v') \tag{3}$$
$$\vdash : \sim q . \phi y . \quad \supset . \sim(\phi y \supset q) .$$
$$\supset . M(y, z, v, z', v') \tag{4}$$

Similarly if $\sim q . \phi u$ or $\sim q . \phi y'$ or $\sim q . \phi u'$. Hence by *8·1·28

$$\vdash : \sim q . \phi y \lor \phi u \lor \phi y' \lor \phi u' . \supset . (\exists a, z, v, z', v') . M(a, z, v, z', v') \tag{5}$$
$$\vdash : \sim \phi y . \sim \phi u . \sim \phi y' . \sim \phi u' . \supset . \phi y \supset \sim \chi z . \phi u \supset \sim \chi v . \phi y' \supset \sim \chi z' . \phi u' \supset \sim \chi v'$$
$$\supset . M(a, z, v, z', v') \tag{6}$$
$$(5) . (6) . \supset \vdash : \sim q . \quad \supset . (\exists a, z, v, z', v') . M(a, z, v, z', v') \tag{7}$$

$\vdash . (1) . (2) . (3) . (7) . \supset \vdash .$ Prop

$*8{\cdot}343.\ \vdash :. (\exists x) . \phi x . \supset . q : \supset : \{(\exists x) . \chi x\} \mid q . \supset . \{(\exists x) . \phi x\} \mid \{(\exists x) . \chi x\}$

Prefix to matrix is $(y, z, u, v, y', z', u', v') : (\exists a, x, x')$.

Call the matrix $f(a, x, x')$.

It is true if $\sim\chi z . \sim\chi v . \sim\chi z' . \sim\chi v'$ (1)

Also $\chi z . q . \supset . f(a, z, z) . \supset . (\exists a, x, x') . f(a, x, x')$ (2)

Similarly if we have $\chi v . q$ or $\chi z' . q$ or $\chi v' . q$ (3)

From (1) . (2) . (3), by $*8{\cdot}28$, $q . \supset . (\exists a, x, x') . f(a, x, x')$ (4)

Now $\phi a . \sim q . \supset . f(a, x, x')$. Hence

$$\phi y . \sim q . \supset . f(y, x, x') . \supset . (\exists a, x, x') . f(a, x, x')$$

Similarly for $\phi z . \sim q$, $\phi y' . \sim q$, $\phi z' . \sim q$. Hence

$$\phi y \vee \phi z \vee \phi y' \vee \phi z' . \sim q . \supset . (\exists a, x, x') . f(a, x, x') \quad (5)$$

But $\sim\phi y . \sim\phi z . \sim\phi y' . \sim\phi z' . \supset . f(a, x, x')$ (6)

By (5) and (6), $\sim q . \supset . (\exists a, x, x') . f(a, x, x')$ (7)

$\vdash . (4) . (7) . *8{\cdot}28 . \supset \vdash .$ Prop

In the next four propositions, q and r are replaced by propositions containing apparent variables, while p remains elementary.

$*8{\cdot}35.\ \vdash :. p . \supset . (x) . \psi x : \supset : \{(x) . \chi x\} \mid \{(x) . \psi x\} . \supset . p \mid \{(x) . \chi x\}$

Putting $q . = . (x) . \psi x$, $s . = . (x) . \psi x$, the proposition is

$$(p \mid \sim q) \mid \sim\{(s \mid q) \mid \sim(p \mid s)\}.$$

We have by the definitions

$$\begin{aligned}
\sim q . &= . (\exists b, c) . \psi b \mid \psi c, \\
p \mid \sim q . &= . (b, c) . p \mid (\psi b \mid \psi c), \\
s \mid q . &= . (\exists x, y) . \chi y \mid \psi x, \\
p \mid s . &= . (\exists z) . p \mid \chi z, \\
\sim(p \mid s) . &= . (z, w) . (p \mid \chi z) \mid (p \mid \chi w), \\
(s \mid q) \mid \sim(p \mid s) . &= : (x, y) : (\exists z, w) . (\chi y \mid \psi x) \mid \{(p \mid \chi z) \mid (p \mid \chi w)\}.
\end{aligned}$$

Put $f(x, y, z, w) . = . (\chi y \mid \psi x) \mid \{(p \mid \chi z) \mid (p \mid \chi w)\}$.

Then $\sim\{(s \mid q) \mid \sim(p \mid s)\} . = : (\exists x, y, x', y') : (z, w, z', w') . f(x, y, z, w) \mid f(x', y', z', w')$,

$(p \mid \sim q) \mid \sim\{(s \mid q) \mid \sim(p \mid s)\} . = : (x, y, x', y') : (\exists b, c, z, w, z', w') .$

$$\{p \mid (\psi b \mid \psi c)\} \mid \{f(x, y, z, w) \mid f(x', y', z', w')\}.$$

Writing $\theta\hat{x}$ for $\sim\chi\hat{x}$, the matrix is equivalent to

$p . \supset . \psi b . \psi c : \supset :. \psi x \supset \theta y . \supset : p . \supset . \theta z . \theta w :. \psi x' \supset \theta y' . \supset : p . \supset . \theta z' . \theta w'$.

This is satisfied by putting $b = x . c = x' . z = w = y . z' = w' = y'$. Hence Prop. The same matrix appears in the next three propositions; only the prefix changes.

$*8{\cdot}351.\ \vdash :. p . \supset . (x) . \psi x : \supset : \{(\exists x) . \chi x\} \mid \{(x) . \psi x\} . \supset . p \mid \{(\exists x) . \chi x\}$

Same matrix as in $*8{\cdot}35$, but prefix $(x, z, w, x', z', w') : (\exists b, c, y, y')$.

Matrix is true if $\theta z . \theta w . \theta z' . \theta w'$.

Assume $\sim\theta z$, and put $y = y' = z . b = x . c = x'$.

We now have $\psi x \supset \theta y . \equiv . \sim \psi x$ and $p . \supset . \theta z . \theta w : \equiv . \sim p$. Hence matrix is equivalent to

$$p . \supset . \psi x . \psi x' : \supset :. \sim \psi x . \supset . \sim p :. \sim \psi x' . \supset : p . \supset . \theta z' . \theta w',$$

which is true. Similarly if $\sim \theta w \vee \sim \theta z' \vee \sim \theta w'$. Hence Prop, by $*8 \cdot 1 \cdot 28$.

$*8 \cdot 352$. $\vdash :. p . \supset . (\exists x) . \psi x : \supset : \{(x) . \chi x\} \mid \{(\exists x) . \psi x\} . \supset . p \mid \{(x) . \chi x\}$

Same matrix, but prefix $(b, c, y, y') : (\exists x, z, w, x', z', w')$.

Satisfied by $x = b . x' = c . z = w = y . z' = w' = y'$. Hence Prop.

$*8 \cdot 353$. $\vdash :. p . \supset . (\exists x) . \psi x : \supset : \{(\exists x) . \chi x\} \mid \{(\exists x) . \psi x\} . \supset . p \mid \{(\exists x) . \chi x\}$

Same matrix, with prefix $(b, c, z, w, z', w') : (\exists x, y, x', y')$.

If ψb is true and θz false, matrix is satisfied by $x = x' = b . y = y' = z$, because these values make $\psi x \supset \theta y$ and $\psi x' \supset \theta y'$ false. Smilarly if ψb is true and θw or $\theta z'$ or $\theta w'$ is false, and if ψc is true and θz, θw, $\theta z'$ or $\theta w'$ is false. It remains to consider $\sim \psi b . \sim \psi c : \vee : \theta z . \theta w . \theta z' . \theta w'$.
The second alternative makes the matrix true, because it gives

$$p . \supset . \theta z . \theta w : p . \supset . \theta z' . \theta w'.$$

The first alternative gives

$$\begin{aligned} &p . \supset . \psi b . \psi c : \supset : \sim p : \\ &\qquad \supset : p . \supset . \theta z . \theta w : p . \supset . \theta z' . \theta w', \end{aligned}$$

so that again the matrix is true. Hence Prop.

This finishes the cases in which one or two of the three constituents of $p \supset q . \supset . s \mid q \supset p \mid s$ remain elementary. It remains to consider the eight cases in which none remains elementary. These all have the same matrix.

$*8 \cdot 36$. $\vdash :. (x) . \phi x . \supset . (x) . \psi x : \supset : \{(x) . \chi x\} \mid \{(x) . \psi x\} . \supset . \{(x) . \phi x\} \mid \{(x) . \chi x\}$

Putting $p . = . (x) . \phi x$, $q . = . (x) . \psi x$, $s . = . (x) . \chi x$, we have

$$\begin{aligned} \sim q . &= . (\exists b, c) . \psi b \mid \psi c, \\ p \mid \sim q . &= : (\exists a) : (b, c) . \phi a \mid (\psi b \mid \psi c), \\ s \mid q . &= . (\exists x, y) . \chi y \mid \psi x, \\ p \mid s . &= . (\exists z, w) . \phi z \mid \chi w, \\ \sim (p \mid s) . &= . (z, w, u, v) . (\phi z \mid \chi w) \mid (\phi u \mid \chi v), \\ (s \mid q) \mid \sim (p \mid s) . &= : (x, y) : (\exists z, w, u, v) . (\chi y \mid \psi x) \mid \{(\phi z \mid \chi w) \mid (\phi u \mid \chi v)\}. \end{aligned}$$

Put $f(x, y, z, w, u, v) . = . (\chi y \mid \psi x) \mid \{(\phi z \mid \chi w) \mid (\phi u \mid \chi v)\}$. Then

$$\begin{aligned} &\sim \{(s \mid q) \mid \sim (p \mid s)\} . = : (\exists x, y, x', y') : (z, w, u, v, z', w', u', v') . \\ &\qquad\qquad f(x, y, z, w, u, v) \mid f(x', y', z', w', u', v'), \\ &(p \mid \sim q) \mid \sim \{(s \mid q) \mid \sim (p \mid s)\} . = : (a, x, y, x', y') : (\exists b, c, z, w, u, v, z', w', u', v') . \\ &\qquad\qquad \{\phi a \mid (\psi b \mid \psi c)\} \mid \{f(x, y, z, w, u, v) \mid f(x', y', z', w', u', v')\}. \end{aligned}$$

Writing $\theta \hat{x}$ for $\sim \chi \hat{x}$, the matrix is equivalent to

$$\begin{aligned} &\phi a . \supset . \psi b . \psi c : \supset :. \psi x \supset \theta y . \supset . \phi z \supset \theta w . \phi u \supset \theta v : \\ &\qquad\qquad \psi x' \supset \theta y' . \supset . \phi z' \supset \theta w' . \phi u' \supset \theta v'. \end{aligned}$$

This is satisfied by $b = x . c = x' . z = u = z' = u' = a . w = v = y . w' = v' = y'$. Hence Prop.

***8·361.** $\vdash :. (x) . \phi x . \supset . (x) . \psi x : \supset : \{(\exists x) . \chi x\} | \{(x) . \psi x\} . \supset . \{(x) . \phi x\} | \{(\exists x) . \chi x\}$

Same matrix, but "all" and "some" are interchanged in arguments to χ, *i.e.* in y, w, v, y', w', v'. The $\exists$-variables are therefore $b, c, y, y', z, z', u, u'$.

If $\sim \phi a$, put $z = u = z' = u' = a$, and matrix is satisfied.

If ϕa is true, matrix is true if $\sim \psi b \vee \sim \psi c$, *i.e.* if $\sim \psi x \vee \sim \psi x'$, since b, c are arbitrary. Assume $\psi x . \psi x'$. Then matrix reduces to

$$\theta y . \supset . \phi z \supset \theta w . \phi u \supset \theta v : \theta y' . \supset . \phi z' \supset \theta w' . \phi u' \supset \theta v'.$$

If $\theta w, \theta v, \theta w', \theta v'$ are all true, this is true.

If $\sim \theta w$, put $y = y' = w$, and matrix is satisfied.

Similarly if $\sim \theta v$, $\sim \theta w'$ or $\sim \theta v'$. Hence Prop.

***8·362.** $\vdash :. (x) . \phi x . \supset . (\exists x) . \psi x : \supset : \{(x) . \chi x\} | \{(\exists x) . \psi x\} . \supset . \{(x) . \phi x\} | \{(x) . \chi x\}$

Matrix as in *8·36. Prefix results from *8·36 by interchanging "all" and "some" among ψ-arguments, *i.e.* b, c, x, x'. Hence Prop results from same substitutions as in *8·36.

***8·363.** $\vdash :. (x) . \phi x . \supset . (\exists x) . \psi x : \supset : \{(\exists x) . \chi x\} | \{(\exists x) . \psi x\} .$
$\supset . \{(x) . \phi x\} | \{(\exists x) . \chi x\}$

Results from interchanging "all" and "some," in *8·361, in the ψ-arguments, viz. b, c, x, x'. The $\exists$-variables are therefore $x, x', y, y', z, z', u, u'$, and the proof proceeds exactly as in *8·361, interchanging x, x' and b, c.

***8·364.** $\vdash :. (\exists x) . \phi x . \supset . (x) . \psi x : \supset : \{(x) . \chi x\} | \{(x) . \psi x\} . \supset . \{(\exists x) . \phi x\} | \{(x) . \chi x\}$

The proposition is what results from *8·36 by interchanging "all" and "some" in the ϕ-arguments, viz. a, z, u, z', u'. Hence the $\exists$-arguments are a, b, c, w, v, w', v'. If θy is true, put $w = v = w' = v' = y$, and the matrix is satisfied. If $\theta y'$ is true, put $w = v = w' = v' = y'$, and the matrix is satisfied. Assume $\sim \theta y . \sim \theta y'$. The matrix is true if $\psi x \supset \theta y$ and $\psi x' \supset \theta y'$ are false, *i.e.*, since $\theta y, \theta y'$ are false, if ψx and $\psi x'$ are true. If ψx is false, put $b = c = x$ and $a = y$; then $\phi a . \supset . \psi b . \psi c$ is false, and the matrix is true. If $\psi x'$ is false, similarly. Hence Prop.

***8·365.** $\vdash :. (\exists x) . \phi x . \supset . (x) . \psi x : \supset : \{(\exists x) . \chi x\} | \{(x) . \psi x\} .$
$\supset . \{(\exists x) . \phi x\} | \{(\exists x) . \chi x\}$

Prop is what results from *8·364 by interchanging "all" and "some" in the χ-arguments, viz. y, w, v, y', w', v'. Hence the $\exists$-arguments are a, b, c, y, y'. Matrix is true if $\theta w . \theta v . \theta w' . \theta v'$. Assume $\sim \theta w$, and put $y = y' = w$. Matrix is true if $\psi x \supset \theta y$ and $\psi x' \supset \theta y'$ are false, *i.e.*, in the present case, if ψx and $\psi x'$ are true. Suppose one of them false, and put $b = x . c = x'$. Then $\psi b . \psi c$ is false. Therefore $\phi a . \supset . \psi b . \psi c$ is false if ϕa is true; therefore the matrix is true if ϕa is true. Therefore if ϕz is true, the matrix is true for $a = z$. Similarly if ϕu, $\phi z'$ or $\phi u'$ is true. But if all are false, matrix is also true. Hence matrix is true when we have $\sim \theta w$ and $\sim \psi x \vee \sim \psi x'$. Similarly for $\sim \theta v$, $\sim \theta w'$ or $\sim \theta v'$ with $\sim \psi x \vee \sim \psi x'$. We saw that matrix can be satisfied

for $\sim\theta w$, $\sim\theta v$, $\sim\theta w'$ or $\sim\theta v'$ with $\psi x \,.\, \psi x'$. Hence it can be satisfied for $\sim\theta w \vee \sim\theta v \vee \sim\theta w' \vee \sim\theta v'$. And we saw that it is true for $\theta w \,.\, \theta v \,.\, \theta w' \,.\, \theta v'$. This completes the cases. Hence Prop.

***8·366.** $\vdash :. (\exists x) \,.\, \phi x \,.\supset.\, (\exists x) \,.\, \psi x : \supset : \{(x) \,.\, \chi x\} \,|\, \{(\exists x) \,.\, \psi x\} \,.$
$\supset . \{(\exists x) \,.\, \phi x\} \,|\, \{(x) \,.\, \chi x\}$

Prop is what results from *8·364 by interchanging "all" and "some" among ψ-arguments, viz. b, c, x, x'. Hence $\exists$-arguments are a, x, x', w, v, w', v'. The proof proceeds as in *8·364, interchanging b, c and x, x'.

***8·367.** $\vdash :. (\exists x) \,.\, \phi x \,.\supset.\, (\exists x) \,.\, \psi x : \supset : \{(\exists x) \,.\, \chi x\} \,|\, \{(\exists x) \,.\, \psi x\} \,.$
$\supset . \{(\exists x) \,.\, \phi x\} \,|\, \{(\exists x) \,.\, \chi x\}$

Prop is what results from *8·365 by interchanging "all" and "some" among ψ-arguments, viz. b, c, x, x'. Hence the $\exists$-arguments are a, x, x', y, y'. The proof proceeds as in *8·365, interchanging b, c and x, x'.

This completes the 26 cases of $p \supset q \,.\supset.\, s|q \supset p|s$. Hence in all the propositions of *1—*5 we can substitute propositions containing one variable. The proofs for propositions containing 2 or 3 or 4 or ... variables are step-by-step the same. Hence the propositions of *1—*5 hold of all first-order propositions.

The extension to second-order propositions, and thence to third-order propositions, and so on, is made by exactly analogous steps. Hence all stroke-functions which can be demonstrated for elementary propositions can be demonstrated for propositions of any order.

It remains to prove $\sim\{(x) \,.\, \phi x\} \,.\equiv.\, (\exists x) \,.\, \sim\phi x$ and similar propositions.

***8·4.** $\vdash : \sim\{(x) \,.\, \phi x\} \,.\equiv.\, (\exists x) \,.\, \sim\phi x$

Dem.

$\vdash .$ *8·1 .	$\supset \vdash : \phi x \,|\, \phi x \,.\supset.\, (\exists y) \,.\, \phi x \,|\, \phi y$	(1)
$\vdash . (1) .$ *8·21 .	$\supset \vdash : (\exists x) \,.\, \phi x \,|\, \phi x \,.\supset.\, (\exists x, y) \,.\, \phi x \,|\, \phi y :$	
[(*8·01·012)]	$\supset \vdash : (\exists x) \,.\, \sim\phi x \,.\supset.\, \sim\{(x) \,.\, \phi x\}$	(2)
We have	$\vdash : p|q \,.\equiv.\, p|p \vee q|q$	(3)
$\vdash . (3) .$	$\supset \vdash : \phi x \,|\, \phi y \,.\equiv.\, \phi x \,|\, \phi x \vee \phi y \,|\, \phi y$	(4)
$\vdash . (4) .$ *8·22·24 .	$\supset \vdash : \phi x \,|\, \phi y \,.\supset.\, (\exists x) \,.\, \phi x \,|\, \phi x$	(5)
[(*8·011)]	$\vdash :. (\exists x, y) \,.\, f(x,y) \,.\supset.\, p : \equiv : (x,y) \,.\, f(x,y) \supset p$	(6)
$\vdash . (5) . (6) .$	$\supset \vdash : (\exists x, y) : \phi x \,|\, \phi y \,.\supset.\, (\exists x) \,.\, \phi x \,|\, \phi x :$	
[(*8·01·012)]	$\supset \vdash : \sim\{(x) \,.\, \phi x\} \,.\supset.\, (\exists x) \,.\, \sim\phi x$	(7)
$\vdash . (2) . (7) .$	$\supset \vdash .$ Prop	

***8·41.** $\vdash : \sim\{(\exists x) \,.\, \phi x\} \,.\equiv.\, (x) \,.\, \sim\phi x$

[Similar proof]

***8·42.** $\vdash :. p \,.\supset.\, (\exists x) \,.\, \phi x : \equiv : (\exists x) \,.\, p \supset \phi x$

Dem.

	$\vdash :. p \,.\supset.\, (\exists x) \,.\, \phi x : \equiv : p \,|\, \{\sim(\exists x) \,.\, \phi x\} :$
[*8·41]	$\equiv : p \,|\, \{(x) \,.\, \sim\phi x\} :$
[(*8·011)]	$\equiv : (\exists x) \,.\, p \,|\, \sim\phi x :$
[*8·21]	$\equiv : (\exists x) \,.\, p \supset \phi x :. \supset \vdash .$ Prop

***8·43.** $\vdash :. p . \supset . (x) . \phi x : \equiv : (x) . p \supset \phi x$

[Similar proof]

Other propositions of this type may be taken for granted.

***8·44.** $\vdash :. (x) . \phi x . \supset : (x) . \psi x . \supset . (x) . \phi x . \psi x$

Dem.

$\vdash :. \phi z . \supset : \psi z . \supset . \phi z . \psi z$ (1)

$\vdash . (1) . *8\cdot 1 . \supset \vdash ::. (\exists x) ::. (\exists y) :: (z) :. \phi x . \supset : \psi y . \supset . \phi z . \psi z$ (2)

$\vdash . (2) . *8\cdot 42\cdot 43 . \supset \vdash .$ Prop

***8·5.** If $F(p, q, r, \ldots)$ is a stroke-function of elementary propositions, and $p, q, r, \ldots$ are replaced by first-order propositions $p_1, q_1, r_1, \ldots$, we shall have

$$p \equiv p_1 . q \equiv q_1 . r \equiv r_1, \ldots \supset : F(p, q, r, \ldots) . \equiv . F(p_1, q_1, r_1, \ldots).$$

This follows from

$$p_1 . = . (x) . \phi x : \supset : p \equiv p_1 . \supset . p_1 | q \equiv p | q . q | p_1 \equiv q | p,$$

$$p_1 . = . (\exists x) . \phi x : \supset : p \equiv p_1 . \supset . p_1 | q \equiv p | q . q | p_1 \equiv q | p,$$

both of which are very easily proved.

APPENDIX B

$*89$. MATHEMATICAL INDUCTION

THE difficulties which arise in connection with mathematical induction when the axiom of reducibility is rejected have been explained in the Introduction to the present edition. Retaining the definition of R_* ($*90{\cdot}01$), we have

$$\vdash :. \, xR_*y \,.\, \equiv : x \in C\text{‘}R : \breve{R}\text{“}\mu \subset \mu \,.\, x \in \mu \,.\, \supset_\mu \,.\, y \in \mu .$$

The "μ" which occurs here as apparent variable must be of some definite order. If κ is a class of classes, and the members of κ are of the order contemplated in the definition of R_*, we cannot infer

$$xR_*y \,.\, \supset : \breve{R}\text{“}p\text{‘}\kappa \subset p\text{‘}\kappa \,.\, x \in p\text{‘}\kappa \,.\, \supset \,.\, y \in p\text{‘}\kappa$$

nor yet $\quad xR_*y \,.\, \supset : \breve{R}\text{“}s\text{‘}\kappa \subset s\text{‘}\kappa \,.\, x \in s\text{‘}\kappa \,.\, \supset \,.\, y \in s\text{‘}\kappa .$

It is necessary, *primâ facie*, to have

$$\alpha \in \kappa \,.\, \supset_\alpha \,.\, \breve{R}\text{“}\alpha \subset \alpha$$

in order to be able to argue from $x \in p\text{‘}\kappa$ to $y \in p\text{‘}\kappa$ or from $x \in s\text{‘}\kappa$ to $y \in s\text{‘}\kappa$. In the following pages, we shall show how to avoid the resulting complications.

Let us denote by "μ_m" a variable class of the mth order, and put

$*89{\cdot}01$. $\quad xR_{*m}y \,.\, = : x \in C\text{‘}R : \breve{R}\text{“}\mu_m \subset \mu_m \,.\, x \in \mu_m \,.\, \supset_{\mu_m} \,.\, y \in \mu_m \quad$ Df

Since every class of a lower order is equal to some class of any given higher order, $R_{*m} \mathrel{\dot\subset} R_{*n}$ if $m > n$. We shall show that

$$m > 5 \,.\, \supset \,.\, R_{*m} = R_{*5} .$$

Hence we take R_{*5} as R_*, and the complications disappear.

In $*90$, substituting R_{*m} for R_* and μ_m for μ and $\phi_m z$ for ϕz, the first proposition involving an invalid induction is $*90{\cdot}17$, where we use the fact that $\overleftarrow{R_*}\text{‘}x$ is hereditary. It is obvious that $\overleftarrow{R_{*m}}\text{‘}x$ is a class of order $m+1$, and therefore, although

$$\breve{R}\text{“}\overleftarrow{R_{*m}}\text{‘}x \subset \overleftarrow{R_{*m}}\text{‘}x ,$$

we cannot infer

$$y \in \overleftarrow{R_{*m}}\text{‘}x \,.\, yR_{*m}z \,.\, \supset \,.\, z \in \overleftarrow{R_{*m}}\text{‘}x .$$

In this case, however, as in many others, there is no difficulty in substituting a valid induction. Put

$$\kappa = \hat{\mu}_m \{ \breve{R}\text{“}\mu_m \subset \mu_m \,.\, x \in \mu_m \} .$$

Then $\overleftarrow{R_{*m}}\text{‘}x = p\text{‘}\kappa$. Now we have not merely $\breve{R}\text{“}p\text{‘}\kappa \subset p\text{‘}\kappa$ but also

$$\mu_m \in \kappa \,.\, \supset \,.\, \breve{R}\text{“}\mu_m \subset \mu_m .$$

Hence the induction is valid.

The proofs of $R_*^2 \subset R_*$ and analogous propositions are easily re-written so as to be valid.

The next difficulty—and this one is more serious—arises in connection with ∗90·31. The present proof uses the fact that

$$x(I \restriction C`R \cup R_* | R) z$$

is a hereditary property of z. But it is a property of a higher order than those by which R_* is defined; *i.e.* if R_* is R_{*m}, then $x(I \restriction C`R \cup R_{*m} | R) z$ is of order $m+1$. Let us prove first

$$R_0 \cup R_* | R \subset R_*,$$

where

∗89·02. $R_0 = I \restriction C`R$ Df

The proof is as follows:

∗89·1. $\vdash . R_0 \cup R_* | R \subset R_*$

Dem.

$\vdash :: x \epsilon \mu . \breve{R}``\mu \subset \mu . \supset :. x = z . \vee . u \epsilon \mu . uRz : \supset . z \epsilon \mu$ (1)

$\vdash . (1) . \text{Comm} . \supset \vdash :. x = z . \vee . u \epsilon \mu . uRz : \supset : x \epsilon \mu . \breve{R}``\mu \subset \mu . \supset . z \epsilon \mu :.$

$\supset \vdash :: x = z :. \vee :. x \epsilon \mu . \breve{R}``\mu \subset \mu . \supset . u \epsilon \mu : uRz :. \supset :$

$x \epsilon \mu . \breve{R}``\mu \subset \mu . \supset . z \epsilon \mu ::$

$\supset \vdash :: x = z :. \vee :. x \epsilon \mu . \breve{R}``\mu \subset \mu . \supset_\mu . u \epsilon \mu : uRz :.$

$\supset : x \epsilon \mu . \breve{R}``\mu \subset \mu . \supset_\mu . z \epsilon \mu ::$

$\supset \vdash :. xR_0z . \vee . xR_*u . uRz : \supset . xR_*z :. \supset \vdash . \text{Prop}$

∗89·101. $\vdash . R_0 \cup R | R_* \subset R_*$ [Proof as in ∗90·311]

∗89·102. $\vdash : R \epsilon \text{Cls} \rightarrow 1 . \supset . R_* = R_0 \cup R | R_*$

Dem.

$\vdash :. \text{Hp} . \breve{R}`x \epsilon \beta . \breve{R}``\beta \subset \beta . \supset : x \epsilon \iota`x \cup \beta . \breve{R}``(\iota`x \cup \beta) \subset \beta :$

$\supset : xR_*y . \supset . y \epsilon \iota`x \cup \beta$ (1)

$\vdash . (1) . \text{Comm} . \supset \vdash :. \text{Hp} . y \neq x . xR_*y . \supset : \breve{R}`x \epsilon \beta . \breve{R}``\beta \subset \beta . \supset . y \epsilon \beta$ (2)

$\vdash . (2) . \supset \vdash : \text{Hp} . xR_*y . x \neq y . \supset . x(R | R_*) y$ (3)

$\vdash . (3) . ∗89·101 . \supset \vdash . \text{Prop}$

∗89·103. $\vdash : R \epsilon 1 \rightarrow \text{Cls} . \supset . R_* = R_0 \cup R_* | R$ $\left[∗89·102 \dfrac{\breve{R}}{R}\right]$

∗89·104. $\vdash :. \kappa = \hat{\alpha}(x \epsilon \alpha . \breve{R}``\alpha \subset \alpha) . \supset : x (R | R_*) z . \supset . z \epsilon p`\breve{R}```\kappa$

Dem.

$\vdash : \text{Hp} . xRy . \supset . y \epsilon p`\breve{R}```\kappa$ (1)

$\vdash : \text{Hp} . \alpha \epsilon \kappa . y \epsilon \breve{R}``\alpha . yR_*z . \supset . z \epsilon \breve{R}``\alpha$ (2)

$\vdash . (2) . \text{Comm} . \supset \vdash : \text{Hp} . y \epsilon p`\breve{R}```\kappa . yR_*z . \supset . z \epsilon p`\breve{R}```\kappa$ (3)

$\vdash . (1) . (3) . \supset \vdash . \text{Prop}$

***89·105.** $\vdash :. \mathrm{Hp} *89\cdot104 . R \in \mathrm{Cls} \rightarrow 1 . \supset : x (R \mid R_*) z . \equiv . z \in p`\breve{R}```\kappa$

Dem.

$\vdash :. \mathrm{Hp} . \breve{R}`x \in \mu . \breve{R}``\mu \subset \mu . \beta = \iota`\breve{R}`x \cup \breve{R}``\mu .$

$\supset : y \in R``\beta \cup - \mathrm{D}`R . \mathrm{E} ! \breve{R}`y . \supset . \breve{R}`y \in R``\beta \cup - \mathrm{D}`R :$

$\supset : \breve{R}``(R``\beta \cup - \mathrm{D}`R) \subset R``\beta \cup - \mathrm{D}`R$ (1)

$\vdash : \mathrm{Hp}(1) . \supset . x \in R``\beta$ (2)

$\vdash . (1) . (2) . \supset \vdash :. \mathrm{Hp}(1) . z \in p`\breve{R}```\kappa . \supset . z \in \breve{R}``(R``\beta \cup - \mathrm{D}`R)$

$\supset . z \in \beta$ (3)

$\vdash : \mathrm{Hp}(1) . \supset . \beta \subset \mu$ (4)

$\vdash . (3) . (4) . \supset \vdash :. \mathrm{Hp} . z \in p`\breve{R}```\kappa . \supset : \breve{R}`x \in \mu . \breve{R}``\mu \subset \mu . \supset . z \in \mu :$

$\supset : x (R \mid R_*) z$ (5)

$\vdash . (5) . *89\cdot104 . \supset \vdash . \mathrm{Prop}$

***89·106.** $\vdash : R \in \mathrm{Cls} \rightarrow 1 . \supset . R_* \mid R \Subset R \mid R_*$

Dem.

$\vdash : x (R_* \mid R) z . \equiv . z \in \breve{R}``p`\kappa$ (1)

$\vdash . (1) . *89\cdot105 . *40\cdot37 . \supset \vdash . \mathrm{Prop}$

It is now necessary to take up the subject of intervals (cf. *121). Our further progress depends upon the fact that in suitable circumstances the R-interval between x and y, *i.e.* $\overleftarrow{R_*}`x \cap \overrightarrow{R_*}`y$, is an inductive class.

***89·11.** $\vdash : R \in \mathrm{Cls} \rightarrow 1 . xRz . zR_*y . \supset . R(x \mapsto y) = \iota`x \cup R(z \mapsto y)$

Dem.

$\vdash . *89\cdot102 . \supset \vdash :: \mathrm{Hp} . \supset :. xR_*u . \equiv : x = u . \vee . zR_*u$ (1)

$\vdash : \mathrm{Hp} . x = u . \supset . u \in R(x \mapsto y)$ (2)

$\vdash :. \mathrm{Hp} . zR_*u . \supset : uR_*y . \supset . u \in R(x \mapsto y)$ (3)

$\vdash . (2) . (3) . \supset \vdash : \mathrm{Hp} . \supset . \iota`x \cup R(z \mapsto y) \subset R(x \mapsto y)$ (4)

$\vdash . (1) . \quad \supset \vdash : \mathrm{Hp} . \supset . R(x \mapsto y) \subset \iota`x \cup R(z \mapsto y)$ (5)

$\vdash . (4) . (5) . \supset \vdash . \mathrm{Prop}$

***89·111.** $\vdash : \sim(zR_*y) . \supset . R(z \mapsto y) = \Lambda$

***89·112.** $\vdash : R \in \mathrm{Cls} \rightarrow 1 . xRz . xR_*y . \sim(zR_*y) . \supset . x = y . R(x \mapsto y) = R(x \mapsto x)$
[*89·102]

***89·113.** $\vdash : R \in \mathrm{Cls} \rightarrow 1 . x \in C`R . \sim(xR \mid R_*x) . \supset . R(x \mapsto x) = \iota`x$

Dem.

$\vdash :. \mathrm{Hp} . \supset : yR_*x . \supset . \sim(xR \mid R_*y)$

$\supset : xR_*y . yR_*x . \supset . xR_*y . \sim(xR \mid R_*y) .$

[*89·102] $\supset . x = y :. \supset \vdash . \mathrm{Prop}$

***89·114.** $\vdash : R \in \mathrm{Cls} \rightarrow 1 . \breve{R}``\alpha \subset \alpha . x \in \alpha - \breve{R}``\alpha . \supset . \sim(xR \mid R_*x)$ [*89·105]

***89·115.** $\vdash :. R \in \mathrm{Cls} \rightarrow 1 . \breve{R}``\alpha \subset \alpha . x \in \alpha - \breve{R}``\alpha . \supset . R(x \mapsto x) = \iota`x$
[*89·113·114]

We now take as the definition of an inductive class the property proved in $*121 \cdot 24$, *i.e.* we put

$$\text{Cls induct} = \hat{\rho}\{\eta \in \mu \,.\supset_{\eta, y}.\, \eta \cup \iota`y \in \mu : \Lambda \in \mu : \supset_\mu .\, \rho \in \mu\} \quad \text{Df.}$$

That is to say, if $\quad M = \hat{\eta}\hat{\zeta}\{(\exists y) \,.\, \zeta = \eta \cup \iota`y\}$,

we put $\quad \text{Cls induct} = \overleftarrow{M_*}`\Lambda \quad \text{Df.}$

There will be different orders of inductive classes according to the order of μ. μ must be at least of the second order, since $\iota`y$ is of the second order; at least, not much could be proved if we took μ to be of the first order. We put

$$\text{Cls induct}_m = \overleftarrow{M_{*m}}`\Lambda \quad \text{Df.}$$

We have $\quad (\exists \mu_2) \,.\, \Lambda = \mu_2 : (\exists \mu_2) \,.\, \eta = \mu_2 \,.\supset.\, (\exists \mu_2) \,.\, \eta \cup \iota`y = \mu_2$.

Now $(\exists \mu_2) \,.\, \eta = \mu_2$ is a third-order property. Hence

$*89 \cdot 12$. $\vdash : \rho \in \text{Cls induct}_3 \,.\supset.\, (\exists \mu_2) \,.\, \rho = \mu_2$

This proposition is fundamental.

$*89 \cdot 13$. $\vdash :. R \in \text{Cls} \to 1 : \eta \in \mu \,.\supset_{\eta, y}.\, \eta \cup \iota`y \in \mu : \Lambda \in \mu : \sim(xR \mid R_* x) \,.\, xRz : \supset :$
$R(z \mapsto y) \in \mu \,.\supset.\, R(x \mapsto y) \in \mu$

$[*89 \cdot 11 \cdot 111 \cdot 112 \cdot 113]$

Put

$*89 \cdot 131$. $R_m(x \mapsto y) = \overleftarrow{R_{*m}}`x \cap \overrightarrow{R_{*m}}`y \quad$ Df

Then

$$\kappa = \hat{\alpha}_m(\breve{R}``\alpha_m \subset \alpha_m \,.\, x \in \alpha_m) \,.\, \lambda = \hat{\beta}_m(R``\beta_m \subset \beta_m \,.\, y \in \beta_m) \,.\supset.$$
$$R_m(x \mapsto y) = p`\kappa \cap p`\lambda.$$

Thus $R_m(x \mapsto y)$ is a class of order $m+1$. Moreover we have

$*89 \cdot 132$. $\vdash :. R \in \text{Cls} \to 1 \,.\, xRy \,.\supset : \sim(yR \mid R_* y) \,.\supset.\, \sim(xR \mid R_* x)$

Dem.

$\vdash : \sim(yR \mid R_* y) \,.\, xRy \,.\supset.\, (\exists \alpha) \,.\, \breve{R}``\alpha \subset \alpha \,.\, y \in \alpha - \breve{R}``\alpha \,.\, xRy \quad (1)$

$\vdash : \text{Hp} \,.\, \breve{R}``\alpha \subset \alpha \,.\, y \in \alpha - \breve{R}``\alpha \,.\, xRy \,.\, \gamma = \iota`x \cup \iota`y \cup \breve{R}``\alpha \,.$
$\supset.\, \breve{R}``\gamma = \iota`y \cup \overleftarrow{R}`y \cup \breve{R}``\breve{R}``\alpha \,. \quad (2)$
$\supset.\, \breve{R}``\gamma \subset \gamma \quad (3)$

$\vdash :. \sim(yR \mid R_* y) \,.\supset : \sim(yRy) :$
$\supset : xRy \,.\supset.\, x \neq y \quad (4)$

$\vdash : y \in \alpha - \breve{R}``\alpha \,.\, xRy \,.\supset.\, x \sim\in \alpha \quad (5)$

$\vdash :. \sim(yR \mid R_* y) \,.\supset : \sim(yR^2 y) :$
$\supset : xRy \,.\supset.\, x \sim\in \overleftarrow{R}`y \quad (6)$

$\vdash .\, (2) \,.\, (4) \,.\, (5) \,.\, (6) \,.\supset \vdash : \text{Hp}(2) \,.\, \sim(yR \mid R_* y) \,.\supset.\, x \in \gamma - \breve{R}``\gamma \quad (7)$

$\vdash .\, (3) \,.\, (7) \,.\supset \vdash : \text{Hp}(2) \,.\supset.\, \sim(xR \mid R_* x) \quad (8)$

$\vdash .\, (1) \,.\, (8) \,.\supset \vdash .\, \text{Prop}$

***89·133.** $\vdash :. R \in \text{Cls} \to 1 : \eta \in \mu \,.\, \supset_{\eta, y} .\, \eta \cup \iota\text{‘}y \in \mu : \Lambda \in \mu : xRz : \supset :$
$$\sim(zR \,|\, R_{*}z) \,.\, R\,(z \mapsto y) \in \mu \,.\, \supset .\, \sim(xR \,|\, R_{*}x) \,.\, R\,(x \mapsto y) \in \mu$$
[*89·13·132]

***89·14.** $\vdash :. R \in \text{Cls} \to 1 \,.\, \sim(yR \,|\, R_{*m}y) \,.\, \supset : xR_{*(m+1)}\,y \,.\, \supset .$
$$R_m\,(x \mapsto y) \in \text{Cls induct}_{m+1}$$

Dem.

By *89·133, $\sim(zR \,|\, R_{*m}z) \,.\, R_m\,(z \mapsto y) \in \mu_{m+1}$ is a hereditary property of z if
$$\eta \in \mu_{m+1} \,.\, \supset_{\eta, y} .\, \eta \cup \iota\text{‘}y \in \mu_{m+1} : \Lambda \in \mu_{m+1}.$$
Moreover this property is of order $m+1$. And by *89·113, y has this property if $\sim(yR \,|\, R_{*m}y)$. Hence x has this property if $xR_{*(m+1)}\,y$. Hence with this hypothesis we have
$$\eta \in \mu_{m+1} \,.\, \supset_{\eta, y} .\, \eta \cup \iota\text{‘}y \in \mu_{m+1} : \Lambda \in \mu_{m+1} : \supset_{\mu_{m+1}} .\, R_m\,(x \mapsto y) \in \mu_{m+1},$$
i.e. $$R_m\,(x \mapsto y) \in \text{Cls induct}_{m+1},$$
which was to be proved.

***89·15.** $\vdash : R \in \text{Cls} \to 1 \,.\, \breve{R}\text{‘‘}\alpha_m \subset \alpha_m \,.\, y \in \alpha_m - \breve{R}\text{‘‘}\alpha_m \,.\, \supset :$
$$xR_{*(m+1)}\,y \,.\, \supset .\, R_m\,(x \mapsto y) \in \text{Cls induct}_{m+1} \qquad [*89·114·14]$$

We have $$R_{m+1}\,(x \mapsto y) \subset R_m\,(x \mapsto y),$$
$$\text{Cls induct}_{m+1} \subset \text{Cls induct}_m.$$
The next point is to prove
$$\rho \in \text{Cls induct}_m \,.\, \gamma \subset \rho \,.\, \supset .\, \gamma \in \text{Cls induct}_m.$$
This can be proved for Cls induct_3, and extended to any other order of inductive classes. The proof is as follows.

***89·16.** $\vdash : \alpha \sim\in \text{Cls induct}_3 \,.\, \gamma \in \text{Cls induct}_3 \,.\, \supset .\, \exists !\, \alpha - \gamma$

Dem.

$\text{Hp} \,.\, \supset : (\exists \mu_3) : \Lambda \in \mu_3 : \beta \in \mu_3 \,.\, \supset_{\beta, y} .\, \beta \cup \iota\text{‘}y \in \mu_3 : \gamma \in \mu_3 \,.\, \alpha \sim\in \mu_3$ (1)

$\Lambda \in \mu_3 : \beta \in \mu_3 \,.\, \supset_{\beta, y} .\, \beta \cup \iota\text{‘}y \in \mu_3 : \gamma \in \mu_3 \,.\, \alpha \sim\in \mu_3 : \supset : \alpha \neq \Lambda \,.\, \Lambda \in \mu_3 :$
$\supset : \exists !\, \alpha - \Lambda \,.\, \Lambda \in \mu_3$ (2)

$\exists !\, \alpha - \beta \,.\, \alpha \subset \beta \cup \iota\text{‘}y \,.\, \supset .\, \alpha = \beta \cup \iota\text{‘}y$ (3)

$(3) \,.\, \supset :. \text{Hp}\,(2) \,.\, \supset : \beta \in \mu_3 \,.\, \alpha \sim\in \mu_3 \,.\, \exists !\, \alpha - \beta \,.\, \supset .$
$\beta \cup \iota\text{‘}y \in \mu_3 \,.\, \alpha \neq \beta \cup \iota\text{‘}y \,.\, \exists !\, \alpha - (\beta \cup \iota\text{‘}y)$ (4)

$(4) \,.\, \supset :. \text{Hp}\,(2) \,.\, \supset : \beta \in \mu_3 \,.\, \exists !\, \alpha - \beta \,.\, \supset .\, \beta \cup \iota\text{‘}y \in \mu_3 \,.\, \exists !\, \alpha - (\beta \cup \iota\text{‘}y)$ (5)

$(2) \,.\, (5) \,.\, \supset \vdash :. \text{Hp}\,(2) \,.\, \supset : \beta \in \text{Cls induct}_3 \,.\, \supset .\, \beta \in \mu_3 \,.\, \exists !\, \alpha - \beta$ (6)

$(1) \,.\, (6) \,.\, \supset \vdash .$ Prop

***89·17.** $\vdash : \gamma \in \text{Cls induct}_3 \,.\, \alpha \subset \gamma \,.\, \supset .\, \alpha \in \text{Cls induct}_3$ [*89·16 . Transp]

It follows that, with the hypothesis of *89·15, $R_m\,(x \mapsto y)$, $R_{m+1}\,(x \mapsto y)$, etc. are all of them inductive classes of the $(m+1)$th or any lower order.

$*89{\cdot}18.\quad \vdash :. R \in \mathrm{Cls} \to 1 \,.\, y, z \in \overleftarrow{R_{*3}}\text{‘}x \,.\, \sim(yR \,|\, R_{*2}y) \,.\, \supset : yR_{*3}z \,.\, \vee \,.\, zR_{*3}y$

Dem.

Put $\xi = R_3(x \mapsto y) \cap R_3(x \mapsto z)$.

$\vdash . *89{\cdot}12{\cdot}14{\cdot}17 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset . \, \xi \in \mathrm{Cls}_2$, *i.e.* ξ is a class of the second order (1)

$\vdash :. \mathrm{Hp} \,.\, \sim(yR_{*3}z) \,.\, \sim(zR_{*3}y) \,.\, \supset : u \in \xi \,.\, \supset . \, uR_{*3}y \,.\, uR_{*3}z \,.\, u \neq y \,.\, u \neq z \,.$

$[*89{\cdot}102] \qquad \supset . \, \breve{R}\text{‘}uR_{*3}y \,.\, \breve{R}\text{‘}uR_{*3}z \,.$

$[\mathrm{Hp}] \qquad \supset . \, \breve{R}\text{‘}u \in \xi$ (2)

$\vdash . (1) . (2) . \supset \vdash : \mathrm{Hp}(2) \,.\, \supset . \, y \in \xi$ (3)

$\vdash : \mathrm{Hp}(2) \,.\, \supset . \, y \sim\in \xi$ (4)

$\vdash . (3) . (4) . \supset \vdash :. \mathrm{Hp} \,.\, \supset : yR_{*3}z \,.\, \vee \,.\, zR_{*3}y :. \supset \vdash . \mathrm{Prop}$

$*89{\cdot}19.\quad \vdash : R \in \mathrm{Cls} \to 1 \,.\, \breve{R}\text{‘‘}\mu_2 \subset \mu_2 \,.\, \lambda = \overleftarrow{R_{*3}}\text{‘}x \cap \mu_2 - \breve{R}\text{‘‘}\mu_2 \,.\, \supset . \, \lambda \in 0 \cup 1$

Dem.

$\vdash : \mathrm{Hp} \,.\, y, z \in \lambda \,.\, y \neq z \,.\, \xi = R_2(x \mapsto y) \cap R_2(x \mapsto z) \,.\, \supset . \, \breve{R}\text{‘‘}\xi \subset \xi \,.\, x \in \xi$ [as above]

$[*89{\cdot}12{\cdot}15{\cdot}17] \qquad \supset . \, y, z \in \xi$ (1)

$\vdash : \mathrm{Hp}(1) \,.\, \supset . \, y, z \sim\in \xi$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

$*89{\cdot}2.\quad \vdash : R \in \mathrm{Cls} \to 1 \,.\, xR_{*3}y \,.\, R_2(y \mapsto y) \in \mathrm{Cls\ induct}_3 \,.\, \supset . \, R_2(x \mapsto y) \in \mathrm{Cls\ induct}_3$

Dem.

As in $*89{\cdot}11{\cdot}111{\cdot}112$,

$\vdash :. R \in \mathrm{Cls} \to 1 \,.\, xRz \,.\, \supset :$

$R(x \mapsto y) = \iota\text{‘}x \cup R(z \mapsto y) \,.\, \vee \,.\, R(x \mapsto y) = \iota\text{‘}x \,.\, \vee \,.\, R(x \mapsto y) = \Lambda$ (1)

$\vdash . (1) . \supset \vdash :. \mathrm{Hp}(1) : \Lambda \in \mu : \alpha \in \mu \,.\, \supset_{\alpha, u} . \, \alpha \cup \iota\text{‘}u \in \mu : \supset :$

$R(z \mapsto y) \in \mu \,.\, \supset . \, R(x \mapsto y) \in \mu$ (2)

$\vdash . (2) . \supset \vdash : R \in \mathrm{Cls} \to 1 \,.\, xR_{*3}y \,.\, R_2(y \mapsto y) \in \mathrm{Cls\ induct}_3 \,.\, \supset .$

$R_2(x \mapsto y) \in \mathrm{Cls\ induct}_3 : \supset \vdash . \mathrm{Prop}$

To deal further with the case in which $y(R \,|\, R_{*2})y$, proceed as follows: Having proved

$R \in \mathrm{Cls} \to 1 \,.\, xR_{*3}y \,.\, R_2(y \mapsto y) \in \mathrm{Cls\ induct}_3 \,.\, \supset . \, R_2(x \mapsto y) \in \mathrm{Cls\ induct}_3$,

we have to prove $R_2(y \mapsto y) \in \mathrm{Cls\ induct}_3$; for this purpose, put

$$S = (-\iota\text{‘}y) \upharpoonleft R.$$

Then $\qquad S \in \mathrm{Cls} \to 1 \,.\, S \mathrel{\subseteq} R.$

Observe that $\qquad yRy \,.\, \supset . \, R(y \mapsto y) = \iota\text{‘}y,$

$\qquad yR^2y \,.\, \supset . \, R(y \mapsto y) = \iota\text{‘}y \cup \iota\text{‘}\breve{R}\text{‘}y.$

Assume, therefore, $\qquad \sim(yRy) \,.\, \sim(yR^2y).$

We have $\breve{S}\text{‘‘}\mu = \breve{R}\text{‘‘}(\mu - \iota\text{‘}y) \,.\, S\text{‘‘}\mu = R\text{‘‘}\mu - \iota\text{‘}y$. Hence

$$\breve{S}\text{‘‘}\mu \subset \mu \,.\, \breve{R}\text{‘}y \in \mu \,.\, \equiv . \, \breve{R}\text{‘‘}\mu \subset \mu \,.\, \breve{R}\text{‘}y \in \mu,$$

$$S\text{‘‘}\mu \subset \mu \,.\, y \in \mu \,.\, \equiv . \, R\text{‘‘}\mu \subset \mu \,.\, y \in \mu.$$

Hence $\qquad \overleftarrow{S_*}\text{‘}\breve{R}\text{‘}y = \overleftarrow{R_*}\text{‘}\breve{R}\text{‘}y \,.\, \overrightarrow{S_*}\text{‘}y = \overrightarrow{R_*}\text{‘}y.$

Hence $\qquad S_2(\breve{R}\text{‘}y \mapsto y) = R_2(\breve{R}\text{‘}y \mapsto y) = R_2(y \mapsto y)$

because $y\,(R \,|\, R_{*2})\,y$.

Moreover we have $\sim(yS \,|\, S_* y)$ because $y \sim\in \mathrm{D}\text{‘}S$.

Hence by $*89{\cdot}14$, $R_2(y \mapsto y) \in \mathrm{Cls\ induct}_3$. Hence generally:

$*89{\cdot}201.\ \vdash : R \in \mathrm{Cls} \to 1 \,.\, xR_{*3}y \,.\, \supset \,.\, R_2(x \mapsto y) \in \mathrm{Cls\ induct}_3$

We have $\qquad R_3(x \mapsto y) \subset R_2(x \mapsto y).$

Hence by $*89{\cdot}17$, $R_2(x \mapsto y) \in \mathrm{Cls\ induct}_3 \,.\, \supset \,.\, R_3(x \mapsto y) \in \mathrm{Cls\ induct}_3$. Hence

$*89{\cdot}21.\ \vdash : R \in \mathrm{Cls} \to 1 \,.\, \supset \,.\, R_3(x \mapsto y) \in \mathrm{Cls\ induct}_3$

because $\qquad \sim(xR_{*3}y) \,.\, \supset \,.\, R_3(x \mapsto y) = \Lambda.$

$*89{\cdot}22.\ \vdash :. R \in \mathrm{Cls} \to 1 \,.\, y, z \in \overleftarrow{R_{*3}}\text{‘}x \,.\, \supset : yR_{*3}z \,.\, \vee \,.\, zR_{*3}y$

[Proof as in $*89{\cdot}18$, using $*89{\cdot}21$ instead of $*89{\cdot}14$]

$*89{\cdot}221.\ \mathrm{Potid}_m\text{‘}R = \overrightarrow{(R_{\mathrm{ts}})_m}\text{‘}R_0 \quad \mathrm{Df}$

$*89{\cdot}23.\ \vdash :. S, T \in \mathrm{Potid}_3\text{‘}R \,.\, \supset : SR_{\mathrm{ts}3}T \,.\, \vee \,.\, TR_{\mathrm{ts}3}S \qquad \left[*89{\cdot}22 \dfrac{\mathrm{Cnv}\text{‘}|\,R}{R}\right]$

$*89{\cdot}24.\ \vdash : R \in \mathrm{Cls} \to 1 \,.\, \breve{R}\text{“}\lambda \subset \lambda \,.\, x \in \lambda \,.\, \supset \,.\, \overleftarrow{R_{*3}}\text{‘}x \subset \lambda$

Here λ is assumed to be of more than the third order.

Dem.

$\vdash :. \mathrm{Hp} \,.\, y \in \overleftarrow{R_{*3}}\text{‘}x - \lambda \,.\, \supset : z \in \lambda \cap R_3(x \mapsto y) \,.\, \supset \,.\, z \neq y \,.$

$\supset \,.\, \breve{R}\text{‘}z \in \lambda \cap R_3(x \mapsto y) \qquad (1)$

$\vdash \,.\, *89{\cdot}21{\cdot}17{\cdot}12 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, (\exists \mu_2) \,.\, \lambda \cap R_3(x \mapsto y) = \mu_2 \qquad (2)$

$\vdash \,.\, (1) \,.\, (2) \,.\, \supset \vdash : \mathrm{Hp}(1) \,.\, \supset \,.\, (\exists \mu_2) \,.\, \lambda \cap R_3(x \mapsto y) = \mu_2 \,.\, \breve{R}\text{“}\mu_2 \subset \mu_2 \,.\, x \in \mu_2 \,.$

$\supset \,.\, \overleftarrow{R_{*3}}\text{‘}x \subset \lambda \cap R_3(x \mapsto y) \qquad (3)$

$\vdash \,.\, (3) \,.\, \supset \vdash :. \mathrm{Hp} \,.\, \supset : y \in \overleftarrow{R_{*3}}\text{‘}x - \lambda \,.\, \supset \,.\, y \in \lambda :$

$\supset : \overleftarrow{R_{*3}}\text{‘}x \subset \lambda :. \supset \vdash \,.\, \mathrm{Prop}$

Hence if λ is an inductive class, it can be used in an induction no matter what its order may be, if $R \in \mathrm{Cls} \to 1$.

$*89{\cdot}25.\ \vdash : R \in 1 \to \mathrm{Cls} \,.\, \supset \,.\, R_3(x \mapsto y) \in \mathrm{Cls\ induct}_3 \qquad \left[*89{\cdot}21 \dfrac{\breve{R}}{R}\right]$

$*89{\cdot}26.\ \vdash :. R \in 1 \to \mathrm{Cls} \,.\, y, z \in \overrightarrow{R_{*3}}\text{‘}x \,.\, \supset : yR_{*3}z \,.\, \vee \,.\, zR_{*3}y \qquad \left[*89{\cdot}22 \dfrac{\breve{R}}{R}\right]$

$*89{\cdot}27.\ \vdash : R \in 1 \to \mathrm{Cls} \,.\, R\text{“}\lambda \subset \lambda \,.\, x \in \lambda \,.\, \supset \,.\, \overrightarrow{R_{*3}}\text{‘}x \subset \lambda \qquad \left[*89{\cdot}24 \dfrac{\breve{R}}{R}\right]$

***89·28.** $\vdash : R \epsilon (1 \to \mathrm{Cls}) \cup (\mathrm{Cls} \to 1) . \supset . R_{*3} = \dot{s}`\mathrm{Potid}_3`R$

Dem.

$\vdash : T \epsilon \mathrm{Potid}_3`R . xTy . yRz . \supset . T|R \epsilon \mathrm{Potid}_3`R . x(T|R)z$

Hence $\vdash : \dot{s}`\mathrm{Potid}_3`R = S . \supset . \breve{R}``\overleftarrow{S}`x \subset \overleftarrow{S}`x$ (1)

$\vdash . (1) . *89·24 . \supset \vdash : R \epsilon \mathrm{Cls} \to 1 . \mathrm{Hp}(1) . \supset . \overleftarrow{R_{*3}}`x \subset \overrightarrow{S}`x$ (2)

$\vdash :. \mathrm{Hp}(1) . \breve{R}``\mu \subset \mu . \supset : \overleftarrow{T}`x \subset \mu . \supset . \breve{R}``\overleftarrow{T}`x \subset \mu :$

$\supset : x \epsilon \mu . \supset . \overleftarrow{S}`x \subset \mu$ (3)

$\vdash . (3) . \supset \vdash : \mathrm{Hp}(1) . \supset . \overleftarrow{S}`x \subset \overleftarrow{R_{*3}}`x$ (4)

$\vdash . (2) . (4) . \supset \vdash : \mathrm{Hp}(2) . \supset . \overleftarrow{S}`x = \overleftarrow{R_{*3}}`x$ (5)

$\vdash . (5) . \supset \vdash : R \epsilon \mathrm{Cls} \to 1 . \supset . R_{*3} = \dot{s}`\mathrm{Potid}_3`R$ (6)

Similarly $\vdash : R \epsilon 1 \to \mathrm{Cls} . \supset . R_{*3} = \dot{s}`\mathrm{Potid}_3`R$ (7)

$\vdash . (6) . (7) . \supset \vdash . \mathrm{Prop}$

***89·29.** $\vdash : R \epsilon (1 \to \mathrm{Cls}) \cup (\mathrm{Cls} \to 1) . \supset . R_{*(3+m)} = R_{*3}$ [*89·24·27]

We have now to obtain an analogous result when R is not one-many or many-one. For this purpose, we use R_ϵ, which is one-many.

We prove $\overrightarrow{R_{*(m+2)}}`x = s`\overrightarrow{(R_\epsilon)_{*m}}`\iota`x$,

whence, since $(R_\epsilon)_{*(3+m)} = (R_\epsilon)_{*3}$,

it follows that $R_{*(5+m)} = R_{*5}$,

so that for a relation which is not one-many or many-one we obtain the advantages of unlimited induction by proceeding to R_{*5}. The proof is as follows.

***89·3.** $\vdash : R_\epsilon = S . \supset . s`\overrightarrow{S_{*m}}`\iota`x \subset \overrightarrow{R_{*m}}`x$

Dem.

$\vdash ::. \mathrm{Hp} . \supset :: \alpha S_* \iota`x . \equiv :. \iota`x \epsilon \mu : \xi \epsilon \mu . \supset_\xi . R``\xi \epsilon \mu : \supset_\mu . \alpha \epsilon \mu :.$

$\supset :. \iota`x \epsilon \mathrm{Cl}`\gamma : \xi \epsilon \mathrm{Cl}`\gamma . \supset_\xi . R``\xi \epsilon \mathrm{Cl}`\gamma : \supset_\gamma . \alpha \epsilon \mathrm{Cl}`\gamma :.$

$\supset :. x \epsilon \gamma . R``\gamma \subset \gamma . \supset_\gamma . \alpha \subset \gamma :.$

$\supset :. \alpha \subset \overrightarrow{R_*}`x ::. \supset \vdash . \mathrm{Prop}$

***89·31.** $\vdash : R_\epsilon = S . \supset . \overrightarrow{R_{*(m+2)}}`x \subset s`\overrightarrow{S_{*m}}`\iota`x$

Dem.

$\vdash . *89·101 . \supset \vdash . S|S_* \subseteq S_* .$

$\supset \vdash . S``\overrightarrow{S_*}`\iota`x \subset \overrightarrow{S_*}`\iota`x .$

$\supset \vdash . s`S``\overrightarrow{S_*}`\iota`x \subset s`\overrightarrow{S_*}`\iota`x$ (1)

$\vdash . (1) . *40·38 . \supset \vdash : \mathrm{Hp} . \supset . R``s`\overrightarrow{S_*}`\iota`x \subset s`\overrightarrow{S_*}`\iota`x$ (2)

$\vdash : \lambda = \hat{\mu}(\iota`x \epsilon \mu . S``\mu \subset \mu) . \supset . s`\overrightarrow{S_*}`\iota`x = s`p`\lambda$ (3)

$\vdash . (2) . \supset \vdash : s`\overrightarrow{S_*}`\iota`x \epsilon \mathrm{Cls}_n . \supset . \overrightarrow{R_{*n}}`x \subset s`\overrightarrow{S_*}`\iota`x$ (4)

$\vdash . (3) . \supset \vdash . s`\overrightarrow{S_{*m}}`\iota`x \epsilon \mathrm{Cls}_{m+2}$ (5)

$\vdash . (4) . (5) . \supset \vdash . \mathrm{Prop}$

$*89{\cdot}32$. $\vdash . \overrightarrow{R_{*5}}`x = s`\overrightarrow{(R_\epsilon)_{*3}}`x$

Dem.

$\vdash . *89{\cdot}3{\cdot}29 . \supset \vdash . s`\overrightarrow{(R_\epsilon^\prime)_{*3}}`x \subset \overrightarrow{R_{*5}}`x$ (1)

$\vdash . (1) . *89{\cdot}31 . \supset \vdash .$ Prop

$*89{\cdot}33$. $\vdash . R_{*(5+m)} = R_{*5}$

Dem.

As in $*89{\cdot}32$, $\vdash . \overrightarrow{R_{*(5+m)}}`x = s`\overrightarrow{(R_\epsilon)_{*(3+m)}}`x$

$[*89{\cdot}29]$ $= s`\overrightarrow{(R_\epsilon)_{*3}}`x$

$[*89{\cdot}32]$ $= \overrightarrow{R_{*5}}`x . \supset \vdash .$ Prop

$*89{\cdot}34$. $\vdash : y R_{*5} x . x \epsilon \lambda . R``\lambda \subset \lambda . \supset . y \epsilon \lambda$ $[*89{\cdot}33]$

Here λ is supposed to be of any order, however high. Hence, so far as mathematical induction is concerned, all proofs remain valid without the axiom of reducibility provided "R_*" is understood to mean "R_{*5}."

APPENDIX C

TRUTH-FUNCTIONS AND OTHERS

IN the Introduction to the present edition we have assumed that a function can only enter into a proposition through its values. We have in fact assumed that a matrix $f ! (\phi ! \hat{z})$ always arises through some stroke-function

$$F(p, q, r, \ldots)$$

by substituting $\phi ! a$, $\phi ! b$, $\phi ! c$, ... for some or all of p, q, r, ..., and that all other functions of functions are derivable from such matrices by generalization —*i.e.* by replacing some or all of a, b, c, ... by variables, and taking "all values" or "some value."

The uses which we have made of this assumption can be validated by definition, even if the assumption is not universally true. That is to say, we can decide that mathematics is to confine itself to functions of functions which obey the above assumption. This amounts to saying that mathematics is essentially extensional rather than intensional. We might, on this ground, abstain from the inquiry whether our assumption is universally true or not. The inquiry, however, is important on its own account, and we shall, in what follows, suggest certain considerations without arriving at a dogmatic conclusion.

There is a prior question, which is simpler, and that is the question whether all functions of propositions are truth-functions. Or, more precisely, can all propositions which do not contain apparent variables be built up from atomic propositions by means of the stroke? If this were the case, we should have, if $f\hat{p}$ is any function of propositions,

$$p \equiv q \,.\supset.\, fp \equiv fq.$$

Consequently, according to the definition $*13{\cdot}01$,

$$p \equiv q \,.\supset.\, p = q.$$

There will thus be only two propositions, one true and one false. This was Frege's point of view, but it is one which cannot easily be accepted. Frege maintained that every proposition is a proper name, either for the true or for the false. On grounds not connected with our present question, we cannot regard propositions as names; but that does not decide the question whether equivalent propositions are identical. It is this latter question that concerns us. That is to say, we have to consider whether, or in what sense, there are functions fp which are true for some true values of p and false for other true values of p.

Two obvious *primâ facie* instances are "A believes p" and "p is about A." We may take these instances as crucial. If A believes p and p is true, it does

not follow that A believes every other true proposition q; nor, if A believes p, and p is false, does it follow that A believes every other false proposition q. Again, the proposition "A is mortal" is about A; but the proposition "B is mortal," which is equally true, is not about A. Thus the function "p is about A" is not a truth-function of p. This instance is important, because the notation "ϕx" is used to denote a proposition about x, and thus the conception involved *seems* to be presupposed in the whole procedure of propositional functions.

We must, to begin with, distinguish between a proposition as a fact and a proposition as a vehicle of truth or falsehood. The following series of black marks: "Socrates is mortal," is a fact of geography. The noise which I should make if I were to say "Socrates is mortal" would be a fact of acoustics. The mental occurrence when I entertain the belief "Socrates is mortal" is a fact of psychology. None of these introduces the notion of truth or falsehood, which is, for logic, the essential characteristic of propositions. We shall return in a moment to the consideration of propositions as facts.

When we say that truth or falsehood is, for logic, the essential characteristic of propositions, we must not be misunderstood. It does not matter, for mathematical logic, what constitutes truth or falsehood; all that matters is that they divide propositions into two classes according to certain rules. Let us take a set of marks

$$x_1, x_2, \ldots x_{2n-1}, x_{2n}.$$

Let us put, as unexplained assertions,

$$T(x_{2m+1}) \quad (m < n),$$
$$F(x_{2m}) \quad (m \leqslant n).$$

Let us further introduce the symbol $x_r \mid x_s$, and assume

$$T(x_r \mid x_s) \text{ if } F(x_r) \text{ or } F(x_s);$$
$$F(x_r \mid x_s) \text{ if } T(x_r) \text{ and } T(x_s).$$

Assume further that, if p, q, s are any one of the x's or any combination of them by means of the stroke, the above rules are to apply to $p \mid q$, etc., and further we are to have:

$$T\{p \mid (p \mid p)\},$$
$$T\{p \supset q \,.\supset.\, s \mid q \supset p \mid s\},$$

where "$p \supset q$" means "$p \mid (q \mid q)$." Further: given $T\{p \mid (q \mid r)\}$ and $T(p)$, we are to have $T(r)$.

Taking the above as mere conventional rules, all the logic of molecular propositions follows, replacing "$\vdash . p$" by "$T(p)$."

Thus from the formal point of view it is irrelevant what constitutes truth or falsehood: all that matters is that propositions are divided into two classes according to certain rules. It does not matter what propositions are, so long as we are content to regard our primitive propositions as defining hypotheses,

not as truths. (From a philosophical point of view, this formal procedure may be shown to presuppose the non-formal interpretation of our primitive propositions; but that does not matter for our present purpose.)

Throughout the logic of molecular propositions, we do not want to know anything about propositions except whether they are true or false. Further, we are concerned only with those combinations of propositions which are true in virtue of the rules, whether their constituent propositions are true or false. That is—to take the simplest illustration—we assert $p \mid (p \mid p)$, but we never assert any proposition p that has not some suitable molecular structure, although we believe that half of such propositions are true. Our assertions depend always upon structure, never upon the mere fact that some proposition is true.

A new situation arises, however, when we replace p by $\phi ! x$. For example,

we have $\quad \vdash . p \mid (p \mid p)$

and we infer $\quad \vdash . \phi ! x \mid (\phi ! x \mid \phi ! x)$.

We cannot *explain* the notation $\phi ! x$ without introducing characteristics of propositions other than their truth or falsehood. Take for example the primitive proposition (∗8·11)

$$\vdash . (\exists x) . \phi ! x \mid (\phi ! a \mid \phi ! b).$$

The truth of this proposition depends upon the *form* of the constituent propositions $\phi ! x$, $\phi ! a$, $\phi ! b$, not simply upon their truth or falsehood. It cannot be replaced by

$$\text{“} \vdash . (\exists p) . p \mid (q \mid r) \text{,”}$$

which is true but does not have the desired consequences. We are therefore compelled to consider what is meant by saying that a proposition is of the form $\phi ! a$ (where a is some constant). This brings us back to "A occurs in p," which we gave above as an example of a function which is not a truth-function. And this, we shall find, brings us back to the proposition as fact, in opposition to the proposition as true or false.

Let us revert to our two instances: "A believes p" and "p is about A." We shall avoid certain psychological difficulties if we take, to begin with, "A asserts p" instead of "A believes p." Suppose "p" is "Socrates is Greek." A word is a class of similar noises. Thus a person who asserts "Socrates is Greek" is a person who makes, in rapid succession, three noises, of which the first is a member of the class "Socrates," the second a member of the class "is," and the third a member of the class "Greek." This series of events is part of the series of events which constitutes the person. If A is the series of events constituting the person, α is the class of noises "Socrates," β the class "is," and γ the class "Greek," then "A asserts that Socrates is Greek" is (omitting the rapidity of the succession)

$$(\exists x, y, z) . x \epsilon \alpha . y \epsilon \beta . z \epsilon \gamma . x \downarrow y \cup x \downarrow z \cup y \downarrow z \subset A.$$

It is obvious that this is not a function of p as p occurs in a truth-function.

If we now take up "A believes p," we find the matter rather more complicated, owing to doubt as to what constitutes belief. Some people maintain that a proposition must be expressed in words before we can believe it; if that were so, there would not, from our point of view, be any vital difference between believing and asserting. But if we adopt a less unorthodox standpoint, we shall say that when a man believes "Socrates is Greek" he has simultaneously two thoughts, one of which "means" Socrates while the other "means" Greek, and these two thoughts are related in the way we call "predication." It is not necessary for our purposes to define "meaning," beyond noticing that two different thoughts may "have the same meaning." The relation "having the same meaning" is symmetrical and transitive; moreover, if two thoughts "have the same meaning," either can replace the other in any belief without altering its truth-value. Thus we have one class of thoughts, called "Socrates," which all "have the same meaning"; call this class α. We have another class of thoughts, called "Greek," which all "have the same meaning"; call this class β. Call the relation of predication between two thoughts P. (This is the relation which holds between our thought of the subject and our thought of the predicate when we believe that the subject has the predicate. It is wholly different from the relation which holds between the subject and the predicate when our belief is true.) Then "A believes that Socrates is Greek" is

$$(\exists x, y) . x \epsilon \alpha . y \epsilon \beta . xPy . x, y \epsilon C^{\prime}A.$$

Here, again, the proposition as it occurs in truth-functions has disappeared.

It is not necessary to lay any stress upon the above analysis of belief, which may be completely mistaken. All that is intended is to show that "A believes p" may very well not be a function of p, in the sense in which p occurs in truth-functions.

We have now to consider "p is about A," *e.g.* "'Socrates is Greek' is about Socrates." Here we have to distinguish (1) the fact, (2) the belief, (3) the verbal proposition. The fact and the belief, however, do not raise separate problems, since it is fairly clear that Socrates is a constituent of the fact in the same sense in which the thought of Socrates is a constituent of the belief. And the verbal proposition raises no difficulty, since each instance of the verbal proposition is a series containing a part which is an instance of "Socrates." That is to say, "Socrates" (the word) is a class of series of noises, say λ; and "Socrates is Greek" is another class of series, say μ; and the fact that "Socrates" occurs in "Socrates is Greek" is

$$P \epsilon \mu . \supset . (\exists Q) . Q \epsilon \lambda . Q \subset P.$$

Thus we are left with the question: What do we mean by saying that Socrates is a constituent of the fact that Socrates is Greek? This raises the whole problem of analysis. But we do not need an ultimate answer; we only need

an answer sufficient to throw light on the question whether there are functions of propositions which are not truth-functions.

There are those who deny the legitimacy of analysis. Without admitting that they are in the right, we can frame a theory which they need not reject. Let us assume that facts are capable of various kinds of resemblances and differences. Two facts may have particular-resemblance; then we shall say that they are about the same particular. Again they may have predicate-resemblance, or dyadic-relation-resemblance, or etc. We shall say that a fact is about only one particular if any two facts which have particular-resemblance to the given fact have particular-resemblance to each other. Given such a fact, we may define its one particular as the class of all facts having particular-resemblance to the given fact. In that case, to say that Socrates is a constituent of the fact that Socrates is Greek (assuming conventionally that Socrates is a particular) is to say that the fact is a member of the class of facts which is Socrates. In the case of a belief about Socrates, which is itself a fact composed of thoughts, we shall say that a belief is about Socrates if it is one of the class of facts constituting a certain idea which "means" Socrates in whatever sense we may give to "meaning." Here an "idea" is taken to be a class of psychical facts, say all the beliefs which "refer to" Socrates.

We can define predicates by a similar procedure. Take a fact which is only capable of two kinds of resemblance such as we are considering, namely particular-resemblance and predicate-resemblance; such a fact will be a subject-predicate fact. The predicate involved in it is the class of facts to which it has predicate-resemblance.

We shall assume also various kinds of difference: particular-difference, predicate-difference, etc. These are not necessarily incompatible with the corresponding kind of resemblance; *e.g.* $R(x, x)$ and $R(x, y)$ have both particular-resemblance in respect of x and particular-difference in respect of y. This enables us to define what is meant by saying that a particular occurs twice in a fact, as x occurs twice in $R(x, x)$. First: $R(x, x)$ is a dyadic-relation-fact because it is capable of dyadic-relation-resemblance to other facts; second: any two facts having particular-resemblance to $R(x, x)$ have particular-resemblance to each other. This is what we mean by saying that $R(x, x)$ is a dyadic-relation-fact in which x occurs twice, not a subject-predicate fact. Take next a triadic-relation-fact $R(x, x, z)$. This is, by definition, a triadic-relation-fact because it is capable of triadic-relation-resemblance. The facts having particular-resemblance to $R(x, x, z)$ can be divided into two groups (not three) such that any two members of one group have particular-resemblance to each other. This shows that there is repetition, but not whether it is x or z that is repeated. The facts of the one group are $R(x, x, c)$ for varying c; the facts of the other are $R(a, b, z)$ for varying a and b. Each fact of the group $R(x, x, c)$ belongs to only two groups constituted by particular-resemblance, whereas

the facts of the group $R(a, b, z)$, except when it happens that $a = b$, belong to three groups constituted by particular-resemblance. This defines what is meant by saying that x occurs twice and z once in the fact $R(x, x, z)$. It is obvious that we can deal with tetradic etc. relations in the same way.

According to the above, when we say that Socrates is a constituent of the fact that Socrates is Greek, we mean that this fact is a member of the class of facts which is Socrates.

When we use the notation "$\phi ! x$" to denote a proposition in which "x" occurs, it is a fact that "x" occurs in "$\phi ! x$," but we do not need to assert the fact; the fact does its work without having to be asserted. It is also a fact that, if "x" occurs in a proposition p, and p asserts a fact, then x is a constituent of that fact. This is not a law of logic, but a law of language. It might be false in some languages. For instance, in former days, when a crime was committed in India, the indictment stated that it was committed "in the manor of East Greenwich." These words did not denote any constituents of the fact. But a logical language avoids fictions of this kind.

The notation for functions is an illustration of Wittgenstein's principle, that a logical symbol must, in certain formal respects, resemble what it symbolizes. All the facts of which x is a constituent, according to the above, constitute a certain class defined by particular-resemblance. The various symbols $\phi x, \psi x, \chi x, \ldots$ also all resemble each other in a certain respect, namely that their right-hand halves are very similar (not *exactly* similar, because no two x's are exactly alike). The symbols $R(x, x)$, $R(x, x, z)$, etc. are appropriate to their meanings for similar reasons. The symbols are *used* before their suitability can be explained. To explain *why* "ϕx" is a suitable symbol for a proposition about x is, as we have seen, a complicated matter. But to use the symbol is not a complicated matter. Our symbolism, as a set of facts, resembles, in certain logical respects, the facts which it is to symbolize. This makes it a good symbolism. But in using it we do not presuppose the explanation of why it is good, which belongs to a later stage. And so the notation "ϕx" can be used without first explaining what we mean by "a proposition about x."

We are now in a position to deal with the difference between propositions considered factually and propositions as vehicles of truth and falsehood. When we say "'Socrates' occurs in the proposition 'Socrates is Greek,'" we are taking the proposition factually. Taken in this way, it is a class of series, and 'Socrates' is another class of series. Our statement is only true when we take the proposition and the name as classes. The particular 'Socrates' that occurs at the beginning of our sentence does not occur in the proposition 'Socrates is Greek'; what is true is that another particular closely resembling it occurs in the proposition. It is therefore absolutely essential to all such statements to take words and propositions as classes of similar occurrences, not as single occurrences. But when we assert a proposition, the single occurrence is all

that is relevant. When I assert "Socrates is Greek," the particular occurrences of the words have meaning, and the assertion is made by the particular occurrence of that sentence. And to say of that sentence "'Socrates' occurs in it" is simply false, if I mean the 'Socrates' that I have just written down, since it was a different 'Socrates' that occurred in it. Thus we conclude:

A proposition as the vehicle of truth or falsehood is a particular occurrence, while a proposition considered factually is a class of similar occurrences. It is the proposition considered factually that occurs in such statements as "A believes p" and "p is about A."

Of course it is possible to make statements about the particular fact "Socrates is Greek." We may say how many centimetres long it is; we may say it is black; and so on. But these are not the statements that a philosopher or logician is tempted to make.

When an assertion occurs, it is made by means of a particular fact, which is an instance of the proposition asserted. But this particular fact is, so to speak, "transparent"; nothing is said about it, but by means of it something is said about something else. It is this "transparent" quality which belongs to propositions as they occur in truth-functions. This belongs to p when p is asserted, but not when we say "p is true." Thus suppose we say: "All that Xenophon said about Socrates is true." Put

$$X(p) \,.\!=.\, \text{Xenophon asserted } p,$$
$$S(p) \,.\!=.\, p \text{ is about Socrates.}$$

Then our statement is

$$X(p) \,.\, S(p) \,.\supset_p .\, p \text{ is true.}$$

Here the occurrence of p is not "transparent." But if we say

$$x \in \alpha \,.\supset_x .\, \phi\,!\,x$$

we are asserting $\phi\,!\,x$ for a whole class of values of x, and yet "$\phi\,!\,x$" still has a "transparent" occurrence. The essential difference is that in the former case we speak *about* the symbol or belief, whereas in the latter we merely use it to speak about something else. This is the point which distinguishes the occurrences of propositions in mathematical logic from their occurrences in non-truth-functions.

Let us endeavour to give greater definiteness to this point. Take the statement "Socrates had all the predicates that Xenophon said he had." Let the series of events which was Xenophon be called X. Then if Xenophon attributed the predicate α to Socrates, we might appear to have (writing $x \downarrow y \downarrow z \downarrow w$ for the series x, y, z, w)

$$\text{Socrates} \downarrow \text{had} \downarrow \text{predicate} \downarrow \alpha \Subset X.$$

Thus our assertion would be

$$\text{Socrates} \downarrow \text{had} \downarrow \text{predicate} \downarrow \alpha \Subset X \,.\supset_\alpha .\, \text{Socrates had predicate } \alpha.$$

Here, however, there is an ambiguity. On the left, "Socrates," "had," "predicate," and "α" occur as noises; on the right they occur as symbols. This

ambiguity amounts to a fallacy. For, in fact, what I write on paper is not the noise that Xenophon made, but a symbol for that noise. Thus I am using one symbol "Socrates" in two senses: (a) to mean the noise that Xenophon made on a certain occasion, (b) to mean a certain man. We must say:

If Xenophon made a series of noises which mean what is meant by "Socrates had the predicate α," then what this means is true.

For example: If Xenophon said "Socrates was wise," then what is meant by "Socrates was wise" is true.

But this does not assert that Socrates was wise. When I actually assert that Socrates was wise, I say something which cannot be said by talking about the words I use in saying it; and when I assert that Socrates was wise, although an instance of the proposition occurs, yet I do not say anything whatever about the proposition—in particular I do not say that it is true. This is an inference, not logical, but linguistic.

If the above considerations in any way approximate to the truth, we see that there is an absolute gulf between the assertion of a proposition and an assertion about the proposition. The p that occurs when we assert p and the p that occurs in "A asserts p" are by no means identical. The occurrence of propositions as asserted is simpler than their occurrence as something spoken about. In the assertion of a proposition, and in the assertion of any molecular function of a proposition, the proposition does not occur, if we mean by the proposition the p that occurs in such propositions as "A asserts p" or "p is about A." When these latter are analysed, they are found not to conflict with the view that propositions, in the sense in which they occur when they are asserted, only occur in truth-functions.

When p is asserted, p does not really occur, but the constituents of p occur, or an instance of p occurs. The same is true when a molecular proposition containing p is asserted. Thus we cannot infer $p = q$, because here p and q occur in a sense in which they do not occur when molecular propositions containing them are asserted.

Similar considerations apply to propositional functions. Suppose there are two predicates α and β which are always found together; we may still say that they are two, on the ground that $\alpha(x)$ and $\beta(x)$ are facts which do not have predicate-resemblance. But the propositional function $\alpha(\hat{x})$ is solely to be used in building up matrices by means of the stroke. The predicate α is a class of facts, whereas the propositional function $\alpha(\hat{x})$ is merely a symbolic convenience in speaking about certain propositions. Thus we may have $\alpha(\hat{x}) = \beta(\hat{x})$ without having $\alpha = \beta$. In this way we escape the *primâ facie* paradoxes of the theory that propositions only occur in truth-functions and propositional functions only occur through their values. The paradoxes rest on the confusion between factual and assertive propositions.

LIST OF DEFINITIONS

1·01. $p \supset q$

2·33. $p \vee q \vee r$

3·01. $p \,.\, q$

3·02. $p \supset q \supset r$

4·01. $p \equiv q$

4·02. $p \equiv q \equiv r$

4·34. $p \,.\, q \,.\, r$

9·01. $\sim\{(x) \,.\, \phi x\}$

9·011. $\sim(x) \,.\, \phi x$

9·02. $\sim\{(\exists x) \,.\, \phi x\}$

9·021. $\sim(\exists x) \,.\, \phi x$

9·03. $(x) \,.\, \phi x \,.\, \vee \,.\, p$

9·04. $p \,.\, \vee \,.\, (x) \,.\, \phi x$

9·05. $(\exists x) \,.\, \phi x \,.\, \vee \,.\, p$

9·06. $p \,.\, \vee \,.\, (\exists x) \,.\, \phi x$

9·07. $(x) \,.\, \phi x \,.\, \vee \,.\, (\exists y) \,.\, \psi y$

9·08. $(\exists y) \,.\, \psi y \,.\, \vee \,.\, (x) \,.\, \phi x$

10·01. $(\exists x) \,.\, \phi x$

10·02. $\phi x \supset_x \psi x$

10·03. $\phi x \equiv_x \psi x$

11·01. $(x, y) \,.\, \phi(x, y)$

11·02. $(x, y, z) \,.\, \phi(x, y, z)$

11·03. $(\exists x, y) \,.\, \phi(x, y)$

11·04. $(\exists x, y, z) \,.\, \phi(x, y, z)$

11·05. $\phi(x, y) \,.\, \supset_{x, y} \,.\, \psi(x, y)$

11·06. $\phi(x, y) \,.\, \equiv_{x, y} \,.\, \psi(x, y)$

13·01. $x = y$

13·02. $x \neq y$

13·03. $x = y = z$

14·01. $[(\iota x)(\phi x)] \,.\, \psi(\iota x)(\phi x)$

14·02. $E!(\iota x)(\phi x)$

14·03. $[(\iota x)(\phi x), (\iota x)(\psi x)] \,.\, f\{(\iota x)(\phi x), (\iota x)(\psi x)\}$

14·04. $[(\iota x)(\psi x)] \,.\, f\{(\iota x)(\phi x), (\iota x)(\psi x)\}$

20·01. $f\{\hat{z}(\psi z)\}$

20·02. $x \in (\phi ! \hat{z})$

20·03. Cls

20·04. $x, y \in \alpha$

20·05. $x, y, z \in \alpha$

20·06. $x \sim\in \alpha$

20·07. $(\alpha) \,.\, f\alpha$

20·071. $(\exists \alpha) \,.\, f\alpha$

20·072. $[(\iota\alpha)(\phi\alpha)] \,.\, f(\iota\alpha)(\phi\alpha)$

20·08. $f\{\hat{\alpha}(\psi\alpha)\}$

20·081. $\alpha \in \psi ! \hat{\alpha}$

21·01. $f\{\hat{x}\hat{y}\psi(x, y)\}$

21·02. $a\{\phi ! (\hat{x}, \hat{y})\} b$

21·03. Rel

21·07. $(R) \,.\, fR$

21·071. $(\exists R) \,.\, fR$

21·072. $[(\iota R)(\phi R)] \,.\, f(\iota R)(\phi R)$

21·08. $f\{\hat{R}\hat{S}\psi(R, S)\}$

21·081. $P\{\phi ! (\hat{R}, \hat{S})\} Q$

21·082. $f\{\hat{R}(\psi R)\}$

21·083. $R \in \phi ! \hat{R}$

22·01. $\alpha \subset \beta$

22·02. $\alpha \cap \beta$

22·03. $\alpha \cup \beta$

22·04. $-\alpha$

22·05. $\alpha - \beta$

22·53. $\alpha \cap \beta \cap \gamma$

22·71. $\alpha \cup \beta \cup \gamma$

23·01. $R \Subset S$

23·02. $R \dot{\cap} S$

23·03. $R \dot{\cup} S$

23·04. $\dot{-} R$

23·05. $R \dot{-} S$

23·53. $R \dot{\cap} S \dot{\cap} T$

23·71. $R \dot{\cup} S \dot{\cup} T$

24·01. V

24·02. Λ

24·03. $\exists ! \alpha$

25·01. $\dot{\mathrm{V}}$

25·02. $\dot{\Lambda}$

25·03. $\dot{\exists} ! R$

30·01. $R\text{‘}y$

30·02. $R\text{‘}S\text{‘}y$

31·01. Cnv

31·02. $\breve{P}$

32·01. $\overrightarrow{R}$

32·02. $\overleftarrow{R}$

32·03. sg

32·04. gs

33·01. D

33·02. ᗡ

33·03. C

33·04. F

34·01. $R \quad S$

34·02. R^2

34·03. R^3

35·01. $\alpha \upharpoonleft R$

35·02. $R \upharpoonright \beta$

35·03. $\alpha \upharpoonleft R \upharpoonright \beta$

35·04. $\alpha \uparrow \beta$

35·05. $R\text{‘}x \uparrow \beta$

35·24 $\alpha \upharpoonleft R \mid S$

35·25. $S \mid R \upharpoonright \beta$

36·01. $P \lfloor \alpha$

37·01. $R\text{‘‘}\beta$

37·02. R_ϵ

37·03. $\breve{R}_\epsilon$

37·04. $R\text{‘‘‘}\kappa$

37·05. $\mathrm{E} !! R\text{‘‘}\beta$

38·01. $x\text{♀}$

38·02. $\text{♀}y$

38·03. $\alpha \underset{,,}{\text{♀}} y$

40·01. $p\text{‘}\kappa$

40·02. $s\text{‘}\kappa$

41·01. $\dot{p}\text{‘}\lambda$

41·02. $\dot{s}\text{‘}\lambda$

43·01. $R \parallel S$

50·01. I

50·02. J

51·01. ι

52·01. 1

54·01. 0

54·02. 2

55·01. $x \downarrow y$

55·02. $R\text{‘}x \downarrow y$

56·01. $\dot{2}$

56·02. 2_r

56·03. 0_r

60·01. Cl

60·02. Cl ex

60·03. Cls^2

60·04. Cls^3

61·01. Rl

61·02. Rl ex

61·03. Rel^2

61·04. Rel^3

62·01. ϵ

63·01. $t\text{‘}x$

63·011. $t^1\text{‘}x$

63·02. $t_0\text{‘}\alpha$

63·03. $t_1\text{‘}\kappa$

63·04. $t^2\text{‘}x$

63·041. $t^3\text{‘}x$

63·05. $t_2\text{‘}\kappa$

63·051. $t_3\text{‘}\kappa$

64·01. $t_{00}\text{‘}\alpha$

64·011. $t^{11}\text{‘}x$

64·012. $t^{12}\text{‘}x$

64·013. $t^{21}\text{‘}x$

64·014. $t^{22}\text{‘}x$

64·02. $t_{01}\text{‘}\alpha$

64·021. $t_{10}\text{‘}\alpha$

64·022. $t_{11}\text{‘}\alpha$

64·03. $t_0{}^1\text{‘}\alpha$

64·031. $t_1{}^1\text{‘}\alpha$

64·04. ${}^1t_0\text{‘}\alpha$

64·041. ${}^1t_1\text{‘}\alpha$

65·01. α_x

65·02. $\alpha(x)$

65·03. R_x

65·04. $R(x)$

65·1. $R_{(x,y)}$

65·11. $R(x_y)$

65·12. $R(x,y)$

70·01. $\alpha \rightarrow \beta$

73·01. $\alpha\,\overline{\text{sm}}\,\beta$

73·02. sm

80·01. P_Δ

84·01. Cls^2 excl

84·02. Cl excl‘γ

84·03. Cls ex^2 excl

85·5. $P \downarrow y$

88·01. Rel Mult

88·02. Cls^2 Mult

88·03. Mult ax

90·01. R_*

90·02. $\breve{R}_*$

91·01. R_{st}

91·02. R_{ts}

91·03. Pot‘R

91·04. Potid‘R

91·05. R_{po}

93·01. B

93·02. min_P

93·021. max_P

93·03. gen‘P

95·01. $(P * Q)$ Dft [*95]

96·01. $I_R\text{‘}x$ Dft [*96]

96·02. $J_R\text{‘}x$ Dft [*96]

97·01. $\overleftrightarrow{R}\text{‘}x$

100·01. Nc

100·02. NC

102·01. $\mathrm{NC}^{\beta}(\alpha)$

103·01. $\mathrm{N_0c}‘\alpha$

103·02. $\mathrm{N_0C}$

104·01. $\mathrm{N^1c}‘\alpha$

104·011. $\mathrm{N^2c}‘\alpha$

104·02. $\mathrm{N^1C}$

104·021. $\mathrm{N^2C}$

104·03. $\mu^{(1)}$

104·031. $\mu^{(2)}$

105·01. $\mathrm{N_1c}‘\alpha$

105·011. $\mathrm{N_2c}‘\alpha$

105·02. $\mathrm{N_1C}$

105·021. $\mathrm{N_2C}$

105·03. $\mu_{(1)}$

105·031. $\mu_{(2)}$

106·01. $\mathrm{N_{00}c}‘\alpha$

106·011. $\mathrm{N^{11}c}‘\alpha$

106·012. $\mathrm{N_{01}c}‘\alpha$

106·02. $\mathrm{N_0{}^1c}‘\alpha$

106·021. $\mathrm{{}^1N_0c}‘\alpha$

106·03. $\mathrm{N_{00}C}$

106·04. $\mu_{(00)}$

106·041. $\mu^{(11)}$

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110·03. $\mathrm{Nc}‘\alpha +_c \mu$

110·04. $\mu +_c \mathrm{Nc}‘\alpha$

110·561. $\mu +_c \nu +_c \varpi$

111·01. $\kappa\,\overline{\mathrm{sm}}\,\overline{\mathrm{sm}}\,\lambda$

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111·03. sm sm

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112·02. $\Sigma\mathrm{Nc}‘\kappa$

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113·04. $\mathrm{Nc}‘\beta\times_c\mu$

113·05. $\mu\times_c\mathrm{Nc}‘\alpha$

113·511. $\alpha\times\beta\times\gamma$

113·541. $\mu\times_c\nu\times_c\varpi$

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115·01. $\mathrm{Prod}‘\kappa$

115·02. $\mathrm{Cls^3\,arithm}$

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116·02. μ^{ν}

116·03. $(\mathrm{Nc}‘\alpha)^{\nu}$

116·04. $\mu^{\mathrm{Nc}‘\beta}$

117·01. $\mu>\nu$

117·02. $\mu>\mathrm{Nc}‘\alpha$

117·03. $\mathrm{Nc}‘\alpha>\nu$

117·04. $\mu<\nu$

117·05. $\mu\geqslant\nu$

117·06. $\mu\leqslant\nu$

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119·02. $\mathrm{Nc}‘\alpha -_c \nu$

119·03. $\gamma -_c \mathrm{Nc}‘\beta$

120·01. NC induct

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120·02. Cls induct

120·021. Cls_ξ induct

120·03. Infin ax

120·04. Infin ax (x)

120·43. $\mathrm{spec}‘\beta$

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121·011. $P(x \dashv y)$

121·012. $P(x \vdash y)$

121·013. $P(x \vdash\!\dashv y)$

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121·03. finid'P

121·031. fin'P

121·04. ν_P

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124·02. NC refl

124·021. Nc'$\rho \in$ NC refl

124·03. NC mult

126·01. NC ind

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150·02. $S \dagger Q$

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150·04. R'$S ; Q$

150·05. $R ; S ; Q$

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155·02. N_0R

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161·02. $x \leftarrow\!\!+ P$

161·212. $P +\!\!\to x +\!\!\to y$

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162·01. Σ'P

163·01. Rel^2 excl

164·01. $P \,\overline{\text{smor}}\,\overline{\text{smor}}\, Q$

164·02. smor smor

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166·421. $P \times Q \times R$

170·01. P_{cl}

170·02. P_{lc}

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171·02. P_{fd}

172·01. Π'P

173·01. Prod'P

174·01. Rel^3 arithm

176·01. P exp Q

176·02. P^Q

180·01. $P + Q$

180·02. $\mu \dot{+} \nu$

180·03. Nr'$P \dot{+} \nu$

180·04. $\mu \dot{+}$ Nr'Q

180·561. $\mu \dot{+} \nu \dot{+} \varpi$

181·01. $P +\!\!\to x$

181·011. $x \leftarrow\!\!+ P$

181·02. $\mu \dot{+} \dot{1}$

181·021. $\dot{1} \dot{+} \mu$

181·03. Nr'$P \dot{+} \dot{1}$

181·031. $\dot{1} \dot{+}$ Nr'P

181·04. $\dot{1} \dot{+} \dot{1}$

181·561. $\mu \dot{+} \dot{1} \dot{+} \dot{1}$

181·571. $\dot{1} \dot{+} \dot{1} \dot{+} \mu$

182·01. $\widehat{\text{♀}}$

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204·01. Ser
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206·02. prec_P
207·01. lt_P
207·02. tl_P
207·03. limax_P
207·04. limin_P
208·01. cror‘P
211·01. sect‘P
212·01. ς‘P
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231·01. $P\overline{R}_{\mathrm{sc}}Q$
231·02. $P\overline{R}_{\mathrm{os}}Q$
232·01. $(P\overline{R}Q)_{\mathrm{sc}}$‘$\alpha$
232·02. $(P\overline{R}Q)_{\mathrm{os}}$‘$\alpha$
233·01. $(P\overline{R}Q)_{\mathrm{lmx}}$
233·02. $R_{\cdot}(PQ)$
234·01. sc (P, Q)‘R
234·02. os (P, Q)‘R
234·03. ct (PQ)‘R
234·04. contin (PQ)‘R
234·05. P contin Q
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250·02. Ω
251·01. NO
254·01. less
254·02. P_{sm}
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255·03. $\mathrm{N}_0\mathrm{O}$
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256·02. N Dft [*256]
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302·03.	(ρ, σ) Prm (μ, ν)	
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ALFRED NORTH WHITEHEAD

AND

BERTRAND RUSSELL

VOLUME II

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CONTENTS OF VOLUME II

PREFATORY STATEMENT OF SYMBOLIC CONVENTIONS

THE purpose of the following observations is to bring together in one discussion various explanations which are required in applying the theory of types to cardinal arithmetic. It is convenient to collect these observations, since otherwise their dispersion throughout the several numbers of Part III makes it difficult to see what is their total effect. But although we have placed these observations at the beginning, they are to be read concurrently with the text of Part III, at least with so much of the text as consists of explanations of definitions. The earlier portion of what follows is merely a *résumé* of previous explanations; it is only in the later portions that the application to cardinal arithmetic is made.

I. *General Observations on Types.*

Three different kinds of typical ambiguity are involved in our propositions, concerning:

(1) the functional hierarchy,
(2) the propositional hierarchy,
(3) the extensional hierarchy.

The relevance of these must be separately considered.

We often speak as though the type represented by small Latin letters were not composed of functions. It is, however, compatible with all we have to say that it should be composed of functions. It is to be observed, further, that, given the number of individuals, there is nothing in our axioms to show how many predicative functions of individuals there are, *i.e.* their number is not a function of the number of individuals: we only know that their number $\geqslant 2^{\text{Nc}'\text{Indiv}}$, where "Indiv" stands for the class of individuals.

In practice, we proceed along the extensional hierarchy after the early numbers of the book. If we have started from individuals, the result of this is to exclude functions wholly from our hierarchy; if we have started with functions of a given type, all functions of other types are excluded. Thus a fresh extensional hierarchy, wholly excluding every other, starts from each type of function. When we speak simply of "*the* extensional hierarchy," we mean the one which starts from individuals.

It is to be observed that when we have the assertion of a propositional function, say "$\vdash . \phi x$," the x must be of some definite type, *i.e.* we only assert that ϕx is true whatever x may be within some one type. Thus *e.g.* "$\vdash . x = x$" does not assert more than that this assertion holds for any x of a given type. It is true that symbolically the same assertion holds in other types, but other

types cannot be included under one assertion-sign, because no variable can travel beyond its type.

The process of rendering the types of variables ambiguous is begun in ∗8 and ∗9, where we take the first step in regard to the *propositional* hierarchy. Before ∗8 and ∗9, our variables are *elementary propositions*. These are such as contain no apparent variables. Hence the only functions that occur are matrices, and these only occur through their values. The assumption involved in the transition from Section A to Section B (Part I) is that, given "$\vdash . fp$," where p is an elementary proposition, we may substitute for p "$\phi ! (x, y, z, \ldots)$," where ϕ is any matrix. Thus instead of "$\vdash . fp$," which contained one variable p of a given type, we have "$\vdash . f\{\phi ! (x, y, z, \ldots)\}$," which contains several variables of several types (any finite number of variables and types is possible). This assumption involves some rather difficult points. It is to be remembered that no *value* of ϕ contains ϕ as a constituent, and therefore ϕ is not a constituent of fp even if p is a value of ϕ. Thus we pass, above, from an assertion containing no function as a constituent to one containing one or more functions as constituents. The assertion "$\vdash . fp$" concerns *any* elementary proposition, whereas "$\vdash . f\{\phi ! (x, y, z, \ldots)\}$" concerns *any of a certain set* of elementary propositions, namely any of those that are values of ϕ. Different types of functions give different sorts of ways of picking out elementary propositions.

Having assumed or proved "$\vdash . fp$," where p is elementary and therefore involves no ambiguity of type, we thus assert

$$\vdash . f\{\phi ! (x, y, z, \ldots)\},$$

where the types of the arguments and the number of them are wholly arbitrary, except that they must belong to the functional hierarchy including individuals. (The assumption that propositions are incomplete symbols excludes the possibility that the arguments to ϕ are propositions.) The noteworthy point is that we thus obtain an assertion in which there may be any finite number of variables and the variables have unlimited typical ambiguity, from an assertion containing one variable of a perfectly definite type. All this is presupposed before we embark on the propositional hierarchy.

It should be observed that all elementary propositions are values of predicative functions of one individual, *i.e.* of $\phi ! \hat{x}$, where $\hat{x}$ is individual. Thus we need not *assume* that elementary propositions form a type; we may replace p by "$\phi ! x$" in "$\vdash . fp$." In this way, propositions as variables wholly disappear.

In extending statements concerning elementary propositions so as formally to apply to first-order propositions, we have to assume afresh the primitive proposition ∗1·11 (∗1·1 is never used), *i.e.* given "$\vdash . \phi x$" and "$\vdash . \phi x \supset \psi x$," we have "$\vdash . \psi x$," which is practically ∗9·12. This was asserted in ∗1·11 for any case in which ϕx and ψx are elementary propositions. There was

here already an ambiguity of type, owing to the fact that x need not be an individual, but might be a function of any order. *E.g.* we might use $*1{\cdot}11$ to pass from

$$\text{“}\vdash . \phi ! a\text{” and “}\vdash . \phi ! a \supset \phi ! b\text{” to “}\vdash . \phi ! b,\text{”}$$

where ϕ replaces the x of $*1{\cdot}11$, and $\hat{\phi} ! a$, $\hat{\phi} ! b$ replace $\phi\hat{x}$ and $\psi\hat{x}$. Thus $*1{\cdot}11$, even before its extension in $*9$, already states a fresh primitive proposition for each fresh type of functions considered. The novelty in $*9$ is that we allow ϕ and ψ to contain one apparent variable. This may be of any functional type (including Indiv); thus we get another set of symbolically identical primitive propositions. In passing, as indicated at the end of $*9$, to more than one apparent variable, we introduce a new batch of primitive propositions with each additional apparent variable.

Similar remarks apply to the other primitive propositions of $*9$.

What makes the above process legitimate is that nothing in the treatment of functions of order n *presupposes* functions of higher order. We can deal with each new type of functions as it arises, without having to take account of the fact that there are later types. From symbolic analogy we "see" that the process can be repeated indefinitely. This possibility rests upon two things:

(1) A fresh interpretation of our constants—$\vee$, $\sim$, $!$, $(x)\,.$, $(\exists x)\,.$—at each fresh stage;

(2) A fresh assumption, symbolically unchanged, of the primitive propositions which we found sufficient at an earlier stage—the possibility of avoiding symbolic change being due to the fresh interpretation of our constants.

The above remarks apply to the axiom of reducibility as well as to our other primitive propositions. If, at any stage, we wish to deal with a class defined by a function of the 30,000th type, we shall have to repeat our arguments and assumptions 30,000 times. But there is still no necessity to speak of the hierarchy as a whole, or to suppose that statements can be made about "all types."

We come now to the extensional hierarchy. This starts from some one point in the functional hierarchy. We usually suppose it to start from individuals, but any other starting-point is equally legitimate. Whatever type of functions (including Indiv) we start from, all higher types of functions are excluded from the extensional hierarchy, and also all lower types (if any). Some complications arise here. Suppose we start from Indiv. Then if $\phi ! \hat{z}$ is any predicative function of individuals, $\hat{z}(\phi ! z) = \phi ! \hat{z}$. But if we adopt the theory of $*20$, as opposed to that suggested in the Introduction to the second edition, identity between a function and a class does not have the usual properties of identity; in fact, though every function is identical with some class, and vice versa, the number of functions is likely to be

greater than the number of classes. This is due to the fact that we may have $\hat{z}(\phi ! z) = \psi ! \hat{z} . \hat{z}(\phi ! z) = \chi ! \hat{z}$ without having $\psi ! \hat{z} = \chi ! \hat{z}$.

In the extensional hierarchy, we prove the extension from classes to classes of classes, and so on, without fresh primitive propositions (∗20, ∗21) The primitive propositions involved are those concerning the *functional* hierarchy.

From all these various modes of extension we "see" that whatever *can be proved* for lower types, whether functional or extensional, can also be proved for higher types*. Hence we assume that it is unnecessary to know the types of our variables, though they must always be confined within some one definite type.

Now although everything that can be proved for lower types can be proved for higher types, the converse does not hold. In Vol. I only two propositions occur which can be proved for higher but not for lower types. These are $\exists ! 2$ and $\exists ! 2_r$. These can be proved for any type except that of individuals. It is to be observed that we do *not* state that whatever is *true* for lower types is *true* for higher types, but only that whatever *can be proved* for lower types *can be proved* for higher types. If, for example, $\text{Nc}\text{‘}\text{Indiv} = \nu$, then this proposition is false for any higher type; but this proposition, $\text{Nc}\text{‘}\text{Indiv} = \nu$, is one which cannot be proved logically; in fact, it is only ascertainable by a census, not by logic. Thus among the propositions which can be proved by logic, there are some which can only be proved for higher types, but none which can only be proved for lower types.

The propositions which can be proved in some types but not in others all are or depend upon existence-theorems for cardinals. We can prove

$\exists ! 0$, $\exists ! 1$, universally,
$\exists ! 2$, except for Indiv,
$\exists ! 3$, $\exists ! 4$, except for Indiv, Cl‘Indiv, Rl‘Indiv; and so on.

Exactly similar remarks would apply to the functional hierarchy. In both cases, the possibility of proving these propositions depends upon the axiom of reducibility and the definition of identity. Suppose there is only one individual, x. Then $\hat{y} = x$, $\hat{y} \neq x$ are two different functions, which, by the axiom of reducibility, are equivalent to two different predicative functions. Hence there are at least two predicative functions of x, and at least two classes $\iota\text{‘}x$, Λ_x. This argument fails both for classes and functions if either we deny the axiom of reducibility or we suppose that there may be two different individuals which agree in all their predicates, *i.e.* that the definition of identity is misleading.

The statement that what can be proved for lower types can be proved for higher types requires certain limitations, or rather, a more exact formulation.

* But cf. next page for a more exact statement of this principle.

Taking Indiv as a primitive idea, put

$$\mathrm{Kl} = \mathrm{Cl}\text{‘}\mathrm{Indiv} \quad \mathrm{Df}, \qquad \mathrm{Kl}^2 = \mathrm{Cl}\text{‘}\mathrm{Kl} \quad \mathrm{Df}, \quad \text{etc.}$$

Then consider the proposition $\mathrm{Nc}\text{‘}\mathrm{Kl} = \Lambda$. We can prove

$$\mathrm{Nc}\text{‘}\mathrm{Kl} \cap t\text{‘}\mathrm{Indiv} = \Lambda \,.\, \exists \,!\, \mathrm{Nc}\text{‘}\mathrm{Kl} \cap t\text{‘}\mathrm{Kl} \,.\, \exists \,!\, \mathrm{Nc}\text{‘}\mathrm{Kl} \cap t\text{‘}\mathrm{Kl}^2 \,.\, \text{etc.}$$

Thus $\mathrm{Nc}\text{‘}\mathrm{Kl} = \Lambda$ can be proved in the lowest type in which it is significant, and disproved in any other. The difficulty, however, is avoided if Indiv is replaced by a variable α, and Kl by $\mathrm{Cl}\text{‘}t_0\text{‘}\alpha$. Then we have

$$\mathrm{Nc}\text{‘}\mathrm{Cl}\text{‘}t_0\text{‘}\alpha \cap t\text{‘}\alpha = \Lambda,$$

and this holds whatever the type of α may be. Thus in order that our principle about lower and higher types may be true, it is necessary that any relation there may be between two types occurring in a proposition should be preserved; in other words, when one constant type is *defined* in terms of another (as Kl and Indiv), the definition must be restored before the type is varied, so that when one type is varied, so is the other. With this proviso, our principle about higher and lower types holds.

With the above proviso, the truth of our statement is manifest. For we have shown that the same primitive propositions, symbolically, which hold for the lowest type concerned in our reasoning, hold also for subsequent types; and therefore all our proofs can be repeated symbolically unchanged.

The importance of this lies in the fact that, when we have proved a proposition for the lowest significant type, we "see" that it holds in any other assigned significant type. Hence every proposition which is proved without the mention of any type is to be regarded as proved for the lowest significant type, and extended by analogy to any other significant type.

By exactly similar considerations we "see" that a proposition which can be proved for some type other than the lowest significant type must hold for any type in the direct descent from this. *E.g.* suppose we can prove a proposition (such as $\exists \,!\, 2$) for the type Kl (where $\mathrm{Kl} = \mathrm{Cl}\text{‘}\mathrm{Indiv}$); then merely writing Cl‘Indiv for Kl, we have a proposition which is proved concerning Indiv, namely $\exists \,!\, 2 \cap t\text{‘}\mathrm{Cl}\text{‘}\mathrm{Indiv}$, and here, by what was said before, Indiv may be replaced by any higher type.

Thus given a typically ambiguous relation R, such that, if τ is a type, $R\text{‘}\tau$ is a type (Cl or Rl is such a relation), we "see" that, if we can prove $\phi(R\text{‘}\mathrm{Indiv})$, we can also prove $\phi(R\text{‘}\tau)$, where τ is any type, and ϕ is composed of typically ambiguous symbols. Similarly if we can prove $\phi(\mathrm{Indiv}, R\text{‘}\mathrm{Indiv})$, we can prove $\phi(\tau, R\text{‘}\tau)$, where τ is any type. But we cannot in general prove $\phi(\mathrm{Indiv}, R\text{‘}\tau)$ or $\phi(\tau, R\text{‘}\mathrm{Indiv})$, and these may be in fact untrue. *E.g.* we have $\exists \,!\, \mathrm{Nc}\,(\mathrm{Kl})\text{‘}\mathrm{Indiv} \,.\, \sim\exists \,!\, \mathrm{Nc}\,(\mathrm{Kl})\text{‘}\mathrm{Kl}^2$.

Thus more generally, when a proposition containing several ambiguities can be proved for the types $R\text{‘}\mathrm{Indiv}$, $S\text{‘}\mathrm{Indiv}$, ..., but not for lower types, it

is to be regarded as a function of Indiv, and then it becomes true for *any* type; that is, given

$$\phi\,(R\text{‘Indiv}, S\text{‘Indiv}, \ldots),$$

we shall also have

$$\phi\,(R\text{‘}\tau, S\text{‘}\tau, \ldots),$$

where τ is any type. In this way, *all* demonstrable propositions are in the first instance about Indiv, and when so expressed remain true if any other type is substituted for Indiv.

When a proposition containing typically ambiguous symbols *can be proved* to be true in the lowest significant type, and we can "see" that symbolically the same proof holds in any other assigned type, we say that the proposition has "permanent truth." (We may also say, loosely, that it is "true in all types.") When a proposition containing typically ambiguous symbols *can be proved* to be false in the lowest significant type, and we can "see" that it is false in any other assigned type, we say that it has "permanent falsehood." Any other proposition containing typically ambiguous symbols is said to be "fluctuating," or to have "fluctuating truth-value," as opposed to "permanent truth-value," which belongs to propositions that have either permanent truth or permanent falsehood.

In what follows, ambiguities concerned with the propositional hierarchy will be ignored, since they never lead to fluctuating propositions. Thus disjunction and negation and their derivatives will not receive explicit typical determination, but only such typical determination as results from assigning the types of the other typically ambiguous symbols involved.

It is convenient to call the symbolic form of a propositional function simply a "*symbolic form.*" Thus, if a symbolic form contains symbols of ambiguous type it represents different propositional functions according as the types of its ambiguous symbols are differently adjusted. The adjustment is of course always limited by the necessity for the preservation of meaning. It is evident that the ideas of "permanent truth-value" and "fluctuating truth-value" apply in reality to symbolic forms and not to propositions or propositional functions. Ambiguity of type can only exist in the process of determination of meaning. When the meaning has been assigned to a symbolic form and a propositional function thereby obtained, all ambiguity of type has vanished.

To "assert a symbolic form" is to assert each of the propositional functions arising for the set of possible typical determinations which are somewhere enumerated. We have in fact enumerated a very limited number of types starting from that of individuals, and we "see" that this process can be indefinitely continued by analogy. The form is always asserted so far as the enumeration has arrived; and this is sufficient for all purposes, since it is essentially impossible to use a type which has not been arrived at by successive enumeration from the lower types.

The only difficulties which arise in Cardinal Arithmetic in connection with the ambiguities of type of the symbols are those which enter through the use of the symbol sm, or of the symbol Nc, which is $\overrightarrow{\mathrm{sm}}$. For it may happen that a class in one type has no class similar to it in some lower type (cf. $*102\cdot72\cdot73$). All fallacious reasoning in cardinal or ordinal arithmetic in connection with types, apart from that due to the mere absence of meaning in symbols, is due to this fact—in other words to the fact that in some types $\exists ! \mathrm{Nc}`\alpha$ is true, and in other types $\exists ! \mathrm{Nc}`\alpha$ may not be true. The fallacy consists in neglecting this latter possibility of the failure of $\exists ! \mathrm{Nc}`\alpha$ for a limited number of types, that is, in taking the "fluctuating" form $\exists ! \mathrm{Nc}`\alpha$ as though it possessed a "permanent" truth-value.

A fluctuating form however often possesses what is here termed a "stable" truth-value, which is as important as the permanent truth-value of other forms. For example, anticipating our definitions of elementary arithmetic, consider $2 +_c 3 = 5$. There is no abstract logical proof that there are two individuals; so suppose 2 and 3 refer to classes of individuals, but 5 refers to classes of a high enough type, then with these determinations $2 +_c 3 = 5$ cannot be proved. But $2 +_c 3 = 5$ has a *stable* truth-value, since it can always be proved when all the types are high enough. In this case the fact that our empirical census of individuals (at least of the "relative" individuals of ordinary life) has outrun the capacity of logical proof, makes the fluctuation in the truth-value of the form to be entirely unimportant.

In order to make this idea precise, it is necessary to have a convention as to the order in which the types of symbols in a symbolic form are assigned. The rule we adopt is that the types of the *real variables* are to be first assigned, and then those of the *constant symbols*. The types of the apparent variables, if any, will then be completely determinate.

A symbolic form has a *stable* truth-value if, after any assignment of types to the real variables, types can be assigned to the constant symbols so that the truth-value of the proposition thus obtained is the same as the truth-value of any proposition obtained by modifying it by the assignment of higher types to some or all of the constant symbols. This truth-value is the *stable* truth-value.

II. *Formal Numbers.*

The conventions, which we shall give below as to the assignment of types, practically restrict our interpretation of fluctuating symbolic forms to types in which the forms possess their stable truth-value. The assumption that these truth-values are stable never enters into the reasoning. But we judge a truth-value to be stable when any method of raising the types of the constant symbols by one step leaves it unaltered.

In practice the fluctuation of truth-values only enters into our consideration through a limited number of symbols called "formal numbers."

Formal numbers may be "constant" or "functional."

A *constant* formal number is any constant symbol for which there is a constant α such that, in whatever type the constant symbol is determined, it is, in that type, identical with $\text{Nc}`\alpha$. In other words if $\sigma$ be a constant symbol, then $\sigma$ is a formal number provided that "truth" is the permanent truth-value of $\sigma = \text{Nc}`\alpha$, for some constant α.

The *functional* formal numbers are defined by enumeration; they are

$$\text{Nc}`\alpha,\quad \Sigma\text{Nc}`\kappa,\quad \Pi\text{Nc}`\kappa,\quad \text{sm}``\mu,\quad \mu +_c \nu,\quad \mu -_c \nu,\quad \mu \times_c \nu,\quad \mu^\nu,$$

where in each formal number the symbols α, κ, μ, ν occurring in it are called the arguments of the functional form even when they are complex symbols. The argument of $\text{Nc}`(\alpha+\beta)$ is $\alpha+\beta$, and those of $\mu +_c (\nu +_c \varpi)$ are μ and $\nu +_c \varpi$, and those of $1 +_c 2$ are 1 and 2.

Thus among the constant formal numbers are

$$0,\ 1,\ 2,\ \ldots,\ \aleph_0,\ 1 +_c 2,\ 2 \times_c \aleph_0,\ 2^2.$$

The references which support this statement are

$$*101{\cdot}11{\cdot}21{\cdot}32 \,.\, *123{\cdot}36 \,.\, *110{\cdot}42 \,.\, *113{\cdot}23 \,.\, *116{\cdot}23.$$

Among the functional formal numbers are

$$\text{Nc}`(\alpha+\beta),\quad \mu +_c (\nu +_c \varpi),\quad (\mu +_c \nu) \times_c \varpi,\quad (\mu +_c \nu)^\varpi.$$

It will be observed that *e.g.* $1 +_c 2$ is both a constant and a functional formal number, so that the two classes are not mutually exclusive. In fact they possess an indefinite number of members in common.

All the formal numbers, with the exception of $\text{sm}``\mu$ and $\mu -_c \nu$, are members of NC without any hypothesis [cf. $*100{\cdot}41{\cdot}01{\cdot}52 \,.\, *110{\cdot}42 \,.\, *112{\cdot}101 \,.\, *113{\cdot}23 \,.\, *114{\cdot}1 \,.\, *116{\cdot}23$, note to $*119{\cdot}12$, and $*120{\cdot}411$].

A functional formal number consists of two parts, namely, its argument or arguments, and the constant "form." An argument of a functional formal number may be a complex symbol, and may be constant or variable. Thus $\mu +_c \nu$ is an argument of $(\mu +_c \nu) +_c \rho$, and of $(\mu +_c \nu) \times_c 1$ and of $(\mu +_c \nu)^\rho$; also $2 +_c 3$ is an argument of $(2 +_c 3) \times_c 1$. The constant form is constituted by the other symbols which are constants. Two occurrences of functional formal numbers are only occurrences of the same formal number if the arguments and also the constant forms are identical in symbolism. Thus two occurrences of $\text{Nc}`\alpha$ are occurrences of the same formal number, even if they are determined to be in different types; but $\text{Nc}`\alpha$ and $\text{Nc}`\beta$ are different formal numbers. Also μ^1 and $\mu \times_c 1$ are different formal numbers because their "forms" are different, though the arguments μ and 1 are the same and (in the same type) the entity denoted is the same. Thus the distinction between formal numbers depends on the symbolism and not on the entity denoted, and in considering them it is symbolic analogy and not denotation which is to be taken into account. For example two different

occurrences of the same formal number will not denote the same entity, if in the two occurrences the ambiguity of type is determined differently.

The functional formal numbers are divided into three sets: (i) the *primary* set consisting of the forms $\mathrm{Nc}'\alpha$, $\Sigma\,\mathrm{Nc}'\kappa$, $\Pi\,\mathrm{Nc}'\kappa$, (ii) the argumental set consisting only of $\mathrm{sm}''\mu$, (iii) the arithmetical set consisting of $\mu +_c \nu$, $\mu \times_c \nu$, μ^ν, and $\mu -_c \nu$.

A functional formal number has at most two arguments. But an argument of a functional formal number may itself be a functional formal number, and will accordingly possess either one or two arguments, which in their turn may be functional formal numbers, and so on. The whole set of arguments and of arguments of arguments, thus obtained, is called the set of *components* of the original formal number. Thus μ, ν, ρ and $\mu +_c \nu$ are components of $(\mu +_c \nu) +_c \rho$; and μ, ν and $\mathrm{sm}''\mu$ are components of $\nu +_c \mathrm{sm}''\mu$; and μ, α and $\mathrm{Nc}'\alpha$ are components of $\mu +_c \mathrm{Nc}'\alpha$. The two arguments of $(\mu +_c \nu) +_c \rho$ are $\mu +_c \nu$ and ρ, and those of $\nu +_c \mathrm{sm}''\mu$ are ν and $\mathrm{sm}''\mu$, and those of $\mu +_c \mathrm{Nc}'\alpha$ are μ and $\mathrm{Nc}'\alpha$.

Addition, multiplication, exponentiation, and subtraction will be called the arithmetical operations; and in $\mu +_c \nu$, $\mu \times_c \nu$, μ^ν, $\mu -_c \nu$, μ and ν will each be said to be subjected to these respective operations. The *arithmetical components* of an arithmetical formal number (*i.e.* one belonging to the arithmetical set) consist of those of its components which do not appear in the capacity of components of a component which does not belong to the arithmetical set. Thus $\mu, \nu, \rho, \mu +_c \nu$ are arithmetical components of $(\mu +_c \nu) +_c \rho$; and ν and $\mathrm{sm}''\mu$ are arithmetical components of $\nu +_c \mathrm{sm}''\mu$, but μ is not one; and μ and $\mathrm{Nc}'\alpha$ are arithmetical components of $\mu +_c \mathrm{Nc}'\alpha$, but α is not one; and μ and $\mathrm{sm}''(\nu +_c \rho)$ are arithmetical components of $\mu +_c \mathrm{sm}''(\nu +_c \rho)$, but $\nu +_c \rho$ and ν and ρ are components of $\mathrm{sm}''(\nu +_c \rho)$ and are therefore not *arithmetical* components of $\mu +_c \mathrm{sm}''(\nu +_c \rho)$. Only arithmetical formal numbers possess arithmetical components.

A formal number of the arithmetical set having no components which are formal numbers of the argumental set is called a *pure* arithmetical formal number. For example $\mu +_c (\nu +_c \rho)$ and $\mu +_c \mathrm{Nc}'\alpha$ are pure, but $\mu +_c \mathrm{sm}''(\nu +_c \rho)$ and $\mu +_c \mathrm{sm}''\mathrm{Nc}'\alpha$ are not pure.

There are many types involved in the consideration of a formal number. For example, in $\mathrm{Nc}'\alpha$ there is the type of $\mathrm{Nc}'\alpha$ and of α; in $\mu +_c \nu$ there is the type of $\mu +_c \nu$, the type of μ, and the type of ν; and so on for more complex formal numbers. The type of a formal number as a whole in any occurrence is called its *actual type*. This is the type of the entity which it then represents.

The other types involved in a formal number in any occurrence are called its subordinate types.

The actual types are not indicated in the symbolism for the various formal numbers as stated above. They can be indicated relatively to the type of the

variable ξ by writing $\text{Nc}(\xi)\text{‘}\alpha$, $\text{sm}_\xi\text{‘‘}\mu$, $(\mu +_c \nu)_\xi$, $(\mu \times_c \nu)_\xi$, $(\mu^\nu)_\xi$, $(\mu -_c \nu)_\xi$, by the notation of $*65$. Even when the actual type of a complex formal number, such as $\mu +_c (\nu +_c \varpi)$, is settled—so for instance that we have $\{\mu +_c (\nu +_c \varpi)\}_\xi$—the meaning of the symbol is not completely determined, for the type of $\nu +_c \varpi$ remains ambiguous. It follows, however, from

$$*100{\cdot}511 \,.\, *110{\cdot}23 \,.\, *113{\cdot}26 \,.\, *119{\cdot}61{\cdot}62,$$

that the subordinate types make no difference to the value of a formal number, so long as the components are not null.

We can therefore make a formal number definite as soon as its actual type is definite by securing that its components are not null. This is done by the convention II T (below) combined with the definitions

$$*110{\cdot}03{\cdot}04 \,.\, *113{\cdot}04{\cdot}05 \,.\, *116{\cdot}03{\cdot}04.$$

When the subordinate types are adjusted in accordance with these definitions and conventions, they will be said to be *normally adjusted.*

But in order to state this convention II T we require a definition of what is here called the *adequacy* of the actual type of a formal number. The general idea of adequacy is simple enough, namely that, given the subordinate types of σ, the actual type of σ should be high enough to enable us logically to prove $\exists ! \sigma$ when such a proof is possible for types which are not too low. For example, all types except the lowest for which it has meaning are adequate for the constant formal number 2. It is rather difficult however to state the meaning of adequacy with precision in a manner adapted to all formal numbers. Fortunately the definition of the lowest type which corresponds to this general idea of adequacy is not important for our purposes. It will be sufficient to define as adequate some types which certainly do have the property in question.

The method of definition which we adopt is to replace the formal number σ by another one σ' so related to σ that with the same actual type for both we can prove $\exists ! \sigma' \,.\, \supset \,.\, \exists ! \sigma$, whenever σ is not equal to Λ in all types. If σ be functional, we need only consider its argument, or its two arguments, and can dismiss from consideration the other components; then we replace these arguments by others so that the σ' has the required property. Thus:

(i) The actual types of $\text{Nc‘}\alpha$, $\Sigma\text{Nc‘}\kappa$, $\Pi\text{Nc‘}\kappa$, and $\text{sm‘‘}\mu$ are adequate when we can logically prove

$$\exists ! \text{Nc‘}t_0\text{‘}\alpha, \;\; \exists ! \Sigma\text{Nc‘}t_0\text{‘}\kappa, \;\; \exists ! \Pi\text{Nc‘}t_0\text{‘}\kappa, \text{ and } \exists ! \text{sm‘‘}t_0\text{‘}\mu;$$

(ii) The actual types of $\mu +_c \nu$, $\mu -_c \nu$, $\mu \times_c \nu$, and μ^ν are adequate when we can logically prove

$$\exists ! \text{N}_0\text{c‘}t_1\text{‘}\mu +_c \text{N}_0\text{c‘}t_1\text{‘}\nu, \quad \exists ! \text{N}_0\text{c‘}t_1\text{‘}\mu -_c 0 \cap t_0\text{‘}\nu,$$
$$\exists ! \text{N}_0\text{c‘}t_1\text{‘}\mu \times_c \text{N}_0\text{c‘}t_1\text{‘}\nu, \text{ and } \exists ! \text{N}_0\text{c‘}t_1\text{‘}\mu^{\text{N}_0\text{c‘}t_1\text{‘}\nu}.$$

It will be noticed that $t_0\text{‘}\alpha$, $t_0\text{‘}\kappa$, and $t_0\text{‘}\mu$ are the greatest classes of the same type as α, κ, and μ respectively, and that $\text{N}_0\text{c‘}t_1\text{‘}\mu$ and $\text{N}_0\text{c‘}t_1\text{‘}\nu$ are the greatest

cardinal numbers of the same type as μ and ν respectively. These definitions hold even when any of α, κ, μ, ν are complex symbols.

The remaining formal numbers which are not functional must certainly be constant. The difficulty which arises here is that if σ be such a formal number and $\aleph_0$ occurs in its symbolism, we have no logical method of deciding as to the truth or falsehood of $\exists ! \aleph_0$ in any type. But we replace $\aleph_0$ by $N_0c`t_1`\aleph_0$ which is the greatest existent cardinal of the same type as $\aleph_0$ in that occurrence. Thus:

(iii) If σ be a formal number which is not functional, an adequate actual type of σ is one for which we can logically prove $\exists ! \sigma'$, where σ' is derived from σ by replacing any occurrence of $\aleph_0$ in σ by $N_0c`t_1`\aleph_0$. Accordingly if $\aleph_0$ does not occur in σ, an adequate type is any actual type for which we can logically prove $\exists ! \sigma$.

In the case of members of the primary and argumental groups we have substituted the V of the appropriate type in the place of each variable. When the actual type is adequate we have

$$(\alpha) . \exists ! Nc`\alpha, \quad (\kappa) . \exists ! \Sigma Nc`\kappa, \quad (\kappa) . \exists ! \Pi Nc`\kappa, \quad (\mu) . \exists ! sm``\mu.$$

In the case of members of the arithmetical group (except in the case of $\mu -_c \nu$), we have substituted for each argument the largest cardinal number which can be obtained in the type of that argument, namely the $N_0c`V$ for the V of the appropriate type. Accordingly we are sure (except in the case of $\mu -_c \nu$) that for all other values of the arguments which are existent cardinal numbers the formal number is not null.

It will be noticed that normal adjustment only concerns the subordinate types. For example $*110{\cdot}03$ secures that in $Nc`\alpha +_c \mu$ the actual type of $Nc`\alpha$ is adequate, and $*110{\cdot}23$ shows that any adequate actual type of $Nc`\alpha$ will do. But nothing is said about the actual type of $Nc`\alpha +_c \mu$. We make the following definition: When the subordinate types of a formal number are normally adjusted, and the actual type is adequate, the types of the formal number are said to be *arithmetically* adjusted.

We notice that for the primary set, the arithmetical adjustment of types means the same thing as the adequate adjustment of the actual type. Also if the arguments of a formal number of the arithmetical set are simple symbols, the two ideas come to the same thing.

In the case of variable formal numbers of the primary set, it follows from $*117{\cdot}22{\cdot}32$ that when their types are arithmetically adjusted they are not equal to Λ for any values of their variables.

Also in the case of those variable formal numbers which are of the pure arithmetical set (excluding $\mu -_c \nu$) it follows from $*100{\cdot}4{\cdot}52{\cdot}42 . *113{\cdot}23 . *116{\cdot}23$ that, working from the ultimate components reached by successive analysis upwards, for all values of such ultimate components which are members

of $NC - \iota`\Lambda$ they can be reduced to the case of the formal numbers of the primary group; and that therefore they are not equal to Λ when their types are arithmetically adjusted. For example in $\mu +_c \{\nu +_c (\rho +_c \sigma)\}$, μ, ν, ρ, σ are these ultimate components; let them be existent cardinal numbers. Hence when the types are arithmetically adjusted, the actual type of $\rho +_c \sigma$ is adequate and $\rho +_c \sigma$ is an existent cardinal; we can therefore substitute $N_0c`\alpha$ for it. By the same reasoning we can substitute $N_0c`\beta$ for $\nu +_c N_0c`\alpha$, and again $N_0c`\gamma$ for $\mu +_c N_0c`\beta$.

A definite standard arithmetical adjustment of types for any formal number can always be found by making every use of sm, whether explicit or concealed in Nc or in some other symbol, to be homogeneous. Proofs which apply to any arithmetical adjustment of types start by dealing with this standard type, and then by the use of *104·21 . *106·21·211·212·213 the extension is made to the adjacent higher classical and relational types. We then "see" that by the analogy of symbolism this extension can always be formally proved at each stage, so that we are dealing with the stable truth-value. For some constant formal numbers a lower existential type can be found than that indicated by this method.

III. *Classification of Occurrences of Formal Numbers.*

A symbolic form of any of the kinds [cf. *117·01·04·05·06]

$$\mu > \nu, \quad \mu < \nu, \quad \mu \geqslant \nu, \quad \mu \leqslant \nu,$$

is called an *arithmetical inequality*.

These forms only arise when we are comparing cardinal numbers in respect to the relation of being "greater than" or "less than." It might seem natural to include equations among these arithmetical inequalities. Their use however, even as between cardinal numbers, is not so exclusively arithmetical, and it is convenient to consider them separately under another heading during our preliminary investigations.

In the arithmetical inequalities as above written, μ and ν, or any symbols replacing μ and ν, are called the *opposed sides* of the inequality, and either of μ or ν is called a *side* of the inequality.

Symbolic forms of the kinds $\sigma = \kappa$ and $\sigma \neq \kappa$, where either σ or κ is a formal number, will be called *equations* and *inequations* respectively; and σ and κ are called the *opposed sides* of the equation or inequation, and either of them is simply a *side* of the equation or inequation.

When we reach the exclusively arithmetical point of view, it will be convenient to put together equations, inequations and arithmetical inequalities as one sort of symbolic form. Their separation here is for the sake of investigations into the exceptions due to the failure of existence theorems in low types. It is unnecessary to consider arithmetical inequalities in this connection.

The ways in which a symbol σ can occur in a symbolic form are named as follows:

The occurrence of σ in $\mathrm{sm}``\sigma$ is called an *argumental* occurrence,

The occurrence of σ as an argument of an arithmetical formal number (which may be a component of another formal number) or as one side of an arithmetical inequality is called an *arithmetical* occurrence,

The occurrence of σ as one side of an equation is called an *equational* occurrence,

The occurrence of σ in "$\xi \epsilon \sigma$" is called an *attributive* occurrence,

Any other occurrence of σ is called a *logical* occurrence, so also is $\sigma = \Lambda$.

It is obvious that a pair of opposed sides of an equation or inequation must be of the same type. Furthermore, if σ be a formal number, and **∗20·18** is applied so as to give

$$\vdash :. \sigma = \kappa . \supset : f(\sigma) . \equiv . f(\kappa),$$

the equational occurrence of σ must be of the same type as its occurrence in $f(\sigma)$, otherwise the inference is fallacious. Accordingly substitution in arithmetical formulae can only be undertaken when the conventions as to the relations of ambiguous types secure this identity. This question is considered later in this prefatory statement, and the result appears in the text as ∗118·01.

At this point some examples will be useful; they will also be referred to subsequently in connection with the conventions limiting ambiguities of type.

∗100·35. $\vdash :. \exists ! \mathrm{Nc}`\alpha . \vee . \exists ! \mathrm{Nc}`\beta : \supset :$

$$\mathrm{Nc}`\alpha = \mathrm{Nc}`\beta . \equiv . \alpha \epsilon \mathrm{Nc}`\beta . \equiv . \beta \epsilon \mathrm{Nc}`\alpha . \equiv . \alpha \,\mathrm{sm}\, \beta$$

Here the formal numbers are $\mathrm{Nc}`\alpha$ and $\mathrm{Nc}`\beta$, each of which has three occurrences. The first occurrence of $\mathrm{Nc}`\alpha$ is logical, its second is equational, and its third is attributive.

∗100·42 (in the demonstration).

$$\vdash : \mu, \nu \epsilon \mathrm{NC} . \exists ! \mu \cap \nu . \supset . (\exists \alpha, \beta) . \mu = \mathrm{Nc}`\alpha . \nu = \mathrm{Nc}`\beta . \mathrm{Nc}`\alpha = \mathrm{Nc}`\beta$$

Here $\mathrm{Nc}`\alpha$ and $\mathrm{Nc}`\beta$ are the only formal numbers, and all their occurrences are equational.

∗100·44 (in the demonstration).

$$\vdash : \mu \epsilon \mathrm{NC} . \exists ! \mathrm{Nc}`\alpha . \alpha \epsilon \mu . \supset . (\exists \beta) . \mu = \mathrm{Nc}`\beta . \mathrm{Nc}`\alpha = \mathrm{Nc}`\beta$$

Here $\mathrm{Nc}`\alpha$ and $\mathrm{Nc}`\beta$ are the only formal numbers; the first occurrence of $\mathrm{Nc}`\alpha$ is logical, its second is equational; both the occurrences of $\mathrm{Nc}`\beta$ are equational.

∗100·511. $\vdash : \exists ! \mathrm{Nc}`\beta . \supset . \mathrm{sm}``\mathrm{Nc}`\beta = \mathrm{Nc}`\beta$

Here the formal numbers are $\mathrm{Nc}`\beta$ and $\mathrm{sm}``\mathrm{Nc}`\beta$. The first occurrence of $\mathrm{Nc}`\beta$ is logical, the second is argumental, the third is equational; the only occurrence of $\mathrm{sm}``\mathrm{Nc}`\beta$ is equational.

∗100·521. $\vdash : \mu \in \mathrm{NC} \,.\, \exists\,!\, \mathrm{sm}``\mu \,.\, \supset \,.\, \mathrm{sm}``\mathrm{sm}``\mu = \mu$

Here sm$``\mu$ and sm$``$sm$``\mu$ are the only formal numbers; sm$``\mu$ has two occurrences, the first logical, the second argumental; sm$``$sm$``\mu$ has one occurrence, which is equational.

∗101·28 (in the demonstration).

$$\vdash : \gamma \in \mathrm{sm}``1 \,.\, \equiv \,.\, (\exists \alpha) \,.\, \alpha \in 1 \,.\, \gamma \,\mathrm{sm}\, \alpha$$

Here the formal numbers are 1 and sm$``$1. The first occurrence of 1 is argumental, the second is attributive; the occurrence of sm$``$1 is attributive.

∗101·38. $\vdash : \exists\,!\, 2 \,.\, \supset \,.\, s`\mathrm{Cl}``2 = 0 \cup 1 \cup 2$

Here the formal numbers are 0, 1, and 2, and their occurrences are all logical.

∗110·54. $\vdash \,.\, (\mathrm{Nc}`\alpha +_c \mathrm{Nc}`\beta) +_c \mathrm{Nc}`\gamma = \mathrm{Nc}`(\alpha + \beta + \gamma)$

Here the formal numbers are

$\mathrm{Nc}`\alpha$, $\mathrm{Nc}`\beta$, $\mathrm{Nc}`\gamma$, $\mathrm{Nc}`(\alpha + \beta + \gamma)$, $\mathrm{Nc}`\alpha +_c \mathrm{Nc}`\beta$, $(\mathrm{Nc}`\alpha +_c \mathrm{Nc}`\beta) +_c \mathrm{Nc}`\gamma$.

The occurrence of $\mathrm{Nc}`(\alpha + \beta + \gamma)$ and that of $(\mathrm{Nc}`\alpha +_c \mathrm{Nc}`\beta) +_c \mathrm{Nc}`\gamma$ are both equational, and they must be of the same type since they are opposed sides of the same equation. The occurrences of the other formal numbers are as arithmetical components of a more complex arithmetical formal number and are therefore arithmetical.

∗116·63. $\vdash \,.\, \mu^{\nu \times_c \varpi} = (\mu^\nu)^\varpi$

The formal numbers are $\nu \times_c \varpi$, μ^ν, $\mu^{\nu \times_c \varpi}$, and $(\mu^\nu)^\varpi$. Each formal number occurs once only. The occurrences of $\nu \times_c \varpi$ and μ^ν are arithmetical, and those of the other two are equational.

∗117·108. $\vdash :. \mathrm{Nc}`\alpha \geqslant \mathrm{Nc}`\beta \,.\, \equiv : \mathrm{Nc}`\alpha > \mathrm{Nc}`\beta \,.\, \mathrm{v} \,.\, \mathrm{Nc}`\alpha = \mathrm{Nc}`\beta$

The formal numbers are $\mathrm{Nc}`\alpha$ and $\mathrm{Nc}`\beta$, each with three occurrences. The first two occurrences of each formal number are arithmetical, the last occurrence of each is equational.

∗120·53 (in the demonstration).

$$\vdash : \beta = \gamma +_c \delta \,.\, \exists\,!\, \beta \,.\, \supset \,.\, \alpha^\beta = \alpha^\gamma \times_c \alpha^\delta$$

Here the formal numbers are $\gamma +_c \delta$, α^β, α^γ, α^δ, $\alpha^\gamma \times_c \alpha^\delta$. Each formal number has one occurrence. Those of $\gamma +_c \delta$, α^β and $\alpha^\gamma \times_c \alpha^\delta$ are equational, and those of α^γ and α^δ are arithmetical.

∗120·53 (in the demonstration).

$$\vdash : \alpha^\beta = \alpha^\gamma \,.\, \beta = \gamma +_c \delta \,.\, \exists\,!\, \alpha^\beta \,.\, \supset \,.\, \alpha^\gamma = \alpha^\gamma \times_c \alpha^\delta$$

Here the formal numbers are α^β, α^γ, α^δ, $\alpha^\gamma \times_c \alpha^\delta$, $\gamma +_c \delta$. The first occurrence of α^β is equational, its second occurrence is logical; the first two occurrences of α^γ are equational, its third occurrence is arithmetical; the only occurrence of α^δ is arithmetical; the only occurrences of $\alpha^\gamma \times_c \alpha^\delta$ and of $\gamma +_c \delta$ are equational.

IV. *The Conventions* IT *and* IIT.

Two occurrences of a formal number with the same actual type are said to be *bound* to each other.

The choice of types for formal numbers, when they are not made definite in terms of variables by the notation of ∗65, is limited by the following conventions, which enable us to dispense largely with the elaboration produced by the definition of types.

IT. *All logical occurrences of the same formal number are in the same type; argumental occurrences are bound to logical and attributive occurrences; and, if there are no argumental occurrences, equational occurrences are bound to logical occurrences.*

This rule only applies, so far as meaning permits, to those types which remain ambiguous after the assignment of types to the real variables.

It will be noticed that if there are no argumental or logical occurrences of a formal number, IT does not in any way apply to the assignment of types to the occurrences in the form of that formal number.

The identification of types in argumental and attributive occurrences by IT is rendered necessary to secure the use of the equivalence

$$\gamma \in \mathrm{sm}``\sigma \,.\equiv .\, (\exists \alpha) \,.\, \alpha \in \sigma \,.\, \gamma \,\mathrm{sm}\, \alpha,$$

where σ is a formal number. Without the convention, this application of ∗37·1 would be fallacious. The only one of our examples to which this part of the convention applies is ∗101·28 (demonstration), where it secures that the two occurrences of 1 are in the same type. It is relevant however to the symbolism in the demonstration of ∗100·521.

It will be found in practice that this convention relates the types of occurrences in the same way as would naturally be done by anyone who was not thinking of the convention at all. To see how the convention works, we will run through the examples which have already been given above.

In ∗100·35, IT directs the logical and equational occurrences of $\mathrm{Nc}`\alpha$ to be in the same type, and similarly for $\mathrm{Nc}`\beta$. Also "meaning" secures that the equational types of $\mathrm{Nc}`\alpha$ and $\mathrm{Nc}`\beta$ are the same. Thus these four occurrences are all in one type, which has no necessary relation to the types of the attributive occurrences of $\mathrm{Nc}`\alpha$ and $\mathrm{Nc}`\beta$. Thus, using the notation of ∗65·04 to secure typical definiteness, ∗100·35 is to mean

$$\vdash :. \exists ! \,\mathrm{Nc}\,(\xi)`\alpha \,.\vee .\, \exists ! \,\mathrm{Nc}\,(\xi)`\beta : \supset :$$
$$\mathrm{Nc}\,(\xi)`\alpha = \mathrm{Nc}\,(\xi)`\beta \,.\equiv .\, \alpha \in \mathrm{Nc}\,(\alpha)`\beta \,.\equiv .\, \beta \in \mathrm{Nc}\,(\beta)`\alpha \,.\equiv .\, \alpha \,\mathrm{sm}\, \beta.$$

The types of these attributive occurrences are settled by the necessity of 'meaning."

In ∗100·42 (demonstration), since all the occurrences of formal numbers are equational, IT produces no limitation of types.

In $*100\cdot44$ (demonstration), IT secures that the two occurrences of $\text{Nc}`\alpha$ are in the same type. Also we notice that the first occurrence of $\text{Nc}`\beta$ is really (cf. $*65\cdot04$) $\text{Nc}\,(\alpha)`\beta$, since "$\alpha \epsilon \mu$" occurs, and thus "meaning" requires this relation of types, and the second occurrence of $\text{Nc}`\beta$ is in the type of the occurrences of $\text{Nc}`\alpha$.

In $*100\cdot511$, IT directs that the logical and argumental occurrences are to have the same type. In $*100\cdot521$, IT directs that the two occurrences of $\text{sm}``\mu$ are to have the same type. In $*101\cdot28$ both occurrences of 1 are to be in the same type. In $*101\cdot38$, IT directs that all the occurrences of 2 are to have the same type.

The convention IT in no way limits the types in $*110\cdot54$, nor in $*116\cdot63$, nor in $*117\cdot108$.

In the first example from $*120\cdot53$ (in the demonstration) convention IT has no application.

In the second example from $*120\cdot53$ (in the demonstration) convention IT directs that the two occurrences of α^β shall be in the same type; and the necessity of "meaning" secures that the first occurrence of α^γ shall also be in this type. The same necessity secures that $\gamma +_0 \delta$ shall be in the same type as β; and it also secures that in "$\alpha^\gamma = \alpha^\gamma \times_0 \alpha^\delta$" the first occurrence of α^γ and that of $\alpha^\gamma \times_0 \alpha^\delta$ shall have a common type, which is otherwise unfettered; also nothing has been decided as to the types of α^γ and α^δ in $\alpha^\gamma \times_0 \alpha^\delta$.

We now come to conventions embodying the outcome of arithmetical ideas. The term "arithmetical" is here used to denote investigations in which the interest lies in the comparison of formal numbers in respect to equality or inequality, excluding the exceptional cases—whenever the cases are exceptional—due to the failure of existence in low types. The thorough-going arithmetical point of view, which we adopt later in the investigation on Ratio and Quantity and also in this volume in $*117$ and $*126$ and some earlier propositions, would sweep aside as uninteresting all investigation of the exact ways in which the failure of existence theorems is relevant to the truth of propositions, thus concentrating attention exclusively on stable truth-values. But the logical investigation has its own intrinsic interest among the principles of the subject. It is obvious however that it should be restrained to a consideration of the theorems of purely logical interest. In practice this extrusion of uninteresting cases of the failure of arithmetical theorems, even amid the logical investigations of the first part of this volume, is effected by securing that *all arithmetical occurrences of formal numbers have their actual types adequate.*

As far as formal numbers of the primary group, *i.e.* $\text{Nc}`\alpha$, $\Sigma\text{Nc}`\kappa$, $\Pi\text{Nc}`\kappa$, are concerned, the arithmetical adjustment of types is secured formally in the symbolism by the definitions $*110\cdot03\cdot04$ for addition, and $*113\cdot04\cdot05$ for

multiplication, and $*116\cdot03\cdot04$ for exponentiation, and $*117\cdot02\cdot03$ for arithmetical inequalities, and $*119\cdot02\cdot03$ for subtraction.

We save the symbolic elaboration which would arise from the extension of similar definitions to other formal numbers by the following convention:

IIT. *Whenever a formal number σ occurs, so that, if it were replaced by* $\text{Nc}`\alpha$, *the actual type of* $\text{Nc}`\alpha$ *would by definition have to be adequate, then the actual type of σ is also to be adequate.*

For example in $\mu +_c (\nu +_c \varpi)$, if $\nu +_c \varpi$ were replaced by $\text{Nc}`\alpha$, then by $*110\cdot04$ the actual type of $\text{Nc}`\dot{\alpha}$ is adequate. Hence by IIT the actual type of $\nu +_c \varpi$ is to be adequate: accordingly so long as ν and ϖ are simple variables and members of $\text{NC} - \iota`\Lambda$, we can always assume $\exists ! (\nu +_c \varpi)$ for the type of the occurrence of $\nu +_c \varpi$ in $\mu +_c (\nu +_c \varpi)$.

It is essential to notice that so long as the argument of an argumental formal number, or the arguments of an arithmetical formal number, are adjusted arithmetically, the exact types chosen make no difference. This follows for argumental formal numbers from $*102\cdot862\cdot87\cdot88$, for addition from $*110\cdot25$, for multiplication from $*113\cdot26$, for exponentiation from $*116\cdot26$, for subtraction from $*119\cdot61\cdot62$. Thus (remembering also $*100\cdot511$) in any definite type a formal number has one definite meaning provided that any subordinate formal number which occurs in its symbolism is determined existentially. The convention IIT directs us always to take this definite meaning for any pure arithmetical formal number.

The convention does not determine completely the meaning of an arithmetical formal number which is not pure. For example, $\mu +_c (\nu +_c \rho)$ is a pure arithmetical formal number when μ, ν, ρ are determined in type; and convention IIT directs that the type of $(\nu +_c \rho)$ is to be adequate. But $\mu +_c \text{sm}``(\nu +_c \rho)$ is an arithmetical formal number which is not pure, and convention IIT directs that the type of the domain of sm is to be adequate, but does not affect the type of $\nu +_c \rho$. Thus it is easy to see that IIT secures the adequacy of the actual types of all *arithmetical* components of any arithmetical formal numbers which occur, but does not affect the actual type of a formal number which occurs as the argument of an argumental formal number. But in this case convention IT will bind the actual type of this occurrence of the argument to any logical or attributive occurrence of the same formal number. For example, if $\exists ! \nu +_c \rho$ and $\mu +_c \text{sm}``(\nu +_c \rho)$ occur in the same form, then these two occurrences of $\nu +_c \rho$ must have the same actual type. In practice argumental formal numbers are useful as components of arithmetical formal numbers for the very purpose of avoiding the automatic adjustment of types directed by IIT.

The meaning of IIT is best explained by examples. Among our previous examples we need only consider those in which arithmetical formal numbers occur.

In $*110{\cdot}54$ the convention or definitions direct us to determine the types of $\text{Nc}`\alpha$ and $\text{Nc}`\beta$ adequately when forming $\text{Nc}`\alpha +_c \text{Nc}`\beta$, also to determine $\text{Nc}`\alpha +_c \text{Nc}`\beta$ and $\text{Nc}`\gamma$ adequately when forming $(\text{Nc}`\alpha +_c \text{Nc}`\beta) +_c \text{Nc}`\gamma$. The convention does not apply to the types of $(\text{Nc}`\alpha +_c \text{Nc}`\beta) +_c \text{Nc}`\gamma$ and $\text{Nc}`(\alpha + \beta + \gamma)$. These types must be identical in order to secure meaning.

In $*116{\cdot}63$ the convention directs us to adjust the types of $\nu \times_c \varpi$ and μ^ν adequately; it does not affect the types of $\mu^{\nu \times_c \varpi}$ and $(\mu^\nu)^\varpi$, which must be identical to secure meaning. If we replace μ, ν, ϖ by formal numbers, by 2, $\aleph_0$, and 1 for example, we get "$\vdash . 2^{\aleph_0 \times_c 1} = (2^{\aleph_0})^1$." The convention now directs that 1 is to be determined adequately. It so happens that any type is adequate for it, since $\exists ! 1$ can be proved in any type. Then adequate types for $\aleph_0 \times_c 1$ and $2^{\aleph_0}$ are types for which we can prove $\exists ! (\text{N}_0\text{c}`t_1`\aleph_0) \times_c 1$ and $\exists ! 2^{\text{N}_0\text{c}`t_1`\aleph_0}$. Thus if τ is the type of $\aleph_0$ in both cases, an adequate type for $\aleph_0 \times_c 1$ is τ, and for $2^{\aleph_0}$ is $\text{Cl}`\tau$.

In $*117{\cdot}108$ we find arithmetical occurrences in arithmetical inequalities. Thus IIT directs us to take the first two occurrences of $\text{Nc}`\alpha$ and the first two of $\text{Nc}`\beta$ with adequate actual types. The type of $\text{Nc}`\alpha$ and $\text{Nc}`\beta$ in $\text{Nc}`\alpha = \text{Nc}`\beta$ is not affected by it. It is evident that the conventions IT, IIT are not sufficient to secure the truth of this proposition as thus symbolized. It is essential that in the equation the type be adjusted adequately for both formal numbers. In fact the general arithmetical convention, that types of equational as well as of arithmetical occurrences are adjusted arithmetically, is here used.

V. *Some Important Principles.*

Principle of Arithmetical Substitution. In $*120{\cdot}53$, the application of IIT needs a consideration of the whole question of arithmetical substitution. Consider the first of the two examples. We have

$$\vdash : \beta = \gamma +_c \delta . \exists ! \beta . \supset . \alpha^\beta = \alpha^\gamma \times_c \alpha^\delta.$$

It is obvious that unless we can pass with practical immediateness from "$\beta = \gamma +_c \delta . \alpha^\beta = \alpha^\beta$" to "$\alpha^\beta = \alpha^{\gamma +_c \delta}$" by $*20{\cdot}18$, arithmetic is made practically impossible by the theory of types. But a difficulty arises from the application of IIT. Suppose we assign the types of our real variables first. Then the types of α, β, γ, δ can be arbitrarily assigned, and there is no necessary connection between them which arises from the preservation of meaning. Thus β may be in a type which is not an adequate type for $\gamma +_c \delta$. Assume that this is the case. But the equational use of $\gamma +_c \delta$ is in the same type as β, and by IIT the arithmetical use of $\gamma +_c \delta$ in $\alpha^{\gamma +_c \delta}$ is in an adequate type. Thus, on the face of it, the reasoning, appealing to $*20{\cdot}18$, by which the substitution was justified, is fallacious; for the two occurrences of $\gamma +_c \delta$ in fact mean different things.

In order to generalize our solution of this difficulty it is convenient to define the term "arithmetical equation." An *arithmetical equation* is an

equation between purely arithmetical formal numbers whose actual types are both determined adequately. Then it is evident that from "$\sigma = \tau . f(\tau)$," where σ and τ are formal numbers and τ occurs arithmetically in $f(\tau)$, we cannot infer $f(\sigma)$ unless the equation $\sigma = \tau$ is arithmetical. For otherwise the τ in the equation cannot be identified with the τ in $f(\tau)$.

When we have "$\beta = \tau . f(\tau)$," where τ is a formal number and β is a number in a definite type, and wish to pass to "$f(\beta)$," or "$\beta = \tau . f(\beta)$" and wish to pass to "$f(\tau)$," the occurrence of τ in $f(\tau)$ being arithmetical, the type of β may not be an adequate type for τ. Accordingly the τ in "$\beta = \tau$" cannot be identified with the τ in $f(\tau)$. The type of the τ in the equation ought to be freed from dependence on that of β. Accordingly the transition is only legitimate when we can write instead

$$\text{"}\beta +_c 0 = \tau . f(\tau)\text{" or "}\beta +_c 0 = \tau . f(\beta)\text{,"}$$

where in both cases the equation is arithmetical. For now all the symbols are subject to the same rules.

If this modification can be made without altering the truth-value of the asserted propositions, the substitution is legitimate, otherwise it is not.

It is obvious that in the above our immediate passage is to or from $f(\beta +_c 0)$. But it is easy to see that, the occurrence of $\beta +_c 0$ being arithmetical, we always have

$$f(\beta) . \equiv . f(\beta +_c 0).$$

In order to prove this, we have only to prove

$$\alpha +_c (\beta +_c 0) = \alpha +_c \beta,$$
$$\alpha \times_c (\beta +_c 0) = \alpha \times_c \beta,$$
$$(\alpha +_c 0)^\beta = \alpha^\beta,$$
$$\alpha^{\beta +_c 0} = \alpha^\beta,$$

and $$\alpha > \beta +_c 0 . \equiv . \alpha > \beta . \equiv . \alpha +_c 0 > \beta.$$

The demonstration of the first of these propositions runs as follows:

$\vdash . *110\cdot4 . \supset \vdash :. \beta \sim \epsilon \mathrm{NC} . \mathrm{v} . \beta = \Lambda : \supset . \beta +_c 0 = \Lambda . \alpha +_c \beta = \Lambda .$

$[*110\cdot4] \qquad \supset . \alpha +_c (\beta +_c 0) = \Lambda = \alpha +_c \beta$ (1)

$\vdash . *110\cdot4 . \supset \vdash :. \alpha \sim \epsilon \mathrm{NC} . \mathrm{v} . \alpha = \Lambda : \supset . \alpha +_c (\beta +_c 0) = \Lambda = \alpha +_c \beta$ (2)

$\vdash . *110\cdot6 . \supset \vdash : \alpha, \beta \epsilon \mathrm{NC} - \iota`\Lambda . \supset . \alpha +_c (\beta +_c 0) = \alpha +_c \mathrm{sm}``\beta$

$\qquad = \alpha +_c \beta$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash : \alpha +_c (\beta +_c 0) = \alpha +_c \beta$

In the above demonstration the step to (3) is legitimate since by the hypothesis β is a determination of $\mathrm{sm}``\beta$ in an adequate type.

Similar proofs hold for the other propositions, using $*113\cdot204$ and $*116\cdot204$ and $*117\cdot12$ and $*103\cdot13$.

We must also consider the circumstances under which we can pass from "$\beta = \tau$" to "$\beta +_c 0 = \tau$," where the latter equation is arithmetical. In other

words, using $*65{\cdot}01$ we require the hypothesis necessary for

$$\exists ! \tau_\eta . \beta = \tau_\xi . \supset . \beta +_c 0 = \tau_\eta .$$

We have

$$\vdash . *20{\cdot}18 . \quad \supset \vdash : \beta = \tau_\xi . \supset . \beta +_c 0 = \tau_\xi +_c 0 \qquad (1)$$

$$\vdash . *110{\cdot}35 . \supset \vdash : \exists ! \tau_\xi . \exists ! \tau_\eta . \supset . \tau_\xi +_c 0 = \tau_\eta +_c 0 \qquad (2)$$

$$\vdash . (1) . (2) . \supset \vdash :. \exists ! \tau_\xi . \exists ! \tau_\eta . \supset : \beta = \tau_\xi . \supset . \beta +_c 0 = \tau_\eta +_c 0 \quad (3)$$

$$\vdash . (3) . \quad \supset \vdash :. \exists ! \beta . \exists ! \tau_\eta . \supset : \beta = \tau_\xi . \supset . \beta +_c 0 = \tau_\eta +_c 0 \quad (4)$$

Now in (4) the occurrences of $\beta +_c 0$ and $\tau_\eta +_c 0$, which are in the same type, may be chosen to be in any type we like. Hence we deduce

$$\vdash . (4) . *110{\cdot}6 . \supset \vdash :. \exists ! \beta . \exists ! \tau_\eta . \supset : \beta = \tau_\xi . \supset . (\beta +_c 0)_\zeta = \mathrm{sm}_\zeta{}^{\text{``}}\tau_\eta .$$

$$[*100{\cdot}511] \qquad \supset . (\beta +_c 0)_\zeta = \tau_\zeta$$

Hence $\exists ! \beta$ is the requisite condition. Now since ζ can be in any type, we can also choose it in any existential type for τ. Thus with IIT applying to the arithmetical occurrence of τ in $f(\tau)$, we have, where τ is a formal number and β is a number in a definite type,

$$\vdash : \exists ! \beta . \beta = \tau . f(\tau) . \supset . f(\beta),$$

$$\vdash : \exists ! \beta . \beta = \tau . f(\beta) . \supset . f(\tau),$$

$$\vdash : \exists ! \sigma . \sigma = \tau . f(\tau) . \supset . f(\sigma).$$

In the last proposition by IT the equation $\sigma = \tau$ is arithmetical These equations are summed up in $*118{\cdot}01$.

These three fundamental theorems embody the principle of arithmetical substitution. The hypothesis $\exists ! \beta$ is really less than is assumed in ordinary life, the usual tacit assumption being $\beta \in \mathrm{NC} - \iota{}^{\backprime}\Lambda$. In fact unless $\beta \in \mathrm{NC}$, $\beta = \tau$ is necessarily false.

Principle of Identification of Types. Suppose we have proved "$\vdash : \mathrm{Hp} . \supset . \phi\sigma$" and "$\vdash : \phi(\sigma_\xi) . \supset . p$," where σ is a formal number whose occurrence in "$\vdash : \mathrm{Hp} . \supset . \phi\sigma$" is in an entirely ambiguous type, and σ_ξ is the same formal number σ with its type related to that of ξ by $*65{\cdot}01$. Then since the type of the σ in "$\vdash : \mathrm{Hp} . \supset . \phi\sigma$" is ambiguous, we can write "$\vdash : \mathrm{Hp} . \supset . \phi(\sigma_\xi)$," and thence infer "$\vdash . p$."

The principle is: An entirely undetermined type in an asserted symbolic form can be identified with any type ambiguous or otherwise in any other asserted symbolic form or in the same symbolic form.

For example in $*100{\cdot}42$ (demonstration) considered above, since $\exists ! \mu \cap \nu$ occurs, the first occurrences of $\mathrm{Nc}{}^{\backprime}\alpha$ and $\mathrm{Nc}{}^{\backprime}\beta$ are of the same type, and so are their second occurrences in $\mathrm{Nc}{}^{\backprime}\alpha = \mathrm{Nc}{}^{\backprime}\beta$. But the two types are not determined by our conventions to have any necessary connection. In fact the type in $\mathrm{Nc}{}^{\backprime}\alpha = \mathrm{Nc}{}^{\backprime}\beta$ is entirely arbitrary. Accordingly it can be identified with the other type, and thus the inference to the next line, *viz.* to "$\vdash : \mathrm{Hp} . \supset . \mu = \nu$," is justified.

In the case of arithmetical equations, it is important to notice that we have

$\vdash . *100{\cdot}321{\cdot}33 . \supset \vdash :. \exists ! \mathrm{Nc}(\xi)\text{‘}\alpha . \supset : \mathrm{Nc}(\xi)\text{‘}\alpha = \mathrm{Nc}(\xi)\text{‘}\beta . \supset . \mathrm{Nc}\text{‘}\alpha = \mathrm{Nc}\text{‘}\beta$.

Hence if σ and τ are formal numbers,

$$\vdash :. \exists ! \sigma_\xi . \supset : \sigma_\xi = \tau_\xi . \supset . \sigma = \tau.$$

Thus if we have "$\vdash : \mathrm{Hp} . \exists ! \sigma . \supset . \sigma = \tau$" and "$\vdash : \mathrm{Hp}' . \sigma_\eta = \tau_\eta . \supset . p$," we can infer from the former proposition "$\vdash : \mathrm{Hp} . \exists ! \sigma . \supset . \sigma_\eta = \tau_\eta$," and from this and the latter proposition, we infer "$\vdash : \mathrm{Hp}' . \mathrm{Hp} . \exists ! \sigma . \supset . p$," so the general principle of identification can be employed when the $\phi(\sigma)$ in the first proposition is an arithmetical equation.

For example, in an example given above, $*100{\cdot}44$ (demonstration), *viz.*

$$\vdash : \mu \in \mathrm{NC} . \exists ! \mathrm{Nc}\text{‘}\alpha . \alpha \in \mu . \supset . (\exists\beta) . \mu = \mathrm{Nc}\text{‘}\beta . \mathrm{Nc}\text{‘}\alpha = \mathrm{Nc}\text{‘}\beta,$$

the equation $\mathrm{Nc}\text{‘}\alpha = \mathrm{Nc}\text{‘}\beta$ is arithmetical. Accordingly we are justified in asserting the propositional function

$$\vdash : \mu \in \mathrm{NC} . \exists ! \mathrm{Nc}\text{‘}\alpha . \alpha \in \mu . \supset . (\exists\beta) . \mu = \mathrm{Nc}(\alpha)\text{‘}\beta . \mathrm{Nc}(\alpha)\text{‘}\alpha = \mathrm{Nc}(\alpha)\text{‘}\beta,$$

where $\mathrm{Nc}(\alpha)\text{‘}\beta$ in "$\mu = \mathrm{Nc}(\alpha)\text{‘}\beta$" has all along been presupposed by the necessity of meaning.

Thus the inference follows,

$$\vdash : \mu \in \mathrm{NC} . \exists ! \mathrm{Nc}\text{‘}\alpha . \alpha \in \mu . \supset . \mathrm{Nc}(\alpha)\text{‘}\alpha = \mu .$$
$$\supset . \mathrm{Nc}\text{‘}\alpha = \mu.$$

This proof loses its point when μ is looked on as a variable with necessarily the same type throughout. For then the proposition collapses into

$$\vdash :. \mu \in \mathrm{NC} . \supset : \alpha \in \mu . \equiv . \mathrm{Nc}(\alpha)\text{‘}\alpha = \mu.$$

But if μ be a formal number necessarily a member of NC, the proposition is really

$$\vdash :. \exists ! \mathrm{Nc}\text{‘}\alpha . \supset : \alpha \in \mu . \equiv . \mathrm{Nc}\text{‘}\alpha = \mu.$$

With this presupposition we should have in the first line of the demonstration

$$\text{“}\vdash : \exists ! \mathrm{Nc}\text{‘}\alpha . \mathrm{Nc}\text{‘}\alpha = \mu . \supset . \alpha \in \mu,\text{”}$$

though with "μ" a single variable, the line is formally correct as it stands in the text.

Recognition of Particular Cases. It is important to notice the conditions under which $\phi\sigma$ can be recognized as a particular case of $\phi\xi$, where ξ is a real variable and σ is a formal number. In the first place obviously we must substitute $\sigma \cap t_0\text{‘}\xi$ for σ, wherever it occurs in $\phi\sigma$, and thus obtain $\phi(\sigma \cap t_0\text{‘}\xi)$. Then we may find that by the application of our conventions, we can replace this by $\phi\sigma$. For example we have

$*100{\cdot}42$. $\vdash : \mu, \nu \in \mathrm{NC} . \exists ! \mu \cap \nu . \supset . \mu = \nu$

Now put $\text{Nc}'\alpha \cap t_0'\mu$ for μ, we obtain

$\vdash : \text{Nc}'\alpha \cap t_0'\mu, \nu \in \text{NC} . \exists ! (\text{Nc}'\alpha \cap t_0'\mu) \cap \nu . \supset . \text{Nc}'\alpha \cap t_0'\mu = \nu$ (1)

$\vdash . (1) . *100{\cdot}41 . \supset \vdash : \nu \in \text{NC} . \exists ! \text{Nc}'\alpha \cap t_0'\mu \cap \nu . \supset . \text{Nc}'\alpha \cap t_0'\mu = \nu$ (2)

Now by IT, even when ν is a formal number, the identity of types of the two occurrences of $\text{Nc}'\alpha$ is equally secured in

$$\vdash : \nu \in \text{NC} . \exists ! \text{Nc}'\alpha \cap \nu . \supset . \text{Nc}'\alpha = \nu.$$

Thus this is a particular case of $*100{\cdot}42$. Such deductions can be made in general without any explicit formal statement.

Ambiguity of NC. It follows (cf. $*100{\cdot}02$ and $*103{\cdot}02$) from the typical ambiguity of Nc that NC is also typically ambiguous. Hence "$\mu \in \text{NC} . \nu \in \text{NC}$" according to our methods of interpretation would not necessitate that μ and ν should be of the same type. We shall always interpret "$\mu, \nu \in \text{NC}$" as standing for "$\mu \in \text{NC} . \nu \in \text{NC}$" and therefore as not necessarily identifying the types of μ and ν. Similarly for N_0C, NC induct, and NC ind. For example

$*110{\cdot}402$. $\vdash : \mu, \nu \in \text{N}_0\text{C} . \supset . \exists ! (\mu +_c \nu) \cap t't'(\mu \uparrow \nu)$

Here the μ and ν need not be of the same type. Again

$*110{\cdot}41$. $\vdash : \mu, \nu \in \text{N}_0\text{C} . t'\mu = t'\nu . \supset . \exists ! (\mu +_c \nu) \cap t'\mu$

Here the identification of the types of μ and ν requires the hypothesis "$t'\mu = t'\nu$."

VI. *Conventions* AT *and* Infin T.

General Arithmetical Convention. Conventions IT and IIT are always applied, but the following convention is not used at first. This convention limits the remaining ambiguity of type by sweeping away the exceptional cases in low types, due to the failure of existence theorems. The convention will be cited as AT.

AT. *All equations involving pure arithmetical formal numbers are to be arithmetical.*

We have seen that from an arithmetical equation the analogous equation in any other type can be deduced. Thus with AT all equations between formal numbers are so determined in type that their truth in "any type" is deducible. Thus in the few early propositions where AT is introduced, the fact is noted by stating that the equations hold "in *any* type." These propositions are $*103{\cdot}16$, $*110{\cdot}71{\cdot}72$.

The effect of applying AT to other propositions in $*100$ is to render some of the hypotheses (usually logical forms affirming existence) unnecessary, but also materially to limit the scope of the propositions. Take for example

$*100{\cdot}35$. $\vdash : \exists ! \text{Nc}'\alpha . \vee . \exists ! \text{Nc}'\beta : \supset :$

$$\text{Nc}'\alpha = \text{Nc}'\beta . \equiv . \alpha \in \text{Nc}'\beta . \equiv . \beta \in \text{Nc}'\alpha . \equiv . \alpha \,\text{sm}\, \beta$$

If we apply AT to this, we can write

$$\vdash : \text{Nc}'\alpha = \text{Nc}'\beta . \equiv . \alpha \in \text{Nc}'\beta . \equiv . \beta \in \text{Nc}'\alpha . \equiv . \alpha \,\text{sm}\, \beta.$$

For the equational occurrences of $Nc'\alpha$ and $Nc'\beta$ are by A T and II T to be with adequate actual types. But if α is a small class in a high type, an adequate actual type for $Nc'\alpha$ will be a high type, whereas $\exists ! Nc'\alpha$ may hold in a low type. Thus with AT, for the sake of simplicity we abandon the statement of the minimum of hypothesis necessary for our propositions. The enunciation of no other proposition in $*100$ is affected.

The enunciation of no proposition in $*101$ is affected by A T, though it would unduly limit the scope of $*101\cdot34$. In $*110$, A T would unduly limit the scope of such propositions as

$$*110\cdot22\cdot23\cdot24\cdot25\cdot251\cdot252\cdot3\cdot31\cdot32\cdot331\cdot34\cdot35\cdot351\cdot44\cdot51\cdot54$$

and of many others, without altering their enunciations. There is no proposition in $*110$ whose enunciation it would alter. AT is already applied to $*110\cdot71\cdot72$; if AT is removed from these propositions, then $\exists ! Nc'\alpha$ must be added as an hypothesis to both of them. The effect of AT on $*113$ and $*116$ is entirely analogous to that on $*110$; in neither of these two numbers is there any proposition to which A T is applied in the text.

As regards $*117$, A T is applied throughout, so that the propositions are all in the form suitable for subsequent investigations in which the interest is purely arithmetical. It is important however to analyse the effect of AT on the enunciations for the sake of logical investigations, especially in connection with $*120$. First, AT can only affect propositions in which equations or inequations occur, and among such propositions it does not affect the enunciations of those in which both sides of the equations are not formal numbers, so that the equations are not arithmetical after the application of A T. These propositions are $*117\cdot104\cdot14\cdot24\cdot241\cdot243\cdot31\cdot551$. These propositions, which are characterized by the presence of a single letter on one side of any equation involved, can be recognized at a glance. The propositions involving arithmetical equations whose enunciations are unaltered by the removal of AT are $*117\cdot21\cdot54\cdot592$. Propositions involving inequations whose enunciations are unaltered by the removal of AT are $*117\cdot26\cdot27$. Finally the only propositions of $*117$ whose enunciations are altered by the removal of AT are $*117\cdot108\cdot211\cdot23\cdot25\cdot3$.

In $*118$ and $*119$ AT is not used.

In $*120$, which is devoted to those properties of inductive cardinals which are of logical interest, AT is never used. None of the propositions $*117\cdot108\cdot211\cdot23\cdot25\cdot3$ are cited in it, except $*117\cdot25$ in the demonstration of $*120\cdot435$ for a use where AT is not relevant. The application of AT to $*120$ would simplify the hypotheses of $*120\cdot31\cdot41\cdot451\cdot53\cdot55$, and limit the scopes of the propositions.

One other convention, which we will call "Infin T," is required in certain propositions where the hypothesis implies that there are types in which every

inductive cardinal exists, *i.e.* in which V is not an inductive class. Among such hypotheses are Infin ax, $\exists ! \text{Prog}$, $\exists ! \aleph_0$ (or typically definite forms of these hypotheses), or $R \in \text{Prog}$ or $\alpha \in \aleph_0$. When such hypotheses occur, we shall assume that NC induct is, whenever significance permits, to be determined in a type in which every inductive cardinal exists, *i.e.* in which the axiom of infinity holds (cf. $*120{\cdot}03{\cdot}04$). The statement of this convention is as follows:

Infin T. *When the hypothesis of a proposition implies that there is a type in which every inductive cardinal exists, every occurrence of* "NC induct" *in this proposition is to be taken (if conditions of significance permit) in a sufficiently high type to insure the existence of every inductive cardinal.*

It is to be observed that this convention would be unnecessary if we confined ourselves to one extensional hierarchy, for in any one such hierarchy all types are inductive or all are non-inductive, so that if every inductive cardinal exists in one type in the hierarchy, the same holds for any other type in the hierarchy. But when we no longer confine ourselves to one extensional hierarchy, this result may not follow. For example, it may be the case that the number of individuals is inductive, but the number of predicative functions of individuals is not inductive; at any rate, no *logical* reason can be given against this possibility, which can only be rejected on empirical grounds, if at all.

The way in which this convention is used may be illustrated by the demonstration of $*122{\cdot}33$. In the second line of this demonstration, we show that the hypothesis implies

$$\text{E}!\,\nu_R \,.\supset .\, \text{E}!\,(\nu +_c 1)_R \qquad (1)$$

where by $*121{\cdot}04$ $\qquad \nu_R = \breve{R}_{\nu -_c 1}{}^{\backprime}B{}^{\backprime}R \quad \text{Df},$

and by $*121{\cdot}02$ $\qquad R_\nu = \hat{x}\hat{y}\,\{\text{N}_0\text{c}{}^{\backprime}R\,(x \mapsto y) = \nu +_c 1\} \quad \text{Df}.$

It will be seen that these definitions do not suffice to determine the type of ν. Hence in (1), the ν on the left may not be of the same type as the $\nu +_c 1$ on the right. Now the use of $*120{\cdot}473$, which occurs in the next line of the demonstration of $*122{\cdot}33$, requires that the ν on the left and the $\nu +_c 1$ on the right should be of the same type. This requires that the ν should not be taken in a type in which we have $\exists !\,\nu \,.\, \nu +_c 1 = \Lambda$. Hence in order to apply $*120{\cdot}473$, we must choose a type in which all inductive cardinals exist. Since "$R \in \text{Prog}$" occurs in the hypothesis, we know that all inductive cardinals exist in the type of $C{}^{\backprime}R$. But it is unnecessary to restrict ourselves to the type of $C{}^{\backprime}R$, since any other type in which all inductive cardinals exist will equally secure the validity of the demonstration. Thus the convention Infin T secures the restriction required, and no more.

The convention Infin T is often relevant when "Infin ax" without any typical determination occurs in the hypothesis. Whenever this is the case,

if "NC induct" occurs in the proposition in a way which leaves its type undetermined so far as conditions of significance are concerned, it is to be taken in a type in which all its members exist.

VII. *Final Working Rule in Arithmetic.*

It is now (whenever AT is used, together with Infin T when necessary) possible finally to sweep aside all consideration of types in connection with inductive numbers. For by combining $*126\cdot121\cdot122$ and $*120\cdot4232\cdot4622$, we see that it is always possible to take the type high enough so that no definitely determined inductive number shall be null (Λ), and that all the inductive reasoning can take place within this type. Furthermore we have already seen that the arithmetical operations are independent of the types of the components, so long as they are existential. Thus, as far as the ordinary arithmetic of finite numbers is concerned, all the conventions (including AT), and the necessity for hypotheses as to the existence of inductive numbers, are finally superseded by the following single rule:

RULE OF INDEFINITE NUMBERS. *The type assigned to any symbol which represents an inductive number is such that the symbol is not equal to* Λ.

We make the definition

$*126\cdot01$. $\text{Nc ind} = \text{Nc induct} - \iota'\Lambda \quad \text{Df}$

Wherever this symbol "Nc ind" for the class of "indefinite inductive cardinal numbers" is used, the above rule is adhered to. In other words, "$\mu \epsilon \text{NC ind}$" can always be replaced by "$\mu = \text{Nc}'\alpha \,.\, \alpha \epsilon \text{Cls induct}$," where $\text{Nc}'\alpha$ is a homogeneous or ascending cardinal, and α is the appropriate constant, or is a variable, as the case may be. In the latter case, a symbolic form such as

$$(\mu) \,.\, f(\mu \epsilon \text{NC ind}, \mu)$$

can be replaced by

$$(\mu, \alpha) \,.\, f(\mu = \text{Nc}'\alpha \,.\, \alpha \epsilon \text{Cls induct}, \mu).$$

Furthermore by $*120\cdot4622$ it follows that with this rule the result of proceeding by induction in one type and then transforming to another type is the same as that of proceeding by induction in the latter type. Thus for example there is no advantage to be gained by discriminating between 2_ξ and 2_η; for $\text{sm}_\eta{}''2_\xi = 2_\eta$, $\text{sm}_\xi{}''2_\eta = 2_\xi$, $\mu +_c 2_\xi = \mu +_c 2_\eta$, $\mu \times_c 2_\xi = \mu \times_c 2_\eta$, $\mu^2{}_\xi = \mu^2{}_\eta$, $2_\xi{}^\mu = 2_\eta{}^\mu$, and $\mu \geqslant 2_\xi \,.\, \equiv \,.\, \mu \geqslant 2_\eta$, and so on.

Hence all discrimination of the types of indefinite inductive numbers may be dropped; and the types are entirely indefinite and irrelevant.

PART III

CARDINAL ARITHMETIC

SUMMARY OF PART III

IN this Part, we shall be concerned, first, with the definition and general logical properties of cardinal numbers (Section A); then with the operations of addition, multiplication and exponentiation, of which the definitions and formal laws do not require any restriction to finite numbers (Section B); then with the theory of finite and infinite, which is rendered somewhat complicated by the fact that there are two different senses of "finite," which cannot (so far as is known) be identified without assuming the multiplicative axiom. The theory of finite and infinite will be resumed, in connection with series, in Part V, Section E.

It is in this Part that the theory of types first becomes practically relevant. It will be found that contradictions concerning the maximum cardinal are solved by this theory. We have therefore devoted our first section in this Part (with the exception of two numbers giving the most elementary properties of cardinals in general, and of 0 and 1 and 2, respectively) to the application of types to cardinals. Every cardinal is typically ambiguous, and we confer typical definiteness by the notations of $*63$, $*64$, and $*65$. It is especially where existence-theorems are concerned that the theory of types is essential. The chief importance of the propositions of the present part lies, not only, as throughout the book, in the hypotheses necessary to secure the conclusions, but also in the typical ambiguity which can be allowed to the symbols consistently with the truth of the propositions in all the cases thereby included.

SECTION A

DEFINITION AND LOGICAL PROPERTIES OF CARDINAL NUMBERS

Summary of Section A.

The Cardinal Number of a class α, which we will denote by "$\text{Nc}{}^{\backprime}\alpha$," is defined as the class of all classes similar to α, *i.e.* as $\hat{\beta}(\beta \text{ sm } \alpha)$. This definition is due to Frege, and was first published in his *Grundlagen der Arithmetik**; its symbolic expression and use are to be found in his *Grundgesetze der Arithmetik*†. The chief merits of this definition are (1) that the formal properties which we expect cardinal numbers to have result from it; (2) that unless we adopt this definition or some more complicated and practically equivalent definition, it is necessary to regard the cardinal number of a class as an indefinable. Hence the above definition avoids a useless indefinable with its attendant primitive propositions.

It will be observed that, if x is any object, 1 is not the cardinal number of x, but that of $\iota{}^{\backprime}x$. This obviates a confusion which otherwise is liable to arise in dealing with classes. Suppose we have a class α consisting of many terms; we say, nevertheless, that it is *one* class. Thus it seems to be at once one and many. But in fact it is α that is many, and $\iota{}^{\backprime}\alpha$ that is one. In regard to zero, the analogous point is still clearer. Suppose we say "there are no Kings of France." This is equivalent to "the class of Kings of France has no members," or, in our language, "the class of Kings of France is a member of the class 0." It is obvious that we cannot say "the King of France is a member of the class 0," because there is no King of France. Thus in the case of 0 and 1, as more evidently in all other cases, a cardinal number appertains to a class, not to the members of the class.

For the purposes of formal definition, we subject the formula

$$\text{Nc}{}^{\backprime}\alpha = \hat{\beta}(\beta \text{ sm } \alpha)$$

to some simplification. It will be seen that, according to this formula, "Nc" is a relation, namely the relation of a cardinal number to any class of which it is the number. Thus for example 1 has to $\iota{}^{\backprime}x$ the relation Nc; so has 2 to $\iota{}^{\backprime}x \cup \iota{}^{\backprime}y$, provided $x \neq y$. The relation Nc is, in fact, the relation $\overrightarrow{\text{sm}}$; for $\overrightarrow{\text{sm}}{}^{\backprime}\alpha = \hat{\beta}(\beta \text{ sm } \alpha)$. Hence for formal purposes of definition we put

$$\text{Nc} = \overrightarrow{\text{sm}} \quad \text{Df.}$$

* Breslau, 1884. Cf. especially pp. 79, 80.

† Jena, Vol. I. 1893, Vol. II. 1903. Cf. Vol. I. §§ 40—42, pp. 57, 58. The grounds in favour of this definition will be found at length in *Principles of Mathematics*, Part II.

The class of cardinal numbers is the class of objects which are the cardinal numbers of something or other, *i.e.* of objects which, for some α, are equal to $\text{Nc}'\alpha$. We call the class of cardinal numbers NC; thus we have

$$\text{NC} = \hat{\mu}\{(\exists\alpha) \,.\, \mu = \text{Nc}'\alpha\}.$$

For purposes of formal definition, we replace this by the simpler formula

$$\text{NC} = \text{D}'\text{Nc} \quad \text{Df.}$$

In the present section, we shall be concerned with what we may call the purely logical properties of cardinal numbers, namely those which do not depend upon the arithmetical operations of addition, multiplication and exponentiation, nor upon the distinction of finite and infinite*. The chief point to be dealt with, as regards both importance and difficulty, is the relation of a cardinal number in one type to the same or an associated cardinal number in another type. When a symbol is ambiguous as to type, we will call it *typically ambiguous*; when, either always or in a given context, it is unambiguous as to type, we will call it *typically definite*. Now the symbol "sm" is typically ambiguous; the only limitation on its type is that its domain and converse domain must both consist of classes. When we have $\alpha \,\text{sm}\, \beta$, α and β need not be of the same type, in fact, in any type of classes, there are classes similar to some of the classes of any other type of classes. For example, we have $\iota'x \,\text{sm}\, \iota'y$, whatever types x and y may belong to. This ambiguity of "sm" is derived from that of $1 \rightarrow 1$, which in turn is derived from that of 1. We denote (cf. $*65{\cdot}01$) by "1_α" all the unit classes which are of the same type as α. Then (according to the definition $*70{\cdot}01$) $1_\alpha \rightarrow 1_\beta$ will be the class of those one-one relations whose domain is of the same type as α and whose converse domain is of the same type as β. Thus "$1_\alpha \rightarrow 1_\beta$" is typically definite as soon as α and β are given. Suppose now, instead of having merely $\gamma \,\text{sm}\, \delta$, we have

$$(\exists R) \,.\, R \in 1_\alpha \rightarrow 1_\beta \,.\, \text{D}'R = \gamma \,.\, \text{Ꮧ}'R = \delta;$$

then we know not only that $\gamma \,\text{sm}\, \delta$, but also that γ belongs to the same type as α, and δ belongs to the same type as β. When the ambiguous symbol "sm" is rendered typically definite by having its domain defined as being of the same type as α, and its converse domain defined as being of the same type as β, we write it "$\text{sm}_{(\alpha,\beta)}$," because generally, in accordance with $*65{\cdot}1$, if R is a typically ambiguous relation, we write $R_{(\alpha,\beta)}$ for the typically definite relation that results when the domain of R is to consist of terms of the same type as α, and the converse domain is to consist of terms of the same type as β. Thus we have

$$\gamma \,\text{sm}_{(\alpha,\beta)}\, \delta \,.\equiv.\, (\exists R) \,.\, R \in 1_\alpha \rightarrow 1_\beta \,.\, \gamma = \text{D}'R \,.\, \delta = \text{Ꮧ}'R.$$

Here everything is typically definite if α and β (or their types) are given.

* The definitions of the arithmetical operations, and of finite and infinite, are really just as purely logical as what precedes them; but if we are to draw a line between logic and arithmetic somewhere, the arithmetical operations seem the natural point at which to place the beginning of arithmetic.

Passing now to the relation "Nc," it will be seen that it shares the typical ambiguity of "sm." In order to render it typically definite, we must derive it from a typically definite "sm." So long as nothing is added to give typical definiteness, "Nc‘γ" will mean all the classes belonging to some one (unspecified) type and similar to γ. If α is a member of the type to which these classes are to belong, then Nc‘γ is contained in the type of α. For this case, it is convenient to introduce the following two notations, already defined in ∗65. When a typically ambiguous relation R is to be rendered typically definite as to its domain only, by deciding that every member of the domain is to be *contained in* the type of α, we write "$R(\alpha)$" in place of R. When we further wish to determine R as having members of the converse domain *contained in* the type of β, we write "$R(\alpha, \beta)$" in place of R; and when we wish members of the converse domain to be *members of* the type of β, we write "$R(\alpha_\beta)$" in place of R. Thus

$$\text{sg}`\{R_{(\alpha,\beta)}\} = \{\text{sg}`R\}(\alpha_\beta)$$

(cf. ∗65·2), and in particular, since $\text{Nc} = \overrightarrow{\text{sm}}$,

$$\text{Nc}(\alpha_\beta) = \text{sg}`\text{sm}_{(\alpha,\beta)}.$$

Thus "$\text{Nc}(\alpha_\beta)`\gamma$" is only significant when γ is of the same type as β, and then it means "classes of the same type as α and similar to γ (which is of the same type as β)."

"$\text{Nc}(\alpha)`\gamma$" will mean "classes of the same type as $\alpha$ and similar to $\gamma$." As soon as the types of $\alpha$ and $\gamma$ are known, this is a typically definite symbol, being in fact equal to $\text{Nc}(\alpha_\gamma)`\gamma$. Hence so long as we only wish to consider "Nc‘γ," typical definiteness is secured by writing "$\text{Nc}(\alpha)$" in place of "Nc."

When we come to the consideration of NC, "$\text{Nc}(\alpha)$" is no longer a sufficient determination, although it suffices to determine the type. Suppose we put

$$\text{NC}^\beta(\alpha) = \text{D}`\text{Nc}(\alpha_\beta) \quad \text{Df};$$

we have also, in virtue of the definitions in ∗65,

$$\text{NC}(\alpha) = \text{NC} \cap t^{2}`\alpha = \text{D}`\text{Nc}(\alpha).$$

Thus $\text{NC}(\alpha)$ is definite as to type, but is the domain of a relation whose converse domain is ambiguous as to type; and it will appear that there are some propositions about $\text{NC}(\alpha)$ whose truth or falsehood depends upon the determination chosen for the converse domain of $\text{Nc}(\alpha)$. Hence if we wish to have a symbol which is completely definite, we must write "$\text{NC}^\beta(\alpha)$."

This point is important in connection with the contradictions as to the maximum cardinal. The following remarks will illustrate it further.

Cantor has shown that, if β is any class, no class contained in β is similar to Cl‘β. Hence in particular if β is a type, no class contained in β is similar

to $\text{Cl}'\beta$, which is the next type above β. Consequently, if $\beta = \alpha \cup -\alpha$, where α is any class, we have

$$\sim(\exists\gamma) \, . \, \gamma \subset \alpha \cup -\alpha \, . \, \gamma \operatorname{sm} \text{Cl}'(\alpha \cup -\alpha).$$

Now (cf. $*63$) we put

$$t_0{}'\alpha = \alpha \cup -\alpha \quad \text{Df},$$

and we have $t'\alpha = \text{Cl}'(\alpha \cup -\alpha)$. Thus we find

$$\sim(\exists\gamma) \, . \, \gamma \subset t_0{}'\alpha \, . \, \gamma \operatorname{sm} t'\alpha.$$

Hence $$\text{Nc}\,(\alpha_{t'\alpha})'t'\alpha = \Lambda.$$

That is to say, no class of the same type as α has as many members as $t'\alpha$ has. Hence also

$$\Lambda \in \text{NC}^{t'\alpha}\,(\alpha).$$

But $$\gamma \subset t_0{}'\alpha \, . \supset . \, \gamma \in \text{Nc}\,(\alpha_\alpha)'\gamma \, . \supset . \, \exists \,!\, \text{Nc}\,(\alpha_\alpha)'\gamma,$$

and "$\text{Nc}\,(\alpha_\alpha)'\gamma$" is only significant when $\gamma \subset t_0{}'\alpha$; hence

$$\mu \in \text{NC}^\alpha\,(\alpha) \, . \supset_\mu . \, \exists \,!\, \mu$$

and $$\Lambda \sim\in \text{NC}^\alpha\,(\alpha).$$

Now the notation "$\text{NC}\,(\alpha)$" will apply with equal justice to $\text{NC}^\alpha\,(\alpha)$ or to $\text{NC}^{t'\alpha}\,(\alpha)$; but we have just seen that in the first case we shall have $\Lambda \sim\in \text{NC}\,(\alpha)$, and in the second we shall have $\Lambda \in \text{NC}\,(\alpha)$. Consequently "$\text{NC}\,(\alpha)$" has not sufficient definiteness to prevent practically important differences between the various determinations of which it is capable.

A converse procedure to the above yields similar results. Let α be a class of classes; then $s'\alpha$ is of lower type than α. Let us consider $\text{NC}^{s'\alpha}\,(\alpha)$. In accordance with $*63$, we write $t_1{}'\alpha$ for the type containing $s'\alpha$, *i.e.* for $s'\alpha \cup -s'\alpha$. Then the greatest number in the class $\text{NC}^{s'\alpha}\,(\alpha)$ will be $\text{Nc}\,(\alpha)'t_1{}'\alpha$; but neither this nor any lesser member of the class will be equal to $\text{Nc}\,(\alpha)'t_0{}'\alpha$, because, as before,

$$\sim(\exists\gamma) \, . \, \gamma \subset t_1{}'\alpha \, . \, \gamma \operatorname{sm} t_0{}'\alpha.$$

Hence $\text{Nc}\,(\alpha)'t_0{}'\alpha$, which is a member of $\text{NC}^\alpha\,(\alpha)$, is not a member of $\text{NC}^{s'\alpha}\,(\alpha)$; but $\text{NC}^\alpha\,(\alpha)$ and $\text{NC}^{s'\alpha}\,(\alpha)$ have an equal right to be called $\text{NC}\,(\alpha)$. Hence again "$\text{NC}\,(\alpha)$" is a symbol not sufficiently definite for many of our purposes.

The solution of the paradox concerning the maximum cardinal is evident in view of what has been said. This paradox is as follows: It results from a theorem of Cantor's that there is no maximum cardinal, since, for all values of α,

$$\text{Nc}'\text{Cl}'\alpha > \text{Nc}'\alpha.$$

But at first sight it would seem that the class which contains everything must be the greatest possible class, and must therefore contain the greatest possible number of terms. We have seen, however, that a class α must always be contained within some one type; hence all that is proved is that there are greater classes in the next type, which is that of $\text{Cl}'\alpha$. Since there is always a next higher type, we thus have a maximum cardinal in each type, without

having any absolutely maximum cardinal. The maximum cardinal in the type of α is

$$\mathrm{Nc}\,(\alpha)\text{‘}(\alpha \cup -\alpha).$$

But if we take the corresponding cardinal in the next type, *i.e.*

$$\mathrm{Nc}\,(\mathrm{Cl}\text{‘}\alpha)\text{‘}(\alpha \cup -\alpha),$$

this is not as great as $\mathrm{Nc}\,(\mathrm{Cl}\text{‘}\alpha)\text{‘}\mathrm{Cl}\text{‘}(\alpha \cup -\alpha)$, and is therefore not the maximum cardinal of its type. This gives the complete solution of the paradox.

For most purposes, what we wish to know in order to have a sufficient amount of typical definiteness is not the absolute types of α and β, as above, but merely what we may call their *relative* types. Thus, for example, α and β may be of the same type; in that case, $\mathrm{Nc}\,(\alpha_\beta)$ and $\mathrm{NC}^\beta\,(\alpha)$ are respectively equal to $\mathrm{Nc}\,(\alpha_\alpha)$ and $\mathrm{NC}^\alpha\,(\alpha)$. We will call cardinals which, for some α, are members of the class $\mathrm{NC}^\alpha\,(\alpha)$, *homogeneous* cardinals, because the "sm" from which they are derived is a homogeneous relation. We shall denote the homogeneous cardinal of α by "$\mathrm{N_0c}\text{‘}\alpha$," and we shall denote the class of homogeneous cardinals (in an unspecified type) by "$\mathrm{N_0C}$"; thus we put

$$\mathrm{N_0c}\text{‘}\alpha = \mathrm{Nc}\text{‘}\alpha \cap t\text{‘}\alpha \quad \mathrm{Df},$$

$$\mathrm{N_0C} = \mathrm{D}\text{‘}\mathrm{N_0c} \qquad \mathrm{Df}.$$

Almost all the properties of $\mathrm{N_0C}$ are the same in different types. When further typical definiteness is required, it can be secured by writing $\mathrm{N_0c}\,(\alpha)$, $\mathrm{N_0C}\,(\alpha)$ in place of $\mathrm{N_0c}$, $\mathrm{N_0C}$. For although $\mathrm{Nc}\,(\alpha)$ and $\mathrm{NC}\,(\alpha)$ were not wholly definite, $\mathrm{N_0c}\,(\alpha)$ and $\mathrm{N_0C}\,(\alpha)$ are wholly definite. Apart from the fact of being of different types, the only property in which $\mathrm{N_0C}\,(\alpha)$ and $\mathrm{N_0C}\,(\beta)$ differ when α and β are of different types is in regard to the magnitude of the cardinals belonging to them. Thus suppose the whole universe consisted (as monists aver) of a single individual. Let us call the type of this individual "Indiv." Then $\mathrm{N_0C}\,(\mathrm{Indiv})$ will consist of 0 and 1, *i.e.*

$$\mathrm{N_0C}\,(\mathrm{Indiv}) = \iota\text{‘}0 \cup \iota\text{‘}1.$$

But in the next higher type, there will be two members, namely Λ and Indiv. Thus

$$\mathrm{N_0C}\,(t\text{‘}\mathrm{Indiv}) = \iota\text{‘}0 \cup \iota\text{‘}1 \cup \iota\text{‘}2.$$

Similarly $\quad \mathrm{N_0C}\,(t\text{‘}t\text{‘}\mathrm{Indiv}) = \iota\text{‘}0 \cup \iota\text{‘}1 \cup \iota\text{‘}2 \cup \iota\text{‘}3 \cup \iota\text{‘}4,$

the members of $t\text{‘}t\text{‘}\mathrm{Indiv}$ being $\Lambda \cap t\text{‘}\mathrm{Indiv}$, $\iota\text{‘}\Lambda$, $\iota\text{‘}\mathrm{Indiv}$, $\iota\text{‘}\Lambda \cup \iota\text{‘}\mathrm{Indiv}$; and so on. (The greatest cardinal in any except the lowest type is always a power of 2.)

The maximum of $\mathrm{N_0C}\,(\alpha)$ is $\mathrm{N_0c}\text{‘}t_0\text{‘}\alpha$; but apart from this difference of maximum and its consequences, $\mathrm{N_0C}\,(\alpha)$ and $\mathrm{N_0C}\,(\beta)$ do not differ in any important properties. Hence for most purposes $\mathrm{N_0C}$ and $\mathrm{N_0c}$ have as much typical definiteness as is necessary.

Among cardinals which are not homogeneous we shall consider three kinds. The first of these we shall call *ascending* cardinals. A cardinal $\mathrm{NC}^\beta\,(\alpha)$ is

called an *ascending* cardinal if the type of β is t'α or t't'α or t't't'α or etc. We write t^2'α for t't'α, t^3'α for t't't'α, and so on. We put

$$N^1c`\alpha = Nc`\alpha \cap t`t`\alpha \quad Df$$

$$N^2c`\alpha = Nc`\alpha \cap t`t^2`\alpha \quad Df$$

$$N^3c`\alpha = Nc`\alpha \cap t`t^3`\alpha \quad Df \quad \text{and so on,}$$

and

$$N^1C = D`N^1c \quad Df$$

$$N^2C = D`N^2c \quad Df$$

$$N^3C = D`N^3c \quad Df \quad \text{and so on.}$$

We then have obviously

$$N^1C\,(t`\alpha) \subset N_0C\,(t`\alpha).$$

We also have (by what was said earlier)

$$N_0c`t`\alpha \sim\epsilon\, N^1C\,(t`\alpha).$$

Hence

$$\exists !\, N_0C\,(t`\alpha) - N^1C\,(t`\alpha).$$

The members of $N_0C\,(t`\alpha) - N^1C\,(t`\alpha)$ will be all cardinals which exceed $Nc`t_0`\alpha$ but do not exceed $Nc`t`\alpha$.

Let us recur in illustration to our previous hypothesis of the universe consisting of a single individual. Then $N^1c`\text{Indiv}$ will consist of those classes which are similar to "Indiv" but of the next higher type. These are $\iota`\Lambda$ and $\iota`\text{Indiv}$. In our case we had $N_0c`\text{Indiv} = 1$. This leads to

$$N^1c`\text{Indiv} = 1 \,.\, N^2c`\text{Indiv} = 1 \text{ etc.}$$

or, introducing typical definiteness,

$$N^1c`\text{Indiv} = 1\,(t`\text{Indiv}) \,.\, N^2c`\text{Indiv} = 1\,(t^2`\text{Indiv}) \text{ etc.}$$

We have then $1\,(t`\text{Indiv}) \,\epsilon\, N^1C\,(t`t`\text{Indiv})$. Also

$$1\,(t`\text{Indiv}) \,\epsilon\, N_0C\,(t`t`\text{Indiv}).$$

And in the case supposed, $1\,(t`\text{Indiv})$ is the maximum of $N^1C\,(t`t`\text{Indiv})$, but $2\,(t`\text{Indiv}) \,\epsilon\, N_0C\,(t`t`\text{Indiv})$. Hence

$$N_0C\,(t`t`\text{Indiv}) - N^1C\,(t`t`\text{Indiv}) = \iota`2.$$

Generalizing, we see that $N^1C\,(t`\alpha)$ consists of the same numbers as $N_0C\,(\alpha)$ each raised one degree in type. Similar propositions hold of $N^2C\,(t^2`\alpha)$, $N^3C\,(t^3`\alpha)$ etc.

It is often useful to have a notation for what we may call "the same cardinal in another type." Suppose μ is a typically definite cardinal; then we will denote by $\mu^{(1)}$ the same cardinal in the next type, *i.e.*

$$sm``\mu \cap t`\mu.$$

Note that, if μ is a cardinal, $sm``\mu \cap \mu = \mu$; and whether μ is a typically definite cardinal or not,

$$sm``\mu \cap t`\alpha$$

is a cardinal in a definite type. If μ is typically definite, then $sm``\mu \cap t`\alpha$ is wholly definite; if $\mu$ is typically ambiguous, $sm``\mu \cap t`\alpha$ has the same kind of

indefiniteness as belongs to $\mathrm{NC}(\alpha)$. The most important case is when μ is typically definite and α has an assigned relation of type to μ. We then put, as observed above,

$$\mu^{(1)} = \mathrm{sm}``\mu \cap t`\mu \quad \mathrm{Df}$$

$$\mu^{(2)} = \mathrm{sm}``\mu \cap t^2`\mu \quad \mathrm{Df\ etc.}$$

If μ is an $\mathrm{N_0C}$, $\mu^{(1)}$ is an $\mathrm{N^1C}$ and $\mu^{(2)}$ is an $\mathrm{N^2C}$ and so on. $\mathrm{N^1C}(t`\alpha)$ will consist of all numbers which are of the form $\mu^{(1)}$ for some μ which is a member of $\mathrm{N_0C}(\alpha)$; *i.e.*

$$\mathrm{N^1C}(t`\alpha) = \hat{\nu}\{(\exists\mu) \,.\, \mu \in \mathrm{N_0C}(\alpha) \,.\, \nu = \mu^{(1)}\}.$$

The second kind of non-homogeneous cardinals to be considered is called the class of "descending cardinals." These are such as go into a lower type; *i.e.* $\mathrm{Nc}(\alpha)`\beta$ is a descending cardinal if α is of a lower type than β. We put

$$\mathrm{N_1c}`\alpha = \mathrm{Nc}`\alpha \cap t`t_1`\alpha \quad \mathrm{Df}$$

$$\mathrm{N_2c}`\alpha = \mathrm{Nc}`\alpha \cap t`t_2`\alpha \quad \mathrm{Df\ etc.}$$

$$\mathrm{N_1C} = \mathrm{D}`\mathrm{N_1c} \quad \mathrm{Df}$$

$$\mathrm{N_2C} = \mathrm{D}`\mathrm{N_2c} \quad \mathrm{Df\ etc.}$$

$$\mu_{(1)} = \mathrm{sm}``\mu \cap t_1`\mu \quad \mathrm{Df}$$

$$\mu_{(2)} = \mathrm{sm}``\mu \cap t_2`\mu \quad \mathrm{Df\ etc.}$$

We have obviously $\qquad \mathrm{N_0c}`\alpha = \mathrm{N_1c}`\iota``\alpha.$

Hence $\qquad \mathrm{N_0C}(\alpha) \subset \mathrm{N_1C}(\alpha).$

Also $\qquad \gamma \in \mathrm{N_1c}`\delta \,.\supset .\, \mathrm{N_1c}`\delta = \mathrm{N_0c}`\gamma,$

whence $\qquad \exists\,!\,\mathrm{N_1c}`\delta \,.\supset .\, \mathrm{N_1c}`\delta \in \mathrm{N_0C},$

whence $\qquad \mathrm{N_1C} - \iota`\Lambda \subset \mathrm{N_0C}.$

Since also $\Lambda \sim\in \mathrm{N_0C}(\alpha)$, we find

$$\mathrm{N_0C} = \mathrm{N_1C} - \iota`\Lambda,$$

this proposition not requiring any further typical definiteness, since it holds however such definiteness may be introduced, remembering that such definiteness is necessarily so introduced as to secure significance. Further, in virtue of the fact that no class contained in $t_0`\alpha$ is similar to $t`\alpha$, we have

$$\Lambda \in \mathrm{N_1C}(\alpha).$$

Consequently $\qquad \mathrm{N_1C} = \mathrm{N_0C} \cup \iota`\Lambda.$

We can prove in just the same way

$$\mathrm{N_2C} = \mathrm{N_0C} \cup \iota`\Lambda.$$

Hence $\qquad \mathrm{N_1C} = \mathrm{N_2C},$

and this result can obviously be extended to all descending cardinals.

The third kind of non-homogeneous cardinals to be considered may be called "relational cardinals." They are those applicable to classes of relations having a given relation of type to a given class. Consider for example $\mathrm{Nc}`\epsilon_\Delta`\kappa$. (We shall take this as the definition of the product of the numbers of the

members of κ.) Suppose now that κ consists of a single term: we want to be able to say

$$\mathrm{Nc}\text{‘}\epsilon_\Delta\text{‘}\kappa = \mathrm{Nc}\text{‘}\breve{\iota}\text{‘}\kappa.$$

We have in this case, if $\kappa = \iota\text{‘}\alpha$,

$$\epsilon_\Delta\text{‘}\kappa = \downarrow \alpha\text{‘‘}\alpha,$$

and we know that $\downarrow \alpha\text{‘‘}\alpha \,\mathrm{sm}\, \alpha$. But if we put simply

$$\mathrm{Nc}\text{‘} \downarrow \alpha\text{‘‘}\alpha = \mathrm{Nc}\text{‘}\alpha,$$

our proposition, though not mistaken, requires care in interpretation. Just as we put $\iota\text{‘‘}\alpha \in \mathrm{N}^1\mathrm{c}\text{‘}\alpha$, so we want a notation giving typical definiteness to the proposition $\downarrow \alpha\text{‘‘}\alpha \in \mathrm{Nc}\text{‘}\alpha$. This is provided as follows.

Using the notation of $*64$, put

$$\mathrm{N}_{00}\mathrm{c}\text{‘}\alpha = \mathrm{Nc}\text{‘}\alpha \cap t\text{‘}t_{00}\text{‘}\alpha \quad \mathrm{Df}$$
$$\mathrm{N}_0{}^1\mathrm{c}\text{‘}\alpha = \mathrm{Nc}\text{‘}\alpha \cap t\text{‘}t_0{}^1\text{‘}\alpha \quad \mathrm{Df\ etc.}$$
$$\mathrm{N}_{00}\mathrm{C} = \mathrm{D}\text{‘}\mathrm{N}_{00}\mathrm{c} \quad \mathrm{Df}$$
$$\mathrm{N}_0{}^1\mathrm{C} = \mathrm{D}\text{‘}\mathrm{N}_0{}^1\mathrm{c} \quad \mathrm{Df\ etc.}$$
$$\mu_{(00)} = \mathrm{sm}\text{‘‘}\mu \cap t\text{‘}t_{00}\text{‘}t_1\text{‘}\mu \quad \mathrm{Df\ etc.}$$

Then we have, for example,

$$\downarrow \alpha\text{‘‘}\alpha \subset t_0{}^1\text{‘}\alpha, \quad i.e. \downarrow \alpha\text{‘‘}\alpha \in t\text{‘}t_0{}^1\text{‘}\alpha.$$

Hence $\downarrow \alpha\text{‘‘}\alpha \in \mathrm{N}_0{}^1\mathrm{c}\text{‘}\alpha$, where $\mathrm{N}_0{}^1\mathrm{c}\text{‘}\alpha = \mathrm{Nc}\text{‘}\alpha \cap t\text{‘}t_0{}^1\text{‘}\alpha$.

Similarly $\quad x \in t\text{‘}\alpha \,.\supset.\, \downarrow x\text{‘‘}\alpha \in \mathrm{N}_{00}\mathrm{c}\text{‘}\alpha.$

Thus the above definitions give us what is required.

In order to complete our notation for types, we should need to be able to express the type of the domain or converse domain of R, or of any relation whose domain and converse domain have respectively given relations of type to the domain and converse domain of R. Thus we might put

$$\mathrm{d}_0\text{‘}R = t_0\text{‘}\mathrm{D}\text{‘}R \quad \mathrm{Df}$$
$$\mathrm{b}_0\text{‘}R = t_0\text{‘}\mathrm{G}\text{‘}R \quad \mathrm{Df}$$

("b" appears here as "d" written backwards)

$$\mathrm{d}_{00}\text{‘}R = t\text{‘}(\mathrm{d}_0\text{‘}R \uparrow \mathrm{b}_0\text{‘}R) \quad \mathrm{Df}$$
$$= t\text{‘}R$$
$$\mathrm{d}^{mn}\text{‘}R = t\text{‘}(t^m\text{‘}\mathrm{d}_0\text{‘}R \uparrow t^n\text{‘}\mathrm{b}_0\text{‘}R) \quad \mathrm{Df\ and\ so\ on.}$$

This notation would enable us to deal with descending relational cardinals. But it is not required in the present work, and is therefore not introduced among the numbered propositions.

When a typically ambiguous symbol, such as "sm" or "Nc," occurs more than once in a given context, it must not be assumed, unless required by the conditions of significance, that it is to receive the same typical determination in each case. Thus *e.g.* we shall write "$\alpha \,\mathrm{sm}\, \beta \,.\supset.\, \beta \,\mathrm{sm}\, \alpha$," although, if α and β are of different types, the two symbols "sm" must receive different typical determinations.

Formulae which are typically ambiguous, or only partially definite as to type, must not be admitted unless every significant interpretation is true. Thus for example we may admit

$$\text{“} \vdash . \alpha \in \text{Nc}`\alpha \text{”}$$

because here "Nc" must mean "$\text{Nc}(\alpha_\alpha)$," so that the only ambiguity remaining is as to the type of α, and the formula holds whatever type α may belong to, provided "$\text{Nc}`\alpha$" is significant, *i.e.* provided $\alpha$ is a class. But we must not, from "$\alpha \in \text{Nc}`\alpha$," allow ourselves to infer

$$\text{“} \exists ! \text{Nc}`\alpha \text{.”}$$

For here the conditions of significance no longer demand that "Nc" should mean "$\text{Nc}(\alpha_\alpha)$": it might just as well mean "$\text{Nc}(\beta_\alpha)$." And as we saw, if β is a lower type than α, and α is sufficiently large of its type, we may have

$$\text{Nc}(\beta_\alpha)`\alpha = \Lambda,$$

so that "$\exists ! \text{Nc}`\alpha$" is not admissible without qualification. Nevertheless, as we shall see in $*100$, there are a certain number of propositions to be made about a wholly ambiguous Nc or NC.

$*100$. DEFINITION AND ELEMENTARY PROPERTIES OF CARDINAL NUMBERS

Summary of $*100$.

In this number we shall be concerned only with such immediate consequences of the definition of cardinal numbers as do not require typical definiteness, beyond what the inherent conditions of significance may bestow. We introduce here the fundamental definitions:

$*100{\cdot}01$. $\text{Nc} = \overrightarrow{\text{sm}}$ Df

$*100{\cdot}02$. $\text{NC} = \text{D}‘\text{Nc}$ Df

The definition "Nc" is required chiefly for the sake of the descriptive function $\text{Nc}‘\alpha$. We have

$*100{\cdot}1$. $\vdash . \text{Nc}‘\alpha = \hat{\beta}(\beta \mathrel{\breve{\text{sm}}} \alpha) = \hat{\beta}(\alpha \mathrel{\text{sm}} \beta)$

This may be stated in various equivalent forms, which are given at the beginning of this number ($*100{\cdot}1$—$\cdot16$). After a few propositions on Nc as a relation, we proceed to the elementary properties of $\text{Nc}‘\alpha$. We have

$*100{\cdot}3$. $\vdash . \alpha \in \text{Nc}‘\alpha$

$*100{\cdot}31$. $\vdash : \alpha \in \text{Nc}‘\beta . \equiv . \beta \in \text{Nc}‘\alpha . \equiv . \alpha \mathrel{\text{sm}} \beta$

$*100{\cdot}321$. $\vdash : \alpha \mathrel{\text{sm}} \beta . \supset . \text{Nc}‘\alpha = \text{Nc}‘\beta$

$*100{\cdot}33$. $\vdash : \exists ! \text{Nc}‘\alpha \cap \text{Nc}‘\beta . \supset . \alpha \mathrel{\text{sm}} \beta$

We proceed next to the elementary properties of NC. We have

$*100{\cdot}4$. $\vdash : \mu \in \text{NC} . \equiv . (\exists \alpha) . \mu = \text{Nc}‘\alpha$

$*100{\cdot}42$. $\vdash : \mu, \nu \in \text{NC} . \exists ! \mu \cap \nu . \supset . \mu = \nu$

$*100{\cdot}45$. $\vdash : \mu \in \text{NC} . \alpha \in \mu . \supset . \text{Nc}‘\alpha = \mu$

$*100{\cdot}51$. $\vdash : \mu \in \text{NC} . \alpha \in \mu . \supset . \text{sm}“\mu = \text{Nc}‘\alpha$

Observe that when we have such a hypothesis as "$\mu \in \text{NC}$," the μ, though it may be of any type, must be of *some* type; hence the μ cannot have the typical ambiguity which belongs to $\text{Nc}‘\alpha$. If we put $\mu = \text{Nc}‘\alpha$, this will hold only in the type of μ; but "$\text{sm}“\mu$" is a typically ambiguous symbol, which will represent in any type the "same" number as μ. Thus "$\text{sm}“\mu = \text{Nc}‘\alpha$" is an equation which is applicable to all possible typical determinations of "sm" and "Nc."

$*100{\cdot}52$. $\vdash : \mu \in \text{NC} . \exists ! \mu . \supset . \text{sm}“\mu \in \text{NC}$

The hypothesis $\exists ! \mu$ is unnecessary, but we cannot prove this till later ($*102$).

We end the number with some propositions ($*100{\cdot}6$—$\cdot64$) stating that various classes (such as $\iota“\alpha$), which have already been proved to be similar to α, have $\text{Nc}‘\alpha$ members.

***100·01.** $\mathrm{Nc} = \overrightarrow{\mathrm{sm}} \quad \mathrm{Df}$

***100·02.** $\mathrm{NC} = \mathrm{D}‘\mathrm{Nc} \quad \mathrm{Df}$

***100·1.** $\vdash . \mathrm{Nc}‘\alpha = \hat{\beta}(\beta \,\mathrm{sm}\, \alpha) = \hat{\beta}(\alpha \,\mathrm{sm}\, \beta) \quad [*32{\cdot}13 . *73{\cdot}31 . (*100{\cdot}01)]$

***100·11.** $\vdash . \mathrm{Nc}‘\alpha = \hat{\beta}\{(\exists R) . R \in 1 \to 1 . \mathrm{D}‘R = \alpha . \text{Ɑ}‘R = \beta\} \quad [*100{\cdot}1 . *73{\cdot}1]$

***100·12.** $\vdash . \mathrm{Nc}‘\alpha = \hat{\beta}\{(\exists R) . R \in 1 \to 1 . \alpha \subset \mathrm{D}‘R . \beta = \breve{R}“\alpha\}$
$[*100{\cdot}1 . *73{\cdot}11]$

***100·13.** $\vdash . \mathrm{Nc}‘\alpha = \text{Ɑ}“(1 \to 1 \cap \overleftarrow{\mathrm{D}}‘\alpha) = \mathrm{D}“(1 \to 1 \cap \overleftarrow{\text{Ɑ}}‘\alpha)$

Dem.

$$\vdash . *100{\cdot}11 . *33{\cdot}6 . \supset \vdash . \mathrm{Nc}‘\alpha = \hat{\beta}\{(\exists R) . R \in 1 \to 1 . R \in \overleftarrow{\mathrm{D}}‘\alpha . \text{Ɑ}‘R = \beta\}$$
$$[*22{\cdot}33 . *37{\cdot}6] \quad = \text{Ɑ}“(1 \to 1 \cap \overleftarrow{\mathrm{D}}‘\alpha) \quad (1)$$
$$\vdash . *100{\cdot}1 . *73{\cdot}1 . *33{\cdot}61 . \supset \vdash . \mathrm{Nc}‘\alpha = \hat{\beta}\{(\exists R) . R \in 1 \to 1 . R \in \overleftarrow{\text{Ɑ}}‘\alpha . \mathrm{D}‘R = \beta\}$$
$$[*22{\cdot}33 . *37{\cdot}6] \quad = \mathrm{D}“(1 \to 1 \cap \overleftarrow{\text{Ɑ}}‘\alpha) \quad (2)$$
$$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$$

***100·14.** $\vdash . \mathrm{Nc}‘\alpha = \hat{\beta}\{(\exists R) . \alpha \subset \text{Ɑ}‘R . R \upharpoonright \alpha \in 1 \to 1 . \beta = R“\alpha\}$
$[*73{\cdot}15 . *100{\cdot}1]$

***100·15.** $\vdash . \mathrm{Nc}‘\alpha = \hat{\beta}\{(\exists R) : \mathrm{E}\,!!\, R“\alpha :$
$x, y \in \alpha . R‘x = R‘y . \supset_{x,y} . x = y : \beta = R“\alpha\}$

Dem.

$$\vdash . *74{\cdot}1{\cdot}11 . \supset$$
$$\vdash :. \mathrm{E}\,!!\, R“\alpha : x, y \in \alpha . R‘x = R‘y . \supset_{x,y} . x = y : \beta = R“\alpha : \equiv :$$
$$R \upharpoonright \alpha \in 1 \to \mathrm{Cls} . \alpha \subset \text{Ɑ}‘R . R \upharpoonright \alpha \in 1 \to 1 . \beta = R“\alpha \quad (1)$$
$$\vdash . (1) . *4{\cdot}71 . *100{\cdot}14 . \supset \vdash . \mathrm{Prop}$$

***100·16.** $\vdash . \mathrm{Nc}‘\alpha = \hat{\beta}\{(\exists R) :. x, y \in \alpha . \supset_{x,y} : R‘x = R‘y . \equiv . x = y :. \beta = R“\alpha\}$

Dem.

$$\vdash . *71{\cdot}59 . \supset$$
$$\vdash :: x, y \in \alpha . \supset_{x,y} : R‘x = R‘y . \equiv . x = y :. \equiv . R \upharpoonright \alpha \in 1 \to 1 . \alpha \subset \text{Ɑ}‘R \quad (1)$$
$$\vdash . (1) . *100{\cdot}14 . \supset \vdash . \mathrm{Prop}$$

***100·2.** $\vdash . \mathrm{E}\,!\, \mathrm{Nc}‘\alpha \quad [*32{\cdot}12 . (*100{\cdot}01)]$

***100·21.** $\vdash . \text{Ɑ}‘\mathrm{Nc} = \mathrm{Cls}$

Dem.

$$\vdash . *37{\cdot}76 . (*100{\cdot}01) . \supset \vdash . \text{Ɑ}‘\mathrm{Nc} \subset \mathrm{Cls} \quad (1)$$
$$\vdash . *33{\cdot}431 . *100{\cdot}2 . \supset \vdash . \mathrm{Cls} \subset \text{Ɑ}‘\mathrm{Nc} \quad (2)$$
$$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$$

***100·22.** $\vdash . \mathrm{Nc} \in 1 \to \mathrm{Cls} \quad [*72{\cdot}12 . (*100{\cdot}01)]$

***100·3.** $\vdash . \alpha \in \mathrm{Nc}‘\alpha \quad [*73{\cdot}3 . *100{\cdot}1]$

Note that it is fallacious to infer $\exists\,!\, \mathrm{Nc}‘\alpha$, for reasons explained in the introduction to the present section.

***100·31.** $\vdash : \alpha \in \text{Nc}`\beta . \equiv . \beta \in \text{Nc}`\alpha . \equiv . \alpha \,\text{sm}\, \beta$ [*32·18 . *73·31 . (*100·01)]

***100·32.** $\vdash : \alpha \in \text{Nc}`\beta . \beta \in \text{Nc}`\gamma . \supset . \alpha \in \text{Nc}`\gamma$ [*100·31 . *73·32]

***100·321.** $\vdash : \alpha \,\text{sm}\, \beta . \supset . \text{Nc}`\alpha = \text{Nc}`\beta$

Dem.

$\vdash . *73 \cdot 37 . \supset \vdash :. \text{Hp} . \supset : \gamma \,\text{sm}\, \alpha . \equiv_\gamma . \gamma \,\text{sm}\, \beta :$

[*100·1] $\supset : \text{Nc}`\alpha = \text{Nc}`\beta :. \supset \vdash . \text{Prop}$

Note that $\text{Nc}`\alpha = \text{Nc}`\beta . \supset . \alpha \,\text{sm}\, \beta$ is not always true. We might be tempted to prove it as follows:

$\vdash . *100 \cdot 1 . \supset \vdash :. \text{Nc}`\alpha = \text{Nc}`\beta . \equiv : \gamma \,\text{sm}\, \alpha . \equiv_\gamma . \gamma \,\text{sm}\, \beta :$

[*10·1] $\supset : \alpha \,\text{sm}\, \alpha . \equiv . \alpha \,\text{sm}\, \beta :$

[*73·3] $\supset : \alpha \,\text{sm}\, \beta$

But the use of *10·1 here is only legitimate when the "sm" concerned is a homogeneous relation. If $\text{Nc}`\alpha$, $\text{Nc}`\beta$ are descending cardinals, we may have $\text{Nc}`\alpha = \Lambda = \text{Nc}`\beta$ without having $\alpha \,\text{sm}\, \beta$.

***100·33.** $\vdash : \exists ! \text{Nc}`\alpha \cap \text{Nc}`\beta . \supset . \alpha \,\text{sm}\, \beta$

Dem.

$\vdash . *100 \cdot 1 . \supset \vdash : \text{Hp} . \supset . (\exists \gamma) . \gamma \,\text{sm}\, \alpha . \gamma \,\text{sm}\, \beta .$

[*73·31] $\supset . (\exists \gamma) . \alpha \,\text{sm}\, \gamma . \gamma \,\text{sm}\, \beta .$

[*73·32] $\supset . \alpha \,\text{sm}\, \beta : \supset \vdash . \text{Prop}$

Note that we do not always have

$$\alpha \,\text{sm}\, \beta . \supset . \exists ! \text{Nc}`\alpha \cap \text{Nc}`\beta .$$

For if the Nc concerned is a descending Nc, and α and β are sufficiently great, $\text{Nc}`\alpha$ and $\text{Nc}`\beta$ may both be Λ. For example, we have

$$\text{Cl}`(\alpha \cup - \alpha) \,\text{sm}\, \text{Cl}`(\alpha \cup - \alpha).$$

But $\text{Nc}\,(\alpha)`\text{Cl}`(\alpha \cup - \alpha) = \Lambda$, so that

$$\sim \exists ! \text{Nc}\,(\alpha)`\text{Cl}`(\alpha \cup - \alpha) \cap \text{Nc}\,(\alpha)`\text{Cl}`(\alpha \cup - \alpha).$$

Thus "$\alpha \,\text{sm}\, \beta . \supset . \exists ! \text{Nc}`\alpha \cap \text{Nc}`\beta$" is not always true when it is significant.

***100·34.** $\vdash : \exists ! \text{Nc}`\alpha \cap \text{Nc}`\beta . \supset . \text{Nc}`\alpha = \text{Nc}`\beta$ [*100·33·321]

***100·35.** $\vdash :. \exists ! \text{Nc}`\alpha . \vee . \exists ! \text{Nc}`\beta : \supset :$

$$\text{Nc}`\alpha = \text{Nc}`\beta . \equiv . \alpha \in \text{Nc}`\beta . \equiv . \beta \in \text{Nc}`\alpha . \equiv . \alpha \,\text{sm}\, \beta$$

Dem.

$\vdash . *22 \cdot 5 . \quad \supset \vdash :. \text{Hp} . \supset : \text{Nc}`\alpha = \text{Nc}`\beta . \supset . \exists ! \text{Nc}`\alpha \cap \text{Nc}`\beta .$

[*100·33] $\supset . \alpha \,\text{sm}\, \beta$ (1)

$\vdash . (1) . *100 \cdot 321 . \supset \vdash :. \text{Hp} . \supset : \text{Nc}`\alpha = \text{Nc}`\beta . \equiv . \alpha \,\text{sm}\, \beta$ (2)

$\vdash . (2) . *100 \cdot 31 . \supset \vdash . \text{Prop}$

Thus the only case in which the implications in *100·321·33·34 cannot be turned into equivalences is the case in which $\text{Nc}`\alpha$ and $\text{Nc}`\beta$ are both Λ.

***100·36.** $\vdash :. \beta \in \text{Nc}`\alpha . \supset : \exists ! \alpha . \equiv . \exists ! \beta$ [*100·31 . *73·36]

***100·4.** $\vdash : \mu \in \text{NC} . \equiv . (\exists \alpha) . \mu = \text{Nc}`\alpha$ [*37·78·79 . (*100·02·01)]

$*100{\cdot}41$. $\vdash . \mathrm{Nc}`\alpha \in \mathrm{NC}$ $[*100{\cdot}4{\cdot}2 . *14{\cdot}204]$

$*100{\cdot}42$. $\vdash : \mu, \nu \in \mathrm{NC} . \exists ! \mu \cap \nu . \supset . \mu = \nu$

Dem.

$\vdash . *100{\cdot}4 . \supset \vdash : \mathrm{Hp} . \supset . (\exists \alpha, \beta) . \mu = \mathrm{Nc}`\alpha . \nu = \mathrm{Nc}`\beta . \exists ! \mathrm{Nc}`\alpha \cap \mathrm{Nc}`\beta .$

$[*100{\cdot}34] \quad \supset . (\exists \alpha, \beta) . \mu = \mathrm{Nc}`\alpha . \nu = \mathrm{Nc}`\beta . \mathrm{Nc}`\alpha = \mathrm{Nc}`\beta .$

$[*14{\cdot}15] \quad \supset . \mu = \nu : \supset \vdash . \mathrm{Prop}$

$*100{\cdot}43$. $\vdash . \mathrm{NC} \in \mathrm{Cls}^2 \,\mathrm{excl}$ $[*100{\cdot}42 . *84{\cdot}11]$

$*100{\cdot}44$. $\vdash :. \mu \in \mathrm{NC} . \exists ! \mathrm{Nc}`\alpha . \supset : \alpha \in \mu . \equiv . \mathrm{Nc}`\alpha = \mu$

Dem.

$\vdash . *100{\cdot}3 . \supset \vdash : \mathrm{Nc}`\alpha = \mu . \supset . \alpha \in \mu \qquad (1)$

$\vdash . *10{\cdot}24 . \supset \vdash : \mu \in \mathrm{NC} . \exists ! \mathrm{Nc}`\alpha . \alpha \in \mu . \supset .$

$\mu \in \mathrm{NC} . \exists ! \mu . \exists ! \mathrm{Nc}`\alpha . \alpha \in \mu .$

$[*100{\cdot}4] \quad \supset . (\exists \beta) . \mu = \mathrm{Nc}`\beta . \exists ! \mathrm{Nc}`\beta . \exists ! \mathrm{Nc}`\alpha . \alpha \in \mathrm{Nc}`\beta .$

$[*100{\cdot}35] \quad \supset . (\exists \beta) . \mu = \mathrm{Nc}`\beta . \mathrm{Nc}`\alpha = \mathrm{Nc}`\beta .$

$[*14{\cdot}15] \quad \supset . \mathrm{Nc}`\alpha = \mu \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

$*100{\cdot}45$. $\vdash : \mu \in \mathrm{NC} . \alpha \in \mu . \supset . \mathrm{Nc}`\alpha = \mu$ $[*100{\cdot}4{\cdot}31{\cdot}321]$

$*100{\cdot}5$. $\vdash : \mu \in \mathrm{NC} . \alpha, \beta \in \mu . \supset . \alpha \,\mathrm{sm}\, \beta$

Dem.

$\vdash . *100{\cdot}4 . \supset \vdash : \mathrm{Hp} . \supset . (\exists \gamma) . \mu = \mathrm{Nc}`\gamma . \alpha, \beta \in \mathrm{Nc}`\gamma .$

$[*100{\cdot}31] \quad \supset . (\exists \gamma) . \alpha \,\mathrm{sm}\, \gamma . \beta \,\mathrm{sm}\, \gamma .$

$[*73{\cdot}31{\cdot}32] \quad \supset . \alpha \,\mathrm{sm}\, \beta : \supset \vdash . \mathrm{Prop}$

$*100{\cdot}51$. $\vdash : \mu \in \mathrm{NC} . \alpha \in \mu . \supset . \mathrm{sm}``\mu = \mathrm{Nc}`\alpha$

Dem.

$\vdash . *100{\cdot}5 . \mathrm{Fact} . \supset \vdash :. \mathrm{Hp} . \supset : \beta \in \mu . \gamma \,\mathrm{sm}\, \beta . \supset . \alpha \,\mathrm{sm}\, \beta . \gamma \,\mathrm{sm}\, \beta .$

$[*73{\cdot}31{\cdot}32] \quad \supset . \alpha \,\mathrm{sm}\, \gamma .$

$[*100{\cdot}31] \quad \supset . \gamma \in \mathrm{Nc}`\alpha \qquad (1)$

$\vdash . (1) . *10{\cdot}11{\cdot}21{\cdot}23 . *37{\cdot}1 . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{sm}``\mu \subset \mathrm{Nc}`\alpha \qquad (2)$

$\vdash . *100{\cdot}31 . \quad \supset \vdash :. \mathrm{Hp} . \supset : \gamma \in \mathrm{Nc}`\alpha . \supset . \gamma \,\mathrm{sm}\, \alpha . \alpha \in \mu .$

$[*37{\cdot}1] \quad \supset . \gamma \in \mathrm{sm}``\mu \qquad (3)$

$\vdash . (2) . (3) . \supset \vdash . \mathrm{Prop}$

$*100{\cdot}511$. $\vdash : \exists ! \mathrm{Nc}`\beta . \supset . \mathrm{sm}``\mathrm{Nc}`\beta = \mathrm{Nc}`\beta$

Here the last "$\mathrm{Nc}`\beta$" may be of a different type from the others: the proposition holds however its type is determined.

Dem.

$\vdash . *100{\cdot}51{\cdot}41 . \supset \vdash : \alpha \in \mathrm{Nc}`\beta . \supset . \mathrm{sm}``\mathrm{Nc}`\beta = \mathrm{Nc}`\alpha$

$[*100{\cdot}31{\cdot}321] \quad = \mathrm{Nc}`\beta \qquad (1)$

$\vdash . (1) . *10{\cdot}11{\cdot}23 . \supset \vdash . \mathrm{Prop}$

∗100·52. $\vdash : \mu \epsilon \mathrm{NC} . \exists ! \mu . \supset . \mathrm{sm}``\mu \epsilon \mathrm{NC}$ [∗100·51·4]

This proposition still holds when $\mu = \Lambda$, but the proof is more difficult, since it depends upon the proof that every null-class of classes is an NC, which in turn depends upon the proof that $\mathrm{Cl}`\alpha$ is not similar to α or to any class contained in α.

∗100·521. $\vdash : \mu \epsilon \mathrm{NC} . \exists ! \mathrm{sm}``\mu . \supset . \mathrm{sm}``\mathrm{sm}``\mu = \mu$

Dem.

$\vdash . ∗37·29 . \mathrm{Transp} . \supset \vdash :. \mathrm{Hp} . \supset : \exists ! \mu :$

[∗100·52] $\supset : \mathrm{sm}``\mu \epsilon \mathrm{NC} :$

[∗100·51.Hp] $\supset : \gamma \epsilon \mathrm{sm}``\mu . \supset . \mathrm{sm}``\mathrm{sm}``\mu = \mathrm{Nc}`\gamma$ (1)

$\vdash . ∗37·1 . \mathrm{Fact} . \supset \vdash : \mathrm{Hp} . \gamma \epsilon \mathrm{sm}``\mu . \supset . (\exists\alpha) . \alpha \epsilon \mu . \mu \epsilon \mathrm{NC} . \gamma \,\mathrm{sm}\, \alpha .$

[∗100·45·321] $\supset . (\exists\alpha) . \mathrm{Nc}`\alpha = \mu . \mathrm{Nc}`\gamma = \mathrm{Nc}`\alpha .$

[∗13·17] $\supset . \mathrm{Nc}`\gamma = \mu$ (2)

$\vdash . (1) . (2) . \supset \vdash :. \mathrm{Hp} . \gamma \epsilon \mathrm{sm}``\mu . \supset . \mathrm{sm}``\mathrm{sm}``\mu = \mu$ (3)

$\vdash . (3) . ∗10·11·23·35 . \supset \vdash . \mathrm{Prop}$

∗100·53. $\vdash :. \exists ! \mu . \exists ! \nu . \supset : \mu \epsilon \mathrm{NC} . \nu = \mathrm{sm}``\mu . \equiv . \nu \epsilon \mathrm{NC} . \mu = \mathrm{sm}``\nu$

Dem.

$\vdash . ∗100·52 . \supset \vdash :. \mathrm{Hp} . \supset : \mu \epsilon \mathrm{NC} . \nu = \mathrm{sm}``\mu . \supset . \nu \epsilon \mathrm{NC}$ (1)

$\vdash . ∗100·521 . \supset \vdash :. \mathrm{Hp} . \supset : \mu \epsilon \mathrm{NC} . \nu = \mathrm{sm}``\mu . \supset . \mu = \mathrm{sm}``\nu$ (2)

$\vdash . (1) . (2) . \supset \vdash :. \mathrm{Hp} . \supset : \mu \epsilon \mathrm{NC} . \nu = \mathrm{sm}``\mu . \supset . \nu \epsilon \mathrm{NC} . \mu = \mathrm{sm}``\nu$ (3)

$\vdash . (3) . (3) \frac{\nu, \mu}{\mu, \nu} . \supset \vdash . \mathrm{Prop}$

∗100·6. $\vdash . \iota``\alpha \epsilon \mathrm{Nc}`\alpha$ [∗73·41 . ∗100·31]

∗100·61. $\vdash . \hat{\beta} \{(\exists y) . y \epsilon \alpha . \beta = \iota`x \cup \iota`y\} \epsilon \mathrm{Nc}`\alpha$ [∗73·27 . ∗54·21 . ∗100·31]

∗100·62. $\vdash . x \downarrow ``\alpha \epsilon \mathrm{Nc}`\alpha$ [∗73·61 . ∗100·31]

∗100·621. $\vdash . \downarrow x``\alpha \epsilon \mathrm{Nc}`\alpha$ [∗73·611 . ∗100·31]

∗100·63. $\vdash . \epsilon_\Delta`\iota`\alpha \epsilon \mathrm{Nc}`\alpha$ [∗83·41 . ∗100·31]

∗100·631. $\vdash . \mathrm{D}``\epsilon_\Delta`\iota`\alpha \epsilon \mathrm{Nc}`\alpha$ [∗83·7 . ∗100·6]

∗100·64. $\vdash : \kappa \epsilon \mathrm{Cls}^2 \mathrm{excl} . \supset . \mathrm{D}``\epsilon_\Delta`\kappa \subset \mathrm{Nc}`\kappa$

Dem.

$\vdash . ∗84·3 . ∗80·14 . \supset \vdash : \mathrm{Hp} . R \epsilon \epsilon_\Delta`\kappa . \supset . R \epsilon 1 \rightarrow 1 . \kappa = \mathrm{Cl}`R .$

[∗73·2.∗100·31] $\supset . \mathrm{D}`R \epsilon \mathrm{Nc}`\kappa : \supset \vdash . \mathrm{Prop}$

*101. ON 0 AND 1 AND 2

Summary of *101.

In the present number, we have to show that 0 and 1 and 2 as previously defined are cardinal numbers in the sense defined in *100, and to add a few elementary propositions to those already given concerning them. We prove (*101·12·241) that 0 and 1 are not null, which cannot be proved, with our axioms, for any other cardinal, except (in the case of finite cardinals) when the type is specified as a sufficiently high one. Thus we prove (*101·42·43) that 2_{Cls} and 2_{Rel} exist; this follows from $\Lambda \neq V$ and $\dot{\Lambda} \neq \dot{V}$. We prove (*101·22·34) that 0 and 1 and 2 are all different from each other. We prove (*101·15·28) that $\mathrm{sm}``0 = 0$ and $\mathrm{sm}``1 = 1$, but we cannot prove $\mathrm{sm}``2 = 2$ unless we assume the existence of at least two individuals, or define the first 2 in "$\mathrm{sm}``2 = 2$" as a 2 of some type other than 2_{Indiv}, where "Indiv" stands for the type of individuals.

It should be observed that, since 0 and 1 and 2 are typically ambiguous, their properties are analogous to those of "$\mathrm{Nc}`\alpha$" rather than to those of μ, where $\mu \in \mathrm{NC}$. For example, we have

***100·511.** $\vdash : \exists ! \, \mathrm{Nc}`\beta \, . \supset . \, \mathrm{sm}``\mathrm{Nc}`\beta = \mathrm{Nc}`\beta$

but we shall not have $\mu \in \mathrm{NC} \, . \exists ! \, \mu \, . \supset . \, \mathrm{sm}``\mu = \mu$ unless the "sm" concerned is homogeneous, since in other cases the symbols do not express a significant proposition. But in *100·511 we may substitute 0 or 1 or 2, and the proposition remains significant and true. In fact we have (*101·1·2·31)

$$\vdash . \, 0 = \mathrm{Nc}`\Lambda \, . \, 1 = \mathrm{Nc}`\iota`x \, . \, 2 = \mathrm{Nc}`(\iota`\iota`x \cup \iota`\Lambda),$$

where 0 and 1 and 2 have an ambiguity corresponding to that of "Nc."

***101·1.** $\vdash . \, 0 = \mathrm{Nc}`\Lambda$ [*73·48 . *100·1]

***101·11.** $\vdash . \, 0 \in \mathrm{NC}$ [*101·1 . *100·4]

***101·12.** $\vdash . \exists ! \, 0$ [*51·161 . (*54·01)]

***101·13.** $\vdash . \exists ! \, 0 \cap \mathrm{Cl}`\alpha \, . \, \Lambda \in 0 \cap \mathrm{Cl}`\alpha$ [*51·16 . *60·3]

***101·14.** $\vdash : \mathrm{Nc}`\gamma = 0 \, . \equiv . \, \gamma = \Lambda$

Dem.

$\vdash .$ *101·1·12 $. \supset \vdash : \mathrm{Nc}`\gamma = 0 \, . \equiv . \, \mathrm{Nc}`\gamma = \mathrm{Nc}`\Lambda \, . \exists ! \, \mathrm{Nc}`\Lambda \, .$

[*13·194] $\equiv . \, \mathrm{Nc}`\gamma = \mathrm{Nc}`\Lambda \, . \exists ! \, \mathrm{Nc}`\Lambda \, . \exists ! \, \mathrm{Nc}`\gamma \, .$

[*100·35] $\equiv . \, \gamma \in \mathrm{Nc}`\Lambda \, . \exists ! \, \mathrm{Nc}`\Lambda \, . \exists ! \, \mathrm{Nc}`\gamma \, .$

[*101·1.*54·102] $\equiv . \, \gamma = \Lambda \, . \exists ! \, \mathrm{Nc}`\Lambda \, . \exists ! \, \mathrm{Nc}`\gamma \, .$

[*101·1·12.*13·194] $\equiv . \, \gamma = \Lambda : \supset \vdash . \, \mathrm{Prop}$

∗101·15. $\vdash . \mathrm{sm}``0 = 0$

Dem.

$\vdash . *37·1 . \supset \vdash : \gamma \in \mathrm{sm}``0 . \equiv . (\exists \alpha) . \alpha \in 0 . \gamma \,\mathrm{sm}\, \alpha .$
$[*54·102] \qquad \equiv . \gamma \,\mathrm{sm}\, \Lambda .$
$[*73·48] \qquad \equiv . \gamma \in 0 : \supset \vdash . \mathrm{Prop}$

∗101·16. $\vdash :. \mu \in \mathrm{NC} - \iota`0 . \supset : \alpha \in \mu . \supset_\alpha . \exists ! \alpha$

Dem.

$\vdash . *100·45 . \quad \supset \vdash : \mu \in \mathrm{NC} . \Lambda \in \mu . \supset . \mu = \mathrm{Nc}`\Lambda$
$[*101·1] \qquad = 0 \quad (1)$
$\vdash . (1) . \mathrm{Transp} . \supset \vdash :. \mu \in \mathrm{NC} - \iota`0 . \supset : \Lambda \sim\in \mu :$
$[*24·63] \qquad \supset : \alpha \in \mu . \supset_\alpha . \exists ! \alpha :. \supset \vdash . \mathrm{Prop}$

∗101·17. $\vdash : \Lambda \in \mathrm{Nc}`\alpha . \equiv . \mathrm{Nc}`\alpha = 0 . \equiv . \mathrm{Nc}`\alpha = \mathrm{Nc}`\Lambda . \equiv . \alpha = \Lambda$

Dem.

$\vdash . *100·31·321 . \supset \vdash : \Lambda \in \mathrm{Nc}`\alpha . \supset . \mathrm{Nc}`\alpha = \mathrm{Nc}`\Lambda .$
$[*101·1] \qquad \supset . \mathrm{Nc}`\alpha = 0 \quad (1)$
$\vdash . *101·13 . \quad \supset \vdash : \mathrm{Nc}`\alpha = 0 . \supset . \Lambda \in \mathrm{Nc}`\alpha \quad (2)$
$\vdash . (1) . (2) . \quad \supset \vdash : \Lambda \in \mathrm{Nc}`\alpha . \equiv . \mathrm{Nc}`\alpha = 0 . \quad (3)$
$[*101·1] \qquad \equiv . \mathrm{Nc}`\alpha = \mathrm{Nc}`\Lambda . \quad (4)$
$[*101·14] \qquad \equiv . \alpha = \Lambda \quad (5)$
$\vdash . (3) . (4) . (5) . \supset \vdash . \mathrm{Prop}$

∗101·2. $\vdash . 1 = \mathrm{Nc}`\iota`x \quad [*73·45 . *100·1]$

∗101·21. $\vdash . 1 \in \mathrm{NC} \quad [*101·2 . *100·4]$

∗101·22. $\vdash . 1 \neq 0$

Dem.

$\vdash . *52·21 . *101·13 . \supset \vdash . \Lambda \sim\in 1 . \Lambda \in 0 .$
$[*13·14] \qquad \supset \vdash . 1 \neq 0$

∗101·23. $\vdash . 1 \cap 0 = \Lambda$

Dem.

$\vdash . *52·21 . \quad \supset \vdash : \alpha \in 1 . \supset . \alpha \neq \Lambda .$
$[*54·102] \qquad \supset . \alpha \sim\in 0 \quad (1)$
$\vdash . (1) . *24·39 . \supset \vdash . \mathrm{Prop}$

∗101·24. $\vdash : \exists ! \alpha . \supset . \exists ! 1 \cap \mathrm{Cl}`\alpha$

Dem.

$\vdash . *52·22 . *60·6 . \supset \vdash : x \in \alpha . \supset . \iota`x \in 1 \cap \mathrm{Cl}`\alpha \quad (1)$
$\vdash . (1) . *10·11·28 . \supset \vdash . \mathrm{Prop}$

***101·241.** $\vdash . \exists ! 1$ [*52·23]

***101·25.** $\vdash : \alpha \in 1 . \beta \subset \alpha . \beta \neq \alpha . \supset . \beta \in 0$

Dem.

$\vdash . *52 \cdot 64 . *22 \cdot 621 . \supset \vdash : \alpha \in 1 . \beta \subset \alpha . \supset . \beta \in 1 \cup 0$ (1)

$\vdash . *52 \cdot 46 . \quad \supset \vdash : \alpha, \beta \in 1 . \beta \subset \alpha . \supset . \beta = \alpha :$

[Transp] $\quad \supset \vdash : \alpha \in 1 . \beta \subset \alpha . \beta \neq \alpha . \supset . \beta \sim \in 1$ (2)

$\vdash . (1) . (2) . \supset \vdash .$ Prop

***101·26.** $\vdash . s\text{‘}\mathrm{Cl}\text{“}1 = 0 \cup 1$

Dem.

$\vdash . *60 \cdot 371 . *40 \cdot 43 . \supset \vdash . s\text{‘}\mathrm{Cl}\text{“}1 \subset 0 \cup 1$ (1)

$\vdash . *60 \cdot 3 \cdot 34 . \quad \supset \vdash . \Lambda \in \mathrm{Cl}\text{‘}\iota\text{‘}x . \iota\text{‘}x \in \mathrm{Cl}\text{‘}\iota\text{‘}x .$

[*52·22.*40·4] $\quad \supset \vdash . \Lambda \in s\text{‘}\mathrm{Cl}\text{“}1 . \iota\text{‘}x \in s\text{‘}\mathrm{Cl}\text{“}1 .$

[*51·2.*52·1] $\quad \supset \vdash . 0 \subset s\text{‘}\mathrm{Cl}\text{“}1 . 1 \subset s\text{‘}\mathrm{Cl}\text{“}1$ (2)

$\vdash . (1) . (2) . \supset \vdash .$ Prop

***101·27.** $\vdash . 1 = \hat{\alpha}\{(\exists x) . x \in \alpha . \alpha - \iota\text{‘}x \in 0\}$

Dem.

$\vdash . *54 \cdot 102 . \supset$

$\vdash : (\exists x) . x \in \alpha . \alpha - \iota\text{‘}x \in 0 . \equiv . (\exists x) . x \in \alpha . \alpha - \iota\text{‘}x = \Lambda .$

[*24·3] $\quad \equiv . (\exists x) . x \in \alpha . \alpha \subset \iota\text{‘}x .$

[*51·2] $\quad \equiv . (\exists x) . \alpha = \iota\text{‘}x .$

[*52·1] $\quad \equiv . \alpha \in 1 : \supset \vdash .$ Prop

***101·28.** $\vdash . \mathrm{sm}\text{“}1 = 1$

Dem.

$\vdash . *37 \cdot 1 . \supset \vdash : \gamma \in \mathrm{sm}\text{“}1 . \equiv . (\exists \alpha) . \alpha \in 1 . \gamma \,\mathrm{sm}\, \alpha .$

[*52·1] $\quad \equiv . (\exists x) . \gamma \,\mathrm{sm}\, \iota\text{‘}x .$

[*73·45] $\quad \equiv . \gamma \in 1 : \supset \vdash .$ Prop

***101·29.** $\vdash : \iota\text{‘}x \in \mathrm{Nc}\text{‘}\alpha . \equiv . \mathrm{Nc}\text{‘}\alpha = 1 . \equiv . \mathrm{Nc}\text{‘}\alpha = \mathrm{Nc}\text{‘}\iota\text{‘}x . \equiv . \alpha \in 1$

Dem.

$\vdash . *100 \cdot 31 \cdot 321 . \quad \supset \vdash : \iota\text{‘}x \in \mathrm{Nc}\text{‘}\alpha . \supset . \mathrm{Nc}\text{‘}\alpha = \mathrm{Nc}\text{‘}\iota\text{‘}x .$

[*101·2] $\quad \supset . \mathrm{Nc}\text{‘}\alpha = 1$ (1)

$\vdash . *52 \cdot 22 . \quad \supset \vdash : \mathrm{Nc}\text{‘}\alpha = 1 . \supset . \iota\text{‘}x \in \mathrm{Nc}\text{‘}\alpha$ (2)

$\vdash . (1) . (2) . \quad \supset \vdash : \iota\text{‘}x \in \mathrm{Nc}\text{‘}\alpha . \equiv . \mathrm{Nc}\text{‘}\alpha = 1 .$ (3)

[*101·2] $\quad \equiv . \mathrm{Nc}\text{‘}\alpha = \mathrm{Nc}\text{‘}\iota\text{‘}x$ (4)

$\vdash . *101 \cdot 2 . *52 \cdot 1 . \supset \vdash : \alpha \in 1 . \supset . \mathrm{Nc}\text{‘}\alpha = 1$ (5)

$\vdash . *100 \cdot 3 . \quad \supset \vdash : \mathrm{Nc}\text{‘}\alpha = 1 . \supset . \alpha \in 1$ (6)

$\vdash . (3) . (4) . (5) . (6) . \supset \vdash .$ Prop

***101·3.** $\vdash : x \neq y \,.\supset .\, 2 = \mathrm{Nc}`(\iota`x \cup \iota`y)$

Dem.

$\vdash .*73\cdot71\cdot43 \,.\, *51\cdot231 \,.\supset \vdash :.\, \mathrm{Hp} \,.\supset : z \neq w \,.\supset .\, (\iota`z \cup \iota`w) \,\mathrm{sm}\, (\iota`x \cup \iota`y) :$

$[*54\cdot101] \quad \supset : \beta \in 2 \,.\supset .\, \beta \,\mathrm{sm}\, (\iota`x \cup \iota`y) :$

$[*100\cdot1] \quad \supset : 2 \subset \mathrm{Nc}`(\iota`x \cup \iota`y)$ (1)

$\vdash .*53\cdot32 \,.\, *71\cdot163 \,.\supset \vdash : R \in 1 \rightarrow 1 \,.\, x, y \in \mathrm{C}`R \,.\supset .$

$R``(\iota`x \cup \iota`y) = \iota`R`x \cup \iota`R`y$ (2)

$\vdash .*71\cdot56 \,.\, \mathrm{Transp} \,.\supset \vdash : \mathrm{Hp} \,.\, R \in 1 \rightarrow 1 \,.\, x, y \in \mathrm{C}`R \,.\supset .\, R`x \neq R`y$ (3)

$\vdash .(2).(3).*54\cdot26 \,.\supset$

$\vdash :.\, \mathrm{Hp} \,.\supset : R \in 1 \rightarrow 1 \,.\, x, y \in \mathrm{C}`R \,.\, \beta = R``(\iota`x \cup \iota`y) \,.\supset .\, \beta \in 2 :$

$[*10\cdot11\cdot21\cdot23 . *51\cdot234] \supset : (\exists R) \,.\, R \in 1 \rightarrow 1 \,.\, \iota`x \cup \iota`y \subset \mathrm{C}`R \,.\, \beta = R``(\iota`x \cup \iota`y) \,.$

$\supset .\, \beta \in 2 :$

$[*73\cdot12 . *100\cdot1] \quad \supset : \mathrm{Nc}`(\iota`x \cup \iota`y) \subset 2$ (4)

$\vdash .(1).(4) \,.\supset \vdash .\, \mathrm{Prop}$

***101·301.** $\vdash .\, 2 = \hat{\alpha}\{(\exists x) \,.\, x \in \alpha \,.\, \alpha - \iota`x \in 1\}$ $[*54\cdot3]$

In comparing *101·31 with *101·1·2·3, it should be observed that $\iota`x$ and Λ are both *classes*, whereas in *101·1·2·3 there was no typical limitation beyond what was imposed by the conditions of significance.

***101·31.** $\vdash .\, 2 = \mathrm{Nc}`(\iota`\iota`x \cup \iota`\Lambda)$

Dem.

$\vdash .*51\cdot161 \,.\quad \supset \vdash .\, \iota`x \neq \Lambda$ (1)

$\vdash .(1).*101\cdot3 \,.\supset \vdash .\, \mathrm{Prop}$

***101·32.** $\vdash .\, 2 \in \mathrm{NC}$ $[*101\cdot31 \,.\, *100\cdot4]$

***101·33.** $\vdash : \alpha, \beta \in 1 \,.\, \alpha \cap \beta = \Lambda \,.\supset .\, \alpha \cup \beta \in 2$ $[*54\cdot43]$

***101·34.** $\vdash .\, 2 \neq 0 \,.\, 2 \neq 1$

Dem.

$\vdash .*101\cdot13 \,. \quad \supset \vdash .\, \Lambda \in 0$ (1)

$\vdash .*101\cdot301 \,. \quad \supset \vdash : \alpha \in 2 \,.\supset .\, \exists ! \alpha :$

$[*24\cdot63] \quad \supset \vdash .\, \Lambda \sim\in 2$ (2)

$\vdash .(1).(2).*13\cdot14 \,. \quad \supset \vdash .\, 2 \neq 0$ (3)

$\vdash .*52\cdot22 \,.\, *54\cdot26 \,.\, *22\cdot56 \,.\supset \vdash .\, \iota`y \in 1 \,.\, \iota`y \sim\in 2 \,.$

$[*13\cdot14] \quad \supset \vdash .\, 1 \neq 2$ (4)

$\vdash .(3).(4) \,.\supset \vdash .\, \mathrm{Prop}$

***101·35.** $\vdash .\, 2 \cap 0 = \Lambda \,.\, 2 \cap 1 = \Lambda$ $[*100\cdot42 \,.\, \mathrm{Transp} \,.\, *101\cdot11\cdot21\cdot32\cdot34]$

***101·36.** $\vdash : \alpha \in 2 \,.\, \beta \subset \alpha \,.\, \beta \neq \alpha \,.\supset .\, \beta \in 0 \cup 1$

Dem.

$\vdash .*54\cdot42 \,. \quad \supset \vdash : \alpha \in 2 \,.\, \beta \subset \alpha \,.\, \exists ! \beta \,.\, \beta \neq \alpha \,.\supset .\, \beta \in 1$ (1)

$\vdash .*54\cdot102 \,.\supset \vdash : \sim\exists ! \beta \,.\supset .\, \beta \in 0$ (2)

$\vdash .(1).(2) \,.\supset \vdash .\, \mathrm{Prop}$

***101·37.** $\vdash . s'\mathrm{Cl}''2 \subset 0 \cup 1 \cup 2$ [*54·411]

***101·38.** $\vdash : \exists ! 2 . \supset . s'\mathrm{Cl}''2 = 0 \cup 1 \cup 2$

Dem.

$\vdash . *60{\cdot}3 . \quad \supset \vdash : \mathrm{Hp} . \supset . (\exists \alpha) . \alpha \epsilon 2 . \Lambda \epsilon \mathrm{Cl}'\alpha .$

$[*40{\cdot}4] \quad \supset . \Lambda \epsilon s'\mathrm{Cl}''2 .$

$[*51{\cdot}2] \quad \supset . 0 \subset s'\mathrm{Cl}''2$ (1)

$\vdash . *60{\cdot}34 . \supset \vdash . 2 \subset s'\mathrm{Cl}''2$ (2)

$\vdash . *54{\cdot}101 . \supset \vdash :: \mathrm{Hp} . \supset :. (\exists x, y) . x \neq y :.$

$[*13{\cdot}171 . \mathrm{Transp}] \quad \supset :. (\exists x, y) :. (z) : z \neq x . \mathrm{v} . z \neq y :.$

$[*54{\cdot}26] \quad \supset :. (\exists x, y) :. (z) : \iota'z \cup \iota'x \epsilon 2 . \mathrm{v} . \iota'z \cup \iota'y \epsilon 2 :.$

$[*11{\cdot}26 . *22{\cdot}58] \quad \supset :. (z) :. (\exists \alpha, \beta) : \alpha \epsilon 2 . \iota'z \epsilon \mathrm{Cl}'\alpha . \mathrm{v} . \beta \epsilon 2 . \iota'z \epsilon \mathrm{Cl}'\beta :.$

$[*40{\cdot}4] \quad \supset :. (z) . \iota'z \epsilon s'\mathrm{Cl}''2 :.$

$[*52{\cdot}1] \quad \supset :. 1 \subset s'\mathrm{Cl}''2$ (3)

$\vdash . (1) . (2) . (3) . *101{\cdot}37 . \supset \vdash . \mathrm{Prop}$

***101·4.** $\vdash : (\exists x, y) . x \neq y . \equiv . \exists ! 2$

Dem.

$\vdash . *54{\cdot}26 . \supset \vdash : x \neq y . \supset . \exists ! 2 :$

$[*11{\cdot}11{\cdot}35] \supset \vdash : (\exists x, y) . x \neq y . \supset . \exists ! 2$ (1)

$\vdash . *54{\cdot}101 . \supset \vdash : \alpha \epsilon 2 . \supset . (\exists x, y) . x \neq y :$

$[*10{\cdot}11{\cdot}23] \supset \vdash : \exists ! 2 . \supset . (\exists x, y) . x \neq y$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

When we are considering the lowest type occurring in a context, our premisses do not suffice to prove $(\exists x, y) . x \neq y$. For every other type, this can be proved. Thus $\Lambda \neq \mathrm{V}$ and $\dot{\Lambda} \neq \dot{\mathrm{V}}$ give the required result for classes and relations respectively.

***101·41.** $\vdash : (\exists x) . \iota'x \neq \mathrm{V} . \equiv . \exists ! 2$

Dem.

$\vdash . *24{\cdot}14 . \mathrm{Transp} . \supset$

$\vdash :. (\exists x) . \iota'x \neq \mathrm{V} . \equiv : (\exists x) : (\exists y) . y \sim\epsilon \iota'x :$

$[*51{\cdot}15] \quad \equiv : (\exists x, y) . x \neq y :$

$[*101{\cdot}4] \quad \equiv : \exists ! 2 :. \supset \vdash . \mathrm{Prop}$

***101·42.** $\vdash . \exists ! 2_{\mathrm{Cls}} . \iota'\Lambda \cup \iota'\mathrm{V} \epsilon 2_{\mathrm{Cls}}$

Dem.

$\vdash . *20{\cdot}41 . *24{\cdot}1 . \supset \vdash . \Lambda, \mathrm{V} \epsilon \mathrm{Cls} . \Lambda \neq \mathrm{V}$ (1)

$\vdash . (1) . *54{\cdot}26 . \quad \supset \vdash . \iota'\Lambda \cup \iota'\mathrm{V} \epsilon 2 . \iota'\Lambda \cup \iota'\mathrm{V} \subset \mathrm{Cls} .$

$[*63{\cdot}371{\cdot}105] \quad \supset \vdash . \iota'\Lambda \cup \iota'\mathrm{V} \epsilon 2 \cap t'\mathrm{Cls} .$

$[(*65{\cdot}01)] \quad \supset \vdash . \iota'\Lambda \cup \iota'\mathrm{V} \epsilon 2_{\mathrm{Cls}} . \supset \vdash . \mathrm{Prop}$

***101·43.** $\vdash . \exists ! 2_{\mathrm{Rel}}$ [Proof as in *101·42]

$*102$. ON CARDINAL NUMBERS OF ASSIGNED TYPES

Summary of $*102$.

In this number, we shall consider a typically definite relation "Nc," *i.e.* we shall consider the relation, to a class δ which is given as of the same type as β, of the class μ of those classes γ which are similar to δ and of the same type as α. We shall then put

$$\mu = \mathrm{Nc}\,(\alpha_\beta)\text{‘}\delta,$$
$$\gamma \,\epsilon\, \mathrm{Nc}\,(\alpha_\beta)\text{‘}\delta,$$
$$\gamma \,\mathrm{sm}_{(\alpha,\,\beta)}\,\delta,$$

and the class of all such numbers as μ for a given α and β we shall call $\mathrm{NC}^\beta\,(\alpha)$, so that

$$\mathrm{NC}^\beta\,(\alpha) = \mathrm{D}\text{‘}\mathrm{Nc}\,(\alpha_\beta).$$

The notations here introduced for giving typical definiteness to "sm" and "Nc" are those defined in $*65$ for any typically ambiguous relation.

By $*63{\cdot}01{\cdot}02$ we have, if α is a typically ambiguous symbol,

$$\vdash .\, \alpha_x = \alpha \cap t\text{‘}x,$$
$$\vdash .\, \alpha\,(x) = \alpha \cap t\text{‘}\iota\text{‘}x.$$

Thus $\vdash .\, \alpha\,(x) = \alpha_{\iota\text{‘}x}$. If we apply the definitions to 1, "1_x" is meaningless unless x is a class; we therefore write a Greek letter in place of x, and we have

$$\vdash .\, 1_\beta = 1 \cap t\text{‘}\beta = 1 \cap (\iota\text{‘}\beta \cup -\iota\text{‘}\beta).$$

If $x \,\epsilon\, \beta$, we shall have $\iota\text{‘}x = \beta \,.\vee.\, \iota\text{‘}x \neq \beta$. Hence

$$\vdash : x \,\epsilon\, \beta \,.\supset.\, \iota\text{‘}x \,\epsilon\, 1_\beta.$$

Similarly $\qquad \vdash : x \sim\epsilon\, \beta \,.\supset.\, \iota\text{‘}x \,\epsilon\, 1_\beta.$

Thus $\qquad \vdash : x \,\epsilon\, t_0\text{‘}\beta \,.\supset.\, \iota\text{‘}x \,\epsilon\, 1_\beta.$

The converse implication also holds, so that

$$\vdash : x \,\epsilon\, t_0\text{‘}\beta \,.\equiv.\, \iota\text{‘}x \,\epsilon\, 1_\beta.$$

Thus 1_β consists of all unit classes whose sole members x either are or are not members of β, *i.e.* for which "$x \,\epsilon\, \beta$" is significant.

In "$x \,\epsilon\, t_0\text{‘}\beta \,.\supset.\, \iota\text{‘}x \,\epsilon\, 1_\beta$," the hypothesis renders explicit the condition of significance; thus "$\iota\text{‘}x \,\epsilon\, 1_\beta$" is always true when significant, and always significant when $x \,\epsilon\, t_0\text{‘}\beta$. On the interpretation of negative statements concerning types, see the note at the end of this number.

It should be noted that all the constant relations introduced in this work are typically ambiguous. Consider *e.g.* $\dot\Lambda$, sg, D, s, $\dot s$, I, ι, ϵ, Cl, Rl. These all have more or less typical ambiguity, though all of them have what we will call *relative* typical definiteness, *i.e.* when the type of the relatum is given, that of the referent is given also. (In regard to D, it is not true that, conversely

when the type of the referent is given, that of the relatum is also given.) But "sm" and "Nc" have not even relative definiteness. When the type of the relatum is given, that of the referent becomes no more definite than before; the only restrictions are that the relatum for "sm" or "Nc" must be a class, that the referent for "sm" must be a class, and that the referent for "Nc" must be a class of classes. When a relation R has relative definiteness, it is enough to fix the type of the relatum; and if further $R \epsilon 1 \rightarrow \mathrm{Cls}$, so that R leads to a descriptive function, "$R`y$" has complete typical definiteness as soon as the type of y is given. Now the constant relations hitherto introduced, with the exception of "sm" and "$\dot{\mathrm{V}}$," have all been one-many relations, and have been used almost exclusively in the form of descriptive functions. Hence no special notation has been required to give typical definiteness, since "$R`y$," in these circumstances, has typical definiteness as soon as $y$ is assigned. But with the consideration of "sm" and "Nc," which do not have even relative definiteness, an explicit means of giving typical definiteness becomes necessary. It should be observed, however, that "$\mathrm{Nc}`\delta$" has typical definiteness, when δ is known, as soon as the *domain* of "Nc" has typical definiteness, since δ must belong to the converse domain. It is for the sake of this and similar cases that we introduced the two definitions in $*65$, which only give typical definiteness to the *domain*.

In virtue of the definitions in $*65$, if R is a typically ambiguous relation, and x is a referent, R becomes R_x; if, further, y is a relatum, R becomes $R_{(x,y)}$. If x is a referent for R, we have $(\exists y) \,.\, x \epsilon \overrightarrow{R}`y$, and $\overrightarrow{R}`y \epsilon \mathrm{D}`\overrightarrow{R}$. Thus $\mathrm{D}`\overrightarrow{R}$ has a member of the type next above that of x, *i.e.* of the type of $\iota`x$. Thus

$$\vdash \,.\, \mathrm{sg}`(R_x) = (\overrightarrow{R})(x)$$

and

$$\vdash \,.\, \mathrm{sg}`\{R_{(x,y)}\} = (\overrightarrow{R})(x_y)$$

as was proved in $*65$. Hence in particular

$$\vdash \,.\, \mathrm{sg}`\{\mathrm{sm}_{(\alpha,\beta)}\} = \mathrm{Nc}\,(\alpha_\beta).$$

It is chiefly for this reason that it is worth while to introduce the definition of $R\,(x_y)$.

We have, in virtue of the above, as will be proved in $*102{\cdot}46$,

$$\vdash : \gamma \epsilon t`\alpha \,.\, \delta \epsilon t`\beta \,.\, \gamma \,\mathrm{sm}\, \delta \,.\equiv .\, \gamma \epsilon \mathrm{Nc}\,(\alpha_\beta)`\delta.$$

With regard to "$\mathrm{Nc}\,(\alpha)$," which is to be interpreted by $*65{\cdot}04$, some caution is necessary. This will mean *some one* of those typically different relations called "Nc" which have their domains composed of terms of the same type as α. But it will not mean the logical sum of all such relations, because these relations are of different types according as their converse domains differ in type, and therefore their logical sum is meaningless. Thus for example if the type of β is lower than or equal to that of α, we shall have

$$\vdash \,.\, \exists !\, \mathrm{Nc}\,(\alpha)`\beta,$$

whence, if "Nc (α)" has its converse domain composed of terms of the same type as β,

$$\vdash . \Lambda \sim\epsilon \text{D}'\text{Nc}(\alpha).$$

But if β is of higher type than α, we shall find

$$\vdash . \Lambda \epsilon \text{D}'\text{Nc}(\alpha).$$

Thus "Nc (α)" is indeterminate in a way that makes a practical difference.

Exactly similar remarks apply to NC (α). We have

$$\vdash . \text{NC}(\alpha) = \text{D}'\text{Nc}(\alpha);$$

thus "NC (α)" shares the ambiguity of "Nc (α)." The question whether $\Lambda \epsilon \text{NC}(\alpha)$ depends upon the decision of this ambiguity. The difficulty is that "NC (α)" stands for the domain of any one determination of "Nc" which has its domain composed of objects of the type of $\iota'\alpha$; but it is the domain of *only one* such determination of "Nc," because different determinations are of different types, and therefore cannot be taken together, even when their domains are all of the same type. In consequence of this ambiguity, "NC (α)" is a symbol which is as a rule better avoided, and "Nc (α)" is not often useful except as a descriptive function, in which case the relatum supplies the requisite typical definiteness.

The peculiarity of "NC (α)" is that it is *typically* definite, and yet is capable of different meanings: it is not *wholly* definite, being defined as the domain of a relation whose converse domain is typically ambiguous. It results that we cannot profitably make "NC" half-definite, as "NC (α)" does, but must make it completely definite, as we do by taking $\text{D}'\text{Nc}(\alpha_\beta)$. For this we adopt the notation $\text{NC}^\beta(\alpha)$. We cannot adopt the notation $\text{NC}(\alpha_\beta)$, because that would conflict with $*65{\cdot}11$, nor $\text{NC}(\alpha)_\beta$, because that would conflict with $*65{\cdot}01$, nor $\text{NC}_\beta(\alpha)$, for the same reason. But $\text{NC}^\beta(\alpha)$ has no previously defined meaning. We may if we like regard "NC^β" as $\text{D}'(\text{Nc} \upharpoonright t'\beta)$. Then the required meaning of "$\text{NC}^\beta(\alpha)$" would result from $*65{\cdot}04$. But as "NC^β" so defined is not required, it is simpler to regard "$\text{NC}^\beta(\alpha)$" as a single symbol. We therefore put

$*102{\cdot}01$. $\text{NC}^\beta(\alpha) = \text{D}'\text{Nc}(\alpha_\beta)$ Df

The present number begins with various propositions ($*102{\cdot}2$—$\cdot27$) on a typically definite relation of similarity, *i.e.* $\text{sm}_{(\alpha,\beta)}$. We then have a set of propositions ($*102{\cdot}3$—$\cdot46$) on "$\text{Nc}(\alpha_\beta)'\delta$." This is only significant if β and δ are of the same type; it then denotes the class of those classes which are similar to δ and of the same type as α. We then have a set of propositions ($*102{\cdot}5$—$\cdot64$) on $\text{NC}^\beta(\alpha)$, *i.e.* on cardinals consisting of classes of the same type as α which are similar to classes of the same type as β. We next prove ($*102{\cdot}71$—$\cdot75$) that no sub-class of α is similar to $\text{Cl}'\alpha$, and therefore (substituting $t_0'\alpha$ for α) no class of the same type as α is similar to $t'\alpha$, and therefore

$*102{\cdot}74$. $\vdash . \Lambda \epsilon \text{NC}^{t'\alpha}(\alpha)$

This proves that Λ is a cardinal, which is a proposition constantly required. The remaining propositions of $*102$ are concerned with $\mathrm{sm}``\mu$ where μ is a typically definite cardinal.

The most useful propositions in this number (apart from $*102\cdot74$) are

$*102\cdot3$. $\vdash : \gamma \,\mathrm{sm}_{(\alpha,\beta)}\, \delta \,.\equiv.\, \gamma \in \mathrm{Nc}\,(\alpha_\beta)`\delta$

$*102\cdot46$. $\vdash : \gamma \in \mathrm{Nc}\,(\alpha_\beta)`\delta \,.\doteq.\, \delta \in \mathrm{Nc}\,(\beta_\alpha)`\gamma \,.\equiv.\, \gamma \,\mathrm{sm}\, \delta \,.\, \gamma \in t`\alpha \,.\, \delta \in t`\beta$

$*102\cdot5$. $\vdash : \mu \in \mathrm{NC}^\beta\,(\alpha) \,.\equiv.\, (\exists\delta) \,.\, \mu = \mathrm{Nc}\,(\alpha_\beta)`\delta$

$*102\cdot6$. $\vdash .\, \mathrm{Nc}\,(\alpha)`\beta = \mathrm{Nc}\,(\alpha_\beta)`\beta = \hat{\gamma}\,(\gamma \,\mathrm{sm}\, \beta \,.\, \gamma \in t`\alpha) = \mathrm{Nc}`\beta \cap t`\alpha$

$*102\cdot72$. $\vdash : \beta \subset \alpha \,.\supset.\, \sim(\beta \,\mathrm{sm}\, \mathrm{Cl}`\alpha)$

This is used in proving $\mu \in \mathrm{NC} \,.\supset.\, 2^\mu > \mu$, which is the proposition from which Cantor deduced that there is no greatest cardinal. (If $\mu = \mathrm{Nc}`\alpha$, $2^\mu = \mathrm{Nc}`\mathrm{Cl}`\alpha$, and thus there is a rise of type.)

$*102\cdot84$. $\vdash : (\exists\gamma) \,.\, \gamma \,\mathrm{sm}\, \alpha \,.\, \gamma \in t`\alpha \,.\, \delta \,\mathrm{sm}\, \gamma \,.\equiv.\, \delta \,\mathrm{sm}\, \alpha$

$*102\cdot85$. $\vdash .\, \mathrm{sm}``\mu \cap t`\beta = \mathrm{sm}_\beta``\mu$

$*102\cdot01$. $\mathrm{NC}^\beta\,(\alpha) = \mathrm{D}`\mathrm{Nc}\,(\alpha_\beta)$ Df

$*102\cdot11$. $\vdash : R \in 1 \to 1 \,.\supset.\, R_{(x,y)} \in 1\,(x) \to 1\,(y)$

Here, if R is a real variable, the conditions of significance require $R = R_{(x,y)}$. But if R is a typically ambiguous constant, such as I or $\dot{\Lambda}$ or sg, $R_{(x,y)}$ is a typically definite constant. It is chiefly for such cases that propositions such as the above are useful.

Dem.

$\vdash .\, *37\cdot402 \,.\, (*65\cdot1) \,.\supset \vdash .\, \mathrm{D}`R_{(x,y)} \subset t`x \,.$
$[*33\cdot15] \qquad \supset \vdash .\, \{\mathrm{sg}`R_{(x,y)}\}`z \subset t`x \,.$
$[*63\cdot5] \qquad \supset \vdash .\, \{\mathrm{sg}`R_{(x,y)}\}`z \in t`t`x \qquad (1)$
$\vdash .\, (1) \,.\, *71\cdot102 \,. \qquad \supset \vdash : \mathrm{Hp} \,.\, z \in \text{Ɑ}`R_{(x,y)} \,.\supset.\, \{\mathrm{sg}`R_{(x,y)}\}`z \in 1 \cap t`t`x \,.$
$[(*65\cdot02)] \qquad \supset.\, \{\mathrm{sg}`R_{(x,y)}\}`z \in 1\,(x) \qquad (2)$
Similarly $\qquad \vdash : \mathrm{Hp} \,.\, w \in \mathrm{D}`R_{(x,y)} \,.\supset.\, \{\mathrm{gs}`R_{(x,y)}\}`w \in 1\,(y) \qquad (3)$
$\vdash .\, (2) \,.\, (3) \,.\, *70\cdot1 \,.\supset \vdash .\, \mathrm{Prop}$

$*102\cdot13$. $\vdash : R \in 1 \to 1 \,.\supset.\, R_x \in 1\,(x) \to 1$ [Proof as in $*102\cdot11$]

$*102\cdot2$. $\vdash : \gamma \,\mathrm{sm}_{(\alpha,\beta)}\, \delta \,.\equiv.\, \gamma \,\mathrm{sm}\, \delta \,.\, \gamma \in t`\alpha \,.\, \delta \in t`\beta$ [$*35\cdot102 \,.\, (*65\cdot1)$]

$*102\cdot21$. $\vdash : \gamma \,\mathrm{sm}_{(\alpha,\beta)}\, \delta \,.\equiv.\, (\exists R) \,.\, R \in 1 \to 1 \,.\, \mathrm{D}`R \in t`\alpha \,.$
$\text{Ɑ}`R \in t`\beta \,.\, \mathrm{D}`R = \gamma \,.\, \text{Ɑ}`R = \delta$ [$*102\cdot2 \,.\, *73\cdot1$]

$*102\cdot22$. $\vdash : \gamma \,\mathrm{sm}\,(x,y)\, \delta \,.\equiv.\, \gamma \,\mathrm{sm}\, \delta \,.\, \gamma \subset t`x \,.\, \delta \subset t`y$ [$*63\cdot5 \,.\, (*65\cdot12)$]

$*102\cdot23$. $\vdash : \gamma \,\mathrm{sm}\,(x,y)\, \delta \,.\equiv.\, (\exists R) \,.\, R \in 1 \to 1 \,.\, \mathrm{D}`R \subset t`x \,.$
$\text{Ɑ}`R \subset t`y \,.\, \mathrm{D}`R = \gamma \,.\, \text{Ɑ}`R = \delta$ [$*102\cdot22 \,.\, *73\cdot1$]

***102·24.** $\vdash : \gamma \operatorname{sm}(x, y)\, \delta \,.\equiv .\, (\exists R) \,.\, R \in 1(x) \rightarrow 1(y) \,.\, \text{D}'R = \gamma \,.\, \text{Ɑ}'R = \delta$

Dem.

$\vdash . *102\cdot23 . *40\cdot5\cdot52\cdot43 . *37\cdot25 . \supset$

$\vdash :. \gamma \operatorname{sm}(x, y)\, \delta \,.\equiv : (\exists R) : R \in 1 \rightarrow 1 : w \in \text{Ɑ}'R \,.\supset_w .\, \overrightarrow{R}'w \subset t'x :$

$z \in \text{D}'R \,.\supset_z .\, \overleftarrow{R}'z \subset t'y : \text{D}'R = \gamma \,.\, \text{Ɑ}'R = \delta :$

$[*63\cdot5] \qquad \equiv : (\exists R) : R \in 1 \rightarrow 1 \,.\, \overrightarrow{R}``\text{Ɑ}'R \subset t't'x \,.\, \overleftarrow{R}``\text{D}'R \subset t't'y \,.$

$\text{D}'R = \gamma \,.\, \text{Ɑ}'R = \delta :$

$[*71\cdot102 . (*65\cdot02)] \equiv : (\exists R) . \overrightarrow{R}``\text{Ɑ}'R \subset 1(x) . \overleftarrow{R}``\text{D}'R \subset 1(y) . \text{D}'R = \gamma . \text{Ɑ}'R = \delta :$

$[*70\cdot1] \qquad \equiv : (\exists R) \,.\, R \in 1(x) \rightarrow 1(y) \,.\, \text{D}'R = \gamma \,.\, \text{Ɑ}'R = \delta :. \supset \vdash . \text{Prop}$

***102·25.** $\vdash : \gamma \operatorname{sm}_{(\alpha, \beta)} \delta \,.\equiv .\, (\exists R) \,.\, R \in 1_\alpha \rightarrow 1_\beta \,.\, \text{D}'R = \gamma \,.\, \text{Ɑ}'R = \delta$

[Proof as in *102·24]

***102·26.** $\vdash : \gamma \operatorname{sm}_{(\alpha, \beta)} \delta \,.\, \gamma' \operatorname{sm}_{(\alpha, \beta)} \delta \,.\supset .\, \gamma \operatorname{sm}_{(\alpha, \alpha)} \gamma'$

Dem.

$\vdash . *102\cdot2 . \supset \vdash : \text{Hp} \,.\supset .\, \gamma \operatorname{sm} \delta \,.\, \gamma' \operatorname{sm} \delta \,.\, \gamma, \gamma' \in t'\alpha \,.$

$[*73\cdot32] \qquad \supset .\, \gamma \operatorname{sm} \gamma' \,.\, \gamma, \gamma' \in t'\alpha \,.$

$[*102\cdot2] \qquad \supset .\, \gamma \operatorname{sm}_{(\alpha, \alpha)} \gamma' : \supset \vdash . \text{Prop}$

***102·27.** $\vdash : \gamma \operatorname{sm}_{(\alpha, \beta)} \delta \,.\, \gamma' \operatorname{sm}_{(\alpha', \beta)} \delta \,.\supset .\, \gamma \operatorname{sm}_{(\alpha, \alpha')} \gamma'$ [Proof as in *102·26]

***102·3.** $\vdash : \gamma \operatorname{sm}_{(\alpha, \beta)} \delta \,.\equiv .\, \gamma \in \text{Nc}\,(\alpha_\beta)'\delta$

Dem.

$\vdash . *32\cdot18 . \supset$

$\vdash : \gamma \operatorname{sm}_{(\alpha, \beta)} \delta \,.\equiv .\, \gamma \in \{\text{sg}'\text{sm}_{(\alpha, \beta)}\}'\delta \,.$

$[*65\cdot2] \qquad \equiv .\, \gamma \in \{(\text{sg}'\text{sm})\,(\alpha_\beta)\}'\delta \,.$

$[(*100\cdot01)] \qquad \equiv .\, \gamma \in \text{Nc}\,(\alpha_\beta)'\delta : \supset \vdash . \text{Prop}$

***102·31.** $\vdash . \text{Nc}\,(\alpha_\beta)'\delta = \text{D}``\{1 \rightarrow 1 \cap \hat{R}\,(\text{D}'R \in t'\alpha \,.\, \text{Ɑ}'R \in t'\beta \,.\, \text{Ɑ}'R = \delta)\}$

Dem.

$\vdash . *102\cdot3\cdot21 . \supset$

$\vdash : \gamma \in \text{Nc}\,(\alpha_\beta)'\delta \,.\equiv .\, (\exists R) \,.\, R \in 1 \rightarrow 1 \,.\, \text{D}'R \in t'\alpha \,.\, \text{Ɑ}'R \in t'\beta \,.\, \text{D}'R = \gamma \,.\, \text{Ɑ}'R = \delta \,.$

$[*33\cdot123 . *37\cdot1] \equiv .\, \gamma \in \text{D}``\{1 \rightarrow 1 \cap \hat{R}\,(\text{D}'R \in t'\alpha \,.\, \text{Ɑ}'R \in t'\beta \,.\, \text{Ɑ}'R = \delta)\} :$

$\supset \vdash . \text{Prop}$

***102·32.** $\vdash . \text{Nc}\,(\alpha_\beta)'\delta = \text{D}``\{(1_\alpha \rightarrow 1_\beta) \cap \overleftarrow{\text{Ɑ}}\,'\delta\}$

Dem.

$\vdash . *102\cdot3\cdot25 . \supset$

$\vdash : \gamma \in \text{Nc}\,(\alpha_\beta)'\delta \,.\equiv .\, (\exists R) \,.\, R \in 1_\alpha \rightarrow 1_\beta \,.\, \text{D}'R = \gamma \,.\, \text{Ɑ}'R = \delta \,.$

$[*33\cdot61] \qquad \equiv .\, (\exists R) \,.\, R \in 1_\alpha \rightarrow 1_\beta \,.\, R \in \overleftarrow{\text{Ɑ}}'\delta \,.\, \text{D}'R = \gamma \,.$

$[*33\cdot123 . *37\cdot1] \equiv .\, \gamma \in \text{D}``\{(1_\alpha \rightarrow 1_\beta) \cap \overleftarrow{\text{Ɑ}}'\delta\} : \supset \vdash . \text{Prop}$

***102·34.** $\vdash . \text{Nc}\,(\alpha, \beta)'\delta = \text{D}``\{1 \rightarrow 1 \cap \hat{R}\,(\text{D}'R \in t'\alpha \,.\, \text{Ɑ}'R \subset t'\beta \,.\, \text{Ɑ}'R = \delta)\}$

[Proof as in *102·31]

∗102·35. $\vdash . \mathrm{Nc}(\alpha, \beta)\text{‘}\delta = \mathrm{D}\text{‘‘}[\{1_\alpha \to 1(\beta)\} \cap \overleftarrow{\text{Cl}}\text{‘}\delta]$ [Proof as in ∗102·32]

∗102·36. $\vdash . \mathrm{E} ! \mathrm{Nc}(\alpha_\beta)\text{‘}\delta$ [∗102·31 . ∗14·21]

This proposition is true whenever it is significant, and is significant whenever $\delta \in t\text{‘}\beta$. When δ belongs to some other type, the above proposition is not significant.

∗102·361. $\vdash . \mathrm{E} ! \mathrm{Nc}(\alpha, \beta)\text{‘}\delta$ [∗102·34 . ∗14·21]

∗102·37. $\vdash . \text{Cl}\text{‘}\mathrm{Nc}(\alpha_\beta) = t\text{‘}\beta$

Dem.

$\vdash . ∗37·402 . (∗65·11) . \supset \vdash . \text{Cl}\text{‘}\mathrm{Nc}(\alpha_\beta) \subset t\text{‘}\beta$ (1)

$\vdash . ∗102·36 . ∗33·43 . \supset \vdash . (\delta) . \delta \in \text{Cl}\text{‘}\mathrm{Nc}(\alpha_\beta) .$

[∗63·14] $\supset \vdash . t_0\text{‘}\text{Cl}\text{‘}\mathrm{Nc}(\alpha_\beta) = \text{Cl}\text{‘}\mathrm{Nc}(\alpha_\beta)$ (2)

$\vdash . (1) . ∗63·21 . \supset \vdash . t_0\text{‘}\text{Cl}\text{‘}\mathrm{Nc}(\alpha_\beta) = t\text{‘}\beta$ (3)

$\vdash . (2) . (3) . \supset \vdash .$ Prop

∗102·4. $\vdash : \gamma \in \mathrm{Nc}(\alpha_\beta)\text{‘}\delta . \gamma' \in \mathrm{Nc}(\alpha_\beta)\text{‘}\delta . \supset . \gamma \in \mathrm{Nc}(\alpha_\alpha)\text{‘}\gamma'$ [∗102·3·26]

∗102·41. $\vdash : \gamma \in \mathrm{Nc}(\alpha_\beta)\text{‘}\delta . \gamma' \in \mathrm{Nc}(\alpha'_\beta)\text{‘}\delta . \supset . \gamma \in \mathrm{Nc}(\alpha_{\alpha'})\text{‘}\gamma'$ [∗102·3·27]

∗102·42. $\vdash . \alpha \in \mathrm{Nc}(\alpha_\alpha)\text{‘}\alpha$ [∗102·3·2 . ∗73·3 . ∗63·103]

∗102·43. $\vdash . \exists ! \mathrm{Nc}(\alpha_\alpha)\text{‘}\alpha$ [∗102·42]

This inference is legitimate because, when α is given, "$\mathrm{Nc}(\alpha_\alpha)\text{‘}\alpha$" is typically definite. The inference from "$\alpha \in \mathrm{Nc}\text{‘}\alpha$" (which is true) to "$\exists ! \mathrm{Nc}\text{‘}\alpha$" is not valid, because "$\exists ! \mathrm{Nc}\text{‘}\alpha$" may hold only for *some* of the possible determinations of the ambiguity of "Nc."

∗102·44. $\vdash : \alpha \,\mathrm{sm}\, \beta . \equiv . \alpha \in \mathrm{Nc}(\alpha_\beta)\text{‘}\beta . \equiv . \beta \in \mathrm{Nc}(\beta_\alpha)\text{‘}\alpha$

Dem.

$\vdash . ∗63·102 . \supset$

$\vdash : \alpha \,\mathrm{sm}\, \beta . \equiv . \alpha \,\mathrm{sm}\, \beta . \alpha \in t\text{‘}\alpha . \beta \in t\text{‘}\beta$ (1)

$\vdash . (1) . ∗102·2·3 . \supset \vdash .$ Prop

∗102·45. $\vdash : \gamma \in \mathrm{Nc}(\alpha_\beta)\text{‘}\delta . \supset . \gamma \in \mathrm{Nc}(\alpha_\alpha)\text{‘}\gamma$

Dem.

$\vdash . ∗102·3·2 . \supset \vdash : \mathrm{Hp} . \supset . \gamma \in t\text{‘}\alpha$ (1)

$\vdash . ∗73·3 . \supset \vdash . \gamma \,\mathrm{sm}\, \gamma$ (2)

$\vdash . (1) . (2) . ∗102·3·2 . \supset \vdash .$ Prop

∗102·46. $\vdash : \gamma \in \mathrm{Nc}(\alpha_\beta)\text{‘}\delta . \equiv . \delta \in \mathrm{Nc}(\beta_\alpha)\text{‘}\gamma . \equiv . \gamma \,\mathrm{sm}\, \delta . \gamma \in t\text{‘}\alpha . \delta \in t\text{‘}\beta$
[∗102·2·3 . ∗73·31]

∗102·5. $\vdash : \mu \in \mathrm{NC}^\beta(\alpha) . \equiv . (\exists \delta) . \mu = \mathrm{Nc}(\alpha_\beta)\text{‘}\delta$ [∗100·22 . ∗71·41 . (∗102·01)]

In using propositions, such as those of ∗100, in which we have a typically ambiguous "Nc" or "NC," any significant typical definiteness may be added, since, when a typically ambiguous proposition is asserted, that includes the assertion of every possible proposition resulting from determining the ambiguity.

$*102{\cdot}501.\ \vdash . \mathrm{Nc}\,(\alpha_\beta)`\delta \in \mathrm{NC}^\beta\,(\alpha)\quad [*102{\cdot}5{\cdot}36]$

$*102{\cdot}51.\ \vdash : \gamma \in \mathrm{Nc}\,(\alpha_\beta)`\delta . \supset . \mathrm{Nc}\,(\alpha_\beta)`\delta = \mathrm{Nc}\,(\alpha_\alpha)`\gamma .$
$\mathrm{Nc}\,(\alpha_\beta)`\delta \in \mathrm{NC}^\beta\,(\alpha) . \mathrm{Nc}\,(\alpha_\alpha)`\gamma \in \mathrm{NC}^\alpha\,(\alpha)$

Dem.

$\vdash . *102{\cdot}3{\cdot}2 . \supset$

$\vdash :. \mathrm{Hp} . \supset : \gamma \,\mathrm{sm}\, \delta . \gamma \in t`\alpha . \delta \in t`\beta :$

$[*73{\cdot}37 . *4{\cdot}73]\ \supset : \xi \,\mathrm{sm}\, \delta . \equiv . \xi \,\mathrm{sm}\, \gamma : \xi \,\mathrm{sm}\, \delta . \equiv . \xi \,\mathrm{sm}\, \delta . \delta \in t`\beta :$
$\xi \,\mathrm{sm}\, \gamma . \equiv . \xi \,\mathrm{sm}\, \gamma . \gamma \in t`\alpha :$

$[*4{\cdot}22]\ \supset : \xi \,\mathrm{sm}\, \delta . \delta \in t`\beta . \equiv . \xi \,\mathrm{sm}\, \gamma . \gamma \in t`\alpha :$

$[\mathrm{Fact}]\ \supset : \xi \,\mathrm{sm}\, \delta . \xi \in t`\alpha . \delta \in t`\beta . \equiv . \xi \,\mathrm{sm}\, \gamma . \xi \in t`\alpha . \gamma \in t`\alpha :$

$[*102{\cdot}2{\cdot}3]\ \supset : \mathrm{Nc}\,(\alpha_\beta)`\delta = \mathrm{Nc}\,(\alpha_\alpha)`\gamma \quad (1)$

$\vdash . (1) . *102{\cdot}501 . \supset \vdash . \mathrm{Prop}$

$*102{\cdot}52.\ \vdash : \exists ! \,\mathrm{Nc}\,(\alpha_\beta)`\delta . \supset . \mathrm{Nc}\,(\alpha_\beta)`\delta \in \mathrm{NC}^\alpha\,(\alpha)\quad [*102{\cdot}51]$

$*102{\cdot}53.\ \vdash . \mathrm{NC}^\beta\,(\alpha) - \iota`\Lambda \subset \mathrm{NC}^\alpha\,(\alpha)$

Dem.

$\vdash . *102{\cdot}52 . \supset \vdash : \mu = \mathrm{Nc}\,(\alpha_\beta)`\delta . \exists ! \,\mu . \supset . \mu \in \mathrm{NC}^\alpha\,(\alpha) \quad (1)$

$\vdash . (1) . *102{\cdot}5 . \supset \vdash . \mathrm{Prop}$

$*102{\cdot}54.\ \vdash : \delta \in \mathrm{Nc}\,(\beta_\alpha)`\gamma . \supset . \mathrm{Nc}\,(\alpha_\beta)`\delta = \mathrm{Nc}\,(\alpha_\alpha)`\gamma\quad [*102{\cdot}51{\cdot}46]$

$*102{\cdot}541.\ \vdash : \exists ! \,\mathrm{Nc}\,(\beta_\alpha)`\gamma . \supset . \mathrm{Nc}\,(\alpha_\alpha)`\gamma \in \mathrm{NC}^\beta\,(\alpha) - \iota`\Lambda$

Dem.

$\vdash . *102{\cdot}54{\cdot}501 . \supset \vdash : \delta \in \mathrm{Nc}\,(\beta_\alpha)`\gamma . \supset . \mathrm{Nc}\,(\alpha_\alpha)`\gamma \in \mathrm{NC}^\beta\,(\alpha) \quad (1)$

$\vdash . *102{\cdot}46{\cdot}45 . \supset \vdash : \delta \in \mathrm{Nc}\,(\beta_\alpha)`\gamma . \supset . \gamma \in \mathrm{Nc}\,(\alpha_\alpha)`\gamma .$

$[*10{\cdot}24]\ \supset . \exists ! \,\mathrm{Nc}\,(\alpha_\alpha)`\gamma \quad (2)$

$\vdash . (1) . (2) . \supset$

$\vdash : \delta \in \mathrm{Nc}\,(\beta_\alpha)`\gamma . \supset . \mathrm{Nc}\,(\alpha_\alpha)`\gamma \in \mathrm{NC}^\beta\,(\alpha) - \iota`\Lambda : \supset \vdash . \mathrm{Prop}$

$*102{\cdot}55.\ \vdash : \Lambda \sim\in \mathrm{NC}^\alpha\,(\beta) . \supset . \mathrm{NC}^\beta\,(\alpha) - \iota`\Lambda = \mathrm{NC}^\alpha\,(\alpha)$

Dem.

$\vdash . *102{\cdot}5 . \supset$

$\vdash :. \mathrm{Hp} . \supset : \mu = \mathrm{Nc}\,(\beta_\alpha)`\gamma . \supset_{\mu,\gamma} . \exists ! \,\mu .$

$[*102{\cdot}541]\ \supset_{\mu,\gamma} . \mathrm{Nc}\,(\alpha_\alpha)`\gamma \in \mathrm{NC}^\beta\,(\alpha) - \iota`\Lambda :$

$[*10{\cdot}23]\ \supset : (\exists \mu) . \mu = \mathrm{Nc}\,(\beta_\alpha)`\gamma . \supset_\gamma . \mathrm{Nc}\,(\alpha_\alpha)`\gamma \in \mathrm{NC}^\beta\,(\alpha) - \iota`\Lambda :$

$[*102{\cdot}36]\ \supset : (\gamma) . \mathrm{Nc}\,(\alpha_\alpha)`\gamma \in \mathrm{NC}^\beta\,(\alpha) - \iota`\Lambda :$

$[*13{\cdot}191]\ \supset : \nu = \mathrm{Nc}\,(\alpha_\alpha)`\gamma . \supset_{\nu,\gamma} . \nu \in \mathrm{NC}^\beta\,(\alpha) - \iota`\Lambda :$

$[*102{\cdot}5]\ \supset : \nu \in \mathrm{NC}^\alpha\,(\alpha) . \supset_\nu . \nu \in \mathrm{NC}^\beta\,(\alpha) - \iota`\Lambda \quad (1)$

$\vdash . (1) . *102{\cdot}53 . \supset \vdash . \mathrm{Prop}$

The above proposition shows that, if every class of the same type as β is similar to some class of the same type as α, then, given a class γ of the same type as α, there is a class δ, of the same type as β, such that the classes similar to δ and of the same type as α are the same as the classes similar to γ and of

the same type as α; and conversely, given any class δ, of the same type as β, and similar to some class of the same type as α, then there is a class γ, of the same type as α, such that the classes similar to γ and of the same type as α are the same as the classes similar to δ and of the same type as α. We may express this by saying that, if the cardinals which go from the type of α to the type of β are never null, then those that go from the type of β to the type of α, with the exception of Λ (if Λ is one of them), are the same as those that begin and end within the type of α. The latter are what we call "homogeneous" cardinals. Thus our proposition is a step towards reducing the general study of cardinals to that of homogeneous cardinals.

***102·6.** $\vdash . \mathrm{Nc}(\alpha)`\beta = \mathrm{Nc}(\alpha_\beta)`\beta = \hat{\gamma}(\gamma \operatorname{sm} \beta . \gamma \in t`\alpha) = \mathrm{Nc}`\beta \cap t`\alpha$

Dem.

$\vdash . *35·1 . (*65·04) . \supset$

$\vdash : \mu = \mathrm{Nc}(\alpha)`\beta . \equiv . \mu = \mathrm{Nc}`\beta . \mu \in t^2`\alpha .$

$[*63·5] \quad \equiv . \mu = \mathrm{Nc}`\beta . \mu \subset t`\alpha .$

$[*65·13] \quad \equiv . \mu = \mathrm{Nc}`\beta \cap t`\alpha . \quad (1)$

$[*100·1] \quad \equiv . \mu = \hat{\gamma}(\gamma \operatorname{sm} \beta . \gamma \in t`\alpha) . \quad (2)$

$[*63·103] \quad \equiv . \mu = \hat{\gamma}(\gamma \operatorname{sm} \beta . \gamma \in t`\alpha . \beta \in t`\beta) .$

$[*102·46] \quad \equiv . \mu = \mathrm{Nc}(\alpha_\beta)`\beta \quad (3)$

$\vdash . (1) . (2) . (3) . *20·2 . *100·1 . \supset \vdash . \mathrm{Prop}$

***102·61.** $\vdash : \delta \in t`\beta . \supset . \mathrm{Nc}(\alpha)`\delta = \mathrm{Nc}(\alpha_\beta)`\delta$

Dem.

$\vdash . *4·73 . \supset \vdash : \mathrm{Hp} . \supset . \hat{\gamma}(\gamma \operatorname{sm} \delta . \gamma \in t`\alpha) = \hat{\gamma}(\gamma \operatorname{sm} \delta . \gamma \in t`\alpha . \delta \in t`\beta)$

$[*102·46] \quad = \mathrm{Nc}(\alpha_\beta)`\delta \quad (1)$

$\vdash . (1) . *102·6 . \supset \vdash . \mathrm{Prop}$

***102·62.** $\vdash . \mathrm{NC}^\beta(\alpha) = \mathrm{Nc}(\alpha)``t`\beta$

Dem.

$\vdash . *37·7 . (*100·01) . \supset$

$\vdash . \mathrm{Nc}(\alpha)``t`\beta = \hat{\mu}\{(\exists \delta) . \delta \in t`\beta . \mu = \mathrm{Nc}(\alpha)`\delta\}$

$[*102·61] \quad = \hat{\mu}\{(\exists \delta) . \delta \in t`\beta . \mu = \mathrm{Nc}(\alpha_\beta)`\delta\}$

$[*102·37] \quad = \mathrm{D}`\mathrm{Nc}(\alpha_\beta)$

$[(*102·01)] \quad = \mathrm{NC}^\beta(\alpha) . \supset \vdash . \mathrm{Prop}$

***102·63.** $\vdash : \mu = \mathrm{Nc}`\gamma . \alpha \in \mu . \supset . \mu = \mathrm{Nc}(\alpha)`\gamma$

Dem.

$\vdash . *63·5 . \supset \vdash : \mathrm{Hp} . \supset . \mu = \mathrm{Nc}`\gamma . \mu \subset t`\alpha .$

$[*65·13] \quad \supset . \mu = \mathrm{Nc}`\gamma \cap t`\alpha .$

$[*102·6] \quad \supset . \mu = \mathrm{Nc}(\alpha)`\gamma : \supset \vdash . \mathrm{Prop}$

***102·64.** $\vdash : \mu \in \mathrm{NC} . \exists ! \mu . \supset . (\exists \alpha, \gamma) . \mu = \mathrm{Nc}(\alpha)`\gamma \quad [*102·63 . *100·4]$

The following propositions are part of Cantor's proof that there is no greatest cardinal. They are inserted here in order to enable us to prove that

Λ is a cardinal, namely what we call a "descending" cardinal, *i.e.* one whose corresponding "sm" goes from a higher to a lower type.

***102·71.** $\vdash : R \epsilon \text{Cls} \rightarrow 1 \,.\, \text{D}'R \subset \alpha \,.\, \text{G}'R \subset \text{Cl}'\alpha \,.\, \supset .\, \exists ! \,\text{Cl}'\alpha - \text{G}'R$

Dem.

$\vdash . *20·33 . *4·73 . \supset$

$\vdash :: \text{Hp} \,.\, \varpi = \hat{x}(x \epsilon \text{D}'R \,.\, x \sim\epsilon \breve{R}'x) \,.\, \supset :.$

$x \epsilon \text{D}'R \,.\, \supset_x : x \epsilon \varpi \,.\, \equiv .\, x \sim\epsilon \breve{R}'x :$

$[*5·18] \quad \supset_x : \sim\{x \epsilon \varpi \,.\, \equiv .\, x \epsilon \breve{R}'x\} :$

$[*20·43.\text{Transp}.*71·164] \supset_x : \varpi \neq \breve{R}'x :.$

$[*71·411.\text{Transp}] \supset :. \varpi \sim\epsilon \text{G}'R \qquad (1)$

$\vdash . *20·33 . *3·26 . \supset \vdash : \text{Hp}(1) \,.\, \supset .\, \varpi \subset \text{D}'R \,.$

$[\text{Hp}] \quad \supset .\, \varpi \subset \alpha \qquad (2)$

$\vdash . (1) . (2) . *13·191 . \supset$

$\vdash : \text{Hp} \,.\, \supset .\, \hat{x}(x \epsilon \text{D}'R \,.\, x \sim\epsilon \breve{R}'x) \epsilon \text{Cl}'\alpha - \text{G}'R : \supset \vdash . \text{Prop}$

***102·72.** $\vdash : \beta \subset \alpha \,.\, \supset .\, \sim(\beta \,\text{sm}\, \text{Cl}'\alpha)$

Dem.

$\vdash . *102·71 . \supset \vdash :. \text{Hp} . \supset : R \epsilon 1 \rightarrow 1 \,.\, \text{D}'R = \beta \,.\, \text{G}'R \subset \text{Cl}'\alpha \,.\, \supset_R .\, \exists ! \,\text{Cl}'\alpha - \text{G}'R :$

$[*24·55.*22·41] \quad \supset : R \epsilon 1 \rightarrow 1 \,.\, \text{D}'R = \beta \,.\, \supset_R .\, \text{G}'R \neq \text{Cl}'\alpha :$

$[*10·51] \quad \supset : \sim(\exists R) \,.\, R \epsilon 1 \rightarrow 1 \,.\, \text{D}'R = \beta \,.\, \text{G}'R = \text{Cl}'\alpha :$

$[*73·1] \quad \supset : \sim(\beta \,\text{sm}\, \text{Cl}'\alpha) :. \supset \vdash . \text{Prop}$

***102·73.** $\vdash . \text{Nc}(\alpha)'t'\alpha = \Lambda$

Dem.

$\vdash . *102·6 . \supset \vdash . \text{Nc}(\alpha)'t'\alpha = \hat{\gamma}(\gamma \,\text{sm}\, t'\alpha \,.\, \gamma \epsilon t'\alpha)$

$[*63·65] \quad = \hat{\gamma}(\gamma \,\text{sm}\, \text{Cl}'t_0'\alpha \,.\, \gamma \subset t_0'\alpha)$

$[*102·72] \quad = \Lambda \,.\, \supset \vdash . \text{Prop}$

This proposition proves that no class of the same type as α is similar to $t'\alpha$. Now $t'\alpha$ is the greatest class of its type; thus there are classes of the type next above that of α which are too great to be similar to any class of the type of α. Thus (as will be explicitly proved later) the maximum cardinal in one type is less than that in the next higher type. Cantor's proposition that there is no maximum cardinal only holds when we are allowed to rise to continually higher types: in each type, there is a maximum for that type, namely the number of members of the type.

***102·74.** $\vdash . \Lambda \epsilon \text{NC}^{t'\alpha}(\alpha)$

Dem.

$\vdash . *102·6·501 . \supset \vdash . \text{Nc}(\alpha)'t'\alpha \epsilon \text{NC}^{t'\alpha}(\alpha) \qquad (1)$

$\vdash . (1) . *102·73 . \supset \vdash . \text{Prop}$

Note on negative statements concerning types. Statements such as "$x \sim \epsilon t`y$" or "$x \sim \epsilon t_0`\alpha$" are always false when they are significant. Hence when an object belongs to one type, there is no significant way of expressing what we mean when we say that it does not belong to some other type. The reason is that, when, for example, $t`\alpha$ and $t_0`\alpha$ are said to be different, the statement is only significant if interpreted as applying to the symbols, *i.e.* as meaning to deny that the two symbols denote the same class. We cannot assert that they denote *different* classes, since "$t`\alpha \neq t_0`\alpha$" is not significant, but we can deny that they denote the same class. Owing to this peculiarity, propositions dealing with types acquire their importance largely from the fact that they can be interpreted as dealing with the symbols rather than directly with the objects denoted by the symbols. Another reason for the importance of typically definite propositions is that, when they are implications of which the hypothesis can be asserted, they can be used for *inference*, *i.e.* for the assertion of the conclusion. Where typically ambiguous symbols occur in implications, on the contrary, the conditions of significance may be different for the hypothesis and the conclusion, so that fallacies may arise from the use of such implications in inference. *E.g.* it is fallacy to infer "$\vdash . \exists ! \mathrm{Nc}`\alpha$" from the (true) propositions "$\vdash : \alpha \epsilon \mathrm{Nc}`\alpha . \supset . \exists ! \mathrm{Nc}`\alpha$" and "$\vdash . \alpha \epsilon \mathrm{Nc}`\alpha$." (The truth of the first of these two requires that "$\mathrm{Nc}`\alpha$" should receive the same typical determination in both its occurrences.) For these two reasons hypotheticals concerning types are often useful, in spite of the fact that their hypotheses are always true when they are significant.

$*103$. HOMOGENEOUS CARDINALS

Summary of $*103$.

In this number, we shall consider cardinals generated by a homogeneous relation of similarity. A "homogeneous" cardinal is to mean all the classes similiar to some class α and of the same type as α. The "homogeneous cardinal of α" will be defined as $\mathrm{Nc}'\alpha \cap t'\alpha$; we shall denote it by "$\mathrm{N_0c}'\alpha$." Then the class of homogeneous cardinals is the class of all such cardinals as "$\mathrm{N_0c}'\alpha$," *i.e.* it is $\mathrm{D}'\mathrm{N_0c}$; this we shall denote by "$\mathrm{N_0C}$." The symbol "$\mathrm{N_0c}'\alpha$" is typically definite as soon as α is assigned; "$\mathrm{N_0C}$," on the contrary, is typically ambiguous: it must be a Cls^3, but otherwise its type may vary indefinitely. Homogeneous cardinals have, however, many properties which do not require that the ambiguity of "$\mathrm{N_0C}$" should be determined, and few which do require this. They are important also as being the simplest kind of cardinals, and as being a kind to which other kinds can usually be reduced.

The chief advantage of homogeneous cardinals is that they are never null ($*103{\cdot}13{\cdot}22$). This enables us to avoid by their means the explicit exclusion of exceptional cases; thus throughout Section B we shall use homogeneous cardinals in defining the arithmetical operations: the arithmetical sum of $\mathrm{Nc}'\alpha$ and $\mathrm{Nc}'\beta$, for example, will be defined by means of $\mathrm{N_0c}'\alpha$ and $\mathrm{N_0c}'\beta$, in order to exclude such a determination of the typical ambiguity of $\mathrm{Nc}'\alpha$ and $\mathrm{Nc}'\beta$ as would make either of them null. It is true that not only homogeneous cardinals, but also ascending cardinals (cf. $*104$), are never null. But homogeneous cardinals are much the simplest kind of cardinals that are never null, and are therefore the most convenient.

The fact that no homogeneous cardinal is null is derived from

$*103{\cdot}12$. $\vdash . \alpha \in \mathrm{N_0c}'\alpha$

Other important propositions in this number are the following:

$*103{\cdot}2$. $\vdash : \mu \in \mathrm{N_0C} . \equiv . (\exists\alpha) . \mu = \mathrm{Nc}'\alpha \cap t'\alpha . \equiv . (\exists\alpha) . \mu = \mathrm{N_0c}'\alpha$

$*103{\cdot}26$. $\vdash :. \mu \in \mathrm{NC} . \supset : \alpha \in \mu . \equiv . \mathrm{N_0c}'\alpha = \mu$

The above proposition is used constantly.

$*103{\cdot}27$. $\vdash : \mu = \mathrm{N_0c}'\alpha . \equiv . \mu \in \mathrm{NC} . \alpha \in \mu$

Thus to say that μ is the homogeneous cardinal of α is equivalent to saying that μ is a cardinal of which α is a member.

$*103{\cdot}301$. $\vdash . \mathrm{NC}^{\alpha}(\alpha) = \mathrm{N_0C}(\alpha)$

$*103{\cdot}34$. $\vdash . \mathrm{NC} - \iota'\Lambda \subset \mathrm{N_0C}$

$*103{\cdot}4$. $\vdash . \mathrm{sm}''\mathrm{N_0c}'\alpha = \mathrm{Nc}'\alpha$

$*103{\cdot}41$. $\vdash . \mathrm{sm}''\mathrm{N_0c}'\alpha \cap t'\beta = \mathrm{Nc}(\beta)'\alpha$

***103·01.** $\mathrm{N_0c}\text{‘}\alpha = \mathrm{Nc}\text{‘}\alpha \cap t\text{‘}\alpha$ Df

***103·02.** $\mathrm{N_0C} = \mathrm{D}\text{‘}\mathrm{N_0c}$ Df

***103·1.** $\vdash . \mathrm{N_0c}\text{‘}\alpha = (\mathrm{Nc}\text{‘}\alpha)_\alpha = \mathrm{Nc}\,(\alpha)\text{‘}\alpha = \mathrm{Nc}\,(\alpha_\alpha)\text{‘}\alpha$ [*102·6 . (*103·01)]

***103·11.** $\vdash : \beta \in \mathrm{N_0c}\text{‘}\alpha . \equiv . \beta \,\mathrm{sm}\, \alpha . \beta \in t\text{‘}\alpha . \equiv . \beta \in \mathrm{Nc}\text{‘}\alpha . \beta \in t\text{‘}\alpha$
[*103·1 . *102·6]

***103·12.** $\vdash . \alpha \in \mathrm{N_0c}\text{‘}\alpha$ [*103·11 . *73·3 . *63·103]

***103·13.** $\vdash . \exists ! \mathrm{N_0c}\text{‘}\alpha$ [*103·12 . *10·24]

This is a legitimate inference from *103·12 because, when α is given, $\mathrm{N_0c}\text{‘}\alpha$ is typically definite.

***103·14.** $\vdash : \mathrm{N_0c}\text{‘}\alpha = \mathrm{N_0c}\text{‘}\beta . \equiv . \alpha \in \mathrm{N_0c}\text{‘}\beta . \equiv . \beta \in \mathrm{N_0c}\text{‘}\alpha . \equiv . \alpha \,\mathrm{sm}\, \beta . \alpha \in t\text{‘}\beta$

Dem.

$\vdash .$ *103·11 $. \supset$
$\vdash :. \mathrm{N_0c}\text{‘}\alpha = \mathrm{N_0c}\text{‘}\beta . \equiv : \gamma \,\mathrm{sm}\, \alpha . \gamma \in t\text{‘}\alpha . \equiv_\gamma . \gamma \,\mathrm{sm}\, \beta . \gamma \in t\text{‘}\beta :$ (1)
[*10·1] $\supset : \alpha \,\mathrm{sm}\, \alpha . \alpha \in t\text{‘}\alpha . \equiv . \alpha \,\mathrm{sm}\, \beta . \alpha \in t\text{‘}\beta :$
[*73·3.*63·103] $\supset : \alpha \,\mathrm{sm}\, \beta . \alpha \in t\text{‘}\beta$ (2)
$\vdash .$ *73·32 . *63·17
$\vdash : \alpha \,\mathrm{sm}\, \beta . \alpha \in t\text{‘}\beta . \gamma \,\mathrm{sm}\, \alpha . \gamma \in t\text{‘}\alpha . \supset . \gamma \,\mathrm{sm}\, \beta . \gamma \in t\text{‘}\beta$ (3)
$\vdash . (3) \dfrac{\beta, \alpha}{\alpha, \beta} .$ *73·31 . *63·16 $. \supset$
$\vdash : \alpha \,\mathrm{sm}\, \beta . \alpha \in t\text{‘}\beta . \gamma \,\mathrm{sm}\, \beta . \gamma \in t\text{‘}\beta . \supset . \gamma \,\mathrm{sm}\, \alpha . \gamma \in t\text{‘}\alpha$ (4)
$\vdash . (3) . (4) . (1) . \supset$
$\vdash : \alpha \,\mathrm{sm}\, \beta . \alpha \in t\text{‘}\beta . \supset . \mathrm{N_0c}\text{‘}\alpha = \mathrm{N_0c}\text{‘}\beta$ (5)
$\vdash . (2) . (5) .$ *103·11 . *73·31 . *63·16 $. \supset \vdash .$ Prop

***103·15.** $\vdash : \exists ! \mathrm{N_0c}\text{‘}\alpha \cap \mathrm{N_0c}\text{‘}\beta . \equiv . \mathrm{N_0c}\text{‘}\alpha = \mathrm{N_0c}\text{‘}\beta$

Dem.

$\vdash .$ *103·13 $. \supset \vdash : \mathrm{N_0c}\text{‘}\alpha = \mathrm{N_0c}\text{‘}\beta . \supset . \exists ! \mathrm{N_0c}\text{‘}\alpha \cap \mathrm{N_0c}\text{‘}\beta$ (1)
$\vdash .$ *103·14 $. \supset \vdash : \gamma \in \mathrm{N_0c}\text{‘}\alpha . \gamma \in \mathrm{N_0c}\text{‘}\beta . \supset . \mathrm{N_0c}\text{‘}\alpha = \mathrm{N_0c}\text{‘}\gamma . \mathrm{N_0c}\text{‘}\beta = \mathrm{N_0c}\text{‘}\gamma .$
[*14·131·144] $\supset . \mathrm{N_0c}\text{‘}\alpha = \mathrm{N_0c}\text{‘}\beta :$
[*10·11·23] $\supset \vdash : \exists ! \mathrm{N_0c}\text{‘}\alpha \cap \mathrm{N_0c}\text{‘}\beta . \supset . \mathrm{N_0c}\text{‘}\alpha = \mathrm{N_0c}\text{‘}\beta$ (2)
$\vdash . (1) . (2) . \supset \vdash .$ Prop

***103·16.** $\vdash : \mathrm{N_0c}\text{‘}\alpha = \mathrm{Nc}\text{‘}\beta . \equiv . \mathrm{Nc}\text{‘}\alpha = \mathrm{Nc}\text{‘}\beta$

In this proposition, the equation "$\mathrm{Nc}\text{‘}\alpha = \mathrm{Nc}\text{‘}\beta$" must be supposed to hold in *any* type for which it is significant. Otherwise, we might find a type for which $\mathrm{Nc}\text{‘}\alpha = \Lambda = \mathrm{Nc}\text{‘}\beta$, without having $\mathrm{N_0c}\text{‘}\alpha = \mathrm{Nc}\text{‘}\beta$.

Dem.

$\vdash .$ *103·12 $. \supset \vdash : \mathrm{N_0c}\text{‘}\alpha = \mathrm{Nc}\text{‘}\beta . \supset . \alpha \in \mathrm{Nc}\text{‘}\beta .$
[*100·31·321] $\supset . \mathrm{Nc}\text{‘}\alpha = \mathrm{Nc}\text{‘}\beta$ (1)
$\vdash .$ *22·481 $. \supset \vdash : \mathrm{Nc}\text{‘}\alpha = \mathrm{Nc}\text{‘}\beta . \supset . \mathrm{Nc}\text{‘}\alpha \cap t\text{‘}\alpha = \mathrm{Nc}\text{‘}\beta \cap t\text{‘}\alpha .$
[*65·13.(*103·01)] $\supset . \mathrm{N_0c}\text{‘}\alpha = \mathrm{Nc}\text{‘}\beta$ (2)
$\vdash . (1) . (2) . \supset \vdash .$ Prop

***103·2.** $\vdash : \mu \in \mathrm{N_0C} . \equiv . (\exists\alpha) . \mu = \mathrm{Nc}`\alpha \cap t`\alpha . \equiv . (\exists\alpha) . \mu = \mathrm{N_0c}`\alpha$
[*71·41 . *100·22 . (*103·01·02)]

***103·21.** $\vdash . \mathrm{N_0c}`\alpha \in \mathrm{N_0C} . \mathrm{N_0c}`\alpha \in \mathrm{NC}$ [*103·2 . *100·2·4 . *14·28 . *65·13]

In adducing a proposition, such as *100·2, which is concerned with an "Nc" entirely undetermined in type, any degree of typical determination may be added to our "Nc," since an asserted proposition containing an ambiguous "Nc" is only legitimate if it is true for every possible determination of the ambiguity.

***103·22.** $\vdash : \mu \in \mathrm{N_0C} . \supset . \exists ! \mu$ [*103·13·2]

***103·23.** $\vdash . \Lambda \sim\in \mathrm{N_0C}$ [*103·22]

***103·24.** $\vdash . \mathrm{N_0C} \in \mathrm{Cls\ ex^2\ excl}$ [*100·43 . *103·23 . *84·13]

***103·25.** $\vdash :. \mu, \nu \in \mathrm{N_0C} . \supset : \exists ! \mu \cap \nu . \equiv . \mu = \nu$ [*103·24 . *84·135]

***103·26.** $\vdash :. \mu \in \mathrm{NC} . \supset : \alpha \in \mu . \equiv . \mathrm{N_0c}`\alpha = \mu$

Dem.

$\vdash . *100{\cdot}45 . \quad \supset \vdash :. \mathrm{Hp} . \supset : \alpha \in \mu . \supset . \mathrm{Nc}`\alpha = \mu$ (1)

$\vdash . *63{\cdot}22 . \quad \supset \vdash : \alpha \in \mu . \supset . \mu \subset t`\alpha$ (2)

$\vdash . (1) . (2) . *22{\cdot}621 . \supset \vdash :. \mathrm{Hp} . \supset : \alpha \in \mu . \supset . \mathrm{Nc}`\alpha \cap t`\alpha = \mu .$

$[(*103{\cdot}01)] \quad \supset . \mathrm{N_0c}`\alpha = \mu$ (3)

$\vdash . *103{\cdot}12 . \quad \supset \vdash : \mathrm{N_0c}`\alpha = \mu . \supset . \alpha \in \mu$ (4)

$\vdash . (3) . (4) . \supset \vdash . \mathrm{Prop}$

***103·27.** $\vdash : \mu = \mathrm{N_0c}`\alpha . \equiv . \mu \in \mathrm{NC} . \alpha \in \mu$

Dem.

$\vdash . *103{\cdot}26 . \supset \vdash : \mu \in \mathrm{NC} . \mu = \mathrm{N_0c}`\alpha . \equiv . \mu \in \mathrm{NC} . \alpha \in \mu$ (1)

$\vdash . (1) . *103{\cdot}21 . \supset \vdash . \mathrm{Prop}$

***103·28.** $\vdash : (\exists\alpha) . \gamma \,\mathrm{sm}\, \alpha . \mu = \mathrm{N_0c}`\alpha . \equiv . \exists ! \mu . \mu = \mathrm{Nc}`\gamma$

Dem.

$\vdash . *103{\cdot}27 . \supset$

$\vdash : (\exists\alpha) . \gamma \,\mathrm{sm}\, \alpha . \mu = \mathrm{N_0c}`\alpha . \equiv . (\exists\alpha) . \gamma \,\mathrm{sm}\, \alpha . \mu \in \mathrm{NC} . \alpha \in \mu .$

$[*100{\cdot}31] \quad \equiv . \mu \in \mathrm{NC} . \exists ! \mu \cap \mathrm{Nc}`\gamma .$

$[*100{\cdot}42{\cdot}41] \quad \equiv . \mu \in \mathrm{NC} . \exists ! \mu \cap \mathrm{Nc}`\gamma . \mu = \mathrm{Nc}`\gamma .$

$[*100{\cdot}41] \quad \equiv . \exists ! \mu . \mu = \mathrm{Nc}`\gamma : \supset \vdash . \mathrm{Prop}$

***103·3.** $\vdash : \beta \in t`\alpha . \supset . \mathrm{N_0c}`\beta = \mathrm{Nc}\,(\alpha)`\beta = \mathrm{Nc}\,(\alpha_\alpha)`\beta = \mathrm{Nc}`\beta \cap t`\alpha$

Dem.

$\vdash . *63{\cdot}16 . \supset \vdash : \mathrm{Hp} . \supset . t`\beta = t`\alpha .$

$[*22{\cdot}481 . (*103{\cdot}01)] \quad \supset . \mathrm{N_0c}`\beta = \mathrm{Nc}`\beta \cap t`\alpha$ (1)

$[*102{\cdot}6] \quad = \mathrm{Nc}\,(\alpha)`\beta$ (2)

$[*102{\cdot}61] \quad = \mathrm{Nc}\,(\alpha_\alpha)`\beta$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash . \mathrm{Prop}$

***103·301.** $\vdash . \text{NC}^\alpha(\alpha) = \text{N}_0\text{C}(\alpha)$

Note that although "$\text{NC}(\alpha)$" is not definite, "$\text{N}_0\text{C}(\alpha)$" is absolutely definite as soon as α is assigned.

Dem.

$\vdash . *103·3 . \supset \vdash : \beta \epsilon t`\alpha . \mu = \text{N}_0\text{c}`\beta . \equiv . \beta \epsilon t`\alpha . \mu = \text{Nc}(\alpha_\alpha)`\beta .$

$[*102·37] \quad \equiv . \mu = \text{Nc}(\alpha_\alpha)`\beta \quad (1)$

$\vdash . *63·5 . (*103·01) . \supset$

$\vdash :. \mu = \text{N}_0\text{c}`\beta . \supset : \beta \epsilon t`\alpha . \equiv . \mu \epsilon t^2`\alpha \quad (2)$

$\vdash . (1) . (2) . \supset \vdash : \mu \epsilon t^2`\alpha . \mu = \text{N}_0\text{c}`\beta . \equiv . \mu = \text{Nc}(\alpha_\alpha)`\beta \quad (3)$

$\vdash . (3) . *10·11·281·35 . \supset$

$\vdash :. \mu \epsilon t^2`\alpha : (\exists\beta) . \mu = \text{N}_0\text{c}`\beta : \equiv . (\exists\beta) . \mu = \text{Nc}(\alpha_\alpha)`\beta .$

$[*102·5] \quad \equiv . \mu \epsilon \text{NC}^\alpha(\alpha) \quad (4)$

$\vdash . (4) . *103·2 . \supset \vdash : \mu \epsilon t^2`\alpha \cap \text{N}_0\text{C} . \equiv . \mu \epsilon \text{NC}^\alpha(\alpha) \quad (5)$

$\vdash . (5) . (*65·02) . \supset \vdash . \text{Prop}$

***103·31.** $\vdash : \exists ! \text{Nc}(\alpha_\beta)`\delta . \supset . \text{Nc}(\alpha_\beta)`\delta \epsilon \text{N}_0\text{C}(\alpha)$

Dem.

$\vdash . *102·52 . \supset \vdash : \text{Hp} . \supset . \text{Nc}(\alpha_\beta)`\delta \epsilon \text{NC}^\alpha(\alpha) .$

$[*103·301] \quad \supset . \text{Nc}(\alpha_\beta)`\delta \epsilon \text{N}_0\text{C}(\alpha) : \supset \vdash . \text{Prop}$

***103·32.** $\vdash . \text{NC}^\beta(\alpha) - \iota`\Lambda \subset \text{N}_0\text{C}(\alpha)$

Dem.

$\vdash . *103·31 . \supset \vdash : \mu = \text{Nc}(\alpha_\beta)`\delta . \exists ! \mu . \supset . \mu \epsilon \text{N}_0\text{C}(\alpha) \quad (1)$

$\vdash . (1) . *102·5 . \supset \vdash . \text{Prop}$

In the above proposition, the "β" may be omitted, and we may write (cf. *103·33, below)

$$\vdash . \text{NC}(\alpha) - \iota`\Lambda \subset \text{N}_0\text{C}(\alpha).$$

For the β is wholly arbitrary, so that any possible determination of $\text{NC}(\alpha)$ makes the above proposition true. We may proceed a step further, and write (*103·34, below)

$$\vdash . \text{NC} - \iota`\Lambda \subset \text{N}_0\text{C}.$$

But although we also have $\text{N}_0\text{C} \subset \text{NC} - \iota`\Lambda$, provided the "NC" on the right is suitably determined, we do not have this always. For example, if "NC" is determined as $\text{NC}^\alpha(t`\alpha)$, and "$\text{N}_0\text{C}$" as $\text{N}_0\text{C}(t`\alpha)$, then $\text{N}_0\text{c}`t`\alpha \epsilon \text{N}_0\text{C} - \text{NC}$.

***103·33.** $\vdash . \text{NC}(\alpha) - \iota`\Lambda \subset \text{N}_0\text{C}(\alpha)$

Dem.

$\vdash . *4·2 . (*65·02) . \supset$

$\vdash :. \mu \epsilon \text{NC}(\alpha) - \iota`\Lambda . \equiv : \mu \epsilon \text{NC} . \mu \epsilon t^2`\alpha . \exists ! \mu :$

$[*100·4.*63·5] \quad \equiv : (\exists\beta) . \mu = \text{Nc}`\beta : \mu \subset t`\alpha . \exists ! \mu :$

$[*65·13] \quad \equiv : (\exists\beta) . \mu = \text{Nc}`\beta \cap t`\alpha : \exists ! \mu :$

$[*102·6] \quad \equiv : (\exists\beta) . \mu = \text{Nc}(\alpha_\beta)`\beta . \exists ! \mu :$

$[*103·31] \quad \supset : \mu \epsilon \text{N}_0\text{C}(\alpha) :. \supset \vdash . \text{Prop}$

$*103{\cdot}34$. $\vdash . \mathrm{NC} - \iota\text{‘}\Lambda \subset \mathrm{N}_0\mathrm{C}$

Dem.

$$\begin{aligned}
&\vdash . *100{\cdot}31{\cdot}321 . *63{\cdot}5 . \supset \\
&\vdash : \mu = \mathrm{Nc}\text{‘}\alpha . \beta \in \mu . \supset . \mu = \mathrm{Nc}\text{‘}\beta \cap t\text{‘}\beta \\
&[(*103{\cdot}01)] \qquad\qquad = \mathrm{N}_0\mathrm{c}\text{‘}\beta . \\
&[*103{\cdot}2] \qquad\qquad \supset . \mu \in \mathrm{N}_0\mathrm{C} \qquad (1) \\
&\vdash . (1) . *100{\cdot}4 . *11{\cdot}11{\cdot}35{\cdot}54 . \supset \vdash . \mathrm{Prop}
\end{aligned}$$

Thus every cardinal except Λ is a homogeneous cardinal in the appropriate type. Note that although of course every homogeneous cardinal is a cardinal, yet "$\mathrm{N}_0\mathrm{C} \subset \mathrm{NC}$" must not be asserted, because it is possible to determine the ambiguity of "NC" in such a way as to make this false. Hence we do not get $\mathrm{NC} - \iota\text{‘}\Lambda = \mathrm{N}_0\mathrm{C}$.

$*103{\cdot}35$. $\vdash : \Lambda \sim\in \mathrm{NC}^\alpha(\beta) . \supset . \mathrm{NC}^\beta(\alpha) - \iota\text{‘}\Lambda = \mathrm{N}_0\mathrm{C}(\alpha)$ [$*102{\cdot}55 . *103{\cdot}301$]

The hypothesis of this proposition is satisfied, as will appear later, if the type of β is in what we may call the direct ascent from that of α, *i.e.* if it can be reached from α by a finite number of steps each of which takes us from a type τ to either $\mathrm{Cl}\text{‘}\tau$ or $\mathrm{Rl}\text{‘}(\tau \uparrow \tau)$. Thus in such a case the cardinals (other than Λ) which go from $t\text{‘}\beta$ to $t\text{‘}\alpha$ are the same as those which begin and end within $t\text{‘}\alpha$. It will also appear that in such a case Λ always is a member of $\mathrm{NC}^\beta(\alpha)$. *If* two cardinals which are not equal must always be one greater and the other less, then $\Lambda \in \mathrm{NC}^\beta(\alpha)$ is the condition for $\mathrm{N}_0\mathrm{c}\text{‘}t\text{‘}\beta > \mathrm{Nc}(\beta)\text{‘}t\text{‘}\alpha$. In that case, we shall have $\Lambda \in \mathrm{NC}^\beta(\alpha) . \supset . \Lambda \sim\in \mathrm{NC}^\alpha(\beta)$. But there is no known proof that of two different cardinals one must be the greater, except by assuming the multiplicative axiom and proving thence (by Zermelo's theorem) that every class can be well-ordered (cf. $*258$).

$*103{\cdot}4$. $\vdash . \mathrm{sm}\text{“}\mathrm{N}_0\mathrm{c}\text{‘}\alpha = \mathrm{Nc}\text{‘}\alpha$

Dem.

$$\begin{aligned}
&\vdash . *37{\cdot}1 . \supset \\
&\vdash : \delta \in \mathrm{sm}\text{“}\mathrm{N}_0\mathrm{c}\text{‘}\alpha . \equiv . (\exists\gamma) . \gamma \,\mathrm{sm}\, \alpha . \gamma \in t\text{‘}\alpha . \delta \,\mathrm{sm}\, \gamma . \\
&[*102{\cdot}84] \qquad \equiv . \delta \,\mathrm{sm}\, \alpha : \supset \vdash . \mathrm{Prop}
\end{aligned}$$

$*103{\cdot}41$. $\vdash . \mathrm{sm}\text{“}\mathrm{N}_0\mathrm{c}\text{‘}\alpha \cap t\text{‘}\beta = \mathrm{Nc}(\beta)\text{‘}\alpha$

Dem.

$$\begin{aligned}
&\vdash . *103{\cdot}4 . \supset \vdash . \mathrm{sm}\text{“}\mathrm{N}_0\mathrm{c}\text{‘}\alpha \cap t\text{‘}\beta = \mathrm{Nc}\text{‘}\alpha \cap t\text{‘}\beta \\
&[*102{\cdot}6] \qquad\qquad = \mathrm{Nc}(\beta)\text{‘}\alpha . \supset \vdash . \mathrm{Prop}
\end{aligned}$$

$*103{\cdot}42$. $\vdash : \beta \,\mathrm{sm}\, \alpha . \equiv . \mathrm{Nc}(\beta)\text{‘}\alpha = \mathrm{N}_0\mathrm{c}\text{‘}\beta$

Dem.

$$\begin{aligned}
&\vdash . *100{\cdot}321 . \supset \vdash : \beta \,\mathrm{sm}\, \alpha . \supset . \mathrm{Nc}\text{‘}\alpha = \mathrm{Nc}\text{‘}\beta . \\
&[*22{\cdot}481] \qquad \supset . \mathrm{Nc}\text{‘}\alpha \cap t\text{‘}\beta = \mathrm{Nc}\text{‘}\beta \cap t\text{‘}\beta . \\
&[*102{\cdot}6 . (*103{\cdot}01)] \qquad \supset . \mathrm{Nc}(\beta)\text{‘}\alpha = \mathrm{N}_0\mathrm{c}\text{‘}\beta \qquad (1) \\
&\vdash . *103{\cdot}12 . \supset \vdash : \mathrm{Nc}(\beta)\text{‘}\alpha = \mathrm{N}_0\mathrm{c}\text{‘}\beta . \supset . \beta \in \mathrm{Nc}(\beta)\text{‘}\alpha . \\
&[*100{\cdot}31] \qquad\qquad \supset . \beta \,\mathrm{sm}\, \alpha \qquad (2) \\
&\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}
\end{aligned}$$

***103·43.** $\vdash : \mu \epsilon \mathrm{NC} . \supset . \mathrm{sm}``\mu \cap t_0`\mu = \mu$

Dem.

$\vdash . *37\cdot29 . \supset \vdash : \mu = \Lambda . \supset . \mathrm{sm}``\mu \cap t_0`\mu = \Lambda$ (1)

$\vdash . *103\cdot27 . \supset \vdash : \mu \epsilon \mathrm{NC} . \alpha \epsilon \mu . \supset . \mu = \mathrm{N_0c}`\alpha . t_0`\mu = t`\alpha .$

[*103·41] $\supset . \mathrm{sm}``\mu \cap t_0`\mu = \mathrm{Nc}(\alpha)`\alpha$

[*103·3·27] $= \mu$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***103·44.** $\vdash :. \mu, \nu \epsilon \mathrm{N_0C} . \supset : \mu = \mathrm{sm}``\nu . \equiv . \nu = \mathrm{sm}``\mu$

Dem.

$\vdash . *100\cdot53 . \supset \vdash :. \exists ! \mu . \exists ! \nu . \mu, \nu \epsilon \mathrm{NC} . \supset : \mu = \mathrm{sm}``\nu . \equiv . \nu = \mathrm{sm}``\mu$ (1)

$\vdash . *103\cdot27\cdot2 . \supset \vdash : \mathrm{Hp} . \supset . \exists ! \mu . \exists ! \nu . \mu, \nu \epsilon \mathrm{NC}$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***103·5.** $\vdash . 0 \epsilon \mathrm{N_0C}$

Dem.

$\vdash . *101\cdot11\cdot12 . \supset \vdash . 0 \epsilon \mathrm{NC} . \exists ! 0 .$

[*103·34] $\supset \vdash . 0 \epsilon \mathrm{N_0C} . \supset \vdash . \mathrm{Prop}$

***103·51.** $\vdash . 1 \epsilon \mathrm{N_0C}$

Dem.

$\vdash . *101\cdot21\cdot241 . \supset \vdash . 1 \epsilon \mathrm{NC} . \exists ! 1 .$

[*103·34] $\supset \vdash . 1 \epsilon \mathrm{N_0C} . \supset \vdash . \mathrm{Prop}$

0 and 1 are the only cardinals of which the above property can be proved universally with our assumptions. If (as is possible so far as our assumptions go) the lowest type is a unit class, we shall have *in that type* (though in no other) $2 = \Lambda$, so that in that type $2 \sim\epsilon \mathrm{N_0C}$.

$*104$. ASCENDING CARDINALS

Summary of $*104$.

In this number we have to consider cardinals derived from a relation of similarity which goes from the type of α to that of $t`\alpha$, or to that of $t^2`\alpha$. The propositions to be proved can be extended, by a mere repetition of the proofs, to $t^3`\alpha$, $t^4`\alpha$, etc. This extension must, however, be made afresh in each instance; we cannot prove that it can be made generally, because mathematical induction cannot be applied to the series

$$t_0`\alpha,\ t`\alpha,\ t^2`\alpha,\ t^3`\alpha,\ \ldots.$$

Ascending cardinals, though less important than homogeneous cardinals, yet have considerable importance in arithmetic, because $\mathrm{Nc}`\alpha \times \mathrm{Nc}`\beta$ and $(\mathrm{Nc}`\alpha)^{\mathrm{Nc}`\beta}$ are defined as the cardinals of classes of higher types than those of α and β, and the same applies to the product of the cardinals of members of a class of classes. In these cases, however, we also need cardinals of relational types, which will be dealt with in $*106$.

We have to deal, in this number, with three different sets of notions, namely

$*104\cdot01$. $\mathrm{N}^1\mathrm{c}`\alpha = \mathrm{Nc}`\alpha \cap t`t`\alpha$ Df

$*104\cdot02$. $\mathrm{N}^1\mathrm{C} = \mathrm{D}`\mathrm{N}^1\mathrm{c}$ Df

$*104\cdot03$. $\mu^{(1)} = \mathrm{sm}``\mu \cap t`\mu$ Df

with similar definitions of $\mathrm{N}^2\mathrm{c}`\alpha$, etc. Thus $\mathrm{N}^1\mathrm{c}`\alpha$ consists of all classes similar to α but of the next higher type, *i.e.* it is the cardinal number of α in the type next above that of $\mathrm{N}_0\mathrm{c}`\alpha$; $\mathrm{N}^1\mathrm{C}$ is the class of all such cardinals as $\mathrm{N}^1\mathrm{c}`\alpha$, and is a typically ambiguous symbol, though $\mathrm{N}^1\mathrm{c}`\alpha$ is typically definite when α is given; $\mu^{(1)}$ (if μ is a cardinal which is not null) is the "same" cardinal in the next higher type, so that, *e.g.*, if μ is 1 determined as consisting of unit classes of *individuals*, $\mu^{(1)}$ will be 1 determined as consisting of unit classes of classes of individuals. (When μ is not an existent cardinal, $\mu^{(1)}$ is unimportant.)

The following are the most useful propositions in the present number:

$*104\cdot12$. $\vdash : \beta \in \mathrm{N}^1\mathrm{c}`\alpha \,.\, \gamma \in \mathrm{N}^1\mathrm{c}`\beta \,.\supset .\, \gamma \in \mathrm{N}^2\mathrm{c}`\alpha$

$*104\cdot2$. $\vdash . \iota``\alpha \in \mathrm{N}^1\mathrm{c}`\alpha$

$*104\cdot21$. $\vdash . \exists ! \mathrm{N}^1\mathrm{c}`\alpha$

$*104\cdot24$. $\vdash : \mu = \mathrm{N}^1\mathrm{c}`\alpha \,.\supset .\, \mu = \mathrm{N}_0\mathrm{c}`\iota``\alpha = \mathrm{N}_0\mathrm{c}`\hat{\beta}\{(\exists y) \,.\, y \in \alpha \,.\, \beta = \iota`x \cup \iota`y\}$

$*104.25$. $\vdash . \mathrm{N}^1\mathrm{C} \subset \mathrm{N}_0\mathrm{C}$

$*104\cdot26$. $\vdash : \mu = \mathrm{N}_0\mathrm{c}`\alpha \,.\supset .\, \mu^{(1)} = \mathrm{N}_0\mathrm{c}`\iota``\alpha = \mathrm{N}^1\mathrm{c}`\alpha$

$*104\cdot265$. $\vdash . \mu^{(1)} = \mathrm{sm}_\mu``\mu$

$*104\cdot27.\ \vdash :. \mu \in \mathrm{NC} . \supset : \mu = \mathrm{N_0c}`\alpha . \equiv . \mu^{(1)} = \mathrm{N^1c}`\alpha$

$*104\cdot35.\ \vdash . \mathrm{N^2C} \subset \mathrm{N^1C} . \mathrm{N^2C} \subset \mathrm{N_0C}$

$*104\cdot43.\ \vdash : t`\alpha = t`\beta . \supset . (\exists \gamma, \delta) . \gamma \in \mathrm{N^1c}`\alpha . \delta \in \mathrm{N^1c}`\beta . \gamma \cap \delta = \Lambda$

$*104\cdot01.\ \mathrm{N^1c}`\alpha = \mathrm{Nc}`\alpha \cap t`t`\alpha \quad \mathrm{Df}$

This defines the cardinal number of α in the next type above that of $\mathrm{N_0c}`\alpha$; thus $\mathrm{N^1c}`\alpha$ consists of all classes similar to α and of the next type above that of α.

$*104\cdot011.\ \mathrm{N^2c}`\alpha = \mathrm{Nc}`\alpha \cap t`t^2`\alpha \quad \mathrm{Df}$

Similar definitions are to be assumed for $\mathrm{N^3c}`\alpha$, etc.

$*104\cdot02.\ \mathrm{N^1C} = \mathrm{D}`\mathrm{N^1c} \quad \mathrm{Df}$

$\mathrm{N^1C}$, like $\mathrm{N_0C}$, is typically ambiguous; but $\mathrm{N^1C}(\alpha)$ is typically definite.

$*104\cdot021.\ \mathrm{N^2C} = \mathrm{D}`\mathrm{N^2c} \quad \mathrm{Df}$

Similar definitions are to be assumed for $\mathrm{N^3C}$, etc.

$*104\cdot03.\ \mu^{(1)} = \mathrm{sm}``\mu \cap t`\mu \quad \mathrm{Df}$

Here, if μ is a cardinal, $\mu^{(1)}$ is the same cardinal in the next higher type. For example, if μ is couples of individuals, $\mu^{(1)}$ is couples of classes of individuals.

$*104\cdot031.\ \mu^{(2)} = \mathrm{sm}``\mu \cap t^2`\mu \quad \mathrm{Df}$

Similar definitions are to be assumed for $\mu^{(3)}$, etc.

$*104\cdot1.\ \vdash : \beta \in \mathrm{N^1c}`\alpha . \equiv . \beta \in \mathrm{Nc}`\alpha . \beta \in t`t`\alpha . \equiv . \beta \in \mathrm{Nc}`\alpha . \beta \subset t`\alpha$
[$*63\cdot5 . (*104\cdot01)$]

$*104\cdot101.\ \vdash : \beta \in \mathrm{N^1c}`\alpha . \equiv . \beta \,\mathrm{sm}\, \alpha . \beta \subset t`\alpha$ [$*100\cdot31 . *104\cdot1$]

$*104\cdot102.\ \vdash . \mathrm{N^1c}`\alpha = \mathrm{Nc}\,(t`\alpha)`\alpha = \mathrm{Nc}\,\{(t`\alpha)_\alpha\}`\alpha$ [$*102\cdot6 . (*104\cdot01)$]

$*104\cdot11.\ \vdash : \beta \in \mathrm{N^2c}`\alpha . \equiv . \beta \in \mathrm{Nc}`\alpha . \beta \in t`t^2`\alpha . \equiv . \beta \in \mathrm{Nc}`\alpha . \beta \subset t^2`\alpha$
[$*63\cdot5 . (*104\cdot011)$]

$*104\cdot111.\ \vdash : \beta \in \mathrm{N^2c}`\alpha . \equiv . \beta \,\mathrm{sm}\, \alpha . \beta \subset t^2`\alpha$ [$*100\cdot31 . *104\cdot11$]

$*104\cdot112.\ \vdash . \mathrm{N^2c}`\alpha = \mathrm{Nc}\,(t^2`\alpha)`\alpha = \mathrm{Nc}\,\{(t^2`\alpha)_\alpha\}`\alpha$ [$*102\cdot6 . (*104\cdot011)$]

$*104\cdot12.\ \vdash : \beta \in \mathrm{N^1c}`\alpha . \gamma \in \mathrm{N^1c}`\beta . \supset . \gamma \in \mathrm{N^2c}`\alpha$

Dem.

$\vdash . *104\cdot1 . \supset \vdash : \mathrm{Hp} . \supset . \beta \in \mathrm{Nc}`\alpha . \beta \in t`t`\alpha . \gamma \in \mathrm{Nc}`\beta . \gamma \in t`t`\beta .$

[$*100\cdot32$] $\supset . \gamma \in \mathrm{Nc}`\alpha . \beta \in t`t`\alpha . \gamma \in t`t`\beta .$

[$*63\cdot16$] $\supset . \gamma \in \mathrm{Nc}`\alpha . t`\beta = t`t`\alpha . \gamma \in t`t`\beta .$

[$*13\cdot12$] $\supset . \gamma \in \mathrm{Nc}`\alpha . \gamma \in t`t`t`\alpha .$

[$*104\cdot11$] $\supset . \gamma \in \mathrm{N^2c}`\alpha : \supset \vdash . \mathrm{Prop}$

***104·121.** $\vdash : \beta \in N^1c`\alpha \,.\, \gamma \in N^2c`\alpha \,.\supset .\, \gamma \in N^1c`\beta$

Dem.

$\vdash . *104·102·112 . \supset \vdash : Hp . \supset . \beta \in Nc\{(t`\alpha)_\alpha\}`\alpha . \gamma \in Nc\{(t^2`\alpha)_\alpha\}`\alpha .$

[*102·41] $\supset . \gamma \in Nc\{(t^2`\alpha)_{t`\alpha}\}`\beta$ (1)

$\vdash . *104·1 . \supset \vdash : Hp . \supset . \beta \in t`t`\alpha .$

[*63·16] $\supset . t`\beta = t`t`\alpha .$

[(*65·11)] $\supset . Nc\{(t^2`\alpha)_{t`\alpha}\} = Nc\{(t`\beta)_\beta\}$ (2)

$\vdash . (1) . (2) . *104·102 . \supset \vdash . Prop$

***104·122.** $\vdash : \beta \in N^1c`\alpha . \supset . N^1c`\beta = N^2c`\alpha$ [*104·12·121]

***104·123.** $\vdash : N_0c`\beta = N^1c`\alpha . \supset . N^1c`\beta = N^2c`\alpha$ [*104·122 . *103·26]

***104·13.** $\vdash : \mu \in N^1C . \equiv . (\exists\alpha) . \mu = N^1c`\alpha$ [*100·22 . *71·41 . (*104·02)]

***104·14.** $\vdash : \delta \in \mu^{(1)} . \equiv . (\exists\gamma) . \gamma \in \mu . \delta \,sm\, \gamma . \delta \in t`\mu . \equiv . (\exists\gamma) . \gamma \in \mu . \delta \,sm\, \gamma . \delta \subset t`\gamma$
[*37·1 . *63·22 . (*104·03)]

***104·141.** $\vdash : \mu \in NC . \exists ! \mu . \supset . \mu^{(1)} \in NC$ [*100·52]

When the hypothesis "$\exists ! \mu$" is omitted, this proposition is still true, but with a difference. *E.g.* let us put

$$\mu = Nc(\alpha)`t`\alpha.$$

Then $\mu = \Lambda . \mu^{(1)} = \Lambda$. Thus $\mu^{(1)} \neq Nc(t`\alpha)`t`\alpha$. But we still have

$$\mu^{(1)} = Nc(t`\alpha)`t^2`\alpha.$$

Thus $\mu^{(1)} \in NC$, but $\mu^{(1)}$ is not the same cardinal as μ in a higher type, *i.e.* there are classes whose cardinal in one type is μ, but whose cardinal in the next higher type is not $\mu^{(1)}$.

***104·142.** $\vdash : \mu \in NC . \exists ! \mu . \supset . \mu^{(2)} \in NC$ [*100·52]

***104·15.** $\vdash : \mu \in N^2C . \equiv . (\exists\alpha) . \mu = N^2c`\alpha$ [*100·22 . *71·41 . (*104·021)]

***104·2.** $\vdash . \iota``\alpha \in N^1c`\alpha$

Dem.

$\vdash . *63·621 . \supset \vdash : x \in \alpha . \supset_x . \iota`x \in t`\alpha :$

[*37·61] $\supset \vdash . \iota``\alpha \subset t`\alpha$ (1)

$\vdash . (1) . *100·6 . *104·1 . \supset \vdash . Prop$

***104·201.** $\vdash : \beta \in N_0c`\alpha . \supset . \iota``\beta \in N^1c`\alpha . N^1c`\alpha = N^1c`\beta$

Dem.

$\vdash . *100·31·321 . \supset \vdash : Hp . \supset . Nc`\alpha = Nc`\beta$ (1)

$\vdash . *103·11 . \supset \vdash : Hp . \supset . \beta \in t`\alpha .$

[*63·16] $\supset . t`\alpha = t`\beta .$

[*30·37] $\supset . t`t`\alpha = t`t`\beta$ (2)

$\vdash . (1) . (2) . (*104·01) . \supset \vdash . N^1c`\alpha = N^1c`\beta$ (3)

$\vdash . (3) . *104·2 . \supset \vdash . Prop$

$*104{\cdot}27.\quad \vdash :. \mu \in NC . \supset : \mu = N_0c`\alpha . \equiv . \mu^{(1)} = N^1c`\alpha$

$*104{\cdot}35.\quad \vdash . N^2C \subset N^1C . N^2C \subset N_0C$

$*104{\cdot}43.\quad \vdash : t`\alpha = t`\beta . \supset . (\exists\gamma, \delta) . \gamma \in N^1c`\alpha . \delta \in N^1c`\beta . \gamma \cap \delta = \Lambda$

$*104{\cdot}01.\quad N^1c`\alpha = Nc`\alpha \cap t`t`\alpha \qquad Df$

This defines the cardinal number of α in the next type above that of $N_0c`\alpha$; thus $N^1c`\alpha$ consists of all classes similar to α and of the next type above that of α.

$*104{\cdot}011.\quad N^2c`\alpha = Nc`\alpha \cap t`t^2`\alpha \qquad Df$

Similar definitions are to be assumed for $N^3c`\alpha$, etc.

$*104{\cdot}02.\quad N^1C = D`N^1c \qquad Df$

N^1C, like N_0C, is typically ambiguous; but $N^1C(\alpha)$ is typically definite.

$*104{\cdot}021.\quad N^2C = D`N^2c \qquad Df$

Similar definitions are to be assumed for N^3C, etc.

$*104{\cdot}03.\quad \mu^{(1)} = sm``\mu \cap t`\mu \qquad Df$

Here, if μ is a cardinal, $\mu^{(1)}$ is the same cardinal in the next higher type. For example, if μ is couples of individuals, $\mu^{(1)}$ is couples of classes of individuals.

$*104{\cdot}031.\quad \mu^{(2)} = sm``\mu \cap t^2`\mu \qquad Df$

Similar definitions are to be assumed for $\mu^{(3)}$, etc.

$*104{\cdot}1.\quad \vdash : \beta \in N^1c`\alpha . \equiv . \beta \in Nc`\alpha . \beta \in t`t`\alpha . \equiv . \beta \in Nc`\alpha . \beta \subset t`\alpha$
$[*63{\cdot}5 . (*104{\cdot}01)]$

$*104{\cdot}101.\quad \vdash : \beta \in N^1c`\alpha . \equiv . \beta \, sm \, \alpha . \beta \subset t`\alpha \qquad [*100{\cdot}31 . *104{\cdot}1]$

$*104{\cdot}102.\quad \vdash . N^1c`\alpha = Nc(t`\alpha)`\alpha = Nc\{(t`\alpha)_\alpha\}`\alpha \qquad [*102{\cdot}6 . (*104{\cdot}01)]$

$*104{\cdot}11.\quad \vdash : \beta \in N^2c`\alpha . \equiv . \beta \in Nc`\alpha . \beta \in t`t^2`\alpha . \equiv . \beta \in Nc`\alpha . \beta \subset t^2`\alpha$
$[*63{\cdot}5 . (*104{\cdot}011)]$

$*104{\cdot}111.\quad \vdash : \beta \in N^2c`\alpha . \equiv . \beta \, sm \, \alpha . \beta \subset t^2`\alpha \qquad [*100{\cdot}31 . *104{\cdot}11]$

$*104{\cdot}112.\quad \vdash . N^2c`\alpha = Nc(t^2`\alpha)`\alpha = Nc\{(t^2`\alpha)_\alpha\}`\alpha \qquad [*102{\cdot}6 . (*104{\cdot}011)]$

$*104{\cdot}12.\quad \vdash : \beta \in N^1c`\alpha . \gamma \in N^1c`\beta . \supset . \gamma \in N^2c`\alpha$

Dem.

$\vdash . *104{\cdot}1 . \supset \vdash : Hp . \supset . \beta \in Nc`\alpha . \beta \in t`t`\alpha . \gamma \in Nc`\beta . \gamma \in t`t`\beta .$
$[*100{\cdot}32] \quad \supset . \gamma \in Nc`\alpha . \beta \in t`t`\alpha . \gamma \in t`t`\beta .$
$[*63{\cdot}16] \quad \supset . \gamma \in Nc`\alpha . t`\beta = t`t`\alpha . \gamma \in t`t`\beta .$
$[*13{\cdot}12] \quad \supset . \gamma \in Nc`\alpha . \gamma \in t`t`t`\alpha .$
$[*104{\cdot}11] \quad \supset . \gamma \in N^2c`\alpha : \supset \vdash . Prop$

∗104·121. $\vdash : \beta \in \mathrm{N^1c}\text{‘}\alpha \,.\, \gamma \in \mathrm{N^2c}\text{‘}\alpha \,.\supset .\, \gamma \in \mathrm{N^1c}\text{‘}\beta$

Dem.

$\vdash .\, {*}104{\cdot}102{\cdot}112 \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, \beta \in \mathrm{Nc}\,\{(t\text{‘}\alpha)_\alpha\}\text{‘}\alpha \,.\, \gamma \in \mathrm{Nc}\,\{(t^2\text{‘}\alpha)_\alpha\}\text{‘}\alpha \,.$

$[{*}102{\cdot}41] \quad \supset .\, \gamma \in \mathrm{Nc}\,\{(t^2\text{‘}\alpha)_{t\text{‘}\alpha}\}\text{‘}\beta \quad (1)$

$\vdash .\, {*}104{\cdot}1 \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, \beta \in t\text{‘}t\text{‘}\alpha \,.$

$[{*}63{\cdot}16] \quad \supset .\, t\text{‘}\beta = t\text{‘}t\text{‘}\alpha \,.$

$[({*}65{\cdot}11)] \quad \supset .\, \mathrm{Nc}\,\{(t^2\text{‘}\alpha)_{t\text{‘}\alpha}\} = \mathrm{Nc}\,\{(t\text{‘}\beta)_\beta\} \quad (2)$

$\vdash .\, (1) \,.\, (2) \,.\, {*}104{\cdot}102 \,.\supset \vdash .\, \mathrm{Prop}$

∗104·122. $\vdash : \beta \in \mathrm{N^1c}\text{‘}\alpha \,.\supset .\, \mathrm{N^1c}\text{‘}\beta = \mathrm{N^2c}\text{‘}\alpha \quad [{*}104{\cdot}12{\cdot}121]$

∗104·123. $\vdash : \mathrm{N_0c}\text{‘}\beta = \mathrm{N^1c}\text{‘}\alpha \,.\supset .\, \mathrm{N^1c}\text{‘}\beta = \mathrm{N^2c}\text{‘}\alpha \quad [{*}104{\cdot}122 \,.\, {*}103{\cdot}26]$

∗104·13. $\vdash : \mu \in \mathrm{N^1C} \,.\equiv .\, (\exists\alpha) \,.\, \mu = \mathrm{N^1c}\text{‘}\alpha \quad [{*}100{\cdot}22 \,.\, {*}71{\cdot}41 \,.\, ({*}104{\cdot}02)]$

∗104·14. $\vdash : \delta \in \mu^{(1)} \,.\equiv .\, (\exists\gamma) \,.\, \gamma \in \mu \,.\, \delta \,\mathrm{sm}\, \gamma \,.\, \delta \in t\text{‘}\mu \,.\equiv .\, (\exists\gamma) \,.\, \gamma \in \mu \,.\, \delta \,\mathrm{sm}\, \gamma \,.\, \delta \subset t\text{‘}\gamma$
$[{*}37{\cdot}1 \,.\, {*}63{\cdot}22 \,.\, ({*}104{\cdot}03)]$

∗104·141. $\vdash : \mu \in \mathrm{NC} \,.\, \exists ! \mu \,.\supset .\, \mu^{(1)} \in \mathrm{NC} \quad [{*}100{\cdot}52]$

When the hypothesis "$\exists ! \mu$" is omitted, this proposition is still true, but with a difference. *E.g.* let us put

$$\mu = \mathrm{Nc}\,(\alpha)\text{‘}t\text{‘}\alpha.$$

Then $\mu = \Lambda \,.\, \mu^{(1)} = \Lambda$. Thus $\mu^{(1)} \neq \mathrm{Nc}\,(t\text{‘}\alpha)\text{‘}t\text{‘}\alpha$. But we still have

$$\mu^{(1)} = \mathrm{Nc}\,(t\text{‘}\alpha)\text{‘}t^2\text{‘}\alpha.$$

Thus $\mu^{(1)} \in \mathrm{NC}$, but $\mu^{(1)}$ is not the same cardinal as μ in a higher type, *i.e.* there are classes whose cardinal in one type is μ, but whose cardinal in the next higher type is not $\mu^{(1)}$.

∗104·142. $\vdash : \mu \in \mathrm{NC} \,.\, \exists ! \mu \,.\supset .\, \mu^{(2)} \in \mathrm{NC} \quad [{*}100{\cdot}52]$

∗104·15. $\vdash : \mu \in \mathrm{N^2C} \,.\equiv .\, (\exists\alpha) \,.\, \mu = \mathrm{N^2c}\text{‘}\alpha \quad [{*}100{\cdot}22 \,.\, {*}71{\cdot}41 \,.\, ({*}104{\cdot}021)]$

∗104·2. $\vdash .\, \iota\text{‘‘}\alpha \in \mathrm{N^1c}\text{‘}\alpha$

Dem.

$\vdash .\, {*}63{\cdot}621 \,.\supset \vdash : x \in \alpha \,.\supset_x .\, \iota\text{‘}x \in t\text{‘}\alpha :$

$[{*}37{\cdot}61] \quad \supset \vdash .\, \iota\text{‘‘}\alpha \subset t\text{‘}\alpha \quad (1)$

$\vdash .\, (1) \,.\, {*}100{\cdot}6 \,.\, {*}104{\cdot}1 \,.\supset \vdash .\, \mathrm{Prop}$

∗104·201. $\vdash : \beta \in \mathrm{N_0c}\text{‘}\alpha \,.\supset .\, \iota\text{‘‘}\beta \in \mathrm{N^1c}\text{‘}\alpha \,.\, \mathrm{N^1c}\text{‘}\alpha = \mathrm{N^1c}\text{‘}\beta$

Dem.

$\vdash .\, {*}100{\cdot}31{\cdot}321 \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, \mathrm{Nc}\text{‘}\alpha = \mathrm{Nc}\text{‘}\beta \quad (1)$

$\vdash .\, {*}103{\cdot}11 \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, \beta \in t\text{‘}\alpha \,.$

$[{*}63{\cdot}16] \quad \supset .\, t\text{‘}\alpha = t\text{‘}\beta \,.$

$[{*}30{\cdot}37] \quad \supset .\, t\text{‘}t\text{‘}\alpha = t\text{‘}t\text{‘}\beta \quad (2)$

$\vdash .\, (1) \,.\, (2) \,.\, ({*}104{\cdot}01) \,.\supset \vdash .\, \mathrm{N^1c}\text{‘}\alpha = \mathrm{N^1c}\text{‘}\beta \quad (3)$

$\vdash .\, (3) \,.\, {*}104{\cdot}2 \,.\supset \vdash .\, \mathrm{Prop}$

***104·21.** $\vdash . \exists ! \, \mathrm{N}^1\mathrm{c}'\alpha \quad [*104\cdot2]$

It follows from this proposition that *ascending* cardinals are never null. The proof has to be made separately for each kind of ascending cardinal, *i.e.* $\mathrm{N}^1\mathrm{C}$, $\mathrm{N}^2\mathrm{C}$, etc.

***104·211.** $\vdash . \exists ! \, \mathrm{N}^1\mathrm{c}'\alpha \cap \mathrm{Cl}'1 \quad [*104\cdot2 . *52\cdot3]$

***104·23.** $\vdash . \hat{\beta}\{(\exists y) . y \in \alpha . \beta = \iota'x \cup \iota'y\} \in \mathrm{N}^1\mathrm{c}'\alpha$

Dem.

$$\begin{aligned}
&\vdash . *51\cdot16 . &&\supset \vdash : y \in \alpha . \supset . y \in \alpha \cap (\iota'x \cup \iota'y) . \\
&[*63\cdot16] &&\supset . \iota'x \cup \iota'y \in t'\alpha &(1)\\
&\vdash . (1) . *10\cdot11\cdot23 . \supset \vdash . \hat{\beta}\{(\exists y) . y \in \alpha . \beta = \iota'x \cup \iota'y\} \subset t'\alpha && &(2)\\
&\vdash . (2) . *100\cdot61 . *104\cdot1 . \supset \vdash . \mathrm{Prop}
\end{aligned}$$

***104·231.** $\vdash : \mathrm{N}^1\mathrm{c}'\alpha = \mathrm{N}^1\mathrm{c}'\beta . \supset . \mathrm{N}_0\mathrm{c}'\alpha = \mathrm{N}_0\mathrm{c}'\beta$

Dem.

$$\begin{aligned}
&\vdash . *104\cdot2 . \supset \vdash : \mathrm{Hp} . \supset . \iota''\beta \in \mathrm{N}_1\mathrm{c}'\alpha . \\
&[*104\cdot101] &&\supset . \iota''\beta \,\mathrm{sm}\, \alpha . \iota''\beta \subset t'\alpha . \\
&[*73\cdot41 . *63\cdot21\cdot64] &&\supset . \beta \,\mathrm{sm}\, \alpha . t'\beta = t'\alpha . \\
&[*103\cdot11 . *63\cdot16] &&\supset . \beta \in \mathrm{N}_0\mathrm{c}'\alpha . \\
&[*103\cdot14] &&\supset . \mathrm{N}_0\mathrm{c}'\alpha = \mathrm{N}_0\mathrm{c}'\beta : \supset \vdash . \mathrm{Prop}
\end{aligned}$$

***104·232.** $\vdash : \mathrm{N}^1\mathrm{c}'\alpha = \mathrm{N}^1\mathrm{c}'\beta . \equiv . \mathrm{N}_0\mathrm{c}'\alpha = \mathrm{N}_0\mathrm{c}'\beta . \equiv . \beta \in \mathrm{N}_0\mathrm{c}'\alpha$
$[*104\cdot231\cdot201 . *103\cdot14]$

***104·24.** $\vdash : \mu = \mathrm{N}^1\mathrm{c}'\alpha . \supset . \mu = \mathrm{N}_0\mathrm{c}'\iota''\alpha = \mathrm{N}_0\mathrm{c}'\hat{\beta}\{(\exists y) . y \in \alpha . \beta = \iota'x \cup \iota'y\}$
$[*104\cdot2\cdot23 . *103\cdot26]$

***104·25.** $\vdash . \mathrm{N}^1\mathrm{C} \subset \mathrm{N}_0\mathrm{C} \quad [*104\cdot24\cdot13]$

This proposition holds for each possible determination of the typical ambiguities, *i.e.* for every α we have

$$\mathrm{N}^1\mathrm{C}\,(t'\alpha) \subset \mathrm{N}_0\mathrm{C}\,(t'\alpha).$$

We do not have
$$\mathrm{N}^1\mathrm{C}\,(t'\alpha) = \mathrm{N}_0\mathrm{C}\,(t'\alpha),$$
because
$$\mathrm{N}_0\mathrm{c}'t'\alpha \in \mathrm{N}_0\mathrm{C}\,(t'\alpha) - \mathrm{N}^1\mathrm{C}\,(t'\alpha).$$

***104·251.** $\vdash . \Lambda \sim\in \mathrm{N}^1\mathrm{C} \quad [*104\cdot25 . *103\cdot23]$

***104·252.** $\vdash . \mathrm{N}^1\mathrm{C} \in \mathrm{Cls\,ex}^2\,\mathrm{excl} \quad [*104\cdot25 . *103\cdot24 . *84\cdot26]$

***104·26.** $\vdash : \mu = \mathrm{N}_0\mathrm{c}'\alpha . \supset . \mu^{(1)} = \mathrm{N}_0\mathrm{c}'\iota''\alpha = \mathrm{N}^1\mathrm{c}'\alpha$

Dem.

$$\begin{aligned}
&\vdash . *104\cdot14 . *103\cdot11 . \supset \\
&\vdash :. \mathrm{Hp} . \supset : \delta \in \mu^{(1)} . \equiv . (\exists \gamma) . \gamma \,\mathrm{sm}\, \alpha . \gamma \in t'\alpha . \delta \,\mathrm{sm}\, \gamma . \delta \subset t'\gamma . &(1)\\
&[*73\cdot32 . *63\cdot16] \quad \supset . \delta \,\mathrm{sm}\, \alpha . \delta \subset t'\alpha . \\
&[*104\cdot101] \quad \supset . \delta \in \mathrm{N}^1\mathrm{c}'\alpha &(2)
\end{aligned}$$

$\vdash . *104\cdot101 . \supset$

$\vdash : \delta \epsilon N^1c`\alpha . \quad \supset . \delta \, \mathrm{sm} \, \alpha . \delta \subset t`\alpha .$

$[*73\cdot3 . *63\cdot103] \supset . \alpha \, \mathrm{sm} \, \alpha . \alpha \epsilon t`\alpha . \delta \, \mathrm{sm} \, \alpha . \delta \subset t`\alpha .$

$[*10\cdot24] \quad \supset . (\exists\gamma) . \gamma \, \mathrm{sm} \, \alpha . \gamma \epsilon t`\alpha . \delta \, \mathrm{sm} \, \gamma . \delta \subset t`\gamma \quad (3)$

$\vdash . (3) . (1) . \supset \vdash :. \mathrm{Hp} . \supset : \delta \epsilon N^1c`\alpha . \supset . \delta \epsilon \mu^{(1)} \quad (4)$

$\vdash . (2) . (4) . *104\cdot24 . \supset \vdash . \mathrm{Prop}$

***104·261.** $\vdash : \mu^{(1)} = N^1c`\alpha . \supset . \mu \subset N_0c`\alpha$

Dem.

$\vdash . *104\cdot14\cdot101 . \supset$

$\vdash :. \mathrm{Hp} . \supset : (\exists\gamma) . \gamma \epsilon \mu . \delta \, \mathrm{sm} \, \gamma . \delta \subset t`\gamma . \equiv_\delta . \delta \, \mathrm{sm} \, \alpha . \delta \subset t`\alpha :$

$[*10\cdot23] \supset : \gamma \epsilon \mu . \delta \, \mathrm{sm} \, \gamma . \delta \subset t`\gamma . \supset_{\gamma,\delta} . \delta \, \mathrm{sm} \, \alpha . \delta \subset t`\alpha .$

$[*4\cdot7] \quad \supset_{\gamma,\delta} . \delta \, \mathrm{sm} \, \alpha . \delta \, \mathrm{sm} \, \gamma . \delta \subset t`\alpha . \delta \subset t`\gamma .$

$[*73\cdot32 . *63\cdot13] \quad \supset_{\gamma,\delta} . \gamma \, \mathrm{sm} \, \alpha . \gamma \epsilon t`\alpha .$

$[*103\cdot11] \quad \supset_{\gamma,\delta} . \gamma \epsilon N_0c`\alpha \quad (1)$

$\vdash . (1) . *10\cdot23\cdot35 . *104\cdot101 . \supset$

$\vdash :. \mathrm{Hp} . \supset : \gamma \epsilon \mu . \exists ! N^1c`\gamma . \supset_\gamma . \gamma \epsilon N_0c`\alpha :$

$[*104\cdot21] \supset : \gamma \epsilon \mu . \supset_\gamma . \gamma \epsilon N_0c`\alpha :. \supset \vdash . \mathrm{Prop}$

***104·262.** $\vdash : \mu \epsilon \mathrm{NC} . \mu^{(1)} = N^1c`\alpha . \supset . \mu = N_0c`\alpha$

Dem.

$\vdash . *104\cdot21 . \supset \vdash : \mathrm{Hp} . \supset . \exists ! \mu^{(1)} .$

$[*37\cdot29 . \mathrm{Transp}] \quad \supset . \exists ! \mu \quad (1)$

$\vdash . *103\cdot26 . \supset \vdash : \mathrm{Hp} . \gamma \epsilon \mu . \supset . \mu = N_0c`\gamma \quad (2)$

$\vdash . (1) . (2) . \supset \vdash : \mathrm{Hp} . \supset . (\exists\gamma) . \mu = N_0c`\gamma .$

$[*104\cdot26 . \mathrm{Hp}] \quad \supset . (\exists\gamma) . \mu = N_0c`\gamma . N^1c`\alpha = N^1c`\gamma .$

$[*104\cdot231] \quad \supset . (\exists\gamma) . \mu = N_0c`\gamma . N_0c`\alpha = N_0c`\gamma .$

$[*13\cdot172] \quad \supset . \mu = N_0c`\alpha : \supset \vdash . \mathrm{Prop}$

***104·263.** $\vdash : \alpha \epsilon \mu . \supset . \iota``\alpha \epsilon \mu^{(1)}$

Dem.

$\vdash . *73\cdot41 . *37\cdot1 . \supset \vdash : \mathrm{Hp} . \supset . \iota``\alpha \epsilon \mathrm{sm}``\mu \quad (1)$

$\vdash . *63\cdot64 . \quad \supset \vdash : \mathrm{Hp} . \supset . \iota``\alpha \epsilon t`\mu \quad (2)$

$\vdash . (1) . (2) . (*104\cdot03) . \supset \vdash . \mathrm{Prop}$

***104·264.** $\vdash : \exists ! \mu . \equiv . \exists ! \mu^{(1)}$

Dem.

$\vdash . *104\cdot263 . \quad \supset \vdash : \exists ! \mu . \supset . \exists ! \mu^{(1)} \quad (1)$

$\vdash . *37\cdot29 . \mathrm{Transp} . (*104\cdot03) . \supset \vdash : \exists ! \mu^{(1)} . \supset . \exists ! \mu \quad (2)$

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***104·265.** $\vdash . \mu^{(1)} = \mathrm{sm}_\mu``\mu$ [*102·85 . (*104·03)]

***104·27.** $\vdash :. \mu \epsilon \mathrm{NC} . \supset : \mu = N_0c`\alpha . \equiv . \mu^{(1)} = N^1c`\alpha$ [*104·26·262]

***104·28.** $\vdash : \mu \epsilon \mathrm{NC} - \iota`\Lambda . \supset . \mu^{(1)} \epsilon N^1C$ [*104·26 . *103·34]

***104·29.** $\vdash : \nu \in \mathrm{N}^1\mathrm{C} . \equiv . (\exists \mu) . \mu \in \mathrm{N}_0\mathrm{C} . \nu = \mu^{(1)}$

Dem.

$\vdash . *104·26 . \quad \supset \vdash : \mu = \mathrm{N}_0\mathrm{c}\text{‘}\alpha . \nu = \mu^{(1)} . \supset . \nu = \mathrm{N}^1\mathrm{c}\text{‘}\alpha :$

$[*10·11·28] \quad \supset \vdash : (\exists \alpha) . \mu = \mathrm{N}_0\mathrm{c}\text{‘}\alpha . \nu = \mu^{(1)} . \supset . (\exists \alpha) . \nu = \mathrm{N}^1\mathrm{c}\text{‘}\alpha :$

$[*103·2.*104·13] \supset \vdash : \mu \in \mathrm{N}_0\mathrm{C} . \nu = \mu^{(1)} . \supset . \nu \in \mathrm{N}^1\mathrm{C}$ (1)

$\vdash . *104·26 . *103·2 . \supset$

$\vdash : \nu = \mathrm{N}^1\mathrm{c}\text{‘}\alpha . \mu = \mathrm{N}_0\mathrm{c}\text{‘}\alpha . \supset . \nu = \mu^{(1)} . \mu \in \mathrm{N}_0\mathrm{C}$ (2)

$\vdash . (2) . *10·11·28·35 . \supset$

$\vdash :. \nu = \mathrm{N}^1\mathrm{c}\text{‘}\alpha : (\exists \mu) . \mu = \mathrm{N}_0\mathrm{c}\text{‘}\alpha : \supset . (\exists \mu) . \mu \in \mathrm{N}_0\mathrm{C} . \nu = \mu^{(1)}$ (3)

$\vdash . (3) . *100·2 . *14·204 . \supset$

$\vdash : \nu = \mathrm{N}^1\mathrm{c}\text{‘}\alpha . \supset . (\exists \mu) . \mu \in \mathrm{N}_0\mathrm{C} . \nu = \mu^{(1)}$ (4)

$\vdash . (4) . *10·11·23 . *104·13 . \supset$

$\vdash : \nu \in \mathrm{N}^1\mathrm{C} . \supset . (\exists \mu) . \mu \in \mathrm{N}_0\mathrm{C} . \nu = \mu^{(1)}$ (5)

$\vdash . (1) . (5) . \supset \vdash .$ Prop

***104·3.** $\vdash . \iota\text{“}\iota\text{“}\alpha \in \mathrm{N}^2\mathrm{c}\text{‘}\alpha$

Dem.

$\vdash . *104·2 . \supset \vdash . \iota\text{“}\alpha \in \mathrm{N}^1\mathrm{c}\text{‘}\alpha . \iota\text{“}\iota\text{“}\alpha \in \mathrm{N}^1\mathrm{c}\text{‘}\iota\text{“}\alpha .$

$[*104·12] \quad \supset \vdash . \iota\text{“}\iota\text{“}\alpha \in \mathrm{N}^2\mathrm{c}\text{‘}\alpha$

***104·31.** $\vdash . \exists ! \mathrm{N}^2\mathrm{c}\text{‘}\alpha$ [*104·3]

***104·311.** $\vdash . \mathrm{N}^2\mathrm{c}\text{‘}\alpha = \mathrm{N}_0\mathrm{c}\text{‘}\iota\text{“}\iota\text{“}\alpha = \mathrm{N}^1\mathrm{c}\text{‘}\iota\text{“}\alpha$ [*104·3·2 . *103·26]

***104·32.** $\vdash : \mu = \mathrm{N}_0\mathrm{c}\text{‘}\alpha . \supset . \mu^{(2)} = \mathrm{N}_0\mathrm{c}\text{‘}\iota\text{“}\iota\text{“}\alpha = \mathrm{N}^1\mathrm{c}\text{‘}\iota\text{“}\alpha = \mathrm{N}^2\mathrm{c}\text{‘}\alpha = \{\mu^{(1)}\}^{(1)}$

Dem.

$\vdash . *104·26 . \supset \vdash : \mathrm{Hp} . \supset . \{\mu^{(1)}\}^{(1)} = \mathrm{N}_0\mathrm{c}\text{‘}\iota\text{“}\iota\text{“}\alpha$ (1)

$[*104·311] \quad = \mathrm{N}^2\mathrm{c}\text{‘}\alpha$ (2)

$\vdash . *103·11 . (*104·031) . \supset$

$\vdash :. \mathrm{Hp} . \supset : \delta \in \mu^{(2)} . \equiv . (\exists \gamma) . \gamma \,\mathrm{sm}\, \alpha . \gamma \in t\text{‘}\alpha . \delta \,\mathrm{sm}\, \gamma . \delta \in t\text{‘}t^2\text{‘}\gamma .$

$[*102·84.*63·16] \quad \equiv . \delta \,\mathrm{sm}\, \alpha . \delta \in t\text{‘}t^2\text{‘}\alpha .$

$[*104·11] \quad \equiv . \delta \in \mathrm{N}^2\mathrm{c}\text{‘}\alpha$ (3)

$\vdash . (1) . (2) . (3) . *104·24 . \supset \vdash .$ Prop

***104·33.** $\vdash :. \mu \in \mathrm{NC} . \supset : \mu = \mathrm{N}_0\mathrm{c}\text{‘}\alpha . \equiv . \mu^{(2)} = \mathrm{N}^2\mathrm{c}\text{‘}\alpha$

Dem.

$\vdash . *104·27 . \supset \vdash :. \mathrm{Hp} . \supset : \mu = \mathrm{N}_0\mathrm{c}\text{‘}\alpha . \equiv . \mu^{(1)} = \mathrm{N}^1\mathrm{c}\text{‘}\alpha .$

$[*104·24] \quad \equiv . \mu^{(1)} = \mathrm{N}_0\mathrm{c}\text{‘}\iota\text{“}\alpha .$

$[*104·27·141.*103·13] \quad \equiv . \{\mu^{(1)}\}^{(1)} = \mathrm{N}^1\mathrm{c}\text{‘}\iota\text{“}\alpha .$

$[*104·32·24] \quad \equiv . \mu^{(2)} = \mathrm{N}^2\mathrm{c}\text{‘}\alpha :. \supset \vdash .$ Prop

***104·34.** $\vdash : \varpi \in \mathrm{N}^2\mathrm{C} . \equiv . (\exists \nu) . \nu \in \mathrm{N}^1\mathrm{C} . \varpi = \nu^{(1)} . \equiv . (\exists \mu) . \mu \in \mathrm{N}_0\mathrm{C} . \varpi = \mu^{(2)}$

Dem.

$\vdash . *104·32 . \supset$

$\vdash : \varpi = \mathrm{N}^2\mathrm{c}\text{‘}\alpha . \mu = \mathrm{N}_0\mathrm{c}\text{‘}\alpha . \supset . \varpi = \mu^{(2)} . \mu \in \mathrm{N}_0\mathrm{C}$ (1)

$\vdash . (1) . *100\cdot2 . *10\cdot11\cdot28\cdot35 . \supset$

$\vdash : (\exists\alpha) . \varpi = N^2c`\alpha . \supset . (\exists\mu) . \mu \epsilon N_0C . \varpi = \mu^{(2)}$ (2)

$\vdash . *104\cdot32 . \quad \supset \vdash : \mu = N_0c`\alpha . \varpi = \mu^{(2)} . \supset . \varpi = N^2c`\alpha .$

$[*104\cdot15.*103\cdot2] \supset \vdash : \mu \epsilon N_0C . \varpi = \mu^{(2)} . \supset . \varpi \epsilon N^2C$ (3)

$\vdash . (2) . (3) . \quad \supset \vdash : \varpi \epsilon N^2C . \equiv . (\exists\mu) . \mu \epsilon N_0C . \varpi = \mu^{(2)} .$ (4)

$[*104\cdot32] \quad \equiv . (\exists\mu) . \mu \epsilon N_0C . \varpi = \{\mu^{(1)}\}^{(1)} .$

$[*13\cdot195] \quad \equiv . (\exists\mu, \nu) . \mu \epsilon N_0C . \nu = \mu^{(1)} . \varpi = \nu^{(1)} .$

$[*104\cdot29] \quad \equiv . (\exists\nu) . \nu \epsilon N^1C . \varpi = \nu^{(1)}$ (5)

$\vdash . (4) . (5) . \supset \vdash .$ Prop

***104·35.** $\vdash . N^2C \subset N^1C . N^2C \subset N_0C \quad [*104\cdot311\cdot13\cdot15]$

***104·36.** $\vdash : \gamma \epsilon N^2c`\alpha . \gamma \epsilon N^1c`\beta . \supset . \beta \epsilon N^1c`\alpha . N^1c`\alpha = N_0c`\beta$

Dem.

$\vdash . *104\cdot1\cdot11 . \supset \vdash : Hp . \supset . \gamma \epsilon Nc`\alpha . \gamma \epsilon t`t^{2}`\alpha . \gamma \epsilon Nc`\beta . \gamma \epsilon t`t`\beta .$

$[*100\cdot34.*63\cdot16] \quad \supset . Nc`\alpha = Nc`\beta . t`t^{2}`\alpha = t`t`\beta .$

$[*63\cdot35\cdot15] \quad \supset . Nc`\alpha = Nc`\beta . t^{2}`\alpha = t`\beta .$

$[(*104\cdot01.*103\cdot01)] \quad \supset . N^1c`\alpha = N_0c`\beta$ (1)

$\vdash . (1) . *103\cdot12 . \supset \vdash .$ Prop

***104·37.** $\vdash : N^2c`\alpha = N^1c`\beta . \equiv . N^1c`\alpha = N_0c`\beta$

Dem.

$\vdash . *104\cdot21 . \supset \vdash : N^2c`\alpha = N^1c`\beta . \supset . \exists ! N^2c`\alpha \cap N^1c`\beta .$

$[*104\cdot36] \quad \supset . N^1c`\alpha = N_0c`\beta$ (1)

$\vdash . (1) . *104\cdot123 . \supset \vdash .$ Prop

The following propositions are concerned with the proof that, given any two cardinals μ and ν, of the same type, we can find two mutually exclusive classes one of which has μ terms while the other has ν terms. The proof requires that we should raise the types of both μ and ν one degree above that in which they were originally given, *i.e.* that we should turn μ and ν into $\mu^{(1)}$ and $\nu^{(1)}$. Thus, for example, suppose the total number of individuals in the universe were finite (a supposition which is consistent with our primitive propositions), and suppose μ were this number. Then unless $\nu = 0$, a class of ν individuals will be an existent sub-class of the only class which consists of μ individuals, and therefore we shall have

$$\alpha \epsilon \mu . \beta \epsilon \nu . \supset_{\alpha,\beta} . \exists ! \alpha \cap \beta .$$

But if we consider classes of μ *classes* and ν *classes*, we shall always be able to find a γ and a δ such that

$$\gamma \epsilon \mu^{(1)} . \delta \epsilon \nu^{(1)} . \gamma \cap \delta = \Lambda .$$

The existence of such a γ and δ is important in connection with the arithmetical operations, and is therefore proved here.

***104·4.** $\vdash :. x \in \alpha . x \neq y . x \neq z . y \neq z : (w) . w_\iota = \hat{a}\hat{u}(\alpha = \iota\text{‘}w \cup \iota\text{‘}u) : \supset .$
$x_\iota\text{“}(\alpha - \iota\text{‘}x) \cup \iota\text{‘}y_\iota\text{‘}z \in \mathrm{N^1c}\text{‘}\alpha \cap \mathrm{Cl}\text{‘}2$

Dem.

$\vdash . *100{\cdot}61 . \supset \vdash : \mathrm{Hp} . \supset . x_\iota\text{“}(\alpha - \iota\text{‘}x) \,\mathrm{sm}\, (\alpha - \iota\text{‘}x)$ (1)

$\vdash . *73{\cdot}43 . \supset \vdash : \mathrm{Hp} . \supset . \iota\text{‘}y_\iota\text{‘}z \,\mathrm{sm}\, \iota\text{‘}x$ (2)

$\vdash . *51{\cdot}232 . \mathrm{Transp} . \supset \vdash : \mathrm{Hp} . \supset . x \sim\in y_\iota\text{‘}z$ (3)

$\vdash . *51{\cdot}232 . \supset \vdash : \mathrm{Hp} . \gamma \in x_\iota\text{“}(\alpha - \iota\text{‘}x) . \supset . x \in \gamma$ (4)

$\vdash . (3) . (4) . \supset \vdash : \mathrm{Hp} . \supset . y_\iota\text{‘}z \sim\in x_\iota\text{“}(\alpha - \iota\text{‘}x) .$

$[*51{\cdot}211] \quad \supset . x_\iota\text{“}(\alpha - \iota\text{‘}x) \cap \iota\text{‘}y_\iota\text{‘}z = \Lambda$ (5)

$\vdash . *51{\cdot}21{\cdot}211 . \supset \vdash . (\alpha - \iota\text{‘}x) \cap \iota\text{‘}x = \Lambda$ (6)

$\vdash . (1) . (2) . (5) . (6) . *73{\cdot}71 . *51{\cdot}221 . \supset$

$\vdash : \mathrm{Hp} . \supset . x_\iota\text{“}(\alpha - \iota\text{‘}x) \cup \iota\text{‘}y_\iota\text{‘}z \,\mathrm{sm}\, \alpha$ (7)

$\vdash . *63{\cdot}101{\cdot}16 . *51{\cdot}232{\cdot}16 . \supset$

$\vdash : \mathrm{Hp} . \quad \supset . t\text{‘}x = t\text{‘}y . x \in \alpha . y \in y_\iota\text{‘}z . y_\iota\text{‘}z \in x_\iota\text{“}(\alpha - \iota\text{‘}x) \cup \iota\text{‘}y_\iota\text{‘}z .$

$[*63{\cdot}53{\cdot}2] \quad \supset . t^2\text{‘}x = t\text{‘}\alpha . t^2\text{‘}y = t_0\text{‘}\{x_\iota\text{“}(\alpha - \iota\text{‘}x) \cup \iota\text{‘}y_\iota\text{‘}z\} . t^2\text{‘}x = t^2\text{‘}y .$

$[*13{\cdot}17] \quad \supset . t\text{‘}\alpha = t_0\text{‘}\{x_\iota\text{“}(\alpha - \iota\text{‘}x) \cup \iota\text{‘}y_\iota\text{‘}z\} .$

$[*63{\cdot}105] \quad \supset . x_\iota\text{“}(\alpha - \iota\text{‘}x) \cup \iota\text{‘}y_\iota\text{‘}z \subset t\text{‘}\alpha$ (8)

$\vdash . *54{\cdot}26 . \supset \vdash : \mathrm{Hp} . \supset . x_\iota\text{“}(\alpha - \iota\text{‘}x) \cup \iota\text{‘}y_\iota\text{‘}z \subset 2$ (9)

$\vdash . (7) . (8) . (9) . *104{\cdot}101 . \supset \vdash . \mathrm{Prop}$

***104·41.** $\vdash :. t\text{‘}\alpha = t\text{‘}\beta : (\exists x, y, z) . x \in \alpha . x \neq y . x \neq z . y \neq z : \supset .$
$(\exists \gamma, \delta) . \gamma \in \mathrm{N^1c}\text{‘}\alpha . \delta \in \mathrm{N^1c}\text{‘}\beta . \gamma \cap \delta = \Lambda$

Dem.

$\vdash . *104{\cdot}4{\cdot}2 . *52{\cdot}3 . \supset$

$\vdash : \mathrm{Hp} . \mathrm{Hp}\, *104{\cdot}4 . \supset . (\exists x, y, z) . x_\iota\text{“}(\alpha - \iota\text{‘}x) \cup \iota\text{‘}y_\iota\text{‘}z \in \mathrm{N^1c}\text{‘}\alpha \cap \mathrm{Cl}\text{‘}2 .$
$\iota\text{“}\beta \in \mathrm{N^1c}\text{‘}\beta \cap \mathrm{Cl}\text{‘}1 .$

$[*13{\cdot}22] \quad \supset . (\exists x, y, z, \gamma, \delta) . \gamma = x_\iota\text{“}(\alpha - \iota\text{‘}x) \cup \iota\text{‘}y_\iota\text{‘}z . \delta = \iota\text{“}\beta .$
$\gamma \in \mathrm{N^1c}\text{‘}\alpha \cap \mathrm{Cl}\text{‘}2 . \delta \in \mathrm{N^1c}\text{‘}\beta \cap \mathrm{Cl}\text{‘}1 .$

$[*11{\cdot}55] \quad \supset . (\exists \gamma, \delta) . \gamma \in \mathrm{N^1c}\text{‘}\alpha \cap \mathrm{Cl}\text{‘}2 . \delta \in \mathrm{N^1c}\text{‘}\beta \cap \mathrm{Cl}\text{‘}1$ (1)

$\vdash . (1) . *101{\cdot}35 . \supset \vdash . \mathrm{Prop}$

This proposition proves the desired conclusions provided $\exists ! \alpha$, and $t_0\text{‘}\alpha$ consists of at least three terms. The following propositions deal with the cases in which this hypothesis is not verified.

***104·411.** $\vdash : t\text{‘}\alpha = t\text{‘}\beta . \alpha \in 0 . \gamma = \Lambda_\alpha . \delta = \iota\text{“}\beta . \supset . \gamma \in \mathrm{N^1c}\text{‘}\alpha . \delta \in \mathrm{N^1c}\text{‘}\beta . \gamma \cap \delta = \Lambda$

Dem.

$\vdash . *73{\cdot}47 . \supset \vdash : \mathrm{Hp} . \supset . \gamma \,\mathrm{sm}\, \alpha$ (1)

$*22{\cdot}43 . (*65{\cdot}01) . \supset \vdash : \mathrm{Hp} . \supset . \gamma \subset t\text{‘}\alpha$ (2)

$\vdash . (1) . (2) . *104{\cdot}101 . \supset \vdash : \mathrm{Hp} . \supset . \gamma \in \mathrm{N^1c}\text{‘}\alpha$ (3)

$\vdash . (3) . *104{\cdot}2 . *24{\cdot}23 . \supset \vdash . \mathrm{Prop}$

***104·412.** $\vdash : t\text{‘}\alpha = t\text{‘}\beta . \alpha = \iota\text{‘}x . \gamma = \iota\text{‘}\Lambda_x . \delta = \iota\text{“}\beta . \supset .$
$\gamma \in \mathrm{N^1c}\text{‘}\alpha . \delta \in \mathrm{N^1c}\text{‘}\beta . \gamma \cap \delta = \Lambda$

Dem.

$\vdash . *73{\cdot}43 . \supset \vdash : \mathrm{Hp} . \supset . \gamma \,\mathrm{sm}\, \alpha$ (1)

$\vdash . *63{\cdot}61{\cdot}103 . \supset \vdash : \mathrm{Hp} . \supset . \alpha \in t^2\text{‘}x$ (2)

$\vdash . *22\cdot43 . (*65\cdot01) . \quad \supset \vdash : \mathrm{Hp} . \xi \epsilon \gamma . \supset_\xi . \xi \subset t'x .$

$[*63\cdot5] \quad \supset_\xi . \xi \epsilon t^{2}{}'x .$

$[(2).*63\cdot13] \quad \supset_\xi . \xi \epsilon t'\alpha \quad (3)$

$\vdash . (1) . (3) . *104\cdot101 . \supset \vdash : \mathrm{Hp} . \supset . \gamma \epsilon \mathrm{N}^1\mathrm{c}'\alpha \quad (4)$

$\vdash . *101\cdot23 . \quad \supset \vdash : \mathrm{Hp} . \supset . \gamma \cap \delta = \Lambda \quad (5)$

$\vdash . (4) . (5) . *104\cdot2 . \supset \vdash . \mathrm{Prop}$

***104·413.** $\vdash : t'\alpha = t'\beta . \alpha = \iota'x \cup \iota'y . x \neq y . \gamma = \iota'\Lambda \cup \iota'(\iota'x \cup \iota'y) . \delta = \iota``\beta . \supset .$
$\gamma \epsilon \mathrm{N}^1\mathrm{c}'\alpha . \delta \epsilon \mathrm{N}^1\mathrm{c}'\beta . \gamma \cap \delta = \Lambda$

Dem.

$\vdash . *54\cdot26 . \quad \supset \vdash : \mathrm{Hp} . \supset . \iota'x \cup \iota'y \epsilon 2 . \quad (1)$

$[*101\cdot35] \quad \supset . \Lambda \neq \iota'x \cup \iota'y .$

$[*54\cdot26] \quad \supset . \iota'\Lambda \cup \iota'(\iota'x \cup \iota'y) \epsilon 2 .$

$[*101\cdot3] \quad \supset . \iota'\Lambda \cup \iota'(\iota'x \cup \iota'y) \epsilon \mathrm{Nc}'(\iota'x \cup \iota'y) \quad (2)$

$\vdash . *51\cdot16 . \quad \supset \vdash : \mathrm{Hp} . \supset . \alpha \epsilon \gamma .$

$[*63\cdot5] \quad \supset . \gamma \subset t'\alpha \quad (3)$

$\vdash . (2) . (3) . *104\cdot1 . \quad \supset \vdash : \mathrm{Hp} . \supset . \gamma \epsilon \mathrm{N}^1\mathrm{c}'\alpha \quad (4)$

$\vdash . *52\cdot21\cdot3 . \quad \supset \vdash . \Lambda \sim\epsilon \iota``\beta \quad (5)$

$\vdash . (1) . *52\cdot3 . *54\cdot25 . \supset \vdash : \mathrm{Hp} . \supset . \iota'x \cup \iota'y \sim\epsilon \iota``\beta \quad (6)$

$\vdash . (5) . (6) . \quad \supset \vdash : \mathrm{Hp} . \supset . \gamma \cap \delta = \Lambda \quad (7)$

$\vdash . (4) . (7) . *104\cdot2 . \supset \vdash . \mathrm{Prop}$

***104·42.** $\vdash : t'\alpha = t'\beta . \alpha \epsilon 0 \cup 1 \cup 2 . \supset . (\exists \gamma, \delta) . \gamma \epsilon \mathrm{N}^1\mathrm{c}'\alpha . \delta \epsilon \mathrm{N}^1\mathrm{c}'\beta . \gamma \cap \delta = \Lambda$
[*104·411·412·413 . *52·1 . *54·101]

***104·43.** $\vdash : t'\alpha = t'\beta . \supset . (\exists \gamma, \delta) . \gamma \epsilon \mathrm{N}^1\mathrm{c}'\alpha . \delta \epsilon \mathrm{N}^1\mathrm{c}'\beta . \gamma \cap \delta = \Lambda$

Dem.

$\vdash . *54\cdot56 . \supset$

$\vdash : \mathrm{Hp} . \alpha \sim\epsilon 0 \cup 1 \cup 2 . \supset . (\exists x, y, z) . x, y, z \epsilon \alpha . x \neq y . x \neq z . y \neq z .$

$[*104\cdot41] \quad \supset . (\exists \gamma, \delta) . \gamma \epsilon \mathrm{N}^1\mathrm{c}'\alpha . \delta \epsilon \mathrm{N}^1\mathrm{c}'\beta . \gamma \cap \delta = \Lambda \quad (1)$

$\vdash . (1) . *104\cdot42 . \supset \vdash . \mathrm{Prop}$

The above proposition gives the desired result. The following propositions re-state this result in other forms.

***104·44.** $\vdash : \mu, \nu \epsilon \mathrm{N}^1\mathrm{C} . t'\mu = t'\nu . \supset . (\exists \gamma, \delta) . \gamma \epsilon \mu . \delta \epsilon \nu . \gamma \cap \delta = \Lambda$
[*104·13·43]

***104·45.** $\vdash : \mu, \nu \epsilon \mathrm{N}_0\mathrm{C} . t'\mu = t'\nu . \supset . (\exists \gamma, \delta) . \gamma \epsilon \mu^{(1)} . \delta \epsilon \nu^{(1)} . \gamma \cap \delta = \Lambda$
[*104·29·44]

***104·46.** $\vdash : \mu, \nu \epsilon \mathrm{NC} - \iota'\Lambda . t'\mu = t'\nu . \supset . (\exists \gamma, \delta) . \gamma \epsilon \mu^{(1)} . \delta \epsilon \nu^{(1)} . \gamma \cap \delta = \Lambda$
[*104·28·44]

$*105$. DESCENDING CARDINALS

Summary of $*105$.

In this number, we consider cardinals generated by a relation of similarity which goes from a higher to a lower type, *i.e.* given any class of classes κ, we consider $\text{Nc}'\kappa$ in the type of members of κ (which we shall call $\text{N}_1\text{c}'\kappa$) or in some lower type. Thus *e.g.* we shall have

$$\kappa = \iota``\alpha \,.\, \supset \,.\, \alpha \in \text{N}_1\text{c}'\kappa,$$

where "$\text{N}_1\text{c}'\kappa$" means "classes similar to κ but of the next lower type." Similarly

$$\kappa = \iota``\iota``\alpha \,.\, \supset \,.\, \alpha \in \text{N}_2\text{c}'\kappa,$$

and so on. We shall have generally

$$\beta \in \text{N}_1\text{c}'\alpha \,.\, \equiv \alpha \in \text{N}^1\text{c}'\beta,$$

$$\beta \in \text{N}_2\text{c}'\alpha \,.\, \equiv \alpha \in \text{N}^2\text{c}'\beta,$$

and so on. The chief difference between ascending and descending cardinals is that Λ is one of the latter, but not one of the former. Otherwise the propositions of the present number are mostly analogous to corresponding propositions of $*104$.

On the analogy of the definitions in $*104$, we put

$$\text{N}_1\text{C} = \text{D}'\dot{\text{N}}_1\text{c} \qquad \text{Df},$$

$$\mu_{(1)} = \text{sm}``\mu \cap t_1{}'\mu \quad \text{Df},$$

with similar definitions for N_2C and $\mu_{(2)}$.

No proposition of the present number is ever referred to in the sequel, and the reader who is not interested in the subject may therefore omit it without detriment to what follows. The principal propositions proved are the following:

$*105\cdot25$. $\vdash \,.\, \text{N}_0\text{C} = \text{N}_1\text{C} - \iota'\Lambda$

$*105\cdot251$. $\vdash \,.\, \text{N}_0\text{C} = \text{N}_2\text{C} - \iota'\Lambda$

$*105\cdot26$. $\vdash \,.\, \text{N}_1\text{c}'t'\alpha = \Lambda$

Thus N_1C or N_2C, in any given type, only differs from N_0C in that type by the addition of Λ.

$*105\cdot3$. $\vdash : \mu = \text{N}_0\text{c}'\alpha \,.\, \supset \,.\, \mu_{(1)} = \text{N}_1\text{c}'\alpha$

$*105\cdot322$. $\vdash :. \exists ! \,\text{N}_1\text{c}'\alpha \,.\, \supset : \text{N}_1\text{c}'\alpha = \text{N}_1\text{c}'\beta \,.\, \equiv \,.\, \text{N}_0\text{c}'\alpha = \text{N}_0\text{c}'\beta$

$*105\cdot34$. $\vdash :. \mu \in \text{NC} \,.\, \exists ! \,\mu_{(1)} \,.\, \supset : \mu_{(1)} = \text{N}_1\text{c}'\alpha \,.\, \equiv \,.\, \mu = \text{N}_0\text{c}'\alpha$

$*105\cdot35$. $\vdash :. \mu \in \text{NC} \,.\, \nu \in \text{N}_0\text{C} \,.\, \supset : \mu = \nu^{(1)} \,.\, \equiv \,.\, \mu_{(1)} = \nu$

$*105\cdot38$. $\vdash \,.\, \{\mu_{(1)}\}_{(1)} = \mu_{(2)}$

$*105{\cdot}01.\quad N_1c`\alpha = Nc`\alpha \cap t`t_1`\alpha \quad Df$

We might write

$$N_1c`\alpha = Nc`\alpha \cap t_0`\alpha \quad Df,$$

which would be equivalent to the above. But we choose the above form for the sake of uniformity. If s is any suffix, we put, provided $t_s`\alpha$ has been defined

$$N_sc`\alpha = Nc`\alpha \cap t`t_s`\alpha \quad Df,$$

and if i is any index for which $t^i`\alpha$ has been defined, we put

$$N^ic`\alpha = Nc`\alpha \cap t`t^i`\alpha \quad Df.$$

Thus for the sake of uniformity it is better, in the above definition $*105{\cdot}01$ to write "$t`t_1`\alpha$" rather than "$t_0`\alpha$."

$*105{\cdot}011.\quad N_2c`\alpha = Nc`\alpha \cap t`t_2`\alpha \quad Df$

$*105{\cdot}02.\quad N_1C = D`N_1c \quad Df$

$*105{\cdot}021.\quad N_2C = D`N_2c \quad Df$

$*105{\cdot}03.\quad \mu_{(1)} = sm``\mu \cap t_1`\mu \quad Df$

$*105{\cdot}031.\quad \mu_{(2)} = sm``\mu \cap t_2`\mu \quad Df$

$*105{\cdot}1.\quad \vdash . N_1c`\alpha = Nc`\alpha \cap t_0`\alpha \quad [*63{\cdot}383 . (*105{\cdot}01)]$

$*105{\cdot}101.\quad \vdash . N_2c`\alpha = Nc`\alpha \cap t_1`\alpha \quad [*63{\cdot}41 . (*105{\cdot}011)]$

$*105{\cdot}11.\quad \vdash : \beta \in N_1c`\alpha . \equiv . \beta \in Nc`\alpha . \beta \in t_0`\alpha . \equiv . \beta \,sm\, \alpha . \beta \in t_0`\alpha . \equiv . \beta \,sm\, \alpha . \beta \subset t_1`\alpha$
$[*105{\cdot}1 . *100{\cdot}31 . *63{\cdot}51]$

$*105{\cdot}111.\quad \vdash : \beta \in N_2c`\alpha . \equiv . \beta \in Nc`\alpha . \beta \in t_1`\alpha . \equiv . \beta \,sm\, \alpha . \beta \in t_1`\alpha . \equiv . \beta \,sm\, \alpha . \beta \subset t_2`\alpha$
$[*105{\cdot}101 . *100{\cdot}31 . *63{\cdot}52]$

$*105{\cdot}12.\quad \vdash : \beta \in N_1c`\alpha . \equiv . \beta \in Nc`\alpha . \alpha \subset t`\beta . \equiv . \beta \,sm\, \alpha . \alpha \subset t`\beta . \equiv . \alpha \in N^1c`\beta$
$[*105{\cdot}11 . *63{\cdot}51 . *104{\cdot}1]$

$*105{\cdot}121.\quad \vdash : \beta \in N_2c`\alpha . \equiv . \beta \in Nc`\alpha . \alpha \subset t^{2}`\beta . \equiv . \beta \,sm\, \alpha . \alpha \subset t^{2}`\beta . \equiv . \alpha \in N^2c`\beta$
$[*105{\cdot}111 . *63{\cdot}52 . *104{\cdot}11]$

$*105{\cdot}13.\quad \vdash . N_1c`\alpha = Nc\,(t_1`\alpha)`\alpha = Nc\,\{(t_1`\alpha)_\alpha\}`\alpha \quad [*102{\cdot}6 . (*105{\cdot}01)]$

$*105{\cdot}131.\quad \vdash . N_2c`\alpha = Nc\,(t_2`\alpha)`\alpha = Nc\,\{(t_2`\alpha)_\alpha\}`\alpha \quad [*102{\cdot}6 . (*105{\cdot}011)]$

$*105{\cdot}14.\quad \vdash : \alpha \in t_0`\beta . \supset . N_1c`\beta = Nc\,(\alpha)`\beta = Nc\,(\alpha_\beta)`\beta$

Dem.

$\vdash . *63{\cdot}22 . \supset \vdash : Hp . \supset . t`\alpha = t_0`\beta .$

$[*105{\cdot}1] \qquad \supset . N_1c`\beta = Nc`\beta \cap t`\alpha \qquad (1)$

$\vdash . (1) . *102{\cdot}6 . \supset \vdash . Prop$

$*105{\cdot}141.\quad \vdash : \alpha \in t_1`\beta . \supset . N_2c`\beta = Nc\,(\alpha)`\beta = Nc\,(\alpha_\beta)`\beta \quad$ [Proof as in $*105{\cdot}14$]

$*105{\cdot}142.\quad \vdash : \beta \subset t`\alpha . \supset . N_1c`\beta = Nc\,(\alpha)`\beta = Nc\,(\alpha_\beta)`\beta \quad [*105{\cdot}14 . *63{\cdot}51]$

$*105{\cdot}143.\quad \vdash : \beta \subset t^{2}`\alpha . \supset . N_2c`\beta = Nc\,(\alpha)`\beta = Nc\,(\alpha_\beta)`\beta \quad [*105{\cdot}141 . *63{\cdot}52]$

$*105{\cdot}15.\quad \vdash : \mu \in N_1C . \equiv . (\exists \alpha) . \mu = N_1c`\alpha \quad [*100{\cdot}22 . *71{\cdot}41 . (*105{\cdot}02)]$

$*105{\cdot}151.\quad \vdash : \mu \in N_2C . \equiv . (\exists \alpha) . \mu = N_2c`\alpha$

∗105·16. $\vdash : \delta \in \mu_{(1)} . \equiv . (\exists \gamma) . \gamma \in \mu . \delta \operatorname{sm} \gamma . \delta \in t_1{}^{\backprime}\mu .$
$\equiv . (\exists \gamma) . \gamma \in \mu . \delta \operatorname{sm} \gamma . \delta \in t_0{}^{\backprime}\gamma .$
$\equiv . (\exists \gamma) . \gamma \in \mu . \delta \operatorname{sm} \gamma . \gamma \subset t{}^{\backprime}\delta$ [∗37·1 . ∗63·51·54]

∗105·161. $\vdash : \delta \in \mu_{(2)} . \equiv . (\exists \gamma) . \gamma \in \mu . \delta \operatorname{sm} \gamma . \delta \in t_2{}^{\backprime}\mu .$
$\equiv . (\exists \gamma) . \gamma \in \mu . \delta \operatorname{sm} \gamma . \delta \in t_1{}^{\backprime}\gamma .$
$\equiv . (\exists \gamma) . \gamma \in \mu . \delta \operatorname{sm} \gamma . \gamma \subset t^2{}^{\backprime}\delta$ [∗37·1 . ∗63·52·55]

In what follows, propositions concerning N_2c or N_2C have proofs exactly analogous to those of the corresponding propositions concerning N_1c or N_1C.

∗105·2. $\vdash . N_0c{}^{\backprime}\alpha = N_1c{}^{\backprime}\iota{}^{\backprime\backprime}\alpha$

Dem.

$\vdash .$ ∗105·12 . ∗104·2 . $\supset \vdash . \alpha \in N_1c{}^{\backprime}\iota{}^{\backprime\backprime}\alpha .$
[∗103·26] $\supset \vdash . N_0c{}^{\backprime}\alpha = N_1c{}^{\backprime}\iota{}^{\backprime\backprime}\alpha$

∗105·201. $\vdash . N_0c{}^{\backprime}\alpha = N_2c{}^{\backprime}\iota{}^{\backprime\backprime}\iota{}^{\backprime\backprime}\alpha$

∗105·21. $\vdash . N_0C \subset N_1C$ [∗105·2·15]

∗105·211. $\vdash . N_0C \subset N_2C$

∗105·22. $\vdash : \gamma \in N_1c{}^{\backprime}\delta . \supset . N_1c{}^{\backprime}\delta = N_0c{}^{\backprime}\gamma$ [∗103·26]

∗105·221. $\vdash : \gamma \in N_2c{}^{\backprime}\delta . \supset . N_2c{}^{\backprime}\delta = N_0c{}^{\backprime}\gamma$

∗105·23. $\vdash : \exists ! N_1c{}^{\backprime}\delta . \supset . N_1c{}^{\backprime}\delta \in N_0C$ [∗105·22]

∗105·231. $\vdash : \exists ! N_2c{}^{\backprime}\delta . \supset . N_2c{}^{\backprime}\delta \in N_0C$

∗105·24. $\vdash . N_1C - \iota{}^{\backprime}\Lambda \subset N_0C$ [∗105·23]

∗105·241. $\vdash . N_2C - \iota{}^{\backprime}\Lambda \subset N_0C$

∗105·25. $\vdash . N_0C = N_1C - \iota{}^{\backprime}\Lambda$ [∗105·21·24 . ∗103·23]

∗105·251. $\vdash . N_0C = N_2C - \iota{}^{\backprime}\Lambda$

∗105·252. $\vdash . N_1c{}^{\backprime}\beta = N_2c{}^{\backprime}\iota{}^{\backprime\backprime}\beta$

Dem.

$\vdash .$ ∗105·111 . $\supset \vdash : \alpha \in N_2c{}^{\backprime}\iota{}^{\backprime\backprime}\beta . \equiv . \alpha \operatorname{sm} \iota{}^{\backprime\backprime}\beta . \alpha \in t_1{}^{\backprime}\iota{}^{\backprime\backprime}\beta .$
[∗73·41.∗63·64·54] $\equiv . \alpha \operatorname{sm} \beta . \alpha \in t_0{}^{\backprime}\beta .$
[∗105·11] $\equiv . \alpha \in N_1c{}^{\backprime}\beta : \supset \vdash .$ Prop

∗105·26. $\vdash . N_1c{}^{\backprime}t{}^{\backprime}\alpha = \Lambda$

Dem.

$\vdash .$ ∗105·142 . $\supset \vdash . N_1c{}^{\backprime}t{}^{\backprime}\alpha = Nc\,(\alpha){}^{\backprime}t{}^{\backprime}\alpha$ (1)
$\vdash .$ (1) . ∗102·73 . $\supset \vdash .$ Prop

∗105·261. $\vdash . N_2c{}^{\backprime}\iota{}^{\backprime\backprime}t{}^{\backprime}\alpha = \Lambda$ [∗105·26·252]

∗105·27. $\vdash . \Lambda \in N_1C$ [∗105·26]

∗105·271. $\vdash . \Lambda \in N_2C$

∗105·28. $\vdash . N_1C = N_0C \cup \iota{}^{\backprime}\Lambda$ [∗105·25·27]

∗105·281. $\vdash . N_2C = N_1C = N_0C \cup \iota{}^{\backprime}\Lambda$

∗105·29. $\vdash . NC \subset N_1C . NC \subset N_2C$ [∗105·281 . ∗103·34]

*105·3. $\vdash : \mu = \mathrm{N_0c}`\alpha . \supset . \mu_{(1)} = \mathrm{N_1c}`\alpha$

Dem.

$\vdash . *103·4 . (*105·03) . \supset \vdash : \mu = \mathrm{N_0c}`\alpha . \supset . \mu_{(1)} = \mathrm{Nc}`\alpha \cap t_1`\mu$ (1)

$\vdash . *103·12 . \supset \vdash : \mu = \mathrm{N_0c}`\alpha . \supset . \alpha \in \mu .$

[*63·105] $\supset . \alpha \in t_0`\mu .$

[*63·54] $\supset . t_0`\alpha = t_1`\mu$ (2)

$\vdash . (1) . (2) . \supset \vdash : \mu = \mathrm{N_0c}`\alpha . \supset . \mu_{(1)} = \mathrm{Nc}`\alpha \cap t_0`\alpha .$

[*105·1] $\supset . \mu_{(1)} = \mathrm{N_1c}`\alpha : \supset \vdash . \mathrm{Prop}$

*105·301. $\vdash : \mu = \mathrm{N_0c}`\alpha . \supset . \mu_{(2)} = \mathrm{N_2c}`\alpha$

*105·31. $\vdash : \mu \in \mathrm{N_0C} . \supset . \mu_{(1)} \in \mathrm{N_1C}$ [*105·3·15 . *103·2]

*105·311. $\vdash : \mu \in \mathrm{N_0C} . \supset . \mu_{(2)} \in \mathrm{N_2C}$

*105·312. $\vdash : \gamma \in \mathrm{N_1c}`\alpha . \supset . \alpha \in \mathrm{N^1c}`\gamma . \mathrm{N^1c}`\gamma = \mathrm{N_0c}`\alpha$ [*105·12 . *103·26]

*105·313. $\vdash : \gamma \in \mathrm{N_2c}`\alpha . \supset . \alpha \in \mathrm{N^2c}`\gamma . \mathrm{N^2c}`\gamma = \mathrm{N_0c}`\alpha$

*105·314. $\vdash : \mathrm{N_1c}`\alpha = \mathrm{N_0c}`\gamma . \supset . \mathrm{N_0c}`\alpha = \mathrm{N^1c}`\gamma$ [*105·312 . *103·12]

*105·315. $\vdash : \mathrm{N_2c}`\alpha = \mathrm{N_0c}`\gamma . \supset . \mathrm{N_0c}`\alpha = \mathrm{N^2c}`\gamma$

*105·316. $\vdash : \exists ! \mathrm{N_1c}`\alpha . \mathrm{N_1c}`\alpha = \mathrm{N_1c}`\beta . \supset . \mathrm{N_0c}`\alpha = \mathrm{N_0c}`\beta$

Dem.

$\vdash . *105·312 . \supset \vdash : \gamma \in \mathrm{N_1c}`\alpha . \mathrm{N_1c}`\alpha = \mathrm{N_1c}`\beta . \supset . \mathrm{N^1c}`\gamma = \mathrm{N_0c}`\alpha . \mathrm{N^1c}`\gamma = \mathrm{N_0c}`\beta .$

[*13·171] $\supset . \mathrm{N_0c}`\alpha = \mathrm{N_0c}`\beta$ (1)

$\vdash . (1) . *10·11·23·35 . \supset \vdash . \mathrm{Prop}$

*105·317. $\vdash : \exists ! \mathrm{N_2c}`\alpha . \mathrm{N_2c}`\alpha = \mathrm{N_2c}`\beta . \supset . \mathrm{N_0c}`\alpha = \mathrm{N_0c}`\beta$

*105·32. $\vdash : \mathrm{N_0c}`\alpha = \mathrm{N_0c}`\beta . \supset . \mathrm{N_1c}`\alpha = \mathrm{N_1c}`\beta$

Dem.

$\vdash . *103·41 . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{Nc}\,(t_1`\alpha)`\alpha = \mathrm{Nc}\,(t_1`\alpha)`\beta$ (1)

$\vdash . *103·14 . \supset \vdash : \mathrm{Hp} . \supset . \beta \in t`\alpha .$

[*63·16·36] $\supset . t_1`\alpha = t_1`\beta$ (2)

$\vdash . (1) . (2) . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{Nc}\,(t_1`\alpha)`\alpha = \mathrm{Nc}\,(t_1`\beta)`\beta .$

[*105·13] $\supset . \mathrm{N_1c}`\alpha = \mathrm{N_1c}`\beta : \supset \vdash . \mathrm{Prop}$

*105·321. $\vdash : \mathrm{N_0c}`\alpha = \mathrm{N_0c}`\beta . \supset . \mathrm{N_2c}`\alpha = \mathrm{N_2c}`\beta$

*105·322. $\vdash :. \exists ! \mathrm{N_1c}`\alpha . \supset : \mathrm{N_1c}`\alpha = \mathrm{N_1c}`\beta . \equiv . \mathrm{N_0c}`\alpha = \mathrm{N_0c}`\beta$ [*105·316·32]

*105·323. $\vdash :. \exists ! \mathrm{N_2c}`\alpha . \supset : \mathrm{N_2c}`\alpha = \mathrm{N_2c}`\beta . \equiv . \mathrm{N_0c}`\alpha = \mathrm{N_0c}`\beta$

*105·324. $\vdash : \exists ! \mu_{(1)} . \supset . \exists ! \mu$ [*37·29 . (*105·03)]

*105·325. $\vdash : \exists ! \mu_{(2)} . \supset . \exists ! \mu$

***105·326.** $\vdash : \mu \in \mathrm{NC} . \mu_{(1)} = \mathrm{N_0c}‘\gamma . \supset . \mu = \mathrm{N^1c}‘\gamma$

Dem.

$\vdash . *103·26 . \quad \supset \vdash : \mathrm{Hp} . \alpha \in \mu . \supset . \mu = \mathrm{N_0c}‘\alpha .$ (1)

$[*105·3] \quad \supset . \mu_{(1)} = \mathrm{N_1c}‘\alpha .$

$[\mathrm{Hp}] \quad \supset . \mathrm{N_1c}‘\alpha = \mathrm{N_0c}‘\gamma .$

$[*105·314] \quad \supset . \mathrm{N_0c}‘\alpha = \mathrm{N^1c}‘\gamma .$

$[(1)] \quad \supset . \mu = \mathrm{N^1c}‘\gamma$ (2)

$\vdash . (2) . *10·11·23·35 . \supset \vdash : \mathrm{Hp} . \exists ! \mu . \supset . \mu = \mathrm{N^1c}‘\gamma$ (3)

$\vdash . (3) . *105·324 . *103·13 . \supset \vdash . \mathrm{Prop}$

***105·327.** $\vdash : \mu \in \mathrm{NC} . \mu_{(2)} = \mathrm{N_0c}‘\gamma . \supset . \mu = \mathrm{N^2c}‘\gamma$

***105·33.** $\vdash : \mu \in \mathrm{NC} . \exists ! \mu_{(1)} . \mu_{(1)} = \mathrm{N_1c}‘\alpha . \supset . \mu = \mathrm{N_0c}‘\alpha$

Dem.

$\vdash . *103·26 . \supset \vdash : \gamma \in \mu_{(1)} . \mu_{(1)} = \mathrm{N_1c}‘\alpha . \supset . \mathrm{N_1c}‘\alpha = \mathrm{N_0c}‘\gamma .$

$[*105·314] \quad \supset . \mathrm{N_0c}‘\alpha = \mathrm{N^1c}‘\gamma$ (1)

$\vdash . (1) . *105·326 . \supset$

$\vdash : \gamma \in \mu_{(1)} . \mu_{(1)} = \mathrm{N_1c}‘\alpha . \mu \in \mathrm{NC} . \supset . \mu = \mathrm{N_0c}‘\alpha$ (2)

$\vdash . (2) . *10·11·23·35 . \supset \vdash . \mathrm{Prop}$

***105·331.** $\vdash : \mu \in \mathrm{NC} . \exists ! \mu_{(2)} . \mu_{(2)} = \mathrm{N_2c}‘\alpha . \supset . \mu = \mathrm{N_0c}‘\alpha$

***105·34.** $\vdash :. \mu \in \mathrm{NC} . \exists ! \mu_{(1)} . \supset : \mu_{(1)} = \mathrm{N_1c}‘\alpha . \equiv . \mu = \mathrm{N_0c}‘\alpha$ [*105·33·3]

***105·341.** $\vdash :. \mu \in \mathrm{NC} . \exists ! \mu_{(2)} . \supset : \mu_{(2)} = \mathrm{N_2c}‘\alpha . \equiv . \mu = \mathrm{N_0c}‘\alpha$

***105·342.** $\vdash . \mu \in \mathrm{NC} . \supset . \mu_{(1)} \in \mathrm{N_1C}$

Dem.

$\vdash . *103·34 . \quad \supset \vdash : \mathrm{Hp} . \exists ! \mu . \supset . \mu \in \mathrm{N_0C} .$

$[*105·31] \quad \supset . \mu_{(1)} \in \mathrm{N_1C}$ (1)

$\vdash . *105·324 . \supset \vdash : \mathrm{Hp} . \sim \exists ! \mu . \supset . \sim \exists ! \mu_{(1)} .$

$[*105·27] \quad \supset . \mu_{(1)} \in \mathrm{N_1C}$ (2)

$\vdash . (1) . (2) . \quad \supset \vdash . \mathrm{Prop}$

***105·343.** $\vdash : \mu \in \mathrm{NC} . \supset . \mu_{(2)} \in \mathrm{N_2C}$

***105·344.** $\vdash : \mu = \mathrm{N^1c}‘\gamma . \supset . \mu_{(1)} = \mathrm{N_0c}‘\gamma$

Dem.

$\vdash . *104·24 . \supset \vdash : \mathrm{Hp} . \supset . \mu = \mathrm{N_0c}‘\iota‘‘\gamma .$

$[*105·3] \quad \supset . \mu_{(1)} = \mathrm{N_1c}‘\iota‘‘\gamma .$

$[*105·2] \quad \supset . \mu_{(1)} = \mathrm{N_0c}‘\gamma : \supset \vdash . \mathrm{Prop}$

***105·345.** $\vdash : \mu = \mathrm{N^2c}‘\gamma . \supset . \mu_{(2)} = \mathrm{N_0c}‘\gamma$

***105·35.** $\vdash :. \mu \in \mathrm{NC} . \nu \in \mathrm{N_0C} . \supset : \mu = \nu^{(1)} . \equiv . \mu_{(1)} = \nu$

Dem.

$\vdash . *105·326 . *104·26 . \supset$

$\vdash : \mu \in \mathrm{NC} . \nu = \mathrm{N_0c}‘\gamma . \mu_{(1)} = \nu . \supset . \mu = \mathrm{N^1c}‘\gamma . \nu^{(1)} = \mathrm{N^1c}‘\gamma .$

$[*13·172] \quad \supset . \mu = \nu^{(1)}$ (1)

$\vdash . *104·26 . \mathrm{Fact} . \supset$

$\vdash : \mu \in \mathrm{NC} . \nu = \mathrm{N_0c}‘\gamma . \mu = \nu^{(1)} . \supset . \mu = \mathrm{N^1c}‘\gamma . \nu = \mathrm{N_0c}‘\gamma .$

[*105·344] $\supset . \mu_{(1)} = \mathrm{N}_0\mathrm{c}`\gamma . \nu = \mathrm{N}_0\mathrm{c}`\gamma .$

[*13·172] $\supset . \mu_{(1)} = \nu$ (2)

$\vdash . (1) . (2) . \supset \vdash :. \mu \in \mathrm{NC} . \nu = \mathrm{N}_0\mathrm{c}`\gamma . \supset : \mu = \nu^{(1)} . \equiv . \mu_{(1)} = \nu$ (3)

$\vdash . (3) . *103·2 . \supset \vdash .$ Prop

***105·351.** $\vdash :. \mu \in \mathrm{NC} . \nu \in \mathrm{N}_0\mathrm{C} . \supset : \mu = \nu^{(2)} . \equiv . \mu_{(2)} = \nu$

***105·352.** $\vdash :. \mu, \nu \in \mathrm{NC} . \exists ! \nu . \supset : \mu = \nu^{(1)} . \equiv . \mu_{(1)} = \nu$ [*105·35 . *103·34]

***105·353.** $\vdash :. \mu, \nu \in \mathrm{NC} . \exists ! \nu . \supset : \mu = \nu^{(2)} . \equiv . \mu_{(2)} = \nu$

***105·354.** $\vdash : \nu \in \mathrm{NC} . \exists ! \nu . \supset . \{\nu^{(1)}\}_{(1)} = \nu$ [*105·352]

***105·355.** $\vdash : \nu \in \mathrm{NC} . \exists ! \nu . \supset . \{\nu^{(2)}\}_{(2)} = \nu$

***105·356.** $\vdash : \mu \in \mathrm{NC} . \exists ! \mu_{(1)} . \supset . \{\mu_{(1)}\}^{(1)} = \mu$ [*105·352]

***105·357.** $\vdash : \mu \in \mathrm{NC} . \exists ! \mu_{(2)} . \supset . \{\mu_{(2)}\}^{(2)} = \mu$

***105·36.** $\vdash : \beta \in \mathrm{N}_1\mathrm{c}`\alpha . \gamma \in \mathrm{N}_1\mathrm{c}`\beta . \supset . \gamma \in \mathrm{N}_2\mathrm{c}`\alpha$

Dem.

$\vdash . *105·11 . \supset \vdash : \mathrm{Hp} . \supset . \beta \,\mathrm{sm}\, \alpha . \beta \in t_0`\alpha . \gamma \,\mathrm{sm}\, \beta . \gamma \in t_0`\beta .$

[*73·32.*63·38] $\supset . \gamma \,\mathrm{sm}\, \alpha . \gamma \in t_1`\alpha .$

[*105·111] $\supset . \gamma \in \mathrm{N}_2\mathrm{c}`\alpha : \supset \vdash .$ Prop

***105·361.** $\vdash : \beta \in \mathrm{N}_1\mathrm{c}`\alpha . \gamma \in \mathrm{N}_2\mathrm{c}`\alpha . \supset . \gamma \in \mathrm{N}_1\mathrm{c}`\beta$

Dem.

$\vdash . *105·11·111 . \supset \vdash : \mathrm{Hp} . \supset . \beta \,\mathrm{sm}\, \alpha . \beta \in t_0`\alpha . \gamma \,\mathrm{sm}\, \alpha . \gamma \in t_1`\alpha .$

[*73·31·32] $\supset . \gamma \,\mathrm{sm}\, \beta . \beta \in t_0`\alpha . \gamma \in t_1`\alpha$ (1)

$\vdash . *63·54 . \quad \supset \vdash : \beta \in t_0`\alpha . \supset . t_0`\beta = t_1`\alpha$ (2)

$\vdash . (1) . (2) . \quad \supset \vdash : \mathrm{Hp} . \supset . \gamma \,\mathrm{sm}\, \beta . \gamma \in t_0`\beta .$

[*105·11] $\supset . \gamma \in \mathrm{N}_1\mathrm{c}`\beta : \supset \vdash .$ Prop

***105·362.** $\vdash : \beta \in \mathrm{N}_1\mathrm{c}`\alpha . \supset . \mathrm{N}_1\mathrm{c}`\beta = \mathrm{N}_2\mathrm{c}`\alpha$ [*105·36·361]

***105·37.** $\vdash : \mathrm{N}_0\mathrm{c}`\beta = \mathrm{N}_1\mathrm{c}`\alpha . \supset . \mathrm{N}_1\mathrm{c}`\beta = \mathrm{N}_2\mathrm{c}`\alpha$ [*105·362 . *103·12]

***105·371.** $\vdash : \exists ! \mu_{(2)} . \supset . \exists ! \mu_{(1)}$

Dem.

$\vdash . *63·381 . (*63·05) . \supset$

$\vdash : \gamma \,\mathrm{sm}\, \alpha . \alpha \in \mu . \gamma \in t_2`\mu . \supset . \gamma \,\mathrm{sm}\, \alpha . \alpha \in \mu . t`\gamma = t_2`\mu .$

[*73·41.*63·64] $\supset . \iota``\gamma \,\mathrm{sm}\, \alpha . \alpha \in \mu . t_0`\iota``\gamma = t_2`\mu .$

[*63·57] $\supset . \iota``\gamma \,\mathrm{sm}\, \alpha . \alpha \in \mu . t`\iota``\gamma = t_1`\mu .$

[*63·103] $\supset . \iota``\gamma \,\mathrm{sm}\, \alpha . \alpha \in \mu . \iota``\gamma \in t_1`\mu .$

[*105·16] $\supset . \iota``\gamma \in \mu_{(1)} .$

[*10·24] $\supset . \exists ! \mu_{(1)}$ (1)

$\vdash . (1) . *10·11·23 . \supset$

$\vdash : (\exists \alpha) . \gamma \,\mathrm{sm}\, \alpha . \alpha \in \mu . \gamma \in t_2`\mu . \supset . \exists ! \mu_{(1)}$ (2)

$\vdash . (2) . *105·161 . \supset \vdash : \gamma \in \mu_{(2)} . \supset . \exists ! \mu_{(1)}$ (3)

$\vdash . (3) . *10·11·23 . \supset \vdash .$ Prop

***105·372.** $\vdash : \mu_{(1)} = \Lambda . \supset . \mu_{(2)} = \Lambda$ [*105·371 . Transp]

***105·38.** $\vdash . \{\mu_{(1)}\}_{(1)} = \mu_{(2)}$

Dem.

$\vdash . *105·16 . \supset \vdash : \gamma \in \{\mu_{(1)}\}_{(1)} . \equiv . (\exists \beta) . \beta \in \mu_{(1)} . \gamma \,\mathrm{sm}\, \beta . \gamma \in t_0{}^{\backprime}\beta .$

[*105·16] $\equiv . (\exists \alpha, \beta) . \alpha \in \mu . \beta \,\mathrm{sm}\, \alpha . \beta \in t_0{}^{\backprime}\alpha . \gamma \,\mathrm{sm}\, \beta . \gamma \in t_0{}^{\backprime}\beta .$ (1)

[*73·32.*63·38] $\supset . (\exists \alpha) . \alpha \in \mu . \gamma \,\mathrm{sm}\, \alpha . \gamma \in t_1{}^{\backprime}\alpha$ (2)

$\vdash . *73·41 . *63·64·53·57 . \supset$

$\vdash : \alpha \in \mu . \gamma \,\mathrm{sm}\, \alpha . \gamma \in t_1{}^{\backprime}\alpha . \supset . \alpha \in \mu . \iota^{\backprime\backprime}\gamma \,\mathrm{sm}\, \alpha . \gamma \,\mathrm{sm}\, \iota^{\backprime\backprime}\gamma . \gamma \in t_0{}^{\backprime}\iota^{\backprime\backprime}\gamma . \iota^{\backprime\backprime}\gamma \in t_0{}^{\backprime}\alpha .$

[(1)] $\supset . \gamma \in \{\mu_{(1)}\}_{(1)}$ (3)

$\vdash . (2) . (3) . \supset \vdash : \gamma \in \{\mu_{(1)}\}_{(1)} . \equiv . (\exists \alpha) . \alpha \in \mu . \gamma \,\mathrm{sm}\, \alpha . \gamma \in t_1{}^{\backprime}\alpha .$

[*105·161] $\equiv . \gamma \in \mu_{(2)} : \supset \vdash . \mathrm{Prop}$

***105·4.** $\vdash : \gamma \in \mathrm{N_2c}{}^{\backprime}\alpha . \supset . \iota^{\backprime\backprime}\gamma \in \mathrm{N_1c}{}^{\backprime}\alpha$

Dem.

$\vdash . *105·111 . *73·41 . *63·64 . \supset \vdash : \mathrm{Hp} . \supset . \iota^{\backprime\backprime}\gamma \,\mathrm{sm}\, \alpha . \gamma \in t_1{}^{\backprime}\alpha . \gamma \in t_0{}^{\backprime}\iota^{\backprime\backprime}\gamma .$

[*63·41·383·16·55] $\supset . \iota^{\backprime\backprime}\gamma \,\mathrm{sm}\, \alpha . t_1{}^{\backprime}\alpha = t_0{}^{\backprime}\iota^{\backprime\backprime}\gamma .$

[*63·54] $\supset . \iota^{\backprime\backprime}\gamma \,\mathrm{sm}\, \alpha . \iota^{\backprime\backprime}\gamma \in t_0{}^{\backprime}\alpha .$

[*105·11] $\supset . \iota^{\backprime\backprime}\gamma \in \mathrm{N_1c}{}^{\backprime}\alpha : \supset \vdash . \mathrm{Prop}$

***105·41.** $\vdash : \exists ! \mathrm{N_2c}{}^{\backprime}\alpha . \supset . \exists ! \mathrm{N_1c}{}^{\backprime}\alpha$ [*105·4]

***105·42.** $\vdash : \mathrm{N_1c}{}^{\backprime}\alpha = \Lambda . \supset . \mathrm{N_2c}{}^{\backprime}\alpha = \Lambda$ [*105·41]

***105·43.** $\vdash : \mu_{(1)} = \mathrm{N_1c}{}^{\backprime}\alpha . \supset . \mu_{(2)} = \mathrm{N_2c}{}^{\backprime}\alpha$

Dem.

$\vdash . *105·11 . \qquad \supset \vdash : \mathrm{Hp} . \beta \in \mu_{(1)} . \supset . \beta \in \mathrm{Nc}{}^{\backprime}\alpha \cap t_0{}^{\backprime}\alpha .$

[*63·54.*100·31·321] $\supset . \mathrm{Nc}{}^{\backprime}\beta = \mathrm{Nc}{}^{\backprime}\alpha . t_0{}^{\backprime}\beta = t_1{}^{\backprime}\alpha .$

[*105·1·101] $\supset . \mathrm{N_1c}{}^{\backprime}\beta = \mathrm{N_2c}{}^{\backprime}\alpha$ (1)

$\vdash . *105·3 . *103·26 . \supset \vdash : \mathrm{Hp} . \beta \in \mu_{(1)} . \supset . \mathrm{N_1c}{}^{\backprime}\beta = \{\mu_{(1)}\}_{(1)}$

[*105·38] $= \mu_{(2)}$ (2)

$\vdash . (1) . (2) . \qquad \supset \vdash : \mathrm{Hp} . \exists ! \mu_{(1)} . \supset . \mu_{(2)} = \mathrm{N_2c}{}^{\backprime}\alpha$ (3)

$\vdash . *105·372·42 . \qquad \supset \vdash : \mathrm{Hp} . \mu_{(1)} = \Lambda . \supset . \mu_{(2)} = \Lambda . \mathrm{N_2c}{}^{\backprime}\alpha = \Lambda$ (4)

$\vdash . (3) . (4) . \supset \vdash . \mathrm{Prop}$

***105·44.** $\vdash . \mathrm{N_2c}{}^{\backprime}t^{2\backprime}\alpha = \Lambda$

Dem.

$\vdash . *105·26 . \supset \vdash . \mathrm{N_1c}{}^{\backprime}t{}^{\backprime}t{}^{\backprime}\alpha = \Lambda .$

[*105·42] $\supset \vdash . \mathrm{N_2c}{}^{\backprime}t{}^{\backprime}t{}^{\backprime}\alpha = \Lambda . \supset \vdash . \mathrm{Prop}$

$*106$. CARDINALS OF RELATIONAL TYPES

Summary of $*106$.

In this number we have to consider the cardinals whose members are classes of relations which have a given relation of type to some given class. For example, we have $\downarrow x``\alpha \operatorname{sm} \alpha$, and $\downarrow x``\alpha$ has a given relation of type to α when x is given. Thus we want a notation for

$$\operatorname{Nc}`\alpha \cap t`\downarrow x``\alpha$$

and all the associated ideas. In this number, we shall deal only with relations in which the referent and relatum have a relation, as to type, which can be expressed by the notations of $*63$, *i.e.* roughly speaking, when, for suitable values of α, m, n, our relations are contained in

$$t^m`\alpha \uparrow t^n`\alpha \text{ or } t_m`\alpha \uparrow t_n`\alpha \text{ or } t^m`\alpha \uparrow t_n`\alpha \text{ or } t_m`\alpha \uparrow t^n`\alpha.$$

Thus if $t_{\mu\nu}`\alpha$ has been defined, we shall put

$$\mathrm{N}_{\mu\nu}\mathrm{c}`\alpha = \operatorname{Nc}`\alpha \cap t`t_{\mu\nu}`\alpha \quad \mathrm{Df},$$
$$\mathrm{N}_{\mu\nu}\mathrm{C} = \mathrm{D}`\mathrm{N}_{\mu\nu}\mathrm{c} \quad \mathrm{Df},$$
$$\xi_{(\mu\nu)} = \operatorname{sm}``\xi \cap t`t_{\mu\nu}`t_1`\xi \quad \mathrm{Df},$$

with analogous definitions for $t^{\mu\nu}`\alpha$, $t^{\mu}{}_{\nu}`\alpha$ and ${}^{\mu}t_{\nu}`\alpha$.

Much the most important case is that of $t_{00}`\alpha$. For this case we have

$*106\cdot1$. $\vdash : \beta \in \mathrm{N}_{00}\mathrm{c}`\alpha . \equiv . \beta \in \operatorname{Nc}`\alpha . \beta \in t`t_{00}`\alpha . \equiv . \beta \operatorname{sm} \alpha . \beta \in t`t`(t_0`\alpha \uparrow t_0`\alpha) .$
$\equiv . \beta \operatorname{sm} \alpha . \beta \subset t`(\alpha \uparrow \alpha)$

Thus $\mathrm{N}_{00}\mathrm{c}`\alpha$ will be the number of a class of relations whose fields are of the same type as $\alpha$, provided this class of relations is similar to $\alpha$. *E.g.* the number of terms such as $x \downarrow x$, where $x \in \alpha$, will be $\mathrm{N}_{00}\mathrm{c}`\alpha$.

We have

$*106\cdot21$. $\vdash . \exists ! \mathrm{N}_{00}\mathrm{c}`\alpha . \mathrm{N}_{00}\mathrm{c}`\alpha \in \mathrm{N}_0\mathrm{C}$

$*106\cdot22$. $\vdash : \lambda \in \mathrm{N}_0{}^1\mathrm{c}`\alpha . \equiv . \operatorname{Cnv}``\lambda \in {}^1\mathrm{N}_0\mathrm{c}`\alpha$

$*106\cdot23$. $\vdash : \beta \in \mathrm{N}^1\mathrm{c}`\alpha . \supset . \mathrm{N}^{11}\mathrm{c}`\alpha = \mathrm{N}_{00}\mathrm{c}`\beta$

$*106\cdot32$. $\vdash : t_0`\alpha = t_0`\beta . \supset . (\exists \gamma, \delta) . \gamma \in \mathrm{N}_{00}\mathrm{c}`\alpha . \delta \in \mathrm{N}_{00}\mathrm{c}`\beta . \gamma \cap \delta = \Lambda$

$*106\cdot4\cdot41\cdot411$. $\vdash : \mu = \mathrm{N}_0\mathrm{c}`\alpha . \supset . \mu_{(00)} = \mathrm{N}_{00}\mathrm{c}`\alpha . \mu^{(11)} = \mathrm{N}^{11}\mathrm{c}`\alpha . \mu_{(11)} = \mathrm{N}_{11}\mathrm{c}`\alpha$

$*106\cdot53$. $\vdash . \operatorname{Nc}(\alpha)`t_{00}`\alpha = \Lambda$

whence it follows that

$*106\cdot54$. $\vdash . \mathrm{N}_0\mathrm{c}`t_{00}`\alpha \sim\in \mathrm{N}_{00}\mathrm{C}$

The propositions of this number, except $*106{\cdot}21$, are never referred to again (except in $*154{\cdot}25{\cdot}251{\cdot}262$, which are themselves never used again), but they have a somewhat greater importance than the propositions of $*105$, owing to the fact that the arithmetical operations are defined by means of classes of relations, *i.e.* the sum of two cardinals (for instance) is defined as the cardinal number of a certain class of relations (cf. $*110$).

$*106{\cdot}01$. $\mathrm{N}_{00}\mathrm{c}\text{‘}\alpha = \mathrm{Nc}\text{‘}\alpha \cap t\text{‘}t_{00}\text{‘}\alpha$ Df

$*106{\cdot}011$. $\mathrm{N}^{11}\mathrm{c}\text{‘}\alpha = \mathrm{Nc}\text{‘}\alpha \cap t\text{‘}t^{11}\text{‘}\alpha$ Df

$*106{\cdot}012$. $\mathrm{N}_{01}\mathrm{c}\text{‘}\alpha = \mathrm{Nc}\text{‘}\alpha \cap t\text{‘}t_{01}\text{‘}\alpha$ Df etc.

$*106{\cdot}02$. $\mathrm{N}_0{}^1\mathrm{c}\text{‘}\alpha = \mathrm{Nc}\text{‘}\alpha \cap t\text{‘}t_0{}^1\text{‘}\alpha$ Df etc.

$*106{\cdot}021$. ${}^1\mathrm{N}_0\mathrm{c}\text{‘}\alpha = \mathrm{Nc}\text{‘}\alpha \cap t\text{‘}{}^1t_0\text{‘}\alpha$ Df etc.

$*106{\cdot}03$. $\mathrm{N}_{00}\mathrm{C} = \mathrm{D}\text{‘}\mathrm{N}_{00}\mathrm{c}$ Df etc.

$*106{\cdot}04$. $\mu_{(00)} = \mathrm{sm}\text{“}\mu \cap t\text{‘}t_{00}\text{‘}t_1\text{‘}\mu$ Df

$*106{\cdot}041$. $\mu^{(11)} = \mathrm{sm}\text{“}\mu \cap t\text{‘}t^{11}\text{‘}t_1\text{‘}\mu$ Df etc.

$*106{\cdot}1$. $\vdash : \beta \in \mathrm{N}_{00}\mathrm{c}\text{‘}\alpha . \equiv . \beta \in \mathrm{Nc}\text{‘}\alpha . \beta \in t\text{‘}t_{00}\text{‘}\alpha .$
$\equiv . \beta \,\mathrm{sm}\, \alpha . \beta \in t\text{‘}t\text{‘}(t_0\text{‘}\alpha \uparrow t_0\text{‘}\alpha) .$
$\equiv . \beta \,\mathrm{sm}\, \alpha . \beta \subset t\text{‘}(\alpha \uparrow \alpha)$
$[*100{\cdot}1 . (*106{\cdot}01 . *64{\cdot}01) . *64{\cdot}11]$

$*106{\cdot}101$. $\vdash : \beta \in \mathrm{N}^{11}\mathrm{c}\text{‘}\alpha . \equiv . \beta \in \mathrm{Nc}\text{‘}\alpha . \beta \in t\text{‘}t^{11}\text{‘}\alpha .$
$\equiv . \beta \,\mathrm{sm}\, \alpha . \beta \in t\text{‘}t\text{‘}(t\text{‘}\alpha \uparrow t\text{‘}\alpha) .$
$\equiv . \beta \,\mathrm{sm}\, \alpha . \beta \subset t\text{‘}(t\text{‘}\alpha \uparrow t\text{‘}\alpha)$

Similar propositions hold for any other double index mn for which $t^{mn}\text{‘}\alpha$ has been defined.

$*106{\cdot}11$. $\vdash : \beta \in \mathrm{N}_{01}\mathrm{c}\text{‘}\alpha . \equiv . \beta \in \mathrm{Nc}\text{‘}\alpha . \beta \in t\text{‘}t_{01}\text{‘}\alpha .$
$\equiv . \beta \,\mathrm{sm}\, \alpha . \beta \in t\text{‘}t\text{‘}(t_0\text{‘}\alpha \uparrow t_1\text{‘}\alpha) .$
$\equiv . \beta \,\mathrm{sm}\, \alpha . \beta \subset t\text{‘}(t_0\text{‘}\alpha \uparrow t_1\text{‘}\alpha)$

Similar propositions hold for any other double suffix mn for which $t_{mn}\text{‘}\alpha$ has been defined.

$*106{\cdot}12$. $\vdash : \beta \in \mathrm{N}_0{}^1\mathrm{c}\text{‘}\alpha . \equiv . \beta \in \mathrm{Nc}\text{‘}\alpha . \beta \in t\text{‘}t_0{}^1\text{‘}\alpha .$
$\equiv . \beta \,\mathrm{sm}\, \alpha . \beta \in t\text{‘}t\text{‘}(t_0\text{‘}\alpha \uparrow t\text{‘}\alpha) .$
$\equiv . \beta \,\mathrm{sm}\, \alpha . \beta \subset t\text{‘}(t_0\text{‘}\alpha \uparrow t\text{‘}\alpha)$

$*106{\cdot}121$. $\vdash : \beta \in {}^1\mathrm{N}_0\mathrm{c}\text{‘}\alpha . \equiv . \beta \in \mathrm{Nc}\text{‘}\alpha . \beta \in t\text{‘}{}^1t_0\text{‘}\alpha .$
$\equiv . \beta \,\mathrm{sm}\, \alpha . \beta \in t\text{‘}t\text{‘}(t\text{‘}\alpha \uparrow t_0\text{‘}\alpha) .$
$\equiv . \beta \,\mathrm{sm}\, \alpha . \beta \subset t\text{‘}(t\text{‘}\alpha \uparrow t_0\text{‘}\alpha)$

Similar propositions hold for any other index and suffix for which $t_m{}^n\text{‘}\alpha$ or ${}^nt_m\text{‘}\alpha$ has been defined.

$*106{\cdot}13$. $\vdash : \mu \in \mathrm{N}_{00}\mathrm{C} . \equiv . (\exists \alpha) . \mu = \mathrm{N}_{00}\mathrm{c}\text{‘}\alpha$ $[*100{\cdot}22 . *71{\cdot}41]$

Similar propositions hold for $\mathrm{N}^{11}\mathrm{C}\text{‘}\alpha$ etc.

***106·14.** $\vdash : \beta \in \mu_{(00)} . \equiv . (\exists \alpha) . \alpha \in \mu . \beta \,\mathrm{sm}\, \alpha . \beta \in t\text{‘}t\text{‘}(t_1\text{‘}\mu \uparrow t_1\text{‘}\mu) .$
$\equiv . (\exists \alpha) . \alpha \in \mu . \beta \,\mathrm{sm}\, \alpha . \beta \in t\text{‘}t_{00}\text{‘}\alpha .$
$\equiv . (\exists \alpha) . \alpha \in \mu . \beta \,\mathrm{sm}\, \alpha . \beta \subset t\text{‘}(\alpha \uparrow \alpha)$ [*64·33·11]

***106·141.** $\vdash : \beta \in \mu_0{}^1 . \equiv . (\exists \alpha) . \alpha \in \mu . \beta \,\mathrm{sm}\, \alpha . \beta \in t\text{‘}t\text{‘}(t_1\text{‘}\mu \uparrow t_0\text{‘}\mu) .$
$\equiv . (\exists \alpha) . \alpha \in \mu . \beta \,\mathrm{sm}\, \alpha . \beta \in t\text{‘}t_0{}^1\text{‘}\alpha .$
$\equiv . (\exists \alpha) . \alpha \in \mu . \beta \,\mathrm{sm}\, \alpha . \beta \subset t\text{‘}(\alpha \uparrow t\text{‘}\alpha)$

Similar propositions hold for ${}^1\mu_0$, μ^{11}, μ_{11} etc.

***106·2.** $\vdash : x \in t_0\text{‘}\alpha . \supset . \downarrow x\text{“}\alpha \in \mathrm{N}_{00}\mathrm{c}\text{‘}\alpha . \downarrow x\text{“}\alpha \in \mathrm{N}_0\mathrm{c}\text{‘} \downarrow x\text{“}\alpha$

Dem.

$\vdash . *55·15 . \supset \vdash : R \in \downarrow x\text{“}\alpha . \supset . \mathrm{D}\text{‘}R \subset \alpha . \text{Ɑ}\text{‘}R = \iota\text{‘}x :$
[*63·105] $\supset \vdash :. x \in t_0\text{‘}\alpha . \supset : R \in \downarrow x\text{“}\alpha . \supset_R . \mathrm{D}\text{‘}R \subset t_0\text{‘}\alpha . \text{Ɑ}\text{‘}R \subset t_0\text{‘}\alpha .$
[*35·83] $\supset_R . R \mathrel{\subset\!\!\!\cdot} t_0\text{‘}\alpha \uparrow t_0\text{‘}\alpha .$
[*64·16·13] $\supset_R . R \in t\text{‘}(\alpha \uparrow \alpha) :$
[*22·1] $\supset : \downarrow x\text{“}\alpha \subset t\text{‘}(\alpha \uparrow \alpha)$ (1)
$\vdash . (1) . *73·611 . *106·1 . *103·12 . \supset \vdash . \mathrm{Prop}$

***106·201.** $\vdash : \beta \in t\text{‘}\alpha . \supset . \downarrow \beta\text{“}\alpha \in \mathrm{N}_0{}^1\mathrm{c}\text{‘}\alpha$

***106·202.** $\vdash : \beta \in t^2\text{‘}\alpha . \supset . \downarrow \beta\text{“}\alpha \in \mathrm{N}_0{}^2\mathrm{c}\text{‘}\alpha$

***106·203.** $\vdash . \downarrow \alpha\text{“}\alpha \in \mathrm{N}_0{}^1\mathrm{c}\text{‘}\alpha$ [*106·201]

***106·204.** $\vdash . \downarrow (\iota\text{“}\alpha)\text{“}\alpha \in \mathrm{N}_0{}^2\mathrm{c}\text{‘}\alpha$ [*106·202]

***106·21.** $\vdash . \exists ! \mathrm{N}_{00}\mathrm{c}\text{‘}\alpha . \mathrm{N}_{00}\mathrm{c}\text{‘}\alpha \in \mathrm{N}_0\mathrm{C}$ [*106·2 . *63·18]

***106·211.** $\vdash . \Lambda \sim\in \mathrm{N}_{00}\mathrm{C} . \mathrm{N}_{00}\mathrm{C} \subset \mathrm{N}_0\mathrm{C} . \mathrm{N}_{00}\mathrm{C} \in \mathrm{Cls}\,\mathrm{ex}^2\,\mathrm{excl}$ [*106·21 . *103·24]

***106·212.** $\vdash . \Lambda \sim\in \mathrm{N}_0{}^1\mathrm{C} . \mathrm{N}_0{}^1\mathrm{C} \subset \mathrm{N}_0\mathrm{C} . \mathrm{N}_0{}^1\mathrm{C} \in \mathrm{Cls}\,\mathrm{ex}^2\,\mathrm{excl}$ [*106·203]

***106·213.** $\vdash . \Lambda \sim\in \mathrm{N}_0{}^2\mathrm{C} . \mathrm{N}_0{}^2\mathrm{C} \subset \mathrm{N}_0\mathrm{C} . \mathrm{N}_0{}^2\mathrm{C} \in \mathrm{Cls}\,\mathrm{ex}^2\,\mathrm{excl}$ [*106·204]

***106·22.** $\vdash : \lambda \in \mathrm{N}_0{}^1\mathrm{c}\text{‘}\alpha . \equiv . \mathrm{Cnv}\text{“}\lambda \in {}^1\mathrm{N}_0\mathrm{c}\text{‘}\alpha$

Dem.

$\vdash . *73·4 . \supset \vdash : \lambda \,\mathrm{sm}\, \alpha . \equiv . \mathrm{Cnv}\text{“}\lambda \,\mathrm{sm}\, \alpha$ (1)
$\vdash . *64·16 . \supset \vdash :. \lambda \subset t\text{‘}(t_0\text{‘}\alpha \uparrow t\text{‘}\alpha) . \equiv : R \in \lambda . \supset_R . R \mathrel{\subset\!\!\!\cdot} t_0\text{‘}\alpha \uparrow t\text{‘}\alpha :$
[*35·84] $\equiv : R \in \lambda . \supset_R . \breve{R} \mathrel{\subset\!\!\!\cdot} t\text{‘}\alpha \uparrow t_0\text{‘}\alpha :$
[*37·63] $\equiv : S \in \mathrm{Cnv}\text{“}\lambda . \supset_S . S \mathrel{\subset\!\!\!\cdot} t\text{‘}\alpha \uparrow t_0\text{‘}\alpha :$
[*64·16] $\equiv : \mathrm{Cnv}\text{“}\lambda \subset t\text{‘}(t\text{‘}\alpha \uparrow t_0\text{‘}\alpha)$ (2)
$\vdash . (1) . (2) . *106·12 . \supset \vdash . \mathrm{Prop}$

The proof requires, in addition to *106·12, its analogue for ${}^1\mathrm{N}_0\mathrm{c}\text{‘}\alpha$. Such analogues will be assumed as required.

***106·221.** $\vdash : \lambda \in \mathrm{N}_0{}^2\mathrm{c}\text{‘}\alpha . \equiv . \mathrm{Cnv}\text{“}\lambda \in {}^2\mathrm{N}_0\mathrm{c}\text{‘}\alpha$

***106·222.** $\vdash . \Lambda \sim\in {}^1\mathrm{N}_0\mathrm{C} . {}^1\mathrm{N}_0\mathrm{C} \subset \mathrm{N}_0\mathrm{C} . {}^1\mathrm{N}_0\mathrm{C} \in \mathrm{Cls}\,\mathrm{ex}^2\,\mathrm{excl}$ [*106·22·212]

***106·223.** $\vdash . \Lambda \sim\epsilon {}^2\mathrm{N}_0\mathrm{C} . {}^2\mathrm{N}_0\mathrm{C} \subset \mathrm{N}_0\mathrm{C} . {}^2\mathrm{N}_0\mathrm{C} \epsilon \mathrm{Cls\,ex}^2\,\mathrm{excl}$

Other propositions of the same kind as the above may be proved by observing that, if m and n are indices for which $t^m{}^\backprime\alpha$ and $t^n{}^\backprime\alpha$ have been defined, we have

$$\gamma \subset t^n{}^\backprime\alpha . \beta \epsilon \mathrm{N}^m\mathrm{c}^\backprime\alpha . \supset . \downarrow \beta^{\backprime\backprime}\gamma \epsilon \mathrm{N}^{mn}\mathrm{c}^\backprime\alpha,$$

of which the proof is direct and simple. Hence, since we always have $\exists ! \mathrm{N}^m\mathrm{c}^\backprime\alpha$, we also always have

$$\exists ! \mathrm{N}^{mn}\mathrm{c}^\backprime\alpha,$$

whence $\qquad \mathrm{N}^{mn}\mathrm{C} \subset \mathrm{N}_0\mathrm{C} . \mathrm{N}^{mn}\mathrm{C} \epsilon \mathrm{Cls\,ex}^2\,\mathrm{excl}.$

We have in like manner

$$\exists ! \mathrm{N}_0{}^m\mathrm{c}^\backprime\alpha . \exists ! {}^m\mathrm{N}_0\mathrm{c}^\backprime\alpha.$$

But we do not always have

$$\exists ! \mathrm{N}_{mn}\mathrm{c}^\backprime\alpha \text{ or } \exists ! \mathrm{N}_n{}^m\mathrm{c}^\backprime\alpha \text{ or } \exists ! {}^m\mathrm{N}_n\mathrm{c}^\backprime\alpha.$$

***106·23.** $\vdash : \beta \epsilon \mathrm{N}^1\mathrm{c}^\backprime\alpha . \supset . \mathrm{N}^{11}\mathrm{c}^\backprime\alpha = \mathrm{N}_{00}\mathrm{c}^\backprime\beta$

Dem.

$\vdash . *64\cdot33 . *104\cdot1 . *63\cdot5 . \supset \vdash : \mathrm{Hp} . \supset . t^{11}{}^\backprime\alpha = t_{00}{}^\backprime\beta$

$\vdash . (1) . (*106\cdot01\cdot011) . *100\cdot321 . \supset \vdash . \mathrm{Prop}$

***106·231.** $\vdash : \beta \epsilon \mathrm{N}_1\mathrm{c}^\backprime\alpha . \supset . \mathrm{N}_{11}\mathrm{c}^\backprime\alpha = \mathrm{N}_{00}\mathrm{c}^\backprime\beta$ [Proof as in *106·23]

***106·24.** $\vdash : \mathrm{N}^1\mathrm{c}^\backprime\alpha = \mathrm{N}_0\mathrm{c}^\backprime\beta . \supset . \mathrm{N}^{11}\mathrm{c}^\backprime\alpha = \mathrm{N}_{00}\mathrm{c}^\backprime\beta$ [*106·23]

***106·241.** $\vdash : \mathrm{N}_1\mathrm{c}^\backprime\alpha = \mathrm{N}_0\mathrm{c}^\backprime\beta . \supset . \mathrm{N}_{11}\mathrm{c}^\backprime\alpha = \mathrm{N}_{00}\mathrm{c}^\backprime\beta$

The analogues of the above propositions for other indices or suffixes are similarly proved.

***106·25.** $\vdash . \mathrm{N}^{11}\mathrm{c}^\backprime\alpha = \mathrm{N}_{00}\mathrm{c}^\backprime\iota^{\backprime\backprime}\alpha$ [*106·23 . *104·2]

***106·251.** $\vdash . \mathrm{N}_{00}\mathrm{c}^\backprime\alpha = \mathrm{N}_{11}\mathrm{c}^\backprime\iota^{\backprime\backprime}\alpha$

***106·31.** $\vdash : x, y \epsilon t_0{}^\backprime\alpha . t_0{}^\backprime\alpha = t_0{}^\backprime\beta . x \neq y . \supset .$

$$\downarrow x^{\backprime\backprime}\alpha \epsilon \mathrm{N}_{00}\mathrm{c}^\backprime\alpha . \downarrow y^{\backprime\backprime}\beta \epsilon \mathrm{N}_{00}\mathrm{c}^\backprime\beta . \downarrow x^{\backprime\backprime}\alpha \cap \downarrow x^{\backprime\backprime}\beta = \Lambda$$

[*106·2 . *55·233]

***106·311.** $\vdash :. x \epsilon t_0{}^\backprime\alpha . t_0{}^\backprime\alpha = t_0{}^\backprime\beta : \alpha = \Lambda . \mathrm{v} . \beta = \Lambda : \supset .$

$$\downarrow x^{\backprime\backprime}\alpha \epsilon \mathrm{N}_{00}\mathrm{c}^\backprime\alpha . \downarrow x^{\backprime\backprime}\beta \epsilon \mathrm{N}_{00}\mathrm{c}^\backprime\beta . \downarrow x^{\backprime\backprime}\alpha \cap \downarrow x^{\backprime\backprime}\beta = \Lambda$$

[*106·2 . *55·232 . Transp]

***106·312.** $\vdash : t_0{}^\backprime\alpha = \iota^\backprime x . \alpha = \beta = \iota^\backprime x . \supset .$

$$\iota^\backprime(\iota^\backprime x \uparrow \iota^\backprime x) \epsilon \mathrm{N}_{00}\mathrm{c}^\backprime\alpha . \iota^\backprime(\Lambda \uparrow \iota^\backprime x) \epsilon \mathrm{N}_{00}\mathrm{c}^\backprime\beta . \iota^\backprime(\iota^\backprime x \uparrow \iota^\backprime x) \cap \iota^\backprime(\Lambda \uparrow \iota^\backprime x) = \Lambda$$

Dem.

$\vdash . *73\cdot43 . \supset \vdash . \iota^\backprime(\iota^\backprime x \uparrow \iota^\backprime x) \,\mathrm{sm}\, \iota^\backprime x . \iota^\backprime(\Lambda \uparrow \iota^\backprime x) \,\mathrm{sm}\, \iota^\backprime x .$

$[*13\cdot12] \quad \supset \vdash : \mathrm{Hp} . \supset . \iota^\backprime(\iota^\backprime x \uparrow \iota^\backprime x) \,\mathrm{sm}\, \alpha . \iota^\backprime(\Lambda \uparrow \iota^\backprime x) \,\mathrm{sm}\, \beta \qquad (1)$

$\vdash . *64\cdot16 . \supset \vdash : \mathrm{Hp} . \supset . \iota^\backprime x \uparrow \iota^\backprime x \epsilon t_{00}{}^\backprime\alpha . \Lambda \uparrow \iota^\backprime x \epsilon t_{00}{}^\backprime\alpha \qquad (2)$

$\vdash . (1) . (2) . *106\cdot1 . *51\cdot161 . *24\cdot54 . *55\cdot202 . \supset \vdash . \mathrm{Prop}$

***106·32.** $\vdash : t_0{}^{\prime}\alpha = t_0{}^{\prime}\beta . \supset . (\exists \gamma, \delta) . \gamma \epsilon \text{N}_{00}\text{c}{}^{\prime}\alpha . \delta \epsilon \text{N}_{00}\text{c}{}^{\prime}\beta . \gamma \cap \delta = \Lambda$

Dem.

$\vdash . *106{\cdot}31 . \supset \vdash :. \text{Hp} : (\exists x, y) . x, y \epsilon t_0{}^{\prime}\alpha . x \neq y : \supset .$

$(\exists \gamma, \delta) . \gamma \epsilon \text{N}_{00}\text{c}{}^{\prime}\alpha . \delta \epsilon \text{N}_{00}\text{c}{}^{\prime}\beta . \gamma \cap \delta = \Lambda$ (1)

$\vdash . *52{\cdot}4 . \quad \supset \vdash : \sim(\exists x, y) . x, y \epsilon t_0{}^{\prime}\alpha . x \neq y . \supset . t_0{}^{\prime}\alpha \epsilon 1 \cup \iota{}^{\prime}\Lambda .$

[*63·18] $\supset . t_0{}^{\prime}\alpha \epsilon 1$ (2)

$\vdash . (2) . *60{\cdot}38 . *63{\cdot}105 . *52{\cdot}46 . \supset \vdash : \sim(\exists x, y) . x, y \epsilon t_0{}^{\prime}\alpha . x \neq y . \exists ! \alpha . \exists ! \beta . \supset .$

$\alpha = \beta = t_0{}^{\prime}\alpha . t_0{}^{\prime}\alpha \epsilon 1 .$

[*106·312] $\supset . (\exists \gamma, \delta) . \gamma \epsilon \text{N}_{00}\text{c}{}^{\prime}\alpha . \delta \epsilon \text{N}_{00}\text{c}{}^{\prime}\beta . \gamma \cap \delta = \Lambda$ (3)

$\vdash . *106{\cdot}311 . *63{\cdot}18 . \supset$

$\vdash :. \text{Hp} : \sim(\exists ! \alpha . \exists ! \beta) : \supset . (\exists \gamma, \delta) . \gamma \epsilon \text{N}_{00}\text{c}{}^{\prime}\alpha . \delta \epsilon \text{N}_{00}\text{c}{}^{\prime}\beta . \gamma \cap \delta = \Lambda$ (4)

$\vdash . (1) . (3) . (4) . \supset \vdash . \text{Prop}$

***106·4.** $\vdash : \mu = \text{N}_0\text{c}{}^{\prime}\alpha . \supset . \mu_{(00)} = \text{N}_{00}\text{c}{}^{\prime}\alpha$

Dem.

$\vdash . *106{\cdot}14 . \supset \vdash :: \text{Hp} . \supset :. \beta \epsilon \mu_{00} . \equiv : (\exists \gamma) . \gamma \epsilon \text{N}_0\text{c}{}^{\prime}\alpha . \beta \,\text{sm}\, \gamma . \beta \epsilon t{}^{\prime}t_{00}{}^{\prime}\gamma :$

[*64·3] $\equiv : (\exists \gamma) . \gamma \epsilon \text{N}_0\text{c}{}^{\prime}\alpha . \beta \,\text{sm}\, \gamma : \beta \epsilon t{}^{\prime}t_{00}{}^{\prime}\alpha :$

[*102·84] $\equiv : \beta \,\text{sm}\, \alpha . \beta \epsilon t{}^{\prime}t_{00}{}^{\prime}\alpha :$

[*106·1] $\equiv : \beta \epsilon \text{N}_{00}\text{c}{}^{\prime}\alpha :: \supset \vdash . \text{Prop}$

***106·401.** $\vdash : \mu = \text{N}^1\text{c}{}^{\prime}\alpha . \supset . \mu_{(00)} = \text{N}^{11}\text{c}{}^{\prime}\alpha$

Dem.

$\vdash . *104{\cdot}24 . *106{\cdot}4 . \supset \vdash : \text{Hp} . \supset . \mu_{(00)} = \text{N}_{00}\text{c}{}^{\prime}\iota{}^{\prime\prime}\alpha$

[*106·25] $= \text{N}^{11}\text{c}{}^{\prime}\alpha : \supset \vdash . \text{Prop}$

***106·402.** $\vdash : \mu = \text{N}_1\text{c}{}^{\prime}\alpha . \exists ! \mu . \supset . \mu_{(00)} = \text{N}_{11}\text{c}{}^{\prime}\alpha$

Dem.

$\vdash . *106{\cdot}231 . \quad \supset \vdash : \text{Hp} . \beta \epsilon \mu . \supset . \text{N}_{11}\text{c}{}^{\prime}\alpha = \text{N}_{00}\text{c}{}^{\prime}\beta$

[*106·4.*103·26] $= \mu_{(00)}$ (1)

$\vdash . (1) . *10{\cdot}11{\cdot}23{\cdot}35 . \supset \vdash : \text{Hp} . \exists ! \mu . \supset . \mu_{(00)} = \text{N}_{11}\text{c}{}^{\prime}\alpha : \supset \vdash . \text{Prop}$

***106·41.** $\vdash : \mu = \text{N}_0\text{c}{}^{\prime}\alpha . \supset . \mu^{(11)} = \text{N}^{11}\text{c}{}^{\prime}\alpha$

Dem.

$\vdash . *63{\cdot}54 . (*106{\cdot}041) . *103{\cdot}27 . \supset$

$\vdash :: \text{Hp} . \supset :. \beta \epsilon \mu^{(11)} . \equiv : (\exists \gamma) . \gamma \epsilon \text{N}_0\text{c}{}^{\prime}\alpha . \beta \,\text{sm}\, \gamma : \beta \epsilon t{}^{\prime}t^{11}{}^{\prime}t_0{}^{\prime}\alpha :$

[*102·84.*64·32] $\equiv : \beta \,\text{sm}\, \alpha . \beta \epsilon t{}^{\prime}t^{11}{}^{\prime}\alpha :$

[(*106·011)] $\equiv : \beta \epsilon \text{N}^{11}\text{c}{}^{\prime}\alpha :: \supset \vdash . \text{Prop}$

***106·411.** $\vdash : \mu = \text{N}_0\text{c}{}^{\prime}\alpha . \supset . \mu_{(11)} = \text{N}_{11}\text{c}{}^{\prime}\alpha$ [Proof as in *106·41]

***106·43.** $\vdash : \mu, \nu \epsilon \text{N}_0\text{C} . t{}^{\prime}\mu = t{}^{\prime}\nu . \supset . (\exists \gamma, \delta) . \gamma \epsilon \mu_{(00)} . \delta \epsilon \nu_{(00)} . \gamma \cap \delta = \Lambda$

Dem.

$\vdash . *103{\cdot}2 . \supset \vdash : \text{Hp} . \supset . (\exists \alpha, \beta) . \mu = \text{N}_0\text{c}{}^{\prime}\alpha . \nu = \text{N}_0\text{c}{}^{\prime}\beta .$

[*106·4] $\supset . (\exists \alpha, \beta) . \mu_{(00)} = \text{N}_{00}\text{c}{}^{\prime}\alpha . \nu_{(00)} = \text{N}_{00}\text{c}{}^{\prime}\beta .$

[*106·32] $\supset . (\exists \gamma, \delta) . \gamma \epsilon \mu_{(00)} . \delta \epsilon \nu_{(00)} . \gamma \cap \delta = \Lambda : \supset \vdash . \text{Prop}$

***106·44.** $\vdash : \mu, \nu \epsilon \text{N}_{00}\text{C} . t{}^{\prime}\mu = t{}^{\prime}\nu . \supset . (\exists \gamma, \delta) . \gamma \epsilon \mu . \delta \epsilon \nu . \gamma \cap \delta = \Lambda$ [*106·32]

The following propositions are analogous to ✱102·71 ff., and similar remarks apply to them.

✱106·5. $\vdash : R \in \text{Cls} \to 1 \,.\, \text{D}`R \subset \alpha \,.\, \text{Cl}`R \subset \text{Rl}`(\alpha \uparrow \alpha) \,.$

$$W = \hat{x}\hat{y}\{x, y \in \alpha \,.\, \sim x(\breve{R}`x)\,y\} \,.\supset.\, W \sim\in \text{Cl}`R \,.\, W \,\text{Є}\, \alpha \uparrow \alpha$$

Dem.

$\vdash .\, ✱4·73 \,.\supset \vdash :: \text{Hp} \,.\supset :.\, x, y \in \alpha \,.\supset_{x,y} : xWy \,.\equiv.\, \sim x(\breve{R}`x)\,y :$

$[✱5·18] \qquad \supset_{x,y} : \sim\{xWy \,.\equiv.\, x(\breve{R}`x)\,y\} :.$

$[✱10·1] \qquad \supset :.\, x \in \alpha \,.\supset_x .\, \sim\{xWx \,.\equiv.\, x(\breve{R}`x)\,x\} \,.$

$[✱21·43.\text{Transp}] \qquad \supset_x .\, W \neq \breve{R}`x :.$

$[\text{Hp}] \qquad \supset :.\, x \in \text{D}`R \,.\supset_x .\, W \neq \breve{R}`x :.$

$[✱71·411.\text{Transp}] \qquad \supset :.\, W \sim\in \text{Cl}`R \qquad (1)$

$\vdash .\, ✱21·33 \,.\, (✱35·04) \,.\supset \vdash : \text{Hp} \,.\supset.\, W \,\text{Є}\, \alpha \uparrow \alpha \qquad (2)$

$\vdash .\, (1) \,.\, (2) \,.\supset \vdash .\, \text{Prop}$

✱106·51. $\vdash : \beta \subset \alpha \,.\supset.\, \sim\{\beta \,\text{sm}\, \text{Rl}`(\alpha \uparrow \alpha)\}$

Dem.

$\vdash .\, ✱106·5 \,.\qquad \supset \vdash : \text{Hp} \,.\, R \in 1 \to 1 \,.\, \text{D}`R = \beta \,.\, \text{Cl}`R \subset \text{Rl}`(\alpha \uparrow \alpha) \,.\supset.$

$$(\exists W) \,.\, W \in \text{Rl}`(\alpha \uparrow \alpha) \,.\, W \sim\in \text{Cl}`R \,.$$

$[✱13·14] \qquad \supset.\, \text{Cl}`R \neq \text{Rl}`(\alpha \uparrow \alpha) \qquad (1)$

$\vdash .\, (1) \,.\, ✱22·41 \,.\supset \vdash :.\, \text{Hp} \,.\supset : R \in 1 \to 1 \,.\, \text{D}`R = \beta \,.\supset_R .\, \text{Cl}`R \neq \text{Rl}`(\alpha \uparrow \alpha) :$

$[✱10·51.✱73·1] \qquad \supset : \sim\{\beta \,\text{sm}\, \text{Rl}`(\alpha \uparrow \alpha)\} :.\supset \vdash .\, \text{Prop}$

✱106·52. $\vdash : \beta \subset t_0`\alpha \,.\supset.\, \beta \sim\in \text{Nc}`t_{00}`\alpha$

Dem.

$\vdash .\, ✱106·51 \,.\supset \vdash : \text{Hp} \,.\supset.\, \sim\{\beta \,\text{sm}\, \text{Rl}`(t_0`\alpha \uparrow t_0`\alpha)\} \,.$

$[✱64·54] \qquad \supset.\, \sim\{\beta \,\text{sm}\, t_{00}`\alpha\} \,.$

$[✱100·1] \qquad \supset.\, \beta \sim\in \text{Nc}`t_{00}`\alpha : \supset \vdash .\, \text{Prop}$

✱106·53. $\vdash .\, \text{Nc}(\alpha)`t_{00}`\alpha = \Lambda \quad [✱106·52 \,.\, ✱102·6 \,.\, ✱63·371]$

✱106·54. $\vdash .\, \text{N}_0\text{c}`t_{00}`\alpha \sim\in \text{N}_{00}\text{C}$

Dem.

$\vdash .\, ✱100·33 \,.\, ✱103·15 \,.\supset$

$\vdash : \text{N}_{00}\text{c}`\beta = \text{N}_0\text{c}`t_{00}`\alpha \,.\supset.\, \beta \,\text{sm}\, t_{00}`\alpha \qquad (1)$

$\vdash .\, ✱103·12 \,.\, (✱106·01) \,.\supset$

$\vdash : \text{N}_{00}\text{c}`\beta = \text{N}_0\text{c}`t_{00}`\alpha \,.\supset.\, t_{00}`\alpha \in t`t_{00}`\beta \,.$

$[✱63·16.(✱64·01)] \qquad \supset.\, t`t`(t_0`\alpha \uparrow t_0`\alpha) = t`t`(t_0`\beta \uparrow t_0`\beta) \,.$

$[✱63·391] \qquad \supset.\, t`(t_0`\alpha \uparrow t_0`\alpha) = t`(t_0`\beta \uparrow t_0`\beta) \,.$

$[✱64·3.(✱64·01)] \qquad \supset.\, t_0`\alpha = t_0`\beta \,.$

$[✱63·105] \qquad \supset.\, \beta \subset t_0`\alpha \qquad (2)$

$\vdash .\, (1) \,.\, (2) \,.\supset \vdash : \text{N}_{00}\text{c}`\beta = \text{N}_0\text{c}`t_{00}`\alpha \,.\supset.\, \beta \in \text{Nc}`t_{00}`\alpha \,.\, \beta \subset t_0`\alpha \qquad (3)$

$\vdash .\, (3) \,.\, \text{Transp} \,.\, ✱106·52 \,.\supset \vdash .\, (\beta) \,.\, \text{N}_{00}\text{c}`\beta \neq \text{N}_0\text{c}`t_{00}`\alpha \,.$

$[✱106·13.\text{Transp}] \qquad \supset \vdash .\, \text{N}_0\text{c}`t_{00}`\alpha \sim\in \text{N}_{00}\text{C} \,.\supset \vdash .\, \text{Prop}$

✱106·55. $\vdash .\, \exists !\, \text{N}_0\text{C} - \text{N}_{00}\text{C} \quad [✱106·54]$

SECTION B

ADDITION, MULTIPLICATION AND EXPONENTIATION

Summary of Section B.

In the present section, we have to consider the arithmetical operations as applied to cardinals, as well as the relation of greater and less between cardinals. Thus the topics to be dealt with in this section are the first that can properly be said to belong to Arithmetic.

The treatment of addition, multiplication and exponentiation to be given in what follows is guided by the desire to secure the greatest possible generality. In the first place, everything to be said generally about the arithmetical operations must apply equally to finite and infinite classes or cardinals. In the second place, we desire such definitions as shall allow the number of summands in a sum or of factors in a product to be infinite. In the third place, we wish to be able to add or multiply two numbers which are not necessarily of the same type. In the fourth place, we wish our definitions to be such that the sum of the cardinal numbers of two or more classes shall depend only upon the cardinal numbers of those classes, and shall be the same when the classes overlap as when they are mutually exclusive; with similar conditions for the product. The desire to obtain definitions fulfilling all these conditions leads to somewhat more complicated definitions than would otherwise be required; but in the outcome, the result is simpler than if we started with simpler definitions, since we avoid vexatious exceptions.

The above observations will become clearer through their applications. Let us begin with the case of arithmetical addition of two classes.

If α and β are mutually exclusive classes, the sum of their cardinal numbers will be the cardinal number of $\alpha \cup \beta$. But in order that α and β may be mutually exclusive, they must have no common members, and this is only significant when they are of the same type. Hence, given two perfectly general classes α and β, we require to find two classes which are mutually exclusive and are respectively similar to α and β; if these two classes are called α' and β', then $\mathrm{Nc}`(\alpha' \cup \beta')$ will be the sum of the cardinal numbers of α and β. We note that $\Lambda \cap \alpha$ and $\Lambda \cap \beta$ indicate respectively the Λ's of the same types as α and β, and accordingly we take as α' and β' the two classes

$$\downarrow (\Lambda \cap \beta)``\iota``\alpha \text{ and } (\Lambda \cap \alpha) \downarrow ``\iota``\beta;$$

these two classes are always of the same type, always mutually exclusive, and always similar to α and β respectively. Hence we define

$$\alpha + \beta = \downarrow (\Lambda \cap \beta)``\iota``\alpha \cup (\Lambda \cap \alpha) \downarrow ``\iota``\beta \quad \text{Df.}$$

The sum of the cardinal numbers of α and β will then be the cardinal number of $\alpha+\beta$; hence we may call $\alpha+\beta$ the *arithmetical* class-sum of two classes, in contradistinction to $\alpha \cup \beta$, which is the *logical* sum. It will be noted that $\alpha+\beta$, unlike $\alpha \cup \beta$, does not require that α and β should be of the same type. Also $\alpha+\alpha$ is not identical with α, but when $\alpha=\Lambda$, $\alpha+\alpha$ is also Λ, though in a different type. Thus the law of tautology does not hold of the arithmetical class-sum of two classes.

If μ and ν are two cardinals of assigned types, we denote their arithmetical sum by $\mu +_c \nu$. (As many kinds of arithmetical addition occur in our work, and as it is essential to our purpose to distinguish them, we effect the distinction by suffixes to the sign of addition. It is, of course, only in dealing with principles that these different symbols are needed: we do not wish to suggest that they should be adopted in ordinary mathematics.) Now if $\mu +_c \nu$ is to have the properties which we commonly associate with the sum of two cardinals, it must be typically ambiguous, and must be the cardinal number of any class which can be divided into two mutually exclusive parts having μ terms and ν terms respectively. Hence we are led to the following definition:

$$\mu +_c \nu = \hat{\xi}\{(\exists \alpha, \beta) . \mu = \mathrm{N_0c}`\alpha . \nu = \mathrm{N_0c}`\beta . \xi \,\mathrm{sm}\, (\alpha+\beta)\} \quad \mathrm{Df.}$$

In this definition, various points should be noted. In the first place, it does not require that μ and ν should be of the same type; $\mu +_c \nu$ is *significant* whenever μ and ν are classes of classes. Thus it is not necessary for significance that μ and ν should be cardinals, though if they are not both cardinals, $\mu +_c \nu = \Lambda$. If they are both cardinals, we find

$$\mu +_c \nu = \hat{\xi}\{(\exists \alpha, \beta) . \alpha \epsilon \mu . \beta \epsilon \nu . \xi \,\mathrm{sm}\, (\alpha+\beta)\}.$$

Thus in this case $\quad \alpha \epsilon \mu . \beta \epsilon \nu . \supset . \alpha+\beta \epsilon \mu +_c \nu.$

Hence if neither μ nor ν is null, and if α has μ terms and β has ν terms, $\alpha+\beta$ is a member of $\mu +_c \nu$. It easily follows that

$$\vdash : \mu = \mathrm{N_0c}`\alpha . \nu = \mathrm{N_0c}`\beta . \supset . \mu +_c \nu = \mathrm{Nc}`(\alpha+\beta).$$

Hence when μ and ν are homogeneous cardinals (*i.e.* when they are cardinals other than Λ), their sum is the number of the arithmetical class-sum of any two classes having μ terms and ν terms respectively.

A few words are necessary to explain why, in the definition, we put $\mu = \mathrm{N_0c}`\alpha . \nu = \mathrm{N_0c}`\beta$ rather than $\mu = \mathrm{Nc}`\alpha . \nu = \mathrm{Nc}`\beta$. The reason is this. Suppose either μ or ν, say μ, is Λ. Then, by $*102{\cdot}73$, $\mu = \mathrm{Nc}(\zeta)`t`\zeta$, if ζ is of the appropriate type. Hence if we had put

$$\mu +_c \nu = \hat{\xi}\{(\exists \alpha, \beta) . \mu = \mathrm{Nc}`\alpha . \nu = \mathrm{Nc}`\beta . \xi \,\mathrm{sm}\, (\alpha+\beta)\} \quad \mathrm{Df,}$$

where the ambiguities of type involved in $\mathrm{Nc}`\alpha$ and $\mathrm{Nc}`\beta$ may be determined as we please, we should have

$$\nu = \mathrm{Nc}`\beta . \supset . t`\zeta + \beta \epsilon \mu +_c \nu,$$

i.e. $\quad \nu = \mathrm{Nc}`\beta . \supset . t`\zeta + \beta \epsilon \Lambda +_c \nu.$

We should also have $t`t`\zeta + \beta \epsilon \Lambda +_c \nu$ and so on. Thus $\Lambda +_c \nu$ would not have a definite value, *i.e.* it would not merely have typical ambiguity, which it ought to have, but it would not have a definite value even when its type was assigned. Thus such a definition would be unsuitable. For the above reasons, we put $\mu = \mathrm{N_0c}`\alpha \,.\, \nu = \mathrm{N_0c}`\beta$ in the definition, and obtain the typical ambiguity which we desire by means of the typical ambiguity of the "sm" in "$\xi \,\mathrm{sm}\,(\alpha + \beta)$." It is always essential to right symbolism that the values of typically ambiguous symbols should be unique as soon as their type is assigned. The scope of these definitions and of the corresponding definitions for multiplication and exponentiation (*113·04·05 . *116·03·04) is extended by convention II T of the prefatory statement.

The above definition of $\mu +_c \nu$ is designed for the case in which μ and ν are typically definite. But we must be able to speak of "$\mathrm{Nc}`\gamma +_c \mathrm{Nc}`\delta$," and this must be a definite cardinal, namely $\mathrm{Nc}`(\gamma + \delta)$. If we simply write $\mathrm{Nc}`\gamma$, $\mathrm{Nc}`\delta$ in place of μ, ν in the definition of $\mu +_c \nu$, we find

$$\mathrm{Nc}`\gamma +_c \mathrm{Nc}`\delta = \hat{\xi}\{(\exists \alpha, \beta) \,.\, \mathrm{Nc}`\gamma = \mathrm{N_0c}`\alpha \,.\, \mathrm{Nc}`\delta = \mathrm{N_0c}`\beta \,.\, \xi \,\mathrm{sm}\,(\alpha + \beta)\}.$$

But this will not always have a definite value when the type of $\mathrm{Nc}`\gamma +_c \mathrm{Nc}`\delta$ is assigned. To take a simple case, write $t`\zeta$ for $\gamma$ and $\iota`y$ for δ. Then

$$\mathrm{Nc}`t`\zeta +_c \mathrm{Nc}`\iota`y = \hat{\xi}\{(\exists \alpha, \beta) \,.\, \mathrm{Nc}`t`\zeta = \mathrm{N_0c}`\alpha \,.\, \mathrm{Nc}`\iota`y = \mathrm{N_0c}`\beta \,.\, \xi \,\mathrm{sm}\,(\alpha + \beta)\},$$

whence we easily obtain

$$\mathrm{Nc}`t`\zeta +_c \mathrm{Nc}`\iota`y = \hat{\xi}\{(\exists \alpha) \,.\, \mathrm{Nc}`t`\zeta = \mathrm{N_0c}`\alpha \,.\, \xi \,\mathrm{sm}\,(\alpha + \iota`y)\}.$$

If we determine the ambiguity of $\mathrm{Nc}`t`\zeta$ to be $\mathrm{N_1c}`t`\zeta$, we find

$$\mathrm{Nc}`t`\zeta +_c \mathrm{Nc}`\iota`y = \Lambda$$

in all types; but if we determine the ambiguity to be $\mathrm{N_0c}`t`\zeta$, we have

$$\mathrm{Nc}`t`\zeta +_c \mathrm{Nc}`\iota`y = \mathrm{Nc}`(t`\zeta + \iota`y),$$

and this exists in the type of $t`\zeta + \iota`y$, if not in lower types. Hence the value of $\mathrm{Nc}`t`\zeta +_c \mathrm{Nc}`\iota`y$ depends upon the determination of the ambiguity of $\mathrm{Nc}`t`\zeta$. It is obvious that we want our definition to yield

$$\mathrm{Nc}`\gamma +_c \mathrm{Nc}`\delta = \mathrm{Nc}`(\gamma + \delta)$$

in all types; but in order to insure that this shall hold even when, for some values of ζ, $\mathrm{Nc}\,(\zeta)`\gamma = \Lambda$, we must introduce two new definitions, namely

$$\mathrm{Nc}`\alpha +_c \mu = \mathrm{N_0c}`\alpha +_c \mu \quad \mathrm{Df},$$

$$\mu +_c \mathrm{Nc}`\alpha = \mu +_c \mathrm{N_0c}`\alpha \quad \mathrm{Df},$$

whence $\quad \vdash : \mathrm{Nc}`\alpha +_c \mathrm{Nc}`\beta = \mathrm{N_0c}`\alpha +_c \mathrm{N_0c}`\beta = \mathrm{Nc}`(\alpha + \beta).$

This definition is to be applied when "$\mathrm{Nc}`\gamma$" and "$\mathrm{Nc}`\delta$" occur without any determination of type. On the other hand, if we have $\mathrm{Nc}\,(\zeta)`\gamma$ and $\mathrm{Nc}\,(\eta)`\delta$, we apply the definition of $\mu +_c \nu$. We shall find that whenever $\mathrm{Nc}\,(\zeta)`\gamma$ and $\mathrm{Nc}\,(\eta)`\delta$ both exist,

$$\mathrm{Nc}\,(\zeta)`\gamma +_c \mathrm{Nc}\,(\eta)`\delta = \mathrm{N_0c}`\gamma +_c \mathrm{N_0c}`\delta.$$

Thus the above definition is only required in order to exclude values of ζ or η for which either $\mathrm{Nc}\,(\zeta)`\gamma$ or $\mathrm{Nc}\,(\eta)`\delta$ is Λ.

The commutative and associative laws of arithmetical addition are easily deduced from the definition of $\alpha + \beta$. We shall have

$$\vdash . \alpha + \beta = \mathrm{Cnv}``(\beta + \alpha),$$

whence $$\vdash . \mathrm{Nc}`\alpha +_c \mathrm{Nc}`\beta = \mathrm{Nc}`\beta +_c \mathrm{Nc}`\alpha,$$

because each $= \mathrm{Nc}`(\alpha + \beta)$. A similar though slightly longer proof shows that

$$\vdash . (\alpha + \beta) + \gamma \,\mathrm{sm}\, \alpha + (\beta + \gamma),$$

whence $$\vdash . (\mathrm{Nc}`\alpha +_c \mathrm{Nc}`\beta) +_c \mathrm{Nc}`\gamma = \mathrm{Nc}`\alpha +_c (\mathrm{Nc}`\beta +_c \mathrm{Nc}`\gamma).$$

The above definition of $\alpha + \beta$ enables us to proceed to the sum of any finite number of classes, and allows any one class to recur in the summation. But it does not enable us to define the sum of an infinite number of classes. For this we need a new definition. Since an infinite number of classes cannot be given by enumeration, but only by intension, we shall have to take a class of classes κ, and define the arithmetical sum of the members of κ. Thus now the classes which are the summands must all be of the same type (since they are all members of κ), and no one class can occur more than once, since each member of κ only counts once. (In order to deal with repetition, we must advance to multiplication, which will be explained shortly.) Thus in removing the limitation to a finite number of summands, we introduce certain other limitations. This is the reason which makes it worth while to introduce the above definition of $\alpha + \beta$ in addition to the definition now to be given.

If κ is a class of classes, the sum of the cardinal numbers of the members of κ will evidently be obtained by constructing a class of mutually exclusive classes whose members have a one-one relation to the members of corresponding members of κ. Suppose α, β are two different members of κ, and suppose x is a member both of α and of β. Then we wish to count x twice over, once as a member of α and once as a member of β. The simplest way to do this is to form the ordinal couples $x \downarrow \alpha$ and $x \downarrow \beta$, which are not identical except when α and β are identical. Thus if we take all such ordinal couples, *i.e.* if we take the class

$$\hat{R}\{(\exists x) . x \epsilon \alpha . R = x \downarrow \alpha\},$$

for every α which is a member of κ, we get a class of mutually exclusive classes, namely the classes of the form $\downarrow \alpha``\alpha$, where $\alpha \epsilon \kappa$, and each of these is similar to the corresponding member of κ. Hence the logical sum of this class of classes, *i.e.*

$$\hat{R}\{(\exists \alpha, x) . \alpha \epsilon \kappa . x \epsilon \alpha . R = x \downarrow \alpha\},$$

has the required number of terms. Now, by $*85{\cdot}601$,

$$\downarrow \alpha``\alpha = \epsilon \,\rotatebox[origin=c]{180}{$\uparrow$}`\alpha.$$

Hence the class whose logical sum we are taking is $\epsilon \,\rotatebox[origin=c]{180}{$\uparrow$}``\kappa$. Hence we put

$$\Sigma`\kappa = s`\epsilon \,\rotatebox[origin=c]{180}{$\uparrow$}``\kappa \quad \mathrm{Df.}$$

$\Sigma`\kappa$ may be called the *arithmetical* sum of $\kappa$, in contradistinction to $s`\kappa$, which is the logical sum. Thus $\Sigma`\kappa$ bears to $s`\kappa$ a relation analogous to that which $\alpha + \beta$ bears to $\alpha \cup \beta$.

We put further $\quad \Sigma\text{Nc}‘\kappa = \text{Nc}‘s‘\epsilon\downarrow``\kappa \quad$ Df.

Thus $\Sigma\text{Nc}‘\kappa$ is the sum of the numbers of members of κ.

It is to be observed that $\Sigma\text{Nc}‘\kappa$ is not in general a function of $\text{Nc}``\kappa$. For, if two members of $\kappa$ have the same cardinal number, this will only count once in $\text{Nc}``\kappa$, whereas it counts twice in $\Sigma\text{Nc}‘\kappa$.

We shall find that, provided $\alpha \neq \beta$,

$$\Sigma\text{Nc}‘(\iota‘\alpha \cup \iota‘\beta) = \text{Nc}‘\alpha +_0 \text{Nc}‘\beta.$$

Thus where a finite number of summands are concerned, the two definitions of addition agree, except that the first allows one class to count several times over, while the second does not.

In dealing with multiplication, our procedure is closely analogous to the procedure for addition. We first define the *arithmetical class-product* of two classes α and β, which is a certain class whose cardinal number is the product of the cardinal numbers of α and β. We write $\beta \times \alpha$ for the arithmetical class-product of β and α, and define it as the class of all ordinal couples of which the referent is a member of α and the relatum a member of β, *i.e.* as

$$\hat{R}\{(\exists x, y) \,.\, x \in \alpha \,.\, y \in \beta \,.\, R = x \downarrow y\}.$$

By *40·7, this class is $s‘\alpha \underset{,,}{\downarrow} ``\beta$. Hence we put

$$\beta \times \alpha = s‘\alpha \underset{,,}{\downarrow} ``\beta \quad \text{Df.}$$

The class $\alpha \underset{,,}{\downarrow} ``\beta$ is similar to $\beta$, and each member of it is similar to $\alpha$; hence if $\text{N}_0\text{c}‘\alpha = \mu$ and $\text{N}_0\text{c}‘\beta = \nu$, $s‘\alpha \underset{,,}{\downarrow} ``\beta$ consists of ν classes having μ members each. The class $\alpha \underset{,,}{\downarrow} ``\beta$ is important also in connection with exponentiation. The product of two cardinals is defined as follows:

$$\mu \times_0 \nu = \hat{\xi}\{(\exists \alpha, \beta) \,.\, \mu = \text{N}_0\text{c}‘\alpha \,.\, \nu = \text{N}_0\text{c}‘\beta \,.\, \xi \,\text{sm}\, (\alpha \times \beta)\} \quad \text{Df.}$$

In regard to types, this definition calls for analogous remarks to those which were made on $\mu +_0 \nu$. Also, as before, we need definitions of $\mu \times_0 \text{Nc}‘\alpha$ and $\text{Nc}‘\alpha \times_0 \mu$, whence we obtain

$$\text{Nc}‘\alpha \times_0 \text{Nc}‘\beta = \text{N}_0\text{c}‘\alpha \times_0 \text{N}_0\text{c}‘\beta \quad \text{Df.}$$

By means of these definitions, we can define the product of any finite number of cardinals; but in order to define products which have an infinite number of factors, we need a new definition.

If κ is a class of classes, we take $\epsilon_\Delta‘\kappa$ as its arithmetical product. In simple cases, it is easy to see the justification of this decision. *E.g.* let κ consist of the three classes $\alpha_1, \alpha_2, \alpha_3$, and let the members of α_1 be x_1, x_2; those of α_2, y_1, y_2; those of α_3, z_1, z_2. Then the members of $\epsilon_\Delta‘\kappa$ are

$$x_1 \downarrow \alpha_1 \cup y_1 \downarrow \alpha_2 \cup z_1 \downarrow \alpha_3,$$
$$x_2 \downarrow \alpha_1 \cup y_1 \downarrow \alpha_2 \cup z_1 \downarrow \alpha_3,$$
$$x_1 \downarrow \alpha_1 \cup y_2 \downarrow \alpha_2 \cup z_1 \downarrow \alpha_3,$$
$$x_2 \downarrow \alpha_1 \cup y_2 \downarrow \alpha_2 \cup z_1 \downarrow \alpha_3,$$

with four more obtained by substituting z_2 for z_1 in the above. Thus $\mathrm{Nc}`\epsilon_\Delta`\kappa = 8 = \mathrm{Nc}`\alpha_1 \times_0 \mathrm{Nc}`\alpha_2 \times_0 \mathrm{Nc}`\alpha_3$. In general, however, the existence of $\epsilon_\Delta`\kappa$ is doubtful, owing to the doubt as to the validity of the multiplicative axiom. (We shall return to this point shortly.) Hence there is no proof that the product of an infinite number of factors cannot be zero unless one of the factors is zero.

When κ is a class of mutually exclusive classes, $\epsilon_\Delta`\kappa$ is similar to $\mathrm{D}``\epsilon_\Delta`\kappa$. On account of its lower type, $\mathrm{D}``\epsilon_\Delta`\kappa$ is often more convenient than $\epsilon_\Delta`\kappa$. Hence we put

$$\mathrm{Prod}`\kappa = \mathrm{D}``\epsilon_\Delta`\kappa \quad \mathrm{Df},$$

or (what comes to the same thing)

$$\mathrm{Prod} = \mathrm{D}_\epsilon | \epsilon_\Delta \quad \mathrm{Df}.$$

For the product of the cardinal numbers of the members of κ, we put

$$\Pi\mathrm{Nc}`\kappa = \mathrm{Nc}`\epsilon_\Delta`\kappa \quad \mathrm{Df}.$$

As in the case of $\Sigma\mathrm{Nc}`\kappa$, $\Pi\mathrm{Nc}`\kappa$ is not in general a function of $\mathrm{Nc}``\kappa$. We shall have

$$\vdash : \alpha \neq \beta \,.\supset .\, \Pi\mathrm{Nc}`(\iota`\alpha \cup \iota`\beta) = \mathrm{Nc}`\alpha \times_c \mathrm{Nc}`\beta.$$

Thus for products of a finite number of different factors, the two definitions of multiplication agree.

It remains to define exponentiation. Since this is not a commutative operation, it essentially involves an order as between the base and the exponent; hence we do not obtain a definition of the exponentiation of a class κ, analogous to $\Sigma\mathrm{Nc}`\kappa$ or $\Pi\mathrm{Nc}`\kappa$, but only a definition of μ^ν, which may be extended to any finite number of exponentiations. We put

$$\alpha \exp \beta = \mathrm{Prod}`\alpha \underset{,,}{\downarrow} ``\beta \quad \mathrm{Df},$$

where $\alpha \underset{,,}{\downarrow} ``\beta$ has the meaning explained above, resulting from $*38{\cdot}03$. It will be observed that, if $\mathrm{N_0c}`\alpha = \mu$ and $\mathrm{N_0c}`\beta = \nu$, $\alpha \underset{,,}{\downarrow} ``\beta$ is a class of ν mutually exclusive classes each of which has μ terms; hence $\alpha \exp \beta$ may suitably be used to define μ^ν. Hence we put

$$\mu^\nu = \hat{\xi}\{(\exists \alpha, \beta) \,.\, \mu = \mathrm{N_0c}`\alpha \,.\, \nu = \mathrm{N_0c}`\beta \,.\, \xi \,\mathrm{sm}\, (\alpha \exp \beta)\} \quad \mathrm{Df},$$

and for the same reasons as before, we put

$$(\mathrm{Nc}`\alpha)^\nu = (\mathrm{N_0c}`\alpha)^\nu \quad \mathrm{Df} \text{ and } \mu^{\mathrm{Nc}`\beta} = \mu^{\mathrm{N_0c}`\beta} \quad \mathrm{Df}.$$

The above definition of exponentiation gives the same value of μ^ν as results from Cantor's definition by means of "Belegungen." The class of Cantor's "Belegungen" is

$$\hat{R}\{R \epsilon 1 \rightarrow \mathrm{Cls} \,.\, \mathrm{D}`R \subset \alpha \,.\, \mathrm{Cl}`R = \beta\},$$

i.e.

$$(\alpha \uparrow \beta)_\Delta`\beta,$$

and it is easily proved that this is similar to $\alpha \exp \beta$.

The usual formal properties of exponentiation result without much difficulty from the above definitions.

The above definition of exponentiation is so framed as to make propositions on exponentiation independent of the multiplicative axiom, except when exponentiation is to be connected with multiplication, *i.e.* when it is to be shown that the product of ν factors, each of which is μ, is μ^ν. This proposition cannot be proved generally without the multiplicative axiom. Similarly, in the theory of multiplication, the proposition that the sum of ν μ's is $\mu \times_c \nu$ requires the multiplicative axiom (as does also the proposition that a product is zero when and only when one of its factors is zero). Otherwise, the theory of multiplication proceeds without the need for employing the multiplicative axiom.

To take first the connection of addition and multiplication: this connection, in the form in which we naturally suppose it to hold, is affirmed in the proposition:

$$\mu, \nu \in \mathrm{NC} \,.\, \kappa \in \nu \cap \mathrm{Cls\,excl}`\mu \,.\supset .\, s`\kappa \in \mu \times_c \nu \qquad \text{(A)}$$

or

$$\mu, \nu \in \mathrm{NC} \,.\, \kappa \in \nu \cap \mathrm{Cl}`\mu \,.\supset .\, \Sigma`\kappa \in \mu \times_c \nu.$$

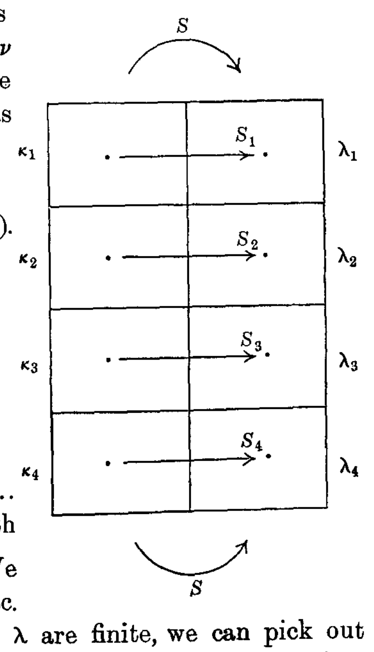
We will take the first of these as being simpler. It affirms that the sum of ν μ's is $\mu \times_c \nu$. This can be proved when ν is finite, whether μ is finite or not; but when ν is infinite, it cannot be proved without the multiplicative axiom. This may be seen as follows. We know that

$$\vdash : \mu, \nu \in \mathrm{NC} \,.\, \alpha \in \mu \,.\, \beta \in \nu \,.\supset .$$
$$\alpha \underset{,,}{\downarrow} \text{``}\beta \in \nu \cap \mathrm{Cls\,excl}`\mu \,.\, s`\alpha \underset{,,}{\downarrow} \text{``}\beta \in \mu \times_c \nu \qquad \text{(B)}.$$

Thus (A) above will result if we can prove

$$\kappa, \lambda \in \nu \cap \mathrm{Cls\,excl}`\mu \,.\supset .\, s`\kappa \,\mathrm{sm}\, s`\lambda,$$

since we shall put $\alpha \underset{,,}{\downarrow} \text{``}\beta$ for λ and use (B).



Since $\kappa, \lambda \in \nu$, we have $\kappa \,\mathrm{sm}\, \lambda$. Assume

$$S \in 1 \rightarrow 1 \,.\, \mathrm{D}`S = \kappa \,.\, \mathrm{Cl}`S = \lambda.$$

Let $\kappa_1, \kappa_2, \ldots$ be members of κ, and let $\lambda_1, \lambda_2, \ldots$ be the members of λ which are correlated with $\kappa_1, \kappa_2, \ldots$ by S, *i.e.* $\lambda_1 = \breve{S}`\kappa_1 \,.\, \lambda_2 = \breve{S}`\kappa_2 \,.$ etc. We have, since $\kappa, \lambda \in \mathrm{Cl}`\mu$, $\kappa_1 \,\mathrm{sm}\, \lambda_1 \,.\, \kappa_2 \,\mathrm{sm}\, \lambda_2 \,.$ etc. Thus $\alpha S \beta \,.\supset_{\alpha,\beta} .\, \alpha \,\mathrm{sm}\, \beta$, *i.e.* $S \subset \mathrm{sm}$. If κ and λ are finite, we can pick out arbitrarily a correlation S_1 for κ_1 and λ_1, another S_2 for κ_2 and λ_2, and so on; then $S_1 \cup S_2 \cup \ldots$ correlates $s`\kappa$ and $s`\lambda$, and therefore $s`\kappa \,\mathrm{sm}\, s`\lambda$. But when κ and λ are infinite, this method is impracticable. In this case, we proceed as follows.

By *73·01, $\quad \alpha \,\overline{\mathrm{sm}}\, \beta = (1 \rightarrow 1) \cap \overleftarrow{\mathrm{D}}`\alpha \cap \overleftarrow{\mathrm{Cl}}`\beta \quad$ Df.

Thus "$\alpha \,\overline{\mathrm{sm}}\, \alpha$" will stand for all the permutations of a class into itself; "$\alpha \,\overline{\mathrm{sm}}\, \beta$" stands for all the permutations of α into β, *i.e.* all the $1 \rightarrow 1$'s whose

domain is α and whose converse domain is β. It is obvious that

$$\vdash : \exists ! \, \alpha \, \overline{\mathrm{sm}} \, \beta \cap \gamma \, \overline{\mathrm{sm}} \, \delta \, . \supset . \, \alpha = \gamma \, . \, \beta = \delta .$$

In the case of the κ and λ above, we know that $\alpha \, \mathrm{sm} \, \beta$ when $\alpha S \beta$; thus

$$\alpha \in \kappa \, . \supset_\alpha . \, \exists ! \, \alpha \, \overline{\mathrm{sm}} \, (\breve{S}\text{‘}\alpha)$$

or

$$\beta \in \lambda \, . \supset_\beta . \, \exists ! \, (S\text{‘}\beta) \, \overline{\mathrm{sm}} \, \beta .$$

Put $$\mathrm{Crp} \, (S)\text{‘}\beta = (S\text{‘}\beta) \, \overline{\mathrm{sm}} \, \beta \quad \mathrm{Df},$$

where "Crp" stands for "correspondence." Thus $\mathrm{Crp}\,(S)\text{‘}\beta$ is the class of all correspondences of $S\text{‘}\beta$ and β; $\mathrm{Crp}\,(S)\text{‘‘}\lambda$ is the class of all such classes of correspondences. If we extract one member out of each of these classes of correspondences, we get a class of relations whose sum is a correlator of $s\text{‘}\kappa$ and $s\text{‘}\lambda$; *i.e.*

$$\varpi \in \mathrm{D}\text{‘‘}\epsilon_\Delta\text{‘}\mathrm{Crp}\,(S)\text{‘‘}\lambda \, . \supset . \, \dot{s}\text{‘}\varpi \in (s\text{‘}\kappa) \, \overline{\mathrm{sm}} \, (s\text{‘}\lambda).$$

Thus the desired result follows whenever

$$\exists ! \, \epsilon_\Delta\text{‘}\mathrm{Crp} \, (S)\text{‘‘}\lambda.$$

Now we have $S \in 1 \to 1 \, . \, S \subseteq \mathrm{sm} \, . \supset . \, \mathrm{Crp}\,(S)\text{‘‘}\lambda \in \mathrm{Cls\,ex^2\,excl}.$

Consequently

$$\mathrm{Mult\,ax} \, . \supset : S \in 1 \to 1 \, . \, S \subseteq \mathrm{sm} \, . \, \mathrm{D}\text{‘}S = \kappa \, . \, \mathrm{Ꮧ}\text{‘}S = \lambda \, . \, \kappa, \lambda \in \mathrm{Cls^2\,excl} \, . \supset . \, s\text{‘}\kappa \, \mathrm{sm} \, s\text{‘}\lambda,$$

whence, by what was said previously,

$$\mathrm{Mult\,ax} \, . \supset : \kappa \in \nu \cap \mathrm{Cls\,excl}\text{‘}\mu \, . \supset . \, s\text{‘}\kappa \in \mu \times_0 \nu \, . \, \Sigma\mathrm{Nc}\text{‘}\kappa = \mu \times_0 \nu.$$

The consideration of $\epsilon_\Delta\text{‘}\mathrm{Crp}\,(S)\text{‘‘}\lambda$ leads similarly to the proposition

$$\vdash :. \, \mathrm{Mult\,ax} \, . \supset : \mu, \nu \in \mathrm{NC} \, . \, \kappa \in \nu \cap \mathrm{Cl}\text{‘}\mu \, . \supset . \, \epsilon_\Delta\text{‘}\kappa \in \mu^\nu \, . \, \Pi\mathrm{Nc}\text{‘}\kappa = \mu^\nu.$$

The proof is closely analogous to that for the connection of addition and multiplication.

It will be seen that, in the above use of the multiplicative axiom, we have two classes of classes κ and λ concerning which we assume

$$(\exists S) \, . \, S \in 1 \to 1 \, . \, S \subseteq \mathrm{sm} \, . \, \mathrm{D}\text{‘}S = \kappa \, . \, \mathrm{Ꮧ}\text{‘}S = \lambda,$$

i.e. we assume that κ and λ are similar classes of similar classes. A slightly modified hypothesis concerning κ and λ will enable us to obtain many results, without the multiplicative axiom, which otherwise might be expected to require this axiom. This is effected as follows.

Put $$\kappa \, \mathrm{sm\,sm} \, \lambda \, . \equiv . \, (\exists T) \, . \, T \in 1 \to 1 \, . \, \mathrm{Ꮧ}\text{‘}T = s\text{‘}\lambda \, . \, \kappa = T_\epsilon\text{‘‘}\lambda,$$

where "sm sm" is a single symbol representing a relation.

When this relation holds between κ and λ, we shall say that κ and λ have "double similarity." In this case, T correlates $s\text{‘}\kappa$ and $s\text{‘}\lambda$, while T_ϵ correlates κ and λ, so that if β is a member of λ, $T_\epsilon\text{‘}\beta$, *i.e.* $T\text{‘‘}\beta$, is its correlate in κ. We shall then have

$$\vdash : \kappa \, \mathrm{sm\,sm} \, \lambda \, . \supset . \, s\text{‘}\kappa \, \mathrm{sm} \, s\text{‘}\lambda,$$

$$\vdash : \kappa \, \mathrm{sm\,sm} \, \lambda \, . \supset . \, \Sigma\mathrm{Nc}\text{‘}\kappa = \Sigma\mathrm{Nc}\text{‘}\lambda,$$

$$\vdash : \kappa \, \mathrm{sm\,sm} \, \lambda \, . \supset . \, \Pi\mathrm{Nc}\text{‘}\kappa = \Pi\mathrm{Nc}\text{‘}\lambda.$$

Also we have

$$\vdash : \kappa \operatorname{sm} \operatorname{sm} \lambda . \supset . (\exists S) . S \in 1 \to 1 . S \mathrel{\subset} \operatorname{sm} . \mathrm{D}`S = \kappa . \mathrm{G}`S = \lambda.$$

Conversely,

$$\vdash : \kappa, \lambda \in \mathrm{Cls}^2 \operatorname{excl} . S \in 1 \to 1 . S \mathrel{\subset} \operatorname{sm} . \mathrm{D}`S = \kappa . \mathrm{G}`S = \lambda .$$
$$\varpi \in \mathrm{D}``\epsilon_\Delta`\mathrm{Crp}(S)``\lambda . T = \dot{s}`\varpi . \supset . T \in 1 \to 1 . \mathrm{G}`T = s`\lambda . \kappa = T_\epsilon``\lambda,$$

whence

$$\vdash :: \text{Mult ax} . \supset :. \kappa, \lambda \in \mathrm{Cls}^2 \operatorname{excl} : (\exists S) . S \in 1 \to 1 . S \mathrel{\subset} \operatorname{sm} . \mathrm{D}`S = \kappa . \mathrm{G}`S = \lambda : \supset . \kappa \operatorname{sm} \operatorname{sm} \lambda.$$

Hence the multiplicative axiom is only required in order to pass from

$$(\exists S) . S \in 1 \to 1 . S \mathrel{\subset} \operatorname{sm} . \mathrm{D}`S = \kappa . \mathrm{G}`S = \lambda$$

to $\kappa \operatorname{sm} \operatorname{sm} \lambda$. It is this fact, and the consequent possibility of diminishing the use of the multiplicative axiom, which has led us to the employment of "sm sm" in the present section.

We treat also, in this section, the relation of greater and less between cardinals. We say that $\mathrm{Nc}`\alpha > \mathrm{Nc}`\beta$ when there is a part of α which is similar to β, but no part of β is similar to α. The principal proposition in this subject is the Schröder-Bernstein theorem, *i.e.*

$$\vdash : \mu \geqslant \nu . \nu \geqslant \mu . \supset . \mu = \nu.$$

This is an immediate consequence of *73·88. It cannot be shown, without assuming the multiplicative axiom, that of any two cardinals one must be the greater, *i.e.*

$$\mu, \nu \in \mathrm{NC} . \mu \neq \nu . \supset : \mu > \nu . \vee . \nu > \mu.$$

If we assume the multiplicative axiom, this results from Zermelo's proof that on that assumption, every class can be well-ordered, together with Cantor's proof that of any two well-ordered series which are not similar, one must be similar to a part of the other. But these propositions cannot be proved till a much later stage (*258).

$*110$. THE ARITHMETICAL SUM OF TWO CLASSES AND OF TWO CARDINALS

Summary of $*110$.

In this number, we start from the definition:

$*110\cdot01$. $\alpha + \beta = \downarrow(\Lambda \cap \beta)\text{“}\iota\text{“}\alpha \cup (\Lambda \cap \alpha)\downarrow\text{“}\iota\text{“}\beta$ Df

$\alpha + \beta$ is called the "arithmetical class-sum" of α and β. The definition is framed so as to give two mutually exclusive classes respectively similar to α and β, so that the number of terms in the logical sum of these two classes is the arithmetical sum of the numbers of terms in α and β respectively. $\alpha + \beta$ is significant whenever α and β are classes, whatever their types may be.

By means of $\alpha + \beta$, we define the arithmetical sum of two cardinals as follows:

$*110\cdot02$. $\mu +_c \nu = \hat{\xi}\{(\exists \alpha, \beta) . \mu = \mathrm{N_0c}\text{‘}\alpha . \nu = \mathrm{N_0c}\text{‘}\beta . \xi \,\mathrm{sm}\, (\alpha + \beta)\}$ Df

This defines the "arithmetical sum of two cardinals." (It is not necessary to *significance* that μ and ν should be cardinals, but only that they should be classes of classes. If, however, either is not a cardinal, $\mu +_c \nu = \Lambda$.) It will be observed that, when μ and ν are typically definite, so are α and β in the above definition; but ξ is typically ambiguous, on account of the ambiguity of "sm." Hence $\mu +_c \nu$ is also typically ambiguous.

It will be shown that $\mu +_c \nu$ is always a cardinal, and that, if

$$\mu = \mathrm{N_0c}\text{‘}\alpha . \nu = \mathrm{N_0c}\text{‘}\beta, \text{ then } \mu +_c \nu = \mathrm{Nc}\text{‘}(\alpha + \beta).$$

Hence whenever μ and ν are cardinals other than Λ, $\mu +_c \nu$ is an existent cardinal in some types, though it may be Λ in others.

Two more definitions are required in this number, namely:

$*110\cdot03$. $\mathrm{Nc}\text{‘}\alpha +_c \mu = \mathrm{N_0c}\text{‘}\alpha +_c \mu$ Df

$*110\cdot04$. $\mu +_c \mathrm{Nc}\text{‘}\alpha = \mu +_c \mathrm{N_0c}\text{‘}\alpha$ Df

These definitions are needed in order to apply the definition of $\mu +_c \nu$ to the case in which μ and ν are replaced by typically ambiguous symbols $\mathrm{Nc}\text{‘}\alpha$ and $\mathrm{Nc}\text{‘}\beta$. It does not make any difference to the value of $\mathrm{Nc}\text{‘}\alpha +_c \mathrm{Nc}\text{‘}\beta$ how the ambiguities of $\mathrm{Nc}\text{‘}\alpha$ and $\mathrm{Nc}\text{‘}\beta$ are determined, so long as they are determined in a way that insures $\exists ! \mathrm{Nc}\text{‘}\alpha . \exists ! \mathrm{Nc}\text{‘}\beta$; but if there are types in which either $\mathrm{Nc}\text{‘}\alpha$ or $\mathrm{Nc}\text{‘}\beta$ is Λ, we get $\mathrm{Nc}\text{‘}\alpha +_c \mathrm{Nc}\text{‘}\beta = \Lambda$ in all types if we determine the ambiguities so that $\mathrm{Nc}\text{‘}\alpha = \Lambda$ or $\mathrm{Nc}\text{‘}\beta = \Lambda$. It is in order to exclude such determinations of the ambiguity that the above definitions are required. Also in connection with these definitions and the corresponding definitions $*113\cdot04\cdot05$ and $*116\cdot03\cdot04$ and $*117\cdot02\cdot03$, the convention II T of the prefatory statement must be noted.

The propositions of the present number begin with the properties of $\alpha + \beta$. We show (∗110·11·12) that $\alpha + \beta$ consists of two mutually exclusive parts, which are respectively similar to α and β; we show (∗110·14) that if α and β are mutually exclusive, $\alpha \cup \beta$ is similar to $\alpha + \beta$, and (∗110·15) that if γ and δ are respectively similar to α and β, then $\gamma + \delta$ is similar to $\alpha + \beta$. We show (∗110·16) that $\mathrm{Nc}`(\alpha + \beta)$ consists of all classes which can be divided into two mutually exclusive parts which are respectively similar to α and β.

We then proceed (∗110·2—·252) to the consideration of $\mu +_c \nu$. Here μ and ν are typically definite, and the definition ∗110·02 applies to any typically definite symbols, such as $\mathrm{N_0c}`\alpha$ or $\mathrm{Nc}\,(\eta)`\alpha$. We prove (∗110·21) that if μ and ν are cardinals, their sum consists of all classes similar to some class of the form $\alpha + \beta$, where $\alpha \in \mu \,.\, \beta \in \nu$; we prove (∗110·22) that the sum of $\mathrm{N_0c}`\alpha$ and $\mathrm{N_0c}`\beta$ is $\mathrm{Nc}`(\alpha + \beta)$, and (∗110·25) that if μ and ν are cardinals, their sum is equal to the sum of the "same" cardinals in any other types in which they are not null, *i.e.*

∗110·25. $\vdash : \mu, \nu \in \mathrm{NC} \,.\, \exists !\, \mathrm{sm}_\eta``\mu \,.\, \exists !\, \mathrm{sm}_\zeta``\nu \,.\, \supset .\, \mu +_c \nu = \mathrm{sm}_\eta``\mu +_c \mathrm{sm}_\zeta``\nu$

We then (∗110·3—·351) consider $\mathrm{Nc}`\alpha +_c \mathrm{Nc}`\beta$, to which we apply the definitions ∗110·03·04. We have

∗110·3. $\vdash .\, \mathrm{Nc}`\alpha +_c \mathrm{Nc}`\beta = \mathrm{N_0c}`\alpha +_c \mathrm{N_0c}`\beta = \mathrm{Nc}`(\alpha + \beta)$

whence the other properties of $\mathrm{Nc}`\alpha +_c \mathrm{Nc}`\beta$ follow from previous propositions.

We then have (∗110·4—·44) various propositions on the type of $\mu +_c \nu$ and its existence and kindred matters. The chief of these are

∗110·4. $\vdash : \exists !\, \mu +_c \nu \,.\, \supset .\, \mu, \nu \in \mathrm{NC} - \iota`\Lambda \,.\, \mu, \nu \in \mathrm{N_0C}$

∗110·42. $\vdash .\, \mu +_c \nu \in \mathrm{NC}$

This proposition requires no hypothesis, because, if μ and ν are not both cardinals, $\mu +_c \nu = \Lambda$, and Λ is a cardinal, by ∗102·74.

Our next set of propositions (∗110·5—·57) are concerned with the permutative and associative laws, which are ∗110·51 and ∗110·56 respectively.

We then (∗110·6—·643) consider the addition of 0 or 1, proving (∗110·61) that a cardinal is unchanged by the addition of 0, and (∗110·643) that $1 +_c 1 = 2$.

∗110·01. $\alpha + \beta = \downarrow (\Lambda \cap \beta)``\iota``\alpha \cup (\Lambda \cap \alpha) \downarrow ``\iota``\beta$ Df

∗110·02. $\mu +_c \nu = \hat{\xi}\{(\exists \alpha, \beta) \,.\, \mu = \mathrm{N_0c}`\alpha \,.\, \nu = \mathrm{N_0c}`\beta \,.\, \xi \,\mathrm{sm}\, (\alpha + \beta)\}$ Df

∗110·03. $\mathrm{Nc}`\alpha +_c \mu = \mathrm{N_0c}`\alpha +_c \mu$ Df

∗110·04. $\mu +_c \mathrm{Nc}`\alpha = \mu +_c \mathrm{N_0c}`\alpha$ Df

These definitions are extended by II T of the prefatory statement.

∗110·1. $\vdash :.\, R \in \alpha + \beta \,.\, \equiv : (\exists x) \,.\, x \in \alpha \,.\, R = (\iota`x) \downarrow (\Lambda \cap \beta) \,.\, \mathrm{v} \,.$
$(\exists y) \,.\, y \in \beta \,.\, R = (\Lambda \cap \alpha) \downarrow (\iota`y)$

[∗38·13·131 . (∗110·01)]

$\ast$110·101. $\vdash.(\iota\text{‘}x)\downarrow(\Lambda\cap\beta)\neq(\Lambda\cap\alpha)\downarrow(\iota\text{‘}y)$

Dem.

$\vdash.\ast55\cdot15.\supset\vdash.\text{D‘}(\iota\text{‘}x)\downarrow(\Lambda\cap\beta)=\iota\text{‘}\iota\text{‘}x.\text{D‘}(\Lambda\cap\alpha)\downarrow(\iota\text{‘}y)=\iota\text{‘}(\Lambda\cap\alpha)$ (1)
$\vdash.\ast51\cdot161.\supset\vdash.\iota\text{‘}x\neq(\Lambda\cap\alpha).$
$[\ast51\cdot23]\quad\supset\vdash.\iota\text{‘}\iota\text{‘}x\neq\iota\text{‘}(\Lambda\cap\alpha)$ (2)
$\vdash.(1).(2).\supset\vdash.\text{D‘}(\iota\text{‘}x)\downarrow(\Lambda\cap\beta)\neq\text{D‘}(\Lambda\cap\alpha)\downarrow(\iota\text{‘}y).\supset\vdash.\text{Prop}$

$\ast$110·11. $\vdash.\downarrow(\Lambda\cap\beta)\text{“}\iota\text{“}\alpha\cap(\Lambda\cap\alpha)\downarrow\text{“}\iota\text{“}\beta=\Lambda$

Dem.

$\vdash.\ast110\cdot101.\supset\vdash:x\in\alpha.R=\downarrow(\Lambda\cap\beta)\text{‘}\iota\text{‘}x.y\in\beta.S=(\Lambda\cap\alpha)\downarrow\text{‘}\iota\text{‘}y.\supset.R\neq S:$
$[\ast37\cdot67]\quad\supset\vdash:R\in\downarrow(\Lambda\cap\beta)\text{“}\iota\text{“}\alpha.S\in(\Lambda\cap\alpha)\downarrow\text{“}\iota\text{“}\beta.\supset.R\neq S$ (1)
$\vdash.(1).\ast24\cdot37.\supset\vdash.\text{Prop}$

$\ast$110·12. $\vdash.\downarrow(\Lambda\cap\beta)\text{“}\iota\text{“}\alpha\,\text{sm}\,\alpha.(\Lambda\cap\alpha)\downarrow\text{“}\iota\text{“}\beta\,\text{sm}\,\beta$ [$\ast$73·41·61·611]

$\ast$110·11·12 give the justification for the use of $\alpha+\beta$ in defining arithmetical addition, since they show that $\alpha+\beta$ consists of two mutually exclusive parts which are respectively similar to α and β.

$\ast$110·13. $\vdash:\gamma\,\text{sm}\,\alpha.\delta\,\text{sm}\,\beta.\gamma\cap\delta=\Lambda.\supset.\gamma\cup\delta\,\text{sm}\,(\alpha+\beta)$

Dem.

$\vdash.\ast110\cdot12.\supset\vdash:\text{Hp}.\supset.\gamma\,\text{sm}\downarrow(\Lambda\cap\beta)\text{“}\iota\text{“}\alpha.\delta\,\text{sm}\,(\Lambda\cap\alpha)\downarrow\text{“}\iota\text{“}\beta$ (1)
$\vdash.(1).\ast110\cdot11.\ast73\cdot71.\supset\vdash.\text{Prop}$

$\ast$110·14. $\vdash:\alpha\cap\beta=\Lambda.\supset.\alpha\cup\beta\,\text{sm}\,(\alpha+\beta)$ [$\ast$110·13.$\ast$73·3]

Thus whenever α and β are mutually exclusive, their logical sum may replace their arithmetical sum in defining the sum of their cardinal numbers.

$\ast$110·15. $\vdash:\gamma\,\text{sm}\,\alpha.\delta\,\text{sm}\,\beta.\supset.\gamma+\delta\,\text{sm}\,\alpha+\beta$

Dem.

$\vdash.\ast110\cdot12.\supset\vdash:\text{Hp}.\supset.\downarrow(\Lambda\cap\delta)\text{“}\iota\text{“}\gamma\,\text{sm}\,\alpha.(\Lambda\cap\gamma)\downarrow\text{“}\iota\text{“}\delta\,\text{sm}\,\beta$ (1)
$\vdash.\ast110\cdot11.\supset\vdash.\downarrow(\Lambda\cap\delta)\text{“}\iota\text{“}\gamma\cap(\Lambda\cap\gamma)\downarrow\text{“}\iota\text{“}\delta=\Lambda$ (2)
$\vdash.(1).(2).\ast110\cdot13.\supset$
$\vdash:\text{Hp}.\supset.\downarrow(\Lambda\cap\delta)\text{“}\iota\text{“}\gamma\cup(\Lambda\cap\gamma)\downarrow\text{“}\iota\text{“}\delta\,\text{sm}\,\alpha+\beta:\supset\vdash.\text{Prop}$

$\ast$110·151. $\vdash:.\alpha\cap\beta=\Lambda.\supset:\xi\,\text{sm}(\alpha\cup\beta).\equiv.(\exists\gamma,\delta).\gamma\,\text{sm}\,\alpha.\delta\,\text{sm}\,\beta.\gamma\cap\delta=\Lambda.\xi=\gamma\cup\delta$

Dem.

$\vdash.\ast73\cdot71.\supset\vdash:.\text{Hp}.\supset:$
$(\exists\gamma,\delta).\gamma\,\text{sm}\,\alpha.\delta\,\text{sm}\,\beta.\gamma\cap\delta=\Lambda.\xi=\gamma\cup\delta.\supset.\xi\,\text{sm}\,(\alpha\cup\beta)$ (1)
$\vdash.\ast72\cdot411.\ast37\cdot25\cdot22.\ast73\cdot22.\supset$
$\vdash:S\in1\to1.\text{D‘}S=\xi.\text{Ɑ‘}S=\alpha\cup\beta.\alpha\cap\beta=\Lambda.\supset.$
$S\text{“}\alpha\cap S\text{“}\beta=\Lambda.\xi=S\text{“}\alpha\cup S\text{“}\beta.S\text{“}\alpha\,\text{sm}\,\alpha.S\text{“}\beta\,\text{sm}\,\beta.$
$[\ast11\cdot36]\supset.(\exists\gamma,\delta).\gamma\,\text{sm}\,\alpha.\delta\,\text{sm}\,\beta.\gamma\cap\delta=\Lambda.\xi=\gamma\cup\delta$ (2)
$\vdash.(2).\ast10\cdot11\cdot23\cdot35.\ast73\cdot1.\supset$
$\vdash:.\text{Hp}.\supset:\xi\,\text{sm}\,(\alpha\cup\beta).\supset.(\exists\gamma,\delta).\gamma\,\text{sm}\,\alpha.\delta\,\text{sm}\,\beta.\gamma\cap\delta=\Lambda.\xi=\gamma\cup\delta$ (3)
$\vdash.(1).(3).\supset\vdash.\text{Prop}$

***110·152.** $\vdash : \xi \operatorname{sm} (\alpha + \beta) . \equiv . (\exists \gamma, \delta) . \gamma \operatorname{sm} \alpha . \delta \operatorname{sm} \beta . \gamma \cap \delta = \Lambda . \xi = \gamma \cup \delta$

Dem.

$\vdash . *110·151·11 . \supset$

$\vdash : \xi \operatorname{sm} (\alpha + \beta) . \equiv . (\exists \gamma, \delta) . \gamma \operatorname{sm} \downarrow (\Lambda \cap \beta) ``\iota``\alpha . \delta \operatorname{sm} (\Lambda \cap \alpha) \downarrow ``\iota``\beta .$
$\gamma \cap \delta = \Lambda . \xi = \gamma \cup \delta .$

$[*73·37 . *110·12] \equiv . (\exists \gamma, \delta) . \gamma \operatorname{sm} \alpha . \delta \operatorname{sm} \beta . \gamma \cap \delta = \Lambda . \xi = \gamma \cup \delta : \supset \vdash . \text{Prop}$

***110·16.** $\vdash . \mathrm{Nc}`(\alpha + \beta) = \hat{\xi} \{(\exists \gamma, \delta) . \gamma \operatorname{sm} \alpha . \delta \operatorname{sm} \beta . \gamma \cap \delta = \Lambda . \xi = \gamma \cup \delta\}$ [*110·152 . *100·1]

***110·17.** $\vdash : \alpha \in t`\beta . \supset . \exists ! \mathrm{Nc} (t`\alpha)`(\alpha + \beta)$

Dem.

$\vdash . *104·43 . \supset$

$\vdash : \mathrm{Hp} . \supset . (\exists \gamma, \delta) . \gamma \operatorname{sm} \alpha . \gamma \subset t`\alpha . \delta \operatorname{sm} \beta . \delta \subset t`\alpha . \gamma \cap \delta = \Lambda .$

$[*22·59] \quad \supset . (\exists \gamma, \delta) . \gamma \operatorname{sm} \alpha . \delta \operatorname{sm} \beta . \gamma \cap \delta = \Lambda . \gamma \cup \delta \subset t`\alpha .$

$[*110·16] \quad \supset . (\exists \xi) . \xi \subset t`\alpha . \xi \in \mathrm{Nc}`(\alpha + \beta) .$

$[*102·6 . *63·5] \supset . \exists ! \mathrm{Nc} (t`\alpha)`(\alpha + \beta) : \supset \vdash . \text{Prop}$

Thus when α and β are of the same type, $\mathrm{Nc}`(\alpha + \beta)$ exists at least in the type next above that of $\alpha$ and $\beta$. We cannot prove that it exists in the type of $\alpha$ and $\beta$. *E.g.* suppose the lowest type contained only one member; then if $x$ were that one member, $\mathrm{Nc}`(\iota`x + \iota`x)$ would not exist in the type to which $\iota`x$ belongs; but would exist in the next type, *i.e.* there would not be two individuals, but there would be two classes, namely $\Lambda$ and $\iota`x$, so that $\iota`\Lambda \cup \iota`\iota`x \in \mathrm{Nc}`(\iota`x + \iota`x)$.

***110·18.** $\vdash . \alpha + \beta \in t`t`(t`\alpha \uparrow t`\beta)$

Dem.

$\vdash . *64·53 . \quad \supset \vdash : x \in \alpha . \supset . \downarrow (\Lambda \cap \beta)`\iota`x \in t`(t`\alpha \uparrow t`\beta)$ (1)

$\vdash . (1) . *37·61 . \supset \vdash . \downarrow (\Lambda \cap \beta)``\iota``\alpha \subset t`(t`\alpha \uparrow t`\beta)$ (2)

Similarly $\quad \vdash . (\Lambda \cap \alpha) \downarrow ``\iota``\beta \subset t`(t`\alpha \uparrow t`\beta)$ (3)

$\vdash . (2) . (3) . \quad \supset \vdash . \alpha + \beta \subset t`(t`\alpha \uparrow t`\beta) .$

$[*63·5] \quad \supset \vdash . \alpha + \beta \in t`t`(t`\alpha \uparrow t`\beta) . \supset \vdash . \text{Prop}$

***110·2.** $\vdash : \xi \in \mu +_c \nu . \equiv . (\exists \alpha, \beta) . \mu = \mathrm{N_0c}`\alpha . \nu = \mathrm{N_0c}`\alpha . \xi \operatorname{sm} (\alpha + \beta)$ [(*110·02)]

***110·201.** $\vdash :. \xi \in \mu +_c \nu . \equiv : \mu, \nu \in \mathrm{NC} : (\exists \alpha, \beta) . \alpha \in \mu . \beta \in \nu . \xi \operatorname{sm} (\alpha + \beta)$ [*103·27 . *110·2]

***110·202.** $\vdash :. \xi \in \mu +_c \nu . \equiv :$
$\exists ! \mu . \exists ! \nu : (\exists \gamma, \delta) . \mu = \mathrm{Nc}`\gamma . \nu = \mathrm{Nc}`\delta . \gamma \cap \delta = \Lambda . \xi = \gamma \cup \delta$

Dem.

$\vdash . *110·2·152 . \supset \vdash :. \xi \in \mu +_c \nu . \equiv :$
$(\exists \alpha, \beta, \gamma, \delta) . \mu = \mathrm{N_0c}`\alpha . \nu = \mathrm{N_0c}`\beta . \gamma \operatorname{sm} \alpha . \delta \operatorname{sm} \beta . \gamma \cap \delta = \Lambda . \xi = \gamma \cup \delta :$

$[*103·28] \equiv : (\exists \gamma, \delta) . \exists ! \mu . \exists ! \nu . \mu = \mathrm{Nc}`\gamma . \nu = \mathrm{Nc}`\delta . \gamma \cap \delta = \Lambda . \xi = \gamma \cup \delta :.$
$\supset \vdash . \text{Prop}$

***110·21.** $\vdash :. \mu, \nu \in \text{NC} . \supset : \xi \in \mu +_c \nu . \equiv . (\exists \alpha, \beta) . \alpha \in \mu . \beta \in \nu . \xi \,\text{sm}\, (\alpha + \beta)$ [*110·201]

***110·211.** $\vdash :. \mu, \nu \in \text{NC} . \supset :. \xi \in \mu +_c \nu . \equiv .$
$(\exists \gamma, \delta) . \gamma \in \text{sm}``\mu . \delta \in \text{sm}``\nu . \gamma \cap \delta = \Lambda . \xi = \gamma \cup \delta$

Dem.

$\vdash . *110·21·152 . \supset \vdash :. \text{Hp} . \supset : \xi \in \mu +_c \nu . \equiv .$
$(\exists \alpha, \beta, \gamma, \delta) . \alpha \in \mu . \beta \in \nu . \gamma \,\text{sm}\, \alpha . \delta \,\text{sm}\, \beta . \gamma \cap \delta = \Lambda . \xi = \gamma \cup \delta .$
[*37·1] $\equiv . (\exists \gamma, \delta) . \gamma \in \text{sm}``\mu . \delta \in \text{sm}``\nu . \gamma \cap \delta = \Lambda . \xi = \gamma \cup \delta :. \supset \vdash . \text{Prop}$

***110·212.** $\vdash :. \mu, \nu \in \text{NC} . \supset : \xi \in \mu +_c \nu . \equiv . (\exists \gamma) . \gamma \in \text{sm}``\mu . \gamma \subset \xi . \xi - \gamma \in \text{sm}``\nu$

Dem.

$\vdash . *110·211 . *24·47 . \supset$
$\vdash :. \text{Hp} . \supset : \xi \in \mu +_c \nu . \equiv . (\exists \gamma, \delta) . \gamma \in \text{sm}``\mu . \delta \in \text{sm}``\nu . \gamma \subset \xi . \delta = \xi - \gamma .$
[*13·195] $\equiv . (\exists \gamma) . \gamma \in \text{sm}``\mu . \gamma \subset \xi . \xi - \gamma \in \text{sm}``\nu :. \supset \vdash . \text{Prop}$

***110·22.** $\vdash . \text{N}_0\text{c}`\alpha +_c \text{N}_0\text{c}`\beta = \text{Nc}`(\alpha + \beta)$

Dem.

$\vdash . *103·4 . *110·211 . \supset$
$\vdash : \xi \in \text{N}_0\text{c}`\alpha +_c \text{N}_0\text{c}`\beta . \equiv . (\exists \gamma, \delta) . \gamma \in \text{Nc}`\alpha . \delta \in \text{Nc}`\beta . \gamma \cap \delta = \Lambda . \xi = \gamma \cup \delta .$
[*100·31] $\equiv . (\exists \gamma, \delta) . \gamma \,\text{sm}\, \alpha . \delta \,\text{sm}\, \beta . \gamma \cap \delta = \Lambda . \xi = \gamma \cup \delta .$
[*110·16] $\equiv . \xi \in \text{Nc}`(\alpha + \beta) : \supset \vdash . \text{Prop}$

***110·221.** $\vdash : \xi \in \text{Nc}(\eta)`\alpha +_c \text{Nc}(\zeta)`\beta . \equiv . \exists ! \text{Nc}(\eta)`\alpha . \exists ! \text{Nc}(\zeta)`\beta . \xi \in \text{Nc}`(\alpha + \beta)$

Dem.

$\vdash . *110·202 . \supset \vdash :. \xi \in \text{Nc}(\eta)`\alpha +_c \text{Nc}(\zeta)`\beta .$
$\equiv : \exists ! \text{Nc}(\eta)`\alpha . \exists ! \text{Nc}(\zeta)`\beta : (\exists \gamma, \delta) . \text{Nc}(\eta)`\alpha = \text{Nc}`\gamma .$
$\text{Nc}(\zeta)`\beta = \text{Nc}`\delta . \gamma \cap \delta = \Lambda . \xi = \gamma \cup \delta :$
[*100·35] $\equiv : \exists ! \text{Nc}(\eta)`\alpha . \exists ! \text{Nc}(\zeta)`\beta : (\exists \gamma, \delta) . \gamma \,\text{sm}\, \alpha . \delta \,\text{sm}\, \beta . \gamma \cap \delta = \Lambda . \xi = \gamma \cup \delta :$
[*110·16] $\equiv : \exists ! \text{Nc}(\eta)`\alpha . \exists ! \text{Nc}(\zeta)`\beta . \xi \in \text{Nc}`(\alpha + \beta) :. \supset \vdash . \text{Prop}$

***110·23.** $\vdash : \exists ! \text{Nc}(\eta)`\alpha . \exists ! \text{Nc}(\zeta)`\beta . \supset .$
$\text{Nc}(\eta)`\alpha +_c \text{Nc}(\zeta)`\beta = \text{Nc}`(\alpha + \beta) = \text{N}_0\text{c}`\alpha +_c \text{N}_0\text{c}`\beta$ [*110·221·22]

Thus $\text{Nc}(\eta)`\alpha +_c \text{Nc}(\zeta)`\beta$ is independent of η and ζ so long as $\text{Nc}`\alpha$ and $\text{Nc}`\beta$ exist in the types of η and ζ respectively.

***110·231.** $\vdash :. \text{Nc}(\eta)`\alpha = \Lambda . \vee . \text{Nc}(\zeta)`\beta = \Lambda : \supset . \text{Nc}(\eta)`\alpha +_c \text{Nc}(\zeta)`\beta = \Lambda$ [*110·221]

***110·24.** $\vdash : \eta \,\text{sm}\, \alpha . \zeta \,\text{sm}\, \beta . \supset . \text{N}_0\text{c}`\eta +_c \text{N}_0\text{c}`\zeta = \text{N}_0\text{c}`\alpha +_c \text{N}_0\text{c}`\beta$

Dem.

$\vdash . *103·42 . \supset \vdash : \text{Hp} . \supset . \text{N}_0\text{c}`\eta = \text{Nc}(\eta)`\alpha . \text{N}_0\text{c}`\zeta = \text{Nc}(\zeta)`\beta \quad (1)$
$\vdash . (1) . *103·13 . \supset \vdash : \text{Hp} . \supset . \exists ! \text{Nc}(\eta)`\alpha . \exists ! \text{Nc}(\zeta)`\beta .$
[*110·23] $\supset . \text{Nc}(\eta)`\alpha +_c \text{Nc}(\zeta)`\beta = \text{N}_0\text{c}`\alpha +_c \text{N}_0\text{c}`\beta \quad (2)$
$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***110·25.** $\vdash : \mu, \nu \in \mathrm{NC} . \exists ! \mathrm{sm}_\eta{}^{\prime\prime}\mu . \exists ! \mathrm{sm}_\zeta{}^{\prime\prime}\nu . \supset . \mu +_c \nu = \mathrm{sm}_\eta{}^{\prime\prime}\mu +_c \mathrm{sm}_\zeta{}^{\prime\prime}\nu$

Dem.

$\vdash . *103·27 . \supset \vdash : \mu, \nu \in \mathrm{NC} . \alpha \in \mu . \beta \in \nu . \exists ! \mathrm{sm}_\eta{}^{\prime\prime}\mu . \exists ! \mathrm{sm}_\zeta{}^{\prime\prime}\nu .$

$\supset . \mu = \mathrm{N_0c}^{\prime}\alpha . \nu = \mathrm{N_0c}^{\prime}\beta . \exists ! \mathrm{sm}_\eta{}^{\prime\prime}\mu . \exists ! \mathrm{sm}_\zeta{}^{\prime\prime}\nu .$

$[*103·41 . *102·85] \supset . \mu = \mathrm{N_0c}^{\prime}\alpha . \nu = \mathrm{N_0c}^{\prime}\beta . \mathrm{sm}_\eta{}^{\prime\prime}\mu = \mathrm{Nc}(\eta)^{\prime}\alpha .$

$\mathrm{sm}_\zeta{}^{\prime\prime}\nu = \mathrm{Nc}(\zeta)^{\prime}\beta . \exists ! \mathrm{Nc}(\eta)^{\prime}\alpha . \exists ! \mathrm{Nc}(\zeta)^{\prime}\beta .$

$[*110·23] \quad \supset . \mu +_c \nu = \mathrm{N_0c}^{\prime}\alpha +_c \mathrm{N_0c}^{\prime}\beta = \mathrm{sm}_\eta{}^{\prime\prime}\mu +_c \mathrm{sm}_\zeta{}^{\prime\prime}\nu \quad (1)$

$\vdash . (1) . *10·11·23·35 . \supset$

$\vdash : \mu, \nu \in \mathrm{NC} . \exists ! \mu . \exists ! \nu . \exists ! \mathrm{sm}_\eta{}^{\prime\prime}\mu . \exists ! \mathrm{sm}_\zeta{}^{\prime\prime}\nu . \supset . \mu +_c \nu = \mathrm{sm}_\eta{}^{\prime\prime}\mu +_c \mathrm{sm}_\zeta{}^{\prime\prime}\nu \quad (2)$

$\vdash . *37·29 . \mathrm{Transp} . \supset \vdash : \exists ! \mathrm{sm}_\eta{}^{\prime\prime}\mu . \exists ! \mathrm{sm}_\zeta{}^{\prime\prime}\nu . \supset . \exists ! \mu . \exists ! \nu \quad (3)$

$\vdash . (2) . (3) . \supset \vdash . \mathrm{Prop}$

***110·251.** $\vdash : \mu, \nu \in \mathrm{NC} . \supset . \mu^{(1)} +_c \nu^{(1)} = \mu +_c \nu$

Dem.

$\vdash . *110·25 . *104·265 . \supset$

$\vdash : \mathrm{Hp} . \exists ! \mu^{(1)} . \exists ! \nu^{(1)} . \supset . \mu^{(1)} +_c \nu^{(1)} = \mu +_c \nu \quad (1)$

$\vdash . *110·202 . \supset \vdash : \sim(\exists ! \mu^{(1)} . \exists ! \nu^{(1)}) . \supset . \mu^{(1)} +_c \nu^{(1)} = \Lambda \quad (2)$

$\vdash . *104·264 . \supset \vdash : \mathrm{Hp}(2) . \supset . \sim(\exists ! \mu . \exists ! \nu) .$

$[*110·202] \quad \supset . \mu +_c \nu = \Lambda .$

$[(2)] \quad \supset . \mu^{(1)} +_c \nu^{(1)} = \mu +_c \nu \quad (3)$

$\vdash . (1) . (3) . \supset \vdash . \mathrm{Prop}$

***110·252.** $\vdash : \mu, \nu \in \mathrm{NC} . \supset . \mu_{(00)} +_c \nu_{(00)} = \mu +_c \nu$ [Proof as in *110·251]

A similar proof applies to $\mu^{(2)}$, $\nu^{(2)}$, etc., and to any such derived cardinals whose existence follows from that of μ and ν. The proposition does not hold generally for $\mu_{(1)}$, $\nu_{(1)}$ and other descending derived cardinals, because they may be null when μ and ν exist.

The following proposition (*110·3) is more often used than any other in this number except *110·4.

***110·3.** $\vdash . \mathrm{Nc}^{\prime}\alpha +_c \mathrm{Nc}^{\prime}\beta = \mathrm{N_0c}^{\prime}\alpha +_c \mathrm{N_0c}^{\prime}\beta = \mathrm{Nc}^{\prime}(\alpha + \beta)$ [*110·22.(*110·03·04)]

***110·31.** $\vdash : \gamma \,\mathrm{sm}\, \alpha . \delta \,\mathrm{sm}\, \beta . \supset . \mathrm{Nc}^{\prime}\gamma +_c \mathrm{Nc}^{\prime}\delta = \mathrm{Nc}^{\prime}\alpha +_c \mathrm{Nc}^{\prime}\beta$ [*110·24·3]

The following proposition is frequently used.

***110·32.** $\vdash : \alpha \cap \beta = \Lambda . \supset . \mathrm{Nc}^{\prime}\alpha +_c \mathrm{Nc}^{\prime}\beta = \mathrm{Nc}^{\prime}(\alpha \cup \beta)$ [*110·3·14]

***110·33.** $\vdash : \xi \in \mathrm{Nc}^{\prime}\alpha +_c \mathrm{Nc}^{\prime}\beta . \equiv . (\exists \gamma, \delta) . \gamma \,\mathrm{sm}\, \alpha . \delta \,\mathrm{sm}\, \beta . \gamma \cap \delta = \Lambda . \xi = \gamma \cup \delta$ [*110·3·16]

The above proposition is used in *110·63. We might have used the above to define arithmetical addition, but this method would have been less convenient than the method adopted in this number, both because there would have been more difficulty in dealing with types, and because the existence of $\mathrm{Nc}^{\prime}\alpha +_c \mathrm{Nc}^{\prime}\beta$ (in the types in which it does exist) is less evident with the above definition than with the definitions given in *110·01·02·03·04.

$*110\cdot331.$ $\vdash . \mathrm{Nc}`\alpha +_0 \mathrm{Nc}`\beta = \hat{\xi}\{(\exists\gamma) . \gamma \,\mathrm{sm}\, \alpha . \xi - \gamma \,\mathrm{sm}\, \beta . \gamma \subset \xi\}$

Dem.

$\vdash . *110\cdot33 . *24\cdot47 . \supset$

$\vdash : \xi \in \mathrm{Nc}`\alpha +_0 \mathrm{Nc}`\beta . \equiv . (\exists\gamma, \delta) . \gamma \,\mathrm{sm}\, \alpha . \delta \,\mathrm{sm}\, \beta . \gamma \subset \xi . \delta = \xi - \gamma .$

$[*13\cdot195] \quad \equiv . (\exists\gamma) . \gamma \,\mathrm{sm}\, \alpha . \xi - \gamma \,\mathrm{sm}\, \beta . \gamma \subset \xi : \supset \vdash . \mathrm{Prop}$

$*110\cdot34.$ $\vdash : \exists ! \mathrm{Nc}(\eta)`\alpha . \exists ! \mathrm{Nc}(\zeta)`\beta . \supset . \mathrm{Nc}(\eta)`\alpha +_0 \mathrm{Nc}(\zeta)`\beta = \mathrm{Nc}`\alpha +_0 \mathrm{Nc}`\beta$
$[*110\cdot23\cdot3]$

$*110\cdot35.$ $\vdash . \mathrm{N}^1\mathrm{c}`\alpha +_0 \mathrm{N}^1\mathrm{c}`\beta = \mathrm{Nc}`\alpha +_0 \mathrm{Nc}`\beta \quad [*104\cdot102\cdot21 . *110\cdot34]$

$*110\cdot351.$ $\vdash . \mathrm{N}_{00}\mathrm{c}`\alpha +_0 \mathrm{N}_{00}\mathrm{c}`\beta = \mathrm{Nc}`\alpha +_0 \mathrm{Nc}`\beta \quad [*106\cdot21 . *110\cdot34]$

Similar propositions will hold generally for *ascending* cardinals.

The following proposition ($*110\cdot4$) is the most used of the propositions in this number. It is useful both in the form given, and in the form resulting from transposition, in which it shows that $\mu +_0 \nu = \Lambda$ unless both μ and ν are existent cardinals. It is chiefly useful in avoiding the necessity of the hypothesis $\mu, \nu \in \mathrm{NC}$ in such propositions as the commutative and associative laws.

$*110\cdot4.$ $\vdash : \exists ! \mu +_0 \nu . \supset . \mu, \nu \in \mathrm{NC} - \iota`\Lambda . \mu, \nu \in \mathrm{N_0C} \quad [*110\cdot201\cdot202\cdot2]$

The following propositions, down to $*110\cdot411$ inclusive, are concerned with types. They are not referred to in the sequel.

$*110\cdot401.$ $\vdash : \mu = \mathrm{N_0c}`\alpha . \nu = \mathrm{N_0c}`\beta . \supset . \alpha + \beta \in t`t`(\mu \uparrow \nu)$

Dem.

$\vdash . *110\cdot18 . *103\cdot12 . \supset \vdash : \mathrm{Hp} . \supset . \alpha + \beta \in t`t`(t`\alpha \uparrow t`\beta) . \alpha \in \mu . \beta \in \nu .$

$[*63\cdot11] \quad \supset . \alpha + \beta \in t`t`(t`\alpha \uparrow t`\beta) . t`\alpha = t_0`\mu . t`\beta = t_0`\nu .$

$[*13\cdot12] \quad \supset . \alpha + \beta \in t`t`(t_0`\mu \uparrow t_0`\nu) .$

$[*64\cdot13] \quad \supset . \alpha + \beta \in t`t`(\mu \uparrow \nu) : \supset \vdash . \mathrm{Prop}$

$*110\cdot402.$ $\vdash : \mu, \nu \in \mathrm{N_0C} . \supset . \exists ! (\mu +_0 \nu) \cap t`t`(\mu \uparrow \nu)$

Dem.

$\vdash . *110\cdot22 . *100\cdot3 . \supset$

$\vdash : \mu = \mathrm{N_0c}`\alpha . \nu = \mathrm{N_0c}`\beta . \supset . \alpha + \beta \in \mu +_0 \nu .$

$[*110\cdot401] \quad \supset . \alpha + \beta \in (\mu +_0 \nu) \cap t`t`(\mu \uparrow \nu) .$

$[*10\cdot24] \quad \supset . \exists ! (\mu +_0 \nu) \cap t`t`(\mu \uparrow \nu) \quad (1)$

$\vdash . (1) . *103\cdot2 . \supset \vdash . \mathrm{Prop}$

$*110\cdot403.$ $\vdash : \mu, \nu \in \mathrm{N_0C} . \equiv . \exists ! (\mu +_0 \nu) \cap t`t`(\mu \uparrow \nu) \quad [*110\cdot402\cdot4]$

$*110\cdot404.$ $\vdash . \exists ! (\mathrm{Nc}`\alpha +_0 \mathrm{Nc}`\beta) \cap t`t`(t`\alpha \uparrow t`\beta) \quad [*110\cdot18\cdot3 . *100\cdot3]$

***110·41.** $\vdash : \mu, \nu \in \mathrm{N_0C} \,.\, t`\mu = t`\nu \,.\, \supset \,.\, \exists ! (\mu +_c \nu) \cap t`\mu$

Dem.

$\vdash . *103·11 . \supset \vdash : \mu = \mathrm{N_0c}`\alpha \,.\, \nu = \mathrm{N_0c}`\beta \,.\, t`\mu = t`\nu \,.\, \supset .$
$\mu \subset t`\alpha \,.\, \nu \subset t`\beta \,.\, t`\mu = t`\nu \,.$
[*63·21·35] $\supset . t_0`\mu = t`\alpha \,.\, t_0`\nu = t`\beta \,.\, t_0`\mu = t_0`\nu \,.$
[*13·16·17] $\supset . t`\alpha = t`\beta = t_0`\mu \,.$
[*110·17] $\supset . \exists ! \mathrm{Nc}`(\alpha + \beta) \cap t`t_0`\mu \,.$
[*110·22.*63·19] $\supset . \exists ! (\mu +_c \nu) \cap t`\mu : \supset \vdash .$ Prop

***110·411.** $\vdash : t`\alpha = t`\beta \,.\, \supset \,.\, \exists ! (\mathrm{Nc}`\alpha +_c \mathrm{Nc}`\beta) \cap t`t`\alpha \,.\, \exists ! \mathrm{Nc}\,(t`\alpha)`(\alpha + \beta)$
[*110·17·3]

It will be observed that the following proposition (*110·42) requires no hypothesis. This is owing to *110·4 and *102·74.

110·42. $\vdash . \mu +_c \nu \in \mathrm{NC}$

Dem.

$\vdash . *110·22 . \quad \supset \vdash : \mu = \mathrm{N_0c}`\alpha \,.\, \nu = \mathrm{N_0c}`\beta \,.\, \supset . \mu +_c \nu = \mathrm{Nc}`(\alpha + \beta) \,.$
[*100·41] $\supset . \mu +_c \nu \in \mathrm{NC}$ (1)
$\vdash . (1) . *103·2 . \quad \supset \vdash : \mu, \nu \in \mathrm{N_0C} \,.\, \supset . \mu +_c \nu \in \mathrm{NC}$ (2)
$\vdash . *110·4 . \mathrm{Transp} . \supset \vdash : \sim(\mu, \nu \in \mathrm{N_0C}) \,.\, \supset . \mu +_c \nu = \Lambda \,.$
[*102·74] $\supset . \mu +_c \nu \in \mathrm{NC}$ (3)
$\vdash . (2) . (3) . \supset \vdash .$ Prop

***110·43.** $\vdash : \mu +_c \nu = \mathrm{N_0c}`\eta \,.\, \equiv . \eta \in \mu +_c \nu$ [*110·42 . *103·26]

***110·44.** $\vdash . \mathrm{sm}``(\mu +_c \nu) = \mu +_c \nu$

Dem.

$\vdash . *37·1 . *110·2 . \supset$
$\vdash : \xi \in \mathrm{sm}``(\mu +_c \nu) \,.\, \equiv . (\exists \eta, \alpha, \beta) \,.\, \mu = \mathrm{N_0c}`\alpha \,.\, \nu = \mathrm{N_0c}`\beta \,.\, \eta \,\mathrm{sm}\, (\alpha + \beta) \,.\, \xi \,\mathrm{sm}\, \eta \,.$
[*73·3·32] $\equiv . (\exists \alpha, \beta) \,.\, \mu = \mathrm{N_0c}`\alpha \,.\, \nu = \mathrm{N_0c}`\beta \,.\, \xi \,\mathrm{sm}\, (\alpha + \beta) \,.$
[*110·2] $\equiv . \xi \in \mu +_c \nu : \supset \vdash .$ Prop

The above proposition depends upon the fact that $\mu +_c \nu$ is typically ambiguous, even when μ and ν are typically definite. It is used in the theory of inductive cardinals (*120·32·41·424).

The following propositions are concerned with the commutative and associative laws for arithmetical addition of cardinals.

***110·5.** $\vdash . \beta + \alpha = \mathrm{Cnv}``(\alpha + \beta)$

Dem.

$\vdash . *55·14 . \supset \vdash . \mathrm{Cnv}``(\alpha + \beta) = \Lambda \cap \beta \downarrow ``\iota``\alpha \cup \downarrow \Lambda \cap \alpha``\iota``\beta$
[(*110·01)] $= \beta + \alpha \,.\, \supset \vdash .$ Prop

***110·501.** $\vdash . \beta + \alpha \,\mathrm{sm}\, \alpha + \beta$ [*110·5 . *73·4]

***110·51.** $\vdash . \mu +_c \nu = \nu +_c \mu$ [*110·2·501 . *73·37]

It is not necessary to the truth of the above proposition that μ and ν should be cardinals. If either is not a cardinal, $\mu +_c \nu$ and $\nu +_c \mu$ are both Λ.

The following propositions lead to the associative law (∗110·56).

∗110·52. $\vdash : \xi \operatorname{sm} (\alpha + \beta) + \gamma . \equiv . (\exists \pi, \rho, \sigma) . \pi \operatorname{sm} \alpha . \rho \operatorname{sm} \beta . \sigma \operatorname{sm} \gamma .$
$\pi \cap \rho = \Lambda . \pi \cap \sigma = \Lambda . \rho \cap \sigma = \Lambda . \xi = \pi \cup \rho \cup \sigma$

Dem.

$\vdash . *110·152 . \supset \vdash :. \xi \operatorname{sm} (\alpha + \beta) + \gamma . \equiv : (\exists \eta, \sigma) . \eta \operatorname{sm} (\alpha + \beta) .$
$\sigma \operatorname{sm} \gamma . \eta \cap \sigma = \Lambda . \xi = \eta \cup \sigma :$

[∗110·152] $\equiv : (\exists \pi, \rho, \eta, \sigma) . \pi \operatorname{sm} \alpha . \rho \operatorname{sm} \beta . \pi \cap \rho = \Lambda . \eta = \pi \cup \rho .$
$\sigma \operatorname{sm} \gamma . \eta \cap \sigma = \Lambda . \xi = \eta \cup \sigma :$

[∗13·195.∗22·68.∗24·32] $\equiv : (\exists \pi, \rho, \sigma) . \pi \operatorname{sm} \alpha . \rho \operatorname{sm} \beta . \sigma \operatorname{sm} \gamma . \pi \cap \rho = \Lambda .$
$\pi \cap \sigma = \Lambda . \rho \cap \sigma = \Lambda . \xi = \pi \cup \rho \cup \sigma :. \supset \vdash .$ Prop

∗110·521. $\vdash : \xi \operatorname{sm} \alpha + (\beta + \gamma) . \equiv . (\exists \pi, \rho, \sigma) . \pi \operatorname{sm} \alpha . \rho \operatorname{sm} \beta . \sigma \operatorname{sm} \gamma .$
$\pi \cap \rho = \Lambda . \pi \cap \sigma = \Lambda . \rho \cap \sigma = \Lambda . \xi = \pi \cup \rho \cup \sigma$ [∗110·501·52]

∗110·53. $\vdash . (\alpha + \beta) + \gamma \operatorname{sm} \alpha + (\beta + \gamma)$ [∗110·52·521]

∗110·531. $\alpha + \beta + \gamma = (\alpha + \beta) + \gamma$ Df

∗110·54. $\vdash . (\mathrm{Nc}`\alpha +_c \mathrm{Nc}`\beta) +_c \mathrm{Nc}`\gamma = \mathrm{Nc}`(\alpha + \beta + \gamma)$

Dem.

$\vdash . *110·3 . \supset \vdash . (\mathrm{Nc}`\alpha +_c \mathrm{Nc}`\beta) +_c \mathrm{Nc}`\gamma = \mathrm{Nc}`(\alpha + \beta) +_c \mathrm{Nc}`\gamma$
[∗110·3.(∗110·531)] $= \mathrm{Nc}`(\alpha + \beta + \gamma) . \supset \vdash .$ Prop

∗110·541. $\vdash . \mathrm{Nc}`\alpha +_c (\mathrm{Nc}`\beta +_c \mathrm{Nc}`\gamma) = \mathrm{Nc}`(\alpha + \beta + \gamma)$

Dem.

$\vdash . *110·3 . \supset \vdash . \mathrm{Nc}`\alpha +_c (\mathrm{Nc}`\beta +_c \mathrm{Nc}`\gamma) = \mathrm{Nc}`\{\alpha + (\beta + \gamma)\}$
[∗110·53.(∗110·531)] $= \mathrm{Nc}`(\alpha + \beta + \gamma) . \supset \vdash .$ Prop

∗110·55. $\vdash . (\mathrm{Nc}`\alpha +_c \mathrm{Nc}`\beta) +_c \mathrm{Nc}`\gamma = \mathrm{Nc}`\alpha +_c (\mathrm{Nc}`\beta +_c \mathrm{Nc}`\gamma)$ [∗110·54·541]

∗110·551. $\vdash . (\mathrm{N_0c}`\alpha +_c \mathrm{N_0c}`\beta) +_c \mathrm{N_0c}`\gamma = \mathrm{N_0c}`\alpha +_c (\mathrm{N_0c}`\beta +_c \mathrm{N_0c}`\gamma)$
[∗110·55 . (∗110·03·04)]

∗110·56. $\vdash . (\mu +_c \nu) +_c \varpi = \mu +_c (\nu +_c \varpi)$

Dem.

$\vdash . *110·551 . *103·2 . \supset$
$\vdash : \mu, \nu, \varpi \in \mathrm{N_0C} . \supset . (\mu +_c \nu) +_c \varpi = \mu +_c (\nu +_c \varpi)$ (1)
$\vdash . *110·4 . \mathrm{Transp} . \supset$
$\vdash : \sim(\mu, \nu, \varpi \in \mathrm{N_0C}) . \supset . (\mu +_c \nu) +_c \varpi = \Lambda . \mu +_c (\nu +_c \varpi) = \Lambda .$
[∗13·171] $\supset . (\mu +_c \nu) +_c \varpi = \mu +_c (\nu +_c \varpi)$ (2)
$\vdash . (1) . (2) . \supset \vdash .$ Prop

This is the associative law for arithmetical addition. It will be seen that, like the commutative law, it does not require that μ, ν, ϖ should be cardinals.

∗110·561. $\mu +_c \nu +_c \varpi = (\mu +_c \nu) +_c \varpi$ Df

∗110·57. $\vdash . (\mu +_c \nu) +_c (\varpi +_c \rho) = \mu +_c \nu +_c \varpi +_c \rho$ [∗110·56 . (∗110·561)]

The following propositions, concerning the addition of 0 or 1, are used frequently in dealing with inductive cardinals (*120).

***110·6.** $\vdash : \mu \epsilon \mathrm{NC} . \supset . \mu +_c 0 = \mathrm{sm}``\mu$

Dem.

$\vdash . *101 \cdot 11 . *110 \cdot 21 . \supset$

$\vdash :. \mathrm{Hp} . \supset : \xi \epsilon \mu +_c 0 . \equiv . (\exists \alpha, \beta) . \alpha \epsilon \mu . \beta \epsilon 0 . \xi \,\mathrm{sm}\, (\alpha + \beta) .$

[*54·102] $\equiv . (\exists \alpha) . \alpha \epsilon \mu . \xi \,\mathrm{sm}\, (\alpha + \Lambda) .$

[*110·152] $\equiv . (\exists \alpha, \gamma, \delta) . \alpha \epsilon \mu . \gamma \,\mathrm{sm}\, \alpha . \delta \,\mathrm{sm}\, \Lambda . \gamma \cap \delta = \Lambda . \xi = \gamma \cup \delta .$

[*73·47] $\equiv . (\exists \alpha, \gamma) . \alpha \epsilon \mu . \gamma \,\mathrm{sm}\, \alpha . \xi = \gamma .$

[*13·195] $\equiv . (\exists \alpha) . \alpha \epsilon \mu . \xi \,\mathrm{sm}\, \alpha .$

[*37·1] $\equiv . \xi \epsilon \mathrm{sm}``\mu :. \supset \vdash . \mathrm{Prop}$

When μ is a typically definite cardinal, $\mathrm{sm}``\mu$ is the same cardinal rendered typically ambiguous; when $\mu$ is a typically ambiguous cardinal, $\mathrm{sm}``\mu$ is μ. In place of the above proposition, we *might* write $\mu \epsilon \mathrm{NC} . \supset . \mu +_c 0 = \mu$; this would be true whenever the ambiguity of $\mu +_c 0$ was so determined as to make it significant. But the above form gives more information.

***110·61.** $\vdash . \mathrm{Nc}`\alpha +_c 0 = \mathrm{Nc}`\alpha$

Dem.

$\vdash . *101 \cdot 1 . \supset \vdash . \mathrm{Nc}`\alpha +_c 0 = \mathrm{Nc}`\alpha +_c \mathrm{Nc}`\Lambda$

[*110·32] $= \mathrm{Nc}`(\alpha \cup \Lambda)$

[*24·24] $= \mathrm{Nc}`\alpha . \supset \vdash . \mathrm{Prop}$

In this proposition, $\mathrm{Nc}`\alpha$ is typically ambiguous; hence we escape the necessity of putting $\mathrm{sm}``\mathrm{Nc}`\alpha$ on the right, as we should have to do if $\mathrm{Nc}`\alpha$ were typically definite. We can deduce *110·61 from *110·6 as follows:

$\vdash . *110 \cdot 3 . \supset \vdash . \mathrm{Nc}`\alpha +_c 0 = \mathrm{N_0c}`\alpha +_c 0$

[*110·6] $= \mathrm{sm}``\mathrm{N_0c}`\alpha$

[*103·4] $= \mathrm{Nc}`\alpha$

We have to travel via $\mathrm{N_0c}`\alpha$ in this proof, in order to avoid the possibility of a typical determination of $\mathrm{Nc}`\alpha$ which would make $\mathrm{Nc}`\alpha = \Lambda$. It is for the same reason that we cannot put "$\mathrm{sm}``\mathrm{Nc}`\alpha = \mathrm{Nc}`\alpha$"; for if the first $\mathrm{Nc}`\alpha$ is determined to a type in which $\mathrm{Nc}`\alpha = \Lambda$, while the second is not, this equation becomes false.

***110·62.** $\vdash : \mu +_c \nu = 0 . \equiv . \mu = 0 . \nu = 0$

Dem.

$\vdash . *103 \cdot 27 . *101 \cdot 11 \cdot 13 . \supset \vdash . 0 = \mathrm{N_0c}`\Lambda$ (1)

$\vdash . (1) . *110 \cdot 43 . \supset$

$\vdash :. \mu +_c \nu = 0 . \equiv : \Lambda \epsilon \mu +_c \nu :$

[*110·202] $\equiv : \exists ! \mu . \exists ! \nu : (\exists \gamma, \delta) . \mu = \mathrm{Nc}`\gamma . \nu = \mathrm{Nc}`\delta . \gamma \cap \delta = \Lambda . \gamma \cup \delta = \Lambda :$

[*24·32.*13·22] $\equiv : \exists ! \mu . \exists ! \nu : \mu = \mathrm{Nc}`\Lambda . \nu = \mathrm{Nc}`\Lambda :$

[*101·1·12] $\equiv : \mu = 0 . \nu = 0 :. \supset \vdash . \mathrm{Prop}$

***110·63.** $\vdash . \mathrm{Nc}`\alpha +_c 1 = \hat{\xi}\{(\exists\gamma, y) . \gamma \,\mathrm{sm}\, \alpha . y \sim\in \gamma . \xi = \gamma \cup \iota`y\}$

Dem.

$\vdash . *101{\cdot}2 . \supset$

$\vdash . \mathrm{Nc}`\alpha +_c 1 = \mathrm{Nc}`\alpha +_c \mathrm{Nc}`\iota`x$

[*110·33] $= \hat{\xi}\{(\exists\gamma, \delta) . \gamma \,\mathrm{sm}\, \alpha . \delta \,\mathrm{sm}\, \iota`x . \gamma \cap \delta = \Lambda . \xi = \gamma \cup \delta\}$

[*73·45] $= \hat{\xi}\{(\exists\gamma, \delta) . \gamma \,\mathrm{sm}\, \alpha . \delta \in 1 . \gamma \cap \delta = \Lambda . \xi = \gamma \cup \delta\}$

[*52·1] $= \hat{\xi}\{(\exists\gamma, \delta, y) . \gamma \,\mathrm{sm}\, \alpha . \delta = \iota`y . \gamma \cap \delta = \Lambda . \xi = \gamma \cup \delta\}$

[*13·195.*51·211] $= \hat{\xi}\{(\exists\gamma, y) . \gamma \,\mathrm{sm}\, \alpha . y \sim\in \gamma . \xi = \gamma \cup \iota`y\} . \supset \vdash . \mathrm{Prop}$

The above proposition is much used in the theory of finite and infinite, both cardinal and ordinal. It connects mathematical induction for inductive cardinals with mathematical induction for inductive classes (cf. *120).

***110·631.** $\vdash : \mu \in \mathrm{NC} . \supset . \mu +_c 1 = \hat{\xi}\{(\exists\gamma, y) . \gamma \in \mathrm{sm}``\mu . y \sim\in \gamma . \xi = \gamma \cup \iota`y\}$

Dem.

$\vdash . *110{\cdot}211 . *101{\cdot}21 . \supset$

$\vdash : \mathrm{Hp} . \supset . \mu +_c 1 = \hat{\xi}\{(\exists\gamma, \delta) . \gamma \in \mathrm{sm}``\mu . \delta \in \mathrm{sm}``1 . \gamma \cap \delta = \Lambda . \xi = \gamma \cup \delta\}$

[*101·28] $= \hat{\xi}\{(\exists\gamma, \delta) . \gamma \in \mathrm{sm}``\mu . \delta \in 1 . \gamma \cap \delta = \Lambda . \xi = \gamma \cup \delta\}$

[*52·1.*51·211] $= \hat{\xi}\{(\exists\gamma, y) . \gamma \in \mathrm{sm}``\mu . y \sim\in \gamma . \xi = \gamma \cup \iota`y\} : \supset \vdash . \mathrm{Prop}$

The proposition

$$\mu \in \mathrm{NC} . \supset . \mu +_c 1 = \hat{\xi}\{(\exists\gamma, y) . \gamma \in \mu . y \sim\in \gamma . \xi \,\mathrm{sm}\, \gamma \cup \iota`y\}$$

which might at first sight seem demonstrable, will only be true universally if the total number of objects in any one type is not finite. For suppose α is a type, and $\mu = \mathrm{N_0c}`\alpha$. Then if $\alpha$ is a finite class, $\mu = \iota`\alpha$. Hence $\gamma \in \mu . \supset_{\gamma, y} . y \in \gamma$. Hence $\hat{\xi}\{(\exists\gamma, y) . \gamma \in \mu . y \sim\in \gamma . \xi \,\mathrm{sm}\, (\gamma \cup \iota`y)\} = \Lambda$ in all types. But $\mu +_c 1$ will exist in all types higher than that of γ. If on the other hand the number of entities in α is infinite, we shall have

$$y \in \alpha . \supset . \alpha - \iota`y \in \mathrm{Nc}`\alpha . y \sim\in \alpha - \iota`y.$$

Hence in this case the above proposition will be true universally.

***110·632.** $\vdash : \mu \in \mathrm{NC} . \supset . \mu +_c 1 = \hat{\xi}\{(\exists y) . y \in \xi . \xi - \iota`y \in \mathrm{sm}``\mu\}$

Dem.

$\vdash . *110{\cdot}631 . *51{\cdot}211{\cdot}22 . \supset$

$\vdash : \mathrm{Hp} . \supset . \mu +_c 1 = \hat{\xi}\{(\exists\gamma, y) . \gamma \in \mathrm{sm}``\mu . y \in \xi . \gamma = \xi - \iota`y\}$

[*13·195] $= \hat{\xi}\{(\exists y) . y \in \xi . \xi - \iota`y \in \mathrm{sm}``\mu\} : \supset \vdash . \mathrm{Prop}$

***110·64.** $\vdash . 0 +_c 0 = 0$ [*110·62]

***110·641.** $\vdash . 1 +_c 0 = 0 +_c 1 = 1$ [*110·51·61 . *101·2]

***110·642.** $\vdash . 2 +_c 0 = 0 +_c 2 = 2$ [*110·51·61 . *101·31]

***110·643.** $\vdash . 1 +_0 1 = 2$

Dem.

$\vdash . *110·632 . *101·21·28 . \supset$

$\vdash . 1 +_0 1 = \hat{\xi}\{(\exists y) . y \in \xi . \xi - \iota`y \in 1\}$

$[*54·3] \quad = 2 . \supset \vdash . \text{Prop}$

The above proposition is occasionally useful. It is used at least three times, in *113·66 and *120·123·472.

*110·7·71 are required for proving *110·72, and *110·72 is used in *117·3, which is a fundamental proposition in the theory of greater and less.

***110·7.** $\vdash : \beta \subset \alpha . \supset . (\exists \mu) . \mu \in \text{NC} . \text{Nc}`\alpha = \text{Nc}`\beta +_0 \mu$

Dem.

$\vdash . *24·411·21 . \supset \vdash : \text{Hp} . \supset . \alpha = \beta \cup (\alpha - \beta) . \beta \cap (\alpha - \beta) = \Lambda .$

$[*110·32] \qquad \supset . \text{Nc}`\alpha = \text{Nc}`\beta +_0 \text{Nc}`(\alpha - \beta) : \supset \vdash . \text{Prop}$

***110·71.** $\vdash : (\exists \mu) . \text{Nc}`\alpha = \text{Nc}`\beta +_0 \mu . \supset . (\exists \delta) . \delta \,\text{sm}\, \beta . \delta \subset \alpha$

Dem.

$\vdash . *100·3 . *110·4 . \supset$

$\vdash : \text{Nc}`\alpha = \text{Nc}`\beta +_0 \mu . \supset . \mu \in \text{NC} - \iota`\Lambda \qquad (1)$

$\vdash . *110·3 . \supset \vdash : \text{Nc}`\alpha = \text{Nc}`\beta +_0 \text{Nc}`\gamma . \equiv . \text{Nc}`\alpha = \text{Nc}`(\beta + \gamma) .$

$[*100·3·31] \qquad \supset . \alpha \,\text{sm}\, (\beta + \gamma) .$

$[*73·1] \qquad \supset . (\exists R) . R \in 1 \rightarrow 1 . \text{D}`R = \alpha . \text{Ɑ}`R = \downarrow \Lambda_\gamma``\iota``\beta \cup \Lambda_\beta \downarrow ``\iota``\gamma .$

$[*37·15] \qquad \supset . (\exists R) . R \in 1 \rightarrow 1 . \downarrow \Lambda_\gamma``\iota``\beta \subset \text{Ɑ}`R . R`` \downarrow \Lambda_\gamma``\iota``\beta \subset \alpha .$

$[*110·12 . *73·22] \supset . (\exists \delta) . \delta \subset \alpha . \delta \,\text{sm}\, \beta \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

The above proof depends upon the fact that "$\text{Nc}`\alpha$" and "$\text{Nc}`\beta +_0 \mu$" are typically ambiguous, and therefore, when they are asserted to be equal, this must hold in *any* type, and therefore, in particular, in that type for which we have $\alpha \in \text{Nc}`\alpha$, *i.e.* for $\text{N}_0\text{c}`\alpha$. This is why the use of *100·3 is legitimate.

***110·72.** $\vdash : (\exists \delta) . \delta \,\text{sm}\, \beta . \delta \subset \alpha . \equiv . (\exists \mu) . \mu \in \text{NC} . \text{Nc}`\alpha = \text{Nc}`\beta +_0 \mu$

Dem.

$\vdash . *100·321 . *110·7 . \supset$

$\vdash :. \delta \,\text{sm}\, \beta . \delta \subset \alpha . \supset : \text{Nc}`\delta = \text{Nc}`\beta : (\exists \mu) . \mu \in \text{NC} . \text{Nc}`\alpha = \text{Nc}`\delta +_0 \mu :$

$[*13·12] \qquad \supset : (\exists \mu) . \mu \in \text{NC} . \text{Nc}`\alpha = \text{Nc}`\beta +_0 \mu \qquad (1)$

$\vdash . (1) . *110·71 . \supset \vdash . \text{Prop}$

$*$111. DOUBLE SIMILARITY

Summary of $*$111.

The arithmetical properties of a class, so far as these do not require or assume that it is a class of classes, are the same for any similar class. But a class of classes has many arithmetical properties which it does not share with all similar classes of classes. For example, if κ is a class of classes, the number of members of $s`\kappa$ is an arithmetical property of κ, but it is obvious that this is not determined by the number of members of κ, but requires also a knowledge of the numbers of members of members of κ. For example, let κ consist of the two members α and β, and let λ consist of γ and δ. Then $\kappa \operatorname{sm} \lambda$; but in order to be able to infer $s`\kappa \operatorname{sm} s`\lambda$, we require $\kappa, \lambda \epsilon \mathrm{Cls}^2\,\mathrm{excl}$ and $\alpha \operatorname{sm} \gamma \,.\, \beta \operatorname{sm} \delta$ or $\alpha \operatorname{sm} \delta \,.\, \beta \operatorname{sm} \gamma$ or some such further datum. The relation of "double similarity," to be defined in the present number, is a relation between classes of classes, which, when it holds between κ and λ, insures that all the arithmetical properties of κ and λ are the same, *e.g.* we have (in particular) $\mathrm{Nc}`s`\kappa = \mathrm{Nc}`s`\lambda$ and $\mathrm{Nc}`\epsilon_\Delta`\kappa = \mathrm{Nc}`\epsilon_\Delta`\lambda$. This relation we denote by "sm sm," which is to be read as one symbol. It is defined as follows: We define first the class of "double correlators" of κ and λ, which we denote by "$\kappa \,\overline{\operatorname{sm}}\,\overline{\operatorname{sm}}\, \lambda$," and of which the definition is

$$*111\cdot01.\quad \kappa \,\overline{\operatorname{sm}}\,\overline{\operatorname{sm}}\, \lambda = (1 \to 1) \cap \overleftarrow{\mathrm{C}}`s`\lambda \cap \hat{T}(\kappa = T_\epsilon``\lambda) \quad \mathrm{Df}$$

so that

$$\vdash : T \epsilon \kappa \,\overline{\operatorname{sm}}\,\overline{\operatorname{sm}}\, \lambda \,.\equiv .\, T \epsilon 1 \to 1 \,.\, \mathrm{C}`T = s`\lambda \,.\, \kappa = T_\epsilon``\lambda.$$

We then define "$\kappa \operatorname{sm} \operatorname{sm} \lambda$" as meaning that $\kappa \,\overline{\operatorname{sm}}\,\overline{\operatorname{sm}}\, \lambda$ is not null, *i.e.* that there is at least one double correlator of κ and λ.

To illustrate the nature of a double correlator, let us suppose that κ consists of the two classes α_1 and α_2, and that α_1 consists of x_{11}, x_{12}, while α_2 consists of x_{21}, x_{22}, x_{23}. Similarly let λ consist of β_1 and β_2, while β_1 consists of y_{11}, y_{12} and β_2 consists of y_{21}, y_{22}, y_{23}. Now let T correlate each x with the y having the same two suffixes. Then T is a one-one, and its converse domain is $s`\lambda$. Moreover $T_\epsilon`\beta_1$ (which is $T``\beta_1) = \alpha_1$, and $T_\epsilon`\beta_2 = \alpha_2$, so that $T_\epsilon``\lambda = \kappa$. Thus T is a double correlator according to the definition.

The essential characteristic of a double correlator T is that (1) T is a correlator of $s`\kappa$ and $s`\lambda$, (2) $T_\epsilon \upharpoonright \lambda$ is a correlator of κ and λ. If we write S in place of $T_\epsilon \upharpoonright \lambda$, then if $\beta \epsilon \lambda$, we have $S`\beta \epsilon \kappa$; moreover $T \upharpoonright \beta$ is a correlator of $S`\beta$ and β. Thus κ and λ are similar classes of similar classes. They are not merely this, however, for we not only know that $S`\beta$ is similar to β, but

we know a particular correlator of $S`\beta$ and β, namely $T \upharpoonright \beta$. This is essential to the use of double similarity, as will appear shortly.

Let us consider the relation between κ and λ which consists in their being similar classes of similar classes. This means that there is a correlator S of κ and λ, such that, if $\beta \epsilon \lambda$, $S`\beta$ is similar to β. That is to say, we are to consider the hypothesis

$$(\exists S) \,.\, S \epsilon 1 \rightarrow 1 \,.\, \mathrm{D}`S = \kappa \,.\, \text{Ꮖ}`S = \lambda \,.\, S \mathrel{\mathsf{C}} \mathrm{sm}$$

or, as it may be more briefly expressed,

$$\exists ! \, \kappa \, \overline{\mathrm{sm}} \, \lambda \cap \mathrm{Rl}`\mathrm{sm}.$$

Let us assume $S \epsilon \kappa \, \overline{\mathrm{sm}} \, \lambda \cap \mathrm{Rl}`\mathrm{sm}$. If we attempt to prove (say) that $s`\kappa$ is similar to $s`\lambda$, we find that we are forced to assume the multiplicative axiom, unless κ and λ are finite. This necessity arises as follows. Let us put

$$\mathrm{Crp}\,(S)`\beta = (S`\beta) \, \overline{\mathrm{sm}} \, \beta,$$

where "Crp" stands for "correspondence." Then we know that whenever $\beta \epsilon \lambda$, $\mathrm{Crp}\,(S)`\beta$ is not null. Further it is easy to prove that, if $\kappa$ and $\lambda$ are classes of mutually exclusive classes, and if we can pick out one representative member of $\mathrm{Crp}\,(S)`\beta$ for each value of β which is a member of λ, then the relational sum of all these representative correlations gives us a correlator of $s`\kappa$ and $s`\lambda$. That is, we have

$$\vdash : \kappa, \lambda \epsilon \mathrm{Cls}^2 \,\mathrm{excl} \,.\, S \epsilon \kappa \, \overline{\mathrm{sm}} \, \lambda \cap \mathrm{Rl}`\mathrm{sm} \,.\, R \epsilon \epsilon_\Delta`\mathrm{Crp}(S)``\lambda \,.\, \supset .\, \dot{s}`\mathrm{D}`R \epsilon (s`\kappa) \, \overline{\mathrm{sm}} \, (s`\lambda).$$

But in order to infer hence $s`\kappa \,\mathrm{sm}\, s`\lambda$, we need $\exists ! \, \epsilon_\Delta`\mathrm{Crp}\,(S)``\lambda$, *i.e.* we need to be able to pick out a particular correlator for each pair of similar classes $S`\beta$ and $\beta$. This, however, cannot be done in general without assuming the multiplicative axiom. It follows that we must not define two classes as having double similarity when $\exists ! \, \kappa \, \overline{\mathrm{sm}} \, \lambda \cap \mathrm{Rl}`\mathrm{sm}$, but must give a definition which enables us to specify a particular correlator for each pair of similar classes. This is what is effected by the above definition of double correlators, where our S is given as of the form $T_\epsilon \upharpoonright \lambda$, where $T \epsilon 1 \rightarrow 1 \,.\, \text{Ꮖ}`T = s`\lambda$. If the multiplicative axiom is assumed, but in general not otherwise, we have (*111·5)

$$\kappa, \lambda \epsilon \mathrm{Cls}^2 \,\mathrm{excl} \,.\, \supset : \kappa \,\mathrm{sm}\,\mathrm{sm}\, \lambda \,.\equiv .\, \exists ! \, \kappa \, \overline{\mathrm{sm}} \, \lambda \cap \mathrm{Rl}`\mathrm{sm}.$$

In the present number, we shall begin with various properties of double correlators. We prove (*111·11) that T is a double correlator of κ and λ when, and only when, T is a correlator of $s`\kappa$ and $s`\lambda$, and $T_\epsilon \upharpoonright \lambda$ is a correlator of κ and λ. We prove (*111·112) that in the same hypothesis, $T_\epsilon \upharpoonright \lambda \epsilon \kappa \, \overline{\mathrm{sm}} \, \lambda \cap \mathrm{Rl}`\mathrm{sm}$. We prove (*111·13) that $I \upharpoonright s`\lambda$ is a double correlator of λ with itself; that (*111·131) if T is a double correlator of κ and λ, $\breve{T}$ is a double correlator of λ and κ; that (*111·132) if S, T are double correlators of κ with λ and of λ with μ respectively, $S | T$ is a double correlator of κ with μ. Hence it follows (*111·45·451·452) that double similarity is reflexive, symmetrical, and transitive.

We then proceed (*111·2—·34) to consider $\text{Crp}(S)``\lambda$, where it is to be supposed that $S$ is a correlator of $S``\lambda$ and λ, and that $S`\beta$ is similar to β if $\beta \epsilon \lambda$. We prove

*111·32. $\vdash : \lambda, S``\lambda \epsilon \text{Cls}^2 \text{excl} . S \epsilon 1 \rightarrow 1 . R \epsilon \epsilon_\Delta`\text{Crp}(S)``\lambda . M = \dot{s}`\text{D}`R . \supset .$
$M \epsilon 1 \rightarrow 1 . \text{ɑ}`M = s`\lambda . S``\lambda = M_\epsilon``\lambda . S \upharpoonright \lambda = M_\epsilon \upharpoonright \lambda$

Thus in the case supposed, M is a double correlator of $S``\lambda$ and λ. Thus

*111·322. $\vdash : \kappa, \lambda \epsilon \text{Cls}^2 \text{excl} . S \epsilon \kappa \,\overline{\text{sm}}\, \lambda . R \epsilon \epsilon_\Delta`\text{Crp}(S)``\lambda . M = \dot{s}`\text{D}`R . \supset .$
$M \epsilon \kappa \,\overline{\text{sm}}\,\overline{\text{sm}}\, \lambda . S = M_\epsilon \upharpoonright \lambda$

We then proceed (*111·4—·47) to various propositions on "sm sm," and finally (*111·5·51·53) state three propositions which assume the multiplicative axiom, namely

*111·5. If $\kappa, \lambda \epsilon \text{Cls}^2 \text{excl}$, then $\kappa \,\text{sm sm}\, \lambda . \equiv . \exists ! \kappa \,\overline{\text{sm}}\, \lambda \cap \text{Rl}`\text{sm}$.

*111·51. In the same case, $\exists ! \kappa \,\overline{\text{sm}}\, \lambda \cap \text{Rl}`\text{sm} . \supset . s`\kappa \,\text{sm}\, s`\lambda$, *i.e.* if κ and λ are similar classes of mutually exclusive similar classes, their sums are similar.

*111·53. In the same case, if $\kappa, \lambda \epsilon \text{Cls}^2 \text{excl}$, $\kappa \,\text{sm sm}\, \lambda$. Hence the multiplicative axiom implies that two classes of μ mutually exclusive classes each of which has ν terms, have the same number of terms in their sum.

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*111·01. $\kappa \,\overline{\text{sm}}\,\overline{\text{sm}}\, \lambda = (1 \rightarrow 1) \cap \overleftarrow{\text{ɑ}}`s`\lambda \cap \hat{T}(\kappa = T_\epsilon``\lambda)$ Df

*111·02. $\text{Crp}(S)`\beta = (S`\beta) \,\overline{\text{sm}}\, \beta$ Df

*111·03. $\text{sm sm} = \hat{\kappa}\hat{\lambda}(\exists ! \kappa \,\overline{\text{sm}}\,\overline{\text{sm}}\, \lambda)$ Df

*111·1. $\vdash : T \epsilon \kappa \,\overline{\text{sm}}\,\overline{\text{sm}}\, \lambda . \equiv . T \epsilon 1 \rightarrow 1 . \text{ɑ}`T = s`\lambda . \kappa = T_\epsilon``\lambda$ [(*111·01)]

*111·11. $\vdash : T \epsilon \kappa \,\overline{\text{sm}}\,\overline{\text{sm}}\, \lambda . \equiv . T \epsilon (s`\kappa) \,\overline{\text{sm}}\, (s`\lambda) . T_\epsilon \upharpoonright \lambda \epsilon \kappa \,\overline{\text{sm}}\, \lambda$

Dem.

$\vdash . *37·25 . \text{Fact} . \supset \vdash : \text{ɑ}`T = s`\lambda . \kappa = T_\epsilon``\lambda . \supset . \text{D}`T = T``s`\lambda . \kappa = T_\epsilon``\lambda .$

[*40·38] $\supset . \text{D}`T = s`T```\lambda . \kappa = T_\epsilon``\lambda .$

[(*37·04)] $\supset . \text{D}`T = s`\kappa$ (1)

$\vdash . *72·451 . *60·57 . *35·65 . \supset$

$\vdash : T \epsilon 1 \rightarrow 1 . \text{ɑ}`T = s`\lambda . \supset . T_\epsilon \upharpoonright \lambda \epsilon 1 \rightarrow 1 . \lambda = \text{ɑ}`(T_\epsilon \upharpoonright \lambda)$ (2)

$\vdash . *37·401 . \qquad \supset \vdash : \kappa = T_\epsilon``\lambda . \equiv . \kappa = \text{D}`(T_\epsilon \upharpoonright \lambda)$ (3)

$\vdash . (1) . (2) . (3) . *4·71 . \supset \vdash : T \epsilon 1 \rightarrow 1 . \text{ɑ}`T = s`\lambda . \kappa = T_\epsilon``\lambda . \equiv .$

$T \epsilon 1 \rightarrow 1 . \text{D}`T = s`\kappa . \text{ɑ}`T = s`\lambda . T_\epsilon \upharpoonright \lambda \epsilon 1 \rightarrow 1 . \text{D}`(T_\epsilon \upharpoonright \lambda) = \kappa . \text{ɑ}`(T_\epsilon \upharpoonright \lambda) = \lambda$ (4)

$\vdash . (4) . *111·1 . *73·03 . \supset \vdash . \text{Prop}$

*111·111. $\vdash : T \epsilon \kappa \,\overline{\text{sm}}\,\overline{\text{sm}}\, \lambda . \supset . T_\epsilon \upharpoonright \lambda \subseteq \text{sm}$

Dem.

$\vdash . *111·1 . *60·57 . \supset \vdash : \text{Hp} . \supset . T \epsilon 1 \rightarrow 1 . \lambda \subset \text{Cl}`\text{ɑ}`T .$

[*73·5] $\supset . T_\epsilon \upharpoonright \lambda \subseteq \text{sm} : \supset \vdash . \text{Prop}$

*111·112. $\vdash : T \epsilon \kappa \,\overline{\text{sm}}\,\overline{\text{sm}}\, \lambda . \supset . T_\epsilon \upharpoonright \lambda \epsilon \kappa \,\overline{\text{sm}}\, \lambda \cap \text{Rl}`\text{sm}$ [*111·11·111]

The two following propositions are useful lemmas for the case when T is replaced (as it often is) by $T \upharpoonright \alpha$.

$*111 \cdot 12$. $\vdash : s`\lambda \subset \alpha . \supset . (T \upharpoonright \alpha)_\epsilon``\lambda = T_\epsilon``\lambda . (T \upharpoonright \alpha)_\epsilon \upharpoonright \lambda = T_\epsilon \upharpoonright \lambda$

Dem.

$\vdash . *37 \cdot 101 \cdot 421 . \supset \vdash : \beta \subset \alpha . \supset . (T \upharpoonright \alpha)_\epsilon`\beta = T_\epsilon`\beta$ (1)

$\vdash . *40 \cdot 13 . \quad \supset \vdash :. \text{Hp} . \supset : \beta \epsilon \lambda . \supset . \beta \subset \alpha$ (2)

$\vdash . (1) . (2) . \quad \supset \vdash :. \text{Hp} . \supset : \beta \epsilon \lambda . \supset . (T \upharpoonright \alpha)_\epsilon`\beta = T_\epsilon`\beta :$

$[*37 \cdot 69 . *35 \cdot 71] \quad \supset : (T \upharpoonright \alpha)_\epsilon``\lambda = T_\epsilon``\lambda . (T \upharpoonright \alpha)_\epsilon \upharpoonright \lambda = T_\epsilon \upharpoonright \lambda :. \supset \vdash . \text{Prop}$

$*111 \cdot 121$. $\vdash . (T \upharpoonright s`\lambda)_\epsilon``\lambda = T_\epsilon``\lambda = (T_\epsilon \upharpoonright \lambda)``\lambda . (T \upharpoonright s`\lambda)_\epsilon \upharpoonright \lambda = T_\epsilon \upharpoonright \lambda$

Dem.

$\vdash . *37 \cdot 421 . \supset \vdash . T_\epsilon``\lambda = (T_\epsilon \upharpoonright \lambda)``\lambda$ (1)

$\vdash . (1) . *111 \cdot 12 \dfrac{s`\lambda}{\alpha} . \supset \vdash . \text{Prop}$

$*111 \cdot 13$. $\vdash . I \upharpoonright s`\lambda \epsilon \lambda \,\overline{\text{sm}}\, \overline{\text{sm}}\, \lambda$

Dem.

$\vdash . *72 \cdot 17 . *50 \cdot 5 \cdot 52 . \supset \vdash . I \upharpoonright s`\lambda \epsilon 1 \rightarrow 1 . \text{Ɑ}`(I \upharpoonright s`\lambda) = s`\lambda$ (1)

$\vdash . *111 \cdot 121 . \quad \supset \vdash . (I \upharpoonright s`\lambda)_\epsilon``\lambda = I_\epsilon``\lambda$

$[*50 \cdot 16 \cdot 17] \quad = \lambda$ (2)

$\vdash . (1) . (2) . *111 \cdot 1 . \supset \vdash . \text{Prop}$

$*111 \cdot 131$. $\vdash : T \epsilon \kappa \,\overline{\text{sm}}\, \overline{\text{sm}}\, \lambda . \equiv . \breve{T} \epsilon \lambda \,\overline{\text{sm}}\, \overline{\text{sm}}\, \kappa$

Dem.

$\vdash . *71 \cdot 212 . \supset \vdash : T \epsilon 1 \rightarrow 1 . \equiv . \breve{T} \epsilon 1 \rightarrow 1$ (1)

$\vdash . *111 \cdot 11 . \supset \vdash : T \epsilon \kappa \,\overline{\text{sm}}\, \overline{\text{sm}}\, \lambda . \supset . \text{D}`T = s`\kappa$ (2)

$\vdash . *111 \cdot 1 . (2) . *60 \cdot 57 . \supset$

$\vdash : T \epsilon \kappa \,\overline{\text{sm}}\, \overline{\text{sm}}\, \lambda . \supset . T \epsilon 1 \rightarrow 1 . \kappa \subset \text{Cl}`\text{D}`T . \lambda \subset \text{Cl}`\text{Ɑ}`T . \kappa = T_\epsilon``\lambda .$

$[*74 \cdot 6] \quad \supset . \lambda = (\breve{T})_\epsilon``\kappa$ (3)

$\vdash . (1) . (2) . (3) . *111 \cdot 1 . \supset \vdash : T \epsilon \kappa \,\overline{\text{sm}}\, \overline{\text{sm}}\, \lambda . \supset . \breve{T} \epsilon \lambda \,\overline{\text{sm}}\, \overline{\text{sm}}\, \kappa$ (4)

$\vdash . (4) \dfrac{\breve{T}}{T} . \quad \supset \vdash : \breve{T} \epsilon \lambda \,\overline{\text{sm}}\, \overline{\text{sm}}\, \kappa . \supset . T \epsilon \kappa \,\overline{\text{sm}}\, \overline{\text{sm}}\, \lambda$ (5)

$\vdash . (4) . (5) . \supset \vdash . \text{Prop}$

$*111 \cdot 132$. $\vdash : S \epsilon \kappa \,\overline{\text{sm}}\, \overline{\text{sm}}\, \lambda . T \epsilon \lambda \,\overline{\text{sm}}\, \overline{\text{sm}}\, \mu . \supset . S|T \epsilon \kappa \,\overline{\text{sm}}\, \overline{\text{sm}}\, \mu$

Dem.

$\vdash . *111 \cdot 11 . *73 \cdot 311 . \supset$

$\vdash : \text{Hp} . \supset . S|T \epsilon (s`\kappa) \,\overline{\text{sm}}\, (s`\mu) . (S_\epsilon \upharpoonright \lambda)|(T_\epsilon \upharpoonright \mu) \epsilon \kappa \,\overline{\text{sm}}\, \mu$ (1)

$\vdash . *35 \cdot 354 . \supset \vdash . (S_\epsilon \upharpoonright \lambda)|(T_\epsilon \upharpoonright \mu) = S_\epsilon|(\lambda \upharpoonleft T_\epsilon \upharpoonright \mu)$ (2)

$\vdash . *74 \cdot 251 . *111 \cdot 1 . \supset \vdash : \text{Hp} . \supset . S_\epsilon|(\lambda \upharpoonleft T_\epsilon \upharpoonright \mu) = S_\epsilon|(T_\epsilon \upharpoonright \mu)$

$[*35 \cdot 23] \quad = (S_\epsilon|T_\epsilon) \upharpoonright \mu$

$[*37 \cdot 34] \quad = (S|T)_\epsilon \upharpoonright \mu$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash : \text{Hp} . \supset . S|T \epsilon (s`\kappa) \,\overline{\text{sm}}\, (s`\mu) . (S|T)_\epsilon \upharpoonright \mu \epsilon \kappa \,\overline{\text{sm}}\, \mu .$

$[*111 \cdot 11] \quad \supset . S|T \epsilon \kappa \,\overline{\text{sm}}\, \overline{\text{sm}}\, \mu : \supset \vdash . \text{Prop}$

$*111{\cdot}14$. $\vdash : T \upharpoonright s`\lambda \in \kappa \,\overline{\mathrm{sm}}\,\overline{\mathrm{sm}}\, \lambda \,.\equiv .\, T \upharpoonright s`\lambda \in 1 \to 1 \,.\, s`\lambda \subset \mathrm{C}`T \,.\, \kappa = T_\epsilon``\lambda$

Dem.

$\vdash . *111{\cdot}1{\cdot}121 . \supset$

$\vdash : T \upharpoonright s`\lambda \in \kappa \,\overline{\mathrm{sm}}\,\overline{\mathrm{sm}}\, \lambda \,.\equiv .\, T \upharpoonright s`\lambda \in 1 \to 1 \,.\, \mathrm{C}`(T \upharpoonright s`\lambda) = s`\lambda \,.\, \kappa = T_\epsilon``\lambda \,.$

$[*35{\cdot}65] \qquad \equiv .\, T \upharpoonright s`\lambda \in 1 \to 1 \,.\, s`\lambda \subset \mathrm{C}`T \,.\, \kappa = T_\epsilon``\lambda : \supset \vdash . \mathrm{Prop}$

$*111{\cdot}15$. $\vdash : T \upharpoonright s`\lambda \in \kappa \,\overline{\mathrm{sm}}\,\overline{\mathrm{sm}}\, \lambda \,.\equiv .\, T \upharpoonright s`\lambda \in (s`\kappa) \,\overline{\mathrm{sm}}\, (s`\lambda) \,.\, T_\epsilon \upharpoonright \lambda \in \kappa \,\overline{\mathrm{sm}}\, \lambda$

Dem.

$\vdash . *111{\cdot}11 . \supset$

$\vdash : T \upharpoonright s`\lambda \in \kappa \,\overline{\mathrm{sm}}\,\overline{\mathrm{sm}}\, \lambda \,.\equiv .\, T \upharpoonright s`\lambda \in (s`\kappa) \,\overline{\mathrm{sm}}\, (s`\lambda) \,.\, (T \upharpoonright s`\lambda)_\epsilon \upharpoonright \lambda \in \kappa \,\overline{\mathrm{sm}}\, \lambda \quad (1)$

$\vdash . (1) . *111{\cdot}121 . \supset \vdash . \mathrm{Prop}$

$*111{\cdot}16$. $\vdash : \exists ! \,\alpha \,\overline{\mathrm{sm}}\, \beta \cap \gamma \,\overline{\mathrm{sm}}\, \delta \,.\supset .\, \alpha = \gamma \,.\, \beta = \delta$

Dem.

$\vdash . *73{\cdot}03 . \supset \vdash : \mathrm{Hp} . \supset . (\exists R) . \mathrm{D}`R = \alpha \,.\, \mathrm{C}`R = \beta \,.\, \mathrm{D}`R = \gamma \,.\, \mathrm{C}`R = \delta \,.$

$[*13{\cdot}171] \qquad \supset . \alpha = \gamma \,.\, \beta = \delta : \supset \vdash . \mathrm{Prop}$

$*111{\cdot}18$. $\vdash . \alpha \,\overline{\mathrm{sm}}\, \beta \subset (\alpha \uparrow \beta)_\Delta`\beta$

Dem.

$\vdash . *35{\cdot}83 . *73{\cdot}03 . \supset \vdash : R \in \alpha \,\overline{\mathrm{sm}}\, \beta . \supset . R \mathrel{\mathsf{G}} \alpha \uparrow \beta \quad (1)$

$\vdash . *73{\cdot}03 . \qquad \supset \vdash : R \in \alpha \,\overline{\mathrm{sm}}\, \beta . \supset . R \in 1 \to \mathrm{Cls} \,.\, \mathrm{C}`R = \beta \quad (2)$

$\vdash . (1) . (2) . *80{\cdot}14 . \supset \vdash . \mathrm{Prop}$

The class $(\alpha \uparrow \beta)_\Delta`\beta$ is important, being the class of Cantor's "Belegungen," used by him to define exponentiation; we have in fact

$$\mathrm{Nc}`(\alpha \uparrow \beta)_\Delta`\beta = (\mathrm{Nc}`\alpha)^{\mathrm{Nc}`\beta}.$$

Thus the above proposition shows that $\mathrm{Nc}`(\alpha \,\overline{\mathrm{sm}}\, \beta)$ is less than or equal to $(\mathrm{Nc}`\alpha)^{\mathrm{Nc}`\beta}$; and since, whenever it is not zero, $\mathrm{Nc}`\alpha = \mathrm{Nc}`\beta$, it is less than or equal to

$$(\mathrm{Nc}`\alpha)^{\mathrm{Nc}`\alpha}.$$

The following propositions lead up to $*111{\cdot}32{\cdot}33{\cdot}34$:

$*111{\cdot}2$. $\vdash : \mathrm{E} ! \,S`\beta . \supset . \mathrm{Crp}\,(S)`\beta = (S`\beta) \,\overline{\mathrm{sm}}\, \beta \qquad [*14{\cdot}28 . (*111{\cdot}02)]$

$*111{\cdot}201$. $\vdash : f\{\mathrm{Crp}\,(S)`\beta\} . \equiv . f\{(S`\beta) \,\overline{\mathrm{sm}}\, \beta\} \qquad [*4{\cdot}2 . (*111{\cdot}02)]$

$*111{\cdot}202$. $\vdash : R \in \mathrm{Crp}\,(S)`\beta . \equiv . R \in 1 \to 1 \,.\, \mathrm{D}`R = S`\beta \,.\, \mathrm{C}`R = \beta$

$[*111{\cdot}201 . *73{\cdot}03]$

$*111{\cdot}21$. $\vdash : \exists ! \,\mathrm{Crp}\,(S)`\beta . \equiv . S`\beta \,\mathrm{sm}\, \beta \qquad [*111{\cdot}201 . *73{\cdot}04]$

$*111{\cdot}211$. $\vdash : \exists ! \,\mathrm{Crp}\,(S)`\beta . \supset . \mathrm{E} ! \,S`\beta \,.\, \beta \in \mathrm{C}`S \qquad [*111{\cdot}21 . *14{\cdot}21 . *33{\cdot}43]$

$*111{\cdot}22$. $\vdash :. \beta \in \mathrm{C}`S . \supset_\beta . \exists ! \,\mathrm{Crp}\,(S)`\beta : \equiv . S \in 1 \to \mathrm{Cls} \,.\, S \mathrel{\mathsf{G}} \mathrm{sm}$

Dem.

$\vdash . *111{\cdot}21 . \supset \vdash :. \beta \in \mathrm{C}`S . \supset_\beta . \exists ! \,\mathrm{Crp}\,(S)`\beta : \equiv : \beta \in \mathrm{C}`S . \supset_\beta . S`\beta \,\mathrm{sm}\, \beta :$

$[*72{\cdot}93] \qquad \equiv : S \in 1 \to \mathrm{Cls} . S \mathrel{\mathsf{G}} \mathrm{sm} :. \supset \vdash . \mathrm{Prop}$

$*111{\cdot}221.\ \vdash :.\ S \in 1 \rightarrow \mathrm{Cls}\,.\, S \Subset \mathrm{sm}\,.\supset :\ \exists !\, \mathrm{Crp}\,(S)\text{‘}\beta\,.\equiv .\,\beta \in \text{Ɑ}\text{‘}S$

Dem.

$\vdash .\, *111{\cdot}22\,.\supset \vdash :.\ \mathrm{Hp}\,.\supset :\beta \in \text{Ɑ}\text{‘}S\,.\supset .\ \exists !\, \mathrm{Crp}\,(S)\text{‘}\beta$ (1)

$\vdash .\,(1)\,.\,*111{\cdot}211\,.\supset \vdash .\ \mathrm{Prop}$

$*111{\cdot}23.\ \vdash : S \in 1 \rightarrow 1\,.\,\beta \in \text{Ɑ}\text{‘}S\,.\supset .\ \mathrm{Crp}\,(S)\text{‘}\beta = \mathrm{Cnv}\text{“}\mathrm{Crp}\,(\breve{S})\text{‘}S\text{‘}\beta$

Dem.

$\vdash .\, *111{\cdot}2\,.\,*71{\cdot}163\,.\supset$

$\vdash :.\ \mathrm{Hp}\,.\supset : \mathrm{Crp}\,(S)\text{‘}\beta = (S\text{‘}\beta)\,\overline{\mathrm{sm}}\,\beta$

$[*73{\cdot}301] \qquad = \mathrm{Cnv}\text{“}(\beta\,\overline{\mathrm{sm}}\,S\text{‘}\beta)$

$[*72{\cdot}241] \qquad = \mathrm{Cnv}\text{“}(\breve{S}\text{‘}S\text{‘}\beta\,\overline{\mathrm{sm}}\,S\text{‘}\beta)$ (1)

$\vdash .\,(1)\,.\,*111{\cdot}201 \dfrac{\breve{S},\, S\text{‘}\beta}{S,\ \beta}\,.\supset \vdash .\ \mathrm{Prop}$

$*111{\cdot}24.\ \vdash : S \in 1 \rightarrow \mathrm{Cls}\,.\,\lambda \subset \text{Ɑ}\text{‘}S\,.\supset .\ \mathrm{Crp}\,(S)\text{“}\lambda \in \mathrm{Cls}^2\,\mathrm{excl}$

Dem.

$\vdash .\, *111{\cdot}2\,.\,*71{\cdot}163\,.\supset$

$\vdash :.\ \mathrm{Hp}\,.\supset : \beta, \gamma \in \lambda\,.\supset_{\beta,\gamma} .\ \mathrm{Crp}\,(S)\text{‘}\beta = (S\text{‘}\beta)\,\overline{\mathrm{sm}}\,\beta\,.\,\mathrm{Crp}\,(S)\text{‘}\gamma = (S\text{‘}\gamma)\,\overline{\mathrm{sm}}\,\gamma\,.$ (1)

$[*111{\cdot}16] \qquad \supset_{\beta,\gamma} .\ \exists !\,\mathrm{Crp}(S)\text{‘}\beta \cap \mathrm{Crp}(S)\text{‘}\gamma\,.\supset .\,\beta = \gamma\,.$

$[(1)\,.\,*30{\cdot}37] \qquad \supset .\,\mathrm{Crp}(S)\text{‘}\beta = \mathrm{Crp}(S)\text{‘}\gamma$ (2)

$\vdash .\,(2)\,.\,*37{\cdot}63\,.\supset \vdash :.\ \mathrm{Hp}\,.\supset : \rho, \sigma \in \mathrm{Crp}\,(S)\text{“}\lambda\,.\,\exists !\,\rho \cap \sigma\,.\supset_{\rho,\sigma} .\,\rho = \sigma :.\supset \vdash .\,\mathrm{Prop}$

$*111{\cdot}25.\ \vdash : S \in 1 \rightarrow \mathrm{Cls}\,.\,S \Subset \mathrm{sm}\,.\,\lambda \subset \text{Ɑ}\text{‘}S\,.\supset .\ \mathrm{Crp}\,(S)\text{“}\lambda \in \mathrm{Cls\ ex}^2\,\mathrm{excl}$

$[*111{\cdot}24{\cdot}22]$

$*111{\cdot}3.\ \vdash : \lambda \in \mathrm{Cls}^2\,\mathrm{excl}\,.\supset .\ \dot{s}\text{“}\mathrm{D}\text{“}\epsilon_\Delta\text{‘}\alpha\,\overline{\mathrm{sm}}\text{“}\lambda \subset (\alpha \uparrow s\text{‘}\lambda)_\Delta\text{‘}s\text{‘}\lambda$

Dem.

$\vdash .\, *37{\cdot}29\,.\,*24{\cdot}12\,.\supset$

$\vdash : \epsilon_\Delta\text{‘}\alpha\,\overline{\mathrm{sm}}\text{“}\lambda = \Lambda\,.\supset .\ \dot{s}\text{“}\mathrm{D}\text{“}\epsilon_\Delta\text{‘}\alpha\,\overline{\mathrm{sm}}\text{“}\lambda \subset (\alpha \uparrow s\text{‘}\lambda)_\Delta\text{‘}s\text{‘}\lambda$ (1)

$\vdash .\, *83{\cdot}1\,.\supset$

$\vdash :.\ \mathrm{Hp}\,.\,\exists !\,\epsilon_\Delta\text{‘}\alpha\,\overline{\mathrm{sm}}\text{“}\lambda\,.\supset : \beta \in \lambda\,.\supset_\beta .\,\exists !\,\alpha\,\overline{\mathrm{sm}}\text{‘}\beta\,.$

$[*111{\cdot}18] \qquad \supset_\beta .\,\exists !\,(\alpha \uparrow \beta)_\Delta\text{‘}\beta\,.$

$[*80{\cdot}15] \qquad \supset_\beta .\,\exists !\,(\alpha \uparrow s\text{‘}\lambda)_\Delta\text{‘}\beta :$

$[*80{\cdot}83] \qquad \supset : \{(\alpha \uparrow s\text{‘}\lambda)_\Delta\text{“}\lambda\} \upharpoonleft (\alpha \uparrow s\text{‘}\lambda)_\Delta \in 1 \rightarrow 1$ (2)

$\vdash .\,(2)\,.\,*111{\cdot}18\,.\,*85{\cdot}72 \dfrac{\alpha\,\overline{\mathrm{sm}},\ (\alpha \uparrow s\text{‘}\lambda)_\Delta}{R,\qquad S}\,.\supset$

$\vdash : \mathrm{Hp}\,.\,\exists !\,\epsilon_\Delta\text{‘}\alpha\,\overline{\mathrm{sm}}\text{“}\lambda\,.\supset .\ \mathrm{D}\text{“}\epsilon_\Delta\text{‘}\alpha\,\overline{\mathrm{sm}}\text{“}\lambda \subset \mathrm{D}\text{“}\epsilon_\Delta\text{‘}(\alpha \uparrow s\text{‘}\lambda)_\Delta\text{“}\lambda\,.$

$[*37{\cdot}2] \qquad \supset .\ \dot{s}\text{“}\mathrm{D}\text{“}\epsilon_\Delta\text{‘}\alpha\,\overline{\mathrm{sm}}\text{“}\lambda \subset \dot{s}\text{“}\mathrm{D}\text{“}\epsilon_\Delta\text{‘}(\alpha \uparrow s\text{‘}\lambda)_\Delta\text{“}\lambda$

$[*85{\cdot}27] \qquad \subset (\alpha \uparrow s\text{‘}\lambda)_\Delta\text{‘}s\text{‘}\lambda$ (3)

$\vdash .\,(1)\,.\,(3)\,.\supset \vdash .\ \mathrm{Prop}$

***111·31.** $\vdash : \lambda, S\text{“}\lambda \,\epsilon\, \text{Cls}^2\,\text{excl} \,.\, S \,\epsilon\, 1 \to 1 \,.\, R \,\epsilon\, \epsilon_\Delta\text{‘}\text{Crp}\,(S)\text{“}\lambda \,.\supset .$
$\dot{s}\text{‘}\text{D}\text{‘}R \,\epsilon\, (s\text{‘}S\text{“}\lambda)\, \overline{\text{sm}}\, (s\text{‘}\lambda)$

Dem.

$\vdash . *83\cdot2 . \supset$

$\vdash :. \text{Hp} . \supset : \beta \,\epsilon\, \lambda . \equiv . R\text{‘}\text{Crp}\,(S)\text{‘}\beta \,\epsilon\, \text{Crp}\,(S)\text{‘}\beta .$

$[*111\cdot202] \quad \equiv . R\text{‘}\text{Crp}\,(S)\text{‘}\beta \,\epsilon\, 1 \to 1 . \text{D}\text{‘}R\text{‘}\text{Crp}\,(S)\text{‘}\beta = S\text{‘}\beta .$
$\text{Ɑ}\text{‘}R\text{‘}\text{Crp}\,(S)\text{‘}\beta = \beta \quad (1)$

$\vdash . (1) . *72\cdot322 . \quad \supset \vdash : \text{Hp} . \supset . \dot{s}\text{‘}R\text{“}\text{Crp}\,(S)\text{“}\lambda \,\epsilon\, 1 \to 1 .$

$[*80\cdot34] \quad \supset . \dot{s}\text{‘}\text{D}\text{‘}R \,\epsilon\, 1 \to 1 \quad (2)$

$\vdash . (1) . *37\cdot68 . *50\cdot17 . \supset \vdash : \text{Hp} . \supset . \text{D}\text{“}R\text{“}\text{Crp}\,(S)\text{“}\lambda = S\text{“}\lambda .$
$\text{Ɑ}\text{“}R\text{“}\text{Crp}\,(S)\text{“}\lambda = \lambda .$

$[*80\cdot34] \quad \supset . \text{D}\text{“}\text{D}\text{‘}R = S\text{“}\lambda . \text{Ɑ}\text{“}\text{D}\text{‘}R = \lambda .$

$[*41\cdot43\cdot44] \quad \supset . \text{D}\text{‘}\dot{s}\text{‘}\text{D}\text{‘}R = s\text{‘}S\text{“}\lambda . \text{Ɑ}\text{‘}\dot{s}\text{‘}\text{D}\text{‘}R = s\text{‘}\lambda \quad (3)$

$\vdash . (2) . (3) . *73\cdot03 . \supset \vdash . \text{Prop}$

***111·311.** $\vdash : \lambda, S\text{“}\lambda \,\epsilon\, \text{Cls}^2\,\text{excl} . S \,\epsilon\, 1 \to 1 . \exists ! \,\epsilon_\Delta\text{‘}\text{Crp}\,(S)\text{“}\lambda . \supset . s\text{‘}S\text{“}\lambda \;\text{sm}\; s\text{‘}\lambda$
$[*111\cdot31 . *73\cdot04]$

***111·313.** $\vdash : \lambda \,\epsilon\, \text{Cls}^2\,\text{excl} . R \,\epsilon\, \epsilon_\Delta\text{‘}\text{Crp}\,(S)\text{“}\lambda . \beta \,\epsilon\, \lambda . M = \dot{s}\text{‘}\text{D}\text{‘}R . \supset .$
$M \upharpoonright \beta = R\text{‘}\text{Crp}\,(S)\text{‘}\beta . M \upharpoonright \beta \,\epsilon\, \text{Crp}\,(S)\text{‘}\beta$

Dem.

$\vdash . *83\cdot2 . \supset \vdash :: \text{Hp} . \supset :. \alpha \,\epsilon\, \lambda . \supset_\alpha : R\text{‘}\text{Crp}\,(S)\text{‘}\alpha \,\epsilon\, \text{Crp}\,(S)\text{‘}\alpha : \quad (1)$

$[*111\cdot202] \quad \supset_\alpha : \text{Ɑ}\text{‘}R\text{‘}\text{Crp}\,(S)\text{‘}\alpha = \alpha :$

$[*33\cdot14 . *4\cdot71] \quad \supset_\alpha : x \{R\text{‘}\text{Crp}(S)\text{‘}\alpha\} y . \equiv . x \{R\text{‘}\text{Crp}(S)\text{‘}\alpha\} y . y \,\epsilon\, \alpha \quad (2)$

$\vdash . *35\cdot101 . *83\cdot23 . *41\cdot11 . \supset$

$\vdash :. \text{Hp} . \supset : x (M \upharpoonright \beta) y . \equiv . (\exists \alpha) . \alpha \,\epsilon\, \lambda . x \{R\text{‘}\text{Crp}\,(S)\text{‘}\alpha\} y . y \,\epsilon\, \beta .$

$[(2)] \quad \equiv . (\exists \alpha) . \alpha \,\epsilon\, \lambda . x \{R\text{‘}\text{Crp}\,(S)\text{‘}\alpha\} y . y \,\epsilon\, \alpha \cap \beta .$

$[*84\cdot11 . *22\cdot5] \quad \equiv . (\exists \alpha) . \alpha \,\epsilon\, \lambda . x \{R\text{‘}\text{Crp}\,(S)\text{‘}\alpha\} y . y \,\epsilon\, \beta . \alpha = \beta .$

$[*13\cdot195] \quad \equiv . \beta \,\epsilon\, \lambda . x \{R\text{‘}\text{Crp}\,(S)\text{‘}\beta\} y . y \,\epsilon\, \beta .$

$[\text{Hp} . *4\cdot73 . (2)] \quad \equiv . x \{R\text{‘}\text{Crp}\,(S)\text{‘}\beta\} y \quad (3)$

$\vdash . (1) . (3) . \supset \vdash . \text{Prop}$

***111·32.** $\vdash : \lambda, S\text{“}\lambda \,\epsilon\, \text{Cls}^2\,\text{excl} . S \,\epsilon\, 1 \to 1 . R \,\epsilon\, \epsilon_\Delta\text{‘}\text{Crp}\,(S)\text{“}\lambda . M = \dot{s}\text{‘}\text{D}\text{‘}R . \supset .$
$M \,\epsilon\, 1 \to 1 . \text{Ɑ}\text{‘}M = s\text{‘}\lambda . S\text{“}\lambda = M_\epsilon\text{“}\lambda . S \upharpoonright \lambda = M_\epsilon \upharpoonright \lambda$

Dem.

$\vdash . *111\cdot31 . *73\cdot03 . \supset \vdash : \text{Hp} . \supset . M \,\epsilon\, 1 \to 1 . \text{Ɑ}\text{‘}M = s\text{‘}\lambda \quad (1)$

$\vdash . *111\cdot313\cdot202 . \quad \supset \vdash :. \text{Hp} . \supset : \beta \,\epsilon\, \lambda . \supset . \text{D}\text{‘}(M \upharpoonright \beta) = S\text{‘}\beta . \text{Ɑ}\text{‘}(M \upharpoonright \beta) = \beta .$

$[*37\cdot25] \quad \supset . (M \upharpoonright \beta)\text{“}\beta = S\text{‘}\beta .$

$[*37\cdot421\cdot11] \quad \supset . M_\epsilon\text{‘}\beta = S\text{‘}\beta :$

$[*35\cdot71 . *37\cdot69] \quad \supset : M_\epsilon \upharpoonright \lambda = S \upharpoonright \lambda . M_\epsilon\text{“}\lambda = S\text{“}\lambda \quad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***111·321.** $\vdash : \lambda, S\text{“}\lambda \,\epsilon\, \text{Cls}^2\,\text{excl} . S \,\epsilon\, 1 \to 1 . \exists ! \,\epsilon_\Delta\text{‘}\text{Crp}\,(S)\text{“}\lambda . \supset .$
$(\exists M) . M \,\epsilon\, 1 \to 1 . \text{Ɑ}\text{‘}M = s\text{‘}\lambda . S\text{“}\lambda = M_\epsilon\text{“}\lambda . S \upharpoonright \lambda = M_\epsilon \upharpoonright \lambda$
$[*111\cdot32]$

∗111·322. $\vdash : \kappa, \lambda \in \text{Cls}^2\,\text{excl} \,.\, S \in \kappa \,\overline{\text{sm}}\, \lambda \,.\, R \in \epsilon_\Delta\text{‘Crp}\,(S)\text{“}\lambda \,.\, M = \dot{s}\text{‘D‘}R \,.\, \supset .$
$M \in \kappa \,\overline{\text{sm}}\,\overline{\text{sm}}\, \lambda \,.\, S = M_\epsilon \upharpoonright \lambda$ [∗111·32·1 . ∗35·66 . ∗73·03]

∗111·33. $\vdash :. \text{Mult ax} \,.\, \supset : S \in 1 \rightarrow 1 \,.\, S \subset \text{sm} \,.\, \kappa, \lambda \in \text{Cls}^2\text{excl} \,.\, \kappa = S\text{“}\lambda \,.\, \lambda \subset \text{C‘}S \,.\, \supset .$
$s\text{‘}\kappa \,\text{sm}\, s\text{‘}\lambda$

Dem.

$\vdash .\, ∗111·221 \,.\, \supset$
$\vdash :. S \in 1 \rightarrow 1 \,.\, S \subset \text{sm} \,.\, \kappa, \lambda \in \text{Cls}^2\,\text{excl} \,.\, \kappa = S\text{“}\lambda \,.\, \lambda \subset \text{C‘}S \,.\, \supset :$
$\beta \in \lambda \,.\, \supset_\beta .\, \exists !\, \text{Crp}\,(S)\text{‘}\beta :$
[∗88·37] $\supset : \text{Mult ax} \,.\, \supset .\, \exists !\, \epsilon_\Delta\text{‘Crp}\,(S)\text{“}\lambda \,.$
[∗111·311] $\supset .\, s\text{‘}\kappa \,\text{sm}\, s\text{‘}\lambda :. \supset \vdash .\, \text{Prop}$

∗111·34. $\vdash :. \text{Mult ax} \,.\, \supset :$
$(\exists S) \,.\, S \in 1 \rightarrow 1 \,.\, S \subset \text{sm} \,.\, \text{D‘}S = \kappa \,.\, \text{C‘}S = \lambda \,.\, \kappa, \lambda \in \text{Cls}^2\,\text{excl} \,.\, \supset .$
$(\exists M) \,.\, M \in 1 \rightarrow 1 \,.\, \text{C‘}M = s\text{‘}\lambda \,.\, \kappa = M_\epsilon\text{“}\lambda$

Dem.

$\vdash .\, ∗111·25 \,.\, \supset$
$\vdash :. S \in 1 \rightarrow 1 \,.\, S \subset \text{sm} \,.\, \text{D‘}S = \kappa \,.\, \text{C‘}S = \lambda \,.\, \kappa, \lambda \in \text{Cls}^2\,\text{excl} \,.\, \supset :$
$\text{Crp}\,(S)\text{“}\lambda \in \text{Cls ex}^2\,\text{excl} :$
[∗88·32] $\supset : \text{Mult ax} \,.\, \supset .\, \exists !\, \epsilon_\Delta\text{‘Crp}\,(S)\text{“}\lambda \,.$
[∗111·321] $\supset .\, (\exists M) \,.\, M \in 1 \rightarrow 1 \,.\, \text{C‘}M = s\text{‘}\lambda \,.\, \kappa = M_\epsilon\text{“}\lambda$ (1)
$\vdash .\, (1) \,.\, ∗10·11·23 \,.\, \text{Comm} \,.\, \supset \vdash .\, \text{Prop}$

The following propositions are concerned with the elementary properties of "sm sm." It will be seen that they are closely analogous to those of "sm."

∗111·4. $\vdash : \kappa \,\text{sm sm}\, \lambda \,.\, \equiv .\, (\exists T) \,.\, T \in 1 \rightarrow 1 \,.\, \text{C‘}T = s\text{‘}\lambda \,.\, \kappa = T_\epsilon\text{“}\lambda \,.\, \equiv .\, \exists !\, \kappa \,\overline{\text{sm}}\,\overline{\text{sm}}\, \lambda$
[∗111·1 . (∗111·03)]

∗111·401. $\vdash : \kappa \,\text{sm sm}\, \lambda \,.\, \equiv .\, (\exists T) \,.\, T \in 1 \rightarrow 1 \,.\, s\text{‘}\lambda \subset \text{C‘}T \,.\, \kappa = T_\epsilon\text{“}\lambda$

Dem.

$\vdash .\, ∗22·42 \,.\, ∗111·4 \,.\, \supset \vdash : \kappa \,\text{sm sm}\, \lambda \,.\, \supset .\, (\exists T) \,.\, T \in 1 \rightarrow 1 \,.\, s\text{‘}\lambda \subset \text{C‘}T \,.\, \kappa = T_\epsilon\text{“}\lambda$ (1)
$\vdash .\, (1) \,.\, ∗111·14 \,.\, \supset \vdash .\, \text{Prop}$

∗111·402. $\vdash : \kappa \,\text{sm sm}\, \lambda \,.\, \equiv .\, (\exists T) \,.\, T \upharpoonright s\text{‘}\lambda \in 1 \rightarrow 1 \,.\, s\text{‘}\lambda \subset \text{C‘}T \,.\, \kappa = T_\epsilon\text{“}\lambda$
[∗111·14·1·121]

∗111·43. $\vdash : \kappa \,\text{sm sm}\, \lambda \,.\, \supset .\, (\exists S) \,.\, S \in 1 \rightarrow 1 \,.\, S \subset \text{sm} \,.\, \text{D‘}S = \kappa \,.\, \text{C‘}S = \lambda$
[∗111·11·111]

∗111·44. $\vdash : \kappa \,\text{sm sm}\, \lambda \,.\, \supset .\, \kappa \,\text{sm}\, \lambda \,.\, s\text{‘}\kappa \,\text{sm}\, s\text{‘}\lambda$ [∗111·11·4 . ∗73·03]

∗111·45. $\vdash .\, \lambda \,\text{sm sm}\, \lambda$ [∗111·13·4]

∗111·451. $\vdash : \kappa \,\text{sm sm}\, \lambda \,.\, \equiv .\, \lambda \,\text{sm sm}\, \kappa$ [∗111·131·4]

∗111·452. $\vdash : \kappa \,\text{sm sm}\, \lambda \,.\, \lambda \,\text{sm sm}\, \mu \,.\, \supset .\, \kappa \,\text{sm sm}\, \mu$ [∗111·132·4]

∗111·46. $\vdash : \lambda, S\text{“}\lambda \in \text{Cls}^2\,\text{excl} \,.\, S \in 1 \rightarrow 1 \,.\, \exists !\, \epsilon_\Delta\text{‘Crp}\,(S)\text{“}\lambda \,.\, \supset .\, S\text{“}\lambda \,\text{sm sm}\, \lambda$
[∗111·32·4]

***111·47.** $\vdash :. \kappa \text{ sm sm } \lambda . \supset : \kappa \in \text{Cls}^2 \text{ excl} . \equiv . \lambda \in \text{Cls}^2 \text{ excl}$

Dem.

$\vdash . *111\cdot4 . \supset \vdash :. \text{Hp} . \supset : (\exists T) . T \in 1 \to 1 . \text{Œ}‘T = s‘\lambda . \kappa = T‘‘‘\lambda :$

$[*84\cdot53] \quad \supset : \lambda \in \text{Cls}^2 \text{ excl} . \supset . \kappa \in \text{Cls}^2 \text{ excl} \quad (1)$

$\vdash . (1) . *111\cdot451 . \supset \vdash :. \text{Hp} . \supset : \kappa \in \text{Cls}^2 \text{ excl} . \supset . \lambda \in \text{Cls}^2 \text{ excl} \quad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***111·5.** $\vdash :: \text{Mult ax} . \supset :. \kappa, \lambda \in \text{Cls}^2 \text{ excl} . \supset :$

$\kappa \text{ sm sm } \lambda . \equiv . (\exists S) . S \in 1 \to 1 . S \subset \text{sm} . \text{D}‘S = \kappa . \text{Œ}‘S = \lambda .$

$\equiv . \exists ! \kappa \overline{\text{sm}} \lambda \cap \text{Rl}‘\text{sm} \quad [*111\cdot34\cdot43\cdot4]$

***111·51.** $\vdash :. \text{Mult ax} . \supset : \kappa, \lambda \in \text{Cls}^2 \text{ excl} . \exists ! \kappa \overline{\text{sm}} \lambda \cap \text{Rl}‘\text{sm} . \supset . s‘\kappa \text{ sm } s‘\lambda$ $[*111\cdot5\cdot44]$

***111·52.** $\vdash : \mu, \nu \in \text{NC} . \kappa, \lambda \in \mu \cap \text{Cl}‘\nu . \supset . \exists ! \kappa \overline{\text{sm}} \lambda \cap \text{Rl}‘\text{sm}$

Dem.

$\vdash . *100\cdot5 . *73\cdot1 . \supset \vdash : \text{Hp} . \supset . (\exists S) . S \in 1 \to 1 . \text{D}‘S = \kappa . \text{Œ}‘S = \lambda \quad (1)$

$\vdash . *100\cdot5 . \supset \vdash :. \text{Hp} . \supset : \alpha \in \kappa . \beta \in \lambda . \supset . \alpha \text{ sm } \beta \quad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***111·53.** $\vdash :. \text{Mult ax} . \supset : \mu, \nu \in \text{NC} . \kappa, \lambda \in \mu \cap \text{Cl excl}‘\nu . \kappa \text{ sm sm } \lambda$ $[*111\cdot52\cdot5]$

*112. THE ARITHMETICAL SUM OF A CLASS OF CLASSES

Summary of *112.

In this number, we return to the arithmetical operations. The definition of addition in *110 was only applicable to a finite number of summands, because the summands had to be enumerated. In the present number, we define the arithmetical sum of a class of classes, so that the summands are given as the members of a class, and do not require to be enumerated. Hence the definition in this number is as applicable to an infinite number of summands as to a finite number.

If κ is a class of mutually exclusive classes, the number of $s\text{‘}\kappa$ will be the sum of the numbers of members of κ; *i.e.* if we write "$\Sigma\mathrm{Nc}\text{‘}\kappa$" for the sum of the numbers of members of κ,

$$\kappa \in \mathrm{Cls}^2\,\mathrm{excl} \,.\supset .\, \mathrm{Nc}\text{‘}s\text{‘}\kappa = \Sigma\mathrm{Nc}\text{‘}\kappa.$$

But when the members of κ are not mutually exclusive, a term x which is a member of two members (say α and β) of κ has to be counted twice over in obtaining the arithmetical sum of κ, whereas in the logical sum x is only counted once. Thus we need a construction which shall duplicate x, taking it first as a member of α, and then as a member of β. This is effected if we replace x first by $x \downarrow \alpha$, and then by $x \downarrow \beta$. In fact, $x \downarrow \alpha$ has the kind of arithmetical properties which we mean to secure when we speak of "x considered as a member of α"—a phrase which, as it stands, does not serve our purpose, for x is simply x however we may choose to consider it. Thus we replace α by $\downarrow \alpha\text{‘‘}\alpha$ and β by $\downarrow \beta\text{‘‘}\beta$ and so on; *i.e.* (using *85·5), we replace α by $\epsilon \,\underline{\downarrow}\, \alpha$ and β by $\epsilon \,\underline{\downarrow}\, \beta$ and so on. These new classes are similar to α and β and so on, and are mutually exclusive. Hence their *logical* sum has the number of terms which is wanted for the *arithmetical* sum of the members of κ. Thus we put

$$\Sigma\text{‘}\kappa = s\text{‘}\epsilon \,\underline{\downarrow}\text{‘‘}\kappa \quad \mathrm{Df},$$

$$\Sigma\mathrm{Nc}\text{‘}\kappa = \mathrm{Nc}\text{‘}\Sigma\text{‘}\kappa \quad \mathrm{Df}.$$

With regard to the second of these definitions, it is to be observed that $\Sigma\mathrm{Nc}\text{‘}\kappa$ is not a function of $\mathrm{Nc}\text{‘‘}\kappa$, unless no two members of κ are similar; for $\mathrm{Nc}\text{‘‘}\kappa$ cannot contain the same number twice over. For the same reason, if λ is a class of cardinals, and we define "$\mathrm{Sum}\text{‘}\lambda$," we do not get what is wanted for arithmetical addition, because our definition will not enable us to deal with summations in which there are numbers that are repeated. We could, if it were worth while, define "$\mathrm{Sum}\text{‘}\lambda$" as follows: Take a class of classes κ, consisting of one class having each number which is a member of λ, *i.e.* let κ

be a selection from λ; then $\Sigma`\kappa$ will have the required number of terms. *I.e.* we might put

$$\text{Sum}`\lambda = \hat{\xi}\{(\exists\kappa) . \kappa \epsilon D``\epsilon_\Delta`\lambda . \xi \text{ sm } \Sigma`\kappa\} \quad \text{Df.}$$

But since this definition is only available for sums in which no number is repeated, it is not worth while to introduce it.

In this number we prove the following propositions among others.

***112·15.** $\vdash : \kappa \epsilon \text{Cls}^2 \text{excl} . \supset . s`\kappa \epsilon \Sigma\text{Nc}`\kappa$

This is an extension of *110·32.

***112·17.** $\vdash : \kappa \text{ sm sm } \lambda . \supset . \Sigma\text{Nc}`\kappa = \Sigma\text{Nc}`\lambda . \Sigma`\kappa \text{ sm } \Sigma`\lambda$

The chief point in the above proposition is that it does not require $\kappa, \lambda \epsilon \text{Cls}^2 \text{excl}$.

*112·2—·24 are concerned with the use of the multiplicative axiom and the propositions of *111 in which it appears as hypothesis. We have

***112·22.** $\vdash :. \text{Mult ax} . \supset : \exists ! (\epsilon \downarrow``\kappa) \overline{\text{sm}} (\epsilon \downarrow``\lambda) \cap \text{Rl}`\text{sm} . \supset . \Sigma\text{Nc}`\kappa = \Sigma\text{Nc}`\lambda$

whence we derive the proposition

***112·24.** $\vdash :. \text{Mult ax} . \supset : \mu, \nu \epsilon \text{NC} . \kappa, \lambda \epsilon \mu \cap \text{Cl}`\nu . \supset . \Sigma\text{Nc}`\kappa = \Sigma\text{Nc}`\lambda$

I.e. assuming the multiplicative axiom, two classes which each consist of μ classes of ν terms each have the same number of terms in their sum. This number would naturally be defined as μ multiplied by ν, but owing to the necessity of the multiplicative axiom in this proposition, we have selected a different definition of multiplication (*113) which does not depend upon the multiplicative axiom. The reader should observe that the similarity of two classes, each of which consists of μ mutually exclusive sets of ν terms, cannot be proved in general without the multiplicative axiom.

The remaining propositions of this number give properties of Σ in special cases. We prove that $\Sigma`\Lambda = \Lambda$ (*112·3), that $\Sigma\text{Nc}`\iota`\alpha = \text{Nc}`\alpha$ (*112·321), that $\alpha \neq \beta . \supset . \Sigma\text{Nc}`(\iota`\alpha \cup \iota`\beta) = \text{Nc}`\alpha +_c \text{Nc}`\beta$ (*112·34), which connects the definition of addition in this number with that in *110. Finally we prove the general associative law for addition, in the following two forms:

***112·41.** $\vdash . s`\Sigma``\lambda = \Sigma`s`\lambda$

***112·43.** $\vdash : \lambda \epsilon \text{Cls}^2 \text{excl} . \supset . \text{Nc}`\Sigma`\Sigma``\lambda = \text{Nc}`\Sigma`s`\lambda$

***112·01.** $\Sigma`\kappa = s`\epsilon \downarrow``\kappa \quad \text{Df}$

***112·02.** $\Sigma\text{Nc}`\kappa = \text{Nc}`\Sigma`\kappa \quad \text{Df}$

***112·1.** $\vdash . \Sigma`\kappa = s`\epsilon \downarrow``\kappa$ [*20·2 . (*112·01)]

***112·101.** $\vdash . \Sigma\text{Nc}`\kappa = \text{Nc}`\Sigma`\kappa = \text{Nc}`s`\epsilon \downarrow``\kappa$ [*20·2 . *112·1 . (*112·02)]

∗112·102. $\vdash . \Sigma`\kappa = \hat{R}\{(\exists\alpha, x) . \alpha \in \kappa . x \in \alpha . R = x \downarrow \alpha\}$

Dem.

$\vdash . *85{\cdot}6 . *40{\cdot}11 . *112{\cdot}1 . \supset$

$\vdash . \Sigma`\kappa = \hat{R}\{(\exists\mu, \alpha) . \alpha \in \kappa . \mu = \downarrow \alpha``\alpha . R \in \mu\}$

$[*13{\cdot}195] = \hat{R}\{(\exists\alpha) . \alpha \in \kappa . R \in \downarrow \alpha``\alpha\}$

$[*55{\cdot}231] = \hat{R}\{(\exists\alpha, x) . \alpha \in \kappa . x \in \alpha . R = x \downarrow \alpha\} . \supset \vdash . \text{Prop}$

∗112·103. $\vdash . \Sigma`\kappa = s`\hat{\mu}\{(\exists\alpha) . \alpha \in \kappa . \mu = \downarrow \alpha``\alpha\}$ $[*112{\cdot}1 . *85{\cdot}6]$

∗112·11. $\vdash : \beta \in \Sigma\text{Nc}`\kappa . \equiv . \beta \text{ sm } s`\epsilon \rfloor``\kappa$ $[*112{\cdot}101]$

∗112·12. $\vdash . s`\epsilon \rfloor``\kappa \in \Sigma\text{Nc}`\kappa$ $[*112{\cdot}11]$

∗112·13. $\vdash : \lambda \text{ sm sm } \epsilon \rfloor``\kappa . \supset . s`\lambda \in \Sigma\text{Nc}`\kappa$ $[*111{\cdot}44 . *112{\cdot}11]$

∗112·14. $\vdash : \kappa \in \text{Cls}^2 \text{excl} . \supset . \epsilon \rfloor``\kappa \text{ sm sm } \kappa$

Dem.

$\vdash . *21{\cdot}33 . \supset \vdash :. \text{Hp} . T = \hat{R}\hat{x}\{(\exists\alpha) . \alpha \in \kappa . x \in \alpha . R = x \downarrow \alpha\} . \supset :$

$xTR . yTR . \supset . (\exists\alpha, \beta) . R = x \downarrow \alpha . R = y \downarrow \beta .$

$[*55{\cdot}31] \quad \supset . x = y :$

$[*71{\cdot}17] \supset : T \in 1 \rightarrow \text{Cls}$ (1)

$\vdash . *21{\cdot}33 . \supset$

$\vdash : \text{Hp}(1) . xTR . xTS . \supset . (\exists\alpha, \beta) . \alpha, \beta \in \kappa . x \in \alpha \cap \beta . R = x \downarrow \alpha . S = x \downarrow \beta .$

$[*84{\cdot}11.\text{Hp}] \quad \supset . (\exists\alpha, \beta) . \alpha = \beta . R = x \downarrow \alpha . S = x \downarrow \beta .$

$[*13{\cdot}195] \quad \supset . R = S :$

$[*71{\cdot}171] \supset : T \in \text{Cls} \rightarrow 1$ (2)

$\vdash . *33{\cdot}131 . \supset \vdash :. \text{Hp}(1) . \supset : x \in \text{Œ}`T . \equiv . (\exists R, \alpha) . \alpha \in \kappa . x \in \alpha . R = x \downarrow \alpha .$

$[*55{\cdot}12] \quad \equiv . x \in s`\kappa$ (3)

$\vdash . *37{\cdot}1{\cdot}11 . \supset$

$\vdash :: \text{Hp} . \supset :. \alpha \in \kappa . \supset : R \in T_\epsilon`\alpha . \equiv . (\exists x, \beta) . x \in \alpha \cap \beta . \beta \in \kappa . R = x \downarrow \beta .$

$[*84{\cdot}11.\text{Hp}] \quad \equiv . (\exists x, \beta) . x \in \alpha \cap \beta . \beta \in \kappa . \alpha = \beta . R = x \downarrow \beta .$

$[*13{\cdot}195] \quad \equiv . (\exists x) . x \in \alpha . R = x \downarrow \beta .$

$[*85{\cdot}601] \quad \equiv . R \in \epsilon \rfloor`\alpha :.$

$[*37{\cdot}69] \supset :. T_\epsilon``\kappa = \epsilon \rfloor``\kappa$ (4)

$\vdash . (1) . (2) . (3) . (4) . *111{\cdot}4 . \supset \vdash . \text{Prop}$

∗112·15. $\vdash : \kappa \in \text{Cls}^2 \text{excl} . \supset . s`\kappa \in \Sigma\text{Nc}`\kappa$ $[*112{\cdot}14{\cdot}11 . *111{\cdot}44]$

∗112·151. $s`\epsilon \rfloor``\lambda = \hat{R}\{(\exists\alpha, x) . \alpha \in \lambda . x \in \alpha . R = x \downarrow \alpha\} . \dot{s}`s`\epsilon \rfloor``\lambda = \epsilon \upharpoonright \lambda$

Dem.

$\vdash . *40{\cdot}11 . (*85{\cdot}5) . \supset$

$\vdash . s`\epsilon \rfloor``\lambda = \hat{R}\{(\exists\alpha) . \alpha \in \lambda . R \in \downarrow \alpha``\alpha\}$

$[*38{\cdot}131] = \hat{R}\{(\exists\alpha, x) . \alpha \in \lambda . x \in \alpha . R = x \downarrow \alpha\}$ (1)

$\vdash . (1) . *41{\cdot}11 . \supset$

$\vdash . \dot{s}`s`\epsilon \rfloor``\lambda = \hat{y}\hat{\beta}\{(\exists R, \alpha, x) . \alpha \in \lambda . x \in \alpha . R = x \downarrow \alpha . yR\beta\}$

$[*13\cdot195 . *55\cdot13] = \hat{y}\hat{\beta}\{(\exists \alpha, x) . \alpha \epsilon \lambda . x \epsilon \alpha . y = x . \beta = \alpha\}$

$[*13\cdot22] \quad = \hat{y}\hat{\beta}\{\beta \epsilon \lambda . y \epsilon \beta\}$

$[*35\cdot101] \quad = \epsilon \upharpoonright \lambda$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

The following proposition is a lemma for $*112\cdot153$, which is required for $*112\cdot16$. $*112\cdot16$ in turn is used in $*112\cdot17$, which is a fundamental proposition in the theory of addition.

$*112\cdot152$. $\vdash : T \epsilon 1 \to \text{Cls} . \beta \subset \text{Ɑ}\text{‘}T . \supset . (T \| \breve{T}_\epsilon)\text{‘‘}\epsilon \overline{\downarrow} \beta = \epsilon \overline{\downarrow} (T\text{‘‘}\beta)$

Dem.

$\vdash . *37\cdot6 . *85\cdot601 . \supset \vdash . (T \| \breve{T}_\epsilon)\text{‘‘}\epsilon \overline{\downarrow} \beta = \hat{R}\{(\exists y) . y \epsilon \beta . R = (T \| \breve{T}_\epsilon)\text{‘}(y \downarrow \beta)\}$ (1)

$\vdash . (1) . *55\cdot61 . \supset$

$\vdash : \text{Hp} . \supset . (T \| \breve{T}_\epsilon)\text{‘‘}\epsilon \overline{\downarrow} \beta = \hat{R}\{(\exists y) . y \epsilon \beta . R = (T\text{‘}y) \downarrow (T_\epsilon\text{‘}\beta)\}$

$[*37\cdot11] \quad = \hat{R}\{(\exists y) . y \epsilon \beta . R = (T\text{‘}y) \downarrow (T\text{‘‘}\beta)\}$

$[*38\cdot131] \quad = \downarrow (T\text{‘‘}\beta)\text{‘‘}(T\text{‘‘}\beta)$

$[*85\cdot601] \quad = \epsilon \overline{\downarrow} (T\text{‘‘}\beta) : \supset \vdash . \text{Prop}$

In the following proposition, we have a double correlator of a sort which will frequently occur in cardinal arithmetic, namely $T \| \breve{T}_\epsilon$ with its converse domain limited, where T is a given double correlator (or single correlator, on other occasions). As appears from the propositions used in the above proof of $*112\cdot152$, if T is a correlator whose converse domain includes β and has y as a member, $(T \| \breve{T}_\epsilon)\text{‘}(y \downarrow \beta) = (T\text{‘}y) \downarrow (T\text{‘‘}\beta)$. Thus $T \| \breve{T}_\epsilon$ is an operation which, when operating on suitable relations of individuals to classes (including selectors), turns the individuals into their correlates and the classes into the classes of their members' correlates. This is why it is a useful relation.

$*112\cdot153$. $T \epsilon \kappa \, \overline{\text{sm}} \, \overline{\text{sm}} \, \lambda . \supset . (T \| \breve{T}_\epsilon) \upharpoonright s\text{‘}\epsilon \overline{\downarrow}\text{‘‘}\lambda \, \epsilon \, (\epsilon \overline{\downarrow}\text{‘‘}\kappa) \, \overline{\text{sm}} \, \overline{\text{sm}} \, (\epsilon \overline{\downarrow}\text{‘‘}\lambda)$

Dem.

$\vdash . *112\cdot151 . *41\cdot43\cdot44 . \supset \vdash . s\text{‘D‘‘}s\text{‘}\epsilon \overline{\downarrow}\text{‘‘}\lambda = \text{D‘}(\epsilon \upharpoonright \lambda) . s\text{‘Ɑ‘‘}s\text{‘}\epsilon \overline{\downarrow}\text{‘‘}\lambda = \text{Ɑ‘}(\epsilon \upharpoonright \lambda) .$

$[*62\cdot41\cdot43] \quad \supset \vdash . s\text{‘D‘‘}s\text{‘}\epsilon \overline{\downarrow}\text{‘‘}\lambda = s\text{‘}\lambda . s\text{‘Ɑ‘‘}s\text{‘}\epsilon \overline{\downarrow}\text{‘‘}\lambda = \lambda - \iota\text{‘}\Lambda$ (1)

$\vdash . (1) . *111\cdot1 . *37\cdot231 . \supset \vdash : \text{Hp} . \supset . s\text{‘D‘‘}s\text{‘}\epsilon \overline{\downarrow}\text{‘‘}\lambda \subset \text{Ɑ‘}T . s\text{‘Ɑ‘‘}s\text{‘}\epsilon \overline{\downarrow}\text{‘‘}\lambda \subset \text{Ɑ‘}T_\epsilon$ (2)

$\vdash . *111\cdot1 . *71\cdot29 . \quad \supset \vdash : \text{Hp} . \supset . T \upharpoonright s\text{‘D‘‘}s\text{‘}\epsilon \overline{\downarrow}\text{‘‘}\lambda \, \epsilon \, 1 \to 1$ (3)

$\vdash . *111\cdot11 . (1) . \quad \supset \vdash : \text{Hp} . \supset . T_\epsilon \upharpoonright s\text{‘Ɑ‘‘}s\text{‘}\epsilon \overline{\downarrow}\text{‘‘}\lambda \, \epsilon \, 1 \to 1$ (4)

$\vdash . (2) . (3) . (4) . *74\cdot775 \dfrac{s\text{‘}\epsilon \overline{\downarrow}\text{‘‘}\lambda, T, T_\epsilon}{\lambda, \quad Q, R} . \supset \vdash : \text{Hp} . \supset . (T \| \breve{T}_\epsilon) \upharpoonright s\text{‘}\epsilon \overline{\downarrow}\text{‘‘}\lambda \, \epsilon \, 1 \to 1$ (5)

$\vdash . *43\cdot302 . \quad \supset \vdash . s\text{‘}\epsilon \overline{\downarrow}\text{‘‘}\lambda \subset \text{Ɑ‘}(T \| \breve{T}_\epsilon)$ (6)

$\vdash . *112\cdot152 . \supset \vdash : \text{Hp} . \supset . (T \| \breve{T}_\epsilon)\text{‘‘‘}\epsilon \overline{\downarrow}\text{‘‘}\lambda = \epsilon \overline{\downarrow}\text{‘‘}T\text{‘‘‘}\lambda .$

$[*37\cdot11] \quad \supset . (T \| \breve{T}_\epsilon)_\epsilon\text{‘‘}\epsilon \overline{\downarrow}\text{‘‘}\lambda = \epsilon \overline{\downarrow}\text{‘‘}T_\epsilon\text{‘‘}\lambda$

$[*111\cdot1 . \text{Hp}] \quad = \epsilon \overline{\downarrow}\text{‘‘}\kappa$ (7)

$\vdash . (5) . (6) . (7) . *111\cdot14 . \supset \vdash . \text{Prop}$

***112·16.** $\vdash : \kappa \,\mathrm{sm}\,\mathrm{sm}\, \lambda \,.\supset .\, \epsilon\downarrow``\kappa \,\mathrm{sm}\,\mathrm{sm}\, \epsilon\downarrow``\lambda$ [*112·153 . *111·4]

***112·17.** $\vdash : \kappa \,\mathrm{sm}\,\mathrm{sm}\, \lambda \,.\supset .\, \Sigma\mathrm{Nc}`\kappa = \Sigma\mathrm{Nc}`\lambda \,.\, \Sigma`\kappa \,\mathrm{sm}\, \Sigma`\lambda$

Dem.

$\vdash .$ *112·16 . *111·44 $. \supset \vdash : \mathrm{Hp} \,.\supset .\, s`\epsilon\downarrow``\kappa \,\mathrm{sm}\, s`\epsilon\downarrow``\lambda$ (1)

$\vdash . (1) .$ *112·1·101 $. \supset \vdash . \mathrm{Prop}$

***112·18.** $\vdash . \Sigma\mathrm{Nc}`\kappa = \Sigma\mathrm{Nc}`\epsilon\downarrow``\kappa$

Dem.

$\vdash .$ *85·61 . *112·15 $. \supset \vdash .\, s`\epsilon\downarrow``\kappa \,\epsilon\, \Sigma\mathrm{Nc}`\epsilon\downarrow``\kappa$ (1)

$\vdash . (1) .$ *112·12 . *100·34 $. \supset \vdash . \mathrm{Prop}$

***112·2.** $\vdash : S \,\epsilon\, 1 \to 1 \,.\, \mathrm{D}`S = \epsilon\downarrow``\kappa \,.\, \mathrm{G}`S = \epsilon\downarrow``\lambda \,.\, \exists !\, \epsilon_\Delta`\mathrm{Crp}(S)``\lambda \,.$
$\supset .\, \Sigma\mathrm{Nc}`\kappa = \Sigma\mathrm{Nc}`\lambda \,.\, \Sigma`\kappa \,\mathrm{sm}\, \Sigma`\lambda$

Dem.

$\vdash .$ *111·311 . *85·61 $. \supset \vdash : \mathrm{Hp} \,.\supset .\, s`\epsilon\downarrow``\kappa \,\mathrm{sm}\, s`\epsilon\downarrow``\lambda$ (1)

$\vdash . (1) .$ *112·1·101 $. \supset \vdash . \mathrm{Prop}$

***112·21.** $\vdash :. \mathrm{Mult\,ax} \,.\supset : (\exists S) \,.\, S \,\epsilon\, 1 \to 1 \,.\, S \subset \mathrm{sm} \,.\, \mathrm{D}`S = \epsilon\downarrow``\kappa \,.\, \mathrm{G}`S = \epsilon\downarrow``\lambda \,.$
$\equiv .\, \epsilon\downarrow``\kappa \,\mathrm{sm}\,\mathrm{sm}\, \epsilon\downarrow``\lambda$ [*111·5 . *85·61]

***112·22.** $\vdash :. \mathrm{Mult\,ax} \,.\supset : \exists !\, (\epsilon\downarrow``\kappa)\, \overline{\mathrm{sm}}\, (\epsilon\downarrow``\lambda) \cap \mathrm{Rl}`\mathrm{sm} \,.\supset .$
$\Sigma\mathrm{Nc}`\kappa = \Sigma\mathrm{Nc}`\lambda$ [*112·17·18·21]

***112·23.** $\vdash :. \mathrm{Mult\,ax} \,.\supset : \kappa, \lambda \,\epsilon\, \mathrm{Cls}^2\,\mathrm{excl} \,.\, \exists !\, \kappa \,\overline{\mathrm{sm}}\, \lambda \cap \mathrm{Rl}`\mathrm{sm} \,.\supset .$
$s`\kappa, s`\lambda \,\epsilon\, \Sigma\mathrm{Nc}`\kappa \,.\, \Sigma\mathrm{Nc}`\kappa = \Sigma\mathrm{Nc}`\lambda$

Dem.

$\vdash .$ *112·15 $. \supset \vdash : \mathrm{Hp} \,.\, \kappa, \lambda \,\epsilon\, \mathrm{Cls}^2\,\mathrm{excl} \,.\supset .\, s`\kappa \,\epsilon\, \Sigma\mathrm{Nc}`\kappa \,.\, s`\lambda \,\epsilon\, \Sigma\mathrm{Nc}`\lambda$ (1)

$\vdash .$ *111·51 $. \supset \vdash : \mathrm{Hp}(1) \,.\, \exists !\, \kappa \,\overline{\mathrm{sm}}\, \lambda \cap \mathrm{Rl}`\mathrm{sm} \,.\supset .\, s`\kappa \,\mathrm{sm}\, s`\lambda$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***112·231.** $\vdash : S \,\epsilon\, \kappa \,\overline{\mathrm{sm}}\, \lambda \cap \mathrm{Rl}`\mathrm{sm} \,.\supset .\, \epsilon\downarrow | S | \mathrm{Cnv}`\epsilon\downarrow \,\epsilon (\epsilon\downarrow``\kappa) \overline{\mathrm{sm}} (\epsilon\downarrow``\lambda) \cap \mathrm{Rl}`\mathrm{sm}$

Dem.

$\vdash .$ *73·63 . *85·601 $. \supset \vdash : S \,\epsilon\, \kappa \,\overline{\mathrm{sm}}\, \lambda \,.\supset .\, \epsilon\downarrow | S | \mathrm{Cnv}`\epsilon\downarrow \,\epsilon (\epsilon\downarrow``\kappa) \overline{\mathrm{sm}} (\epsilon\downarrow``\lambda)$ (1)

$\vdash .$ *85·601 . *73·33·34 $. \supset \vdash : S \subset \mathrm{sm} \,.\supset .\, \epsilon\downarrow | S | \mathrm{Cnv}`\epsilon\downarrow \subset \mathrm{sm}$ (2)

$\vdash . (1) . (2) . \supset \vdash : S \,\epsilon\, \kappa \,\overline{\mathrm{sm}}\, \lambda \cap \mathrm{Rl}`\mathrm{sm} \,.\supset .\, \epsilon\downarrow | S | \mathrm{Cnv}`\epsilon\downarrow \,\epsilon (\epsilon\downarrow``\kappa)\, \overline{\mathrm{sm}}\, (\epsilon\downarrow``\lambda) \cap \mathrm{Rl}`\mathrm{sm} :$
$\supset \vdash . \mathrm{Prop}$

***112·24.** $\vdash :. \mathrm{Mult\,ax} \,.\supset : \mu, \nu \,\epsilon\, \mathrm{NC} \,.\, \kappa, \lambda \,\epsilon\, \mu \cap \mathrm{Cl}`\nu \,.\supset .\, \Sigma\mathrm{Nc}`\kappa = \Sigma\mathrm{Nc}`\lambda$

Dem.

$\vdash .$ *111·52 $. \supset \vdash : \mu, \nu \,\epsilon\, \mathrm{NC} \,.\, \kappa, \lambda \,\epsilon\, \mu \cap \mathrm{Cl}`\nu \,.\supset .\, \exists !\, \kappa \,\overline{\mathrm{sm}}\, \lambda \cap \mathrm{Rl}`\mathrm{sm} \,.$

[*112·231] $\supset .\, \exists !\, (\epsilon\downarrow``\kappa) \overline{\mathrm{sm}} (\epsilon\downarrow``\lambda) \cap \mathrm{Rl}`\mathrm{sm}$ (1)

$\vdash . (1) .$ *111·51 . *85·61 $. \supset$

$\vdash :. \mathrm{Mult\,ax} \,.\supset : \mu, \nu \,\epsilon\, \mathrm{NC} \,.\, \kappa, \lambda \,\epsilon\, \mu \cap \mathrm{Cl}`\nu \,.\supset .\, s`\epsilon\downarrow``\kappa \,\mathrm{sm}\, s`\epsilon\downarrow``\lambda \,.$

[*112·101] $\supset .\, \Sigma\mathrm{Nc}`\kappa = \Sigma\mathrm{Nc}`\lambda :. \supset \vdash . \mathrm{Prop}$

***112·3.** $\vdash . \Sigma'\Lambda = \Lambda$ [*37·29 . *40·21 . *112·1]

***112·301.** $\vdash . \Sigma'\iota'\Lambda = \Lambda$

Dem.

$\vdash . *112\cdot102 . \supset \vdash . \Sigma'\iota'\Lambda = \hat{R}\{(\exists\alpha, x) . \alpha \epsilon \iota'\Lambda . x \epsilon \alpha . R = x \downarrow \alpha\}$

$[*51\cdot15] \qquad = \hat{R}\{(\exists x) . x \epsilon \Lambda . R = x \downarrow \Lambda\}$

$[*24\cdot15] \qquad = \Lambda . \supset \vdash . \text{Prop}$

***112·302.** $\vdash . \Sigma'\kappa = \Sigma'(\kappa - \iota'\Lambda)$

Dem.

$\vdash . *112\cdot102 . \supset \vdash . \Sigma'\kappa = \hat{R}\{(\exists\alpha, x) . \alpha \epsilon \kappa . x \epsilon \alpha . R = x \downarrow \alpha\}$

$[*10\cdot24] \qquad = \hat{R}\{(\exists\alpha, x) . \alpha \epsilon \kappa . \exists ! \alpha . x \epsilon \alpha . R = x \downarrow \alpha\}$

$[*53\cdot52] \qquad = \hat{R}\{(\exists\alpha, x) . \alpha \epsilon \kappa - \iota'\Lambda . x \epsilon \alpha . R = x \downarrow \alpha\}$

$[*112\cdot102] \qquad = \Sigma'(\kappa - \iota'\Lambda) . \supset \vdash . \text{Prop}$

Thus if Λ is a member of a class of classes, it does not affect the value of their arithmetical sum.

***112·303.** $\vdash : \kappa \cap \lambda = \Lambda . \supset . \Sigma'\kappa \cap \Sigma'\lambda = \Lambda$

Dem.

$\vdash . *112\cdot102 . \supset$

$\vdash : R \epsilon \Sigma'\kappa \cap \Sigma'\lambda . \equiv . (\exists\alpha, \beta, x, y) . \alpha \epsilon \kappa . \beta \epsilon \lambda . x \epsilon \alpha . y \epsilon \beta . R = x \downarrow \alpha = y \downarrow \beta .$

$[*55\cdot202] \qquad \supset . (\exists\alpha, x) . \alpha \epsilon \kappa \cap \lambda . x \epsilon \alpha .$

$[*24\cdot5] \qquad \supset . \exists ! \kappa \cap \lambda \qquad (1)$

$\vdash . (1) . \text{Transp} . \supset \vdash . \text{Prop}$

***112·304.** $\vdash : \Sigma'\kappa = \Lambda . \equiv . s'\kappa = \Lambda$

Dem.

$\vdash . *112\cdot3\cdot301 . *53\cdot24 . \supset \vdash : s'\kappa = \Lambda . \supset . \Sigma'\kappa = \Lambda \qquad (1)$

$\vdash . *112\cdot102 . \qquad \supset \vdash : \alpha \epsilon \kappa . x \epsilon \alpha . \supset . x \downarrow \alpha \epsilon \Sigma'\kappa :$

$[*10\cdot24 . *40\cdot11] \qquad \supset \vdash : \exists ! s'\kappa . \supset . \exists ! \Sigma'\kappa \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***112·31.** $\vdash . \Sigma'(\kappa \cup \lambda) = \Sigma'\kappa \cup \Sigma'\lambda$

Dem.

$\vdash . *112\cdot1 . \supset \vdash . \Sigma'(\kappa \cup \lambda) = s'\epsilon\,\overline{\downarrow}``(\kappa \cup \lambda)$

$[*40\cdot31] \qquad = s'\epsilon\,\overline{\downarrow}``\kappa \cup s'\epsilon\,\overline{\downarrow}``\lambda$

$[*112\cdot1] \qquad = \Sigma'\kappa \cup \Sigma'\lambda . \supset \vdash . \text{Prop}$

***112·311.** $\vdash : \kappa \cap \lambda = \Lambda . \supset . \Sigma\text{Nc}'(\kappa \cup \lambda) = \Sigma\text{Nc}'\kappa +_c \Sigma\text{Nc}'\lambda$

Dem.

$\vdash . *112\cdot303 . *110\cdot32 . \supset$

$\vdash : \text{Hp} . \supset . \text{Nc}'(\Sigma'\kappa \cup \Sigma'\lambda) = \text{Nc}'\Sigma'\kappa +_c \text{Nc}'\Sigma'\lambda$

$[*112\cdot101] \qquad = \Sigma\text{Nc}'\kappa +_c \Sigma\text{Nc}'\lambda \qquad (1)$

$\vdash . (1) . *112\cdot31 . \supset \vdash . \text{Prop}$

***112·32.** $\vdash . \Sigma\text{'}\iota\text{'}\alpha = \epsilon \downarrow \alpha$

Dem.

$\vdash . *53·31 . *112·1 . \supset \vdash . \Sigma\text{'}\iota\text{'}\alpha = s\text{'}\iota\text{'}\epsilon \downarrow \alpha$

[*53·02] $= \epsilon \downarrow \alpha . \supset \vdash . \text{Prop}$

***112·321.** $\vdash . \Sigma\text{Nc}\text{'}\iota\text{'}\alpha = \text{Nc}\text{'}\alpha$ [*112·32·101 . *85·601]

***112·33.** $\vdash . \Sigma\text{'}(\iota\text{'}\alpha \cup \iota\text{'}\beta) = \epsilon \downarrow \alpha \cup \epsilon \downarrow \beta$ [*112·32·31]

***112·331.** $\vdash . \Sigma\text{'}(\kappa \cup \iota\text{'}\beta) = \Sigma\text{'}\kappa \cup \epsilon \downarrow \beta$ [*112·31·32]

***112·34.** $\vdash : \alpha \neq \beta . \supset . \Sigma\text{Nc}\text{'}(\iota\text{'}\alpha \cup \iota\text{'}\beta) = \text{Nc}\text{'}\alpha +_c \text{Nc}\text{'}\beta$

Dem.

$\vdash . *51·231 . *112·311 . \supset$

$\vdash : \text{Hp} . \supset . \Sigma\text{Nc}\text{'}(\iota\text{'}\alpha \cup \iota\text{'}\beta) = \Sigma\text{Nc}\text{'}\iota\text{'}\alpha +_c \Sigma\text{Nc}\text{'}\iota\text{'}\beta$

[*112·321] $= \text{Nc}\text{'}\alpha +_c \text{Nc}\text{'}\beta : \supset \vdash . \text{Prop}$

This proposition establishes the agreement of the two definitions of addition, namely that in *110 and that in *112. It will be seen that the definition of *112 is inapplicable to the addition of a class to itself, if this is to give the double of the class, instead of (like logical addition) simply reproducing the class. Hence the need of the condition $\alpha \neq \beta$ in the above proposition.

***112·341.** $\vdash : \beta \sim\epsilon \kappa . \supset . \Sigma\text{Nc}\text{'}(\kappa \cup \iota\text{'}\beta) = \Sigma\text{Nc}\text{'}\kappa +_c \text{Nc}\text{'}\beta$

Dem.

$\vdash . *51·211 . \supset \vdash : \text{Hp} . \supset . \kappa \cap \iota\text{'}\beta = \Lambda .$

[*112·311] $\supset . \Sigma\text{Nc}\text{'}(\kappa \cup \iota\text{'}\beta) = \Sigma\text{Nc}\text{'}\kappa +_c \Sigma\text{Nc}\text{'}\iota\text{'}\beta$

[*112·321] $= \Sigma\text{Nc}\text{'}\kappa +_c \text{Nc}\text{'}\beta : \supset \vdash . \text{Prop}$

***112·35.** $\vdash : \alpha \neq \beta . \alpha \neq \gamma . \beta \neq \gamma . \supset . \Sigma\text{Nc}\text{'}(\iota\text{'}\alpha \cup \iota\text{'}\beta \cup \iota\text{'}\gamma) = \text{Nc}\text{'}\alpha +_c \text{Nc}\text{'}\beta +_c \text{Nc}\text{'}\gamma$

Dem.

$\vdash . *51·231 . *112·311 . \supset$

$\vdash : \text{Hp} . \supset . \Sigma\text{Nc}\text{'}(\iota\text{'}\alpha \cup \iota\text{'}\beta \cup \iota\text{'}\gamma) = \Sigma\text{Nc}\text{'}(\iota\text{'}\alpha \cup \iota\text{'}\beta) +_c \Sigma\text{Nc}\text{'}\iota\text{'}\gamma$

[*112·34·321] $= \text{Nc}\text{'}\alpha +_c \text{Nc}\text{'}\beta +_c \text{Nc}\text{'}\gamma : \supset \vdash . \text{Prop}$

Similar propositions can obviously be proved for any finite number of summands.

***112·4.** $\vdash : s\text{'}\kappa, s\text{''}\kappa \,\epsilon\, \text{Cls}^2 \text{excl} . \supset . \Sigma\text{Nc}\text{'}s\text{'}\kappa = \Sigma\text{Nc}\text{'}s\text{''}\kappa$

Dem.

$\vdash . *112·15 . \supset \vdash : \text{Hp} . \supset . \Sigma\text{Nc}\text{'}s\text{'}\kappa = \text{Nc}\text{'}s\text{'}s\text{'}\kappa$

[*42·1] $= \text{Nc}\text{'}s\text{'}s\text{''}\kappa$

[*112·15] $= \Sigma\text{Nc}\text{'}s\text{''}\kappa : \supset \vdash . \text{Prop}$

***112·41.** $\vdash . s`\Sigma``\lambda = \Sigma`s`\lambda$

Dem.

$\vdash . *112·1 . \supset \vdash . s`\Sigma``\lambda = s`s``\epsilon \downarrow```\lambda$

[*42·1] $= s`s`\epsilon \downarrow```\lambda$

[*40·38] $= s`\epsilon \downarrow``s`\lambda$

[*112·1] $= \Sigma`s`\lambda . \supset \vdash . \text{Prop}$

***112·42.** $\vdash : \lambda \epsilon \text{Cls}^2 \text{excl} . \supset . \Sigma``\lambda \epsilon \text{Cls}^2 \text{excl}$

Dem.

$\vdash . *112·303 . \supset \vdash :. \lambda \epsilon \text{Cls}^2 \text{excl} . \supset : \beta, \gamma \epsilon \lambda . \beta \neq \gamma . \supset_{\beta,\gamma} . \Sigma`\beta \cap \Sigma`\gamma = \Lambda :$

[*30·37.Transp.*37·63] $\supset : \mu, \nu \epsilon \Sigma``\lambda . \mu \neq \nu . \supset_{\mu,\nu} . \mu \cap \nu = \Lambda :$

[*84·1] $\supset : \Sigma``\lambda \epsilon \text{Cls}^2 \text{excl} :. \supset \vdash . \text{Prop}$

***112·43.** $\vdash : \lambda \epsilon \text{Cls}^2 \text{excl} . \supset . \text{Nc}`\Sigma`\Sigma``\lambda = \text{Nc}`\Sigma`s`\lambda$

Dem.

$\vdash . *112·15·42 . \supset \vdash : \text{Hp} . \supset . \text{Nc}`\Sigma`\Sigma``\lambda = \text{Nc}`s`\Sigma``\lambda$

[*112·41] $= \text{Nc}`\Sigma`s`\lambda : \supset \vdash . \text{Prop}$

The above is the associative law for arithmetical addition.

$*113$. ON THE ARITHMETICAL PRODUCT OF TWO CLASSES OR OF TWO CARDINALS

Summary of $*113$.

In this number, we give a definition of multiplication which can be extended to any finite number of factors, but not to an infinite number of factors. We define first the arithmetical class-product of two classes α and β, and thence the product of two cardinals μ and ν as the number of terms in the product of α and β when α has μ terms and β has ν terms. In $*114$, we shall give a definition of multiplication which is not restricted to a finite number of factors. The advantages of the definition to be given in this number are, that it does not require the factors to be of the same type, and that it enables us to multiply a class by itself without (as in logical addition and multiplication) simply reproducing the class in question. The disadvantage of the definition in this number is the impossibility of extending it to an infinite number of factors.

The arithmetical class-product of two classes α and β, which we denote by $\beta \times \alpha$[*], is the class of all ordinal couples which take their referent from α and their relatum from β, *i.e.* it is the class of all such relations as $x \downarrow y$, where $x \in \alpha$ and $y \in \beta$. For a given y, the class of couples we obtain is $\downarrow y``\alpha$, which is similar to α; and the number of such classes, for varying y, is $\mathrm{Nc}`\beta$. Thus we have $\mathrm{Nc}`\beta$ classes of $\mathrm{Nc}`\alpha$ couples, and $\beta \times \alpha$ is the logical sum of these classes of couples. The class of such classes as $\downarrow y``\alpha$, where $y \in \beta$, is important again in connection with exponentiation; we have $\downarrow y``\alpha = \alpha \underset{,,}{\downarrow} y$, whence the class of such classes, when y is varied among the β's, is $\alpha \underset{,,}{\downarrow} ``\beta$, and

$$\beta \times \alpha = s`\alpha \underset{,,}{\downarrow} ``\beta \quad (\text{cf. } *40\cdot 7),$$

which we take as the definition of $\beta \times \alpha$.

We represent the arithmetical product of μ and ν by $\mu \times_c \nu$. This, as well as $\mathrm{Nc}`\alpha \times_c \mathrm{Nc}`\beta$, is defined in terms of $\alpha \times \beta$ exactly as, in $*110$, the sum was defined in terms of $\alpha + \beta$.

The present number contains many propositions which belong to the theory of $\alpha \underset{,,}{\downarrow} ``\beta$ rather than (specially) of $\beta \times \alpha$; and many propositions are rather logical than arithmetical in their nature, *i.e.* they might have been given in $*55$. The line is, however, so hard to draw that it has seemed better to deal simultaneously with all propositions on $\alpha \underset{,,}{\downarrow} ``\beta$ or on its sum, which is $\beta \times \alpha$. Thus in the present number, the early propositions, down to $*113\cdot 118$, deal mainly with logical properties of $\alpha \underset{,,}{\downarrow} ``\beta$ and $\beta \times \alpha$; the following propositions,

[*] We define this as $\beta \times \alpha$, rather than $\alpha \times \beta$, for the sake of certain analogies with products in relation-arithmetic. Cf. $*166$.

down to $*113{\cdot}13$, deal mainly with arithmetical properties of $\alpha \underset{\text{,,}}{\downarrow} \text{``}\beta$; the propositions $*113{\cdot}14$—$\cdot191$ are concerned mainly with arithmetical properties of $\beta \times \alpha$; $*113{\cdot}2$—$\cdot27$ deal with the simpler properties of $\mu \times_c \nu$; $*113{\cdot}3$—$\cdot34$ give propositions involving the multiplicative axiom, and exhibiting the connection (assuming this axiom) of addition and multiplication; $*113{\cdot}4$—$\cdot491$ are concerned with various forms of the distributive law; $*113{\cdot}5$—$\cdot541$ deal with the associative law of multiplication, and the remaining propositions deal with multiplication by 0 or 1 or 2.

The most important propositions in the present number are the following:

$*113{\cdot}101$. $\vdash : R \in \beta \times \alpha \,.\equiv .\, (\exists x, y) \,.\, x \in \alpha \,.\, y \in \beta \,.\, R = x \downarrow y$

This merely embodies the definition of $\beta \times \alpha$.

$*113{\cdot}105$. $\vdash : \exists ! \alpha \,.\supset .\, \alpha \underset{\text{,,}}{\downarrow} \in 1 \rightarrow 1$

This proposition is especially useful in dealing with exponentiation ($*116$).

$*113{\cdot}114$. $\vdash :.\, \alpha = \Lambda \,.\vee .\, \beta = \Lambda : \equiv .\, \beta \times \alpha = \Lambda$

It is in virtue of this proposition that a product of a finite number of factors only vanishes when one of its factors vanishes.

$*113{\cdot}118$. $\vdash .\, s\text{`}D\text{``}(\beta \times \alpha) \subset \alpha \,.\, s\text{`}\text{Ɑ}\text{``}(\beta \times \alpha) \subset \beta$

This proposition is chiefly useful in the analogous theory of ordinal products ($*165$, $*166$), where it enables us to apply $*74{\cdot}773$. Unless $\beta = \Lambda$, we have $s\text{`}D\text{``}(\beta \times \alpha) = \alpha$, and unless $\alpha = \Lambda$, $s\text{`}\text{Ɑ}\text{``}(\beta \times \alpha) = \beta$ ($*113{\cdot}116$).

$*113{\cdot}12$. $\vdash : \exists ! \alpha \,.\supset .\, \alpha \underset{\text{,,}}{\downarrow} \text{``}\beta \in \text{Nc}\text{`}\beta \cap \text{Cl excl}\text{`}\text{Nc}\text{`}\alpha$

I.e. unless α is null, $\alpha \underset{\text{,,}}{\downarrow} \text{``}\beta$ consists of $\text{Nc}\text{`}\beta$ mutually exclusive classes each having $\text{Nc}\text{`}\alpha$ members.

$*113{\cdot}127$. $\vdash : R \upharpoonright \gamma \in \alpha \,\overline{\text{sm}}\, \gamma \,.\, S \upharpoonright \delta \in \beta \,\overline{\text{sm}}\, \delta \,.\supset .$

$$(R \parallel \breve{S}) \upharpoonright (\delta \times \gamma) \in (\alpha \underset{\text{,,}}{\downarrow} \text{``}\beta) \,\overline{\text{sm}}\, \overline{\text{sm}}\, (\gamma \underset{\text{,,}}{\downarrow} \text{``}\delta)$$

This is an important proposition, since it gives a double correlator of $\alpha \underset{\text{,,}}{\downarrow} \text{``}\beta$ with $\gamma \underset{\text{,,}}{\downarrow} \text{``}\delta$ whenever simple correlators of α with γ and of β with δ are given. It leads at once to

$*113{\cdot}13$. $\vdash : \alpha \,\text{sm}\, \gamma \,.\, \beta \,\text{sm}\, \delta \,.\supset .\, \alpha \underset{\text{,,}}{\downarrow} \text{``}\beta \,\text{sm sm}\, \gamma \underset{\text{,,}}{\downarrow} \text{``}\delta \,.\, (\beta \times \alpha) \,\text{sm}\, (\delta \times \gamma)$

This proposition is fundamental in the theory of multiplication, since it shows that the number of members of $\beta \times \alpha$ depends only upon the numbers of members of α and β. It is also fundamental in the theory of exponentiation, as will appear in $*116$.

$*113{\cdot}141$. $\vdash .\, \text{Nc}\text{`}(\alpha \times \beta) = \text{Nc}\text{`}(\beta \times \alpha)$

This is the source of the commutative law of multiplication ($*113{\cdot}27$).

$*113{\cdot}146$. $\vdash : \alpha \neq \beta \,.\supset .\, \alpha \times \beta \,\text{sm}\, \epsilon_{\Delta}\text{`}(\iota\text{`}\alpha \cup \iota\text{`}\beta)$

This connects our present theory of multiplication with the theory of selections.

We come next to propositions concerning $\mu \times_0 \nu$. We have

***113·204.** $\vdash :. \mu = \Lambda . \mathbf{v} . \nu = \Lambda . \mathbf{v} . \sim(\mu, \nu \, \epsilon \, \text{NC}) : \supset . \mu \times_0 \nu = \Lambda$

The use of this proposition, like that of *110·4, is for avoiding trivial exceptions.

***113·23.** $\vdash . \mu \times_0 \nu \, \epsilon \, \text{NC}$

***113·25.** $\vdash . \text{Nc}`\gamma \times_0 \text{Nc}`\delta = \text{Nc}`(\gamma \times \delta)$

This proposition enables us to infer propositions on products of cardinals from propositions on products of classes, and is therefore constantly used.

***113·27.** $\vdash . \mu \times_0 \nu = \nu \times_0 \mu$

This is the commutative law of cardinal multiplication.

The chief proposition using the multiplicative axiom is

***113·31.** $\vdash :. \text{Mult ax} . \supset : \mu, \nu \, \epsilon \, \text{NC} . \kappa \, \epsilon \, \nu \cap \text{Cl}`\mu . \supset . \Sigma`\kappa \, \epsilon \, \mu \times_0 \nu$

I.e. assuming the multiplicative axiom, the sum of the numbers of members in ν classes of μ terms is $\mu \times_0 \nu$. If we had taken this sum as *defining* $\mu \times_0 \nu$, almost all propositions on multiplication would have required the multiplicative axiom. The advantage of $\alpha \underset{\prime\prime}{\downarrow} ``\beta$ is that, given $\alpha \text{ sm } \gamma$ and $\beta \text{ sm } \delta$, we can construct a double correlator of $\alpha \underset{\prime\prime}{\downarrow} ``\beta$ with $\gamma \underset{\prime\prime}{\downarrow} ``\delta$, without using the multiplicative axiom. This is proved in *113·127 (mentioned above).

The distributive law, which is next considered, has various forms. We have, to begin with,

***113·4.** $\vdash . (\beta \cup \gamma) \times \alpha = (\beta \times \alpha) \cup (\gamma \times \alpha)$

whence, using also the commutative law, we easily deduce

***113·43.** $\vdash . (\nu +_0 \varpi) \times_0 \mu = \mu \times_0 (\nu +_0 \varpi) = (\mu \times_0 \nu) +_0 (\mu \times_0 \varpi)$

But the distributive law also holds when, instead of enumerated summands β, γ or ν, ϖ, the summands are given as the members of a class κ, which may be infinite. We have

***113·48.** $\vdash . s`\alpha \times ``\kappa = \alpha \times s`\kappa = \text{Cnv}``\{(s`\kappa) \times \alpha\}$

whence, using the definitions of *112, we find

***113·491.** $\vdash : \kappa \, \epsilon \, \text{Cls}^2 \text{ excl} . \supset . \Sigma\text{Nc}`\alpha \times ``\kappa = \text{Nc}`(\alpha \times \Sigma`\kappa) = \text{Nc}`\alpha \times_0 \Sigma\text{Nc}`\kappa$

This is an extension of the distributive law to the case where the number of summands may be infinite.

The associative law

***113·54.** $\vdash . (\mu \times_0 \nu) \times_0 \varpi = \mu \times_0 (\nu \times_0 \varpi)$

is proved without any difficulty.

We prove next that $\mu \times_0 \nu = 0$ when, and only when, $\mu = 0$ or $\nu = 0$, μ, ν being existent cardinals (*113·602); that a cardinal is unchanged when it is multiplied by 1 (*113·62·621); that $\mu \times_0 2 = \mu +_0 \mu$ (*113·66) and that $\mu \times_0 (\nu +_0 1) = (\mu \times_0 \nu) +_0 \mu$ (*113·671).

***113·02.** $\beta \times \alpha = s‘\alpha \underset{,,}{\downarrow} “\beta$ Df

***113·03.** $\mu \times_0 \nu = \hat{\xi}\{(\exists \alpha, \beta) . \mu = \mathrm{N_0c}‘\alpha . \nu = \mathrm{N_0c}‘\beta . \xi \,\mathrm{sm}\, (\alpha \times \beta)\}$ Df

***113·04.** $\mathrm{Nc}‘\beta \times_0 \mu = \mathrm{N_0c}‘\beta \times_0 \mu$ Df

***113·05.** $\mu \times_0 \mathrm{Nc}‘\alpha = \mu \times_0 \mathrm{N_0c}‘\alpha$ Df

In relation to types, *113·03·04·05 call for similar remarks to those made in *110 for addition.

***113·1.** $\vdash . \beta \times \alpha = s‘\alpha \underset{,,}{\downarrow} “\beta$ [(*113·02)]

***113·101.** $\vdash : R \epsilon \beta \times \alpha . \equiv . (\exists x, y) . x \epsilon \alpha . y \epsilon \beta . R = x \downarrow y$ [*40·7 . *113·1]

***113·102.** $\vdash : y \epsilon \beta . \supset . \alpha \underset{,,}{\downarrow} y = (\alpha \uparrow \beta)_\Delta ‘\iota‘y$

Dem. $\vdash . *35·103 . \supset \vdash :. \mathrm{Hp} . \supset : x (\alpha \uparrow \beta) y . \equiv . x \epsilon \alpha :$

[*85·51] $\supset : (\alpha \uparrow \beta)_\Delta ‘\iota‘y = \downarrow y “\alpha$

[(*38·03)] $= \alpha \underset{,,}{\downarrow} y : \supset \vdash . \mathrm{Prop}$

***113·103.** $\vdash . \alpha \underset{,,}{\downarrow} “\beta = (\alpha \uparrow \beta)_\Delta “\iota“\beta = (\alpha \uparrow \beta) \bar{\downarrow} “\beta$ [*113·102 . *85·52]

***113·104.** $\vdash . E ! \alpha \underset{,,}{\downarrow} ‘y$ [*38·12]

***113·105.** $\vdash : \exists ! \alpha . \supset . \alpha \underset{,,}{\downarrow} \epsilon 1 \rightarrow 1$

Dem.

$\vdash . *113·104 . *71·166 . \supset \vdash . \alpha \underset{,,}{\downarrow} \epsilon 1 \rightarrow \mathrm{Cls}$ (1)

$\vdash . *38·131 . \supset \vdash : \alpha \underset{,,}{\downarrow} ‘y = \alpha \underset{,,}{\downarrow} ‘z . x \epsilon \alpha . \supset . x \downarrow y \epsilon \alpha \underset{,,}{\downarrow} ‘z .$

[*38·131] $\supset . (\exists x') . x' \epsilon \alpha . x \downarrow y = x' \downarrow z .$

[*55·202] $\supset . y = z$ (2)

$\vdash . (2) . *10·11·23·35 . \supset \vdash : \exists ! \alpha . \alpha \underset{,,}{\downarrow} ‘y = \alpha \underset{,,}{\downarrow} ‘z . \supset . y = z$ (3)

$\vdash . (1) . (3) . *71·54 . \supset \vdash . \mathrm{Prop}$

***113·106.** $\vdash : x \epsilon \alpha . y \epsilon \beta . \supset . x \downarrow y \epsilon \beta \times \alpha$ [*113·101]

***113·107.** $\vdash : \exists ! \alpha . \exists ! \beta . \supset . \exists ! \beta \times \alpha$ [*113·106]

***113·11.** $\vdash : \exists ! \alpha . \supset . \alpha \underset{,,}{\downarrow} “\beta \epsilon \mathrm{Nc}‘\beta : (y) . \alpha \underset{,,}{\downarrow} y \epsilon \mathrm{Nc}‘\alpha$

Dem. $\vdash . *113·105·104 . *73·26 . \supset \vdash : \exists ! \alpha . \supset . \alpha \underset{,,}{\downarrow} “\beta \,\mathrm{sm}\, \beta$ (1)

$\vdash . *38·2 . *73·611 . \supset \vdash . \alpha \underset{,,}{\downarrow} y \,\mathrm{sm}\, \alpha$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***113·111.** $\vdash . \alpha \underset{,,}{\downarrow} “\beta \epsilon \mathrm{Cls}^2 \,\mathrm{excl}$ [*113·103 . *85·55]

***113·112.** $\vdash : \alpha = \Lambda . \exists ! \beta . \supset . \alpha \underset{,,}{\downarrow} “\beta = \iota‘\Lambda$

Dem. $\vdash . *38·3 . \supset \vdash : \mathrm{Hp} . \supset . \alpha \underset{,,}{\downarrow} “\beta = \hat{\mu} \{(\exists y) . y \epsilon \beta . \mu = \downarrow y “\Lambda\}$

[*37·29] $= \hat{\mu} \{(\exists y) . y \epsilon \beta . \mu = \Lambda\}$

[Hp] $= \iota‘\Lambda$

***113·113.** $\vdash : \beta = \Lambda . \supset . \alpha \underset{\text{,,}}{\downarrow} ``\beta = \Lambda$ [*37·29]

***113·114.** $\vdash :. \alpha = \Lambda . \vee . \beta = \Lambda : \equiv . \beta \times \alpha = \Lambda$ [*113·1·112·113·107 . *53·24]

***113·115.** $\vdash . \dot{s}`(\beta \times \alpha) = \alpha \uparrow \beta$

Dem.

$\vdash . *113\cdot101 . *41\cdot11 . \supset$

$\vdash : u \{\dot{s}`(\beta \times \alpha)\} v . \equiv . (\exists R, x, y) . x \epsilon \alpha . y \epsilon \beta . R = x \downarrow y . vRv .$

[*13·195.*55·13] $\equiv . (\exists x, y) . x \epsilon \alpha . y \epsilon \beta . u = x . v = y .$

[*13·22] $\equiv . u \epsilon \alpha . v \epsilon \beta .$

[*35·103] $\equiv . u (\alpha \uparrow \beta) v : \supset \vdash .$ Prop

***113·116.** $\vdash : \exists ! \beta . \supset . s`\mathrm{D}``(\beta \times \alpha) = \alpha : \exists ! \alpha . \supset . s`\mathrm{Q}``(\beta \times \alpha) = \beta$
[*113·115 . *41·43·44 . *35·85·86]

***113·117.** $\vdash :. \alpha = \Lambda . \vee . \beta = \Lambda : \supset . s`\mathrm{D}``(\beta \times \alpha) = \Lambda . s`\mathrm{Q}``(\beta \times \alpha) = \Lambda$
[*113·115 . *41·43·44 . *35·88]

***113·118.** $\vdash . s`\mathrm{D}``(\beta \times \alpha) \subset \alpha . s`\mathrm{Q}``(\beta \times \alpha) \subset \beta$ [*113·116·117]

***113·12.** $\vdash : \exists ! \alpha . \supset . \alpha \underset{\text{,,}}{\downarrow} ``\beta \epsilon \mathrm{Nc}`\beta \cap \mathrm{Cl\ excl}`\mathrm{Nc}`\alpha$ [*113·11·111]

***113·121.** $\vdash . \Sigma`\alpha \underset{\text{,,}}{\downarrow} ``\beta \mathrm{sm}\, \beta \times \alpha$ [*112·15 . *113·111·1]

***113·122.** $\vdash : R \upharpoonright \gamma, S \upharpoonright \delta \epsilon \mathrm{Cls} \to 1 . \gamma \subset \mathrm{Q}`R . \delta \subset \mathrm{Q}`S . \supset . (R \| \breve{S}) \upharpoonright (\delta \times \gamma) \epsilon 1 \to 1$
[*74·773 . *113·118]

***113·123.** $\vdash : R \upharpoonright \gamma, S \upharpoonright \delta \epsilon 1 \to \mathrm{Cls} . \gamma \subset \mathrm{Q}`R . \delta \subset \mathrm{Q}`S . z \epsilon \gamma . w \epsilon \delta . \supset .$
$(R \| \breve{S})`(z \downarrow w) = (R`z) \downarrow (S`w)$ [*55·61]

***113·124.** $\vdash : R \upharpoonright \gamma, S \upharpoonright \delta \epsilon 1 \to \mathrm{Cls} . \gamma \subset \mathrm{Q}`R . \delta \subset \mathrm{Q}`S . w \epsilon \delta . \supset .$
$(R \| \breve{S})``\gamma \underset{\text{,,}}{\downarrow} w = (R``\gamma) \underset{\text{,,}}{\downarrow} (S`w)$

Dem.

$\vdash . *113\cdot123 . *38\cdot131 . \supset \vdash : \mathrm{Hp} . \supset . (R \| \breve{S})`` \downarrow w``\gamma = \downarrow (S`w)``R``\gamma .$

[*38·2] $\supset . (R \| \breve{S})``\gamma \underset{\text{,,}}{\downarrow} w = (R``\gamma) \underset{\text{,,}}{\downarrow} (S`w) : \supset \vdash .$ Prop

***113·125.** $\vdash : R \upharpoonright \gamma, S \upharpoonright \delta \epsilon 1 \to \mathrm{Cls} . \gamma \subset \mathrm{Q}`R . \delta \subset \mathrm{Q}`S . \supset .$
$(R \| \breve{S})_\epsilon``\gamma \underset{\text{,,}}{\downarrow} ``\delta = (R``\gamma) \underset{\text{,,}}{\downarrow} ``(S``\delta)$ [*113·124]

***113·126.** $\vdash : \mathrm{Hp} *113\cdot125 . \supset . (R \| \breve{S})``(\delta \times \gamma) = (S``\delta) \times (R``\gamma)$

Dem.

$\vdash . *113\cdot1 . *40\cdot38 . \supset \vdash . (R \| \breve{S})``(\delta \times \gamma) = s`(R \| \breve{S})```\gamma \underset{\text{,,}}{\downarrow} ``\delta$ (1)

$\vdash . (1) . *113\cdot125 . \supset \vdash : \mathrm{Hp} . \supset . (R \| \breve{S})``(\delta \times \gamma) = s`(R``\gamma) \underset{\text{,,}}{\downarrow} ``(S``\delta)$

[*113·1] $= (S``\delta) \times (R``\gamma) : \supset \vdash .$ Prop

***113·127.** $\vdash : R \upharpoonleft \gamma \in \alpha \,\overline{\mathrm{sm}}\, \gamma \,.\, S \upharpoonleft \delta \in \beta \,\overline{\mathrm{sm}}\, \delta \,.\, \supset .$

$$(R \parallel \breve{S}) \upharpoonleft (\delta \times \gamma) \in (\alpha \underset{\text{''}}{\downarrow} \text{“}\beta) \,\overline{\mathrm{sm}}\,\overline{\mathrm{sm}}\, (\gamma \underset{\text{''}}{\downarrow} \text{“}\delta)$$

[*113·122·125 . *43·302 . *73·142 . *111·14]

***113·128.** $\vdash : \mathrm{Hp}\, {*}113{\cdot}127 \,.\, \supset . (R \parallel \breve{S}) \upharpoonleft (\delta \times \gamma) \in (\beta \times \alpha) \,\overline{\mathrm{sm}}\, (\delta \times \gamma) \,.$

$$(R \parallel \breve{S})_\epsilon \upharpoonleft (\gamma \underset{\text{''}}{\downarrow} \text{“}\delta) \in (\alpha \underset{\text{''}}{\downarrow} \text{“}\beta) \,\overline{\mathrm{sm}}\, (\gamma \underset{\text{''}}{\downarrow} \text{“}\delta) \quad [{*}113{\cdot}127 \,.\, {*}111{\cdot}15]$$

***113·13.** $\vdash : \alpha \,\mathrm{sm}\, \gamma \,.\, \beta \,\mathrm{sm}\, \delta \,.\, \supset . \alpha \underset{\text{''}}{\downarrow} \text{“}\beta \,\mathrm{sm}\,\mathrm{sm}\, \gamma \underset{\text{''}}{\downarrow} \text{“}\delta \,.\, (\beta \times \alpha) \,\mathrm{sm}\, (\delta \times \gamma)$

[*113·127 . *111·4·44 . *113·1]

***113·14.** $\vdash . \alpha \times \beta = \mathrm{Cnv}\text{“}(\beta \times \alpha)$ [*113·101 . *55·14]

***113·141.** $\vdash . \mathrm{Nc}\text{‘}(\alpha \times \beta) = \mathrm{Nc}\text{‘}(\beta \times \alpha)$ [*113·14 . *73·4]

***113·142.** $\vdash : \exists ! \beta \,.\, \supset . \mathrm{D}\text{“}(\beta \times \alpha) = \iota\text{“}\alpha : \exists ! \alpha \,.\, \supset . \text{Ɑ}\text{“}(\beta \times \alpha) = \iota\text{“}\beta$

Dem.

$\vdash . {*}55{\cdot}261 . {*}2{\cdot}02 . \supset \vdash : y \in \beta \,.\, \supset . \mathrm{D}\text{“}\alpha \underset{\text{''}}{\downarrow} y = \iota\text{“}\alpha$

$[{*}37{\cdot}63] \qquad \supset \vdash : \gamma \in \mathrm{D}\text{“‘}\alpha \underset{\text{''}}{\downarrow} \text{“}\beta \,.\, \supset . \gamma = \iota\text{“}\alpha \qquad (1)$

$\vdash . {*}37{\cdot}45 . \qquad \supset \vdash : \exists ! \beta \,.\, \supset . \exists ! \mathrm{D}\text{“‘}\alpha \underset{\text{''}}{\downarrow} \text{“}\beta \qquad (2)$

$\vdash . (1) . (2) . {*}51{\cdot}141 . \supset \vdash : \exists ! \beta \,.\, \supset . \mathrm{D}\text{“‘}\alpha \underset{\text{''}}{\downarrow} \text{“}\beta = \iota\text{‘}\iota\text{“}\alpha \,.$

$[{*}40{\cdot}38 . {*}53{\cdot}02] \qquad \supset . \mathrm{D}\text{“}s\text{‘}\alpha \underset{\text{''}}{\downarrow} \text{“}\beta = \iota\text{“}\alpha \qquad (3)$

$\vdash . {*}55{\cdot}251 . \qquad \supset \vdash : \exists ! \alpha \,.\, \supset . \text{Ɑ}\text{“}\alpha \underset{\text{''}}{\downarrow} y = \iota\text{‘}\iota\text{‘}y \,.$

$[{*}37{\cdot}355] \qquad \supset . \text{Ɑ}\text{“‘}\alpha \underset{\text{''}}{\downarrow} \text{“}\beta = \iota\text{“}\iota\text{“}\beta \,.$

$[{*}40{\cdot}38 . {*}53{\cdot}22] \qquad \supset . \text{Ɑ}\text{“}s\text{‘}\alpha \underset{\text{''}}{\downarrow} \text{“}\beta = \iota\text{“}\beta \qquad (4)$

$\vdash . (3) . (4) . {*}113{\cdot}1 . \supset \vdash . \mathrm{Prop}$

***113·143.** $\vdash : \alpha \neq \beta \,.\, P = x \downarrow y \,.\, R = x \downarrow \alpha \cup y \downarrow \beta \,.\, \supset .$

$$P = (R\text{‘}\alpha) \downarrow (R\text{‘}\beta) \,.\, R = \mathrm{D}\text{‘}P \uparrow \iota\text{‘}\alpha \cup \text{Ɑ}\text{‘}P \uparrow \iota\text{‘}\beta$$

Dem.

$\vdash . {*}55{\cdot}62 . \supset \vdash : \mathrm{Hp} \,.\, \supset . R\text{‘}\alpha = x \,.\, R\text{‘}\beta = y \,.$

$[{*}30{\cdot}19 . {*}13{\cdot}15] \qquad \supset . P = (R\text{‘}\alpha) \downarrow (R\text{‘}\beta) \qquad (1)$

$\vdash . {*}55{\cdot}15 . \supset \vdash : \mathrm{Hp} \,.\, \supset . \mathrm{D}\text{‘}P = \iota\text{‘}x \,.\, \text{Ɑ}\text{‘}P = \iota\text{‘}y \,.$

$[{*}55{\cdot}1] \qquad \supset . R = \mathrm{D}\text{‘}P \uparrow \iota\text{‘}\alpha \cup \text{Ɑ}\text{‘}P \uparrow \iota\text{‘}\beta \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***113·144.** $\vdash : \alpha \neq \beta \,.\, T = \hat{P}\hat{R}\,\{(\exists x, y) . x \in \alpha . y \in \beta . P = x \downarrow y . R = x \downarrow \alpha \cup y \downarrow \beta\} \,.$

$$\supset . T \in 1 \to 1 \,.\, \mathrm{D}\text{‘}T = \beta \times \alpha \,.\, \text{Ɑ}\text{‘}T = \epsilon_\Delta\text{‘}(\iota\text{‘}\alpha \cup \iota\text{‘}\beta)$$

Dem.

$\vdash . {*}21{\cdot}33 . \supset \vdash :. \mathrm{Hp} \,.\, \supset :$

$PTR \,.\, QTR \,.\, \supset . (\exists x, y, z, w) . x, z \in \alpha . y, w \in \beta . P = x \downarrow y . Q = z \downarrow w \,.$

$$R = x \downarrow \alpha \cup y \downarrow \beta = z \downarrow \alpha \cup w \downarrow \beta \,.$$

[$*113\cdot143$] $\supset . P = (R`\alpha) \downarrow (R`\beta) . Q = (R`\alpha) \downarrow (R`\beta)$.

[$*13\cdot172$] $\supset . P = Q$ (1)

$\vdash . *21\cdot33 . \supset \vdash :. \text{Hp} . \supset : PTQ . PTR . \supset .$

$(\exists x, y, z, w) . x, z \epsilon \alpha . y, w \epsilon \beta . P = x \downarrow y = w \downarrow z . Q = x \downarrow \alpha \cup y \downarrow \beta . R = z \downarrow \alpha \cup w \downarrow \beta$.

[$*113\cdot143$] $\supset . Q = \text{D}`P \uparrow \iota`\alpha \cup \text{Cl}`P \uparrow \iota`\beta . R = \text{D}`P \uparrow \iota`\alpha \cup \text{Cl}`P \uparrow \iota`\beta$.

[$*13\cdot172$] $\supset . Q = R$ (2)

$\vdash . *33\cdot13 . \supset \vdash : \text{Hp} . \supset .$

$$\text{D}`T = \hat{P} \{(\exists R, x, y) . x \epsilon \alpha . y \epsilon \beta . P = x \downarrow y . R = x \downarrow \alpha \cup y \downarrow \beta\}$$

[$*11\cdot55 . *13\cdot19$] $= \hat{P} \{(\exists x, y) . x \epsilon \alpha . y \epsilon \beta . P = x \downarrow y\}$

[$*113\cdot101$] $= \beta \times \alpha$ (3)

$\vdash . *33\cdot131 . \supset \vdash : \text{Hp} . \supset .$

$$\text{Cl}`T = \hat{R} \{(\exists P, x, y) . x \epsilon \alpha . y \epsilon \beta . P = x \downarrow y . R = x \downarrow \alpha \cup y \downarrow \beta\}$$

[$*11\cdot55 . *13\cdot19$] $= \hat{R} \{(\exists x, y) . x \epsilon \alpha . y \epsilon \beta . R = x \downarrow \alpha \cup y \downarrow \beta\}$

[$*80\cdot9$] $= \epsilon_\Delta`(\iota`\alpha \cup \iota`\beta)$ (4)

$\vdash . (1) . (2) . (3) . (4) . \supset \vdash . \text{Prop}$

Note to $*113\cdot144$. In virtue of $*113\cdot143$ and $*55\cdot61$ we have

$\vdash :. \text{Hp} *113\cdot144 . \supset : PTR . \equiv . R \epsilon \epsilon_\Delta`(\iota`\alpha \cup \iota`\beta) . P = (R \parallel \breve{R})`(\alpha \downarrow \beta)$.

At a later stage (in $*150$) we shall put

$$R \dagger S = (R \parallel \breve{R})`S \quad \text{Df.}$$

Thus we shall have, anticipating this notation,

$$\vdash : \text{Hp} *113\cdot144 . \supset . T = \{\dagger(\alpha \downarrow \beta)\} \upharpoonright \epsilon_\Delta`(\iota`\alpha \cup \iota`\beta).$$

Hence we have

$$\vdash : \alpha \neq \beta . \supset . \{\dagger(\alpha \downarrow \beta)\} \upharpoonright \epsilon_\Delta`(\iota`\alpha \cup \iota`\beta) \epsilon (\beta \times \alpha) \overline{\text{sm}} \, \epsilon_\Delta`(\iota`\alpha \cup \iota`\beta).$$

$*113\cdot145$. $\vdash : \alpha \neq \beta . \supset . \beta \times \alpha \, \text{sm} \, \epsilon_\Delta`(\iota`\alpha \cup \iota`\beta)$ [$*113\cdot144$]

$*113\cdot146$. $\vdash : \alpha \neq \beta . \supset . \alpha \times \beta \, \text{sm} \, \epsilon_\Delta`(\iota`\alpha \cup \iota`\beta)$ [$*113\cdot141\cdot145$]

$*113\cdot147$. $\vdash : \text{Hp} *113\cdot144 . \beta \times \alpha = \mu . \supset .$

$$T = \hat{P}\hat{R} \{P \epsilon \mu . R = \text{D}`P \uparrow \iota`s`\text{D}``\mu \cup \text{Cl}`P \uparrow \iota`s`\text{Cl}``\mu\}$$

Dem.

$\vdash . *113\cdot114 . \text{Transp} . \supset \vdash : \text{Hp} . P \epsilon \mu . \supset . \exists ! \alpha . \exists ! \beta .$

[$*113\cdot142 . *53\cdot22$] $\supset . \alpha = s`\text{D}``\mu . \beta = s`\text{Cl}``\mu$ (1)

$\vdash . *113\cdot101\cdot143 . \supset \vdash :. \text{Hp} . P \epsilon \mu . \supset : PTR . \equiv . R = \text{D}`P \uparrow \iota`\alpha \cup \text{Cl}`P \uparrow \iota`\beta$ (2)

$\vdash . *113\cdot144 . \supset \vdash : \text{Hp} . PTR . \supset . P \epsilon \mu$ (3)

$\vdash . (1) . (2) . (3) . *113\cdot101 . \supset \vdash . \text{Prop}$

The advantage of this proposition is that it exhibits the correlator of $\beta \times \alpha$ and $\epsilon_\Delta`(\iota`\alpha \cup \iota`\beta)$ as a function of $\beta \times \alpha$.

$*113\cdot148$. $\vdash : \alpha \cap \beta = \Lambda . \supset . C \upharpoonright (\alpha \times \beta) \in 1 \to 1$

Dem.

$\vdash . *113\cdot101 . *55\cdot15 . \supset$

$\vdash :. \text{Hp} . \supset : R, S \in \alpha \times \beta . C\text{‘}R = C\text{‘}S . \equiv .$

$(\exists x, x', y, y') . x, x' \in \alpha . y, y' \in \beta . R = y \downarrow x , S = y' \downarrow x' . \iota\text{‘}x \cup \iota\text{‘}y = \iota\text{‘}x' \cup \iota\text{‘}y' .$

$[*54\cdot6] \quad \supset . (\exists x, x', y, y') . x, x' \in \alpha . y, y' \in \beta . R = y \downarrow x . S = y' \downarrow x' . x = x' . y = y' .$

$[*13\cdot22\cdot172] \quad \supset . R = S$ (1)

$\vdash . (1) . *71\cdot55 . \supset \vdash . \text{Prop}$

$*113\cdot15$. $\vdash . C\text{“}(\alpha \times \beta) = C\text{“}(\beta \times \alpha) = \hat{\xi}\{(\exists x, y) . x \in \alpha . y \in \beta . \xi = \iota\text{‘}x \cup \iota\text{‘}y\}$

Dem.

$\vdash . *113\cdot1 . *40\cdot38 . \supset \vdash . C\text{“}(\beta \times \alpha) = s\text{‘}C\text{““}\alpha \underset{,,}{\downarrow}\text{“}\beta$

$[*40\cdot4] \qquad = \hat{\xi}\{(\exists y) . y \in \beta . \xi \in C\text{“}\alpha \underset{,,}{\downarrow} y\}$

$[*55\cdot27 . *38\cdot2] \qquad = \hat{\xi}\{(\exists x, y) . x \in \alpha . y \in \beta . \xi = \iota\text{‘}x \cup \iota\text{‘}y\}$ (1)

$\vdash . (1) \dfrac{\beta, \alpha}{\alpha, \beta} . \quad \supset \vdash . C\text{“}(\alpha \times \beta) = \hat{\xi}\{(\exists x, y) . x \in \alpha . y \in \beta . \xi = \iota\text{‘}x \cup \iota\text{‘}y\}$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

$*113\cdot151$. $\vdash : \alpha \neq \beta . \supset . C\text{“}(\alpha \times \beta) = \text{D}\text{“}\epsilon_\Delta\text{‘}(\iota\text{‘}\alpha \cup \iota\text{‘}\beta)$ $[*113\cdot15 . *80\cdot92]$

$*113\cdot152$. $\vdash : \alpha \cap \beta = \Lambda . \supset . C\text{“}(\alpha \times \beta) \operatorname{sm} (\alpha \times \beta) . \text{D}\text{“}\epsilon_\Delta\text{‘}(\iota\text{‘}\alpha \cup \iota\text{‘}\beta) \operatorname{sm} (\alpha \times \beta)$

Dem.

$\vdash . *84\cdot41\cdot62 . \supset \vdash : \text{Hp} . \alpha \neq \beta . \supset . \text{D}\text{“}\epsilon_\Delta\text{‘}(\iota\text{‘}\alpha \cup \iota\text{‘}\beta) \operatorname{sm} \epsilon_\Delta\text{‘}(\iota\text{‘}\alpha \cup \iota\text{‘}\beta)$ (1)

$\vdash . (1) . *113\cdot146\cdot151 . \supset$

$\vdash : \text{Hp} . \alpha \neq \beta . \supset . C\text{“}(\alpha \times \beta) \operatorname{sm} (\alpha \times \beta) . \text{D}\text{“}\epsilon_\Delta\text{‘}(\iota\text{‘}\alpha \cup \iota\text{‘}\beta) \operatorname{sm} (\alpha \times \beta)$ (2)

$\vdash . *24\cdot38 . \supset \vdash : \text{Hp} . \alpha = \beta . \supset . \alpha = \Lambda . \beta = \Lambda .$

$[*113\cdot114 . *83\cdot11 . *37\cdot29] \quad \supset . \alpha \times \beta = \Lambda . \text{D}\text{“}\epsilon_\Delta\text{‘}(\iota\text{‘}\alpha \cup \iota\text{‘}\beta) = \Lambda . C\text{“}(\alpha \times \beta) = \Lambda .$

$[*73\cdot47] \quad \supset . C\text{“}(\alpha \times \beta) \operatorname{sm} (\alpha \times \beta) . \text{D}\text{“}\epsilon_\Delta\text{‘}(\iota\text{‘}\alpha \cup \iota\text{‘}\beta) \operatorname{sm} (\alpha \times \beta)$ (3)

$\vdash . (2) . (3) . \supset \vdash . \text{Prop}$

The following proposition is only significant when λ and μ are classes of relations. It is used in relation-arithmetic ($*172\cdot34$).

$*113\cdot153$. $\vdash : s\text{‘}\lambda \mathbin{\dot\cap} s\text{‘}\mu = \dot\Lambda . \supset . \dot{s} \mid C \upharpoonright (\lambda \times \mu) \in (s\text{‘}\lambda \underset{,,}{\mathbin{\dot\cup}}\text{“}\mu) \overline{\operatorname{sm}} (\lambda \times \mu) . s\text{‘}\lambda \underset{,,}{\mathbin{\dot\cup}}\text{“}\mu \operatorname{sm} \lambda \times \mu$

Dem.

$\vdash . *55\cdot15 . *53\cdot13 . \supset \vdash : R = T \downarrow S . \supset . \dot{s}\text{‘}C\text{‘}R = S \mathbin{\dot\cup} T$ (1)

$\vdash . (1) . *113\cdot101 . \supset$

$\vdash : R, R' \in \lambda \times \mu . \dot{s}\text{‘}C\text{‘}R = \dot{s}\text{‘}C\text{‘}R' . \supset .$

$(\exists S, S', T, T') . S, S' \in \lambda . T, T' \in \mu . R = T \downarrow S . R' = T' \downarrow S' . S \mathbin{\dot\cup} T = S' \mathbin{\dot\cup} T'$ (2)

$\vdash . (2) . *25\cdot48 . *41\cdot13 . \supset$

$\vdash :. \text{Hp} . \supset : R, R' \in \lambda \times \mu . \dot{s}\text{‘}C\text{‘}R = \dot{s}\text{‘}C\text{‘}R' . \supset . R = R'$ (3)

$\vdash . (1) . *113\cdot101 . \supset \vdash . \dot{s}\text{“}C\text{“}(\lambda \times \mu) = \hat{M}\{(\exists S, T) . S \in \lambda . T \in \mu . M = S \mathbin{\dot\cup} T\}$

$[*40\cdot7] \qquad = s\text{‘}\lambda \underset{,,}{\mathbin{\dot\cup}}\text{“}\mu$ (4)

$\vdash . (3) . (4) . *73\cdot25 . \supset \vdash . \text{Prop}$

***113·16.** $\vdash : t`\alpha = t`\beta . \supset . \mathrm{Nc}`(\alpha \times \beta) =$

$$\hat{\xi}\{(\exists \gamma, \delta) . \gamma \in \mathrm{N}^1\mathrm{c}`\alpha . \delta \in \mathrm{N}^1\mathrm{c}`\beta . \gamma \cap \delta = \Lambda . \xi \,\mathrm{sm}\, \mathrm{D}``\epsilon_\Delta`(\iota`\gamma \cup \iota`\delta)\}$$

Dem.

$\vdash . *113·152 . \supset \vdash :. \gamma \in \mathrm{N}^1\mathrm{c}`\alpha . \delta \in \mathrm{N}^1\mathrm{c}`\beta . \gamma \cap \delta = \Lambda . \supset :$

$\xi \,\mathrm{sm}\, \mathrm{D}``\epsilon_\Delta`(\iota`\gamma \cup \iota`\delta) . \equiv . \xi \,\mathrm{sm}\, (\gamma \times \delta) .$

[*113·13.*104·101] $\equiv . \xi \,\mathrm{sm}\, (\alpha \times \beta) .$

[*100·31] $\equiv . \xi \in \mathrm{Nc}`(\alpha \times \beta)$ (1)

$\vdash . (1) . *5·32 . *11·11·341 . \supset$

$\vdash :. (\exists \gamma, \delta) . \gamma \in \mathrm{N}^1\mathrm{c}`\alpha . \delta \in \mathrm{N}^1\mathrm{c}`\beta . \gamma \cap \delta = \Lambda . \xi \,\mathrm{sm}\, \mathrm{D}``\epsilon_\Delta`(\iota`\gamma \cup \iota`\delta) . \equiv :$

$(\exists \gamma, \delta) . \gamma \in \mathrm{N}^1\mathrm{c}`\alpha . \delta \in \mathrm{N}^1\mathrm{c}`\beta . \gamma \cap \delta = \Lambda . \xi \in \mathrm{Nc}`(\alpha \times \beta) :$

[*11·45] $\equiv : (\exists \gamma, \delta) . \gamma \in \mathrm{N}^1\mathrm{c}`\alpha . \delta \in \mathrm{N}^1\mathrm{c}`\beta . \gamma \cap \delta = \Lambda : \xi \in \mathrm{Nc}`(\alpha \times \beta)$ (2)

$\vdash . (2) . *104·43 . \supset \vdash .$ Prop

***113·17.** $\vdash . \beta \times \alpha \in t`t`(\alpha \uparrow \beta)$

Dem.

$\vdash . *113·115 . *41·13 . \supset \vdash : R \in \beta \times \alpha . \supset . R \Subset \alpha \uparrow \beta .$

[*64·201] $\supset . R \in t`(\alpha \uparrow \beta)$ (1)

$\vdash . (1) . *63·5 . \supset \vdash .$ Prop

***113·171.** $\vdash : \alpha \cap \beta = \Lambda . \supset . \exists ! \mathrm{Nc}\,(t`\alpha)`(\alpha \times \beta)$

Dem.

$\vdash . *113·152·15 . \supset \vdash : \mathrm{Hp} . \supset . \hat{\xi}\{(\exists x, y) . x \in \alpha . y \in \beta . \xi = \iota`x \cup \iota`y\} \in \mathrm{Nc}`(\alpha \times \beta)$ (1)

$\vdash . *51·16 . \quad \supset \vdash : x \in \alpha . y \in \beta . \xi = \iota`x \cup \iota`y . \supset . x \in \alpha . x \in \xi .$

[*63·13] $\supset . \xi \in t`\alpha$ (2)

$\vdash . (2) . *11·11·35 . \supset$

$\vdash . \hat{\xi}\{(\exists x, y) . x \in \alpha . y \in \beta . \xi = \iota`x \cup \iota`y\} \subset t`\alpha .$

[*63·5] $\supset \vdash . \hat{\xi}\{(\exists x, y) . x \in \alpha . y \in \beta . \xi = \iota`x \cup \iota`y\} \in t`t`\alpha$ (3)

$\vdash . (1) . (3) . \supset \vdash : \mathrm{Hp} . \supset . \exists ! \mathrm{Nc}`(\alpha \times \beta) \cap t`t`\alpha$ (4)

$\vdash . (4) . *102·6 . \supset \vdash .$ Prop

Note that the hypothesis $\alpha \cap \beta = \Lambda$ is only significant when α and β are of the same type.

***113·172.** $\vdash : \alpha \in t`\beta . \supset . \exists ! \mathrm{Nc}\,(t^{2}`\alpha)`(\alpha \times \beta)$

Dem.

$\vdash . *113·16 . \quad \supset \vdash :. \mathrm{Hp} . \supset : \gamma \in \mathrm{N}^1\mathrm{c}`\alpha . \delta \in \mathrm{N}^1\mathrm{c}`\beta . \gamma \cap \delta = \Lambda . \supset .$

$\mathrm{D}``\epsilon_\Delta`(\iota`\gamma \cup \iota`\delta) \in \mathrm{Nc}`(\alpha \times \beta)$ (1)

$\vdash . (1) . *104·43 . \supset \vdash : \mathrm{Hp} . \supset .$

$(\exists \gamma, \delta) . \gamma \in \mathrm{N}^1\mathrm{c}`\alpha . \delta \in \mathrm{N}^1\mathrm{c}`\beta . \mathrm{D}``\epsilon_\Delta`(\iota`\gamma \cup \iota`\delta) \in \mathrm{Nc}`(\alpha \times \beta)$ (2)

$\vdash . *104·1 . \supset \vdash : \gamma \in \mathrm{N}^1\mathrm{c}`\alpha . \supset . \gamma \in t^{2}`\alpha .$

[*63·61·621] $\supset . \iota`\gamma \cup \iota`\delta \in t`t^{2}`\alpha .$

[*83·81] $\supset . \mathrm{D}``\epsilon_\Delta`(\iota`\gamma \cup \iota`\delta) \in t`t^{2}`\alpha$ (3)

$\vdash . (2) . (3) . \supset \vdash : \mathrm{Hp} . \supset . \exists ! \mathrm{Nc}`(\alpha \times \beta) \cap t`t^{2}`\alpha$ (4)

$\vdash . (4) . *102·6 . \supset \vdash .$ Prop

***113·18.** $\vdash : \exists ! \alpha . \exists ! \beta . \alpha \times \beta = \alpha' \times \beta' . \supset . \alpha = \alpha' . \beta = \beta'$

Dem.

$\vdash . *113·114 . \supset \vdash : \mathrm{Hp} . \supset . \exists ! \alpha' \times \beta' .$

$[*113·114] \quad \supset . \exists ! \alpha' . \exists ! \beta' \quad (1)$

$\vdash . *30·37 . \quad \supset \vdash : \mathrm{Hp} . \supset . s`\text{Ɑ}``(\alpha \times \beta) = s`\text{Ɑ}``(\alpha' \times \beta') .$

$[*113·142.(1)] \quad \supset . s`\iota``\alpha = s`\iota``\alpha' .$

$[*53·22] \quad \supset . \alpha = \alpha' \quad (2)$

Similarly $\quad \vdash : \mathrm{Hp} . \supset . \beta = \beta' \quad (3)$

$\vdash . (2) . (3) . \supset \vdash . \mathrm{Prop}$

***113·181.** $\vdash : \exists ! \alpha . \exists ! \alpha' . \alpha \times \beta = \alpha' \times \beta' . \supset . \beta = \beta'$

Dem.

$\vdash . *13·172 . \supset \vdash : \beta = \Lambda . \beta' = \Lambda . \supset . \beta = \beta' \quad (1)$

$\vdash . *113·18 . \supset \vdash : \mathrm{Hp} . \sim(\beta = \Lambda . \beta' = \Lambda) . \supset . \beta = \beta' \quad (2)$

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***113·182.** $\vdash : \exists ! \beta . \exists ! \beta' . \alpha \times \beta = \alpha' \times \beta' . \supset . \alpha = \alpha'$

[Proof as in *113·181]

***113·183.** $\vdash : \exists ! \alpha . \exists ! \beta . \supset . F``(\alpha \times \beta) = s`C``(\alpha \times \beta) = \alpha \cup \beta$

Dem.

$\vdash . *40·57 . \quad \supset \vdash . s`C``(\alpha \times \beta) = s`\mathrm{D}``(\alpha \times \beta) \cup s`\text{Ɑ}``(\alpha \times \beta) \quad (1)$

$\vdash . *40·56 . \quad \supset \vdash . F``(\alpha \times \beta) = s`C``(\alpha \times \beta) \quad (2)$

$\vdash . *113·142 . \supset \vdash : \mathrm{Hp} . \supset . s`\text{Ɑ}``(\alpha \times \beta) = s`\iota``\alpha$

$[*53·22] \quad = \alpha \quad (3)$

$\vdash . *113·142 . \supset \vdash : \mathrm{Hp} . \supset . s`\mathrm{D}``(\alpha \times \beta) = s`\iota``\beta$

$[*53·22] \quad = \beta \quad (4)$

$\vdash . (3) . (4) . \quad \supset \vdash : \mathrm{Hp} . \supset . s`\mathrm{D}``(\alpha \times \beta) \cup s`\text{Ɑ}``(\alpha \times \beta) = \alpha \cup \beta \quad (5)$

$\vdash . (1) . (2) . (5) . \supset \vdash . \mathrm{Prop}$

***113·19.** $\vdash : \exists ! (\alpha \times \beta) \cap (\gamma \times \delta) . \equiv . \exists ! \alpha \cap \gamma . \exists ! \beta \cap \delta$

Dem.

$\vdash . *113·101 . \supset \vdash :. \exists ! (\alpha \times \beta) \cap (\gamma \times \delta) . \equiv :$

$\quad (\exists x, y, z, w) . x \in \alpha . y \in \beta . z \in \gamma . w \in \delta . x \downarrow y = w \downarrow z :$

$[*55·202] \equiv : (\exists x, y, z, w) . x \in \alpha . y \in \beta . z \in \gamma . w \in \delta . x = z . y = w :$

$[*13·22] \quad \equiv : (\exists x, y) . x \in \alpha \cap \gamma . y \in \beta \cap \delta :. \supset \vdash . \mathrm{Prop}$

***113·191.** $\vdash :. \exists ! \alpha . \supset : \exists ! \alpha \underset{,,}{\downarrow} ``\beta \cap \alpha \underset{,,}{\downarrow} ``\gamma . \equiv . \exists ! \beta \cap \gamma$

Dem.

$\vdash . *37·6 . \supset \vdash : \exists ! \alpha \underset{,,}{\downarrow} ``\beta \cap \alpha \underset{,,}{\downarrow} ``\gamma . \equiv . (\exists y, z) . y \in \beta . z \in \gamma . \alpha \underset{,,}{\downarrow} y = \alpha \underset{,,}{\downarrow} z \quad (1)$

$\vdash . *113\cdot105 . *71\cdot57 . \supset \vdash :. \mathrm{Hp} . \supset : \alpha \underset{,,}{\downarrow} y = \alpha \underset{,,}{\downarrow} z . \equiv . y = z :$

[(1)] $\supset : \exists ! \alpha \underset{,,}{\downarrow} {}^{\text{``}}\beta \cap \alpha \underset{,,}{\downarrow} {}^{\text{``}}\gamma . \equiv . (\exists y, z) . y \epsilon \beta . z \epsilon \gamma . y = z .$

[*13·195] $\equiv . \exists ! \alpha \cap \beta :. \supset \vdash . \mathrm{Prop}$

***113·2.** $\vdash : \xi \epsilon \mu \times_0 \nu . \equiv . (\exists \alpha, \beta) . \mu = \mathrm{N_0c}`\alpha . \nu = \mathrm{N_0c}`\beta . \xi \,\mathrm{sm}\, (\alpha \times \beta)$ [(*113·03)]

***113·201.** $\vdash :. \xi \epsilon \mu \times_0 \nu . \equiv : \mu, \nu \epsilon \mathrm{NC} : (\exists \alpha, \beta) . \alpha \epsilon \mu . \beta \epsilon \nu . \xi \,\mathrm{sm}\, (\alpha \times \beta)$ [*113·2 . *103·27]

***113·202.** $\vdash :. \xi \epsilon \mu \times_0 \nu . \equiv : \exists ! \mu . \exists ! \nu : (\exists \gamma, \delta) . \mu = \mathrm{Nc}`\gamma . \nu = \mathrm{Nc}`\delta . \xi \,\mathrm{sm} (\gamma \times \delta)$

Dem.

$\vdash . *113\cdot201 . *100\cdot4 . \supset$

$\vdash :. \xi \epsilon \mu \times_0 \nu . \equiv : (\exists \alpha, \beta, \gamma, \delta) . \mu = \mathrm{Nc}`\gamma . \nu = \mathrm{Nc}`\delta . \alpha \epsilon \mu . \beta \epsilon \nu . \xi \,\mathrm{sm}\, (\alpha \times \beta) .$

[*100·31] $\equiv : (\exists \alpha, \beta, \gamma, \delta) . \mu = \mathrm{Nc}`\gamma . \nu = \mathrm{Nc}`\delta . \alpha \,\mathrm{sm}\, \gamma . \beta \,\mathrm{sm}\, \delta . \xi \,\mathrm{sm} (\alpha \times \beta) .$

[*113·13 . *73·37] $\equiv : (\exists \alpha, \beta, \gamma, \delta) . \mu = \mathrm{Nc}`\gamma . \nu = \mathrm{Nc}`\delta . \alpha \,\mathrm{sm}\, \gamma . \beta \,\mathrm{sm}\, \delta . \xi \,\mathrm{sm} (\gamma \times \delta) .$

[*100·31] $\equiv : (\exists \alpha, \beta, \gamma, \delta) . \mu = \mathrm{Nc}`\gamma . \nu = \mathrm{Nc}`\delta . \alpha \epsilon \mu . \beta \epsilon \nu . \xi \,\mathrm{sm}\, (\gamma \times \delta) .$

[*10·35] $\equiv : \exists ! \mu . \exists ! \nu : (\exists \gamma, \delta) . \mu = \mathrm{Nc}`\gamma . \nu = \mathrm{Nc}`\delta . \xi \,\mathrm{sm}\, (\gamma \times \delta) :.$

$\supset \vdash . \mathrm{Prop}$

***113·203.** $\vdash : \exists ! \mu \times_0 \nu . \supset . \mu, \nu \epsilon \mathrm{NC} - \iota`\Lambda . \mu, \nu \epsilon \mathrm{N_0C}$ [*113·201·202·2]

***113·204.** $\vdash :. \mu = \Lambda . \vee . \nu = \Lambda . \vee . \sim(\mu, \nu \epsilon \mathrm{NC}) : \supset . \mu \times_0 \nu = \Lambda$ [*113·203]

***113·205.** $\vdash : \sim(\mu, \nu \epsilon \mathrm{N_0C}) . \supset . \mu \times_0 \nu = \Lambda$ [*113·203]

***113·21.** $\vdash :. \mu, \nu \epsilon \mathrm{NC} . \supset : \xi \epsilon \mu \times_0 \nu . \equiv . (\exists \alpha, \beta) . \alpha \epsilon \mu . \beta \epsilon \nu . \xi \,\mathrm{sm}\, (\alpha \times \beta)$ [*113·201]

***113·22.** $\vdash : \xi \epsilon \mathrm{Nc}\, (\eta)`\gamma \times_0 \mathrm{Nc}\, (\zeta)`\delta . \equiv . \exists ! \mathrm{Nc}\, (\eta)`\gamma . \exists ! \mathrm{Nc}\, (\zeta)`\delta . \xi \,\mathrm{sm} (\gamma \times \delta)$

Dem.

$\vdash . *113\cdot21 . *100\cdot41 . \supset \vdash : \xi \epsilon \mathrm{Nc}\, (\eta)`\gamma \times_0 \mathrm{Nc}\, (\zeta)`\delta . \equiv .$

$(\exists \alpha, \beta) . \alpha \epsilon \mathrm{Nc}\, (\eta)`\gamma . \beta \epsilon \mathrm{Nc}\, (\zeta)`\delta . \xi \,\mathrm{sm}\, (\alpha \times \beta) .$

[*102·6] $\equiv . (\exists \alpha, \beta) . \alpha \epsilon \mathrm{Nc}(\eta)`\gamma . \beta \epsilon \mathrm{Nc}(\zeta)`\delta . \alpha \,\mathrm{sm}\, \gamma . \beta \,\mathrm{sm}\, \delta . \xi \,\mathrm{sm} (\alpha \times \beta) .$

[*113·13 . *73·37] $\equiv . (\exists \alpha, \beta) . \alpha \epsilon \mathrm{Nc}(\eta)`\gamma . \beta \epsilon \mathrm{Nc}(\zeta)`\delta . \alpha \,\mathrm{sm}\, \gamma . \beta \,\mathrm{sm}\, \delta . \xi \,\mathrm{sm} (\gamma \times \delta) .$

[*102·6] $\equiv . (\exists \alpha, \beta) . \alpha \epsilon \mathrm{Nc}\, (\eta)`\gamma . \beta \epsilon \mathrm{Nc}\, (\zeta)`\delta . \xi \,\mathrm{sm}\, (\gamma \times \delta) .$

[*10·35] $\equiv . \exists ! \mathrm{Nc}\, (\eta)`\gamma . \exists ! \mathrm{Nc}\, (\zeta)`\delta . \xi \,\mathrm{sm}\, (\gamma \times \delta) : \supset \vdash . \mathrm{Prop}$

***113·221.** $\vdash : \exists ! \mathrm{Nc}\, (\eta)`\gamma . \exists ! \mathrm{Nc}\, (\zeta)`\delta . \supset . \mathrm{Nc}\, (\eta)`\gamma \times_0 \mathrm{Nc}\, (\zeta)`\delta = \mathrm{Nc}`(\gamma \times \delta)$ [*113·22]

***113·222.** $\vdash . \mathrm{N_0c}`\gamma \times_0 \mathrm{N_0c}`\delta = \mathrm{Nc}`(\gamma \times \delta)$

Dem.

$\vdash . *103\cdot1\cdot13 . \supset \vdash . \mathrm{N_0c}`\gamma = \mathrm{Nc}\, (\gamma)`\gamma . \mathrm{N_0c}`\delta = \mathrm{Nc}\, (\delta)`\delta . \exists ! \mathrm{N_0c}`\gamma . \exists ! \mathrm{N_0c}`\delta .$

[*113·221] $\supset \vdash . \mathrm{N_0c}`\gamma \times_0 \mathrm{N_0c}`\delta = \mathrm{Nc}`(\gamma \times \delta) . \supset \vdash . \mathrm{Prop}$

***113·23.** $\vdash . \mu \times_0 \nu \in \text{NC}$

Dem.

$\vdash . *113·222 . *100·41 . \supset \vdash : \mu, \nu \in \text{N}_0\text{C} . \supset . \mu \times_0 \nu \in \text{NC}$ (1)

$\vdash . *113·205 . *102·74 . \supset \vdash : \sim(\mu, \nu \in \text{N}_0\text{C}) . \supset . \mu \times_0 \nu \in \text{NC}$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***113·24.** $\vdash . \text{Nc}`\gamma \times_0 \text{Nc}`\delta = \text{N}_0\text{c}`\gamma \times_0 \text{N}_0\text{c}`\delta$ [(*113·04·05)]

***113·25.** $\vdash . \text{Nc}`\gamma \times_0 \text{Nc}`\delta = \text{Nc}`(\gamma \times \delta)$ [*113·24·222]

This proposition constitutes part of the reason for our definitions. It is obvious that such definitions ought, if possible, to be chosen as will yield this proposition.

***113·251.** $\vdash . \gamma \times \delta \in \text{Nc}`\gamma \times_0 \text{Nc}`\delta$ [*113·25 . *100·3]

***113·26.** $\vdash : \mu, \nu \in \text{NC} . \exists ! \text{sm}_\eta``\mu . \exists ! \text{sm}_\zeta``\nu . \supset . \mu \times_0 \nu = \text{sm}_\eta``\mu \times_0 \text{sm}_\zeta``\nu$

Dem.

$\vdash . *37·29 . \text{Transp} . \supset \vdash : \text{Hp} . \supset . \exists ! \mu . \exists ! \nu .$

$[*102·64] \qquad \supset . (\exists \alpha, \beta, \gamma, \delta) . \mu = \text{Nc}(\alpha)`\gamma . \nu = \text{Nc}(\beta)`\delta$ (1)

$\vdash . *102·88 . \supset \vdash : \mu = \text{Nc}(\alpha)`\gamma . \nu = \text{Nc}(\beta)`\delta . \exists ! \text{sm}_\eta``\mu . \exists ! \text{sm}_\zeta``\nu . \supset .$

$\text{sm}_\eta``\mu = \text{Nc}(\eta)`\gamma . \text{sm}_\zeta``\nu = \text{Nc}(\zeta)`\delta . \exists ! \text{Nc}(\eta)`\gamma . \exists ! \text{Nc}(\zeta)`\delta .$

$[*113·221] \supset . \text{sm}_\eta``\mu \times_0 \text{sm}_\zeta``\nu = \text{Nc}`(\gamma \times \delta)$ (2)

$\vdash . *37·29 . \text{Transp} . *113·221 . \supset$

$\vdash : \mu = \text{Nc}(\alpha)`\gamma . \nu = \text{Nc}(\beta)`\delta . \exists ! \text{sm}_\eta``\mu . \exists ! \text{sm}_\zeta``\nu . \supset . \mu \times_0 \nu = \text{Nc}`(\gamma \times \delta)$ (3)

$\vdash . (2) . (3) . \supset \vdash : \mu = \text{Nc}(\alpha)`\gamma . \nu = \text{Nc}(\beta)`\delta . \exists ! \text{sm}_\eta``\mu . \exists ! \text{sm}_\zeta``\nu . \supset .$

$\mu \times_0 \nu = \text{sm}_\eta``\mu \times_0 \text{sm}_\zeta``\nu$ (4)

$\vdash . (4) . *11·11·35·45 . (1) . \supset \vdash . \text{Prop}$

***113·261.** $\vdash : \mu, \nu \in \text{NC} . \supset . \mu \times_0 \nu = \mu^{(1)} \times_0 \nu^{(1)} = \mu_{(00)} \times_0 \nu_{(00)} = \text{etc.}$

Here "etc." includes all *ascending* derivatives of μ. We shall only prove the result for $\mu^{(1)}$ and $\nu^{(1)}$, since it is proved in just the same way for the other cases. $\mu^{(1)} \times_0 \nu^{(2)}$ or $\mu^{(1)} \times_0 \nu_{(00)}$ or etc. will serve equally well; *i.e.* it is not necessary to take the same derivative of μ as of ν.

Dem.

$\vdash . *104·264·265 . \supset$

$\vdash : \text{Hp} . \exists ! \mu . \exists ! \nu . \supset . \mu^{(1)} = \text{sm}_\mu``\mu . \nu^{(1)} = \text{sm}_\nu``\mu . \exists ! \mu^{(1)} . \exists ! \nu^{(1)} .$

$[*113·26] \qquad \supset . \mu \times_0 \nu = \mu^{(1)} \times_0 \nu^{(1)}$ (1)

$\vdash . *104·264 . *113·204 . \supset$

$\vdash : \sim(\exists ! \mu . \exists ! \nu) . \supset . \mu \times_0 \nu = \Lambda . \mu^{(1)} \times_0 \nu^{(1)} = \Lambda$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

As appears in the above proof, if μ^i and ν^j are any derivatives of μ and ν, the above proposition holds provided we have

$$\exists ! \mu . \exists ! \nu . \supset . \exists ! \mu^i . \exists ! \nu^j.$$

Thus it holds for all *ascending* derivatives, but not always for descending derivatives.

***113·27.** $\vdash . \mu \times_0 \nu = \nu \times_0 \mu$

Dem.

$\vdash . *113·2·141 . \supset$

$\vdash : \xi \in \mu \times_0 \nu . \equiv . (\exists \alpha, \beta) . \mu = N_0c`\alpha . \nu = N_0c`\beta . \xi \,\mathrm{sm}\, (\beta \times \alpha) .$

$[*113·2] \quad \equiv . \xi \in \nu \times_0 \mu : \supset \vdash . \mathrm{Prop}$

Note that this proposition is not confined to the case in which μ and ν are cardinals. When either or both are not cardinals,

$$\mu \times_0 \nu = \Lambda = \nu \times_0 \mu.$$

***113·3.** $\vdash :. \mathrm{Mult\,ax} . \supset : \kappa \in \mathrm{Nc}`\beta \cap \mathrm{Cl}`\mathrm{Nc}`\alpha . \supset . \Sigma`\kappa \in \mathrm{Nc}`\alpha \times_0 \mathrm{Nc}`\beta$

Dem.

$\vdash . *112·24 . *113·12 . \supset$

$\vdash :. \mathrm{Mult\,ax} . \exists ! \alpha . \supset : \kappa \in \mathrm{Nc}`\beta \cap \mathrm{Cl}`\mathrm{Nc}`\alpha . \supset . \Sigma`\kappa \,\mathrm{sm}\, \Sigma`\alpha \downarrow_{,,} ``\beta .$

$[*113·121] \quad \supset . \Sigma`\kappa \,\mathrm{sm}\, \beta \times \alpha .$

$[*113·141·25] \quad \supset . \Sigma`\kappa \in \mathrm{Nc}`\alpha \times_0 \mathrm{Nc}`\beta \qquad (1)$

$\vdash . *113·114·25 . \supset \vdash : \alpha = \Lambda . \supset . \mathrm{Nc}`\alpha \times_0 \mathrm{Nc}`\beta = 0 \qquad (2)$

$\vdash . *101·14 . \supset \vdash :. \alpha = \Lambda . \kappa \in \mathrm{Nc}`\beta \cap \mathrm{Cl}`\mathrm{Nc}`\alpha . \supset : \kappa \in \mathrm{Cl}`\iota`\Lambda :$

$[*60·362] \quad \supset : \kappa = \iota`\Lambda . \vee . \kappa = \Lambda :$

$[*112·3·301] \quad \supset : \Sigma`\kappa = \Lambda \qquad (3)$

$\vdash . (2) . (3) . *54·102 . \supset \vdash : \alpha = \Lambda . \kappa \in \mathrm{Nc}`\beta \cap \mathrm{Cl}`\mathrm{Nc}`\alpha . \supset . \Sigma`\kappa \in \mathrm{Nc}`\alpha \times_0 \mathrm{Nc}`\beta \quad (4)$

$\vdash . (1) . (4) . \supset \vdash . \mathrm{Prop}$

***113·31.** $\vdash :. \mathrm{Mult\,ax} . \supset : \mu, \nu \in \mathrm{NC} . \kappa \in \nu \cap \mathrm{Cl}`\mu . \supset . \Sigma`\kappa \in \mu \times_0 \nu$ [*113·3]

***113·32.** $\vdash :. \mathrm{Mult\,ax} . \supset : \mu, \nu \in \mathrm{NC} . \kappa \in \nu \cap \mathrm{Cl\,excl}`\mu . \supset . s`\kappa \in \mu \times_0 \nu$
[*112·15 . *113·31·23]

***113·33.** $\vdash :. \mathrm{Mult\,ax} . \supset : \mu, \nu \in \mathrm{NC} . \kappa \in \nu \cap \mathrm{Cl}`\mu . \lambda \in \mu \cap \mathrm{Cl}`\nu . \supset .$
$\Sigma\mathrm{Nc}`\kappa = \Sigma\mathrm{Nc}`\lambda = \mu \times_0 \nu$ [*113·31·27·23]

***113·34.** $\vdash :. \mathrm{Mult\,ax} . \supset : \mu, \nu \in \mathrm{NC} . \kappa \in \nu \cap \mathrm{Cl\,excl}`\mu . \lambda \in \mu \cap \mathrm{Cl\,excl}`\nu . \supset .$
$\mathrm{Nc}`s`\kappa = \mathrm{Nc}`s`\lambda = \mu \times_0 \nu$ [*113·32·27]

The above propositions give the connection of addition and multiplication.

The following propositions are concerned with various forms of the distributive law.

***113·4.** $\vdash . (\beta \cup \gamma) \times \alpha = (\beta \times \alpha) \cup (\gamma \times \alpha)$

Dem.

$\vdash . *113·1 . \supset \vdash . (\beta \cup \gamma) \times \alpha = s`\alpha \downarrow_{,,} ``(\beta \cup \gamma)$

$[*40·31] \quad = s`\alpha \downarrow_{,,} ``\beta \cup s`\alpha \downarrow_{,,} ``\gamma$

$[*113·1] \quad = (\beta \times \alpha) \cup (\gamma \times \alpha) . \supset \vdash . \mathrm{Prop}$

***113·401.** $\vdash : \beta \cap \gamma = \Lambda . \supset . (\beta \times \alpha) \cap (\gamma \times \alpha) = \Lambda$ [*113·19 . Transp]

$*113{\cdot}41$. $\vdash . \mathrm{Nc}`(\beta+\gamma) \times_c \mathrm{Nc}`\alpha = \mathrm{Nc}`\{(\beta+\gamma)\times\alpha\} = \mathrm{Nc}`\{(\beta\times\alpha)+(\gamma\times\alpha)\}$
$= \mathrm{Nc}`(\beta\times\alpha) +_c \mathrm{Nc}`(\gamma\times\alpha)$

Dem.

$\vdash . *113{\cdot}25 . *110{\cdot}3 . \quad \supset \vdash . \mathrm{Nc}`(\beta+\gamma) \times_c \mathrm{Nc}`\alpha = \mathrm{Nc}`\{(\beta+\gamma)\times\alpha\} .$
$\mathrm{Nc}`\{(\beta\times\alpha)+(\gamma\times\alpha)\} = \mathrm{Nc}`(\beta\times\alpha) +_c \mathrm{Nc}`(\gamma\times\alpha)$ (1)

$\vdash . *113{\cdot}4 . (*110{\cdot}01) . \quad \supset \vdash . (\beta+\gamma)\times\alpha = (\downarrow\Lambda_\gamma``\iota``\beta\times\alpha) \cup (\Lambda_\beta\downarrow``\iota``\gamma\times\alpha)$ (2)

$\vdash . *113{\cdot}13 . *110{\cdot}12 . \quad \supset \vdash . \downarrow\Lambda_\gamma``\iota``\beta\times\alpha \,\mathrm{sm}\, \beta\times\alpha . \Lambda_\beta\downarrow``\iota``\gamma\times\alpha \,\mathrm{sm}\, \gamma\times\alpha$ (3)

$\vdash . *113{\cdot}401 . *110{\cdot}11 . \quad \supset \vdash . (\downarrow\Lambda_\gamma``\iota``\beta\times\alpha) \cap (\Lambda_\beta\downarrow``\iota``\gamma\times\alpha) = \Lambda$ (4)

$\vdash . *110{\cdot}152 . (2) . (3) . (4) . \supset \vdash . (\beta+\gamma)\times\alpha \,\mathrm{sm}\, \{(\beta\times\alpha)+(\gamma\times\alpha)\}$ (5)

$\vdash . (1) . (5) . \supset \vdash . \mathrm{Prop}$

$*113{\cdot}42$. $\vdash . (\mathrm{Nc}`\beta +_c \mathrm{Nc}`\gamma) \times_c \mathrm{Nc}`\alpha = \mathrm{Nc}`(\beta+\gamma) \times_c \mathrm{Nc}`\alpha$
$= (\mathrm{Nc}`\beta \times_c \mathrm{Nc}`\alpha) +_c (\mathrm{Nc}`\gamma \times_c \mathrm{Nc}`\alpha)$
$[*110{\cdot}3 . *113{\cdot}25 . *113{\cdot}41]$

$*113{\cdot}421$. $\vdash . \mathrm{Nc}`\alpha \times_c (\mathrm{Nc}`\beta +_c \mathrm{Nc}`\gamma) = \mathrm{Nc}`\alpha \times_c \mathrm{Nc}`(\beta+\gamma)$
$= (\mathrm{Nc}`\alpha \times_c \mathrm{Nc}`\beta) +_c (\mathrm{Nc}`\alpha \times_c \mathrm{Nc}`\gamma)$ $[*113{\cdot}42{\cdot}27]$

$*113{\cdot}43$. $\vdash . (\nu +_c \varpi) \times_c \mu = \mu \times_c (\nu +_c \varpi) = (\mu \times_c \nu) +_c (\mu \times_c \varpi)$

Dem.

$\vdash . *113{\cdot}27{\cdot}421 . \supset \vdash : \mu, \nu, \varpi \in \mathrm{NC} . \supset . (\nu +_c \varpi) \times_c \mu = \mu \times_c (\nu +_c \varpi)$
$= (\mu \times_c \nu) +_c (\mu \times_c \varpi)$ (1)

$\vdash . *113{\cdot}204 . *110{\cdot}4 . \supset$

$\vdash : \sim(\mu, \nu, \varpi \in \mathrm{NC}) . \supset . (\nu +_c \varpi) \times_c \mu = \Lambda . \mu \times_c (\nu +_c \varpi) = \Lambda .$
$(\mu \times_c \nu) +_c (\mu \times_c \varpi) = \Lambda$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

The following propositions are concerned with various forms of the distributive law, when the summands are not enumerated, but given as the members of a class.

The first of them ($*113{\cdot}44$) gives the distributive law with regard to arithmetical class-multiplication and logical addition of classes.

$*113{\cdot}44$. $\vdash . (s`\kappa) \times \alpha = s`(\times\alpha)``\kappa$

Dem.

$\vdash . *113{\cdot}1 . \supset \vdash . s`(\times\alpha)``\kappa = s`s``\alpha\downarrow``` \kappa$

$[*42{\cdot}1] \qquad = s`s`\alpha\downarrow``` \kappa$

$[*40{\cdot}38] \qquad = s`\alpha\downarrow``s`\kappa$

$[*113{\cdot}1] \qquad = (s`\kappa)\times\alpha . \supset \vdash . \mathrm{Prop}$

$*113{\cdot}45$. $\vdash : \kappa \in \mathrm{Cls}^2\,\mathrm{excl} . \supset . \times\alpha``\kappa \in \mathrm{Cls}^2\,\mathrm{excl}$

Dem.

$\vdash . *113{\cdot}19 . \quad \supset \vdash : \exists ! \times\alpha`\beta \cap \times\alpha`\gamma . \supset . \exists ! \beta \cap \gamma$ (1)

$\vdash . (1) . *84{\cdot}11 . \supset \vdash :. \mathrm{Hp} . \supset : \beta, \gamma \in \kappa . \exists ! \times\alpha`\beta \cap \times\alpha`\gamma . \supset_{\beta,\gamma} . \beta = \gamma .$

$[*30{\cdot}37] \qquad \supset_{\beta,\gamma} . \times\alpha`\beta = \times\alpha`\gamma :$

$[*37{\cdot}63] \qquad \supset : \rho, \sigma \in \times\alpha``\kappa . \exists ! \rho \cap \sigma . \supset_{\rho,\sigma} . \rho = \sigma$ (2)

$\vdash . (2) . *84{\cdot}11 . \supset \vdash . \mathrm{Prop}$

***113·46.** $\vdash : \kappa \in \mathrm{Cls}^2\,\mathrm{excl} \,.\supset .\, \Sigma{}^{\backprime}\times\alpha{}^{\backprime\backprime}\kappa \,\mathrm{sm}\, (\Sigma{}^{\backprime}\kappa)\times\alpha$

Dem.

$\vdash .\, *112{\cdot}15 \,.\quad \supset \vdash : \mathrm{Hp} \,.\supset .\, \Sigma{}^{\backprime}\kappa \,\mathrm{sm}\, s{}^{\backprime}\kappa \,.$

$[*113{\cdot}13] \quad \supset .\, (\Sigma{}^{\backprime}\kappa)\times\alpha \,\mathrm{sm}\, (s{}^{\backprime}\kappa)\times\alpha \quad (1)$

$\vdash .\, *112{\cdot}15 \,.\, *113{\cdot}45 \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, \Sigma{}^{\backprime}\times\alpha{}^{\backprime\backprime}\kappa \,\mathrm{sm}\, s{}^{\backprime}\times\alpha{}^{\backprime\backprime}\kappa \quad (2)$

$\vdash .\, (1) \,.\, (2) \,.\, *113{\cdot}44 \,.\supset \vdash .\, \mathrm{Prop}$

***113·47.** $\vdash : \kappa \in \mathrm{Cls}^2\,\mathrm{excl} \,.\supset .\, \Sigma\mathrm{Nc}{}^{\backprime}\times\alpha{}^{\backprime\backprime}\kappa = \mathrm{Nc}{}^{\backprime}\{(\Sigma{}^{\backprime}\kappa)\times\alpha\} = \Sigma\mathrm{Nc}{}^{\backprime}\kappa \times_0 \mathrm{Nc}{}^{\backprime}\alpha$
$[*113{\cdot}46]$

This is the distributive law for arithmetical multiplication and arithmetical addition of the kind defined in *112.

***113·48.** $\vdash .\, s{}^{\backprime}\alpha \times{}^{\backprime\backprime}\kappa = \alpha\times s{}^{\backprime}\kappa = \mathrm{Cnv}{}^{\backprime\backprime}\{(s{}^{\backprime}\kappa)\times\alpha\}$

Dem.

$\vdash .\, *113{\cdot}14 \,.\supset \vdash .\, s{}^{\backprime}\alpha \times{}^{\backprime\backprime}\kappa = s{}^{\backprime}\mathrm{Cnv}{}^{\backprime\backprime\backprime}\times\alpha{}^{\backprime\backprime}\kappa$

$[*40{\cdot}38] \quad = \mathrm{Cnv}{}^{\backprime\backprime}s{}^{\backprime}\times\alpha{}^{\backprime\backprime}\kappa$

$[*113{\cdot}44] \quad = \mathrm{Cnv}{}^{\backprime\backprime}\{(s{}^{\backprime}\kappa)\times\alpha\} \quad (1)$

$[*113{\cdot}14] \quad = \alpha\times s{}^{\backprime}\kappa \quad (2)$

$\vdash .\, (1) \,.\, (2) \,.\supset \vdash .\, \mathrm{Prop}$

***113·49.** $\vdash : \kappa \in \mathrm{Cls}^2\,\mathrm{excl} \,.\supset .\, \Sigma{}^{\backprime}\alpha \times{}^{\backprime\backprime}\kappa \,\mathrm{sm}\, \alpha\times(\Sigma{}^{\backprime}\kappa)$

Dem.

$\vdash .\, *113{\cdot}14 \,.\supset \vdash .\, \alpha \times{}^{\backprime\backprime}\kappa = \mathrm{Cnv}{}^{\backprime\backprime\backprime}\times\alpha{}^{\backprime\backprime}\kappa \quad (1)$

$\vdash .\, (1) \,.\, *113{\cdot}45 \,.\, *72{\cdot}11 \,.\, *84{\cdot}53 \,.\supset$

$\vdash : \mathrm{Hp} \,.\supset .\, \alpha \times{}^{\backprime\backprime}\kappa \in \mathrm{Cls}^2\,\mathrm{excl} \,.$

$[*112{\cdot}15] \quad \supset .\, \Sigma{}^{\backprime}\alpha \times{}^{\backprime\backprime}\kappa \,\mathrm{sm}\, s{}^{\backprime}\alpha \times{}^{\backprime\backprime}\kappa \,.$

$[*113{\cdot}48] \quad \supset .\, \Sigma{}^{\backprime}\alpha \times{}^{\backprime\backprime}\kappa \,\mathrm{sm}\, \alpha\times(s{}^{\backprime}\kappa) \,.$

$[*112{\cdot}15 . *113{\cdot}13] \quad \supset .\, \Sigma{}^{\backprime}\alpha \times{}^{\backprime\backprime}\kappa \,\mathrm{sm}\, \alpha\times(\Sigma{}^{\backprime}\kappa) : \supset \vdash .\, \mathrm{Prop}$

***113·491.** $\vdash : \kappa \in \mathrm{Cls}^2\,\mathrm{excl} \,.\supset .\, \Sigma\mathrm{Nc}{}^{\backprime}\alpha \times{}^{\backprime\backprime}\kappa = \mathrm{Nc}{}^{\backprime}(\alpha\times\Sigma{}^{\backprime}\kappa) = \mathrm{Nc}{}^{\backprime}\alpha \times_0 \Sigma\mathrm{Nc}{}^{\backprime}\kappa$
$[*113{\cdot}49{\cdot}25]$

The following propositions are concerned with the associative law for arithmetical multiplication.

***113·5.** $\vdash .\, (\gamma\times\beta)\times\alpha = \hat{R}\{(\exists x,y,z) \,.\, x\in\alpha \,.\, y\in\beta \,.\, z\in\gamma \,.\, R = x\downarrow(y\downarrow z)\}$

Dem.

$\vdash .\, *113{\cdot}101 \,.\supset$

$\vdash .\, (\gamma\times\beta)\times\alpha = \hat{R}\{(\exists x,P) \,.\, x\in\alpha \,.\, P\in(\gamma\times\beta) \,.\, R = x\downarrow P\}$

$[*113{\cdot}101] \quad = \hat{R}\{(\exists x,y,z) \,.\, x\in\alpha \,.\, y\in\beta \,.\, z\in\gamma \,.\, R = x\downarrow(y\downarrow z)\} \,.\supset \vdash .\, \mathrm{Prop}$

***113·51.** $\vdash .\, (\alpha\times\beta)\times\gamma \,\mathrm{sm}\, \alpha\times(\beta\times\gamma)$

Dem.

$\vdash .\, *113{\cdot}141 \,.\supset \vdash .\, \alpha\times(\beta\times\gamma) \,\mathrm{sm}\, (\beta\times\gamma)\times\alpha \quad (1)$

$\vdash .\, *113{\cdot}5 \,.\quad \supset \vdash .\, (\alpha\times\beta)\times\gamma = \hat{R}\{(\exists x,y,z) \,.\, x\in\alpha \,.\, y\in\beta \,.\, z\in\gamma \,.\, R = z\downarrow(y\downarrow x)\} \,.$

$(\beta\times\gamma)\times\alpha = \hat{P}\{(\exists x,y,z) \,.\, x\in\alpha \,.\, y\in\beta \,.\, z\in\gamma \,.\, P = x\downarrow(z\downarrow y)\} \quad (2)$

$\vdash . (2) . \supset \vdash : T = \hat{R}\hat{P} \{(\exists x,y,z) . x \epsilon \alpha . y \epsilon \beta . z \epsilon \gamma . R = z \downarrow (y \downarrow x) . P = x \downarrow (y \downarrow z)\} . \supset .$
$\qquad D‘T = (\alpha \times \beta) \times \gamma . \text{Ɑ}‘T = (\beta \times \gamma) \times \alpha$ (3)

$\vdash . *21{\cdot}33 . \supset \vdash : \text{Hp}(3) . RTP . RTQ . \supset$
$\quad (\exists x, x', y, y', z, z') . x, x' \epsilon \alpha . y, y' \epsilon \beta . z, z' \epsilon \gamma . R = z \downarrow (y \downarrow x) = z' \downarrow (y' \downarrow x') .$
$\qquad P = x \downarrow (z \downarrow y) . Q = x' \downarrow (z' \downarrow y') .$

$[*55{\cdot}202] \supset . P = Q$ (4)

Similarly $\vdash : \text{Hp}(3) . RTP . QTP . \supset . R = Q$ (5)

$\vdash . (3) . (4) . (5) . \supset \vdash . (\alpha \times \beta) \times \gamma \,\text{sm}\, (\beta \times \gamma) \times \alpha$ (6)

$\vdash . (1) . (6) . \supset \vdash . \text{Prop}$

***113·511.** $\alpha \times \beta \times \gamma = (\alpha \times \beta) \times \gamma$ Df

***113·52.** $\vdash . (\text{Nc}‘\alpha \times_c \text{Nc}‘\beta) \times_c \text{Nc}‘\gamma = \text{Nc}‘(\alpha \times \beta \times \gamma)$ [*113·25]

***113·53.** $\vdash . (\text{Nc}‘\alpha \times_c \text{Nc}‘\beta) \times_c \text{Nc}‘\gamma = \text{Nc}‘\alpha \times_c (\text{Nc}‘\beta \times_c \text{Nc}‘\gamma)$

Dem

$\vdash . *113{\cdot}52{\cdot}51 . \supset$
$\vdash . (\text{Nc}‘\alpha \times_c \text{Nc}‘\beta) \times_c \text{Nc}‘\gamma = \text{Nc}‘\{\alpha \times (\beta \times \gamma)\}$
$[*113{\cdot}25] \qquad = \text{Nc}‘\alpha \times_c (\text{Nc}‘\beta \times_c \text{Nc}‘\gamma) . \supset \vdash . \text{Prop}$

***113·531.** $\vdash . (\text{N}_0\text{c}‘\alpha \times_c \text{N}_0\text{c}‘\beta) \times_c \text{N}_0\text{c}‘\gamma = \text{N}_0\text{c}‘\alpha \times_c (\text{N}_0\text{c}‘\beta \times_c \text{N}_0\text{c}‘\gamma)$
[*113·53 . (*113·04·05)]

***113·54.** $\vdash . (\mu \times_c \nu) \times_c \varpi = \mu \times_c (\nu \times_c \varpi)$

Dem.

$\vdash . *113{\cdot}531 . *103{\cdot}2 . \supset$
$\vdash : \mu, \nu \varpi \epsilon \text{N}_0\text{C} . \supset . (\mu \times_c \nu) \times_c \varpi = \mu \times_c (\nu \times_c \varpi)$ (1)
$\vdash . *113{\cdot}204 . \supset$
$\vdash : \sim(\mu, \nu, \varpi \epsilon \text{N}_0\text{C}) . \supset . (\mu \times_c \nu) \times_c \varpi = \Lambda . \mu \times_c (\nu \times_c \varpi) = \Lambda$ (2)
$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***113·541.** $\mu \times_c \nu \times_c \varpi = (\mu \times_c \nu) \times_c \varpi$ Df

***113·6.** $\vdash . \text{Nc}‘\alpha \times_c 0 = 0$

Dem.

$\vdash . *113{\cdot}25 . *101{\cdot}1 . \supset \vdash . \text{Nc}‘\alpha \times_c 0 = \text{Nc}‘(\alpha \times \Lambda)$
$[*113{\cdot}114 . *101{\cdot}1] \qquad = 0 . \supset \vdash . \text{Prop}$

***113·601.** $\vdash : \mu \epsilon \text{NC} - \iota‘\Lambda . \supset . \mu \times_c 0 = 0$

Dem.

$\vdash . *103{\cdot}26 . \supset \vdash : \text{Hp} . \supset . (\exists \alpha) . \mu = \text{N}_0\text{c}‘\alpha$ (1)
$\vdash . *101{\cdot}11{\cdot}13 . *103{\cdot}27 . \supset \vdash . 0 = \text{N}_0\text{c}‘\Lambda$ (2)
$\vdash . (1) . (2) . \supset \vdash : \text{Hp} . \supset . (\exists \alpha) . \mu \times_c 0 = \text{N}_0\text{c}‘\alpha \times_c \text{N}_0\text{c}‘\Lambda$
$[*113{\cdot}222] \qquad = \text{Nc}‘(\alpha \times \Lambda)$
$[*113{\cdot}114 . *101{\cdot}1] \qquad = 0 : \supset \vdash . \text{Prop}$

***113·602.** $\vdash :. \mu \times_0 \nu = 0 . \equiv : \mu, \nu \in \mathrm{NC} - \iota`\Lambda : \mu = 0 . \mathbf{v} . \nu = 0$

Dem.

$\vdash . *113·203 . *101·12 . \supset$

$\vdash : \mu \times_0 \nu = 0 . \supset . \mu, \nu \in \mathrm{NC} - \iota`\Lambda$ (1)

$\vdash . (1) . *113·201 . \supset$

$\vdash :: \mu \times_0 \nu = 0 . \supset :. \xi \in 0 . \equiv_\xi : (\exists \alpha, \beta) . \alpha \in \mu . \beta \in \nu . \xi \,\mathrm{sm}\, (\alpha \times \beta) :.$

[*54·102] $\supset :. \xi = \Lambda . \equiv_\xi : (\exists \alpha, \beta) . \alpha \in \mu . \beta \in \nu . \xi \,\mathrm{sm}\, (\alpha \times \beta) :.$

[*10·1.*13·15] $\supset :. (\exists \alpha, \beta) . \alpha \in \mu . \beta \in \nu . \Lambda \,\mathrm{sm}\, (\alpha \times \beta) :.$

[*73·47] $\supset :. (\exists \alpha, \beta) . \alpha \in \mu . \beta \in \nu . \alpha \times \beta = \Lambda :.$

[*113·114] $\supset :. (\exists \alpha, \beta) : \alpha \in \mu . \beta \in \nu : \alpha = \Lambda . \mathbf{v} . \beta = \Lambda :.$

[*13·195] $\supset :. \Lambda \in \mu . \mathbf{v} . \Lambda \in \nu :.$

[(1).*100·45] $\supset :. \mu = \mathrm{Nc}`\Lambda . \mathbf{v} . \nu = \mathrm{Nc}`\Lambda :.$

[*101·1] $\supset :. \mu = 0 . \mathbf{v} . \nu = 0$ (2)

$\vdash . *113·601·27 . \supset \vdash :. \mu, \nu \in \mathrm{NC} - \iota`\Lambda : \mu = 0 . \mathbf{v} . \nu = 0 : \supset . \mu \times_0 \nu = 0$ (3)

$\vdash . (2) . (3) . \supset \vdash . \mathrm{Prop}$

The following propositions are concerned with multiplication by a unit class or by 1 or 2.

***113·61.** $\vdash . \iota`z \times \alpha = \downarrow z``\alpha$

Dem.

$\vdash . *113·1 . \supset \vdash . \iota`z \times \alpha = s`\alpha \underset{''}{\downarrow} ``\iota`z$

[*53·31·02] $= \alpha \underset{''}{\downarrow} `z$

[*38·2] $= \downarrow z``\alpha . \supset \vdash . \mathrm{Prop}$

***113·611.** $\vdash . \iota`z \times \alpha \,\mathrm{sm}\, \alpha$ [*113·61 . *73·611]

***113·612.** $\vdash . \alpha \times \iota`z \,\mathrm{sm}\, \alpha$ [*113·611·141]

***113·62.** $\vdash . \mathrm{Nc}`\alpha \times_0 1 = \mathrm{Nc}`\alpha$

Dem.

$\vdash . *101·2 . \supset \vdash . \mathrm{Nc}`\alpha \times_0 1 = \mathrm{Nc}`\alpha \times_0 \mathrm{Nc}`\iota`z$

[*113·25] $= \mathrm{Nc}`(\alpha \times \iota`z)$

[*113·612] $= \mathrm{Nc}`\alpha . \supset \vdash . \mathrm{Prop}$

***113·621.** $\vdash : \mu \in \mathrm{NC} . \supset . \mu \times_0 1 = \mathrm{sm}``\mu$

Dem.

$\vdash . *113·204 . \supset \vdash : \mu = \Lambda . \supset . \mu \times_0 1 = \Lambda$

[*37·29] $= \mathrm{sm}``\mu$ (1)

$\vdash . *103·26 . \supset \vdash : \mathrm{Hp} . \alpha \in \mu . \supset . \mu = \mathrm{N_0c}`\alpha .$ (2)

[(*113·04)] $\supset . \mu \times_0 1 = \mathrm{Nc}`\alpha \times_0 1$

[*113·62] $= \mathrm{Nc}`\alpha$

[*103·4.(2)] $= \mathrm{sm}``\mu$ (3)

$\vdash . (2) . *10·11·23·35 . \supset \vdash : \mathrm{Hp} . \exists ! \mu . \supset . \mu \times_0 1 = \mathrm{sm}``\mu$ (4)

$\vdash . (1) . (4) . \supset \vdash . \mathrm{Prop}$

Observe that if μ is a typically definite cardinal, $\text{sm}``\mu$ is the "same" cardinal rendered typically ambiguous; while if $\mu$ is typically ambiguous, $\mu = \text{sm}``\mu$ in every type.

***113·63.** $\vdash : z \sim\epsilon\, \alpha . \supset . \downarrow z``\alpha \,\text{sm}\, \text{D}``\epsilon_\Delta`(\iota`\alpha \cup \iota`\iota`z)$

Dem.

$\vdash . *113·152 . \supset \vdash : \text{Hp} . \supset . \text{D}``\epsilon_\Delta`(\iota`\alpha \cup \iota`\iota`z) \,\text{sm}\, \alpha \times \iota`z \quad (1)$

$\vdash . (1) . *113·61·141 . \supset \vdash . \text{Prop}$

***113·64.** $\vdash . \downarrow z``\alpha \times \downarrow z``\beta \,\text{sm}\, \alpha \times \beta . \downarrow z``\alpha \times \downarrow z``\beta \,\text{sm} \downarrow z``(\alpha \times \beta)$

Dem.

$\vdash . *73·611 . *113·13 . \supset \vdash . \downarrow z``\alpha \times \downarrow z``\beta \,\text{sm}\, \alpha \times \beta \quad (1)$

$\vdash . (1) . *73·611 . \quad \supset \vdash . \downarrow z``\alpha \times \downarrow z``\beta \,\text{sm} \downarrow z``(\alpha \times \beta) \quad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***113·65.** $\vdash . \downarrow z``\alpha \times \downarrow z``\beta = (\downarrow z \,\|\, \text{Cnv}` \downarrow z)``(\alpha \times \beta)$

Dem.

$\vdash . *72·184 . *55·21 . \supset \vdash . \downarrow z \in 1 \rightarrow 1 . \alpha \subset \text{Cl}` \downarrow z . \beta \subset \text{Cl}` \downarrow z .$

$[*113·126] \quad \supset \vdash . \downarrow z``\alpha \times \downarrow z``\beta = (\downarrow z \,\|\, \text{Cnv}` \downarrow z)``(\alpha \times \beta) .$

$\supset \vdash . \text{Prop}$

***113·66.** $\vdash . \mu \times_c 2 = \mu +_c \mu$

Dem.

$\vdash . *110·643 . \supset \vdash . \mu \times_c 2 = \mu \times_c (1 +_c 1)$

$[*113·43] \quad = (\mu \times_c 1) +_c (\mu \times_c 1) \quad (1)$

$\vdash . (1) . \quad \supset \vdash :. \mu = \text{N}_0\text{c}`\alpha . \supset . \mu \times_c 2 = (\text{N}_0\text{c}`\alpha \times_c 1) +_c (\text{N}_0\text{c}`\alpha \times_c 1)$

$[*113·62.(*113·04)] \quad = \text{Nc}`\alpha +_c \text{Nc}`\alpha$

$[*110·3] \quad = \mu +_c \mu \quad (2)$

$\vdash . (2) . *103·2 . \quad \supset \vdash : \mu \in \text{N}_0\text{C} . \supset . \mu \times_c 2 = \mu +_c \mu \quad (3)$

$\vdash . *113·205 . *110·4 . \supset \vdash : \mu \sim\epsilon\, \text{N}_0\text{C} . \supset . \mu \times_c 2 = \Lambda . \mu +_c \mu = \Lambda \quad (4)$

$\vdash . (3) . (4) . \supset \vdash . \text{Prop}$

***113·67.** $\vdash . \text{Nc}`\alpha \times_c \text{Nc}`(\beta + \iota`y) = (\text{Nc}`\alpha \times_c \text{Nc}`\beta) +_c \text{Nc}`\alpha$

Dem.

$\vdash . *113·421 . *101·2 . \supset$

$\vdash . \text{Nc}`\alpha \times_c \text{Nc}`(\beta + \iota`y) = (\text{Nc}`\alpha \times_c \text{Nc}`\beta) +_c (\text{Nc}`\alpha \times_c 1)$

$[*113·62] \quad = (\text{Nc}`\alpha \times_c \text{Nc}`\beta) +_c \text{Nc}`\alpha . \supset \vdash . \text{Prop}$

***113·671.** $\vdash . \mu \times_c (\nu +_c 1) = (\mu \times_c \nu) +_c \mu \quad [*113·67·205 . *110·4]$

$*114$. THE ARITHMETICAL PRODUCT OF A CLASS OF CLASSES

Summary of $*114$.

The kind of multiplication defined in $*113$ cannot be extended beyond a finite number of factors. We therefore, as in the case of addition, introduce another definition, defining the product of the numbers of a class of classes, and capable of being applied to an infinite number of factors. We define the product of the numbers of members of κ as $\text{Nc}`\epsilon_\Delta`\kappa$; thus we put

$$\Pi\text{Nc}`\kappa = \text{Nc}`\epsilon_\Delta`\kappa \quad \text{Df.}$$

It is to be observed that $\Pi\text{Nc}`\kappa$ is not a function of $\text{Nc}``\kappa$, because, if two members of $\kappa$ have the same number, this will count only once in $\text{Nc}``\kappa$, but will count twice in $\Pi\text{Nc}`\kappa$.

It is very easy to see that, in case κ is finite, $\text{Nc}`\epsilon_\Delta`\kappa$ will be what we should ordinarily regard as the product of the numbers of members of κ. For suppose (*e.g.*)

$$\kappa = \iota`\alpha \cup \iota`\beta \cup \iota`\gamma,$$

where $\alpha \neq \beta \,.\, \alpha \neq \gamma \,.\, \beta \neq \gamma$. Then

$$\epsilon_\Delta`\kappa = \hat{R}\{(\exists x, y, z) \,.\, R = x \downarrow \alpha \,\dot{\cup}\, y \downarrow \beta \,\dot{\cup}\, z \downarrow \gamma \,.\, x \in \alpha \,.\, y \in \beta \,.\, z \in \gamma\}.$$

Thus if R is a member of $\epsilon_\Delta`\kappa$, R is determinate when x, y, z are given, x, y, z being the referents to α, β, γ. Whether α, β, γ overlap or not, the choice of any one of x, y, z is entirely independent of the choice of the other two, and therefore the total number of choices possible is obviously the product of the numbers of α, β, γ. Thus our definition will not conflict with what is commonly understood by a product.

The propositions of this number are less numerous and less important than those of $*113$. We shall deal first with products of a single factor, and products in which one factor is null ($*114{\cdot}2$—$\cdot27$). We shall then deal ($*114{\cdot}3$—$\cdot36$) with the relations between the sort of multiplication here defined and the sort defined in $*113$. Then we have a few propositions ($*114{\cdot}4$—$\cdot43$) showing that unit factors make no difference to the value of a product. Then we prove ($*114{\cdot}5$—$\cdot52$) that the value of the product is the same for two classes having double similarity, and then ($*114{\cdot}53$—$\cdot571$) we give extensions of this result which depend upon the multiplicative axiom. Finally, we give some new forms of the associative law of multiplication.

Among the more important propositions in this number are the following:

$*114{\cdot}21$. $\vdash \,.\, \Pi\text{Nc}`\iota`\alpha = \text{Nc}`\alpha$

I.e. a product of one factor is equal to that factor.

***114·23.** $\vdash : \Lambda \in \kappa . \supset . \Pi\text{Nc}`\kappa = 0$

I.e. a product vanishes if one of its factors is zero. The converse requires the multiplicative axiom, as appears from the proposition

***114·26.** $\vdash :. \text{Mult ax} . \equiv : \Pi\text{Nc}`\kappa = 0 . \equiv_\kappa . \Lambda \in \kappa$

I.e. the multiplicative axiom is equivalent to the assumption that a product vanishes when, and only when, one of its factors is zero.

***114·301.** $\vdash : \kappa \cap \lambda = \Lambda . \supset . \epsilon_\Delta`(\kappa \cup \lambda) \,\text{sm}\, \epsilon_\Delta`\kappa \times \epsilon_\Delta`\lambda$

whence

***114·31.** $\vdash : \kappa \cap \lambda = \Lambda . \supset . \Pi\text{Nc}`\kappa \times_c \Pi\text{Nc}`\lambda = \Pi\text{Nc}`(\kappa \cup \lambda)$

which is a form of the associative law, and

***114·35.** $\vdash : \alpha \neq \beta . \supset . \Pi\text{Nc}`(\iota`\alpha \cup \iota`\beta) = \text{Nc}`\alpha \times_c \text{Nc}`\beta$

which connects the two sorts of multiplication.

***114·41.** $\vdash : \lambda \subset 1 . \supset . \Pi\text{Nc}`(\kappa \cup \lambda) = \Pi\text{Nc}`\kappa$

I.e. unit factors make no difference to the value of a product.

***114·51.** $\vdash : T \upharpoonright s`\lambda \in \kappa \,\overline{\text{sm}}\,\overline{\text{sm}}\, \lambda . \supset . (T \| \breve{T}_\epsilon) \upharpoonright \epsilon_\Delta`\lambda \in (\epsilon_\Delta`\kappa) \,\overline{\text{sm}}\, (\epsilon_\Delta`\lambda)$

This proposition gives a correlator of $\epsilon_\Delta`\kappa$ and $\epsilon_\Delta`\lambda$ as a function of a double correlator of κ and λ, and thus leads to

***114·52.** $\vdash : \kappa \,\text{sm sm}\, \lambda . \supset . \Pi\text{Nc}`\kappa = \Pi\text{Nc}`\lambda . \epsilon_\Delta`\kappa \,\text{sm}\, \epsilon_\Delta`\lambda$

Hence, by the propositions of *111, we infer

***114·571.** $\vdash :. \text{Mult ax} . \supset : \mu, \nu \in \text{NC} . \kappa, \lambda \in \mu \cap \text{Cl}`\nu . \supset . \Pi\text{Nc}`\kappa = \Pi\text{Nc}`\lambda$

I.e. assuming the multiplicative axiom, if κ and λ each consist of μ classes of ν terms each, their products are equal.

We have next various forms of the associative law, beginning with

***114·6.** $\vdash : \kappa \in \text{Cls}^2 \,\text{excl} . \supset . \Pi\text{Nc}`\epsilon_\Delta``\kappa = \Pi\text{Nc}`s`\kappa$

which is an immediate consequence of *85·44. The other form is

***114·632.** $\vdash : S \upharpoonright \gamma \in 1 \rightarrow 1 . \gamma \subset \text{G}`S . \gamma \cap S``\gamma = \Lambda . \supset .$
$$\epsilon_\Delta`\hat{\mu}\{(\exists \alpha) . \alpha \in \gamma . \mu = \alpha \times S`\alpha\} \,\text{sm}\, \epsilon_\Delta`(\gamma \cup S``\gamma)$$

As to the sense in which this is a form of the associative law, see the observations following *114·6.

***114·01.** $\Pi\text{Nc}`\kappa = \text{Nc}`\epsilon_\Delta`\kappa$ Df

***114·1.** $\vdash . \Pi\text{Nc}`\kappa = \text{Nc}`\epsilon_\Delta`\kappa$ [(*114·01)]

***114·11.** $\vdash : \beta \in \Pi\text{Nc}`\kappa . \equiv . \beta \,\text{sm}\, \epsilon_\Delta`\kappa . \equiv . \beta \in \text{Nc}`\epsilon_\Delta`\kappa$ [*114·1 . *100·31]

***114·12.** $\vdash . \epsilon_\Delta`\kappa \in \Pi\text{Nc}`\kappa$ [*100·3 . *114·1]

***114·2.** $\vdash . \Pi\text{Nc}`\Lambda = 1$ [*83·15 . *101·2]

Thus a product of no factors is 1. This is the source of $\mu^0 = 1$, as we shall see later.

***114·21.** $\vdash . \Pi Nc`\iota`\alpha = Nc`\alpha$ [*83·41]

***114·22.** $\vdash . \Pi Nc`\iota`\Lambda = 0$ [*114·21 . *101·1]

***114·23.** $\vdash : \Lambda \epsilon \kappa . \supset . \Pi Nc`\kappa = 0$ [*83·11 . *101·1]

Thus an arithmetical product is zero if any of its factors is zero. To prove the converse, we have to assumé the multiplicative axiom, which, in fact, is equivalent to the proposition that an arithmetical product is only zero when at least one of its factors is zero.

***114·24.** $\vdash : \Pi Nc`\lambda \neq 0 . \kappa \subset \lambda . \supset . \Pi Nc`\kappa \neq 0$

Dem.

$\vdash . *114·1 . *101·1 . \supset \vdash : \Pi Nc`\lambda \neq 0 . \supset . \exists ! \epsilon_\Delta`\lambda$ (1)

$\vdash . (1) . *80·6 . \quad \supset \vdash : \Pi Nc`\lambda \neq 0 . \kappa \subset \lambda . \supset . \exists ! \epsilon_\Delta`\kappa .$

[*114·1.*101·1] $\quad \supset . \Pi Nc`\kappa \neq 0 : \supset \vdash . \text{Prop}$

***114·25.** $\vdash :. \text{Mult ax} . \equiv : \Pi Nc`\kappa = 0 . \supset_\kappa . \Lambda \epsilon \kappa$

Dem.

$\vdash . *88·37 . \text{Transp} . \supset$

$\vdash :. \text{Mult ax} . \quad \equiv : \epsilon_\Delta`\kappa = \Lambda . \supset_\kappa . \Lambda \epsilon \kappa :$

[*114·1.*101·1] $\equiv : \Pi Nc`\kappa = 0 . \supset_\kappa . \Lambda \epsilon \kappa :. \supset \vdash . \text{Prop}$

Note that $\Lambda \epsilon \kappa . \equiv . 0 \epsilon Nc``\kappa$.

***114·26.** $\vdash :. \text{Mult ax} . \equiv : \Pi Nc`\kappa = 0 . \equiv_\kappa . \Lambda \epsilon \kappa$ [*88·372 . *101·1]

***114·261.** $\vdash :. \text{Mult ax} . \equiv : \Pi Nc`\kappa = 0 . \equiv_\kappa . 0 \epsilon Nc``\kappa$ [*114·26 . *101·1]

***114·27.** $\vdash :: \text{Mult ax} . \equiv :. \alpha \epsilon \kappa . \supset_\alpha . \exists ! \alpha : \equiv_\kappa . \Pi Nc`\kappa \neq 0$
[*114·26 . Transp . *24·63]

***114·3.** $\vdash : \kappa \neq \lambda . \supset . \epsilon_\Delta`(\iota`\epsilon_\Delta`\kappa \cup \iota`\epsilon_\Delta`\lambda) \text{ sm } \epsilon_\Delta`\kappa \times \epsilon_\Delta`\lambda$

Dem.

$\vdash . *113·146 . \supset \vdash : \epsilon_\Delta`\kappa \neq \epsilon_\Delta`\lambda . \supset . \epsilon_\Delta`(\iota`\epsilon_\Delta`\kappa \cup \iota`\epsilon_\Delta`\lambda) \text{ sm } \epsilon_\Delta`\kappa \times \epsilon_\Delta`\lambda$ (1)

$\vdash . *80·81 . \quad \supset \vdash :. \exists ! \epsilon_\Delta`\kappa . \vee . \exists ! \epsilon_\Delta`\lambda : \kappa \neq \lambda : \supset . \epsilon_\Delta`\kappa \neq \epsilon_\Delta`\lambda$ (2)

$\vdash . *83·903 . *113·114 . \supset$

$\vdash : \epsilon_\Delta`\kappa = \Lambda . \epsilon_\Delta`\lambda = \Lambda . \supset . \epsilon_\Delta`(\iota`\epsilon_\Delta`\kappa \cup \iota`\epsilon_\Delta`\lambda) = \Lambda . \epsilon_\Delta`\kappa \times \epsilon_\Delta`\lambda = \Lambda$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash . \text{Prop}$

***114·301.** $\vdash : \kappa \cap \lambda = \Lambda . \supset . \epsilon_\Delta`(\kappa \cup \lambda) \text{ sm } \epsilon_\Delta`\kappa \times \epsilon_\Delta`\lambda$

Dem.

$\vdash . *85·45 . *114·3 . \supset$

$\vdash : \kappa \cap \lambda = \Lambda . \kappa \neq \lambda . \supset . \epsilon_\Delta`(\kappa \cup \lambda) \text{ sm } \epsilon_\Delta`\kappa \times \epsilon_\Delta`\lambda$ (1)

$\vdash . *22·5 . \supset \vdash : \kappa \cap \lambda = \Lambda . \kappa = \lambda . \supset . \kappa = \Lambda . \lambda = \Lambda .$

[*83·15] $\quad \supset . \epsilon_\Delta`(\kappa \cup \lambda) = \iota`\dot{\Lambda} . \epsilon_\Delta`\kappa = \iota`\dot{\Lambda} . \epsilon_\Delta`\lambda = \iota`\dot{\Lambda} .$

[*113·611] $\quad \supset . \epsilon_\Delta`(\kappa \cup \lambda) \text{ sm } \epsilon_\Delta`\kappa \times \epsilon_\Delta`\lambda$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***114·31.** $\vdash : \kappa \cap \lambda = \Lambda . \supset . \Pi\text{Nc}`\kappa \times_c \Pi\text{Nc}`\lambda = \Pi\text{Nc}`(\kappa \cup \lambda)$
[*114·301·1 . *113·25]

The above is one form of the associative law of multiplication.

***114·311.** $\vdash . \Pi\text{Nc}`(\kappa \cup \lambda) = \Pi\text{Nc}`\kappa \times_c \Pi\text{Nc}`(\lambda - \kappa)$ [*114·31 . *22·91]

***114·32.** $\vdash : \Pi\text{Nc}`(\kappa \cup \lambda) \neq 0 . \equiv . \Pi\text{Nc}`\kappa \neq 0 . \Pi\text{Nc}`\lambda \neq 0$

Dem.

$\vdash .$ *114·311 . *113·602 . $\supset$

$\vdash : \Pi\text{Nc}`(\kappa \cup \lambda) \neq 0 . \supset . \Pi\text{Nc}`\kappa \neq 0$ (1)

$\vdash . (1) \dfrac{\lambda, \kappa}{\kappa, \lambda} . \supset \vdash : \Pi\text{Nc}`(\kappa \cup \lambda) \neq 0 . \supset . \Pi\text{Nc}`\lambda \neq 0$ (2)

$\vdash .$ *114·24 . $\supset \vdash : \Pi\text{Nc}`\lambda \neq 0 . \supset . \Pi\text{Nc}`(\lambda - \kappa) \neq 0 :$

[Fact] $\supset \vdash : \Pi\text{Nc}`\kappa \neq 0 . \Pi\text{Nc}`\lambda \neq 0 . \supset . \Pi\text{Nc}`\kappa \neq 0 . \Pi\text{Nc}`(\lambda - \kappa) \neq 0 .$

[*113·602.*114·311] $\supset . \Pi\text{Nc}`(\kappa \cup \lambda) \neq 0$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash .$ Prop

***114·33.** $\vdash : \alpha \sim\epsilon\, \kappa . \supset . \Pi\text{Nc}`(\kappa \cup \iota`\alpha) = \Pi\text{Nc}`\kappa \times_c \text{Nc}`\alpha$ [*114·31·21]

***114·34.** $\vdash : \Pi\text{Nc}`\kappa \neq 0 . \exists ! \alpha . \equiv . \Pi\text{Nc}`(\kappa \cup \iota`\alpha) \neq 0$
[*114·32·21 . *101·14]

***114·35.** $\vdash : \alpha \neq \beta . \supset . \Pi\text{Nc}`(\iota`\alpha \cup \iota`\beta) = \text{Nc}`\alpha \times_c \text{Nc}`\beta$ [*114·33·21]

***114·36.** $\vdash : \alpha \neq \beta . \alpha \neq \gamma . \beta \neq \gamma . \supset . \Pi\text{Nc}`(\iota`\alpha \cup \iota`\beta \cup \iota`\gamma) = \text{Nc}`\alpha \times_c \text{Nc}`\beta \times_c \text{Nc}`\gamma$
[*114·33·35]

***114·4.** $\vdash : \lambda \subset 1 . \supset . \Pi\text{Nc}`\lambda = 1$ [*83·44]

***114·41.** $\vdash : \lambda \subset 1 . \supset . \Pi\text{Nc}`(\kappa \cup \lambda) = \Pi\text{Nc}`\kappa$ [*83·57]

***114·42.** $\vdash . \Pi\text{Nc}`\kappa = \Pi\text{Nc}`(\kappa - 1)$

Dem.

$\vdash .$ *24·41 . $\supset \vdash . \kappa = (\kappa - 1) \cup (\kappa \cap 1)$ (1)

$\vdash . (1) .$ *114·41 . $\supset \vdash .$ Prop

***114·43.** $\vdash . \Pi\text{Nc}`(\kappa \cup \iota``\alpha) = \Pi\text{Nc}`\kappa$ [*114·41 . *52·3]

***114·5.** $\vdash : T \epsilon \kappa \,\overline{\text{sm}}\, \overline{\text{sm}}\, \lambda . \supset . (T \| \breve{T}_\epsilon) \upharpoonright \epsilon_\Delta`\lambda \,\epsilon\, (\epsilon_\Delta`\kappa) \,\overline{\text{sm}}\, (\epsilon_\Delta`\lambda)$

Dem.

$\vdash .$ *111·1·11 . $\supset \vdash : \text{Hp} . \supset . T, T_\epsilon \upharpoonright \lambda \,\epsilon\, 1 \to 1$ (1)

$\vdash .$ *80·14 . *83·21 . $\supset \vdash . s`\text{D}``\epsilon_\Delta`\lambda \subset s`\lambda . s`\text{Cl}``\epsilon_\Delta`\lambda \subset \lambda$ (2)

$\vdash . (1) . (2) .$ *74·773 . $\supset$

$\vdash : \text{Hp} . \supset . (T \| \breve{T}_\epsilon) \upharpoonright \epsilon_\Delta`\lambda \,\epsilon\, \{(T \| \breve{T}_\epsilon)``\epsilon_\Delta`\lambda\} \,\overline{\text{sm}}\, (\epsilon_\Delta`\lambda)$ (3)

$\vdash .$ *82·43 $\dfrac{\epsilon, T_\epsilon}{P, Q}$. *62·3 . $\supset$

$\vdash : T, T_\epsilon \upharpoonright \lambda \,\epsilon\, 1 \to 1 . s`\lambda \subset \text{Cl}`T . \lambda \subset \text{Cl}`T_\epsilon . \kappa = T_\epsilon``\lambda . \supset .$

$(T \mid \epsilon \upharpoonright \lambda \mid \breve{T}_\epsilon)_\Delta`\kappa = (T \| \breve{T}_\epsilon)``\epsilon_\Delta`\lambda$ (4)

$\vdash . (4) . (1) .$ *111·1 . *37·111 . $\supset \vdash : \text{Hp} . \supset . (T \mid \epsilon \upharpoonright \lambda \mid \breve{T}_\epsilon)_\Delta`\kappa = (T \| \breve{T}_\epsilon)``\epsilon_\Delta`\lambda$ (5)

$\vdash . *34\cdot1 . *37\cdot101 . \supset$

$\vdash : x(T \mid \epsilon \restriction \lambda \mid \breve{T}_\epsilon)\alpha . \equiv . (\exists y, \beta) . xTy . y \epsilon \beta . \beta \epsilon \lambda . \alpha = T``\beta \quad (6)$

$\vdash . (6) . *72\cdot52 . *111\cdot1 . \supset$

$\vdash :. \text{Hp} . \supset : x(T \mid \epsilon \restriction \lambda \mid \breve{T}_\epsilon)\alpha . \equiv . (\exists y, \beta) . xTy . y \epsilon \beta . \beta \epsilon \lambda . \beta = \breve{T}``\alpha . \alpha \subset \text{D}`T .$

$[*111\cdot1\cdot131 . *13\cdot195] \quad \equiv . (\exists y) . xTy . y \epsilon \breve{T}``\alpha . \alpha \epsilon \kappa .$

$[*37\cdot1] \quad \equiv . x \epsilon T``\breve{T}``\alpha . \alpha \epsilon \kappa .$

$[*72\cdot502 . *111\cdot1] \quad \equiv . x(\epsilon \restriction \kappa)\alpha \quad (7)$

$\vdash . (5) . (7) . \supset \vdash : \text{Hp} . \supset . (\epsilon \restriction \kappa)_\Delta`\kappa = (T \| \breve{T}_\epsilon)``\epsilon_\Delta`\lambda .$

$[*83\cdot12] \quad \supset . \epsilon_\Delta`\kappa = (T \| \breve{T}_\epsilon)``\epsilon_\Delta`\lambda \quad (8)$

$\vdash . (3) . (8) . \supset \vdash . \text{Prop}$

$*114\cdot501$. $\vdash : S = T \restriction s`\lambda . \supset . (S \| \breve{S}_\epsilon) \restriction \epsilon_\Delta`\lambda = (T \| \breve{T}_\epsilon) \restriction \epsilon_\Delta`\lambda$

Dem.

$\vdash . *80\cdot14 . *83\cdot21 . \supset$

$\vdash :. R \epsilon \epsilon_\Delta`\lambda . \supset : yR\beta . \supset . y \epsilon s`\lambda . \beta \epsilon \lambda .$

$[*40\cdot13] \quad \supset . y \epsilon s`\lambda . \beta \subset s`\lambda : \quad (1)$

$[*4\cdot71 . \text{Fact}] \supset : xTy . yR\beta . \beta\breve{T}_\epsilon\alpha . \equiv . xTy . y \epsilon s`\lambda . yR\beta . \beta\breve{T}_\epsilon\alpha . \beta \subset s`\lambda .$

$[*37\cdot101 . *22\cdot621] \quad \equiv . x(T \restriction s`\lambda)y . yR\beta . \alpha = T``\beta . \beta = \beta \cap s`\lambda .$

$[(1) . *37\cdot412] \quad \equiv . x(T \restriction s`\lambda)y . yR\beta . \alpha = (T \restriction s`\lambda)``\beta \quad (2)$

$\vdash . (2) . \supset \vdash :. \text{Hp} . \supset : R \epsilon \epsilon_\Delta`\kappa . \supset . T \mid R \mid \breve{T}_\epsilon = S \mid R \mid \breve{S}_\epsilon :$

$[*35\cdot71] \quad \supset : (T \| \breve{T}_\epsilon) \restriction \epsilon_\Delta`\kappa = (S \| \breve{S}_\epsilon) \restriction \epsilon_\Delta`\kappa :. \supset \vdash . \text{Prop}$

$*114\cdot51$. $\vdash : T \restriction s`\lambda \epsilon \kappa \,\overline{\text{sm}}\, \overline{\text{sm}}\, \lambda . \supset . (T \| \breve{T}_\epsilon) \restriction \epsilon_\Delta`\lambda \epsilon (\epsilon_\Delta`\kappa) \,\overline{\text{sm}}\, (\epsilon_\Delta`\lambda)$
$[*114\cdot5\cdot501]$

$*114\cdot52$. $\vdash : \kappa \,\text{sm}\, \text{sm}\, \lambda . \supset . \Pi\text{Nc}`\kappa = \Pi\text{Nc}`\lambda . \epsilon_\Delta`\kappa \,\text{sm}\, \epsilon_\Delta`\lambda \quad [*114\cdot51 . *111\cdot4]$

$*114\cdot53$. $\vdash :: \text{Mult ax} . \supset :. \kappa, \lambda \epsilon \text{Cls}^2 \text{excl} :$
$(\exists S) . S \epsilon 1 \to 1 . S \Subset \text{sm} . \text{D}`S = \kappa . \text{Q}`S = \lambda : \supset . \Pi\text{Nc}`\kappa = \Pi\text{Nc}`\lambda$
$[*114\cdot52 . *111\cdot5]$

$*114\cdot54$. $\vdash :. \text{Mult ax} . \supset : \mu, \nu \epsilon \text{NC} . \kappa, \lambda \epsilon \mu \cap \text{Cl excl}`\nu . \supset . \Pi\text{Nc}`\kappa = \Pi\text{Nc}`\lambda$
$[*114\cdot52 . *111\cdot53]$

The condition $\kappa, \lambda \epsilon \text{Cls}^2 \text{excl}$, which is involved in the hypothesis of $*114\cdot54$ (through $\kappa, \lambda \epsilon \text{Cl excl}`\nu$), is not necessary. The following propositions enable us to remove it. We first prove

$$\epsilon_\Delta`\kappa \,\text{sm}\, \epsilon_\Delta`\epsilon \downarrow``\kappa$$

and then we use $*114\cdot54$ to take us from $\epsilon_\Delta`\epsilon \downarrow``\kappa$ to $\epsilon_\Delta`\epsilon \downarrow``\lambda$. Thence we arrive at $\epsilon_\Delta`\kappa \,\text{sm}\, \epsilon_\Delta`\lambda$.

$*114\cdot56$. $\vdash . \epsilon_\Delta`\kappa \,\text{sm}\, \epsilon_\Delta`\epsilon \downarrow``\kappa . \Pi\text{Nc}`\kappa = \Pi\text{Nc}`\epsilon \downarrow``\kappa \quad [*85\cdot54]$

$*114\cdot561$. $\vdash : S \epsilon \kappa \,\overline{\text{sm}}\, \lambda \cap \text{Rl}`\text{sm} . \supset . \epsilon \downarrow \mid S \mid \text{Cnv}`(\epsilon \downarrow) \epsilon (\epsilon \downarrow``\kappa) \,\overline{\text{sm}}\, (\epsilon \downarrow``\lambda) \cap \text{Rl}`\text{sm}$
$[*73\cdot63 . *85\cdot601 . *38\cdot12 . *33\cdot432]$

***114·562.** $\vdash :. \mathrm{Mult\,ax} . \supset :$

$$(\exists S) . S \in 1 \to 1 . S \mathrel{\dot{\subset}} \mathrm{sm} . \mathrm{D}‘S = \kappa . \text{Ɑ}‘S = \lambda . \supset . \epsilon\!\downarrow‘‘\kappa \ \mathrm{sm\,sm}\ \epsilon\!\downarrow‘‘\lambda$$

Dem.

$\vdash . *114·561 . *85·61 . \supset$

$\vdash :. (\exists S) . S \in 1 \to 1 . S \mathrel{\dot{\subset}} \mathrm{sm} . \mathrm{D}‘S = \kappa . \text{Ɑ}‘S = \lambda . \supset :$

$\epsilon\!\downarrow‘‘\kappa, \epsilon\!\downarrow‘‘\lambda \in \mathrm{Cls^2\,excl} : (\exists T) . T \in 1 \to 1 . T \mathrel{\dot{\subset}} \mathrm{sm} . \mathrm{D}‘T = \epsilon\!\downarrow‘‘\kappa . \text{Ɑ}‘T = \epsilon\!\downarrow‘‘\lambda :$

[*111·5] $\supset : \mathrm{Mult\,ax} . \supset . \epsilon\!\downarrow‘‘\kappa \ \mathrm{sm\,sm}\ \epsilon\!\downarrow‘‘\lambda :. \supset \vdash . \mathrm{Prop}$

***114·57.** $\vdash :. \mathrm{Mult\,ax} . \supset :$

$$(\exists S) . S \in 1 \to 1 . S \mathrel{\dot{\subset}} \mathrm{sm} . \mathrm{D}‘S = \kappa . \text{Ɑ}‘S = \lambda . \supset . \Pi\mathrm{Nc}‘\kappa = \Pi\mathrm{Nc}‘\lambda$$

Dem.

$\vdash . *114·562·52 . \supset$

$\vdash :. \mathrm{Mult\,ax} . \supset : (\exists S) . S \in 1 \to 1 . S \mathrel{\dot{\subset}} \mathrm{sm} . \mathrm{D}‘S = \kappa . \text{Ɑ}‘S = \lambda . \supset .$

$\Pi\mathrm{Nc}‘\epsilon\!\downarrow‘‘\kappa = \Pi\mathrm{Nc}‘\epsilon\!\downarrow‘‘\lambda .$

[*114·56] $\supset . \Pi\mathrm{Nc}‘\kappa = \Pi\mathrm{Nc}‘\lambda :. \supset \vdash . \mathrm{Prop}$

***114·571.** $\vdash :. \mathrm{Mult\,ax} . \supset : \mu, \nu \in \mathrm{NC} . \kappa, \lambda \in \mu \cap \mathrm{Cl}‘\nu . \supset . \Pi\mathrm{Nc}‘\kappa = \Pi\mathrm{Nc}‘\lambda$ [*111·52 . *114·57]

***114·6.** $\vdash : \kappa \in \mathrm{Cls^2\,excl} . \supset . \Pi\mathrm{Nc}‘\epsilon_\Delta‘‘\kappa = \Pi\mathrm{Nc}‘s‘\kappa$ [*85·44]

This is the most general form of the associative law for arithmetical multiplication.

Owing to the fact that we have two kinds of multiplication, namely $\alpha \times \beta$ and $\epsilon_\Delta‘\kappa$, we have four forms of the associative law of multiplication, namely:

(1) *114·6, above,

(2) *113·54, *i.e.* $\vdash . (\mu \times_c \nu) \times_c \varpi = \mu \times_c (\nu \times_c \varpi)$,

(3) *114·31, *i.e.* $\vdash : \kappa \cap \lambda = \Lambda . \supset . \Pi\mathrm{Nc}‘\kappa \times_c \Pi\mathrm{Nc}‘\lambda = \Pi\mathrm{Nc}‘(\kappa \cup \lambda)$,

(4) a form of the associative law which has not yet been proved, which may be explained as follows.

Suppose we have a number of pairs of classes, *e.g.* (α_1, β_1), (α_2, β_2), (α_3, β_3), ... Suppose we form the products $\alpha_1 \times \beta_1$, $\alpha_2 \times \beta_2$, $\alpha_3 \times \beta_3, \ldots$ and multiply all these products together. We wish to prove that (with a suitable hypothesis) the result is similar to the product of all the α's and all the β's taken together as one class; *i.e.* if we call λ the class of products $\alpha_1 \times \beta_1, \alpha_2 \times \beta_2, \alpha_3 \times \beta_3, \ldots$, and μ the class whose members are $\alpha_1, \alpha_2, \alpha_3, \ldots, \beta_1, \beta_2, \beta_3, \ldots$, we wish to prove

$$\Pi\mathrm{Nc}‘\lambda = \Pi\mathrm{Nc}‘\mu.$$

In order to express this proposition in symbols, let S be the correlator of the α's and β's, so that $\beta_\nu = S‘\alpha_\nu$. (The suffix ν will not be used further, since it implies that the number of α's and of β's is finite or denumerable.) Then our class of products of the form $\alpha \times \beta$ is

$$\hat{\mu}\{(\exists \alpha) . \alpha \in \gamma . \mu = \alpha \times S‘\alpha\},$$

where γ is the class of all the α's; and the product of this class of products is

$$\epsilon_\Delta\text{‘}\hat{\mu}\{(\exists\alpha) \,.\, \alpha\in\gamma \,.\, \mu=\alpha\times S\text{‘}\alpha\}.$$

On the other hand, the class of all the α's and β's is $\gamma\cup S\text{‘‘}\gamma$, and the product of this class is

$$\epsilon_\Delta\text{‘}(\gamma\cup S\text{‘‘}\gamma).$$

Thus what we have to prove (with a suitable hypothesis) is

$$\epsilon_\Delta\text{‘}\hat{\mu}\{(\exists\alpha) \,.\, \alpha\in\gamma \,.\, \mu=\alpha\times S\text{‘}\alpha\} \operatorname{sm} \epsilon_\Delta\text{‘}(\gamma\cup S\text{‘‘}\gamma).$$

The hypothesis required is

$$S\upharpoonright\gamma\in 1\to 1 \,.\, \gamma\subset\text{Q‘}S \,.\, \gamma\cap S\text{‘‘}\gamma=\Lambda.$$

A smaller hypothesis suffices, however, for a proposition which, in virtue of $*114{\cdot}301$, is closely allied to the above, namely

$$\epsilon_\Delta\text{‘}\gamma\times\epsilon_\Delta\text{‘}S\text{‘‘}\gamma \operatorname{sm} \epsilon_\Delta\text{‘}\hat{\mu}\{(\exists\alpha) \,.\, \alpha\in\gamma \,.\, \mu=\alpha\times S\text{‘}\alpha\}.$$

For this, a sufficient hypothesis is

$$S\upharpoonright\gamma\in 1\to 1 \,.\, \gamma\subset\text{Q‘}S.$$

Thus *e.g.* we may write I for S, and we find

$$\vdash \,.\, \epsilon_\Delta\text{‘}\gamma\times\epsilon_\Delta\text{‘}\gamma \operatorname{sm} \epsilon_\Delta\text{‘}\hat{\mu}\{(\exists\alpha) \,.\, \alpha\in\gamma \,.\, \mu=\alpha\times\alpha\}.$$

We shall now prove the above propositions. What follows, down to $*114{\cdot}621$, consists of lemmas.

For convenience, we write $S_\times\text{‘}\alpha$ for $\alpha\times S\text{‘}\alpha$ in the course of these lemmas; this notation is introduced in the hypotheses of the lemmas.

$*114{\cdot}601$. $\vdash :. S\upharpoonright\gamma\in 1\to 1 \,.\, \gamma\subset\text{Q‘}S \,.\, \Lambda\sim\in\gamma \,.\, S_\times=\hat{\mu}\hat{\alpha}(\alpha\in\gamma \,.\, \mu=\alpha\times S\text{‘}\alpha) \,.\, \supset :$
$S_\times\in 1\to 1 \,.\, \text{Q‘}S_\times=\gamma \,.\, \text{D‘}S_\times=\hat{\mu}\{(\exists\alpha) \,.\, \alpha\in\gamma \,.\, \mu=\alpha\times S\text{‘}\alpha\} :$
$\alpha\in\gamma \,.\, \supset_\alpha \,.\, S_\times\text{‘}\alpha=\alpha\times S\text{‘}\alpha$

Dem.

$\vdash \,.\, *33{\cdot}11 \,.\quad \supset\vdash : \text{Hp} \,.\, \supset \,.\, \text{D‘}S_\times=\hat{\mu}\{(\exists\alpha) \,.\, \alpha\in\gamma \,.\, \mu=\alpha\times S\text{‘}\alpha\}$ (1)

$\vdash \,.\, *21{\cdot}33 \,.\quad \supset\vdash :. \text{Hp} \,.\, \alpha\in\gamma \,.\, \supset : \mu(S_\times)\alpha \,.\, \equiv_\mu \,.\, \mu=\alpha\times S\text{‘}\alpha :$

$[*30{\cdot}3]\quad \supset : S_\times\text{‘}\alpha=\alpha\times S\text{‘}\alpha$ (2)

$\vdash \,.\, (2) \,.\, *14{\cdot}204 \,.\quad \supset\vdash :. \text{Hp} \,.\, \supset : \alpha\in\gamma \,.\, \supset_\alpha \,.\, \text{E}\,!\,S_\times\text{‘}\alpha \,.$ (3)

$[*33{\cdot}43]\quad \supset_\alpha \,.\, \alpha\in\text{Q‘}S_\times$ (4)

$\vdash \,.\, *21{\cdot}33 \,.\, *33{\cdot}131 \,.\, \supset\vdash :. \text{Hp} \,.\, \supset : \alpha\in\text{Q‘}S_\times \,.\, \supset_\alpha \,.\, \alpha\in\gamma$ (5)

$\vdash \,.\, (4) \,.\, (5) \,.\quad \supset\vdash : \text{Hp} \,.\, \supset \,.\, \text{Q‘}S_\times=\gamma \,.$ (6)

$[(3).*71{\cdot}16]\quad \supset \,.\, S_\times\in 1\to\text{Cls}$ (7)

$\vdash \,.\, *113{\cdot}181 \,.\quad \supset\vdash :. \text{Hp} \,.\, \supset : \alpha,\alpha'\in\gamma \,.\, \alpha\times S\text{‘}\alpha=\alpha'\times S\text{‘}\alpha' \,.\, \supset \,.\, S\text{‘}\alpha=S\text{‘}\alpha' \,.$

$[*71{\cdot}59]\quad \supset \,.\, \alpha=\alpha'$ (8)

$\vdash \,.\, (8) \,.\, *71{\cdot}55 \,.\, (2) \,.\, (6) \,.\, (7) \,.\, \supset\vdash : \text{Hp} \,.\, \supset \,.\, S_\times\in 1\to 1$ (9)

$\vdash \,.\, (1) \,.\, (2) \,.\, (6) \,.\, (9) \,.\, \supset\vdash \,.\, \text{Prop}$

***114·602.** $\vdash : \mathrm{Hp} *114{\cdot}601 . A = \hat{R}\hat{\alpha}\{\alpha \epsilon \gamma . R = (S\text{‘}\alpha) \downarrow \alpha\} . \supset . A \epsilon 1 \to 1 . \text{Ɑ}\text{‘}A = \gamma$

Dem.

As in *114·601, we prove

$\vdash : \mathrm{Hp} . \supset . A \epsilon 1 \to \mathrm{Cls} . \text{Ɑ}\text{‘}A = \gamma$ (1)

$\vdash . *21{\cdot}33 . *13{\cdot}171 . \supset \vdash :. \mathrm{Hp} . \supset : RA\alpha . RA\beta . \supset . (S\text{‘}\alpha) \downarrow \alpha = (S\text{‘}\beta) \downarrow \beta .$

$[*55{\cdot}202] \qquad \supset . \alpha = \beta$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***114·603.** $\vdash : \mathrm{Hp} *114{\cdot}602 . X \epsilon \epsilon_\Delta\text{‘}\gamma . Y \epsilon \epsilon_\Delta\text{‘}S\text{‘‘}\gamma . P = (Y \| \breve{X}) | A | \breve{S}_\times . \supset . P \epsilon \epsilon_\Delta\text{‘}\mathrm{D}\text{‘}S_\times$

Dem.

$\vdash . *43{\cdot}122 . *71{\cdot}166 . *114{\cdot}601{\cdot}602 . \supset \vdash : \mathrm{Hp} . \supset . P \epsilon 1 \to \mathrm{Cls}$ (1)

$\vdash . *43{\cdot}122 . *37{\cdot}32{\cdot}322 . *33{\cdot}431 . \quad \supset \vdash : \mathrm{Hp} . \supset . \text{Ɑ}\text{‘}P = S_\times\text{‘‘}\text{Ɑ}\text{‘}A$

$[*114{\cdot}601{\cdot}602] \qquad = \mathrm{D}\text{‘}S_\times$ (2)

$\vdash . *34{\cdot}1 . \supset :. \mathrm{Hp} . \supset :$

$MP\mu . \equiv . (\exists R, \alpha) . M = Y | R | \breve{X} . R = (S\text{‘}\alpha) \downarrow \alpha . \alpha \epsilon \gamma . \mu = S_\times\text{‘}\alpha .$

$[*113{\cdot}123 . *80{\cdot}14] \quad \equiv . (\exists \alpha) . M = (Y\text{‘}S\text{‘}\alpha) \downarrow (X\text{‘}\alpha) . \mu = S_\times\text{‘}\alpha . \alpha \epsilon \gamma .$

$[*13{\cdot}195 . *114{\cdot}601] \quad \equiv . (\exists \alpha, \beta) . \beta = S\text{‘}\alpha . \alpha \epsilon \gamma . M = (Y\text{‘}\beta) \downarrow (X\text{‘}\alpha) . \mu = \alpha \times \beta .$

$[*83{\cdot}2] \qquad \supset . (\exists \alpha, \beta, u, v) . \beta = S\text{‘}\alpha . \alpha \epsilon \gamma . u \epsilon \alpha . v \epsilon \beta .$

$M = (v \downarrow u) . \mu = \alpha \times \beta .$

$[*113{\cdot}101] \qquad \supset . M \epsilon \mu$ (3)

$\vdash . (1) . (2) . (3) . *80{\cdot}14 . \supset \vdash . \mathrm{Prop}$

***114·604.** $\vdash : \mathrm{Hp} *114{\cdot}602 . T = \hat{P}\hat{Q}\{(\exists X, Y) . X \epsilon \epsilon_\Delta\text{‘}\gamma . Y \epsilon \epsilon_\Delta\text{‘}S\text{‘‘}\gamma .$

$Q = Y \downarrow X . P = (Y \| \breve{X}) | A | \breve{S}_\times\} .$

$\supset . T \epsilon 1 \to \mathrm{Cls} . \text{Ɑ}\text{‘}T = \epsilon_\Delta\text{‘}\gamma \times \epsilon_\Delta\text{‘}S\text{‘‘}\gamma . \mathrm{D}\text{‘}T \subset \epsilon_\Delta\text{‘}\mathrm{D}\text{‘}S_\times$

The relation T here defined is the correlator required for proving

$$\epsilon_\Delta\text{‘}\hat{\mu}\{(\exists \alpha) . \alpha \epsilon \gamma . \mu = \alpha \times S\text{‘}\alpha\} \,\mathrm{sm}\, \epsilon_\Delta\text{‘}\gamma \times \epsilon_\Delta\text{‘}S\text{‘‘}\gamma.$$

Besides what is proved in the present proposition, we shall have to prove

$$T \epsilon \mathrm{Cls} \to 1 . \epsilon_\Delta\text{‘}\mathrm{D}\text{‘}S_\times \subset \mathrm{D}\text{‘}T.$$

The proof of the present proposition is as follows.

Dem.

$\vdash . *21{\cdot}33 . *13{\cdot}171 . \supset \vdash :. \mathrm{Hp} . \supset :$

$PTQ . P'TQ . \supset . (\exists X, Y, X', Y') . Y \downarrow X = Y' \downarrow X' . P = (Y \| \breve{X}) | A | \breve{S}_\times .$

$P' = (Y' \| \breve{X}') | A | \breve{S}_\times .$

$[*55{\cdot}202] \qquad \supset . P = P'$ (1)

$\vdash . *21{\cdot}33 . *114{\cdot}603 . \supset \vdash :. \mathrm{Hp} . \supset : PTQ . \supset . P \epsilon \epsilon_\Delta\text{‘}\mathrm{D}\text{‘}S_\times$ (2)

$\vdash . (1) . (2) . *113{\cdot}101 . \supset \vdash . \mathrm{Prop}$

***114·605.** $\vdash : \mathrm{Hp} *114{\cdot}604 \,.\supset .\, T \in \mathrm{Cls} \to 1$

Dem.

$\vdash . *114{\cdot}601 \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, S_\times \in 1 \to 1$ (1)

$\vdash . (1) . *74{\cdot}71 . *114{\cdot}601{\cdot}602 \,.\supset$

$\vdash :. \mathrm{Hp} \,.\, X, X' \in \epsilon_\Delta{}^\backprime\gamma \,.\, Y, Y' \in \epsilon_\Delta{}^\backprime S^{\backprime\backprime}\gamma \,.\, (Y \| \breve{X}) | A | \breve{S}_\times = (Y' \| \breve{X}') | A | \breve{S}_\times \,.\supset :$

$(Y \| \breve{X}) | A = (Y' \| \breve{X}') | A :$

[*74·7] $\supset : (Y \| \breve{X}) \upharpoonright \mathrm{D}{}^\backprime A = (Y' \| \breve{X}') \upharpoonright \mathrm{D}{}^\backprime A :$

[*114·602] $\supset : \alpha \in \gamma \,.\supset_\alpha .\, (Y \| \breve{X}){}^\backprime(S{}^\backprime\alpha) \downarrow \alpha = (Y' \| \breve{X}'){}^\backprime(S{}^\backprime\alpha) \downarrow \alpha \,.$

[*113·123] $\supset_\alpha .\, (Y{}^\backprime S{}^\backprime\alpha) \downarrow (X{}^\backprime\alpha) = (Y'{}^\backprime S{}^\backprime\alpha) \downarrow (X'{}^\backprime\alpha) \,.$

[*55·202] $\supset_\alpha .\, X{}^\backprime\alpha = X'{}^\backprime\alpha \,.\, Y{}^\backprime S{}^\backprime\alpha = Y'{}^\backprime S{}^\backprime\alpha :$

[*80·14.*33·45] $\supset : X = X' \,.\, Y = Y' :$

[*55·202] $\supset : Y \downarrow X = Y' \downarrow X'$ (2)

$\vdash . (2) . *13{\cdot}22 . *21{\cdot}33 \,.\supset \vdash :. \mathrm{Hp} \,.\supset : PTQ \,.\, PTQ' \,.\supset .\, Q = Q' :. \supset \vdash . \mathrm{Prop}$

The following propositions are required for proving that, with the same hypothesis, $\epsilon_\Delta{}^\backprime \mathrm{D}{}^\backprime S_\times \subset \mathrm{D}{}^\backprime T$.

***114·61.** $\vdash : \mathrm{Hp} *114{\cdot}602 \,.\, P \in \epsilon_\Delta{}^\backprime \mathrm{D}{}^\backprime S_\times \,.\, X = \breve{\iota} | \text{Ꮯ} | P | S_\times \,.\, Y = \breve{\iota} | \mathrm{D} | P | S_\times | \breve{S} \,.\supset .$
$X \in \epsilon_\Delta{}^\backprime\gamma \,.\, Y \in \epsilon_\Delta{}^\backprime S^{\backprime\backprime}\gamma$

Dem.

$\vdash . *72{\cdot}181{\cdot}13{\cdot}131 . *80{\cdot}14 . *114{\cdot}601 \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, X, Y \in 1 \to \mathrm{Cls}$ (1)

$\vdash . *72{\cdot}2{\cdot}181{\cdot}13{\cdot}131 . *80{\cdot}14 . *114{\cdot}601 \,.\supset$

$\vdash :. \mathrm{Hp} \,.\supset : xX\alpha \,.\equiv .\, x = \breve{\iota}{}^\backprime \text{Ꮯ}{}^\backprime P{}^\backprime S_\times{}^\backprime\alpha \,.$ (2)

[*51·53] $\supset .\, x \in \text{Ꮯ}{}^\backprime P{}^\backprime S_\times{}^\backprime\alpha \,.$

[*83·2.*114·601] $\supset .\, (\exists R) \,.\, R \in \alpha \times S{}^\backprime\alpha \,.\, x \in \text{Ꮯ}{}^\backprime R \,.$

[*113·142] $\supset .\, x \in \alpha$ (3)

$\vdash . *114{\cdot}601 \,.\supset \vdash :. \mathrm{Hp} \,.\supset : \alpha \in \gamma \,.\equiv .\, S_\times{}^\backprime\alpha \in \mathrm{D}{}^\backprime S_\times \,.$

[*83·2] $\equiv .\, \mathrm{E}\,!\, P{}^\backprime S_\times{}^\backprime\alpha$ (4)

$\vdash . *83{\cdot}2 \,.\quad \supset \vdash :. \mathrm{Hp} \,.\supset : \mathrm{E}\,!\, P{}^\backprime S_\times{}^\backprime\alpha \,.\equiv .\, P{}^\backprime S_\times{}^\backprime\alpha \in S_\times{}^\backprime\alpha \,.$

[*113·142] $\supset .\, \text{Ꮯ}{}^\backprime P{}^\backprime S_\times{}^\backprime\alpha \in 1 \,.$

[*52·15] $\supset .\, \mathrm{E}\,!\, \breve{\iota}{}^\backprime \text{Ꮯ}{}^\backprime P{}^\backprime S^{\times}{}^\backprime\alpha$ (5)

$\vdash . (2) . (4) . (5) \,.\quad \supset \vdash : \mathrm{Hp} \,.\supset .\, \gamma \subset \text{Ꮯ}{}^\backprime X$ (6)

$\vdash . *34{\cdot}36 . *114{\cdot}601 \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, \text{Ꮯ}{}^\backprime X \subset \gamma$ (7)

$\vdash . (1) . (3) . (6) . (7) \,.\quad \supset \vdash : \mathrm{Hp} \,.\supset .\, X \in \epsilon_\Delta{}^\backprime\gamma$ (8)

Similarly $\quad \vdash : \mathrm{Hp} \,.\supset .\, Y \in \epsilon_\Delta{}^\backprime S^{\backprime\backprime}\gamma$ (9)

$\vdash . (8) . (9) \,.\supset \vdash . \mathrm{Prop}$

***114·611.** $\vdash :. \mathrm{Hp} *114{\cdot}61 \,.\supset : \alpha \in \gamma \,.\supset .\, (Y{}^\backprime S{}^\backprime\alpha) \downarrow (X{}^\backprime\alpha) = P{}^\backprime S_\times{}^\backprime\alpha$

Dem.

$\vdash . *72{\cdot}2 \,.\supset \vdash : \mathrm{Hp} \,.\, \alpha \in \gamma \,.\supset .\, X{}^\backprime\alpha = \breve{\iota}{}^\backprime \text{Ꮯ}{}^\backprime P{}^\backprime S_\times{}^\backprime\alpha \,.\, Y{}^\backprime S{}^\backprime\alpha = \breve{\iota}{}^\backprime \mathrm{D}{}^\backprime P{}^\backprime S_\times{}^\backprime\alpha \,.$

[*55·16.*51·51] $\supset .\, (Y{}^\backprime S{}^\backprime\alpha) \downarrow (X{}^\backprime\alpha) = P{}^\backprime S_\times{}^\backprime\alpha : \supset \vdash . \mathrm{Prop}$

∗114·612. $\vdash : \mathrm{Hp}\, {*}114{\cdot}61 . \supset . (Y \| \breve{X}) \,|\, A \,|\, \breve{S}_\times = P$

Dem.

$\vdash . {*}83{\cdot}15 . \supset \vdash : \mathrm{Hp} . \dot{\exists} \,!\, P . \supset . \exists \,!\, \mathrm{D}\text{‘}S_\times .$

$[{*}114{\cdot}601] \qquad \supset . \exists \,!\, \gamma \qquad (1)$

$\vdash . {*}34{\cdot}1 . \supset \vdash :. \mathrm{Hp} . \supset : M \{(Y \| \breve{X}) \,|\, A \,|\, \breve{S}_\times\} \mu . \equiv .$

$(\exists Q, \alpha) . M = (Y \| \breve{X})\text{‘}Q . QA\alpha . \mu = S_\times\text{‘}\alpha .$

$[{*}114{\cdot}601{\cdot}602] \qquad \equiv . (\exists \alpha) . M = (Y \| \breve{X})\text{‘}(S\text{‘}\alpha) \downarrow \alpha . \mu = S_\times\text{‘}\alpha . \alpha \epsilon \gamma .$

$[{*}113{\cdot}123] \qquad \equiv . (\exists \alpha) . M = (Y\text{‘}S\text{‘}\alpha) \downarrow (X\text{‘}\alpha) . \mu = S_\times\text{‘}\alpha . \alpha \epsilon \gamma .$

$[{*}114{\cdot}611] \qquad \equiv . (\exists \alpha) . M = P\text{‘}S_\times\text{‘}\alpha . \mu = S_\times\text{‘}\alpha . \alpha \epsilon \gamma .$

$[{*}13{\cdot}193 . {*}114{\cdot}601 . {*}71{\cdot}16] \equiv . M = P\text{‘}\mu . \exists \,!\, \gamma .$

$[{*}71{\cdot}36 . {*}80{\cdot}14 . (1)] \qquad \equiv . MP\mu :. \supset \vdash . \mathrm{Prop}$

∗114·613. $\vdash : \mathrm{Hp}\, {*}114{\cdot}61 . \mathrm{Hp}\, {*}114{\cdot}604 . \supset .$

$P = T\text{‘}(Y \downarrow X) . (Y \downarrow X) \epsilon \epsilon_\Delta\text{‘}\gamma \times \epsilon_\Delta\text{‘}S\text{‘‘}\gamma$

Dem.

$\vdash . {*}21{\cdot}33 . {*}114{\cdot}604 . \supset \vdash :. \mathrm{Hp}\, {*}114{\cdot}604 . \supset :$

$X \epsilon \epsilon_\Delta\text{‘}\gamma . Y \epsilon \epsilon_\Delta\text{‘}S\text{‘‘}\gamma . \supset . T\text{‘}(Y \downarrow X) = (Y \| \breve{X}) \,|\, A \,|\, \breve{S}_\times \quad (1)$

$\vdash . (1) . {*}114{\cdot}61{\cdot}612 . {*}113{\cdot}106 . \supset \vdash . \mathrm{Prop}$

∗114·614. $\vdash : \mathrm{Hp}\, {*}114{\cdot}604 . \supset . \epsilon_\Delta\text{‘}\mathrm{D}\text{‘}S_\times \subset \mathrm{D}\text{‘}T$

Dem.

$\vdash . {*}114{\cdot}613 . \supset \vdash :. \mathrm{Hp} . \supset : P \epsilon \epsilon_\Delta\text{‘}\mathrm{D}\text{‘}S_\times . \supset . (\exists Q) . P = T\text{‘}Q .$

$[{*}33{\cdot}43] \qquad \supset . P \epsilon \mathrm{D}\text{‘}T :. \supset \vdash . \mathrm{Prop}$

∗114·62. $\vdash : \mathrm{Hp}\, {*}114{\cdot}604 . \supset . T \epsilon 1 \to 1 . \mathrm{D}\text{‘}T = \epsilon_\Delta\text{‘}\mathrm{D}\text{‘}S_\times . \mathrm{G}\text{‘}T = \epsilon_\Delta\text{‘}\gamma \times \epsilon_\Delta\text{‘}S\text{‘‘}\gamma$

$[{*}114{\cdot}604{\cdot}605{\cdot}614]$

∗114·621. $\vdash : S \restriction \gamma \epsilon 1 \to 1 . \gamma \subset \mathrm{G}\text{‘}S . \Lambda \sim\epsilon \gamma . \supset .$

$\epsilon_\Delta\text{‘}\hat{\mu}\{(\exists \alpha) . \alpha \epsilon \gamma . \mu = \alpha \times S\text{‘}\alpha\} \,\mathrm{sm}\, \epsilon_\Delta\text{‘}\gamma \times \epsilon_\Delta\text{‘}S\text{‘‘}\gamma$

$[{*}114{\cdot}62{\cdot}601]$

The hypothesis $\Lambda \sim\epsilon \gamma$ is not necessary, since, when $\Lambda \epsilon \gamma$,

$$\epsilon_\Delta\text{‘}\hat{\mu}\{(\exists \alpha) . \alpha \epsilon \gamma . \mu = \alpha \times S\text{‘}\alpha\} \text{ and } \epsilon_\Delta\text{‘}\gamma \times \epsilon_\Delta\text{‘}S\text{‘‘}\gamma$$

are both Λ. This is proved in ∗114·63.

∗114·63. $\vdash : S \restriction \gamma \epsilon 1 \to 1 . \gamma \subset \mathrm{G}\text{‘}S . \supset .$

$\epsilon_\Delta\text{‘}\hat{\mu}\{(\exists \alpha) . \alpha \epsilon \gamma . \mu = \alpha \times S\text{‘}\alpha\} \,\mathrm{sm}\, \epsilon_\Delta\text{‘}\gamma \times \epsilon_\Delta\text{‘}S\text{‘‘}\gamma$

Dem.

$\vdash . {*}10{\cdot}24 . {*}83{\cdot}11 . \supset$

$\vdash : \mathrm{Hp} . \Lambda \epsilon \gamma . \supset . \Lambda \times S\text{‘}\Lambda \epsilon \hat{\mu}\{(\exists \alpha) . \alpha \epsilon \gamma . \mu = \alpha \times S\text{‘}\alpha\} . \epsilon_\Delta\text{‘}\gamma = \Lambda .$

$[{*}113{\cdot}114] \qquad \supset . \Lambda \epsilon \hat{\mu}\{(\exists \alpha) . \alpha \epsilon \gamma . \mu = \alpha \times S\text{‘}\alpha\} . \epsilon_\Delta\text{‘}\gamma \times \epsilon_\Delta\text{‘}S\text{‘‘}\gamma = \Lambda .$

$[{*}83{\cdot}11] \qquad \supset . \epsilon_\Delta\text{‘}\hat{\mu}\{(\exists \alpha) . \alpha \epsilon \gamma . \mu = \alpha \times S\text{‘}\alpha\} = \Lambda . \epsilon_\Delta\text{‘}\gamma \times \epsilon_\Delta\text{‘}S\text{‘‘}\gamma = \Lambda \quad (1)$

$\vdash . (1) . {*}73{\cdot}47 . {*}114{\cdot}621 . \supset \vdash . \mathrm{Prop}$

The above is one of the two variants of the associative law for ϵ_Δ and $\times$.

***114·631.** $\vdash . \epsilon_\Delta{}^{\backprime}\hat{\mu}\{(\exists\alpha) . \alpha \in \gamma . \mu = \alpha \times \alpha\} \operatorname{sm} \epsilon_\Delta{}^{\backprime}\alpha \times \epsilon_\Delta{}^{\backprime}\alpha$ $\quad \left[*114·63 \dfrac{I}{S}\right]$

***114·632.** $\vdash : S \upharpoonright \gamma \in 1 \rightarrow 1 . \gamma \subset \text{Ɑ}{}^{\backprime}S . \gamma \cap S^{\backprime\backprime}\gamma = \Lambda . \supset .$

$\epsilon_\Delta{}^{\backprime}\hat{\mu}\{(\exists\alpha) . \alpha \in \gamma . \mu = \alpha \times S^{\backprime}\alpha\} \operatorname{sm} \epsilon_\Delta{}^{\backprime}(\gamma \cup S^{\backprime\backprime}\gamma)$ [*114·63·301]

This is the second variant of the associative law for ϵ_Δ and $\times$.

***114·64.** $\vdash : (R^{\backprime\backprime}\gamma) \rceil R, S \upharpoonright \gamma \in 1 \rightarrow 1 . \gamma \subset \text{Ɑ}{}^{\backprime}R . \gamma \subset \text{Ɑ}{}^{\backprime}S . \supset .$

$\epsilon_\Delta{}^{\backprime}R^{\backprime\backprime}\gamma \times \epsilon_\Delta{}^{\backprime}S^{\backprime\backprime}\gamma \operatorname{sm} \epsilon_\Delta{}^{\backprime}\hat{\mu}\{(\exists z) . z \in \gamma . \mu = R^{\backprime}z \times S^{\backprime}z\}$

Dem.

$\vdash . *114·63 \dfrac{S \mid \breve{R}, R^{\backprime\backprime}\gamma}{S, \quad \gamma} . \supset$

$\vdash : S \mid \breve{R} \upharpoonright R^{\backprime\backprime}\gamma \in 1 \rightarrow 1 . R^{\backprime\backprime}\gamma \subset \text{Ɑ}{}^{\backprime}(S \mid \breve{R}) . \supset .$

$\epsilon_\Delta{}^{\backprime}R^{\backprime\backprime}\gamma \times \epsilon_\Delta{}^{\backprime}S^{\backprime\backprime}\breve{R}^{\backprime\backprime}R^{\backprime\backprime}\gamma \operatorname{sm} \epsilon_\Delta{}^{\backprime}\hat{\mu}\{(\exists\alpha) . \alpha \in R^{\backprime\backprime}\gamma . \mu = \alpha \times (S \mid \breve{R})^{\backprime}\alpha\}$ (1)

$\vdash . *74·14 . *35·354 . \supset \vdash : \text{Hp} . \supset . S \mid \breve{R} \upharpoonright R^{\backprime\backprime}\gamma = S \upharpoonright \gamma \mid \gamma \rceil \breve{R} . \breve{R} \upharpoonright R^{\backprime\backprime}\gamma = \gamma \rceil \breve{R} .$

[*71·252] $\supset . S \mid \breve{R} \upharpoonright R^{\backprime\backprime}\gamma \in 1 \rightarrow 1$ (2)

$\vdash . *37·2 . \quad \supset \vdash : \text{Hp} . \supset . R^{\backprime\backprime}\gamma \subset R^{\backprime\backprime}\text{Ɑ}{}^{\backprime}S .$

[*37·32] $\supset . R^{\backprime\backprime}\gamma \subset \text{Ɑ}{}^{\backprime}(S \mid \breve{R})$ (3)

$\vdash . *74·171 . \quad \supset \vdash : \text{Hp} . \supset . \breve{R}^{\backprime\backprime}R^{\backprime\backprime}\gamma = \gamma$ (4)

$\vdash . (4) . *74·14 . \supset \vdash : \text{Hp} . \supset . (R^{\backprime\backprime}\gamma) \rceil R = R \upharpoonright \gamma .$

[*35·7.*71·4] $\supset . \hat{\mu}\{(\exists\alpha) . \alpha \in R^{\backprime\backprime}\gamma . \mu = \alpha \times (S \mid \breve{R})^{\backprime}\alpha\}$

$= \hat{\mu}\{(\exists z) . z \in \gamma . \mu = R^{\backprime}z \times S^{\backprime}\breve{R}^{\backprime}R^{\backprime}z\}$

[*74·53] $= \hat{\mu}\{(\exists z) . z \in \gamma . \mu = R^{\backprime}z \times S^{\backprime}z\}$ (5)

$\vdash . (1) . (2) . (3) . (4) . (5) . \supset \vdash .$ Prop

In the above proposition, the hypothesis has to be such as to yield $\breve{R}^{\backprime\backprime}R^{\backprime\backprime}\gamma = \gamma$. Various other forms of hypothesis will secure this result, and will give other forms of the above proposition. This subject is treated in *74, above.

***114·65.** $\vdash :. (R^{\backprime\backprime}\gamma) \rceil R, S \upharpoonright \gamma \in 1 \rightarrow 1 . \gamma \subset \text{Ɑ}{}^{\backprime}R . \gamma \subset \text{Ɑ}{}^{\backprime}S . R^{\backprime\backprime}\gamma \cap S^{\backprime\backprime}\gamma = \Lambda . \supset .$

$\epsilon_\Delta{}^{\backprime}(R^{\backprime\backprime}\gamma \cup S^{\backprime\backprime}\gamma) \operatorname{sm} \epsilon_\Delta{}^{\backprime}\hat{\mu}\{(\exists z) . z \in \gamma . \mu = R^{\backprime}z \times S^{\backprime}z\}$

[*114·64·301]

$*115$. MULTIPLICATIVE CLASSES AND ARITHMETICAL CLASSES

Summary of $*115$.

Whenever κ is a class of mutually exclusive classes, $\epsilon_\Delta{}^\backprime\kappa$ is similar to $\mathrm{D}{}^{\backprime\backprime}\epsilon_\Delta{}^\backprime\kappa$; hence

$$\Pi\mathrm{Nc}{}^\backprime\kappa = \mathrm{Nc}{}^\backprime\mathrm{D}{}^{\backprime\backprime}\epsilon_\Delta{}^\backprime\kappa.$$

Now $\mathrm{D}{}^{\backprime\backprime}\epsilon_\Delta{}^\backprime\kappa$ is of the same type as κ; and when κ is a class of mutually exclusive classes, $\mathrm{D}{}^{\backprime\backprime}\epsilon_\Delta{}^\backprime\kappa$ consists of all classes formed by selecting one representative from each member of κ. It often happens that $\mathrm{D}{}^{\backprime\backprime}\epsilon_\Delta{}^\backprime\kappa$ is easier to deal with than $\epsilon_\Delta{}^\backprime\kappa$; hence when possible (*i.e.* when $\kappa \in \mathrm{Cls}^2\,\mathrm{excl}$), it is convenient to use $\mathrm{D}{}^{\backprime\backprime}\epsilon_\Delta{}^\backprime\kappa$, rather than $\epsilon_\Delta{}^\backprime\kappa$, as the standard member of $\Pi\mathrm{Nc}{}^\backprime\kappa$. We therefore put

$$\mathrm{Prod}{}^\backprime\kappa = \mathrm{D}{}^{\backprime\backprime}\epsilon_\Delta{}^\backprime\kappa \quad \mathrm{Df.}$$

We shall call $\mathrm{Prod}{}^\backprime\kappa$ the "multiplicative class" of κ.

The associative law,

$$\mathrm{Prod}{}^\backprime s{}^\backprime\kappa \;\mathrm{sm}\; \mathrm{Prod}{}^\backprime\mathrm{Prod}{}^{\backprime\backprime}\kappa,$$

requires not merely $\kappa \in \mathrm{Cls}^2\,\mathrm{excl}$, but also $s{}^\backprime\kappa \in \mathrm{Cls}^2\,\mathrm{excl}$. The combination of these two hypotheses gives a completely disjointed class of classes of classes, *i.e.* a class of classes of classes κ which can be obtained by dividing a given class ($s{}^\backprime s{}^\backprime\kappa$) into mutually exclusive portions, and then dividing each of those portions into mutually exclusive portions. For example, take a square (a class of points) and divide it by horizontal lines, and then divide each of the resulting rectangles by vertical lines; then the resulting rows of little rectangles form such a class, each row of rectangles being one member of the class. Such a class we call an "arithmetical" class, and denote by "Cls^3 arithm."

The present number is concerned with the properties of multiplicative classes and arithmetical classes. Some of these properties will be useful in dealing with exponentiation.

The present number begins with various propositions concerning $\mathrm{Prod}{}^\backprime\kappa$ which are merely repetitions of previous propositions of $*83$, $*84$, $*85$ or $*113$. Thus we have

$*115{\cdot}141$. $\vdash : \exists ! \,\mathrm{Prod}{}^\backprime\kappa \,.\supset .\, s{}^\backprime\mathrm{Prod}{}^\backprime\kappa = s{}^\backprime\kappa$ by $*83{\cdot}66$,

$*115{\cdot}142$. $\vdash .\, \mathrm{Prod}{}^\backprime\iota{}^\backprime\alpha = \iota{}^{\backprime\backprime}\alpha$ by $*83{\cdot}7$,

$*115{\cdot}143$. $\vdash .\, \mathrm{Prod}{}^\backprime\iota{}^{\backprime\backprime}\alpha = \iota{}^\backprime\alpha$ by $*83{\cdot}71$,

$*115{\cdot}16$. $\vdash .\, \kappa \in \mathrm{Cls}^2\,\mathrm{excl} \,.\supset .\, \mathrm{Prod}{}^\backprime\kappa \subset \mathrm{Nc}{}^\backprime\kappa$ by $*100{\cdot}64$,

and various other properties.

We then proceed to consider Cls^3 arithm. We prove

***115·22.** $\vdash :. \kappa \in \mathrm{Cls}^3 \text{arithm} . \supset : s``\kappa \in \mathrm{Cls}^2 \text{excl} : \alpha, \beta \in \kappa . \exists ! s`\alpha \cap s`\beta . \supset_{\alpha,\beta} . \alpha = \beta$

and *115·23 gives a similar proposition substituting "Prod" for s.

After a few more propositions on Cls^3 arithm, we proceed to the associative law for Prod (*115·34), *i.e.*

$$\vdash : \kappa \in \mathrm{Cls}^3 \text{arithm} . \supset . \text{Prod}`\text{Prod}``\kappa \text{ sm Prod}`s`\kappa.$$

(This proposition, *115·34, also states that, with the same hypothesis, $\text{Prod}`s`\kappa \text{ sm } \epsilon_\Delta`s`\kappa$.) Hence we have

***115·35.** $\vdash : \kappa \in \mathrm{Cls}^3 \text{arithm} . \supset . \text{Nc}`\text{Prod}`\text{Prod}``\kappa = \text{Nc}`\text{Prod}`s`\kappa = \Pi \text{Nc}`\text{Prod}``\kappa$
$$= \Pi \text{Nc}`\epsilon_\Delta``\kappa = \Pi \text{Nc}`s`\kappa$$

We have also

***115·42.** $\vdash : \kappa \in \mathrm{Cls}^3 \text{arithm} . \supset . \text{Prod}`\text{Prod}``\kappa = \text{D}```\text{Prod}`\epsilon_\Delta``\kappa = \text{D}```\text{D}``\epsilon_\Delta`\epsilon_\Delta``\kappa$

***115·44.** $\vdash : \kappa \in \mathrm{Cls}^3 \text{arithm} . \supset . \text{Prod}`s`\kappa = s``\text{Prod}`\text{Prod}``\kappa$

We have next to prove that if two classes of classes have double similarity, so have their multiplicative classes. The proof is simple, since the double correlator is the same as for the original classes, *i.e.*

***115·502.** $\vdash : T \upharpoonright s`\lambda \in \kappa \,\overline{\text{sm}}\,\overline{\text{sm}}\, \lambda . \supset . T \upharpoonright s`\text{Prod}`\lambda \in (\text{Prod}`\kappa) \,\overline{\text{sm}}\,\overline{\text{sm}}\, (\text{Prod}`\lambda)$

whence

***115·51.** $\vdash : \kappa \text{ sm sm } \lambda . \supset . \text{Prod}`\kappa \text{ sm sm Prod}`\lambda$

The number ends with some propositions which result from *114·64·65 and are analogous to them. One of these is used in the following number in proving $\mu^\varpi \times_c \nu^\varpi = (\mu \times_c \nu)^\varpi$, namely,

***115·6.** $\vdash : (R``\gamma) \mathbin{\rceil} R, S \upharpoonright \gamma \in 1 \to 1 . \gamma \subset \text{ᗡ}`R . \gamma \subset \text{ᗡ}`S . R``\gamma, S``\gamma \in \mathrm{Cls}^2 \text{excl} . \supset .$
$$\text{Prod}`R``\gamma \times \text{Prod}`S``\gamma \text{ sm } \epsilon_\Delta`\hat{\mu}\{(\exists z) . z \in \gamma . \mu = R`z \times S`z\}$$

The subject of this number will be useful in dealing with exponentiation, since we shall define μ^ν by means of $\text{Prod}`\alpha \underset{,,}{\downarrow} ``\beta$, where $\mu = \text{N}_0\text{c}`\alpha$ and $\nu = \text{N}_0\text{c}`\beta$.

***115·01.** $\text{Prod}`\kappa = \text{D}``\epsilon_\Delta`\kappa$ Df

***115·02.** $\mathrm{Cls}^3 \text{arithm} = \hat{\kappa}(\kappa, s`\kappa \in \mathrm{Cls}^2 \text{excl})$ Df

***115·1.** $\vdash . \text{Prod}`\kappa = \text{D}``\epsilon_\Delta`\kappa$ [(*115·01)]

***115·101.** $\vdash :. \alpha \in \kappa . \supset_\alpha . \varpi \cap \alpha \in 1 : \varpi \subset s`\kappa : \supset . \varpi \in \text{Prod}`\kappa$ [*84·411]

***115·11.** $\vdash :: \kappa \in \mathrm{Cls}^2 \text{excl} . \supset :. \varpi \in \text{Prod}`\kappa . \equiv : \alpha \in \kappa . \supset_\alpha . \varpi \cap \alpha \in 1 : \varpi \subset s`\kappa$ [*84·412]

Owing to this proposition, $\text{Prod}`\kappa$ can be treated without any reference to $\epsilon_\Delta`\kappa$ whenever $\kappa \in \mathrm{Cls}^2$ excl.

$*115\cdot12$. $\vdash : \kappa \in \text{Cls}^2 \text{excl} . \supset . \text{Prod}'\kappa \in \Pi\text{Nc}'\kappa . \text{Prod}'\kappa \text{ sm } \epsilon_\Delta{}'\kappa$ [$*84\cdot41$]

It is this proposition that makes the notation $\text{Prod}'\kappa$ appropriate for the multiplicative class.

$*115\cdot13$. $\vdash : \alpha \cap \beta = \Lambda . \supset . \text{Prod}'(\iota'\alpha \cup \iota'\beta) \text{ sm } (\alpha \times \beta)$ [$*113\cdot152$]

$*115\cdot131$. $\vdash : \alpha \neq \beta . \supset . \text{Prod}'(\iota'\alpha \cup \iota'\beta) = C``(\alpha \times \beta)$ [$*113\cdot151$]

$*115\cdot14$. $\vdash :. \kappa \cap \lambda = \Lambda . \vee . s'\kappa \cap s'\lambda = \Lambda : \supset :$

$\varpi \in \text{Prod}'(\kappa \cup \lambda) . \equiv . (\exists \rho, \sigma) . \rho \in \text{Prod}'\kappa . \sigma \in \text{Prod}'\lambda . \varpi = \rho \cup \sigma$ [$*83\cdot64\cdot641$]

$*115\cdot141$. $\vdash : \exists ! \text{Prod}'\kappa . \supset . s'\text{Prod}'\kappa = s'\kappa$ [$*83\cdot66$]

$*115\cdot142$. $\vdash . \text{Prod}'\iota'\alpha = \iota``\alpha$ [$*83\cdot7$]

$*115\cdot143$. $\vdash . \text{Prod}'\iota``\alpha = \iota'\alpha$ [$*83\cdot71$]

$*115\cdot144$. $\vdash : \kappa \subset 1 . \supset . \text{Prod}'\kappa = \iota's'\kappa$ [$*83\cdot72$]

$*115\cdot145$. $\vdash :. \kappa \in \text{Cls}^2 \text{excl} . \alpha \in \kappa . \mu \cap \alpha \in 1 . \supset : \mu - \alpha \in \text{Prod}'(\kappa - \iota'\alpha) . \equiv . \mu \in \text{Prod}'\kappa$ [$*84\cdot422$]

$*115\cdot15$. $\vdash :. \kappa, \lambda \in \text{Cls}^2 \text{excl} . s'\kappa = s'\lambda . \supset : \kappa \subset \text{Prod}'\lambda . \equiv . \lambda \subset \text{Prod}'\kappa$ [$*84\cdot43$]

$*115\cdot151$. $\vdash : \kappa \in \text{Cls}^2 \text{excl} . \supset . \epsilon_\Delta{}'s'\kappa = \dot{s}``\text{Prod}'\epsilon_\Delta``\kappa$ [$*85\cdot28$]

$*115\cdot152$. $\vdash . P_\Delta{}'\alpha \text{ sm Prod}'P \downarrow``\alpha$ [$*85\cdot55$]

$*115\cdot153$. $\vdash . \epsilon_\Delta{}'\kappa \text{ sm Prod}'\epsilon \downarrow``\kappa$ [$*115\cdot152$]

$*115\cdot154$. $\vdash . \text{Prod}'\epsilon \downarrow``\kappa \in \Pi\text{Nc}'\kappa$ [$*115\cdot153$]

$*115\cdot16$. $\vdash : \kappa \in \text{Cls}^2 \text{excl} . \supset . \text{Prod}'\kappa \subset \text{Nc}'\kappa$ [$*100\cdot64$]

The following proposition is used in the theory of well-ordered series ($*250\cdot5$).

$*115\cdot17$. $\vdash : \exists ! \epsilon_\Delta{}'\text{Cl ex}'\alpha . \supset . \text{Prod}'\text{Cl ex}'\alpha = \iota'\alpha$

Dem.

$\vdash . *80\cdot14 . *115\cdot1 . *37\cdot45 . \supset \vdash : \text{Hp} . \supset . \exists ! \text{Prod}'\text{Cl ex}'\alpha$ (1)

$\vdash . *60\cdot61 . \text{Fact} . \supset$

$\vdash :. R \in 1 \to \text{Cls} . R \Subset \epsilon . \text{G}'R = \text{Cl ex}'\alpha . \supset : R \in 1 \to \text{Cls} . R \Subset \epsilon . \iota``\alpha \subset \text{G}'R :$

[$*51\cdot15$] $\supset : x \in \alpha . \supset_x . xR(\iota'x) :$

[$*33\cdot14$] $\supset : \alpha \subset \text{D}'R$ (2)

$\vdash . *83\cdot21 . \supset \vdash : \text{Hp}(2) . \supset . \text{D}'R \subset s'\text{Cl ex}'\alpha .$

[$*60\cdot501$] $\supset . \text{D}'R \subset \alpha$ (3)

$\vdash . (2) . (3) . \supset \vdash : R \in 1 \to \text{Cls} . R \Subset \epsilon . \text{G}'R = \text{Cl ex}'\alpha . \supset . \text{D}'R = \alpha$ (4)

$\vdash . (4) . *115\cdot1 . *80\cdot14 . \supset \vdash . \text{Prod}'\text{Cl ex}'\alpha \subset \iota'\alpha$ (5)

$\vdash . (1) . (5) . *51\cdot4 . \supset \vdash . \text{Prop}$

∗115·18. $\vdash . t\text{‘}\mathrm{Prod}\text{‘}\kappa = t\text{‘}\kappa$ [∗83·81]

∗115·2. $\vdash : \kappa \in \mathrm{Cls}^3\,\mathrm{arithm} . \equiv . \kappa, s\text{‘}\kappa \in \mathrm{Cls}^2\,\mathrm{excl}$ [(∗115·02)]

∗115·21. $\vdash :. \kappa \in \mathrm{Cls}^3\,\mathrm{arithm} . \equiv : \alpha, \beta \in \kappa . \exists ! \alpha \cap \beta . \supset_{\alpha,\beta} . \alpha = \beta :$

$\alpha, \beta \in \kappa . \rho \in \alpha . \sigma \in \beta . \exists ! \rho \cap \sigma . \supset_{\alpha,\beta,\rho,\sigma} . \rho =$

[∗115·2 . ∗84·11]

∗115·211. $\vdash : \kappa \in \mathrm{Cls}^3\,\mathrm{arithm} . \alpha, \beta \in \kappa . \rho \in \alpha . \sigma \in \beta . \exists ! \rho \cap \sigma . \supset . \alpha = \beta$

Dem.

$\vdash . *115{\cdot}21 . \supset \vdash : \mathrm{Hp} . \supset . \rho = \sigma . \rho \in \alpha . \sigma \in \beta .$

[∗13·13] $\supset . \rho \in \alpha \cap \beta .$

[∗115·21] $\supset . \alpha = \beta : \supset \vdash . \mathrm{Prop}$

∗115·22. $\vdash :. \kappa \in \mathrm{Cls}^3\,\mathrm{arithm} . \supset : s\text{‘‘}\kappa \in \mathrm{Cls}^2\,\mathrm{excl} : \alpha, \beta \in \kappa . \exists ! s\text{‘}\alpha \cap s\text{‘}\beta . \supset_{\alpha,\beta} . \alpha = \beta$

Dem.

$\vdash . *40{\cdot}11 . \supset \vdash : \exists ! s\text{‘}\alpha \cap s\text{‘}\beta . \equiv . (\exists x, \rho, \sigma) . \rho \in \alpha . \sigma \in \beta . x \in \rho . x \in \sigma .$

[∗10·35] $\equiv . (\exists \rho, \sigma) . \rho \in \alpha . \sigma \in \beta . \exists ! \rho \cap \sigma$ (1)

$\vdash . (1) . *115{\cdot}211 . \supset$

$\vdash :. \mathrm{Hp} . \supset : \alpha, \beta \in \kappa . \exists ! s\text{‘}\alpha \cap s\text{‘}\beta . \supset . \alpha = \beta .$ (2)

[∗30·37] $\supset . s\text{‘}\alpha = s\text{‘}\beta$ (3)

$\vdash . (2) . (3) . *84{\cdot}11 . \supset \vdash . \mathrm{Prop}$

Observe that, although "$s\text{‘‘}\kappa \in \mathrm{Cls}^2\,\mathrm{excl}$" follows from

$$\text{“}\alpha, \beta \in \kappa . \exists ! s\text{‘}\alpha \cap s\text{‘}\beta . \supset_{\alpha,\beta} . \alpha = \beta,\text{”}$$

the converse implication does not hold. If there were two different classes α and β having the same sum, we might have $\exists ! s\text{‘}\alpha \cap s\text{‘}\beta$, *i.e.* $\exists ! s\text{‘}\alpha$, without having $\alpha = \beta$, in spite of "$s\text{‘‘}\kappa \in \mathrm{Cls}^2\,\mathrm{excl}$." In proofs, less use can be made of "$s\text{‘‘}\kappa \in \mathrm{Cls}^2\,\mathrm{excl}$" than of "$\alpha, \beta \in \kappa . \exists ! s\text{‘}\alpha \cap s\text{‘}\beta . \supset_{\alpha,\beta} . \alpha = \beta$." If $\Lambda \sim\in \kappa$ or $\iota\text{‘}\Lambda \sim\in \kappa$, the latter implies $s \upharpoonright \kappa \in 1 \to 1$.

∗115·23. $\vdash :. \kappa \in \mathrm{Cls}^3\,\mathrm{arithm} . \supset :$

$\mathrm{Prod}\text{‘‘}\kappa \in \mathrm{Cls}^2\,\mathrm{excl} : \alpha, \beta \in \kappa . \exists ! \mathrm{Prod}\text{‘}\alpha \cap \mathrm{Prod}\text{‘}\beta . \supset_{\alpha,\beta} . \alpha = \beta$

Dem.

$\vdash . *83{\cdot}62 . \quad \supset \vdash : \varpi \in \mathrm{Prod}\text{‘}\alpha \cap \mathrm{Prod}\text{‘}\beta . \supset . \varpi \subset s\text{‘}\alpha \cap s\text{‘}\beta$ (1)

$\vdash . (1) . *24{\cdot}58 . \quad \supset \vdash : \varpi \in \mathrm{Prod}\text{‘}\alpha \cap \mathrm{Prod}\text{‘}\beta . \exists ! \varpi . \supset . \exists ! s\text{‘}\alpha \cap s\text{‘}\beta$ (2)

$\vdash . (2) . *115{\cdot}22 . \quad \supset \vdash : \mathrm{Hp} . \alpha, \beta \in \kappa . \varpi \in \mathrm{Prod}\text{‘}\alpha \cap \mathrm{Prod}\text{‘}\beta . \exists ! \varpi . \supset . \alpha = \beta$ (3)

$\vdash . *83{\cdot}16 . \mathrm{Transp} . \supset \vdash : \Lambda \in \mathrm{Prod}\text{‘}\alpha \cap \mathrm{Prod}\text{‘}\beta . \supset . \alpha = \Lambda . \beta = \Lambda$ (4)

$\vdash . (3) . (4) . \supset \vdash :. \mathrm{Hp} . \supset : \alpha, \beta \in \kappa . \exists ! \mathrm{Prod}\text{‘}\alpha \cap \mathrm{Prod}\text{‘}\beta . \supset . \alpha = \beta .$ (5)

[∗30·37] $\supset . \mathrm{Prod}\text{‘}\alpha = \mathrm{Prod}\text{‘}\beta$ (6)

$\vdash . (5) . (6) . *84{\cdot}11 . \supset \vdash . \mathrm{Prop}$

∗115·24. $\vdash : \kappa \in \mathrm{Cls}^3\,\mathrm{arithm} . \equiv . \epsilon \upharpoonright \kappa, \epsilon \upharpoonright s\text{‘}\kappa \in \mathrm{Cls} \to 1$ [∗115·2 . ∗84·14]

∗115·25. $\vdash : \kappa \in \mathrm{Cls}^3\,\mathrm{arithm} . \supset . \epsilon_\Delta\text{‘}\kappa \subset 1 \to 1 . \epsilon_\Delta\text{‘}s\text{‘}\kappa \subset 1 \to 1$ [∗84·3 . ∗115·2]

***115·26.** $\vdash : \kappa \in \text{Cls}^3 \text{ arithm} . \supset .$

$$\epsilon_\Delta{}^\text{‘}s^\text{‘‘}\kappa \subset 1 \to 1 . \epsilon_\Delta{}^\text{‘}\epsilon_\Delta{}^\text{‘‘}\kappa \subset 1 \to 1 . \epsilon_\Delta{}^\text{‘}\text{Prod}^\text{‘‘}\kappa \subset 1 \to 1$$

[*84·3 . *115·22 . *84·55 . *115·23]

In the above proposition, $\epsilon_\Delta{}^\text{‘}\epsilon_\Delta{}^\text{‘‘}\kappa \subset 1 \to 1$ does not require the hypothesis $\kappa \in \text{Cls}^3$ arithm, being true always. It is merely included here for convenience of reference.

***115·27.** $\vdash : \kappa \in \text{Cls}^3 \text{ arithm} . \supset . \kappa \subset \text{Cls}^2 \text{ excl}$ [*115·2 . *84·25 . *40·13]

We have now to prove the associative law for "Prod," *i.e.*

$$\kappa \in \text{Cls}^3 \text{ arithm} . \supset . \text{Prod}^\text{‘}s^\text{‘}\kappa \text{ sm Prod}^\text{‘}\text{Prod}^\text{‘‘}\kappa.$$

In virtue of *115·12, we have only to prove (under the hypothesis)

$$\epsilon_\Delta{}^\text{‘}s^\text{‘}\kappa \text{ sm } \epsilon_\Delta{}^\text{‘}\text{Prod}^\text{‘‘}\kappa$$

which, by *85·44, will follow from

$$\epsilon_\Delta{}^\text{‘}\epsilon_\Delta{}^\text{‘‘}\kappa \text{ sm } \epsilon_\Delta{}^\text{‘}\text{Prod}^\text{‘‘}\kappa$$

which, by *114·52, will follow from

$$\epsilon_\Delta{}^\text{‘‘}\kappa \text{ sm sm Prod}^\text{‘‘}\kappa.$$

Now $\qquad \text{Prod}^\text{‘‘}\kappa = \text{D}_\epsilon{}^\text{‘‘}\epsilon_\Delta{}^\text{‘‘}\kappa.$

Thus the correlator which will give our proposition will be $\text{D} \upharpoonright s^\text{‘}\epsilon_\Delta{}^\text{‘‘}\kappa$. We have only to prove that this is a $1 \to 1$, and the rest follows.

***115·3.** $\vdash : \kappa \in \text{Cls}^3 \text{ arithm} . R, S \in s^\text{‘}\epsilon_\Delta{}^\text{‘‘}\kappa . \text{D}^\text{‘}R = \text{D}^\text{‘}S . \supset . R = S$

Dem.

$\vdash . *115·23 . \supset \vdash : \kappa \in \text{Cls}^3\text{arithm} . \alpha, \beta \in \kappa . R \in \epsilon_\Delta{}^\text{‘}\alpha . S \in \epsilon_\Delta{}^\text{‘}\beta . \text{D}^\text{‘}R = \text{D}^\text{‘}S . \supset . \alpha = \beta$ (1)

$\vdash . *115·27 . *84·4 . \supset \vdash : \kappa \in \text{Cls}^3\text{arithm} . \alpha \in \kappa . R, S \in \epsilon_\Delta{}^\text{‘}\alpha . \text{D}^\text{‘}R = \text{D}^\text{‘}S . \supset . R = S$ (2)

$\vdash . (1) . (2) . \supset \vdash : \kappa \in \text{Cls}^3\text{arithm} . \alpha, \beta \in \kappa . R \in \epsilon_\Delta{}^\text{‘}\alpha . S \in \epsilon_\Delta{}^\text{‘}\beta . \text{D}^\text{‘}R = \text{D}^\text{‘}S . \supset . R = S$ (3)

$\vdash . (3) . *10·11·23·35 . *40·11 . \supset \vdash . \text{Prop}$

***115·31.** $\vdash : \kappa \in \text{Cls}^3 \text{ arithm} . \supset . \text{Prod}^\text{‘‘}\kappa \text{ sm sm } \epsilon_\Delta{}^\text{‘‘}\kappa$

Dem.

$\vdash . *115·3 . *71·55 . *72·13 . \supset \vdash : \text{Hp} . \supset . \text{D} \upharpoonright s^\text{‘}\epsilon_\Delta{}^\text{‘‘}\kappa \in 1 \to 1$ (1)

$\vdash . *33·431 . \qquad \supset \vdash . s^\text{‘}\epsilon_\Delta{}^\text{‘‘}\kappa \subset \text{Cl}^\text{‘}\text{D}$ (2)

$\vdash . *37·11 . *115·1 . \qquad \supset \vdash . \text{Prod}^\text{‘‘}\kappa = \text{D}_\epsilon{}^\text{‘‘}\epsilon_\Delta{}^\text{‘‘}\kappa$ (3)

$\vdash . (1) . (2) . (3) . *111·402 . \supset \vdash . \text{Prop}$

***115·32.** $\vdash : \kappa \in \text{Cls}^3 \text{ arithm} . \supset . \epsilon_\Delta{}^\text{‘}\text{Prod}^\text{‘‘}\kappa \text{ sm } \epsilon_\Delta{}^\text{‘}\epsilon_\Delta{}^\text{‘‘}\kappa$ [*115·31 . *114·52]

***115·33.** $\vdash : \kappa \in \text{Cls}^3 \text{ arithm} . \supset . \epsilon_\Delta{}^\text{‘}\text{Prod}^\text{‘‘}\kappa \text{ sm } \epsilon_\Delta{}^\text{‘}s^\text{‘}\kappa$ [*115·32 . *85·44]

***115·34.** $\vdash : \kappa \in \text{Cls}^3 \text{ arithm} . \supset . \text{Prod}^\text{‘}\text{Prod}^\text{‘‘}\kappa \text{ sm Prod}^\text{‘}s^\text{‘}\kappa . \text{Prod}^\text{‘}s^\text{‘}\kappa \text{ sm } \epsilon_\Delta{}^\text{‘}s^\text{‘}\kappa$
[*115·33·12·23]

This proposition gives the associative law for "Prod."

The following proposition embodies the last three propositions.

***115·35.** $\vdash : \kappa \in \text{Cls}^3\,\text{arithm} \,.\, \supset .$

$$\text{Nc}`\text{Prod}`\text{Prod}``\kappa = \text{Nc}`\text{Prod}`s`\kappa = \Pi\text{Nc}`\text{Prod}``\kappa = \Pi\text{Nc}`\epsilon_\Delta``\kappa = \Pi\text{Nc}`s`\kappa$$

[*115·34·33·32]

In connection with $\text{Prod}`s`\kappa$ and $\text{Prod}`\text{Prod}``\kappa$, there remain two propositions of sufficient interest to deserve proof, namely

$$\kappa \in \text{Cls}^3\,\text{arithm} \,.\, \supset .\, \text{Prod}`s`\kappa = s``\text{Prod}`\text{Prod}``\kappa$$

and

$$\kappa \in \text{Cls}^3\,\text{arithm} \,.\, \supset .\, \text{Prod}`\text{Prod}``\kappa = \text{D}```\text{D}``\epsilon_\Delta`\epsilon_\Delta``\kappa.$$

Of these, the first is deduced from the second, while the second is proved by means of *114·51, putting D for the T which appears in that proposition, and $\epsilon_\Delta``\kappa$ for the λ of that proposition.

***115·4.** $\vdash : T \upharpoonright s`\lambda \in 1 \to 1 \,.\, s`\lambda \subset \text{Ɑ}`T \,.\, \supset .\, \text{Prod}`T```\lambda = T```\text{Prod}`\lambda$

Dem.

$\vdash . \text{*}111{\cdot}14 . \text{*}37{\cdot}103 . \supset \vdash : \text{Hp} \,.\, \kappa = T```\lambda \,.\, \supset .\, T \upharpoonright s`\lambda \in \kappa\, \overline{\text{sm}}\, \overline{\text{sm}}\, \lambda \,.$

$[\text{*}114{\cdot}51 . \text{*}73{\cdot}142] \quad \supset .\, \epsilon_\Delta`\kappa = (T \,\|\, \breve{T}_\epsilon)``\epsilon_\Delta`\lambda \quad (1)$

$\vdash . (1) . \text{*}115{\cdot}1 . \quad \supset \vdash : \text{Hp} \,.\, \supset .\, \text{Prod}`T```\lambda = \text{D}``(T \,\|\, \breve{T}_\epsilon)``\epsilon_\Delta`\lambda \quad (2)$

$\vdash . \text{*}37{\cdot}321{\cdot}231 . \quad \supset \vdash .\, \text{D}`(T \,|\, R \,|\, \breve{T}_\epsilon) = \text{D}`(T \,|\, R)$

$[\text{*}37{\cdot}32] \quad = T``\text{D}`R \quad (3)$

$\vdash . (3) . \text{*}43{\cdot}112 . \quad \supset \vdash .\, \text{D}``(T \,\|\, \breve{T}_\epsilon)``\epsilon_\Delta`\lambda = T```\text{D}``\epsilon_\Delta`\lambda$

$[\text{*}115{\cdot}1] \quad = T```\text{Prod}`\lambda \quad (4)$

$\vdash . (2) . (4) . \supset \vdash . \text{Prop}$

***115·41.** $\vdash :. R, S \in s`\lambda \,.\, \text{D}`R = \text{D}`S \,.\, \supset_{R,S} .\, R = S : \supset .\, \text{Prod}`\text{D}```\lambda = \text{D}```\text{Prod}`\lambda$

$\left[\text{*}115{\cdot}4\dfrac{\text{D}}{T} . \text{*}71{\cdot}55 . \text{*}72{\cdot}13\right]$

***115·42.** $\vdash : \kappa \in \text{Cls}^3\,\text{arithm} \,.\, \supset .\, \text{Prod}`\text{Prod}``\kappa = \text{D}```\text{Prod}`\epsilon_\Delta``\kappa$

$\qquad = \text{D}```\text{D}``\epsilon_\Delta`\epsilon_\Delta``\kappa$

Dem.

$\vdash . \text{*}115{\cdot}1 . \quad \supset \vdash .\, \text{Prod}`\text{Prod}``\kappa = \text{Prod}`\text{D}```\epsilon_\Delta``\kappa \quad (1)$

$\vdash . \text{*}115{\cdot}3{\cdot}41 . \supset \vdash : \text{Hp} \,.\, \supset .\, \text{Prod}`\text{D}```\epsilon_\Delta``\kappa = \text{D}```\text{Prod}`\epsilon_\Delta``\kappa \quad (2)$

$[\text{*}115{\cdot}1] \quad = \text{D}```\text{D}``\epsilon_\Delta`\epsilon_\Delta``\kappa \quad (3)$

$\vdash . (1) . (2) . (3) . \supset \vdash . \text{Prop}$

***115·43.** $\vdash : \kappa \in \text{Cls}^2\,\text{excl} \,.\, \supset .\, \text{Prod}`s`\kappa = s``\text{D}```\text{D}``\epsilon_\Delta`\epsilon_\Delta``\kappa$

Dem.

$\vdash . \text{*}115{\cdot}1 . \text{*}85{\cdot}28 . \supset$

$\vdash : \text{Hp} \,.\, \supset .\, \text{Prod}`s`\kappa = \text{D}``\dot{s}``\text{D}``\epsilon_\Delta`\epsilon_\Delta``\kappa$

$[\text{*}41{\cdot}43] \quad = s``\text{D}```\text{D}``\epsilon_\Delta`\epsilon_\Delta``\kappa : \supset \vdash . \text{Prop}$

***115·44.** $\vdash : \kappa \in \text{Cls}^3\,\text{arithm} \,.\, \supset .\, \text{Prod}`s`\kappa = s``\text{Prod}`\text{Prod}``\kappa$ [*115·43·42]

The following proposition is a lemma for *115·46.

***115·45.** $\vdash :. \alpha, \beta \in \kappa . \exists ! s'\alpha \cap s'\beta . \supset_{\alpha,\beta} . \alpha = \beta : \supset .$

$(s \mid D) \upharpoonright \epsilon_\Delta{}'\kappa \in 1 \rightarrow 1 . s \upharpoonright \text{Prod}'\kappa \in 1 \rightarrow 1$

Dem.

$\vdash . *83·2 . *40·13 . \supset \vdash : R \in \epsilon_\Delta{}'\kappa . \alpha \in \kappa . \supset . R'\alpha \subset s'\alpha$ (1)

$\vdash . *83·2 . *33·43 . \supset \vdash : R \in \epsilon_\Delta{}'\kappa . \alpha \in \kappa . \supset . R'\alpha \subset s'D'R$ (2)

$\vdash . *83·23 . \supset$

$\vdash : R \in \epsilon_\Delta{}'\kappa . \alpha \in \kappa . x \in (s'D'R \cap s'\alpha) . \supset . (\exists \beta) . \beta \in \kappa . x \in R'\beta . x \in s'\alpha .$

[(1)] $\supset . (\exists \beta) . \beta \in \kappa . x \in R'\beta . x \in s'\beta . x \in s'\alpha$ (3)

$\vdash . (3) . \supset \vdash :. \text{Hp} . R \in \epsilon_\Delta{}'\kappa . \alpha \in \kappa . \supset : x \in (s'D'R \cap s'\alpha) . \supset . (\exists \beta) . x \in R'\beta . \beta = \alpha .$

[*13·195] $\supset . x \in R'\alpha$ (4)

$\vdash . (1) . (2) . (4) . \supset \vdash :. \text{Hp} . \supset : R \in \epsilon_\Delta{}'\kappa . \alpha \in \kappa . \supset . R'\alpha = s'D'R \cap s'\alpha$ (5)

$\vdash . (5) . \supset \vdash :: \text{Hp} . \supset :. R, S \in \epsilon_\Delta{}'\kappa . s'D'R = s'D'S . \supset : \alpha \in \kappa . \supset_\alpha . R'\alpha = S'\alpha :$

[*33·45.*80·14] $\supset : R = S :.$ (6)

[*71·55.*72·13·161] $\supset :. (s \mid D) \upharpoonright \epsilon_\Delta{}'\kappa \in 1 \rightarrow 1$ (7)

$\vdash . (6) . *37·63 . *115·1 . *30·37 . \supset \vdash :. \text{Hp} . \supset : \mu, \nu \in \text{Prod}'\kappa . s'\mu = s'\nu . \supset . \mu = \nu :$

[*71·55.*72·161] $\supset : s \upharpoonright \text{Prod}'\kappa \in 1 \rightarrow 1$ (8)

$\vdash . (7) . (8) . \supset \vdash . \text{Prop}$

***115·46.** $\vdash : \kappa \in \text{Cls}^3 \text{arithm} . \supset . s \upharpoonright \text{Prod}'\text{Prod}``\kappa \in 1 \rightarrow 1$

Dem.

$\vdash . *115·141 . \supset$

$\vdash : \alpha, \beta \in \kappa . \exists ! s'\text{Prod}'\alpha \cap s'\text{Prod}'\beta . \supset . \exists ! s'\alpha \cap s'\beta$ (1)

$\vdash . (1) . *115·22 . \supset$

$\vdash :. \kappa \in \text{Cls}^3 \text{arithm} . \supset : \alpha, \beta \in \kappa . \exists ! s'\text{Prod}'\alpha \cap s'\text{Prod}'\beta . \supset . \alpha = \beta .$

[*30·37] $\supset . \text{Prod}'\alpha = \text{Prod}'\beta :$

[*37·63] $\supset : \mu, \nu \in \text{Prod}``\kappa . \exists ! s'\mu \cap s'\nu . \supset . \mu = \nu :$

[*115·45] $\supset : s \upharpoonright \text{Prod}'\text{Prod}``\kappa \in 1 \rightarrow 1 :. \supset \vdash . \text{Prop}$

The above proposition is used in dealing with products in relation-arithmetic (*174·42).

***115·5.** $\vdash : T \upharpoonright s'\lambda \in \kappa \,\overline{\text{sm}}\, \overline{\text{sm}}\, \lambda . \supset . \text{Prod}'\kappa = T_\epsilon``\text{Prod}'\lambda$ [*115·4 . *111·14]

***115·501.** $\vdash : T \upharpoonright s'\lambda \in \kappa \,\overline{\text{sm}}\, \overline{\text{sm}}\, \lambda . \exists ! \text{Prod}'\lambda . \supset . T \upharpoonright s'\lambda \in (\text{Prod}'\kappa) \,\overline{\text{sm}}\, \overline{\text{sm}}\, (\text{Prod}'\lambda)$

[*115·5·141 . *111·14]

***115·502.** $\vdash : T \upharpoonright s'\lambda \in \kappa \,\overline{\text{sm}}\, \overline{\text{sm}}\, \lambda . \supset . T \upharpoonright s'\text{Prod}'\lambda \in (\text{Prod}'\kappa) \,\overline{\text{sm}}\, \overline{\text{sm}}\, (\text{Prod}'\lambda)$

Dem.

$\vdash . *35·75 . \supset \vdash : \sim \exists ! \text{Prod}'\lambda . \supset . T \upharpoonright s'\text{Prod}'\lambda = \dot{\Lambda}$ (1)

$\vdash . *115·5 . *37·29 . \supset \vdash : \text{Hp} . \sim \exists ! \text{Prod}'\lambda . \supset . \text{Prod}'\kappa = \Lambda .$

[*37·29.*40·21] $\supset . s'\text{Prod}'\kappa = (T \upharpoonright s'\text{Prod}'\lambda)``s'\text{Prod}'\lambda$ (2)

$\vdash . (1) . *72·1 . (2) . *115·5 . *111·1 . \supset$

$\vdash : \text{Hp} . \sim \exists ! \text{Prod}'\lambda . \supset . T \upharpoonright s'\text{Prod}'\lambda \in (\text{Prod}'\kappa) \,\overline{\text{sm}}\, \overline{\text{sm}}\, (\text{Prod}'\lambda)$ (3)

$\vdash . (3) . *115·501·141 . \supset \vdash . \text{Prop}$

***115·51.** $\vdash : \kappa \operatorname{sm} \operatorname{sm} \lambda . \supset . \operatorname{Prod}`\kappa \operatorname{sm} \operatorname{sm} \operatorname{Prod}`\lambda$ [*115·502]

The above propositions show how, in certain respects, Prod$`\kappa$ is more convenient than $\epsilon_\Delta`\kappa$. We cannot have $\epsilon_\Delta`\kappa \operatorname{sm} \operatorname{sm} \epsilon_\Delta`\lambda$, because $\epsilon_\Delta`\kappa$ is a class of *relations*, not a class of classes; and the correlator of $\epsilon_\Delta`\kappa$ and $\epsilon_\Delta`\lambda$ is by no means so simple a function of the correlator of $\kappa$ and $\lambda$ as $T_\epsilon \upharpoonright \operatorname{Prod}`\lambda$, which correlates Prod$`\kappa$ and Prod$`\lambda$, in virtue of *115·502.

The following propositions are a continuation of those given in *114·601 ff.

***115·6.** $\vdash : (R``\gamma) \rceil R, S \upharpoonright \gamma \in 1 \to 1 . \gamma \subset \text{Cl}`R . \gamma \subset \text{Cl}`S . R``\gamma, S``\gamma \in \text{Cls}^2\text{excl} . \supset .$
$\operatorname{Prod}`R``\gamma \times \operatorname{Prod}`S``\gamma \operatorname{sm} \epsilon_\Delta`\hat{\mu}\{(\exists z) . z \in \gamma . \mu = R`z \times S`z\}$

Dem.

$\vdash . *115\cdot12 . *113\cdot13 . \supset$
$\vdash : \text{Hp} . \supset . \operatorname{Prod}`R``\gamma \times \operatorname{Prod}`S``\gamma \operatorname{sm} \epsilon_\Delta`R``\gamma \times \epsilon_\Delta`S``\gamma \quad (1)$
$\vdash . (1) . *114\cdot64 . \supset \vdash . \text{Prop}$

***115·601.** $\vdash : (R``\gamma) \rceil R, S \upharpoonright \gamma \in 1 \to 1 . \gamma \subset \text{Cl}`R . \gamma \subset \text{Cl}`S . R``\gamma \in \text{Cls}^2\text{excl} . \supset .$
$\hat{\mu}\{(\exists z) . z \in \gamma . \mu = R`z \times S`z\} \in \text{Cls}^2\text{excl}$

Dem.

$\vdash . *113\cdot19 . \supset \vdash :. \text{Hp} . \supset :$
$z, w \in \gamma . \exists ! (R`z \times S`z) \cap (R`w \times S`w) . \supset . \exists ! R`z \cap R`w .$
[*84·11] $\supset . R`z = R`w .$
[*74·53.*30·37] $\supset . z = w .$
[*30·37] $\supset . R`z \times S`z = R`w \times S`w \quad (1)$
$\vdash . (1) . *84\cdot11 . \supset \vdash . \text{Prop}$

***115·602.** $\vdash : (R``\gamma) \rceil R, S \upharpoonright \gamma \in 1 \to 1 . \gamma \subset \text{Cl}`R . \gamma \subset \text{Cl}`S . S``\gamma \in \text{Cls}^2\text{excl} . \supset .$
$\hat{\mu}\{(\exists z) . z \in \gamma . \mu = R`z \times S`z\} \in \text{Cls}^2\text{excl}$
[Proof as in *115·601]

***115·61.** $\vdash :. (R``\gamma) \rceil R, S \upharpoonright \gamma \in 1 \to 1 . \gamma \subset \text{Cl}`R . \gamma \subset \text{Cl}`S . R``\gamma \cap S``\gamma = \Lambda :$
$R``\gamma \in \text{Cls}^2\text{excl} . \vee . S``\gamma \in \text{Cls}^2\text{excl} : \supset .$
$\epsilon_\Delta`(R``\gamma \cup S``\gamma) \operatorname{sm} \operatorname{Prod}`\hat{\mu}\{(\exists z) . z \in \gamma . \mu = R`z \times S`z\}$
[*115·601·602·12 . *114·65]

***115·62.** $\vdash : (R``\gamma) \rceil R, S \upharpoonright \gamma \in 1 \to 1 . \gamma \subset \text{Cl}`R . \gamma \subset \text{Cl}`S . R``\gamma \cap S``\gamma = \Lambda .$
$(R``\gamma \cup S``\gamma) \in \text{Cls}^2\text{excl} . \supset .$
$\operatorname{Prod}`(R``\gamma \cup S``\gamma) \operatorname{sm} \operatorname{Prod}`\hat{\mu}\{(\exists z) . z \in \gamma . \mu = R`z \times S`z\}$
[*115·61·12 . *84·25]

***115·63.** $\vdash : (R``\gamma) \rceil R, S \upharpoonright \gamma \in 1 \to 1 . \gamma \subset \text{Cl}`R . \gamma \subset \text{Cl}`S . R``\gamma, S``\gamma \in \text{Cls}^2\text{excl} . \supset .$
$\operatorname{Prod}`R``\gamma \times \operatorname{Prod}`S``\gamma \operatorname{sm} \operatorname{Prod}`\hat{\mu}\{(\exists z) . z \in \gamma . \mu = R`z \times S`z\}$
[*115·6·601·12]

$*116$. EXPONENTIATION

Summary of $*116$.

In this number, we define "$\alpha \exp \beta$," meaning "α to the exponent β," where α and β are classes, as

$$\text{Prod}`\alpha \underset{,,}{\downarrow} ``\beta.$$

Now $\text{Prod}`\alpha \underset{,,}{\downarrow} ``\beta$ consists of all ways of selecting one each from the members of $\alpha \underset{,,}{\downarrow} ``\beta$, *i.e.* from the classes $\downarrow y``\alpha$, where $y \in \beta$. Thus to get a member of $\text{Prod}`\alpha \underset{,,}{\downarrow} ``\beta$, take a set of couples $x \downarrow y$, where x is always an α, and there is only one x for a given y, and y is each member of β in succession. Thus for each member of β, we have $\text{Nc}`\alpha$ possible referents; hence it is plain that the number of possible sets of couples consists of $\text{Nc}`\beta$ factors each equal to $\text{Nc}`\alpha$, and is therefore fit to be taken as defining $(\text{Nc}`\alpha)^{\text{Nc}`\beta}$.

The definitions of μ^ν and $(\text{Nc}`\alpha)^{\text{Nc}`\beta}$ are derived from the definition of $\alpha \exp \beta$ exactly as the definitions of $\mu +_c \nu$ and $\text{Nc}`\alpha +_c \text{Nc}`\beta$, or of $\mu \times_c \nu$ and $\text{Nc}`\alpha \times_c \text{Nc}`\beta$, were derived respectively from $\alpha + \beta$ and $\alpha \times \beta$.

The chief difficulty in this number lies in the proof of the three formal laws of exponentiation, namely

$$\mu^\nu \times_c \mu^\varpi = \mu^{\nu +_c \varpi},$$

$$\mu^\varpi \times_c \nu^\varpi = (\mu \times_c \nu)^\varpi,$$

and

$$(\mu^\nu)^\varpi = \mu^{\nu \times_c \varpi}.$$

The proofs of the second and third of these, in particular, require various lemmas; but there is no difficulty involved except the complexity of the classes and relations concerned.

The definition of μ^ν is so framed as to minimize the necessity for the multiplicative axiom (see the note on $*113\cdot31$ in the introduction to $*113$). We have

$*116\cdot36$. $\vdash :. \text{Mult ax} . \supset : \mu, \nu \in \text{NC} - \iota`\Lambda . \kappa \in \nu \cap \text{Cl}`\mu . \supset . \Pi\text{Nc}`\kappa = \mu^\nu$

that is, assuming the multiplicative axiom, the product of ν factors each equal to μ is μ^ν (assuming μ and ν to be cardinals which are not null). If we had *defined* μ^ν as the product of ν factors each equal to μ, we should have required the multiplicative axiom for almost all propositions on μ^ν; but by taking the particular class $\alpha \underset{,,}{\downarrow} ``\beta$, we avoid the multiplicative axiom except in a few propositions. Among these few is the above proposition connecting exponentiation with multiplication.

Cantor has defined μ^ν by means of the class of "Belegungen," *i.e.* the class

$$\hat{R}(R \in 1 \to \text{Cls} . \text{D}`R \subset \alpha . \text{Ͻ}`R = \beta)$$

which (*116·12) $= (\alpha \uparrow \beta)_\Delta{}^\backprime\beta$. By *85·53 and *113·103, this class is equal to $\dot{s}^{\backprime\backprime}(\alpha \exp \beta)$ (as is proved in *116·13), whence, since $\dot{s} \upharpoonright \alpha \exp \beta \epsilon 1 \rightarrow 1$, it follows (*116·15) that the class of "Belegungen" is similar to $\alpha \exp \beta$. Hence our definition gives the same value of μ^ν as Cantor's.

The propositions of the present number begin with various simple properties of $\alpha \exp \beta$. Its existence follows from

***116·152.** $\vdash : x \epsilon \alpha . \supset . x \downarrow {}^{\backprime\backprime}\beta \epsilon (\alpha \exp \beta)$

whence (*116·16) $\vdash . \mathrm{Cnv}^{\backprime\backprime\backprime}\beta \underset{,,}{\downarrow} {}^{\backprime\backprime}\alpha \subset \alpha \exp \beta$, and

***116·18.** $\vdash :. \exists ! \alpha . \mathrm{v} . \beta = \Lambda : \equiv . \exists ! \alpha \exp \beta$

We have

***116·19.** $\vdash : \alpha \,\mathrm{sm}\, \gamma . \beta \,\mathrm{sm}\, \delta . \supset . (\alpha \exp \beta) \,\mathrm{sm}\,\mathrm{sm}\, (\gamma \exp \delta)$

in virtue of *113·13 and *115·51. *116·192 shows that, if $R \upharpoonright \gamma$ correlates α with γ, and $S \upharpoonright \delta$ correlates β with δ, then $(R \,\|\, \breve{S}) \upharpoonright (\delta \times \gamma)$ is a double correlator of $(\alpha \exp \beta)$ with $(\gamma \exp \delta)$.

We then proceed to a set of propositions on μ^ν, which are analogous to *113·2 ff. on $\mu \times_0 \nu$. We have

***116·203.** $\vdash : \exists ! \mu^\nu . \supset . \mu, \nu \epsilon \mathrm{NC} - \iota^\backprime\Lambda . \mu, \nu \epsilon \mathrm{N_0C}$

***116·25.** $\vdash . (\mathrm{Nc}^\backprime\gamma)^{\mathrm{Nc}^\backprime\delta} = \mathrm{Nc}^\backprime(\gamma \exp \delta)$

and various other less useful propositions.

We then have various propositions on 0 and 1 and 2. We prove

***116·301.** $\vdash : \mu \epsilon \mathrm{NC} - \iota^\backprime\Lambda . \supset . \mu^0 = 1$

***116·311.** $\vdash : \nu \epsilon \mathrm{NC} - \iota^\backprime\Lambda - \iota^\backprime 0 . \supset . 0^\nu = 0$

***116·321.** $\vdash : \mu \epsilon \mathrm{NC} - \iota^\backprime\Lambda . \supset . \mu^1 = \mathrm{sm}^{\backprime\backprime}\mu$

(Observe that $\mathrm{sm}^{\backprime\backprime}\mu$ is the same cardinal as μ, but rendered typically ambiguous.)

***116·331.** $\vdash : \mu \epsilon \mathrm{NC} - \iota^\backprime\Lambda . \supset . 1^\mu = 1$

***116·34.** $\vdash . \mu^2 = \mu \times_0 \mu$

(This proposition does not require that μ should be a cardinal.)

After the proposition (*116·36) already quoted, on the connection of exponentiation and multiplication, we proceed to a set of propositions on the case where a number of classes are all given as similar (by assignable correlations) to a given class. In *116·411, we prove that if κ is a class of mutually exclusive classes, each of which is similar to a given class γ, and if, when $\alpha \epsilon \kappa$, $M^\backprime\alpha$ is a correlator of α and γ, and T is the sum of $M^{\backprime\backprime}\kappa$, then

$$\mathrm{Nc}^\backprime\epsilon_\Delta{}^\backprime\overrightarrow{T}^{\backprime\backprime}\gamma = \mathrm{Nc}^\backprime T_\Delta{}^\backprime\gamma = \mathrm{Nc}^\backprime(\kappa \exp \gamma) = (\mathrm{Nc}^\backprime\kappa)^{\mathrm{Nc}^\backprime\gamma}.$$

This is a further connection of multiplication and exponentiation. (On the purport of this and following propositions, see the explanation preceding $*116{\cdot}4$.) In $*116{\cdot}43$, the hypothesis is somewhat modified. We still have a set κ of classes which are all similar to γ, but the correlator for a given class α is not given as $M`\alpha$, but is given as $M`w$, where w is a member of a class δ which is similar to κ. Then $\kappa = \text{D}``M``\delta$. We assume that $M \upharpoonright \delta$ is a one-one, and that if $M`w$ and $M`v$ have domains which overlap, then $w = v$. Thus κ is a class of mutually exclusive classes, each of which has $\text{Nc}`\gamma$ terms, while $\kappa$ has $\text{Nc}`\delta$ terms. Then it is proved in $*116{\cdot}43$ that

$$\text{Prod}`\text{D}``M``\delta \text{ sm sm } (\gamma \exp \delta) \,.\, \Pi\text{Nc}`\text{D}``M``\delta = (\text{Nc}`\gamma)^{\text{Nc}`\delta}.$$

This proposition and another ($*116{\cdot}45$) which follows from it are useful in proving the formal laws of exponentiation. The proof of these occupies the following propositions from $*116{\cdot}5$ to $*116{\cdot}68$. We have

$*116{\cdot}52$. $\vdash \,.\, \mu^{\nu} \times_0 \mu^{\varpi} = \mu^{\nu +_c \varpi}$

$*116{\cdot}55$. $\vdash \,.\, \mu^{\varpi} \times_0 \nu^{\varpi} = (\mu \times_0 \nu)^{\varpi}$

$*116{\cdot}63$. $\vdash \,.\, \mu^{\nu \times_0 \varpi} = (\mu^{\nu})^{\varpi}$

An extension of the first of these is

$*116{\cdot}661$. $\vdash \,.\, \Pi\text{Nc}`(\alpha \exp)``\kappa = (\text{Nc}`\alpha)^{\Sigma\text{Nc}`\kappa}$

Here the number of members of κ need not be finite. The purport of the proposition is as follows: Let $\beta, \gamma, \delta, \ldots$ be the members of κ; form $\alpha \exp \beta$, $\alpha \exp \gamma$, $\alpha \exp \delta$, $\ldots$, and take the product of the numbers of all these; then the resulting number is the same as if we first took the sum of the numbers of all the members of κ, thus obtaining (say) a number μ, and raised $\text{Nc}`\alpha$ to the μth power.

An extension of $*116{\cdot}55$ is given by $*116{\cdot}68$, where we prove

$$\vdash : \kappa \in \text{Cls}^2 \text{ excl} \,.\, \supset \,.\, \Pi\text{Nc}`\exp \gamma``\kappa = (\Pi\text{Nc}`\kappa)^{\text{Nc}`\gamma}.$$

There is no analogous extension of $*116{\cdot}63$.

We prove next Cantor's proposition (which is very useful)

$*116{\cdot}72$. $\vdash \,.\, \text{Nc}`\text{Cl}`\alpha = 2^{\text{Nc}`\alpha}$

I.e. the number of combinations of μ things any number at a time is 2^{μ}. (Observe that μ need not be finite.) The remainder of the number is concerned with consequences of this proposition.

$*116{\cdot}01$. $\alpha \exp \beta = \text{Prod}`\alpha \underset{;}{\downarrow} ``\beta$ Df

$*116{\cdot}02$. $\mu^{\nu} = \hat{\gamma}\{(\exists \alpha, \beta) \,.\, \mu = \text{N}_0\text{c}`\alpha \,.\, \nu = \text{N}_0\text{c}`\beta \,.\, \gamma \text{ sm } (\alpha \exp \beta)\}$ Df

$*116{\cdot}03$. $(\text{Nc}`\alpha)^{\nu} = (\text{N}_0\text{c}`\alpha)^{\nu}$ Df

$*116{\cdot}04$. $\mu^{\text{Nc}`\beta} = \mu^{\text{N}_0\text{c}`\beta}$ Df

***116·1.** $\vdash : \xi \in (\alpha \exp \beta) . \equiv . (\exists R) . R \in \epsilon_\Delta\text{‘}\alpha \underset{,,}{\downarrow}\text{‘‘}\beta . \xi = \mathrm{D}\text{‘}R$
[*115·1 . (*116·01)]

***116·11.** $\vdash :. \xi \in (\alpha \exp \beta) . \equiv : y \in \beta . \supset_y . \alpha \cap \hat{x}(x \downarrow y \in \xi) \in 1 : \xi \subset \beta \times \alpha$

Dem.

$\vdash . *113\cdot111 . *115\cdot11 . \supset$

$\vdash :. \xi \in (\alpha \exp \beta) . \equiv : \rho \in \alpha \underset{,,}{\downarrow}\text{‘‘}\beta . \supset_\rho . \rho \cap \xi \in 1 : \xi \subset s\text{‘}\alpha \underset{,,}{\downarrow}\text{‘‘}\beta :$

[*38·2.*113·1] $\equiv : y \in \beta . \supset_y . \downarrow y\text{‘‘}\alpha \cap \xi \in 1 : \xi \subset \beta \times \alpha$ (1)

$\vdash . *37\cdot6 . \supset$

$\vdash : \downarrow y\text{‘‘}\alpha \cap \xi \in 1 . \equiv . \hat{R}\{(\exists x) . x \in \alpha . R = x \downarrow y . R \in \xi\} \in 1 .$

[*13·193] $\equiv . \hat{R}\{(\exists x) . x \in \alpha . x \downarrow y \in \xi . R = x \downarrow y\} \in 1 .$

[*37·6] $\equiv . \downarrow y\text{‘‘}\hat{x}(x \in \alpha . x \downarrow y \in \xi) \in 1 .$

[*73·611·44] $\equiv . \hat{x}(x \in \alpha . x \downarrow y \in \xi) \in 1$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***116·12.** $\vdash . (\alpha \uparrow \beta)_\Delta\text{‘}\beta = \hat{R}\{R \in 1 \rightarrow \mathrm{Cls} . \mathrm{D}\text{‘}R \subset \alpha . \text{Ɑ}\text{‘}R = \beta\}$

Dem.

$\vdash . *80\cdot14 . \supset \vdash : R \in (\alpha \uparrow \beta)_\Delta\text{‘}\beta . \equiv . R \in 1 \rightarrow \mathrm{Cls} . R \Subset \alpha \uparrow \beta . \text{Ɑ}\text{‘}R = \beta .$

[*35·83] $\equiv . R \in 1 \rightarrow \mathrm{Cls} . \mathrm{D}\text{‘}R \subset \alpha . \text{Ɑ}\text{‘}R \subset \beta . \text{Ɑ}\text{‘}R = \beta .$

[*22·42] $\equiv . R \in 1 \rightarrow \mathrm{Cls} . \mathrm{D}\text{‘}R \subset \alpha . \text{Ɑ}\text{‘}R = \beta : \supset \vdash . \mathrm{Prop}$

***116·13.** $\vdash . \dot{s}\text{‘‘}(\alpha \exp \beta) = (\alpha \uparrow \beta)_\Delta\text{‘}\beta$

Dem.

$\vdash . *85\cdot53 . \supset \vdash . (\alpha \uparrow \beta)_\Delta\text{‘}\beta = \dot{s}\text{‘‘}\mathrm{D}\text{‘‘}\epsilon_\Delta\text{‘}(\alpha \uparrow \beta) \underline{\downarrow}\text{‘‘}\beta$

[*113·103] $= \dot{s}\text{‘‘}\mathrm{D}\text{‘‘}\epsilon_\Delta\text{‘}\alpha \underset{,,}{\downarrow}\text{‘‘}\beta$

[*115·1.(*116·01)] $= \dot{s}\text{‘‘}(\alpha \exp \beta) . \supset \vdash . \mathrm{Prop}$

$(\alpha \uparrow \beta)_\Delta\text{‘}\beta$ is the class of one-many relations whose converse domain is β and whose domain is contained in α. This is what Cantor calls the "Belegungsmenge," and is used by him as the definition of exponentiation. In virtue of *116·15, his definition gives the same results as ours.

***116·131.** $\vdash . \dot{s} \upharpoonright (\alpha \exp \beta) \in \{(\alpha \uparrow \beta)_\Delta\text{‘}\beta\} \,\overline{\mathrm{sm}}\, (\alpha \exp \beta)$

Dem.

$\vdash . *84\cdot241 . *113\cdot103 . \supset \vdash . \iota\text{‘‘}\beta \in \mathrm{Cls}^2\,\mathrm{excl} . \alpha \underset{,,}{\downarrow}\text{‘‘}\beta = (\alpha \uparrow \beta)_\Delta\text{‘‘}\iota\text{‘‘}\beta$ (1)

$\vdash . (1) . *85\cdot42 . \supset \vdash : M, N \in \epsilon_\Delta\text{‘}\alpha \underset{,,}{\downarrow}\text{‘‘}\beta . \dot{s}\text{‘}\mathrm{D}\text{‘}M = \dot{s}\text{‘}\mathrm{D}\text{‘}N . \supset . M = N .$

[*30·37] $\supset . \mathrm{D}\text{‘}M = \mathrm{D}\text{‘}N$ (2)

$\vdash . (2) . *37\cdot63 . *115\cdot1 . (*116\cdot01) . \supset \vdash : \mu, \nu \in (\alpha \exp \beta) . \dot{s}\text{‘}\mu = \dot{s}\text{‘}\nu . \supset . \mu = \nu :$

[*71·55.*72·163] $\supset \vdash . \dot{s} \upharpoonright (\alpha \exp \beta) \in 1 \rightarrow 1$ (3)

$\vdash . (3) . *116\cdot13 . \supset \vdash . \mathrm{Prop}$

***116·14.** $\vdash . (\alpha \exp \beta)\, \mathrm{sm}\, \epsilon_\Delta\text{‘}\alpha \underset{,,}{\downarrow}\text{‘‘}\beta$ [*115·12 . *113·111]

***116·15.** $\vdash . (\alpha \exp \beta) \operatorname{sm} (\alpha \uparrow \beta)_\Delta{}^\prime\beta$ [*116·131]

*116·151 is a lemma for *116·152.

***116·151.** $\vdash : x \in \alpha . \supset . x \downarrow | \mathrm{Cnv}^\prime(\alpha \underset{''}{\downarrow} \upharpoonright \beta) \in \epsilon_\Delta{}^\prime \alpha \underset{''}{\downarrow}{}^{\prime\prime}\beta$

Dem.

$\vdash . *113{\cdot}105 . *72{\cdot}184 . \supset \vdash : \mathrm{Hp} . \supset . x \downarrow | \mathrm{Cnv}^\prime(\alpha \underset{''}{\downarrow} \upharpoonright \beta) \in 1 \to \mathrm{Cls}$ (1)

$\vdash . *34{\cdot}1 . *38{\cdot}1 . \quad \supset \vdash :. \mathrm{Hp} . \supset : R \{x \downarrow | \mathrm{Cnv}^\prime(\alpha \underset{''}{\downarrow} \upharpoonright \beta)\} \lambda . \equiv .$

$(\exists y) . R = x \downarrow y . y \in \beta . \lambda = \alpha \underset{''}{\downarrow} y . x \in \alpha .$

[*38·21] $\quad \supset . R \in \lambda$ (2)

$\vdash . *37{\cdot}322{\cdot}401 . \quad \supset \vdash . \text{Ɑ}^\prime\{x \downarrow | \mathrm{Cnv}^\prime(\alpha \underset{''}{\downarrow} \upharpoonright \beta)\} = \alpha \underset{''}{\downarrow}{}^{\prime\prime}\beta$ (3)

$\vdash . (1) . (2) . (3) . *80{\cdot}14 . \supset \vdash . \mathrm{Prop}$

***116·152.** $\vdash : x \in \alpha . \supset . x \downarrow {}^{\prime\prime}\beta \in (\alpha \exp \beta)$

Dem.

$\vdash . *37{\cdot}32 . *35{\cdot}65 . \supset \vdash . \mathrm{D}^\prime\{x \downarrow | \mathrm{Cnv}^\prime(\alpha \underset{''}{\downarrow} \upharpoonright \beta)\} = x \downarrow {}^{\prime\prime}\beta$ (1)

$\vdash . (1) . *116{\cdot}151{\cdot}1 . \supset \vdash . \mathrm{Prop}$

***116·16.** $\vdash . \mathrm{Cnv}^{\prime\prime\prime}\beta \underset{''}{\downarrow}{}^{\prime\prime}\alpha \subset \alpha \exp \beta$

Dem.

$\vdash . *116{\cdot}152 . *55{\cdot}14 . \supset \vdash : x \in \alpha . \supset . \mathrm{Cnv}^{\prime\prime} \downarrow x^{\prime\prime}\beta \in (\alpha \exp \beta) .$

[*38·2] $\quad \supset . \mathrm{Cnv}^{\prime\prime}\beta \underset{''}{\downarrow} x \in (\alpha \exp \beta) : \supset \vdash . \mathrm{Prop}$

The above propositions are useful in establishing existence-theorems, as appears in the following propositions.

***116·17.** $\vdash : \exists ! \beta \underset{''}{\downarrow}{}^{\prime\prime}\alpha . \supset . \exists ! \alpha \exp \beta$ [*116·16 . *37·47]

***116·171.** $\vdash :. \exists ! \alpha . \vee . \beta = \Lambda : \supset . \exists ! \alpha \exp \beta$

Dem.

$\vdash . *113{\cdot}113 . *83{\cdot}15 . *51{\cdot}161 . \supset \vdash : \beta = \Lambda . \supset . \exists ! \alpha \exp \beta$ (1)

$\vdash . *116{\cdot}152 . \quad \supset \vdash : \exists ! \alpha . \supset . \exists ! \alpha \exp \beta$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***116·172.** $\vdash :. \exists ! \alpha \exp \beta . \supset : \exists ! \alpha . \vee . \beta = \Lambda$

Dem.

$\vdash . *83{\cdot}11 . \supset \vdash :. \mathrm{Hp} . \supset : \Lambda \sim\in \alpha \underset{''}{\downarrow}{}^{\prime\prime}\beta :$

[*113·112] $\quad \supset : \sim(\alpha = \Lambda . \exists ! \beta) :$

[*24·51] $\quad \supset : \exists ! \alpha . \vee . \beta = \Lambda :. \supset \vdash . \mathrm{Prop}$

***116·18.** $\vdash :. \exists ! \alpha . \vee . \beta = \Lambda : \equiv . \exists ! \alpha \exp \beta$ [*116·171·172]

***116·181.** $\vdash . \alpha \exp \Lambda = \iota^\prime\Lambda$

Dem.

$\vdash . *113{\cdot}113 . \supset \vdash . \alpha \exp \Lambda = \mathrm{Prod}^\prime\Lambda$

[*83·15.*33·241] $\quad = \iota^\prime\Lambda . \supset \vdash . \mathrm{Prop}$

***116·182.** $\vdash : \exists ! \beta . \supset . \Lambda \exp \beta = \Lambda$ [*113·112 . *83·11]

***116·183.** $\vdash . s\text{‘}(\alpha \exp \beta) = \beta \times \alpha$

Dem.

$\vdash . *115\cdot141 . *116\cdot18 . \supset \vdash :. \exists ! \alpha . \vee . \beta = \Lambda : \supset . s\text{‘}(\alpha \exp \beta) = s\text{‘}\underset{,,}{\alpha \downarrow}\text{“}\beta$

$[*113\cdot1] \qquad = \beta \times \alpha \qquad (1)$

$\vdash . *116\cdot182 . \qquad \supset \vdash :. \alpha = \Lambda . \exists ! \beta . \supset . s\text{‘}(\alpha \exp \beta) = \Lambda$

$[*113\cdot114] \qquad = \beta \times \alpha \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***116·19.** $\vdash : \alpha \,\text{sm}\, \gamma . \beta \,\text{sm}\, \delta . \supset . (\alpha \exp \beta) \,\text{sm sm}\, (\gamma \exp \delta)$

Dem.

$\vdash . *113\cdot13 . \supset \vdash : \text{Hp} . \supset . \underset{,,}{\alpha \downarrow}\text{“}\beta \,\text{sm sm}\, \underset{,,}{\gamma \downarrow}\text{“}\delta .$

$[*115\cdot51] \qquad \supset . (\alpha \exp \beta) \,\text{sm sm}\, (\gamma \exp \delta) : \supset \vdash . \text{Prop}$

***116·191.** $\vdash : R \in \alpha \,\overline{\text{sm}}\, \gamma . S \in \beta \,\overline{\text{sm}}\, \delta . \supset . (R \parallel \breve{S}) \upharpoonright (\delta \times \gamma) \in (\alpha \exp \beta) \,\overline{\text{sm}}\,\overline{\text{sm}}\, (\gamma \exp \delta) .$
$(R \parallel \breve{S})_\epsilon\text{“}(\gamma \exp \delta) = \alpha \exp \beta$

[*113·127 . *115·502 . *116·183]

***116·192.** $\vdash : R \upharpoonright \gamma \in \alpha \,\overline{\text{sm}}\, \gamma . S \upharpoonright \delta \in \beta \,\overline{\text{sm}}\, \delta . \supset .$
$(R \parallel \breve{S}) \upharpoonright (\delta \times \gamma) \in (\alpha \exp \beta) \,\overline{\text{sm}}\,\overline{\text{sm}}\, (\gamma \exp \delta) .$
$(R \parallel \breve{S})_\epsilon \upharpoonright (\gamma \exp \delta) \in (\alpha \exp \beta) \,\overline{\text{sm}}\, (\gamma \exp \delta)$

[*113·127 . *115·502 . *116·183 . *111·15]

***116·194.** $\vdash : R \upharpoonright \gamma \in \alpha \,\overline{\text{sm}}\, \gamma . S \upharpoonright \delta \in \beta \,\overline{\text{sm}}\, \delta . \supset .$
$(R \parallel \breve{S}) \upharpoonright \{(\gamma \uparrow \delta)_\Delta\text{‘}\delta\} \in \{(\alpha \uparrow \beta)_\Delta\text{‘}\beta\} \,\overline{\text{sm}}\, \{(\gamma \uparrow \delta)_\Delta\text{‘}\delta\}$

Dem.

$\vdash . *116\cdot12 . \supset \vdash : \text{Hp} . \supset . s\text{‘}\text{D}\text{“}(\gamma \uparrow \delta)_\Delta\text{‘}\delta \subset \gamma . s\text{‘}\text{Œ}\text{“}(\gamma \uparrow \delta)_\Delta\text{‘}\delta \subset \delta .$

$[*74\cdot773 . *73\cdot142] \qquad \supset . (R \parallel \breve{S}) \upharpoonright \{(\gamma \uparrow \delta)_\Delta\text{‘}\delta\} \in$
$\{(R \parallel \breve{S})\text{“}(\gamma \uparrow \delta)_\Delta\text{‘}\delta\} \,\overline{\text{sm}}\, \{(\gamma \uparrow \delta)_\Delta\text{‘}\delta\} \qquad (1)$

$\vdash . *116\cdot192 . *111\cdot14 . \supset \vdash : \text{Hp} . \supset . \alpha \exp \beta = (R \parallel \breve{S})_\epsilon\text{“}(\gamma \exp \delta) .$

$[*116\cdot13] \qquad \supset . (\alpha \uparrow \beta)_\Delta\text{‘}\beta = \dot{s}\text{“}(R \parallel \breve{S})_\epsilon\text{“}(\gamma \exp \delta)$

$[*43\cdot43] \qquad = (R \parallel \breve{S})\text{“}\dot{s}\text{“}(\gamma \exp \delta)$

$[*116\cdot13] \qquad = (R \parallel \breve{S})\text{“}(\gamma \uparrow \delta)_\Delta\text{‘}\delta \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

The following propositions (down to *116·27 exclusive) are the analogues of propositions with the same decimal part in *113.

***116·2.** $\vdash : \xi \in \mu^\nu . \equiv . (\exists \alpha, \beta) . \mu = \text{N}_0\text{c}\text{‘}\alpha . \nu = \text{N}_0\text{c}\text{‘}\beta . \xi \,\text{sm}\, (\alpha \exp \beta)$ [(*116·02)]

***116·201.** $\vdash :. \xi \in \mu^\nu . \equiv : \mu, \nu \in \text{NC} : (\exists \alpha, \beta) . \alpha \in \mu . \beta \in \nu . \xi \,\text{sm}\, (\alpha \exp \beta)$

[*116·2 . *103·27]

***116·202.** $\vdash :. \xi \in \mu^\nu . \equiv : \exists ! \mu . \exists ! \nu : (\exists \alpha, \beta) . \mu = \text{Nc}`\alpha . \nu = \text{Nc}`\beta . \xi \,\text{sm}\, (\alpha \exp \beta)$
[Proof as in *113·202]

***116·203.** $\vdash : \exists ! \mu^\nu . \supset . \mu, \nu \in \text{NC} - \iota`\Lambda . \mu, \nu \in \text{N}_0\text{C}$ [*116·201·202·2]

***116·204.** $\vdash :. \mu = \Lambda . \vee . \nu = \Lambda . \vee . \sim(\mu, \nu \in \text{NC}) : \supset . \mu^\nu = \Lambda$ [*116·203]

***116·205.** $\vdash : \sim(\mu, \nu \in \text{N}_0\text{C}) . \supset . \mu^\nu = \Lambda$ [*116·203]

***116·21.** $\vdash :. \mu, \nu \in \text{NC} . \supset : \xi \in \mu^\nu . \equiv . (\exists \alpha, \beta) . \alpha \in \mu . \beta \in \nu . \xi \,\text{sm}\, (\alpha \exp \beta)$ [*116·201]

***116·22.** $\vdash : \xi \in \{\text{Nc}(\eta)`\gamma\}^{\text{Nc}(\zeta)`\delta} . \equiv . \exists ! \text{Nc}(\eta)`\gamma . \exists ! \text{Nc}(\zeta)`\delta . \xi \,\text{sm}\, (\gamma \exp \delta)$
[Proof as in *113·22, using *116·19 in place of *113·13]

***116·221.** $\vdash : \exists ! \text{Nc}(\eta)`\gamma . \exists ! \text{Nc}(\zeta)`\delta . \supset . \{\text{Nc}(\eta)`\gamma\}^{\text{Nc}(\zeta)`\delta} = \text{Nc}`(\gamma \exp \delta)$
[*116·22]

***116·222.** $\vdash . (\text{N}_0\text{c}`\gamma)^{\text{N}_0\text{c}`\delta} = \text{Nc}`(\gamma \exp \delta)$ [Proof as in *113·222]

***116·23.** $\vdash . \mu^\nu \in \text{NC}$ [Proof as in *113·23]

***116·24.** $\vdash . (\text{Nc}`\gamma)^{\text{Nc}`\delta} = (\text{N}_0\text{c}`\gamma)^{\text{N}_0\text{c}`\delta}$ [(*116·03·04)]

***116·25.** $\vdash . (\text{Nc}`\gamma)^{\text{Nc}`\delta} = \text{Nc}`(\gamma \exp \delta)$ [*116·24·222]

***116·251.** $\vdash . (\gamma \exp \delta) \in (\text{Nc}`\gamma)^{\text{Nc}`\delta}$ [*116·25 . *100·3]

***116·26.** $\vdash : \mu, \nu \in \text{NC} . \exists ! \text{sm}_\eta``\mu . \exists ! \text{sm}_\zeta``\nu . \supset . \mu^\nu = (\text{sm}_\eta``\mu)^{\text{sm}_\zeta``\nu}$
[Proof as in *113·26]

This proposition shows that we may raise or lower the types of μ and ν as we please, without affecting the value of μ^ν, provided μ and ν, or rather $\text{sm}``\mu$ and $\text{sm}``\nu$, exist in the new types.

***116·261.** $\vdash : \mu, \nu \in \text{NC} . \supset . \mu^\nu = \{\mu^{(1)}\}^{\nu^{(1)}} = \{\mu_{(00)}\}^{\nu_{(00)}} = \text{etc.}$ [Proof as in *113·261]

Here "etc." covers any derivative of μ or ν whose existence follows from that of μ or ν.

***116·27.** $\vdash . \mu^\nu = \hat{\xi} \{(\exists \alpha, \beta) . \mu = \text{N}_0\text{c}`\alpha . \nu = \text{N}_0\text{c}`\beta . \xi \,\text{sm}\, (\alpha \uparrow \beta)_\Delta `\beta\}$
[*116·15 . *73·37 . (*116·02)]

***116·271.** $\vdash : \mu, \nu \in \text{NC} . \alpha \in \mu . \beta \in \nu . \supset . (\alpha \exp \beta) \in \mu^\nu$ [*116·21]

***116·3.** $\vdash . (\text{Nc}`\alpha)^0 = 1$

Dem.

$$\begin{aligned}
&\vdash . *101·1 . *116·25 . \supset \vdash . (\text{Nc}`\alpha)^0 = \text{Nc}`(\alpha \exp \Lambda) \\
&[*116·181] \qquad\qquad = \text{Nc}`\iota`\Lambda \\
&[*101·2] \qquad\qquad = 1 . \supset \vdash . \text{Prop}
\end{aligned}$$

***116·301.** $\vdash : \mu \in \text{NC} - \iota`\Lambda . \supset . \mu^0 = 1$ [Proof as in *113·601]

***116·31.** $\vdash : \beta \neq \Lambda . \supset . 0^{\text{Nc}`\beta} = 0$

Dem.

$$\begin{aligned}
&\vdash . *101·1 . *116·25 . \supset \vdash . 0^{\text{Nc}`\beta} = \text{Nc}`(\Lambda \exp \beta) . \\
&[*116·182] \qquad \supset \vdash : \text{Hp} . \supset . 0^{\text{Nc}`\beta} = \text{Nc}`\Lambda \\
&[*101·1] \qquad\qquad = 0 : \supset \vdash . \text{Prop}
\end{aligned}$$

***116·311.** $\vdash : \nu \in \text{NC} - \iota'\Lambda - \iota'0 \,.\supset .\, 0^\nu = 0$

Dem. $\vdash . *103{\cdot}34 . *101{\cdot}1 . \supset \vdash : \text{Hp} . \supset . (\exists\beta) . \beta \neq \Lambda . \nu = \text{N}_0\text{c}'\beta .$

$[*13{\cdot}12{\cdot}15] \qquad \supset . (\exists\beta) . \beta \neq \Lambda . 0^\nu = 0^{\text{N}_0\text{c}'\beta}$

$[*116{\cdot}31.(*116{\cdot}04)] \qquad = 0 : \supset \vdash . \text{Prop}$

***116·32.** $\vdash . (\text{Nc}'\alpha)^1 = \text{Nc}'\alpha$

Dem. $\vdash . *116{\cdot}25 . *101{\cdot}2 . \supset \vdash . (\text{Nc}'\alpha)^1 = \text{Nc}'\{\alpha \exp (\iota'x)\}$

$[(*116{\cdot}01)] \qquad = \text{Nc}'\text{Prod}'\alpha \underset{,,}{\downarrow} {}^{\prime\prime}\iota'x$

$[*115{\cdot}142.*53{\cdot}31] \qquad = \text{Nc}'\iota^{\prime\prime}\alpha \underset{,,}{\downarrow} x$

$[*113{\cdot}11.*100{\cdot}6] \qquad = \text{Nc}'\alpha . \supset \vdash . \text{Prop}$

***116·321.** $\vdash : \mu \in \text{NC} - \iota'\Lambda . \supset . \mu^1 = \text{sm}^{\prime\prime}\mu \quad [*116{\cdot}32]$

It would not be an error to write "$\mu^1 = \mu$" instead of "$\mu^1 = \text{sm}^{\prime\prime}\mu$" in the above proposition. For if the "sm" is typically determined so that $\text{sm}^{\prime\prime}\mu \in t'\mu$, then $\text{sm}^{\prime\prime}\mu = \mu$. Thus in virtue of *116·321, $\mu^1 = \mu$ is true whenever it is significant. But the above form gives more information, since it preserves the typical ambiguity of μ^1 and $\text{sm}^{\prime\prime}\mu$.

***116·33.** $\vdash . 1^{\text{Nc}'\beta} = 1$

Dem. $\vdash . *113{\cdot}11 . \qquad \supset \vdash : \alpha \in 1 . \supset . \alpha \underset{,,}{\downarrow} {}^{\prime\prime}\beta \subset 1 .$

$[*115{\cdot}144.*101{\cdot}2] \qquad \supset . \text{Nc}'\text{Prod}'\alpha \underset{,,}{\downarrow} {}^{\prime\prime}\beta = 1 \qquad (1)$

$\vdash . (1) . *101{\cdot}2 . \qquad \supset \vdash . \text{Nc}'\{(\iota'x) \exp \beta\} = 1 \qquad (2)$

$\vdash . *101{\cdot}2 . *116{\cdot}25 . \supset \vdash . 1^{\text{Nc}'\beta} = \text{Nc}'\{(\iota'x) \exp \beta\} \qquad (3)$

$\vdash . (2) . (3) . \supset \vdash . \text{Prop}$

***116·331.** $\vdash : \mu \in \text{NC} - \iota'\Lambda . \supset . 1^\mu = 1$

Dem. $\vdash . *103{\cdot}34 . \supset \vdash : \text{Hp} . \supset . (\exists\beta) . \mu = \text{N}_0\text{c}'\beta .$

$[*13{\cdot}12{\cdot}15] \qquad \supset . (\exists\beta) . 1^\mu = 1^{\text{N}_0\text{c}'\beta} .$

$[(*116{\cdot}04)] \qquad \supset . (\exists\beta) . 1^\mu = 1^{\text{Nc}'\beta} .$

$[*116{\cdot}33] \qquad \supset . 1^\mu = 1 : \supset \vdash . \text{Prop}$

***116·34.** $\vdash . \mu^2 = \mu \times_0 \mu$

Dem.

$\vdash . *24{\cdot}1 . *101{\cdot}3 . \supset \vdash . \iota'\Lambda \cup \iota'\text{V} \in 2 .$

$[*116{\cdot}222] \qquad \supset \vdash : \mu = \text{N}_0\text{c}'\alpha . \supset . \mu^2 = \text{Nc}'\text{Prod}'\alpha \underset{,,}{\downarrow} {}^{\prime\prime}(\iota'\Lambda \cup \iota'\text{V})$

$[*53{\cdot}32] \qquad = \text{Nc}'\text{Prod}'(\iota'\alpha \underset{,,}{\downarrow} \Lambda \cup \iota'\alpha \underset{,,}{\downarrow} \text{V})$

$[*115{\cdot}13.*55{\cdot}233.*38{\cdot}2] \qquad = \text{Nc}'(\alpha \underset{,,}{\downarrow} \Lambda \times \alpha \underset{,,}{\downarrow} \text{V})$

$[*113{\cdot}11{\cdot}25{\cdot}13] \qquad = \text{Nc}'\alpha \times_0 \text{Nc}'\alpha$

$[*113{\cdot}24] \qquad = \mu \times_0 \mu \qquad (1)$

$\vdash . (1) . *103{\cdot}2 . \supset \vdash : \mu \in \text{N}_0\text{C} . \supset . \mu^2 = \mu \times_0 \mu \qquad (2)$

$\vdash . *116{\cdot}205 . \quad \supset \vdash : \mu \sim\in \text{N}_0\text{C} . \supset . \mu^2 = \Lambda$

$[*113{\cdot}205] \qquad = \mu \times_0 \mu \qquad (3)$

$\vdash . (2) . (3) . \supset \vdash . \text{Prop}$

***116·35.** $\vdash : \mu^\nu = 0 . \equiv . \mu = 0 . \nu \in \mathrm{NC} - \iota`0 - \iota`\Lambda$

Dem.

$\vdash . *116\cdot311 . \supset \vdash : \mu = 0 . \nu \in \mathrm{NC} - \iota`0 - \iota`\Lambda . \supset . \mu^\nu = 0$ (1)

$\vdash . *101\cdot12 . \supset \vdash : \mu^\nu = 0 . \supset . \exists ! \mu^\nu .$

$[*116\cdot203] \qquad \supset . \mu, \nu \in \mathrm{NC} - \iota`\Lambda$ (2)

$\vdash . (2) . *116\cdot21 . *54\cdot102 . \supset$

$\vdash :. \mu^\nu = 0 . \supset : \xi = \Lambda . \equiv . (\exists \alpha, \beta) . \alpha \in \mu . \beta \in \nu . \xi \,\mathrm{sm}\, (\alpha \exp \beta) :$

$[*73\cdot47] \qquad \supset : (\exists \alpha, \beta) . \alpha \in \mu . \beta \in \nu . \alpha \exp \beta = \Lambda :$

$[*116\cdot18] \qquad \supset : (\exists \alpha, \beta) . \alpha \in \mu . \beta \in \nu . \alpha = \Lambda . \beta \neq \Lambda :$

$[*13\cdot195] \qquad \supset : \Lambda \in \mu . \nu \neq \iota`\Lambda . \exists ! \nu :$

$[*101\cdot1 . *100\cdot45 . (2)] \supset : \mu = 0 . \nu \in \mathrm{NC} - \iota`\Lambda - \iota`0$ (3)

$\vdash . (1) . (3) . \supset \vdash . \mathrm{Prop}$

***116·351.** $\vdash : \mu \in \mathrm{NC} - \iota`\Lambda . \kappa = \Lambda . \nu = 0 . \supset . \mu^\nu = \Pi\mathrm{Nc}`\kappa = 1$

$[*116\cdot301 . *114\cdot2]$

***116·352.** $\vdash : \mu = 0 . \nu \in \mathrm{NC} - \iota`\Lambda . \kappa \in \nu . \Lambda \in \kappa . \supset . \mu^\nu = \Pi\mathrm{Nc}`\kappa = 0$

$[*116\cdot311 . *114\cdot23]$

***116·353.** $\vdash : \mu = 0 . \nu \in \mathrm{NC} - \iota`\Lambda . \kappa \in \nu \cap \mathrm{Cl}`\mu . \supset . \mu^\nu = \Pi\mathrm{Nc}`\kappa$

Dem.

$\vdash . *60\cdot362 . *54\cdot1 . \supset \vdash :. \mathrm{Hp} . \supset : \kappa = \Lambda . \vee . \kappa = \iota`\Lambda$ (1)

$\vdash . *100\cdot45 . *101\cdot1 . \supset \vdash : \mathrm{Hp} . \kappa = \Lambda . \supset . \nu = 0 .$

$[*116\cdot351] \qquad \supset . \mu^\nu = \Pi\mathrm{Nc}`\kappa$ (2)

$\vdash . *51\cdot16 . \qquad \supset \vdash : \mathrm{Hp} . \kappa = \iota`\Lambda . \supset . \Lambda \in \kappa .$

$[*116\cdot352] \qquad \supset . \mu^\nu = \Pi\mathrm{Nc}`\kappa$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash . \mathrm{Prop}$

***116·36.** $\vdash :. \mathrm{Mult\,ax} . \supset : \mu, \nu \in \mathrm{NC} - \iota`\Lambda . \kappa \in \nu \cap \mathrm{Cl}`\mu . \supset . \Pi\mathrm{Nc}`\kappa = \mu^\nu$

Dem.

$\vdash . *113\cdot12 . *100\cdot45 . \supset \vdash : \mu, \nu \in \mathrm{NC} . \alpha \in \mu . \beta \in \nu . \exists ! \alpha . \supset . \alpha \underset{;}{\downarrow} {}``\beta \in \nu \cap \mathrm{Cl}`\mu$ (1)

$\vdash . (1) . *114\cdot571 . \quad \supset \vdash :. \mathrm{Mult\,ax} . \supset :$

$\mu, \nu \in \mathrm{NC} . \alpha \in \mu . \beta \in \nu . \exists ! \alpha . \kappa \in \nu \cap \mathrm{Cl}`\mu . \supset . \Pi\mathrm{Nc}`\kappa = \Pi\mathrm{Nc}`\alpha \underset{;}{\downarrow} {}``\beta$

$[*116\cdot14 . *114\cdot1] \qquad = \mathrm{Nc}`(\alpha \exp \beta)$

$[*116\cdot271] \qquad = \mu^\nu$ (2)

$\vdash . (2) . \supset$

$\vdash :. \mathrm{Mult\,ax} . \supset : \mu, \nu \in \mathrm{NC} - \iota`\Lambda . \exists ! \mu - \iota`\Lambda . \kappa \in \nu \cap \mathrm{Cl}`\mu . \supset . \Pi\mathrm{Nc}`\kappa = \mu^\nu$ (3)

$\vdash . *51\cdot4 . *54\cdot1 . \supset \vdash : \mu \in \mathrm{NC} - \iota`\Lambda . \sim \exists ! \mu - \iota`\Lambda . \supset . \mu = 0$ (4)

$\vdash . (4) . *116\cdot353 . \supset$

$\vdash : \mu, \nu \in \mathrm{NC} - \iota`\Lambda . \sim \exists ! \mu - \iota`\Lambda . \kappa \in \nu \cap \mathrm{Cl}`\mu . \supset . \Pi\mathrm{Nc}`\kappa = \mu^\nu$ (5)

$\vdash . (3) . (5) . \supset \vdash . \mathrm{Prop}$

In the above proposition, "$\nu \in \mathrm{NC}$" is sufficient hypothesis as to ν, since "$\nu \neq \Lambda$" is implied by $\kappa \in \nu \cap \mathrm{Cl}`\mu$. But $\mu \neq \Lambda$ is essential, since if $\mu = \Lambda$, $\mu^{\nu} = \Lambda$ and $\kappa = \Lambda$ (provided $\nu = 0$), whence $\Pi\mathrm{Nc}`\kappa = 1$.

The above proposition connects exponentiation with multiplication.

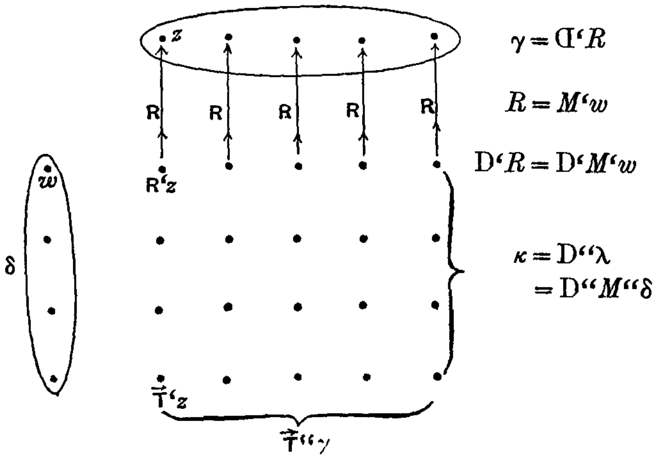
***116·361.** $\vdash :. \mathrm{Mult\,ax} . \supset : \mu, \nu \in \mathrm{NC} - \iota`\Lambda . \kappa \in \nu \cap \mathrm{Cl\,excl}`\mu . \supset . \mathrm{Prod}`\kappa \in \mu^{\nu}$

Dem.

$\vdash . *115{\cdot}12 . \supset \vdash : \kappa \in \nu \cap \mathrm{Cl\,excl}`\mu . \supset . \mathrm{Prod}`\kappa \in \Pi\mathrm{Nc}`\kappa$ (1)

$\vdash . (1) . *116{\cdot}36 . \supset \vdash . \mathrm{Prop}$

The following propositions, which illustrate certain generalizations of the relations of rows and columns, may be made clearer by the accompanying



figure, in which, for the sake of simplicity, all the classes concerned are taken to be finite.

Let κ be a set of classes, constituted by four rows of five dots in the figure, which are each given as similar to a given class γ, represented by the top row of five dots in the figure, namely the row enclosed in an oval. We assume that an actual correlating relation is given correlating each member of κ with γ. Let λ be the class of these relations, and assume that λ consists of one correlator for each member of κ, and that $\kappa \in \mathrm{Cls}^2\,\mathrm{excl}$. Thus $\mathrm{D}``\lambda = \kappa$, and $R \in \lambda . \supset . \text{Ɑ}`R = \gamma$. Put $T = \breve{s}`\lambda$. Then, if $z \in \gamma$, T relates to z every member of the column below z, *i.e.* $\overrightarrow{T}`z$ consists of the four dots which are vertically below z; assuming, what in the circumstances is possible, that each dot is placed below its correlate in γ. Thus $\overrightarrow{T}``\gamma$ represents the columns, while $\mathrm{D}``\lambda$ represents the rows.

We prove, in *116·41, that $\overrightarrow{T}``\gamma$, the class of rows, has double similarity with $\lambda \underset{\prime\prime}{\downarrow} ``\gamma$, or, what comes to the same thing, with $\kappa \underset{\prime\prime}{\downarrow} ``\gamma$. Hence it follows that $T``\gamma$, which is the whole class of dots, is similar to $\gamma \times \lambda$ or $\gamma \times \kappa$, and that $\mathrm{Nc}`\epsilon_{\Delta}`\overrightarrow{T}``\gamma$, which is the product of the numbers of the columns, is equal

to $(\mathrm{Nc}\text{‘}\lambda)^{\mathrm{Nc}\text{‘}\gamma}$ or $(\mathrm{Nc}\text{‘}\kappa)^{\mathrm{Nc}\text{‘}\gamma}$. The correlator which is used for proving these propositions is W, where, if R is a member of λ and z is a member of γ, W correlates $R\text{‘}z$ with $R \downarrow z$.

Similarly, by correlating $R\text{‘}z$ with $z \downarrow R$, calling the correlator U, we have $U\text{‘‘} \downarrow R\text{‘‘}\gamma = R\text{‘‘}\gamma$, *i.e.* $U_\epsilon\text{‘}\gamma \underset{,,}{\downarrow} R = \mathrm{D}\text{‘}R$, whence $U_\epsilon\text{‘‘}\gamma \underset{,,}{\downarrow} \text{‘‘}\lambda = \mathrm{D}\text{‘‘}\lambda$. Hence $\mathrm{D}\text{‘‘}\lambda$, *i.e.* the class of rows, has double similarity with $\gamma \underset{,,}{\downarrow} \text{‘‘}\lambda$ or $\gamma \underset{,,}{\downarrow} \text{‘‘}\kappa$, whence the product of the numbers of the rows is $(\mathrm{Nc}\text{‘}\gamma)^{\mathrm{Nc}\text{‘}\lambda}$ or $(\mathrm{Nc}\text{‘}\gamma)^{\mathrm{Nc}\text{‘}\kappa}$.

Finally, we take a class δ similar to κ or λ (illustrated in the figure by the column of dots enclosed in an oval), and calling M a correlator of λ and δ, we replace λ by $M\text{‘‘}\delta$ and κ by $\mathrm{D}\text{‘‘}M\text{‘‘}\delta$. We thus find that, if $M \upharpoonright \delta$ correlates with δ a class of relations whose domains are mutually exclusive, and which each correlate their domains with a given class γ, then $\mathrm{D}\text{‘‘}M\text{‘‘}\delta$ has double similarity with $\gamma \underset{,,}{\downarrow} \text{‘‘}\delta$, whence the same results as before with δ in place of κ or λ.

The following propositions are useful in connecting multiplication with exponentiation, and in proving the formal laws of exponentiation.

$*116{\cdot}4{\cdot}401$ are lemmas for $*116{\cdot}41$.

$*116{\cdot}4$. $\vdash :. \lambda \subset 1 \to 1 : R, S \in \lambda \,.\, \exists \,!\, \mathrm{D}\text{‘}R \cap \mathrm{D}\text{‘}S \,.\, \supset_{R,S} .\, R = S :$
$\text{Ɑ}\text{‘‘}\lambda \subset \iota\text{‘}\gamma \,.\, W = \hat{x}\hat{P}\{(\exists R, z) \,.\, R \in \lambda \,.\, x = R\text{‘}z \,.\, P = R \downarrow z\} :$
$\supset .\, W \in 1 \to 1 \,.\, \text{Ɑ}\text{‘}W = \gamma \times \lambda \,.\, \mathrm{D}\text{‘}W = \mathrm{D}\text{‘}\dot{s}\text{‘}\lambda$

Dem.

$\vdash .\, *21{\cdot}33 \,.\, \supset \vdash :. \mathrm{Hp} \,.\, \supset : xWP \,.\, xWQ \,.\, \equiv .$
$(\exists R, S, z, w) \,.\, R, S \in \lambda \,.\, x = R\text{‘}z = S\text{‘}w \,.\, P = R \downarrow z \,.\, Q = S \downarrow w \,.$
$[*33{\cdot}43]$ $\equiv .\, (\exists R, S, z, w) \,.\, R, S \in \lambda \,.\, x = R\text{‘}z = S\text{‘}w \,.\, x \in \mathrm{D}\text{‘}R \cap \mathrm{D}\text{‘}S \,.$
$P = R \downarrow z \,.\, Q = S \downarrow w \,.$
$[\mathrm{Hp}.*13{\cdot}195]$ $\supset .\, (\exists R, z, w) \,.\, R \in \lambda \,.\, x = R\text{‘}z = R\text{‘}w \,.\, P = R \downarrow z \,.\, Q = R \downarrow w \,.$
$[*71{\cdot}532.*13{\cdot}195]$ $\supset .\, (\exists R, z) \,.\, R \in \lambda \,.\, x = R\text{‘}z \,.\, P = R \downarrow z \,.\, Q = R \downarrow z \,.$
$[*13{\cdot}172]$ $\supset .\, P = Q$ (1)
$\vdash .\, *21{\cdot}33 \,.\, \supset \vdash :. \mathrm{Hp} \,.\, \supset : xWP \,.\, yWP \,.\, \equiv .$
$(\exists R, S, z, w) \,.\, R, S \in \lambda \,.\, x = R\text{‘}z \,.\, y = S\text{‘}w \,.\, P = R \downarrow z = Q \downarrow w \,.$
$[*55{\cdot}202]$ $\supset .\, (\exists R, S, z, w) \,.\, R, S \in \lambda \,.\, x = R\text{‘}z \,.\, y = S\text{‘}w \,.\, R = S \,.\, z = w \,.$
$[*13{\cdot}22{\cdot}172]$ $\supset .\, x = y$ (2)
$\vdash .\, *33{\cdot}131 \,.\, \supset \vdash :. \mathrm{Hp} \,.\, \supset : P \in \text{Ɑ}\text{‘}W \,.\, \equiv .\, (\exists x, R, z) \,.\, R \in \lambda \,.\, x = R\text{‘}z \,.\, P = R \downarrow z \,.$
$[*71{\cdot}411]$ $\equiv .\, (\exists R, z) \,.\, R \in \lambda \,.\, z \in \text{Ɑ}\text{‘}R \,.\, P = R \downarrow z \,.$
$[\mathrm{Hp}]$ $\equiv .\, (\exists R, z) \,.\, R \in \lambda \,.\, z \in \gamma \,.\, P = R \downarrow z \,.$
$[*113{\cdot}101]$ $\equiv .\, P \in \gamma \times \lambda$ (3)
$\vdash .\, *33{\cdot}13 \,.\, \supset \vdash :. \mathrm{Hp} \,.\, \supset : x \in \mathrm{D}\text{‘}W \,.\, \equiv .\, (\exists P, R, z) \,.\, R \in \lambda \,.\, x = R\text{‘}z \,.\, P = R \downarrow z \,.$
$[*55{\cdot}12.*71{\cdot}36]$ $\equiv .\, (\exists R, z) \,.\, R \in \lambda \,.\, xRz \,.$
$[*41{\cdot}11.*33{\cdot}13]$ $\equiv .\, x \in \mathrm{D}\text{‘}\dot{s}\text{‘}\lambda$ (4)
$\vdash .\, (1) \,.\, (2) \,.\, (3) \,.\, (4) \,.\, \supset \vdash .\, \mathrm{Prop}$

***116·401.** $\vdash : \mathrm{Hp} * 116{\cdot}4 \,.\, T = \dot{s}`\lambda \,.\supset .\, \overrightarrow{T}``\gamma = W_\epsilon``\lambda \underset{,,}{\downarrow} ``\gamma$

Dem. $\vdash . *37{\cdot}11{\cdot}1 . *38{\cdot}2 . \supset \vdash :. \mathrm{Hp} . z \in \gamma . \supset :$

$x \in W_\epsilon`\lambda \underset{,,}{\downarrow} z . \equiv . (\exists R) . R \in \lambda . x W (R \downarrow z) .$

$[*21{\cdot}33] \quad \equiv . (\exists R, S, w) . R, S \in \lambda . x = S`w . R \downarrow z = S \downarrow w .$

$[*55{\cdot}202 . *13{\cdot}22] \quad \equiv . (\exists R) . R \in \lambda . x = R`z .$

$[*71{\cdot}36] \quad \equiv . (\exists R) . R \in \lambda . x R z .$

$[*41{\cdot}11] \quad \equiv . x (\dot{s}`\lambda) z .$

$[\mathrm{Hp} . *32{\cdot}18] \quad \equiv . x \in \overrightarrow{T}`z$ (1)

$\vdash . (1) . *37{\cdot}68 . \supset \vdash . \mathrm{Prop}$

***116·41.** $\vdash :. \lambda \subset 1 \rightarrow 1 . \text{G}``\lambda \subset \iota`\gamma : R, S \in \lambda . \exists ! \mathrm{D}`R \cap \mathrm{D}`S . \supset_{R,S} . R = S : T = \dot{s}`\lambda :$

$\supset . \overrightarrow{T}``\gamma \,\mathrm{sm}\,\mathrm{sm}\, \lambda \underset{,,}{\downarrow} ``\gamma . \overrightarrow{T}``\gamma \,\mathrm{sm}\, \gamma \times \lambda . T \in \mathrm{Cls} \rightarrow 1 . \overrightarrow{T}``\gamma \in \mathrm{Cls}^2\,\mathrm{excl} .$

$\mathrm{Nc}`\epsilon_\Delta`\overrightarrow{T}``\gamma = \mathrm{Nc}`T_\Delta`\gamma = \mathrm{Nc}`\mathrm{Prod}`\overrightarrow{T}``\gamma = \mathrm{Nc}`(\lambda \exp \gamma) = (\mathrm{Nc}`\lambda)^{\mathrm{Nc}`\gamma}$

Dem.

$\vdash . *116{\cdot}4{\cdot}401 . *111{\cdot}4 . *113{\cdot}1 . \supset \vdash : \mathrm{Hp} . \supset . \overrightarrow{T}``\gamma \,\mathrm{sm}\,\mathrm{sm}\, \lambda \underset{,,}{\downarrow} ``\gamma .$ (1)

$[*111{\cdot}44 . *40{\cdot}5] \quad \supset . \overrightarrow{T}``\gamma \,\mathrm{sm}\, \gamma \times \lambda$ (2)

$\vdash . *72{\cdot}321 . *85{\cdot}14 . \supset \vdash : \mathrm{Hp} . \supset . T \in \mathrm{Cls} \rightarrow 1 . \mathrm{Nc}`\epsilon_\Delta`\overrightarrow{T}``\gamma = \mathrm{Nc}`T_\Delta`\gamma$ (3)

$\vdash . (3) . *84{\cdot}51 . \quad \supset \vdash : \mathrm{Hp} . \supset . \overrightarrow{T}``\gamma \in \mathrm{Cls}^2\,\mathrm{excl} .$ (4)

$[*115{\cdot}12] \quad \supset . \mathrm{Nc}`\epsilon_\Delta`\overrightarrow{T}``\gamma = \mathrm{Nc}`\mathrm{Prod}`\overrightarrow{T}``\gamma$ (5)

$\vdash . (1) . *114{\cdot}52 . \quad \supset \vdash : \mathrm{Hp} . \supset . \mathrm{Nc}`\epsilon_\Delta`\overrightarrow{T}``\gamma = \mathrm{Nc}`\epsilon_\Delta`\lambda \underset{,,}{\downarrow} ``\gamma$

$[*116{\cdot}14] \quad = \mathrm{Nc}`(\lambda \exp \gamma)$ (6)

$[*116{\cdot}25] \quad = (\mathrm{Nc}`\lambda)^{\mathrm{Nc}`\gamma}$ (7)

$\vdash . (1) . (2) . (3) . (4) . (5) . (6) . (7) . \supset \vdash . \mathrm{Prop}$

The following proposition is merely another form of *116·41.

***116·411.** $\vdash :. \kappa \in \mathrm{Cls}^2\,\mathrm{excl} : \alpha \in \kappa . \supset_\alpha . M`\alpha \in \alpha \,\overline{\mathrm{sm}}\, \gamma : T = \dot{s}`M``\kappa : \supset .$

$\overrightarrow{T}``\gamma \,\mathrm{sm}\,\mathrm{sm}\, \kappa \underset{,,}{\downarrow} ``\gamma . \overrightarrow{T}``\gamma \,\mathrm{sm}\, \gamma \times \kappa . T \in \mathrm{Cls} \rightarrow 1 . \overrightarrow{T}``\gamma \in \mathrm{Cls}^2\,\mathrm{excl} .$

$\mathrm{Nc}`\epsilon_\Delta`\overrightarrow{T}``\gamma = \mathrm{Nc}`T_\Delta`\gamma = \mathrm{Nc}`\mathrm{Prod}`\overrightarrow{T}``\gamma = \mathrm{Nc}`(\kappa \exp \gamma) = (\mathrm{Nc}`\kappa)^{\mathrm{Nc}`\gamma}$

Dem.

$\vdash . *73{\cdot}03 . \quad \supset \vdash : \mathrm{Hp} . \supset . M``\kappa \subset 1 \rightarrow 1 . \text{G}``M``\kappa \subset \iota`\gamma$ (1)

$\vdash . *111{\cdot}16 . \supset \vdash :. \mathrm{Hp} . \supset : \alpha, \beta \in \kappa . M`\alpha = M`\beta . \supset . \alpha = \beta$ (2)

$\vdash . *14{\cdot}21 . \quad \supset \vdash :. \mathrm{Hp} . \supset : \alpha \in \kappa . \supset . \mathrm{E} ! M`\alpha$ (3)

$\vdash . (2) . (3) . *73{\cdot}24 . \supset \vdash : \mathrm{Hp} . \supset . M``\kappa \,\mathrm{sm}\, \kappa$ (4)

$\vdash . *73{\cdot}03 . \quad \supset \vdash :. \mathrm{Hp} . \supset : \alpha \in \kappa . \supset . \mathrm{D}`M`\alpha = \alpha :$

$[*13{\cdot}12] \quad \supset : \alpha, \beta \in \kappa . \exists ! \mathrm{D}`M`\alpha \cap \mathrm{D}`M`\beta . \supset . \exists ! \alpha \cap \beta .$

$[*84{\cdot}11] \quad \supset . \alpha = \beta .$

$[*30{\cdot}37 . (3)] \quad \supset . M`\alpha = M`\beta :$

$[*37{\cdot}63] \quad \supset : R, S \in M``\kappa . \exists ! \mathrm{D}`R \cap \mathrm{D}`S . \supset . R = S$ (5)

$\vdash . (1) . (4) . (5) . *116{\cdot}41 \dfrac{M``\kappa}{\lambda} . *113{\cdot}13 . *116{\cdot}19 . \supset \vdash . \mathrm{Prop}$

$*116{\cdot}412{\cdot}413$ are lemmas for $*116{\cdot}414$.

$*116{\cdot}412$. $\vdash :. \lambda \subset 1 \to 1 : R, S \in \lambda . \exists ! \text{D}'R \cap \text{D}'S . \supset_{R,S} . R = S : \text{C}``\lambda \subset \iota'\gamma :$
$U = \hat{x}\hat{P}\{(\exists R, z) . R \in \lambda . x = R'z . P = z \downarrow R\} :$
$\supset . U \in (s'\text{D}``\lambda)\, \overline{\text{sm}}\, (\lambda \times \gamma)$ [Proof as in $*116{\cdot}4$]

$*116{\cdot}413$. $\vdash : \text{Hp}\, *116{\cdot}412 . \supset . \text{D}``\lambda = U_\epsilon``\gamma \underset{\text{''}}{\downarrow} ``\lambda$ [Proof as in $*116{\cdot}401$]

$*116{\cdot}414$. $\vdash : \text{Hp}\, *116{\cdot}412 . \supset . U \in (\text{D}``\lambda)\, \overline{\text{sm}}\,\overline{\text{sm}}\, (\gamma \underset{\text{''}}{\downarrow} ``\lambda) . (\text{D}``\lambda) \,\text{sm}\,\text{sm}\, (\gamma \underset{\text{''}}{\downarrow} ``\lambda)$
[$*116{\cdot}412{\cdot}413$]

$*116{\cdot}42$. $\vdash :. \lambda \subset 1 \to 1 : R, S \in \lambda . \exists ! \text{D}'R \cap \text{D}'S . \supset_{R,S} . R = S : \text{C}``\lambda \subset \iota'\gamma :$
$\supset . \text{D}``\lambda \,\text{sm}\,\text{sm}\, (\gamma \underset{\text{''}}{\downarrow} ``\lambda) . (\text{D}'s'\lambda) \,\text{sm}\, (\lambda \times \gamma) . (\epsilon_\Delta{}'\text{D}``\lambda) \,\text{sm}\, (\gamma \exp \lambda) .$
$\text{Nc}'\text{Prod}'\text{D}``\lambda = \Pi\text{Nc}'\text{D}``\lambda = (\text{Nc}'\gamma)^{\text{Nc}'\lambda}$
[$*116{\cdot}414{\cdot}25 . *115{\cdot}51 . *111{\cdot}44 . *41{\cdot}43$]

$*116{\cdot}422$. $\vdash :. M \upharpoonright \delta \in 1 \to 1 : w, v \in \delta . \exists ! \text{D}'M'w \cap \text{D}'M'v . \supset_{w,v} . w = v :$
$w \in \delta . \supset_w . M'w \in 1 \to 1 . \text{C}'M'w = \gamma : \supset . \text{D}``M``\delta \,\text{sm}\,\text{sm}\, \gamma \underset{\text{''}}{\downarrow} ``\delta$

Dem.

$\vdash . *116{\cdot}42 \dfrac{M``\delta}{\lambda} . \supset$

$\vdash :. M``\delta \subset 1 \to 1 : R, S \in M``\delta . \exists ! \text{D}'R \cap \text{D}'S . \supset_{R,S} . R = S : \text{C}``M``\delta \subset \iota'\gamma :$
$\supset . \text{D}``M``\delta \,\text{sm}\,\text{sm}\, \gamma \underset{\text{''}}{\downarrow} ``M``\delta$ (1)

$\vdash . *14{\cdot}21 . \quad \supset \vdash :. \text{Hp} . \supset : w \in \delta . \supset . \text{E}!\, M'w :$ (2)
[$*33{\cdot}43$] $\quad \supset : \delta \subset \text{C}'M :$
[$*73{\cdot}15$] $\quad \supset : (M``\delta) \,\text{sm}\, \delta$ (3)
$\vdash . *51{\cdot}15 . \quad \supset \vdash :. \text{Hp} . \supset : w \in \delta . \supset . \text{C}'M'w \in \iota'\gamma :$
[$*37{\cdot}61$] $\quad \supset : \text{C}``M``\delta \subset \iota'\gamma$ (4)
$\vdash . (2) . *30{\cdot}37 . \supset \vdash :. \text{Hp} . \supset : w, v \in \delta . \exists ! \text{D}'M'w \cap \text{D}'M'v . \supset_{w,v} . M'w = M'v :$
[$*37{\cdot}63$] $\quad \supset : R, S \in M``\delta . \exists ! \text{D}'R \cap \text{D}'S . \supset_{R,S} . R = S$ (5)
$\vdash . (1) . (4) . (5) . \supset \vdash : \text{Hp} . \supset . \text{D}``M``\delta \,\text{sm}\,\text{sm}\, \gamma \underset{\text{''}}{\downarrow} ``M``\delta .$
[$(3) . *113{\cdot}13$] $\quad \supset . \text{D}``M``\delta \,\text{sm}\,\text{sm}\, \gamma \underset{\text{''}}{\downarrow} ``\delta : \supset \vdash . \text{Prop}$

$*116{\cdot}43$. $\vdash :. M \upharpoonright \delta \in 1 \to 1 : w, v \in \delta . \exists ! \text{D}'M'w \cap \text{D}'M'v . \supset_{w,v} . w = v :$
$w \in \delta . \supset_w . M'w \in 1 \to 1 . \text{C}'M'w = \gamma :$
$\supset . \text{Prod}'\text{D}``M``\delta \,\text{sm}\,\text{sm}\, (\gamma \exp \delta) . \Pi\text{Nc}'\text{D}``M``\delta = (\text{Nc}'\gamma)^{\text{Nc}'\delta}$

Dem.

$\vdash . *115{\cdot}51 . *116{\cdot}422 . \supset \vdash : \text{Hp} . \supset . \text{Prod}'\text{D}``M``\delta \,\text{sm}\,\text{sm}\, (\gamma \exp \delta)$ (1)
$\vdash . *116{\cdot}422 . *114{\cdot}52 . \supset \vdash : \text{Hp} . \supset . \Pi\text{Nc}'\text{D}``M``\delta = \Pi\text{Nc}'\gamma \underset{\text{''}}{\downarrow} ``\delta$ (2)
$\vdash . *116{\cdot}14{\cdot}25 . \quad \supset \vdash . \Pi\text{Nc}'\gamma \underset{\text{''}}{\downarrow} ``\delta = (\text{Nc}'\gamma)^{\text{Nc}'\delta}$ (3)
$\vdash . (1) . (2) . (3) . \supset \vdash . \text{Prop}$

The above proposition is used in $*116{\cdot}534{\cdot}61$.

$*116\cdot44.\ \vdash :.\ \exists ! \gamma : (z) . M\text{‘}z \in 1 \to 1 . \text{Ɑ}\text{‘}M\text{‘}z = \mathrm{V} :$

$w, v \in \delta . \exists ! (M\text{‘}w)\text{“}\gamma \cap (M\text{‘}v)\text{“}\gamma . \supset_{w,v} . w = v :$

$\supset . \mathrm{D}\text{“}\upharpoonright\gamma\text{“}M\text{“}\delta \ \mathrm{sm\ sm}\ \gamma \underset{,,}{\downarrow}\text{“}\delta . \mathrm{Prod}\text{‘}\mathrm{D}\text{“}\upharpoonright\gamma\text{“}M\text{“}\delta \ \mathrm{sm\ sm}\ (\gamma \exp \delta)$

Dem.

$\vdash . *71\cdot29 . *35\cdot65 . \supset$

$\vdash :.\ \mathrm{Hp} : (z) . N\text{‘}z = (M\text{‘}z) \upharpoonright \gamma : \supset . (z) . N\text{‘}z \in 1 \to 1 . \text{Ɑ}\text{‘}N\text{‘}z = \gamma \qquad (1)$

$\vdash . *37\cdot401 . \supset \vdash :.\ \mathrm{Hp} . \mathrm{Hp}(1) . \supset : w, v \in \delta . \exists ! \mathrm{D}\text{‘}N\text{‘}w \cap \mathrm{D}\text{‘}N\text{‘}v . \supset_{w,v} . w = v \qquad (2)$

$\vdash . *35\cdot7 . \quad \supset \vdash :.\ \mathrm{Hp} . \mathrm{Hp}(1) . \supset : x \in \gamma . w, v \in \delta . N\text{‘}w = N\text{‘}v . \supset . (N\text{‘}w)\text{‘}x = (N\text{‘}v)\text{‘}x .$

$[(2)] \qquad \supset . w = v \qquad (3)$

$\vdash . (3) . *10\cdot11\cdot23\cdot35 . \supset \vdash :.\ \mathrm{Hp} . \mathrm{Hp}(1) . \supset : w, v \in \delta . N\text{‘}w = N\text{‘}v . \supset . w = v :$

$[*71\cdot55\cdot166] \qquad \supset : N \upharpoonright \delta \in 1 \to 1 \qquad (4)$

$\vdash . (1) . (2) . (4) . *116\cdot422 . *115\cdot51 . \supset$

$\vdash : \mathrm{Hp} . \mathrm{Hp}(1) . \supset . \mathrm{D}\text{“}N\text{“}\delta \ \mathrm{sm\ sm}\ \gamma \underset{,,}{\downarrow}\text{“}\delta . \mathrm{Prod}\text{‘}\mathrm{D}\text{“}N\text{“}\delta \ \mathrm{sm\ sm}\ (\gamma \exp \delta) \qquad (5)$

$\vdash . *38\cdot11 . \supset \vdash : \mathrm{Hp} . \mathrm{Hp}(1) . \supset . \mathrm{D}\text{‘}N\text{‘}z = \mathrm{D}\text{‘}\upharpoonright\gamma\text{‘}M\text{‘}z .$

$[*37\cdot353] \qquad \supset . \mathrm{D}\text{“}N\text{“}\delta = \mathrm{D}\text{“}\upharpoonright\gamma\text{“}M\text{“}\delta \qquad (6)$

$\vdash . (5) . (6) . \supset \vdash . \mathrm{Prop}$

$*116\cdot45.\ \vdash :.\ (z) . M\text{‘}z \in 1 \to 1 . \text{Ɑ}\text{‘}M\text{‘}z = \mathrm{V} :$

$w, v \in \delta . \exists ! (M\text{‘}w)\text{“}\gamma \cap (M\text{‘}v)\text{“}\gamma . \supset_{w,v} . w = v : \supset . \mathrm{Prod}\text{‘}\mathrm{D}\text{“}\upharpoonright\gamma\text{“}M\text{“}\delta \ \mathrm{sm}\ (\gamma \exp \delta)$

Dem.

$\vdash . *116\cdot182 . *115\cdot142 . *37\cdot29 . \supset$

$\vdash : \mathrm{Hp} . \gamma = \Lambda . \exists ! \delta . \supset . \mathrm{Prod}\text{‘}\mathrm{D}\text{“}\upharpoonright\gamma\text{“}M\text{“}\delta = \Lambda . \gamma \exp \delta = \Lambda \qquad (1$

$\vdash . *115\cdot1 . *83\cdot15 . *116\cdot181 . \supset$

$\vdash : \mathrm{Hp} . \delta = \Lambda . \supset . \mathrm{Prod}\text{‘}\mathrm{D}\text{“}\upharpoonright\gamma\text{“}M\text{“}\delta = \iota\text{‘}\Lambda . \gamma \exp \delta = \iota\text{‘}\Lambda \qquad (2$

$\vdash . (1) . (2) . *116\cdot44 . \supset \vdash . \mathrm{Prop}$

The above proposition is used in $*116\cdot676$.

We have now to prove the three formal laws of exponentiation, namely

$$\mu^\nu \times_c \mu^\varpi = \mu^{\nu +_c \varpi},$$
$$\mu^\varpi \times_c \nu^\varpi = (\mu \times_c \nu)^\varpi,$$
and
$$(\mu^\nu)^\varpi = \mu^{\nu \times_c \varpi}.$$

Of these the first is an immediate consequence of the distributive law, while the second and third result from forms of the associative law of multiplication.

$*116\cdot5.\ \vdash : \beta \cap \gamma = \Lambda . \supset . (\alpha \exp \beta) \times (\alpha \exp \gamma)\ \mathrm{sm}\ \alpha \exp (\beta \cup \gamma)$

Dem.

$\vdash . *113\cdot191 . \supset$

$\vdash : \mathrm{Hp} . \exists ! \alpha . \supset . \alpha \underset{,,}{\downarrow}\text{“}\beta \cap \alpha \underset{,,}{\downarrow}\text{“}\gamma = \Lambda .$

$[*114\cdot301] \qquad \supset . \epsilon_\Delta\text{‘}\alpha \underset{,,}{\downarrow}\text{“}\beta \times \epsilon_\Delta\text{‘}\alpha \underset{,,}{\downarrow}\text{“}\gamma \ \mathrm{sm}\ \epsilon_\Delta\text{‘}(\alpha \underset{,,}{\downarrow}\text{“}\beta \cup \alpha \underset{,,}{\downarrow}\text{“}\gamma) .$

$[*116\cdot14 . *113\cdot13] \supset . (\alpha \exp \beta) \times (\alpha \exp \gamma)\ \mathrm{sm}\ \epsilon_\Delta\text{‘}(\alpha \underset{,,}{\downarrow}\text{“}\beta \cup \alpha \underset{,,}{\downarrow}\text{“}\gamma) .$

$[*37\cdot22] \qquad \supset . (\alpha \exp \beta) \times (\alpha \exp \gamma)\ \mathrm{sm}\ \epsilon_\Delta\text{‘}\alpha \underset{,,}{\downarrow}\text{“}(\beta \cup \gamma) .$

$[*116\cdot14] \qquad \supset . (\alpha \exp \beta) \times (\alpha \exp \gamma)\ \mathrm{sm}\ \alpha \exp (\beta \cup \gamma) \qquad (1)$

$\vdash . *116{\cdot}182 . \supset \vdash : \alpha = \Lambda . \exists ! \beta . \supset . \alpha \exp \beta = \Lambda .$

$[*113{\cdot}114] \qquad \supset . (\alpha \exp \beta) \times (\alpha \exp \gamma) = \Lambda \qquad (2)$

$\vdash . *116{\cdot}182 . *24{\cdot}56 . \supset \vdash : \alpha = \Lambda . \exists ! \beta . \supset . \alpha \exp (\beta \cup \gamma) = \Lambda \qquad (3)$

$\vdash . (2) . (3) . \quad \supset \vdash : \alpha = \Lambda . \exists ! \beta . \supset . (\alpha \exp \beta) \times (\alpha \exp \gamma) \operatorname{sm} \alpha \exp (\beta \cup \gamma) \qquad (4)$

Similarly $\quad \vdash : \alpha = \Lambda . \exists ! \gamma . \supset . (\alpha \exp \beta) \times (\alpha \exp \gamma) \operatorname{sm} \alpha \exp (\beta \cup \gamma) \qquad (5)$

$\vdash . *116{\cdot}181 . \supset \vdash : \alpha = \Lambda . \beta = \Lambda . \gamma = \Lambda . \supset . (\alpha \exp \beta) \times (\alpha \exp \gamma) = \iota{`}\Lambda \times \iota{`}\Lambda \quad (6)$

$\vdash . *116{\cdot}181 . \supset \vdash : \alpha = \Lambda . \beta = \Lambda . \gamma = \Lambda . \supset . \alpha \exp (\beta \cup \gamma) = \iota{`}\Lambda \qquad (7)$

$\vdash . (6) . (7) . *113{\cdot}611 . *73{\cdot}43 . \supset$

$\vdash : \alpha = \Lambda . \beta = \Lambda . \gamma = \Lambda . \supset . (\alpha \exp \beta) \times (\alpha \exp \gamma) \operatorname{sm} \alpha \exp (\beta \cup \gamma) \qquad (8)$

$\vdash . (1) . (4) . (5) . (8) . \supset \vdash . \text{Prop}$

In the last line of the above proof, $*73{\cdot}43$ is required because the two Λ's involved have not been proved to be of the same type. They are in fact of the same type, but it is unnecessary to prove this.

$*116{\cdot}51$. $\vdash . (\alpha \exp \beta) \times (\alpha \exp \gamma) \operatorname{sm} \alpha \exp (\beta + \gamma)$

Dem.

$\vdash . *116{\cdot}19 . *110{\cdot}12 . \supset \vdash . (\alpha \exp \beta) \operatorname{sm} (\alpha \exp \downarrow \Lambda_\gamma {``} \iota {``} \beta) .$

$\qquad (\alpha \exp \gamma) \operatorname{sm} (\alpha \exp \Lambda_\beta \downarrow {``} \iota {``} \gamma) .$

$[*113{\cdot}13] \qquad \supset \vdash . (\alpha \exp \beta) \times (\alpha \exp \gamma) \operatorname{sm}$

$\qquad (\alpha \exp \downarrow \Lambda_\gamma {``} \iota {``} \beta) \times (\alpha \exp \Lambda_\beta \downarrow {``} \iota {``} \gamma) .$

$[*110{\cdot}11 . *116{\cdot}5] \qquad \supset \vdash . (\alpha \exp \beta) \times (\alpha \exp \gamma) \operatorname{sm}$

$\qquad \alpha \exp (\downarrow \Lambda_\gamma {``} \iota {``} \beta \cup \Lambda_\beta \downarrow {``} \iota {``} \gamma) \qquad (1)$

$\vdash . (1) . (*110{\cdot}01) . \supset \vdash . \text{Prop}$

$*116{\cdot}52$. $\vdash . \mu^\nu \times_c \mu^\varpi = \mu^{\nu +_c \varpi}$

Dem.

$\vdash . *116{\cdot}51 . *110{\cdot}22 . \supset$

$\vdash . (N_0c{`}\alpha)^{N_0c{`}\beta} \times_c (N_0c{`}\alpha)^{N_0c{`}\gamma} = (N_0c{`}\alpha)^{N_0c{`}\beta +_c N_0c{`}\gamma} \qquad (1)$

$\vdash . (1) . *103{\cdot}2 . \supset \vdash : \mu, \nu, \varpi \in N_0C . \supset . \mu^\nu \times_c \mu^\varpi = \mu^{\nu +_c \varpi} \qquad (2)$

$\vdash . *116{\cdot}205 . *113{\cdot}204 . \supset$

$\vdash : \mu \sim\in N_0C . \supset . \mu^\nu \times_c \mu^\varpi = \Lambda = \mu^{\nu +_c \varpi} \qquad (3)$

$\vdash . *116{\cdot}205 . *113{\cdot}204 . \supset$

$\vdash : \sim(\nu, \varpi \in N_0C) . \supset . \mu^\nu \times_c \mu^\varpi = \Lambda \qquad (4)$

$\vdash . *110{\cdot}4 . *116{\cdot}204 . \supset \vdash : \sim(\nu, \varpi \in N_0C) . \supset . \mu^{\nu +_c \varpi} = \Lambda \qquad (5)$

$\vdash . (4) . (5) . \supset \vdash : \sim(\nu, \varpi \in N_0C) . \supset . \mu^\nu \times_c \mu^\varpi = \mu^{\nu +_c \varpi} \qquad (6)$

$\vdash . (2) . (3) . (6) . \supset \vdash . \text{Prop}$

The following propositions are lemmas for

$$\mu^\varpi \times_c \nu^\varpi = (\mu \times_c \nu)^\varpi .$$

The principal previous propositions used in the proof are $*115{\cdot}6$ and $*116{\cdot}43$. The proof proceeds as follows.

$(\alpha \exp \gamma) \times (\beta \exp \gamma)$ is $\text{Prod}\text{‘}\alpha\underset{,,}{\downarrow}\text{“}\gamma \times \text{Prod}\text{‘}\beta\underset{,,}{\downarrow}\text{“}\gamma$. This, using $*115\cdot6$, and putting $\alpha\underset{,,}{\downarrow}$, $\beta\underset{,,}{\downarrow}$ in place of R and S of that proposition, is similar to $\epsilon_\Delta\text{‘}\hat{\mu}\{(\exists z) . z \in \gamma . \mu = \alpha\underset{,,}{\downarrow} z \times \beta\underset{,,}{\downarrow} z\}$, *i.e.* to $\epsilon_\Delta\text{‘}\hat{\mu}\{(\exists z) . z \in \gamma . \mu = \downarrow z\text{“}\alpha \times \downarrow z\text{“}\beta\}$.

Now by $*113\cdot65$, putting $R\dagger = R \| \breve{R}$ Dft, $\downarrow z\text{“}\alpha \times \downarrow z\text{“}\beta = (\downarrow z)\dagger\text{“}(\alpha \times \beta)$. We now apply $*116\cdot43$, taking $(\downarrow z)\dagger$ as the $M\text{‘}z$ of that proposition, or rather, taking $(\downarrow z)\dagger \upharpoonright (\alpha \times \beta)$. Thus we find

$$\epsilon_\Delta\text{‘}\hat{\mu}\{(\exists z) . z \in \gamma . \mu = \downarrow z\text{“}\alpha \times \downarrow z\text{“}\beta\} \text{ sm } (\alpha \times \beta) \exp \gamma.$$

Hence our proposition follows.

$*116\cdot529$. $R\dagger = R \| \breve{R}$ Dft [$*116$]

In $*150$, this notation will be introduced as a permanent definition. For the present, we only introduce it to avoid $(\downarrow z \| \text{Cnv}\text{‘}\downarrow z)$, which is awkward.

$*116\cdot53$. $\vdash : \exists ! \alpha . \exists ! \beta . \supset .$

$$(\alpha \exp \gamma) \times (\beta \exp \gamma) \text{ sm } \epsilon_\Delta\text{‘}\hat{\mu}\{(\exists z) . z \in \gamma . \mu = \downarrow z\text{“}\alpha \times \downarrow z\text{“}\beta\}$$

Dem.

$\vdash . *113\cdot104\cdot111 . \supset \vdash . \gamma \subset \text{Ɑ}\text{‘}\alpha\underset{,,}{\downarrow} . \gamma \subset \text{Ɑ}\text{‘}\beta\underset{,,}{\downarrow} . \alpha\underset{,,}{\downarrow}\text{“}\gamma, \beta\underset{,,}{\downarrow}\text{“}\gamma \in \text{Cls}^2\text{excl}$ (1)

$\vdash . *113\cdot105 . \quad \supset \vdash : \text{Hp} . \supset . (\alpha\underset{,,}{\downarrow}\text{“}\gamma) \rceil \alpha\underset{,,}{\downarrow}, \beta\underset{,,}{\downarrow} \upharpoonright \gamma \in 1 \to 1$ (2)

$\vdash . (1) . (2) . *115\cdot6 \dfrac{\alpha\underset{,,}{\downarrow}, \beta\underset{,,}{\downarrow}}{R, \ S} . \supset$

$\vdash : \text{Hp} . \supset . (\alpha \exp \gamma) \times (\beta \exp \gamma) \text{ sm } \epsilon_\Delta\text{‘}\hat{\mu}\{(\exists z) . z \in \gamma . \mu = \alpha\underset{,,}{\downarrow} z \times \beta\underset{,,}{\downarrow} z\}$ (3)

$\vdash . (3) . *38\cdot2 . \supset \vdash . \text{Prop}$

The hypothesis $\exists ! \alpha . \exists ! \beta$ is not necessary in the above proposition; but the proof is simpler with the hypothesis, and we do not need the proposition without the hypothesis.

$*116\cdot531$. $\vdash :. M = \hat{R}\hat{z}\{z \in \gamma . R = (\downarrow z)\dagger \upharpoonright (\alpha \times \beta)\} . \supset :$

$$z \in \gamma . \supset_z . M\text{‘}z = (\downarrow z)\dagger \upharpoonright (\alpha \times \beta) . M\text{‘}z \in 1 \to 1 . \text{Ɑ}\text{‘}M\text{‘}z = \alpha \times \beta$$

Dem $\vdash . *74\cdot772 . *55\cdot12 . *72\cdot184 . \supset \vdash . (\downarrow z)\dagger \in 1 \to 1$ (1)

$\vdash . *21\cdot33 . \supset \vdash :. \text{Hp} . z \in \gamma . \supset : RMz . \equiv . R = (\downarrow z)\dagger \upharpoonright (\alpha \times \beta) :$

[$*30\cdot3$] $\supset : M\text{‘}z = (\downarrow z)\dagger \upharpoonright (\alpha \times \beta) :$ (2)

[(1).$*43\cdot122$] $\supset : M\text{‘}z \in 1 \to 1 . \text{Ɑ}\text{‘}M\text{‘}z = \alpha \times \beta$ (3)

$\vdash . (2) . (3) . \supset \vdash . \text{Prop}$

$*116\cdot532$. $\vdash : \text{Hp } *116\cdot531 . \exists ! \alpha . \exists ! \beta . \supset . M \in 1 \to 1 . \text{Ɑ}\text{‘}M = \gamma$

Dem. $\vdash . *116\cdot531 . *14\cdot21 . *71\cdot16 . \supset \vdash : \text{Hp} . \supset . M \in 1 \to \text{Cls}$ (1)

$\vdash . *116\cdot531 . \supset \vdash :. \text{Hp} . z, w \in \gamma . M\text{‘}z = M\text{‘}w . \supset :$

$$(\downarrow z)\dagger \upharpoonright (\alpha \times \beta) = (\downarrow w)\dagger \upharpoonright (\alpha \times \beta) :$$

[$*71\cdot35$] $\supset : R \in (\alpha \times \beta) . \supset . (\downarrow z)\dagger\text{‘}R = (\downarrow w)\dagger\text{‘}R :$

[$*113\cdot101$] $\supset : x \in \alpha . y \in \beta . \supset . (\downarrow z)\dagger\text{‘}(y \downarrow x) = (\downarrow w)\dagger\text{‘}(y \downarrow x) .$

[$*113\cdot123$] $\supset . (y \downarrow z) \downarrow (x \downarrow z) = (y \downarrow w) \downarrow (x \downarrow w) .$

[$*55\cdot202$] $\supset . z = w$ (2)

$\vdash . (2) . \quad \supset \vdash :. \mathrm{Hp} . \supset : z, w \in \gamma . M'z = M'w . \supset . z = w$ (3)

$\vdash . *116\cdot531 . *14\cdot21 . *33\cdot43 . \supset \vdash : \mathrm{Hp} . z \in \gamma . \supset . z \in \text{Ɑ}'R$ (4)

$\vdash . *21\cdot33 . \supset \vdash :. \mathrm{Hp} . \supset : RMz . \supset_{R,z} . z \in \gamma :$

[$*33\cdot351$] $\supset : \text{Ɑ}'R \subset \gamma$ (5)

$\vdash . (1) . (3) . (4) . (5) . *71\cdot55 . \supset \vdash . \mathrm{Prop}$

$*116\cdot533$. $\vdash :. \mathrm{Hp} *116\cdot531 . \supset : \mathrm{D}''M''\gamma = \hat{\mu}\{(\exists z) . z \in \gamma . \mu = \downarrow z''\alpha \times \downarrow z''\beta\} :$
$z, w \in \gamma . \exists ! \mathrm{D}'M'z \cap \mathrm{D}'M'w . \supset_{z,w} . z = w$

Dem.

$\vdash . *116\cdot531 . \supset \vdash : \mathrm{Hp} . z \in \gamma . \supset . \mathrm{D}'M'z = \mathrm{D}'\{(\downarrow z) \dagger \upharpoonright (\alpha \times \beta)\}$

[$*37\cdot401$] $= (\downarrow z) \dagger''(\alpha \times \beta)$

[$*113\cdot65$] $= \downarrow z''\alpha \times \downarrow z''\beta$ (1)

$\vdash . (1) . *37\cdot6 . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{D}''M''\gamma = \hat{\mu}\{(\exists z) . z \in \gamma . \mu = \downarrow z''\alpha \times \downarrow z''\beta\}$ (2)

$\vdash . *113\cdot19 . \quad \supset \vdash : \exists ! (\downarrow z''\alpha \times \downarrow z''\beta) \cap (\downarrow w''\alpha \times \downarrow w''\beta) . \supset .$
$\exists ! \downarrow z''\alpha \cap \downarrow w''\alpha .$

[$*55\cdot232$] $\supset . z = w$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash . \mathrm{Prop}$

$*116\cdot534$. $\vdash : \mathrm{Hp} *116\cdot532 . \supset . \epsilon_\Delta{}'\mathrm{D}''M''\gamma \ \mathrm{sm} \ (\alpha \times \beta) \exp \gamma$

Dem.

$\vdash . *116\cdot531\cdot532\cdot533 . \supset$

$\vdash :. \mathrm{Hp} . \supset : M \in 1 \to 1 : z, w \in \gamma . \exists ! \mathrm{D}'M'z \cap \mathrm{D}'M'w . \supset_{z,w} . z = w :$
$z \in \gamma . \supset_z . M'z \in 1 \to 1 . \text{Ɑ}'M'z = \alpha \times \beta :$

[$*116\cdot43$] $\supset : \mathrm{Prod}'\mathrm{D}''M''\gamma \ \mathrm{sm} \ (\alpha \times \beta) \exp \gamma :$

[$*115\cdot12 . *30\cdot37 . *84\cdot11$] $\supset : \epsilon_\Delta{}'\mathrm{D}''M''\gamma \ \mathrm{sm} \ (\alpha \times \beta) \exp \gamma :. \supset \vdash . \mathrm{Prop}$

$*116\cdot535$. $\vdash : \exists ! \alpha . \exists ! \beta . \supset . (\alpha \exp \gamma) \times (\beta \exp \gamma) \ \mathrm{sm} \ (\alpha \times \beta) \exp \gamma$
[$*116\cdot53\cdot533\cdot534$]

The hypothesis $\exists ! \alpha . \exists ! \beta$ is not necessary, as we shall now prove.

$*116\cdot54$. $\vdash . (\alpha \exp \gamma) \times (\beta \exp \gamma) \ \mathrm{sm} \ (\alpha \times \beta) \exp \gamma$

Dem.

$\vdash . *116\cdot182 . \supset \vdash : \alpha = \Lambda . \exists ! \gamma . \supset . \alpha \exp \gamma = \Lambda .$

[$*113\cdot114$] $\supset . (\alpha \exp \gamma) \times (\beta \exp \gamma) = \Lambda$ (1)

$\vdash . *113\cdot114 . *116\cdot182 . \supset \vdash : \alpha = \Lambda . \exists ! \gamma . \supset . (\alpha \times \beta) \exp \gamma = \Lambda$ (2)

$\vdash . (1) . (2) . \quad \supset \vdash : \alpha = \Lambda . \exists ! \gamma . \supset . (\alpha \exp \gamma) \times (\beta \exp \gamma) \ \mathrm{sm} \ (\alpha \times \beta) \exp \gamma$ (3)

Similarly $\vdash : \beta = \Lambda . \exists ! \gamma . \supset . (\alpha \exp \gamma) \times (\beta \exp \gamma) \ \mathrm{sm} \ (\alpha \times \beta) \exp \gamma$ (4)

$\vdash . *116\cdot181 . \supset \vdash : \gamma = \Lambda . \supset . (\alpha \exp \gamma) \times (\beta \exp \gamma) = \iota'\Lambda \times \iota'\Lambda .$

[$*113\cdot611 . *73\cdot43$] $\supset . (\alpha \exp \gamma) \times (\beta \exp \gamma) \ \mathrm{sm} \ \iota'\Lambda$ (5)

$\vdash . *116\cdot181 . \supset \vdash : \gamma = \Lambda . \supset . (\alpha \times \beta) \exp \gamma = \iota'\Lambda .$

[(5)] $\supset . (\alpha \exp \gamma) \times (\beta \exp \gamma) \ \mathrm{sm} \ (\alpha \times \beta) \exp \gamma$ (6)

$\vdash . (3) . (4) . (6) . *116\cdot535 . \supset \vdash . \mathrm{Prop}$

In obtaining (5), we use $*73\cdot43$ as well as $*113\cdot611$, because Λ's of different types are involved.

***116·55.** $\vdash . \mu^{\varpi} \times_c \nu^{\varpi} = (\mu \times_c \nu)^{\varpi}$

Dem.

$\vdash . *116\cdot54\cdot222 . *113\cdot222 . \supset \vdash . (\mathrm{N_0c}'\alpha)^{\mathrm{N_0c}'\gamma} \times_c (\mathrm{N_0c}'\beta)^{\mathrm{N_0c}'\gamma}$
$\qquad = (\mathrm{N_0c}'\alpha \times_c \mathrm{N_0c}'\beta)^{\mathrm{N_0c}'\gamma}$ (1)

$\vdash . (1) . *103\cdot2 . \quad \supset \vdash : \mu, \nu, \varpi \in \mathrm{N_0C} . \supset . \mu^{\varpi} \times_c \nu^{\varpi} = (\mu \times_c \nu)^{\varpi}$ (2)

$\vdash . *116\cdot205 . *113\cdot204 . \supset \vdash : \varpi \sim\in \mathrm{N_0C} . \supset . \mu^{\varpi} \times_c \nu^{\varpi} = \Lambda = (\mu \times_c \nu)^{\varpi}$ (3)

$\vdash . *116\cdot205 . *113\cdot204 . \supset \vdash : \sim(\mu, \nu \in \mathrm{N_0C}) . \supset . \mu^{\varpi} \times_c \nu^{\varpi} = \Lambda$ (4)

$\vdash . *113\cdot204 . *116\cdot204 . \supset \vdash : \sim(\mu, \nu \in \mathrm{N_0C}) . \supset . (\mu \times_c \nu)^{\varpi} = \Lambda$ (5)

$\vdash . (4) . (5) . \quad \supset \vdash : \sim(\mu, \nu \in \mathrm{N_0C}) . \supset . \mu^{\varpi} \times_c \nu^{\varpi} = (\mu \times_c \nu)^{\varpi}$ (6)

$\vdash . (2) . (3) . (6) . \quad \supset \vdash . \mathrm{Prop}$

This completes the proof of the second of the formal laws of exponentiation. The following propositions are lemmas for the third of these laws, namely

$$(\mu^{\nu})^{\varpi} = \mu^{\nu \times_c \varpi}.$$

***116·6.** $\vdash : \exists ! \alpha . \supset . \alpha \exp (\beta \times \gamma) \operatorname{sm} \mathrm{Prod}'\mathrm{Prod}``\alpha \underset{''}{\downarrow} ```\beta \underset{''}{\downarrow} ``\gamma .$
$\qquad \alpha \underset{''}{\downarrow} ```\beta \underset{''}{\downarrow} ``\gamma \in \mathrm{Cls^3\,arithm}$

Dem.

$\vdash . *113\cdot105 . *84\cdot53 \dfrac{\alpha \underset{''}{\downarrow}}{R} . *113\cdot111 . \supset \vdash : \mathrm{Hp} . \supset . \alpha \underset{''}{\downarrow} ```\beta \underset{''}{\downarrow} ``\gamma \in \mathrm{Cls^2\,excl}$ (1)

$\vdash . *40\cdot38 . \quad \supset \vdash . s'\alpha \underset{''}{\downarrow} ```\beta \underset{''}{\downarrow} ``\gamma = \alpha \underset{''}{\downarrow} ``s'\beta \underset{''}{\downarrow} ``\gamma$ (2)

$[*113\cdot111] \quad \supset \vdash . s'\alpha \underset{''}{\downarrow} ```\beta \underset{''}{\downarrow} ``\gamma \in \mathrm{Cls^2\,excl}$ (3)

$\vdash . (1) . (3) . *115\cdot2 . \quad \supset \vdash : \mathrm{Hp} . \supset . \alpha \underset{''}{\downarrow} ```\beta \underset{''}{\downarrow} ``\gamma \in \mathrm{Cls^3\,arithm}$ (4)

$\vdash . *113\cdot141 . *116\cdot19 . \supset \vdash . \mathrm{Nc}'\{\alpha \exp (\beta \times \gamma)\} = \mathrm{Nc}'\{\alpha \exp (\gamma \times \beta)\}$

$[(*116\cdot01 . *113\cdot02)] \qquad = \mathrm{Nc}'\mathrm{Prod}'\alpha \underset{''}{\downarrow} ``s'\beta \underset{''}{\downarrow} ``\gamma$

$[(2)] \qquad = \mathrm{Nc}'\mathrm{Prod}'s'\alpha \underset{''}{\downarrow} ```\beta \underset{''}{\downarrow} ``\gamma$

$[*115\cdot35 . (4)] \qquad = \mathrm{Nc}'\mathrm{Prod}'\mathrm{Prod}``\alpha \underset{''}{\downarrow} ```\beta \underset{''}{\downarrow} ``\gamma$ (5)

$\vdash . (4) . (5) . \supset \vdash . \mathrm{Prop}$

***116·601.** $\vdash . |(\mathrm{Cnv}' \downarrow z) \in 1 \to 1 \quad [*74\cdot774 . *72\cdot184]$

***116·602.** $\vdash :. M = \hat{R}\hat{z} [z \in \gamma . R = \{|(\mathrm{Cnv}' \downarrow z)\}_{\epsilon} \upharpoonright (\alpha \exp \beta)] . \supset :$
$\qquad z \in \gamma . \supset . M'z = \{|(\mathrm{Cnv}' \downarrow z)\}_{\epsilon} \upharpoonright (\alpha \exp \beta) : \mathrm{C\!\!\!I}'M = \gamma$

Dem.

$\vdash . *21\cdot33 . \supset$

$\vdash :: \mathrm{Hp} . \supset :. z \in \gamma . \supset : RMz . \equiv . R = \{|(\mathrm{Cnv}' \downarrow z)\}_{\epsilon} \upharpoonright (\alpha \exp \beta)$ (1)

$\vdash . (1) . *30\cdot3 . \supset \vdash :. \mathrm{Hp} . \supset : z \in \gamma . \supset . M'z = \{|(\mathrm{Cnv}' \downarrow z)\}_{\epsilon} \upharpoonright (\alpha \exp \beta)$ (2)

$\vdash . *21\cdot33 . *33\cdot131 . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{C\!\!\!I}'M = \gamma$ (3)

$\vdash . (2) . (3) . \supset \vdash . \mathrm{Prop}$

***116·603.** $\vdash :. \mathrm{Hp}\, *116\cdot602 . \supset : z \in \gamma . \supset . \mathrm{C\!\!\!I}'M'z = \alpha \exp \beta$
$\qquad [*116\cdot602 . *37\cdot231 . *35\cdot65]$

***116·604.** $\vdash :. \text{Hp} * 116\cdot602 . \supset : z \in \gamma . \supset . \text{D}‘M‘z = \text{Prod}‘\alpha \underset{,,}{\downarrow} “\beta \underset{,,}{\downarrow} z$

Dem.

$\vdash . * 37\cdot401 . * 116\cdot602 . \supset$

$\vdash : \text{Hp} . z \in \gamma . \supset . \text{D}‘M‘z \quad = \{ | (\text{Cnv}‘ \downarrow z)\}_\epsilon “(\alpha \exp \beta)$

$[* 115\cdot4 . * 116\cdot601 . * 43\cdot301] = \text{Prod}‘\{ | (\text{Cnv}‘ \downarrow z)\}_\epsilon “\alpha \underset{,,}{\downarrow} “\beta$

$[* 113\cdot125 \frac{I}{R} . * 50\cdot75\cdot16] \quad = \text{Prod}‘\alpha \underset{,,}{\downarrow} “ \downarrow z “\beta$

$[* 38\cdot2] \quad = \text{Prod}‘\alpha \underset{,,}{\downarrow} “\beta \underset{,,}{\downarrow} z : \supset \vdash . \text{Prop}$

***116·605.** $\vdash :. \text{Hp} * 116\cdot602 . \supset : z \in \gamma . \supset . M‘z \in 1 \rightarrow 1$

Dem.

$\vdash . * 116\cdot601 . * 72\cdot451 . \supset$

$\vdash . \{ | (\text{Cnv}‘ \downarrow z)\}_\epsilon \upharpoonright \text{Cl}‘\text{ꓷ}‘ | (\text{Cnv}‘ \downarrow z) \in 1 \rightarrow 1 .$

$[* 43\cdot301] \supset \vdash . \{ | (\text{Cnv}‘ \downarrow z)\}_\epsilon \upharpoonright (\alpha \exp \beta) \in 1 \rightarrow 1 \qquad (1)$

$\vdash . (1) . * 116\cdot602 . \supset \vdash . \text{Prop}$

***116·606.** $\vdash :. \text{Hp} * 116\cdot602 . \exists ! \alpha . \exists ! \beta . \supset :$

$M \in 1 \rightarrow 1 : z, w \in \gamma . \text{D}‘M‘z = \text{D}‘M‘w . \supset_{z,w} . z = w$

Dem.

$\vdash . * 116\cdot602 . * 14\cdot21 . \supset \vdash :. \text{Hp} . \supset : z \in \text{ꓷ}‘M . \supset_z . \text{E} ! M‘z :$

$[* 71\cdot16] \quad \supset : M \in 1 \rightarrow \text{Cls} \qquad (1)$

$\vdash . * 30\cdot37 . \quad \supset \vdash :. \text{Hp} . \supset : z, w \in \gamma . M‘z = M‘w . \supset . \text{D}‘M‘z = \text{D}‘M‘w \qquad (2)$

$\vdash . * 116\cdot604 . \supset \vdash :. \text{Hp} . \supset : z, w \in \gamma . \text{D}‘M‘z = \text{D}‘M‘w . \supset .$

$\text{Prod}‘\alpha \underset{,,}{\downarrow} “\beta \underset{,,}{\downarrow} z = \text{Prod}‘\alpha \underset{,,}{\downarrow} “\beta \underset{,,}{\downarrow} w .$

$[* 30\cdot37] \quad \supset . s‘\text{Prod}‘\alpha \underset{,,}{\downarrow} “\beta \underset{,,}{\downarrow} z = s‘\text{Prod}‘\alpha \underset{,,}{\downarrow} “\beta \underset{,,}{\downarrow} w .$

$[* 116\cdot171 . * 115\cdot141 . (* 116\cdot01)] \quad \supset . s‘\alpha \underset{,,}{\downarrow} “\beta \underset{,,}{\downarrow} z = s‘\alpha \underset{,,}{\downarrow} “\beta \underset{,,}{\downarrow} w .$

$[* 113\cdot1] \quad \supset . \beta \underset{,,}{\downarrow} z \times \alpha = \beta \underset{,,}{\downarrow} w \times \alpha .$

$[* 113\cdot182] \quad \supset . \beta \underset{,,}{\downarrow} z = \beta \underset{,,}{\downarrow} w .$

$[* 113\cdot105 . \text{Hp}] \quad \supset . z = w \qquad (3)$

$\vdash . (1) . (2) . (3) . * 71\cdot55 . * 116\cdot602 . \supset \vdash . \text{Prop}$

***116·607.** $\vdash :. \text{Hp} * 116\cdot602 . \exists ! \alpha . \exists ! \beta . \supset :$

$M \in 1 \rightarrow 1 . \text{D}“M“\gamma = \text{Prod}“\alpha \underset{,,}{\downarrow} “‘\beta \underset{,,}{\downarrow} “\gamma :$

$z, w \in \gamma . \text{D}‘M‘z = \text{D}‘M‘w . \supset_{z,w} . z = w :$

$z \in \gamma . \supset_z . M‘z \in 1 \rightarrow 1 . \text{ꓷ}‘M‘z = \alpha \exp \beta \quad [* 116\cdot606\cdot604\cdot605\cdot603]$

***116·61.** $\vdash : \exists ! \alpha . \exists ! \beta . \supset . \text{Prod}‘\text{Prod}“\alpha \underset{,,}{\downarrow} “‘\beta \underset{,,}{\downarrow} “\gamma \ \text{sm} \ (\alpha \exp \beta) \exp \gamma$

$[* 116\cdot607\cdot43]$

***116·611.** $\vdash : \exists ! \alpha . \exists ! \beta . \supset . \alpha \exp (\beta \times \gamma) \ \text{sm} \ (\alpha \exp \beta) \exp \gamma \quad [* 116\cdot6\cdot61]$

***116·62.** $\vdash . \alpha \exp (\beta \times \gamma) \operatorname{sm} (\alpha \exp \beta) \exp \gamma$

Dem.

$\vdash . *116{\cdot}181 . *113{\cdot}114 . \supset \vdash : \beta = \Lambda . \supset . \alpha \exp (\beta \times \gamma) = \iota`\Lambda$ (1)

$\vdash . *116{\cdot}181 . \quad \supset \vdash : \beta = \Lambda . \supset . (\alpha \exp \beta) \exp \gamma = (\iota`\Lambda) \exp \gamma$ (2)

$\vdash . *116{\cdot}33{\cdot}25 . \quad \supset \vdash . \mathrm{Nc}`\{(\iota`\Lambda) \exp \gamma\} = 1$ (3)

$\vdash . (1) . (2) . (3) . *52{\cdot}22 . *100{\cdot}31 . \supset \vdash : \beta = \Lambda . \supset . \alpha \exp (\beta \times \gamma) \operatorname{sm} (\alpha \exp \beta) \exp \gamma$ (4)

$\vdash . *113{\cdot}107 . *116{\cdot}182 . \supset \vdash : \alpha = \Lambda . \exists ! \beta . \exists ! \gamma . \supset . \alpha \exp (\beta \times \gamma) = \Lambda$ (5)

$\vdash . *116{\cdot}182 . \quad \supset \vdash : \alpha = \Lambda . \exists ! \beta . \exists ! \gamma . \supset . (\alpha \exp \beta) \exp \gamma = \Lambda$ (6)

$\vdash . (5) . (6) . \supset \vdash : \alpha = \Lambda . \exists ! \beta . \exists ! \gamma . \supset . \alpha \exp (\beta \times \gamma) \operatorname{sm} (\alpha \exp \beta) \exp \gamma$ (7)

$\vdash . *113{\cdot}114 . *116{\cdot}181 . \supset \vdash : \gamma = \Lambda . \supset . \alpha \exp (\beta \times \gamma) = \iota`\Lambda . (\alpha \exp \beta) \exp \gamma = \iota`\Lambda .$

$[*73{\cdot}43] \quad \supset . \alpha \exp (\beta \times \gamma) \operatorname{sm} (\alpha \exp \beta) \exp \gamma$ (8)

$\vdash . (4) . (7) . (8) . \quad \supset \vdash :. \alpha = \Lambda . \vee . \beta = \Lambda . \vee . \gamma = \Lambda : \supset .$

$\alpha \exp (\beta \times \gamma) \operatorname{sm} (\alpha \exp \beta) \exp \gamma$ (9)

$\vdash . (9) . *116{\cdot}611 . \supset \vdash . \mathrm{Prop}$

***116·63.** $\vdash . \mu^{\nu \times_0 \varpi} = (\mu^\nu)^\varpi$

Dem.

$\vdash . *113{\cdot}222 . \supset \vdash . (\mathrm{N_0c}`\alpha)^{\mathrm{N_0c}`\beta \times_0 \mathrm{N_0c}`\gamma} = (\mathrm{N_0c}`\alpha)^{\mathrm{Nc}`(\beta \times \gamma)}$

$[*116{\cdot}222 . (*116{\cdot}04)] = \mathrm{Nc}`\{\alpha \exp (\beta \times \gamma)\}$

$[*116{\cdot}62] \quad = \mathrm{Nc}`\{(\alpha \exp \beta) \exp \gamma\}$

$[*116{\cdot}222] \quad = \{\mathrm{N_0c}`(\alpha \exp \beta)\}^{\mathrm{N_0c}`\gamma}$

$[*116{\cdot}222 . (*116{\cdot}03)] = \{(\mathrm{N_0c}`\alpha)^{\mathrm{N_0c}`\beta}\}^{\mathrm{N_0c}`\gamma}$ (1)

$\vdash . (1) . *103{\cdot}2 . \quad \supset \vdash : \mu, \nu, \varpi \in \mathrm{N_0C} . \supset . \mu^{\nu \times_0 \varpi} = (\mu^\nu)^\varpi$ (2)

$\vdash . *116{\cdot}204{\cdot}205 . \quad \supset \vdash : \sim(\mu, \nu \in \mathrm{N_0C}) . \supset . (\mu^\nu)^\varpi = \Lambda$ (3)

$\vdash . *113{\cdot}205 . *116{\cdot}204{\cdot}205 . \supset \vdash : \sim(\mu, \nu \in \mathrm{N_0C}) . \supset . \mu^{\nu \times_0 \varpi} = \Lambda$ (4)

$\vdash . *116{\cdot}205 . \quad \supset \vdash : \varpi \sim\in \mathrm{N_0C} . \supset . (\mu^\nu)^\varpi = \Lambda$ (5)

$\vdash . *113{\cdot}205 . *116{\cdot}204 . \supset \vdash : \varpi \sim\in \mathrm{N_0C} . \supset . \mu^{\nu \times_0 \varpi} = \Lambda$ (6)

$\vdash . (3) . (4) . (5) . (6) . \quad \supset \vdash : \sim(\mu, \nu, \varpi \in \mathrm{N_0C}) . \supset . \mu^{\nu \times_0 \varpi} = (\mu^\nu)^\varpi$ (7)

$\vdash . (2) . (7) . \quad \supset \vdash . \mathrm{Prop}$

This completes the proof of the third of the formal laws of exponentiation.

***116·64.** $\vdash . (\mu^\nu)^\varpi = (\mu^\varpi)^\nu \quad [*116{\cdot}63 . *113{\cdot}27]$

***116·651.** $\vdash : Q \in \mathrm{Cls} \to 1 . \kappa \in \mathrm{Cls}^2 \mathrm{excl} . \supset . \epsilon_\Delta`P_\Delta``Q```\kappa \operatorname{sm} P_\Delta`Q``s`\kappa$

Dem.

$\vdash . *84{\cdot}53 . \supset \vdash : \mathrm{Hp} . \supset . Q```\kappa \in \mathrm{Cls}^2 \mathrm{excl} .$

$[*85{\cdot}43] \quad \supset . \epsilon_\Delta`P_\Delta``Q```\kappa \operatorname{sm} P_\Delta`s`Q```\kappa .$

$[*40{\cdot}38] \quad \supset . \epsilon_\Delta`P_\Delta``Q```\kappa \operatorname{sm} P_\Delta`Q``s`\kappa : \supset \vdash . \mathrm{Prop}$

***116·652.** $\vdash : Q \in \mathrm{Cls} \to 1 . \kappa \in \mathrm{Cls}^2 \mathrm{excl} . \supset . \epsilon_\Delta`\epsilon_\Delta``Q```\kappa \operatorname{sm} \epsilon_\Delta`Q``s`\kappa$

$\left[*116{\cdot}651 \dfrac{\epsilon}{P}\right]$

The following propositions are lemmas for $*116{\cdot}661$, which is an extension of $*116{\cdot}52$.

$*116{\cdot}653$. $\vdash : \kappa \in \text{Cls}^2\,\text{excl} \,.\supset .\, \alpha \underset{,,}{\downarrow} \text{‘‘‘}\kappa \in \text{Cls}^2\,\text{arithm}$

Dem.

$\vdash . *113{\cdot}105 . *84{\cdot}53 . \supset \vdash : \text{Hp} . \exists ! \alpha . \supset . \alpha \underset{,,}{\downarrow} \text{‘‘‘}\kappa \in \text{Cls}^2\,\text{excl}$ (1)

$\vdash . *113{\cdot}111 . \quad \supset \vdash . \alpha \underset{,,}{\downarrow} \text{‘‘}s\text{‘}\kappa \in \text{Cls}^2\,\text{excl} .$

$[*40{\cdot}38] \quad \supset \vdash . s\text{‘}\alpha \underset{,,}{\downarrow} \text{‘‘‘}\kappa \in \text{Cls}^2\,\text{excl}$ (2)

$\vdash . *113{\cdot}112{\cdot}113 . \supset \vdash :. \alpha = \Lambda . \supset : \beta \in \kappa . \exists ! \beta . \supset . \alpha \underset{,,}{\downarrow} \text{‘‘}\beta = \iota\text{‘}\Lambda :$

$\beta \in \kappa . \beta = \Lambda . \supset . \alpha \underset{,,}{\downarrow} \text{‘‘}\beta = \Lambda$ (3)

$\vdash . (3) . \supset \vdash :. \alpha = \Lambda . \supset : \alpha \underset{,,}{\downarrow} \text{‘‘‘}\kappa \subset \iota\text{‘}\iota\text{‘}\Lambda \cup \iota\text{‘}\Lambda :$

$[*24{\cdot}43{\cdot}561] \quad \supset : \rho, \sigma \in \alpha \underset{,,}{\downarrow} \text{‘‘‘}\kappa . \exists ! \rho \cap \sigma . \supset . \rho, \sigma \in \iota\text{‘}\iota\text{‘}\Lambda .$

$[*51{\cdot}15] \quad \supset . \rho = \sigma :$

$[*84{\cdot}11] \quad \supset : \alpha \underset{,,}{\downarrow} \text{‘‘‘}\kappa \in \text{Cls}^2\,\text{excl}$ (4)

$\vdash . (1) . (4) . \supset \vdash : \text{Hp} . \supset . \alpha \underset{,,}{\downarrow} \text{‘‘‘}\kappa \in \text{Cls}^2\,\text{excl}$ (5)

$\vdash . (2) . (5) . \supset \vdash . \text{Prop}$

$*116{\cdot}654$. $\vdash : \kappa \in \text{Cls}^2\,\text{excl} . \supset . \{\text{Prod}\text{‘}(\alpha\,\text{exp})\text{‘‘}\kappa\}\ \text{sm}\ \{\alpha\,\text{exp}\,(s\text{‘}\kappa)\}$

Dem.

$\vdash . *38{\cdot}13 . (*116{\cdot}01) . \supset \vdash . \text{Prod}\text{‘}(\alpha\,\text{exp})\text{‘‘}\kappa = \text{Prod}\text{‘}\text{Prod}\text{‘‘}\alpha \underset{,,}{\downarrow} \text{‘‘‘}\kappa$ (1)

$\vdash . (1) . *116{\cdot}653 . *115{\cdot}34 . \supset$

$\vdash . \{\text{Prod}\text{‘}(\alpha\,\text{exp})\text{‘‘}\kappa\}\ \text{sm}\ \{\text{Prod}\text{‘}s\text{‘}\alpha \underset{,,}{\downarrow} \text{‘‘‘}\kappa\}$ (2)

$\vdash . (2) . *40{\cdot}38 . (*116{\cdot}01) . \supset \vdash . \text{Prop}$

$*116{\cdot}655$. $\vdash : \kappa \in \text{Cls}^2\,\text{excl} . \supset . \Pi\text{Nc}\text{‘}(\alpha\,\text{exp})\text{‘‘}\kappa = (\text{Nc}\text{‘}\alpha)^{\Sigma\text{Nc}\text{‘}\kappa}$ $[*116{\cdot}654]$

This proposition is an extension of $*116{\cdot}5$.

The hypothesis $\kappa \in \text{Cls}^2\,\text{excl}$ is unnecessary in the above proposition, as we shall now prove.

$*116{\cdot}656$. $\vdash : \exists ! \alpha\,\text{exp}\,\beta \cap \alpha\,\text{exp}\,\gamma . \supset . \beta = \gamma$

Dem.

$\vdash . *116{\cdot}11 . *52{\cdot}16 . \supset$

$\vdash :. \mu \in (\alpha\,\text{exp}\,\beta) \cap (\alpha\,\text{exp}\,\gamma) . \supset : y \in \beta . \supset . (\exists x) . x \in \alpha . x \downarrow y \in \mu : \mu \subset \gamma \times \alpha :$

$[*113{\cdot}101] \quad \supset : y \in \beta . \supset . (\exists x) . x \in \alpha . x \downarrow y \in \mu : x \downarrow y \in \mu . \supset . y \in \gamma :$

$[\text{Syll}] \quad \supset : y \in \beta . \supset . (\exists x) . x \in \alpha . x \downarrow y \in \mu . y \in \gamma .$

$[*10{\cdot}35] \quad \supset . y \in \gamma$ (1)

Similarly $\vdash :. \mu \in \alpha\,\text{exp}\,\beta \cap \alpha\,\text{exp}\,\gamma . \supset : y \in \gamma . \supset . y \in \beta$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***116·657.** $\vdash . (\alpha \exp)``\kappa \in \text{Cls}^2 \text{excl}$ [*116·656]

***116·658.** $\vdash . \alpha \exp (\epsilon \downarrow\!\!\!\!\rceil\, \beta) = \{|(\text{Cnv}`\downarrow \beta)\}_\epsilon``(\alpha \exp \beta)$

Dem.

$\vdash . *116·602·604 . *37·401 . \supset \vdash . \{|(\text{Cnv}`\downarrow \beta)\}_\epsilon``(\alpha \exp \beta) = \alpha \exp (\beta \underset{,,}{\downarrow} \beta)$

[*85·601] $= \alpha \exp (\epsilon \downarrow\!\!\!\!\rceil\, \beta) . \supset \vdash . \text{Prop}$

***116·659.** $T = \hat{\nu}\hat{\mu}\{(\exists \beta) . \beta \in \kappa . \mu \in \alpha \exp \beta . \nu = |(\text{Cnv}`\downarrow \beta)``\mu\} . \supset .$
$T \in (\alpha \exp)``\epsilon \downarrow\!\!\!\!\rceil\, ``\kappa \,\overline{\text{sm}}\, \overline{\text{sm}}\, (\alpha \exp)``\kappa$

Dem.

$\vdash . *40·4 . \supset \vdash : \text{Hp} . \supset . \text{G}`T = s`(\alpha \exp)``\kappa$ (1)

$\vdash . *21·33 . \supset \vdash :. \text{Hp} . \supset : \nu T \mu . \varpi T \mu . \supset .$
$(\exists \beta, \gamma) . \beta, \gamma \in \kappa . \mu \in \alpha \exp \beta . \mu \in \alpha \exp \gamma .$
$\nu = |(\text{Cnv}`\downarrow \beta)``\mu . \varpi = |(\text{Cnv}`\downarrow \gamma)``\mu .$

[*116·656] $\supset . (\exists \beta, \gamma) . \beta = \gamma . \nu = |(\text{Cnv}`\downarrow \beta)``\mu . \varpi = |(\text{Cnv}`\downarrow \gamma)``\mu .$

[*13·195] $\supset . \nu = \varpi$ (2)

$\vdash . *21·33 . \supset \vdash :. \text{Hp} . \supset : \varpi T \mu . \varpi T \nu . \supset .$
$(\exists \beta, \gamma) . \beta, \gamma \in \kappa . \mu \in \alpha \exp \beta . \nu \in \alpha \exp \gamma .$
$\varpi = |(\text{Cnv}`\downarrow \beta)``\mu = |(\text{Cnv}`\downarrow \gamma)``\nu .$

[*116·658] $\supset . (\exists \beta, \gamma) . \beta, \gamma \in \kappa . \mu \in \alpha \exp \beta . \nu \in \alpha \exp \gamma .$
$\varpi = |(\text{Cnv}`\downarrow \beta)``\mu = |(\text{Cnv}`\downarrow \gamma)``\nu .$
$\varpi \in \alpha \exp (\epsilon \downarrow\!\!\!\!\rceil\, \beta) \cap \alpha \exp (\epsilon \downarrow\!\!\!\!\rceil\, \gamma) .$

[*116·656] $\supset . (\exists \beta, \gamma) . \beta, \gamma \in \kappa . |(\text{Cnv}`\downarrow \beta)``\mu = |(\text{Cnv}`\downarrow \gamma)``\nu .$
$\epsilon \downarrow\!\!\!\!\rceil\, \beta = \epsilon \downarrow\!\!\!\!\rceil\, \gamma .$

[*85·601] $\supset . (\exists \beta) . \beta \in \kappa . |(\text{Cnv}`\downarrow \beta)``\mu = |(\text{Cnv}`\downarrow \beta)``\nu .$

[*116·601.*72·441] $\supset . \mu = \nu$ (3)

$\vdash . (2) . (3) . \supset \vdash : \text{Hp} . \supset . T \in 1 \rightarrow 1$ (4)

$\vdash . *116·658 . \supset \vdash :. \text{Hp} . \supset : \beta \in \kappa . \supset . T``(\alpha \exp \beta) = \alpha \exp (\epsilon \downarrow\!\!\!\!\rceil\, \beta) :$

[*37·69] $\supset : T_\epsilon``(\alpha \exp)``\kappa = (\alpha \exp)``\epsilon \downarrow\!\!\!\!\rceil\, ``\kappa$ (5)

$\vdash . (1) . (4) . (5) . *111·1 . \supset \vdash . \text{Prop}$

***116·66.** $\vdash . \text{Prod}`(\alpha \exp)``\kappa \,\text{sm}\, \{\alpha \exp (\Sigma`\kappa)\}$

Dem.

$\vdash . *116·659 . *115·51 . \supset \vdash . \text{Prod}`(\alpha \exp)``\kappa \,\text{sm}\, \text{Prod}`(\alpha \exp)``\epsilon \downarrow\!\!\!\!\rceil\, ``\kappa$ (1)

$\vdash . *85·61 . *116·654 . \supset \vdash . \text{Prod}`(\alpha \exp)``\epsilon \downarrow\!\!\!\!\rceil\, ``\kappa \,\text{sm}\, \{\alpha \exp (s`\epsilon \downarrow\!\!\!\!\rceil\, ``\kappa)\}$ (2)

$\vdash . (1) . (2) . *112·1 . \supset \vdash . \text{Prop}$

***116·661.** $\vdash . \Pi \text{Nc}`(\alpha \exp)``\kappa = (\text{Nc}`\alpha)^{\Sigma \text{N}}$ [*116·66·657 . *115·12 . *112·101]

This proposition is an extension of *116·52.

The following propositions are concerned in proving $*116{\cdot}68$, which is an extension of $*116{\cdot}54$, where the α and β of that proposition are replaced by the members of a class κ.

$*116{\cdot}67$. $\vdash :. \rho = \hat{\lambda}\{(\exists \alpha) . \alpha \in \kappa . \lambda = \alpha \underset{,,}{\downarrow} {}``\gamma\} . \supset : \kappa \in \mathrm{Cls}^2\,\mathrm{excl} . \supset . \rho \in \mathrm{Cls}^2\mathrm{arithm}$

Dem.

$\vdash . *20{\cdot}3 . \supset \vdash : \mathrm{Hp} . \lambda, \mu \in \rho . \exists ! \lambda \cap \mu . \supset .$

$(\exists \alpha, \beta) . \alpha, \beta \in \kappa . \lambda = \alpha \underset{,,}{\downarrow} {}``\gamma . \mu = \beta \underset{,,}{\downarrow} {}``\gamma . \exists ! \lambda \cap \mu .$

$[*37{\cdot}6] \quad \supset . (\exists \alpha, \beta, z, w) . \alpha, \beta \in \kappa . z, w \in \gamma . \alpha \underset{,,}{\downarrow} z = \beta \underset{,,}{\downarrow} w . \lambda = \alpha \underset{,,}{\downarrow} {}``\gamma .$

$\mu = \beta \underset{,,}{\downarrow} {}``\gamma .$

$[*55{\cdot}262 . *38{\cdot}2] \supset . (\exists \alpha, z, w) . \alpha \in \kappa . z, w \in \gamma . \lambda = \alpha \underset{,,}{\downarrow} {}``\gamma . \mu = \alpha \underset{,,}{\downarrow} {}``\gamma .$

$[*13{\cdot}172] \quad \supset . \lambda = \mu \quad (1)$

$\vdash . *37{\cdot}6 . *40{\cdot}11 . \supset$

$\vdash : \mathrm{Hp} . \xi, \eta \in s`\rho . \exists ! \xi \cap \eta . \supset .$

$(\exists \alpha, \beta, z, w) . \alpha, \beta \in \kappa . z, w \in \gamma . \xi = \alpha \underset{,,}{\downarrow} z . \eta = \beta \underset{,,}{\downarrow} w . \exists ! \xi \cap \eta .$

$[*55{\cdot}232 . *38{\cdot}2] \supset . (\exists \alpha, \beta, z) . \alpha, \beta \in \kappa . z \in \gamma . \xi = \alpha \underset{,,}{\downarrow} z . \eta = \beta \underset{,,}{\downarrow} z . \exists ! \alpha \cap \beta \quad (2)$

$\vdash . (2) . *84{\cdot}11 . \supset$

$\vdash : \mathrm{Hp} . \kappa \in \mathrm{Cls}^2\,\mathrm{excl} . \xi, \eta \in s`\rho . \exists ! \xi \cap \eta . \supset . (\exists \alpha, \beta, z) . \xi = \alpha \underset{,,}{\downarrow} z . \eta = \beta \underset{,,}{\downarrow} z . \alpha = \beta .$

$[*13{\cdot}195{\cdot}172] \quad \supset . \xi = \eta \quad (3)$

$\vdash . (1) . (3) . \supset \vdash . \mathrm{Prop}$

$*116{\cdot}671$. $\vdash :. \sigma = \hat{\mu}\{(\exists z) . z \in \gamma . \mu = \downarrow z```\kappa\} . \supset : \mathrm{Hp}\, *116{\cdot}67 . \supset . s`\rho = s`\sigma$

Dem.

$\vdash . *40{\cdot}11 . \supset \vdash :. \mathrm{Hp}\, *116{\cdot}67 . \supset : \xi \in s`\rho . \equiv . (\exists \alpha) . \alpha \in \kappa . \xi \in \alpha \underset{,,}{\downarrow} {}``\gamma .$

$[*38{\cdot}3] \quad \equiv . (\exists \alpha, z) . \alpha \in \kappa . z \in \gamma . \xi = \downarrow z``\alpha .$

$[*37{\cdot}103] \quad \equiv . (\exists z) . z \in \gamma . \xi \in \downarrow z```\kappa .$

$[*40{\cdot}11] \quad \equiv . \xi \in s`\hat{\mu}\{(\exists z) . z \in \gamma . \mu = \downarrow z```\kappa\} :.$

$\supset \vdash . \mathrm{Prop}$

$*116{\cdot}672$. $\vdash : \mathrm{Hp}\, *116{\cdot}671 . \kappa \in \mathrm{Cls}^2\,\mathrm{excl} . \Lambda \sim\in \kappa . \supset . \sigma \in \mathrm{Cls}^2\,\mathrm{excl}$

Dem.

$\vdash . *37{\cdot}103 . \supset \vdash : \mathrm{Hp} . \mu, \nu \in \sigma . \exists ! \mu \cap \nu . \supset .$

$(\exists z, w, \alpha, \beta) . z, w \in \gamma . \alpha, \beta \in \kappa . \downarrow z``\alpha = \downarrow w``\beta .$

$\mu = \downarrow z```\alpha . \nu = \downarrow w```\beta .$

$[*55{\cdot}262] \quad \supset . (\exists z, w, \alpha) . z, w \in \gamma . \alpha \in \kappa . \downarrow z``\alpha = \downarrow w``\alpha .$

$\mu = \downarrow z```\alpha . \nu = \downarrow w```\alpha .$

$[*113{\cdot}105 . *38{\cdot}2 . \mathrm{Hp}] \supset . (\exists z, \alpha) . \mu = \downarrow z```\alpha . \nu = \downarrow z```\alpha .$

$[*13{\cdot}172] \quad \supset . \mu = \nu : \supset \vdash . \mathrm{Prop}$

***116·673.** $\vdash : \mathrm{Hp} *116{\cdot}672 . \supset . \epsilon_\Delta{}^\backprime(\exp\gamma){}^{\backprime\backprime}\kappa \,\mathrm{sm}\, \epsilon_\Delta{}^\backprime\epsilon_\Delta{}^{\backprime\backprime}\sigma$

Dem.

$\vdash . *38{\cdot}131 . (*116{\cdot}01) . \supset \vdash . \epsilon_\Delta{}^\backprime(\exp\gamma){}^{\backprime\backprime}\kappa = \epsilon_\Delta{}^\backprime\hat{\xi}\{(\exists\alpha) . \alpha\epsilon\kappa . \xi = \mathrm{Prod}{}^\backprime\alpha \downarrow_{,,} {}^{\backprime\backprime}\gamma\}$

$[*37{\cdot}6] \qquad = \epsilon_\Delta{}^\backprime\mathrm{Prod}{}^{\backprime\backprime}\hat{\lambda}\{(\exists\alpha) . \alpha\epsilon\kappa . \lambda = \alpha \downarrow_{,,} {}^{\backprime\backprime}\gamma\} \qquad (1)$

$\vdash . (1) . *115{\cdot}33 . *116{\cdot}67 . \supset$

$\vdash : \mathrm{Hp} . \supset . \epsilon_\Delta{}^\backprime(\exp\gamma){}^{\backprime\backprime}\kappa \,\mathrm{sm}\, \epsilon_\Delta{}^\backprime s{}^\backprime\hat{\lambda}\{(\exists\alpha) . \alpha\epsilon\kappa . \lambda = \alpha \downarrow_{,,} {}^{\backprime\backprime}\gamma\} .$

$[*116{\cdot}671] \qquad \supset . \epsilon_\Delta{}^\backprime(\exp\gamma){}^{\backprime\backprime}\kappa \,\mathrm{sm}\, \epsilon_\Delta{}^\backprime s{}^\backprime\sigma .$

$[*85{\cdot}44 . *116{\cdot}672] \supset . \epsilon_\Delta{}^\backprime(\exp\gamma){}^{\backprime\backprime}\kappa \,\mathrm{sm}\, \epsilon_\Delta{}^\backprime\epsilon_\Delta{}^{\backprime\backprime}\sigma : \supset \vdash . \mathrm{Prop}$

***116·674.** $\vdash :. M = \hat{R}\hat{z}\{R = (\downarrow z) \| \mathrm{Cnv}{}^\backprime(\downarrow z)_\epsilon\} . \supset :$
$(z) . M{}^\backprime z \,\epsilon\, 1 \to 1 . \mathrm{D}{}^\backprime(M{}^\backprime z) \upharpoonright \epsilon_\Delta{}^\backprime\kappa = \epsilon_\Delta{}^\backprime \downarrow z{}^{\backprime\backprime\backprime}\kappa$

Dem.

$\vdash . *30{\cdot}3 . \qquad \supset \vdash : \mathrm{Hp} . \supset . M{}^\backprime z = (\downarrow z) \| \mathrm{Cnv}{}^\backprime(\downarrow z)_\epsilon \qquad (1)$

$\vdash . *72{\cdot}184 . *111{\cdot}14 . \supset \vdash . \downarrow z \upharpoonright \kappa \,\epsilon\, (\downarrow z{}^{\backprime\backprime\backprime}\kappa) \,\overline{\mathrm{sm}}\, \overline{\mathrm{sm}}\, \kappa .$

$[*114{\cdot}51] \qquad \supset \vdash . \{\downarrow z \| \mathrm{Cnv}{}^\backprime(\downarrow z)_\epsilon\} \upharpoonright \epsilon_\Delta{}^\backprime\kappa \,\epsilon\, (\epsilon_\Delta{}^\backprime \downarrow z{}^{\backprime\backprime\backprime}\kappa) \,\overline{\mathrm{sm}}\, (\epsilon_\Delta{}^\backprime\kappa) \qquad (2)$

$\vdash . (1) . (2) . *73{\cdot}03 . \quad \supset \vdash . \mathrm{Prop}$

***116·675.** $\vdash :. \mathrm{Hp} *116{\cdot}674 . \exists ! s{}^\backprime\kappa . \supset : \exists ! (M{}^\backprime w){}^{\backprime\backprime}\epsilon_\Delta{}^\backprime\kappa \cap (M{}^\backprime v){}^{\backprime\backprime}\epsilon_\Delta{}^\backprime\kappa . \supset . w = v$

Dem.

$\vdash . *116{\cdot}674 . \supset \vdash :. \mathrm{Hp} . \supset : \exists ! (M{}^\backprime w){}^{\backprime\backprime}\epsilon_\Delta{}^\backprime\kappa \cap (M{}^\backprime v){}^{\backprime\backprime}\epsilon_\Delta{}^\backprime\kappa . \supset .$
$\exists ! \epsilon_\Delta{}^\backprime \downarrow w{}^{\backprime\backprime\backprime}\kappa \cap \epsilon_\Delta{}^\backprime \downarrow v{}^{\backprime\backprime\backprime}\kappa .$

$[*80{\cdot}32] \qquad \supset . \downarrow w{}^{\backprime\backprime\backprime}\kappa = \downarrow v{}^{\backprime\backprime\backprime}\kappa .$

$[*40{\cdot}38] \qquad \supset . \downarrow w{}^{\backprime\backprime}s{}^\backprime\kappa = \downarrow v{}^{\backprime\backprime}s{}^\backprime\kappa .$

$[*113{\cdot}105 . *38{\cdot}2] \qquad \supset . w = v :. \supset \vdash . \mathrm{Prop}$

***116·676.** $\vdash : \mathrm{Hp} *116{\cdot}672{\cdot}675 . \supset . \mathrm{Prod}{}^\backprime\mathrm{D}{}^{\backprime\backprime} \upharpoonright (\epsilon_\Delta{}^\backprime\kappa){}^{\backprime\backprime}M{}^{\backprime\backprime}\gamma \,\mathrm{sm}\, (\epsilon_\Delta{}^\backprime\kappa) \exp\gamma .$
$\mathrm{D}{}^{\backprime\backprime} \upharpoonright (\epsilon_\Delta{}^\backprime\kappa){}^{\backprime\backprime}M{}^{\backprime\backprime}\gamma = \epsilon_\Delta{}^{\backprime\backprime}\sigma$

Dem.

$\vdash . *116{\cdot}674{\cdot}675 . *116{\cdot}45 \dfrac{\epsilon_\Delta{}^\backprime\kappa, \gamma}{\gamma, \quad \delta} . \supset$

$\vdash : \mathrm{Hp} . \supset . \mathrm{Prod}{}^\backprime\mathrm{D}{}^{\backprime\backprime} \upharpoonright (\epsilon_\Delta{}^\backprime\kappa){}^{\backprime\backprime}M{}^{\backprime\backprime}\gamma \,\mathrm{sm}\, (\epsilon_\Delta{}^\backprime\kappa) \exp\gamma \qquad (1)$

$\vdash . *116{\cdot}674 . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{D}{}^{\backprime\backprime} \upharpoonright (\epsilon_\Delta{}^\backprime\kappa){}^{\backprime\backprime}M{}^{\backprime\backprime}\gamma = \hat{\mu}\{(\exists z) . z\epsilon\gamma . \mu = \epsilon_\Delta{}^\backprime \downarrow z{}^{\backprime\backprime\backprime}\kappa\}$

$[*37{\cdot}6 . \mathrm{Hp}] \qquad = \epsilon_\Delta{}^{\backprime\backprime}\sigma \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***116·68.** $\vdash : \kappa \,\epsilon\, \mathrm{Cls}^2\,\mathrm{excl} . \supset . \epsilon_\Delta{}^\backprime(\exp\gamma){}^{\backprime\backprime}\kappa \,\mathrm{sm}\, (\epsilon_\Delta{}^\backprime\kappa) \exp\gamma .$
$\Pi\mathrm{Nc}{}^\backprime(\exp\gamma){}^{\backprime\backprime}\kappa = (\Pi\mathrm{Nc}{}^\backprime\kappa)^{\mathrm{Nc}{}^\backprime\gamma}$

Dem.

$\vdash . *115{\cdot}12 . *84{\cdot}55 . \supset \vdash . \mathrm{Prod}{}^\backprime\epsilon_\Delta{}^{\backprime\backprime}\sigma \,\mathrm{sm}\, \epsilon_\Delta{}^\backprime\epsilon_\Delta{}^{\backprime\backprime}\sigma \qquad (1)$

$\vdash . (1) . *116{\cdot}673{\cdot}676 . \supset$

$\vdash : \mathrm{Hp} . \Lambda \sim\epsilon\, \kappa . \exists ! s{}^\backprime\kappa . \supset . \epsilon_\Delta{}^\backprime(\exp\gamma){}^{\backprime\backprime}\kappa \,\mathrm{sm}\, (\epsilon_\Delta{}^\backprime\kappa) \exp\gamma \qquad (2)$

$\vdash . *53\cdot24 . \supset \vdash : \Lambda \sim\epsilon \kappa . \sim\exists ! s`\kappa . \supset . \kappa = \Lambda .$

[*83·15.*116·33] $\supset . \epsilon_\Delta`(\exp \gamma)``\kappa = \iota`\Lambda . (\epsilon_\Delta`\kappa) \exp \gamma \epsilon 1 .$

[*73·45] $\supset . \epsilon_\Delta`(\exp \gamma)``\kappa \text{ sm } (\epsilon_\Delta`\kappa) \exp \gamma$ (3)

$\vdash . *83\cdot11 . *116\cdot182 . \supset \vdash : \Lambda \epsilon \kappa . \exists ! \gamma . \supset . \epsilon_\Delta`\kappa = \Lambda . \Lambda \epsilon (\exp \gamma)``\kappa .$

[*116·182.*83·11] $\supset . (\epsilon_\Delta`\kappa) \exp \gamma = \Lambda . \epsilon_\Delta`(\exp \gamma)``\kappa = \Lambda$ (4)

$\vdash . *116\cdot181 . \supset \vdash : \Lambda \epsilon \kappa . \gamma = \Lambda . \supset . (\epsilon_\Delta`\kappa) \exp \gamma = \iota`\Lambda$ (5)

$\vdash . *116\cdot181 . \supset \vdash : \Lambda \epsilon \kappa . \gamma = \Lambda . \supset . (\exp \gamma)``\kappa = \iota`\iota`\Lambda .$

[*83·41] $\supset . \epsilon_\Delta`(\exp \gamma)``\kappa \text{ sm } \iota`\Lambda$ (6)

$\vdash . (4) . (5) . (6) . \supset \vdash : \Lambda \epsilon \kappa . \supset . \epsilon_\Delta`(\exp \gamma)``\kappa \text{ sm } (\epsilon_\Delta`\kappa) \exp \gamma$ (7)

$\vdash . (2) . (3) . (7) . *114\cdot1 . *116\cdot25 . \supset \vdash . \text{Prop}$

The above proposition is an extension of *116·54·55.

The following propositions are lemmas for

$$\text{Nc}`\text{Cl}`\alpha = 2^{\text{Nc}`\alpha}.$$

The proposition and its proof are due to Cantor.

***116·7.** $\vdash . \text{Nc}`\{(\iota`\Lambda \cup \iota`\text{V}) \uparrow \alpha\}_\Delta`\alpha = 2^{\text{Nc}`\alpha}$

Dem.

$\vdash . *24\cdot1 . *101\cdot3 . \supset \vdash . \text{Nc}`(\iota`\Lambda \cup \iota`\text{V}) = 2$ (1)

$\vdash . *116\cdot15 . \supset \vdash . \text{Nc}`\{(\iota`\Lambda \cup \iota`\text{V}) \uparrow \alpha\}_\Delta`\alpha = \text{Nc}`\{(\iota`\Lambda \cup \iota`\text{V}) \exp \alpha\}$

[*116·25] $= \{\text{Nc}`(\iota`\Lambda \cup \iota`\text{V})\}^{\text{Nc}`\alpha}$

[(1)] $= 2^{\text{Nc}`\alpha} . \supset \vdash . \text{Prop}$

In this and following propositions, the class $\iota`\Lambda \cup \iota`\text{V}$ is introduced solely as a known class consisting of two terms. Any other class of two terms will serve equally well.

***116·71.** $\vdash : R \epsilon \{(\iota`\Lambda \cup \iota`\text{V}) \uparrow \alpha\}_\Delta`\alpha . \supset . \overleftarrow{R}`\text{V} = \alpha - \overleftarrow{R}`\Lambda$

Dem.

$\vdash . *116\cdot12 . \supset \vdash : \text{Hp} . \supset . R \epsilon 1 \rightarrow \text{Cls} . \text{D}`R \subset \iota`\Lambda \cup \iota`\text{V} . \text{C}\!\text{I}`R = \alpha$ (1)

[*37·271] $\supset . \alpha = \breve{R}``(\iota`\Lambda \cup \iota`\text{V})$

[*53·302] $= \overleftarrow{R}`\Lambda \cup \overleftarrow{R}`\text{V}$ (2)

$\vdash . (1) . *71\cdot18 . \supset \vdash : \text{Hp} . \supset . \overleftarrow{R}`\Lambda \cap \overleftarrow{R}`\text{V} = \Lambda$ (3)

$\vdash . (2) . (3) . *24\cdot47 . \supset \vdash . \text{Prop}$

***116·711.** $\vdash : R, S \epsilon \{(\iota`\Lambda \cup \iota`\text{V}) \uparrow \alpha\}_\Delta`\alpha . \overleftarrow{R}`\Lambda = \overleftarrow{S}`\Lambda . \supset . R = S$

Dem.

$\vdash . *116\cdot71 . \quad \supset \vdash : \text{Hp} . \supset . \overleftarrow{R}`\text{V} = \overleftarrow{S}`\text{V}$ (1)

$\vdash :. (1) . *116\cdot12 . \supset \vdash :. \text{Hp} . \supset : \gamma \epsilon \text{D}`R \cup \text{D}`S . \supset_\gamma . \overleftarrow{R}`\gamma = \overleftarrow{S}`\gamma :$

[*33·48] $\supset : R = S :. \supset \vdash . \text{Prop}$

***116·712.** $\vdash :. T = \hat{\mu}\hat{R}[R \in \{(\iota`\Lambda \cup \iota`\mathrm{V}) \uparrow \alpha\}_\Delta`\alpha \,.\, \mu = \overleftarrow{R}`\Lambda] \,.\, \supset :$

$R \in \{(\iota`\Lambda \cup \iota`\mathrm{V}) \uparrow \alpha\}_\Delta`\alpha \,.\supset .\, T`R = \overleftarrow{R}`\Lambda : \mathrm{G}`T = \{(\iota`\Lambda \cup \iota`\mathrm{V}) \uparrow \alpha\}_\Delta`\alpha$

Dem.

$\vdash . *21{\cdot}33 . \supset \vdash :: \mathrm{Hp} . \supset :. R \in \{(\iota`\Lambda \cup \iota`\mathrm{V}) \uparrow \alpha\}_\Delta`\alpha . \supset : \mu T R . \equiv_\mu . \mu = \overleftarrow{R}`\Lambda :$

$[*30{\cdot}3] \quad \supset : T`R = \overleftarrow{R}`\Lambda \quad (1)$

$\vdash . (1) . *14{\cdot}21 . \supset \vdash :. \mathrm{Hp} . \supset : R \in \{(\iota`\Lambda \cup \iota`\mathrm{V}) \uparrow \alpha\}_\Delta`\alpha . \supset . \mathrm{E}\,!\, T`R .$

$[*33{\cdot}43] \quad \supset . R \in \mathrm{G}`T \quad (2)$

$\vdash . *21{\cdot}33 . *33{\cdot}131 . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{G}`T \subset \{(\iota`\Lambda \cup \iota`\mathrm{V}) \uparrow \alpha\}_\Delta`\alpha \quad (3)$

$\vdash . (1) . (2) . (3) . \supset \vdash . \mathrm{Prop}$

***116·713.** $\vdash : \mathrm{Hp}\, *116{\cdot}712 . \supset . T \in 1 \to 1$

Dem.

$\vdash . *116{\cdot}712 . *14{\cdot}21 . \supset \vdash :. \mathrm{Hp} . \supset : R \in \mathrm{G}`T . \supset . \mathrm{E}\,!\, T`R :$

$[*71{\cdot}16] \quad \supset : R \in 1 \to \mathrm{Cls} \quad (1)$

$\vdash . *116{\cdot}712{\cdot}711 . \quad \supset \vdash :. \mathrm{Hp} . \supset : R, S \in \mathrm{G}`T . T`R = T`S . \supset . R = S \quad (2)$

$\vdash . (1) . (2) . *71{\cdot}55 . \quad \supset \vdash . \mathrm{Prop}$

***116·714.** $\vdash : \mathrm{Hp}\, *116{\cdot}712 . \mu \in \mathrm{Cl}`\alpha . R = \hat{\gamma}\hat{x}\, \{\gamma = \Lambda . x \in \mu . \mathbf{v} . \gamma = \mathrm{V} . x \in \alpha - \mu\} . \supset .$

$R \in \{(\iota`\Lambda \cup \iota`\mathrm{V}) \uparrow \alpha\}_\Delta`\alpha . \mu = T`R$

Dem.

$\vdash . *21{\cdot}33 . *33{\cdot}13 . \quad \supset \vdash :. \mathrm{Hp} . \supset : \gamma \in \mathrm{D}`R . \supset . \gamma \in \iota`\Lambda \cup \iota`\mathrm{V} \quad (1)$

$\vdash . *21{\cdot}33 . *33{\cdot}131 . \supset \vdash :: \mathrm{Hp} . \supset :. x \in \mathrm{G}`R . \equiv :$

$(\exists\gamma) : \gamma = \Lambda . x \in \mu . \mathbf{v} . \gamma = \mathrm{V} . x \in \alpha - \mu :$

$[*10{\cdot}42 . *13{\cdot}19] \equiv : x \in \mu . \mathbf{v} . x \in \alpha - \mu : \quad (2)$

$[*24{\cdot}411 . \mathrm{Hp}] \quad \equiv : x \in \alpha \quad (3)$

$\vdash . *21{\cdot}33 . *30{\cdot}3 . \supset \vdash :. \mathrm{Hp} . \supset : x \in \mu . \supset_x . R`x = \Lambda : x \in \alpha - \mu . \supset_x . R`x = \mathrm{V} :$

$[(2) . *14{\cdot}21] \quad \supset : x \in \mathrm{G}`R . \supset_x . \mathrm{E}\,!\, R`x :$

$[*71{\cdot}16] \quad \supset : R \in 1 \to \mathrm{Cls} \quad (4)$

$\vdash . *21{\cdot}33 . \quad \supset \vdash :: \mathrm{Hp} . \supset :. \gamma = \Lambda . \supset : \gamma R x . \equiv_x . x \in \mu :$

$[*32{\cdot}181] \quad \supset : \overleftarrow{R}`\gamma = \mu \quad (5)$

$\vdash . (1) . (3) . (4) . *116{\cdot}12 . \supset \vdash : \mathrm{Hp} . \supset . R \in \{(\iota`\Lambda \cup \iota`\mathrm{V}) \uparrow \alpha\}_\Delta`\alpha \quad (6)$

$\vdash . (5) . (6) . *116{\cdot}712 . \quad \supset \vdash : \mathrm{Hp} . \supset . \mu = T`R \quad (7)$

$\vdash . (6) . (7) . \supset \vdash . \mathrm{Prop}$

***116·715.** $\vdash : \mathrm{Hp}\, *116{\cdot}712 . \supset . \mathrm{D}`T = \mathrm{Cl}`\alpha$

Dem.

$\vdash . *116{\cdot}714 . *33{\cdot}43 . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{Cl}`\alpha \subset \mathrm{D}`T \quad (1)$

$\vdash . *21{\cdot}33 . *33{\cdot}13 . \supset$

$\vdash :. \mathrm{Hp} . \supset : \mu \in \mathrm{D}`T . \supset . (\exists R) . R \in \{(\iota`\Lambda \cup \iota`\mathrm{V}) \uparrow \alpha\}_\Delta`\alpha . \mu = \overleftarrow{R}`\Lambda .$

$[*33{\cdot}151] \quad \supset . (\exists R) . R \in \{(\iota`\Lambda \cup \iota`\mathrm{V}) \uparrow \alpha\}_\Delta`\alpha . \mu \subset \mathrm{G}`R .$

$[*80{\cdot}14] \quad \supset . \mu \subset \alpha \quad (2)$

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***116·72.** $\vdash . \text{Nc‘Cl‘}\alpha = 2^{\text{Nc‘}\alpha}$

Dem.

$\vdash . *116\cdot712\cdot713\cdot715 . \supset \vdash . \text{Cl‘}\alpha \,\text{sm}\, \{(\iota\text{‘}\Lambda \cup \iota\text{‘V}) \uparrow \alpha\}_\Delta\text{‘}\alpha$ (1)

$\vdash . (1) . *116\cdot7 . \supset \vdash . \text{Prop}$

***116·8.** $\vdash . \text{Rl‘}(\rho \uparrow \sigma) = \dot{s}\text{‘‘Cl‘}(\sigma \times \rho)$

Dem.

$\vdash . *60\cdot2 . \supset \vdash :. R \in \dot{s}\text{‘‘Cl‘}(\sigma \times \rho) . \equiv : (\exists\lambda) . \lambda \subset \sigma \times \rho . R = \dot{s}\text{‘}\lambda :$

$[*113\cdot101] \equiv : (\exists\lambda) : P \in \lambda . \supset_P . (\exists x, y) . x \in \rho . y \in \sigma . P = x \downarrow y : R = \dot{s}\text{‘}\lambda :$

$[*41\cdot11] \quad \equiv : (\exists\lambda) : P \in \lambda . \supset_P . (\exists x, y) . x \in \rho . y \in \sigma . P = x \downarrow y :$

$uRv . \equiv_{u,v} . (\exists P) . P \in \lambda . uPv :$

$[*10\cdot56] \quad \supset : uRv . \supset_{u,v} . (\exists x, y) . x \in \rho . y \in \sigma . u (x \downarrow y) v :$

$[*55\cdot13] \quad \supset : uRv . \supset_{u,v} . u \in \rho . v \in \sigma :$

$[*35\cdot103] \quad \supset : R \mathrel{\dot{\subset}} \rho \uparrow \sigma$ (1)

$\vdash . *35\cdot103 . *113\cdot101 . \supset$

$\vdash : R \mathrel{\dot{\subset}} \rho \uparrow \sigma . \lambda = \hat{P}\{(\exists x, y) . xRy . P = x \downarrow y\} . \supset . \lambda \subset \sigma \times \rho$ (2)

$\vdash . *41\cdot11 . *13\cdot195 . \supset$

$\vdash :. \text{Hp}(2) . \supset : u(\dot{s}\text{‘}\lambda) v . \equiv . (\exists x, y) . xRy . u (x \downarrow y) v .$

$[*55\cdot13] \quad \equiv . uRv$ (3)

$\vdash . (2) . (3) . \supset \vdash : R \mathrel{\dot{\subset}} \rho \uparrow \sigma . \supset . (\exists\lambda) . \lambda \subset \sigma \times \rho . R = \dot{s}\text{‘}\lambda$ (4)

$\vdash . (1) . (4) . \supset \vdash . \text{Prop}$

***116·81.** $\vdash . \dot{s} \upharpoonright \text{Cl‘}(\sigma \times \rho) \in 1 \to 1$

Dem.

$\vdash . *41\cdot13 . \supset \vdash :. \alpha, \beta \in \text{Cl‘}(\sigma \times \rho) . \dot{s}\text{‘}\alpha = \dot{s}\text{‘}\beta . x \downarrow y \in \alpha . \supset : x \downarrow y \mathrel{\dot{\subset}} \dot{s}\text{‘}\beta :$

$[*41\cdot11] \quad \supset : (\exists P) . P \in \beta . x \downarrow y \mathrel{\dot{\subset}} P :$

$[*113\cdot101.\text{Hp}] \supset : (\exists P, u, v) . P \in \beta . P = u \downarrow v . x \downarrow y \mathrel{\dot{\subset}} u \downarrow v :$

$[*55\cdot134\cdot34] \quad \supset : (\exists P, u, v) . P \in \beta . P = u \downarrow v . x \downarrow y = u \downarrow v :$

$[*13\cdot172\cdot13] \quad \supset : x \downarrow y \in \beta$ (1)

$\vdash . (1) \dfrac{\beta, \alpha}{\alpha, \beta} . \supset \vdash : \alpha, \beta \in \text{Cl‘}(\sigma \times \rho) . \dot{s}\text{‘}\alpha = \dot{s}\text{‘}\beta . x \downarrow y \in \beta . \supset . x \downarrow y \in \alpha$ (2)

$\vdash . (1) . (2) . *113\cdot101 . \supset \vdash : \alpha, \beta \in \text{Cl‘}(\sigma \times \rho) . \dot{s}\text{‘}\alpha = \dot{s}\text{‘}\beta . \supset . \alpha = \beta$ (3)

$\vdash . (3) . *71\cdot55 . *72\cdot163 . \supset \vdash . \text{Prop}$

***116·82.** $\vdash . \text{Rl‘}(\rho \uparrow \sigma) \,\text{sm}\, \text{Cl‘}(\sigma \times \rho)$ $[*116\cdot8\cdot81]$

***116·83.** $\vdash . \text{Nc‘Rl‘}(\rho \uparrow \sigma) = 2^{\text{Nc‘}\rho \times_\circ \text{Nc‘}\sigma}$ $[*116\cdot82\cdot72 . *113\cdot25]$

***116·9.** $\vdash : \text{Nc‘}t\text{‘}x = \mu . \supset . \text{Nc‘}t^2\text{‘}x = 2^\mu$ $[*116\cdot72 . *63\cdot66]$

***116·901.** $\vdash : \text{Nc‘}t_0\text{‘}\alpha = \mu . \supset . \text{Nc‘}t\text{‘}\alpha = 2^\mu$ $[*116\cdot72 . *63\cdot65]$

***116·91.** $\vdash : \text{Nc‘}t_0\text{‘}\alpha = \mu . \supset . \text{Nc‘}t_{00}\text{‘}\alpha = 2^{\mu^2}$ $[*116\cdot83 . *64\cdot5\cdot11]$

***116·92.** $\vdash : \text{Nc‘}t_0\text{‘}\alpha = \mu . \supset . \text{Nc‘}t_0{}^1\text{‘}\alpha = 2^{\mu \times_\circ 2^\mu} . \text{Nc‘}t^{11}\text{‘}\alpha = 2^{2^\mu \times_\circ 2^\mu}$. etc.
$[*116\cdot83 . *64\cdot16 . *116\cdot901]$

$*117$. GREATER AND LESS

Summary of $*117$.

A cardinal μ is said to be greater than another cardinal ν when there is a class α which has μ terms and has a part which has ν terms, while there is no class β which has ν terms and has a part which has μ terms. The relation "greater than" is transitive and asymmetrical; and by the Schröder-Bernstein theorem, if μ is greater than or equal to ν, and ν is greater than or equal to μ, then $\mu = \nu$. But we cannot prove that of any two cardinals one must be the greater, unless we assume the multiplicative axiom. The proof then follows from Zermelo's theorem that on that assumption every class can be well-ordered. This subject will be dealt with at a later stage.

The form of the definitions is so arranged as to allow of the inequality of two cardinals in different types. The relevant considerations are the same as for the definitions of addition, multiplication and exponentiation.

Our definition of "$\mu > \nu$" is

$*117\cdot01$. $\mu > \nu \,.=.\, (\exists \alpha, \beta) \,.\, \mu = \mathrm{N_0c}`\alpha \,.\, \nu = \mathrm{N_0c}`\beta \,.$
$\qquad\qquad \exists ! \,\mathrm{Cl}`\alpha \cap \mathrm{Nc}`\beta \,.\, \sim\exists ! \,\mathrm{Cl}`\beta \cap \mathrm{Nc}`\alpha$ Df

We also define "$\mu > \mathrm{Nc}`\alpha$" as meaning "$\mu > \mathrm{N_0c}`\alpha$," and "$\mathrm{Nc}`\alpha > \nu$" as meaning "$\mathrm{N_0c}`\alpha > \nu$," for the reasons explained in $*110$. It then easily follows that if $\mu > \nu$, μ and ν must be homogeneous cardinals (this is part of $*117\cdot15$); that if μ and ν are homogeneous cardinals, and $\mu > \nu$, the same holds if we substitute $\mathrm{sm}``\mu$ and $\mathrm{sm}``\nu$ for one or both of μ and ν ($*117\cdot16$); that

$*117\cdot13$. $\vdash : \mathrm{Nc}`\alpha > \mathrm{Nc}`\beta \,.\equiv.\, \exists ! \,\mathrm{Cl}`\alpha \cap \mathrm{Nc}`\beta \,.\, \sim\exists ! \,\mathrm{Cl}`\beta \cap \mathrm{Nc}`\alpha$

and that

$*117\cdot14$. $\vdash : \mu > \nu \,.\equiv.\, (\exists \alpha, \beta) \,.\, \mu = \mathrm{N_0c}`\alpha \,.\, \nu = \mathrm{N_0c}`\beta \,.\, \mathrm{Nc}`\alpha > \mathrm{Nc}`\beta$

We cannot define "$\mu \geqslant \nu$" as "$\mu > \nu \,.\vee.\, \mu = \nu$," because "$\mu = \nu$" restricts μ and ν too much by requiring that they should be of the same type, and restricts them too little by not requiring that they should both be existent cardinals. To avoid both these inconveniences, we put

$*117\cdot05$. $\mu \geqslant \nu \,.=:\, \mu > \nu \,.\vee.\, \mu, \nu \in \mathrm{N_0C} \,.\, \mu = \mathrm{sm}``\nu$ Df

The use of this definition is chiefly through the propositions

$*117\cdot108$. $\vdash :. \mathrm{Nc}`\alpha \geqslant \mathrm{Nc}`\beta \,.\equiv:\, \mathrm{Nc}`\alpha > \mathrm{Nc}`\beta \,.\vee.\, \mathrm{Nc}`\alpha = \mathrm{Nc}`\beta$

$*117\cdot24$. $\vdash : \mu \geqslant \nu \,.\equiv.\, (\exists \alpha, \beta) \,.\, \mu = \mathrm{N_0c}`\alpha \,.\, \nu = \mathrm{N_0c}`\beta \,.\, \mathrm{Nc}`\alpha \geqslant \mathrm{Nc}`\beta$

In $*117{\cdot}2$, we repeat the Schröder-Bernstein theorem ($*73{\cdot}88$), which is required in most of the remaining propositions of this number. It leads at once to the propositions

$*117{\cdot}22$. $\vdash : \exists ! \, \mathrm{Cl}`\alpha \cap \mathrm{Nc}`\beta \, . \equiv . \, \mathrm{Nc}`\alpha \geqslant \mathrm{Nc}`\beta$

(which practically supersedes the definition of "$\geqslant$")

$*117{\cdot}221$. $\vdash : \mathrm{Nc}`\alpha \geqslant \mathrm{Nc}`\beta \, . \equiv . \, (\exists \rho) \, . \, \rho \subset \alpha \, . \, \rho \, \mathrm{sm} \, \beta$

$*117{\cdot}222$. $\vdash : \beta \subset \alpha \, . \supset . \, \mathrm{Nc}`\alpha \geqslant \mathrm{Nc}`\beta$

$*117{\cdot}23$. $\vdash : \mathrm{Nc}`\alpha \geqslant \mathrm{Nc}`\beta \, . \, \mathrm{Nc}`\beta \geqslant \mathrm{Nc}`\alpha \, . \equiv . \, \mathrm{Nc}`\alpha = \mathrm{Nc}`\beta$

This last proposition may be called the Schröder-Bernstein theorem with as much propriety as $*73{\cdot}88$; the two are scarcely different.

If we now revert to the definition of $\mu > \nu$, or to $*117{\cdot}13$, and apply $*117{\cdot}22$, we see ($*117{\cdot}26$) that "$\mathrm{Nc}`\alpha > \mathrm{Nc}`\beta$" may be conveniently regarded as asserting $\mathrm{Nc}`\alpha \geqslant \mathrm{Nc}`\beta \, . \sim (\mathrm{Nc}`\beta \geqslant \mathrm{Nc}`\alpha)$; in fact, the best ideas to work with are $\geqslant$ and its converse $\leqslant$, which for practical purposes we regard as defined by $*117{\cdot}22$, and from which we derive $>$ and $<$. The relation $>$ will be the product of $\geqslant$ into the negation of its converse; this holds for μ and ν ($*117{\cdot}281$) as well as for $\mathrm{Nc}`\alpha$ and $\mathrm{Nc}`\beta$.

$*117{\cdot}3{\cdot}31$ constitute an important use of $*110{\cdot}72$, namely to prove that one existent cardinal is greater than another or equal to it when the first can be obtained by adding to the second (where what is added must be a cardinal). That is to say, we have

$*117{\cdot}3$. $\vdash : \mathrm{Nc}`\alpha \geqslant \mathrm{Nc}`\beta \, . \equiv . \, (\exists \varpi) \, . \, \varpi \in \mathrm{NC} \, . \, \mathrm{Nc}`\alpha = \mathrm{Nc}`\beta +_c \varpi$

$*117{\cdot}31$. $\vdash :. \, \mu \geqslant \nu \, . \equiv : \mu, \nu \in \mathrm{N_0C} : (\exists \varpi) \, . \, \varpi \in \mathrm{NC} \, . \, \mu = \nu +_c \varpi$

$*117{\cdot}4$—$\cdot 471$ are concerned in proving that $>$ and $\geqslant$ are transitive, that $>$ is asymmetrical ($*117{\cdot}42$), and allied propositions.

Our next set of propositions is concerned with 0 and 1 and 2. We prove that a homogeneous cardinal is whatever is greater than or equal to 0 ($*117{\cdot}501$); that a homogeneous cardinal other than 0 is whatever is greater than 0 ($*117{\cdot}511$); that a homogeneous cardinal other than 0 is whatever is greater than or equal to 1 ($*117{\cdot}531$); and that a homogeneous cardinal other than 0 and 1 is whatever is greater than 1 ($*117{\cdot}55$), and is whatever is greater than or equal to 2 ($*117{\cdot}551$).

We next prove a set of propositions concerning $\geqslant$ which have no analogues for $>$, except when the cardinals concerned are finite. Thus *e.g.* we prove

$*117{\cdot}561$. $\vdash : \mu \geqslant \nu \, . \, \varpi \in \mathrm{N_0C} \, . \supset . \, \mu +_c \varpi \geqslant \nu +_c \varpi$

If we substitute $>$ for $\geqslant$, this no longer holds. Thus *e.g.* put $\mu = 2$, $\nu = 1$, $\varpi = \aleph_0$ (cf. $*123$); then $\mu > \nu$, but $\mu +_c \varpi = \nu +_c \varpi = \varpi$. Similar remarks apply to the analogous propositions ($*117{\cdot}571{\cdot}581{\cdot}591$) on multiplication and exponentiation.

We prove next that a sum is greater than or equal to either of its summands (∗117·6); that a product neither of whose factors vanishes is greater than or equal to either of its factors (∗117·62); that, assuming μ and ν are existent cardinals, then if they are neither 0 nor 1, their product is greater than or equal to their sum (∗117·631), and if μ is neither 0 nor 1, then $\mu^\nu \geqslant \mu \times_c \nu$ (∗117·652).

The last important proposition in this number is Cantor's theorem

∗117·661. $\vdash : \mu \in N_0C . \supset . 2^\mu > \mu$

which follows immediately from ∗102·72 and ∗116·72.

The propositions of this number are much used in the following section, on finite and infinite.

∗117·01. $\mu > \nu . = . (\exists \alpha, \beta) . \mu = N_0c`\alpha . \nu = N_0c`\beta .$
$\exists ! Cl`\alpha \cap Nc`\beta . \sim \exists ! Cl`\beta \cap Nc`\alpha$ Df

∗117·02. $\mu > Nc`\alpha . = . \mu > N_0c`\alpha$ Df

∗117·03. $Nc`\alpha > \nu . = . N_0c`\alpha > \nu$ Df

∗117·04. $\mu < \nu . = . \nu > \mu$ Df

∗117·05. $\mu \geqslant \nu . = : \mu > \nu . \vee . \mu, \nu \in N_0C . \mu = sm``\nu$ Df

∗117·06. $\mu \leqslant \nu . = . \nu \geqslant \mu$ Df

The analogues of ∗117·02·03 are to be applied also to ∗117·04·05·06.

∗117·1. $\vdash : \mu > \nu . \equiv . (\exists \alpha, \beta) . \mu = N_0c`\alpha . \nu = N_0c`\beta .$
$\exists ! Cl`\alpha \cap Nc`\beta . \sim \exists ! Cl`\beta \cap Nc`\alpha$ [(∗117·01)]

∗117·101. $\vdash : \mu > Nc`\beta . \equiv . \mu > N_0c`\beta$ [(∗117·02)]

∗117·102. $\vdash : Nc`\alpha > \nu . \equiv . N_0c`\alpha > \nu$ [(∗117·03)]

∗117·103. $\vdash : \mu < \nu . \equiv . \nu > \mu$ [(∗117·04)]

∗117·104. $\vdash :. \mu \geqslant \nu . \equiv : \mu > \nu . \vee . \mu, \nu \in N_0C . \mu = sm``\nu$ [(∗117·05)]

∗117·105. $\vdash : \mu \leqslant \nu . \equiv . \nu \geqslant \mu$ [(∗117·06)]

∗117·106. $\vdash : Nc`\alpha > Nc`\beta . \equiv . N_0c`\alpha > N_0c`\beta$ [∗117·101·102]

∗117·107. $\vdash : Nc`\alpha \geqslant Nc`\beta . \equiv . N_0c`\alpha \geqslant N_0c`\beta$

Dem.

$\vdash .$ ∗117·104·106 $. \supset$

$\vdash :. Nc`\alpha \geqslant Nc`\beta . \equiv : N_0c`\alpha > N_0c`\beta . \vee . Nc`\alpha, Nc`\beta \in N_0C . Nc`\alpha = sm``Nc`\beta :$

[∗100·511.∗103·22] $\equiv : N_0c`\alpha > N_0c`\beta . \vee . Nc`\alpha, Nc`\beta \in N_0C . Nc`\alpha = Nc`\beta :$

[∗103·16] $\equiv : N_0c`\alpha > N_0c`\beta . \vee . Nc`\alpha, Nc`\beta \in N_0C . Nc`\alpha = N_0c`\beta :$

[∗103·21] $\equiv : N_0c`\alpha > N_0c`\beta . \vee . Nc`\beta \in N_0C . Nc`\alpha = N_0c`\beta :$

[∗103·16] $\equiv : N_0c`\alpha > N_0c`\beta . \vee . Nc`\beta \in N_0C . N_0c`\alpha = Nc`\beta :$

[∗103·2] $\equiv : N_0c`\alpha > N_0c`\beta . \vee . N_0c`\alpha = Nc`\beta :$

[∗103·4] $\equiv : N_0c`\alpha > N_0c`\beta . \vee . N_0c`\alpha = sm``N_0c`\beta :$

[∗103·21.∗117·104] $\equiv : N_0c`\alpha \geqslant N_0c`\beta :. \supset \vdash .$ Prop

***117·108.** $\vdash :. \mathrm{Nc}\text{‘}\alpha \geqslant \mathrm{Nc}\text{‘}\beta . \equiv : \mathrm{Nc}\text{‘}\alpha > \mathrm{Nc}\text{‘}\beta . \mathrm{v} . \mathrm{Nc}\text{‘}\alpha = \mathrm{Nc}\text{‘}\beta$
[*117·107·106·104 . *103·16·4]

***117·11.** $\vdash :. \alpha \,\mathrm{sm}\, \alpha' . \beta \,\mathrm{sm}\, \beta' . \supset : \exists ! \mathrm{Cl}\text{‘}\alpha \cap \mathrm{Nc}\text{‘}\beta . \equiv . \exists ! \mathrm{Cl}\text{‘}\alpha' \cap \mathrm{Nc}\text{‘}\beta'$

Dem.

$\vdash . *100\cdot321 . \supset \vdash :. \mathrm{Hp} . \supset : \exists ! \mathrm{Cl}\text{‘}\alpha' \cap \mathrm{Nc}\text{‘}\beta . \equiv . \exists ! \mathrm{Cl}\text{‘}\alpha' \cap \mathrm{Nc}\text{‘}\beta'$ (1)

$\vdash . *73\cdot21 . \supset$

$\vdash : R \in 1 \rightarrow 1 . \mathrm{D}\text{‘}R = \alpha . \mathrm{G}\text{‘}R = \alpha' . \gamma \subset \alpha . \gamma \in \mathrm{Nc}\text{‘}\beta . \supset .$

$\breve{R}\text{‘‘}\gamma \subset \alpha' . \breve{R}\text{‘‘}\gamma \in \mathrm{Nc}\text{‘}\beta .$

$[*60\cdot2] \supset . \exists ! \mathrm{Cl}\text{‘}\alpha' \cap \mathrm{Nc}\text{‘}\beta$ (2)

$\vdash . (2) . *10\cdot11\cdot23\cdot35 . *73\cdot1 . \supset$

$\vdash : \alpha \,\mathrm{sm}\, \alpha' . \exists ! \mathrm{Cl}\text{‘}\alpha \cap \mathrm{Nc}\text{‘}\beta . \supset . \exists ! \mathrm{Cl}\text{‘}\alpha' \cap \mathrm{Nc}\text{‘}\beta$ (3)

$\vdash . (3) \dfrac{\alpha', \alpha}{\alpha, \alpha'} . \supset \vdash : \alpha \,\mathrm{sm}\, \alpha' . \exists ! \mathrm{Cl}\text{‘}\alpha' \cap \mathrm{Nc}\text{‘}\beta . \supset . \exists ! \mathrm{Cl}\text{‘}\alpha \cap \mathrm{Nc}\text{‘}\beta$ (4)

$\vdash . (3) . (4) . \supset \vdash :. \alpha \,\mathrm{sm}\, \alpha' . \supset : \exists ! \mathrm{Cl}\text{‘}\alpha \cap \mathrm{Nc}\text{‘}\beta . \equiv . \exists ! \mathrm{Cl}\text{‘}\alpha' \cap \mathrm{Nc}\text{‘}\beta$ (5)

$\vdash . (1) . (5) . \supset \vdash . \mathrm{Prop}$

***117·12.** $\vdash :. \mu > \nu . \equiv : \mu, \nu \in \mathrm{N_0C} :$

$\gamma \in \mu . \delta \in \nu . \supset_{\gamma, \delta} . \exists ! \mathrm{Cl}\text{‘}\gamma \cap \mathrm{Nc}\text{‘}\delta . \sim \exists ! \mathrm{Cl}\text{‘}\delta \cap \mathrm{Nc}\text{‘}\gamma$

Dem.

$\vdash . *117\cdot1\cdot11 . \supset$

$\vdash :. \mu > \nu . \equiv : (\exists \alpha, \beta) : \mu = \mathrm{N_0c}\text{‘}\alpha . \nu = \mathrm{N_0c}\text{‘}\beta . \exists ! \mathrm{Cl}\text{‘}\alpha \cap \mathrm{Nc}\text{‘}\beta . \sim \exists ! \mathrm{Cl}\text{‘}\beta \cap \mathrm{Nc}\text{‘}\alpha :$

$\gamma \in \mu . \delta \in \nu . \supset_{\gamma, \delta} . \exists ! \mathrm{Cl}\text{‘}\gamma \cap \mathrm{Nc}\text{‘}\delta . \sim \exists ! \mathrm{Cl}\text{‘}\delta \cap \mathrm{Nc}\text{‘}\gamma :$

$[*103\cdot12] \quad \equiv : (\exists \alpha, \beta) : \mu = \mathrm{N_0c}\text{‘}\alpha . \nu = \mathrm{N_0c}\text{‘}\beta . \alpha \in \mu . \beta \in \nu . \exists ! \mathrm{Cl}\text{‘}\alpha \cap \mathrm{Nc}\text{‘}\beta .$

$\sim \exists ! \mathrm{Cl}\text{‘}\beta \cap \mathrm{Nc}\text{‘}\alpha :$

$\gamma \in \mu . \delta \in \nu . \supset_{\gamma, \delta} . \exists ! \mathrm{Cl}\text{‘}\gamma \cap \mathrm{Nc}\text{‘}\delta . \sim \exists ! \mathrm{Cl}\text{‘}\delta \cap \mathrm{Nc}\text{‘}\gamma :$

$[*10\cdot55] \quad \equiv : (\exists \alpha, \beta) : \mu = \mathrm{N_0c}\text{‘}\alpha . \nu = \mathrm{N_0c}\text{‘}\beta . \alpha \in \mu . \beta \in \nu :$

$\gamma \in \mu . \delta \in \nu . \supset_{\gamma, \delta} . \exists ! \mathrm{Cl}\text{‘}\gamma \cap \mathrm{Nc}\text{‘}\delta . \sim \exists ! \mathrm{Cl}\text{‘}\delta \cap \mathrm{Nc}\text{‘}\gamma :$

$[*103\cdot12\cdot2] \equiv : \mu, \nu \in \mathrm{N_0C} : \gamma \in \mu . \delta \in \nu . \supset_{\gamma, \delta} . \exists ! \mathrm{Cl}\text{‘}\gamma \cap \mathrm{Nc}\text{‘}\delta . \sim \exists ! \mathrm{Cl}\text{‘}\delta \cap \mathrm{Nc}\text{‘}\gamma :.$

$\supset \vdash . \mathrm{Prop}$

***117·121.** $\vdash :. \mu > \nu . \equiv : \mu, \nu \in \mathrm{N_0C} :$

$\alpha \in \mu . \supset_\alpha . (\exists \beta) . \beta \in \nu . \exists ! \mathrm{Cl}\text{‘}\alpha \cap \mathrm{Nc}\text{‘}\beta . \sim \exists ! \mathrm{Cl}\text{‘}\beta \cap \mathrm{Nc}\text{‘}\alpha$

Dem.

$\vdash . *117\cdot1\cdot11 . \supset$

$\vdash :. \mu > \nu . \equiv : (\exists \alpha, \beta) : \mu = \mathrm{N_0c}\text{‘}\alpha . \nu = \mathrm{N_0c}\text{‘}\beta . \exists ! \mathrm{Cl}\text{‘}\alpha \cap \mathrm{Nc}\text{‘}\beta .$

$\sim \exists ! \mathrm{Cl}\text{‘}\beta \cap \mathrm{Nc}\text{‘}\alpha :$

$\gamma \in \mu . \supset_\gamma . (\exists \delta) . \delta \in \nu . \exists ! \mathrm{Cl}\text{‘}\gamma \cap \mathrm{Nc}\text{‘}\delta . \sim \exists ! \mathrm{Cl}\text{‘}\delta \cap \mathrm{Nc}\text{‘}\gamma$

$[*103\cdot12 . *10\cdot55] \equiv : (\exists \alpha, \beta) : \mu = \mathrm{N_0c}\text{‘}\alpha . \nu = \mathrm{N_0c}\text{‘}\beta . \alpha \in \mu :$

$\gamma \in \mu . \supset_\gamma . (\exists \delta) . \delta \in \nu . \exists ! \mathrm{Cl}\text{‘}\gamma \cap \mathrm{Nc}\text{‘}\delta . \sim \exists ! \mathrm{Cl}\text{‘}\delta \cap \mathrm{Nc}\text{‘}\gamma$

$[*103\cdot12\cdot2] \quad \equiv : \mu, \nu \in \mathrm{N_0C} : \gamma \in \mu . \supset_\gamma . (\exists \delta) . \delta \in \nu . \exists ! \mathrm{Cl}\text{‘}\gamma \cap \mathrm{Nc}\text{‘}\delta .$

$\sim \exists ! \mathrm{Cl}\text{‘}\delta \cap \mathrm{Nc}\text{‘}\gamma :. \supset \vdash . \mathrm{Prop}$

The above proof is given shortly because it proceeds on the same lines as *117·12. In applying *10·55, the ϕx of that proposition is replaced by $\alpha \in \mu$,

and the ψx is replaced by

$$(\exists\beta) \,.\, \beta \in \nu \,.\, \exists ! \,\mathrm{Cl}‘\alpha \cap \mathrm{Nc}‘\beta \,.\, \sim\exists ! \,\mathrm{Cl}‘\beta \cap \mathrm{Nc}‘\alpha.$$

***117·13.** $\vdash : \mathrm{Nc}‘\alpha > \mathrm{Nc}‘\beta \,.\equiv .\, \exists ! \,\mathrm{Cl}‘\alpha \cap \mathrm{Nc}‘\beta \,.\, \sim\exists ! \,\mathrm{Cl}‘\beta \cap \mathrm{Nc}‘\alpha$

Dem.

$\vdash . *117·106 . \supset$

$\vdash :. \mathrm{Nc}‘\alpha > \mathrm{Nc}‘\beta \,.\equiv : \mathrm{N_0c}‘\alpha > \mathrm{N_0c}‘\beta :$

[*103·2.*117·12] $\equiv : \gamma \in \mathrm{N_0c}‘\alpha \,.\, \delta \in \mathrm{N_0c}‘\beta \,.\supset_{\gamma,\delta} .$
$\exists ! \,\mathrm{Cl}‘\gamma \cap \mathrm{Nc}‘\delta \,.\, \sim\exists ! \,\mathrm{Cl}‘\delta \cap \mathrm{Nc}‘\gamma :$

[*100·31.*117·11] $\equiv : \gamma \in \mathrm{N_0c}‘\alpha \,.\, \delta \in \mathrm{N_0c}‘\beta \,.\supset_{\gamma,\delta} .$
$\exists ! \,\mathrm{Cl}‘\alpha \cap \mathrm{Nc}‘\beta \,.\, \sim\exists ! \,\mathrm{Cl}‘\beta \cap \mathrm{Nc}‘\alpha :$

[*10·23] $\equiv : \exists ! \,\mathrm{N_0c}‘\alpha \,.\, \exists ! \,\mathrm{N_0c}‘\beta \,.\supset .$
$\exists ! \,\mathrm{Cl}‘\alpha \cap \mathrm{Nc}‘\beta \,.\, \sim\exists ! \,\mathrm{Cl}‘\beta \cap \mathrm{Nc}‘\alpha :$

[*103·13] $\equiv : \exists ! \,\mathrm{Cl}‘\alpha \cap \mathrm{Nc}‘\beta \,.\, \sim\exists ! \,\mathrm{Cl}‘\beta \cap \mathrm{Nc}‘\alpha :. \supset \vdash .$ Prop

***117·14.** $\vdash : \mu > \nu \,.\equiv .\, (\exists\alpha, \beta) \,.\, \mu = \mathrm{N_0c}‘\alpha \,.\, \nu = \mathrm{N_0c}‘\beta \,.\, \mathrm{Nc}‘\alpha > \mathrm{Nc}‘\beta$
[*117·1·13]

***117·15.** $\vdash : \mu > \nu \,.\equiv .\, \mu, \nu \in \mathrm{N_0C} \,.\, \exists ! \,s‘\mathrm{Cl}“\mu \cap \mathrm{sm}“\nu \,.\, \sim\exists ! \,s‘\mathrm{Cl}“\nu \cap \mathrm{sm}“\mu$

Dem.

$\vdash . *103·4 . *117·1 . \supset$

$\vdash :. \mu > \nu \,.\equiv : (\exists\alpha, \beta) \,.\, \mu = \mathrm{N_0c}‘\alpha \,.\, \nu = \mathrm{N_0c}‘\beta \,.\, \exists ! \,\mathrm{Cl}‘\alpha \cap \mathrm{sm}“\nu \,.$
$\sim\exists ! \,\mathrm{Cl}‘\beta \cap \mathrm{sm}“\mu :$

[*103·2·26] $\equiv : \mu, \nu \in \mathrm{N_0C} : (\exists\alpha, \beta) \,.\, \alpha \in \mu \,.\, \beta \in \nu \,.\, \exists ! \,\mathrm{Cl}‘\alpha \cap \mathrm{sm}“\nu \,.$
$\sim\exists ! \,\mathrm{Cl}‘\beta \cap \mathrm{sm}“\mu :$

[*117·11] $\equiv : \mu, \nu \in \mathrm{N_0C} : (\exists\alpha, \beta) \,.\, \alpha \in \mu \,.\, \beta \in \nu \,.\, \exists ! \,\mathrm{Cl}‘\alpha \cap \mathrm{sm}“\nu :$
$\delta \in \nu \,.\supset_\delta .\, \sim\exists ! \,\mathrm{Cl}‘\delta \cap \mathrm{sm}“\mu :$

[*103·13.*10·51] $\equiv : \mu, \nu \in \mathrm{N_0C} : (\exists\alpha) \,.\, \alpha \in \mu \,.\, \exists ! \,\mathrm{Cl}‘\alpha \cap \mathrm{sm}“\nu :$
$\sim(\exists\delta) \,.\, \delta \in \nu \,.\, \exists ! \,\mathrm{Cl}‘\delta \cap \mathrm{sm}“\mu :$

[*40·4.*60·2] $\equiv : \mu, \nu \in \mathrm{N_0C} \,.\, \exists ! \,s‘\mathrm{Cl}“\mu \cap \mathrm{sm}“\nu \,.$
$\sim\exists ! \,s‘\mathrm{Cl}“\nu \cap \mathrm{sm}“\mu :. \supset \vdash .$ Prop

The advantage of this proposition is that it expresses "$\mu > \nu$" in terms of μ and ν alone, without the auxiliary α and β of the definition.

***117·16.** $\vdash :. \mu, \nu \in \mathrm{N_0C} \,.\supset : \mu > \nu \,.\equiv .\, \mathrm{sm}“\mu > \nu \,.\equiv .\, \mu > \mathrm{sm}“\nu \,.\equiv .\, \mathrm{sm}“\mu > \mathrm{sm}“\nu$
[*117·14 . *103·4]

***117·2.** $\vdash : \alpha \,\mathrm{sm}\, \alpha' \,.\, \beta \,\mathrm{sm}\, \beta' \,.\, \beta' \subset \alpha \,.\, \alpha' \subset \beta \,.\supset .\, \alpha \,\mathrm{sm}\, \beta$ [*73·88]

This proposition (which is the Schröder-Bernstein theorem) is fundamental in the theory of greater and less.

***117·21.** $\vdash : \exists ! \,\mathrm{Cl}‘\alpha \cap \mathrm{Nc}‘\beta \,.\, \exists ! \,\mathrm{Cl}‘\beta \cap \mathrm{Nc}‘\alpha \,.\supset .\, \mathrm{Nc}‘\alpha = \mathrm{Nc}‘\beta$
[*117·2 . *100·321]

***117·211.** $\vdash : \exists ! \,\mathrm{Cl}‘\alpha \cap \mathrm{Nc}‘\beta \,.\, \exists ! \,\mathrm{Cl}‘\beta \cap \mathrm{Nc}‘\alpha \,.\equiv .\, \mathrm{Nc}‘\alpha = \mathrm{Nc}‘\beta$

Dem.

$\vdash . *100·3 . *60·34 . \supset \vdash : \mathrm{Nc}‘\alpha = \mathrm{Nc}‘\beta \,.\supset .\, \alpha \in \mathrm{Cl}‘\alpha \cap \mathrm{Nc}‘\beta \,.\, \beta \in \mathrm{Cl}‘\beta \cap \mathrm{Nc}‘\alpha$ (1)

$\vdash . (1) . *117·21 . \supset \vdash .$ Prop

***117·22.** $\vdash : \exists ! \mathrm{Cl}\text{‘}\alpha \cap \mathrm{Nc}\text{‘}\beta . \equiv . \mathrm{Nc}\text{‘}\alpha \geqslant \mathrm{Nc}\text{‘}\beta$

Dem.

$\vdash . *117{\cdot}13 . \supset \vdash : \mathrm{Hp} . \sim\exists ! \mathrm{Cl}\text{‘}\beta \cap \mathrm{Nc}\text{‘}\alpha . \equiv . \mathrm{Nc}\text{‘}\alpha > \mathrm{Nc}\text{‘}\beta \quad (1)$

$\vdash . *117{\cdot}211 . \supset \vdash : \mathrm{Hp} . \exists ! \mathrm{Cl}\text{‘}\beta \cap \mathrm{Nc}\text{‘}\alpha . \equiv . \mathrm{Nc}\text{‘}\alpha = \mathrm{Nc}\text{‘}\beta \quad (2)$

$\vdash . (1) . (2) . *117{\cdot}108 . \supset \vdash . \mathrm{Prop}$

***117·221.** $\vdash : \mathrm{Nc}\text{‘}\alpha \geqslant \mathrm{Nc}\text{‘}\beta . \equiv . (\exists \rho) . \rho \subset \alpha . \rho \mathrm{sm} \beta$ [*117·22 . *60·2 . *100·1]

***117·222.** $\vdash : \beta \subset \alpha . \supset . \mathrm{Nc}\text{‘}\alpha \geqslant \mathrm{Nc}\text{‘}\beta$ [*117·221]

***117·23.** $\vdash : \mathrm{Nc}\text{‘}\alpha \geqslant \mathrm{Nc}\text{‘}\beta . \mathrm{Nc}\text{‘}\beta \geqslant \mathrm{Nc}\text{‘}\alpha . \equiv . \mathrm{Nc}\text{‘}\alpha = \mathrm{Nc}\text{‘}\beta$ [*117·211·22]

***117·24.** $\vdash : \mu \geqslant \nu . \equiv . (\exists \alpha, \beta) . \mu = \mathrm{N_0c}\text{‘}\alpha . \nu = \mathrm{N_0c}\text{‘}\beta . \mathrm{Nc}\text{‘}\alpha \geqslant \mathrm{Nc}\text{‘}\beta$

Dem.

$\vdash . *117{\cdot}104{\cdot}14 . \supset \vdash :. \mu \geqslant \nu . \equiv : (\exists \alpha, \beta) . \mu = \mathrm{N_0c}\text{‘}\alpha . \nu = \mathrm{N_0c}\text{‘}\beta . \mathrm{Nc}\text{‘}\alpha > \mathrm{Nc}\text{‘}\beta . \vee .$
$(\exists \alpha, \beta) . \mu = \mathrm{N_0c}\text{‘}\alpha . \nu = \mathrm{N_0c}\text{‘}\beta . \mu = \mathrm{sm}\text{“}\nu :$

[*103·4.*13·193] $\equiv : (\exists \alpha, \beta) . \mu = \mathrm{N_0c}\text{‘}\alpha . \nu = \mathrm{N_0c}\text{‘}\beta . \mathrm{Nc}\text{‘}\alpha > \mathrm{Nc}\text{‘}\beta . \vee .$
$(\exists \alpha, \beta) . \mu = \mathrm{N_0c}\text{‘}\alpha . \nu = \mathrm{N_0c}\text{‘}\beta . \mathrm{N_0c}\text{‘}\alpha = \mathrm{Nc}\text{‘}\beta :$

[*103·16] $\equiv : (\exists \alpha, \beta) . \mu = \mathrm{N_0c}\text{‘}\alpha . \nu = \mathrm{N_0c}\text{‘}\beta . \mathrm{Nc}\text{‘}\alpha > \mathrm{Nc}\text{‘}\beta . \vee .$
$(\exists \alpha, \beta) . \mu = \mathrm{N_0c}\text{‘}\alpha . \nu = \mathrm{N_0c}\text{‘}\beta . \mathrm{Nc}\text{‘}\alpha = \mathrm{Nc}\text{‘}\beta :$

[*11·41.*117·108] $\equiv : (\exists \alpha, \beta) . \mu = \mathrm{N_0c}\text{‘}\alpha . \nu = \mathrm{N_0c}\text{‘}\beta . \mathrm{Nc}\text{‘}\alpha \geqslant \mathrm{Nc}\text{‘}\beta :.$
$\supset \vdash . \mathrm{Prop}$

***117·241.** $\vdash : \mu \geqslant \nu . \equiv . (\exists \alpha, \beta) . \mu = \mathrm{N_0c}\text{‘}\alpha . \nu = \mathrm{N_0c}\text{‘}\beta . \exists ! \mathrm{Cl}\text{‘}\alpha \cap \mathrm{Nc}\text{‘}\beta$
[*117·24·22]

***117·242.** $\vdash :. \mu, \nu \in \mathrm{NC} . \supset : \mu \geqslant \nu . \equiv . (\exists \alpha, \beta) . \alpha \in \mu . \beta \in \nu . \exists ! \mathrm{Cl}\text{‘}\alpha \cap \mathrm{Nc}\text{‘}\beta$
[*117·241 . *103·26]

***117·243.** $\vdash :. \mu \geqslant \nu . \equiv : (\exists \alpha, \beta) : \mu = \mathrm{N_0c}\text{‘}\alpha . \nu = \mathrm{N_0c}\text{‘}\beta : (\exists \rho) . \rho \subset \alpha . \rho \mathrm{sm} \beta$
[*117·24·221]

***117·244.** $\vdash :. \mu, \nu \in \mathrm{N_0C} . \supset : \mu \geqslant \nu . \equiv . \mathrm{sm}\text{“}\mu \geqslant \nu . \equiv . \mu \geqslant \mathrm{sm}\text{“}\nu . \equiv .$
$\mathrm{sm}\text{“}\mu \geqslant \mathrm{sm}\text{“}\nu$ [*117·24 . *103·4]

***117·25.** $\vdash : \mu \geqslant \nu . \nu \geqslant \mu . \equiv . \mu, \nu \in \mathrm{N_0C} . \mathrm{sm}\text{“}\mu = \mathrm{sm}\text{“}\nu$

Dem.

$\vdash . *117{\cdot}24 . \supset$

$\vdash : \mu \geqslant \nu . \nu \geqslant \mu . \equiv . (\exists \alpha, \beta, \gamma, \delta) . \mu = \mathrm{N_0c}\text{‘}\alpha = \mathrm{N_0c}\text{‘}\gamma . \nu = \mathrm{N_0c}\text{‘}\beta = \mathrm{N_0c}\text{‘}\delta .$
$\mathrm{Nc}\text{‘}\alpha \geqslant \mathrm{Nc}\text{‘}\beta . \mathrm{Nc}\text{‘}\delta \geqslant \mathrm{Nc}\text{‘}\gamma .$

[*117·107] $\equiv . (\exists \alpha, \beta, \gamma, \delta) . \mu = \mathrm{N_0c}\text{‘}\alpha = \mathrm{N_0c}\text{‘}\gamma . \nu = \mathrm{N_0c}\text{‘}\beta = \mathrm{N_0c}\text{‘}\delta .$
$\mathrm{N_0c}\text{‘}\alpha \geqslant \mathrm{N_0c}\text{‘}\beta . \mathrm{N_0c}\text{‘}\delta \geqslant \mathrm{N_0c}\text{‘}\gamma .$

[*13·193] $\equiv . (\exists \alpha, \beta, \gamma, \delta) . \mu = \mathrm{N_0c}\text{‘}\alpha = \mathrm{N_0c}\text{‘}\gamma . \nu = \mathrm{N_0c}\text{‘}\beta = \mathrm{N_0c}\text{‘}\delta .$
$\mathrm{N_0c}\text{‘}\alpha \geqslant \mathrm{N_0c}\text{‘}\beta . \mathrm{N_0c}\text{‘}\beta \geqslant \mathrm{N_0c}\text{‘}\alpha .$

[*117·107·23] $\equiv . (\exists \alpha, \beta, \gamma, \delta) . \mu = \mathrm{N_0c}\text{‘}\alpha = \mathrm{N_0c}\text{‘}\gamma . \nu = \mathrm{N_0c}\text{‘}\beta = \mathrm{N_0c}\text{‘}\delta .$
$\mathrm{Nc}\text{‘}\alpha = \mathrm{Nc}\text{‘}\beta .$

[*11·45.*103·2] $\equiv . (\exists \alpha, \beta) . \mu = \mathrm{N_0c}\text{‘}\alpha . \nu = \mathrm{N_0c}\text{‘}\beta . \mu, \nu \in \mathrm{N_0C} \,\dot{:}\, \mathrm{Nc}\text{‘}\alpha = \mathrm{Nc}\text{‘}\beta .$

[*103·4] $\equiv . (\exists \alpha, \beta) . \mu = \mathrm{N_0c}\text{‘}\alpha . \nu = \mathrm{N_0c}\text{‘}\beta . \mu, \nu \in \mathrm{N_0C} . \mathrm{sm}\text{“}\mu = \mathrm{sm}\text{“}\nu .$

[*11·45.*103·2] $\equiv . \mu, \nu \in \mathrm{N_0C} . \mathrm{sm}\text{“}\mu = \mathrm{sm}\text{“}\nu : \supset \vdash . \mathrm{Prop}$

$*117{\cdot}26.\quad \vdash : \mathrm{Nc}`\alpha > \mathrm{Nc}`\beta . \equiv . \mathrm{Nc}`\alpha \geqslant \mathrm{Nc}`\beta . \mathrm{Nc}`\alpha \neq \mathrm{Nc}`\beta$

Dem.

$\vdash . *117{\cdot}13 . *13{\cdot}12 . \mathrm{Transp} . \supset \vdash : \mathrm{Nc}`\alpha > \mathrm{Nc}`\beta . \supset . \mathrm{Nc}`\alpha \neq \mathrm{Nc}`\beta :$

$[*117{\cdot}108] \quad \supset \vdash : \mathrm{Nc}`\alpha > \mathrm{Nc}`\beta . \supset . \mathrm{Nc}`\alpha \geqslant \mathrm{Nc}`\beta . \mathrm{Nc}`\alpha \neq \mathrm{Nc}`\beta \quad (1)$

$\vdash . *117{\cdot}108 . *5{\cdot}6 . \supset \vdash : \mathrm{Nc}`\alpha \geqslant \mathrm{Nc}`\beta . \mathrm{Nc}`\alpha \neq \mathrm{Nc}`\beta . \supset . \mathrm{Nc}`\alpha > \mathrm{Nc}`\beta \quad (2)$

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

$*117{\cdot}27.\quad \vdash : \mathrm{Nc}`\alpha < \mathrm{Nc}`\beta . \equiv . \mathrm{Nc}`\alpha \leqslant \mathrm{Nc}`\beta . \mathrm{Nc}`\alpha \neq \mathrm{Nc}`\beta$
$[*117{\cdot}26{\cdot}103{\cdot}105]$

$*117{\cdot}28.\quad \vdash : \mathrm{Nc}`\alpha > \mathrm{Nc}`\beta . \equiv . \mathrm{Nc}`\alpha \geqslant \mathrm{Nc}`\beta . \sim(\mathrm{Nc}`\beta \geqslant \mathrm{Nc}`\alpha)$
$[*117{\cdot}22{\cdot}13]$

$*117{\cdot}281.\quad \vdash : \mu > \nu . \equiv . \mu \geqslant \nu . \sim(\nu \geqslant \mu) \quad [*117{\cdot}14{\cdot}28{\cdot}24]$

$*117{\cdot}29.\quad \vdash : \mathrm{Nc}`\alpha < \mathrm{Nc}`\beta . \equiv . \mathrm{Nc}`\alpha \leqslant \mathrm{Nc}`\beta . \sim(\mathrm{Nc}`\beta \leqslant \mathrm{Nc}`\alpha) \quad [*117{\cdot}28]$

$*117{\cdot}291.\quad \vdash : \mu < \nu . \equiv . \mu \leqslant \nu . \sim(\nu \leqslant \mu) \quad [*117{\cdot}281]$

$*117{\cdot}3.\quad \vdash : \mathrm{Nc}`\alpha \geqslant \mathrm{Nc}`\beta . \equiv . (\exists \varpi) . \varpi \in \mathrm{NC} . \mathrm{Nc}`\alpha = \mathrm{Nc}`\beta +_c \varpi$

Dem.

$\vdash . *117{\cdot}221 . \supset \vdash : \mathrm{Nc}`\alpha \geqslant \mathrm{Nc}`\beta . \equiv . (\exists \delta) . \delta \,\mathrm{sm}\, \beta . \delta \subset \alpha .$

$[*110{\cdot}72] \quad \equiv . (\exists \varpi) . \varpi \in \mathrm{NC} . \mathrm{Nc}`\alpha = \mathrm{Nc}`\beta +_c \varpi :$
$\supset \vdash . \mathrm{Prop}$

$*117{\cdot}31.\quad \vdash :. \mu \geqslant \nu . \equiv : \mu, \nu \in \mathrm{N_0C} : (\exists \varpi) . \varpi \in \mathrm{NC} . \mu = \nu +_c \varpi$

Dem.

$\vdash . *117{\cdot}24{\cdot}3 . \supset$

$\vdash :. \mu \geqslant \nu . \equiv : (\exists \alpha, \beta, \varpi) . \mu = \mathrm{N_0c}`\alpha . \nu = \mathrm{N_0c}`\beta . \mathrm{Nc}`\alpha = \mathrm{Nc}`\beta +_c \varpi :$

$[(*110{\cdot}03)] \quad \equiv : (\exists \alpha, \beta, \varpi) . \mu = \mathrm{N_0c}`\alpha . \nu = \mathrm{N_0c}`\beta . \mathrm{Nc}`\alpha = \nu +_c \varpi :$

$[*103{\cdot}16 . *110{\cdot}42] \equiv : (\exists \alpha, \beta, \varpi) . \mu = \mathrm{N_0c}`\alpha . \nu = \mathrm{N_0c}`\beta . \mu = \nu +_c \varpi :$

$[*103{\cdot}2] \quad \equiv : \mu, \nu \in \mathrm{N_0C} : (\exists \varpi) . \mu = \nu +_c \varpi :. \supset \vdash . \mathrm{Prop}$

$*117{\cdot}32.\quad \vdash : \mu \geqslant \nu . \exists ! \,\mathrm{sm}``\mu \cap t`\alpha . \supset . \exists ! \,\mathrm{sm}``\nu \cap t`\alpha$

Dem.

$\vdash . *117{\cdot}241 . *103{\cdot}4 . \supset$

$\vdash : \mathrm{Hp} . \supset . (\exists \beta, \gamma) . \mu = \mathrm{N_0c}`\beta . \nu = \mathrm{N_0c}`\gamma . \exists ! \mathrm{Cl}`\beta \cap \mathrm{Nc}`\gamma . \mathrm{sm}``\mu = \mathrm{Nc}`\beta .$
$\mathrm{sm}``\nu = \mathrm{Nc}`\gamma \quad (1)$

$\vdash . *63{\cdot}105{\cdot}371 . *73{\cdot}12 . \supset$

$\vdash : R \in \rho \,\overline{\mathrm{sm}}\, \beta . \rho \in t`\alpha . \sigma \subset \beta . \sigma \,\mathrm{sm}\, \gamma . \supset . R``\sigma \in t`\alpha . R``\sigma \,\mathrm{sm}\, \gamma \quad (2)$

$\vdash . (2) . *73{\cdot}04 . \supset \vdash : \rho \in \mathrm{Nc}`\beta \cap t`\alpha . \sigma \in \mathrm{Cl}`\beta \cap \mathrm{Nc}`\gamma . \supset . \exists ! \mathrm{Nc}`\gamma \cap t`\alpha \quad (3)$

$\vdash . (1) . (3) . \supset \vdash . \mathrm{Prop}$

The above proposition shows that if a cardinal μ exists in a given type, so do all smaller cardinals.

***117·4.** $\vdash : \mu \geqslant \nu . \nu \geqslant \varpi . \supset . \mu \geqslant \varpi$

Dem.

$\vdash . *117{\cdot}243 . \supset \vdash :. \text{Hp} . \supset : (\exists \alpha, \beta, \gamma) : \mu = \text{N}_0\text{c}`\alpha . \nu = \text{N}_0\text{c}`\beta . \varpi = \text{N}_0\text{c}`\gamma :$
$(\exists \rho) . \rho \subset \alpha . \rho \,\text{sm}\, \beta : (\exists \sigma) . \sigma \subset \beta . \sigma \,\text{sm}\, \gamma :$

$[*117{\cdot}11.*60{\cdot}2.*100{\cdot}1] \quad \supset : (\exists \alpha, \beta, \gamma) : \mu = \text{N}_0\text{c}`\alpha . \nu = \text{N}_0\text{c}`\beta . \varpi = \text{N}_0\text{c}`\gamma :$
$(\exists \rho, \tau) . \rho \subset \alpha . \rho \,\text{sm}\, \beta . \tau \subset \rho . \tau \,\text{sm}\, \gamma :$

$[*22{\cdot}44] \quad \supset : (\exists \alpha, \gamma) : \mu = \text{N}_0\text{c}`\alpha . \varpi = \text{N}_0\text{c}`\gamma : (\exists \tau) . \tau \subset \alpha . \tau \,\text{sm}\, \gamma :$

$[*117{\cdot}243] \quad \supset : \mu \geqslant \varpi :. \supset \vdash . \text{Prop}$

***117·41.** $\vdash : \mu \leqslant \nu . \nu \leqslant \varpi . \supset . \mu \leqslant \varpi \quad [*117{\cdot}4]$

***117·42.** $\vdash : \sim(\mu > \mu) . \sim(\mu < \mu)$

Dem.

$\vdash . *117{\cdot}15 . *13{\cdot}12 . \text{Transp} . \supset \vdash : \mu > \nu . \supset . \mu \neq \nu : \supset \vdash . \text{Prop}$

***117·43.** $\vdash : \mu \geqslant \nu . \sim(\mu \geqslant \varpi) . \supset . \sim(\nu \geqslant \varpi) \quad [*117{\cdot}4 . \text{Transp}]$

***117·44.** $\vdash : \nu \geqslant \varpi . \sim(\mu \geqslant \varpi) . \supset . \sim(\mu \geqslant \nu) \quad [*117{\cdot}4 . \text{Transp}]$

***117·45.** $\vdash : \mu \geqslant \nu . \nu > \varpi . \supset . \mu > \varpi$

Dem.

$\vdash . *117{\cdot}281 . \supset \vdash : \text{Hp} . \supset . \mu \geqslant \nu . \nu \geqslant \varpi . \sim(\varpi \geqslant \nu) .$

$[*117{\cdot}4{\cdot}44] \quad \supset . \mu \geqslant \varpi . \sim(\varpi \geqslant \mu) .$

$[*117{\cdot}281] \quad \supset . \mu > \varpi : \supset \vdash . \text{Prop}$

***117·46.** $\vdash : \mu > \nu . \nu \geqslant \varpi . \supset . \mu > \varpi$ [Proof as in *117·45]

***117·47.** $\vdash : \mu > \nu . \nu > \varpi . \supset . \mu > \varpi \quad [*117{\cdot}45{\cdot}104]$

***117·471.** $\vdash : \mu < \nu . \nu < \varpi . \supset . \mu < \varpi \quad [*117{\cdot}47{\cdot}103]$

***117·5.** $\vdash : \mu \in \text{N}_0\text{C} . \supset . \mu \geqslant 0$

Dem.

$\vdash . *60{\cdot}3 . *100{\cdot}3 . \supset \vdash . \exists ! \text{Cl}`\alpha \cap \text{Nc}`\Lambda .$

$[*117{\cdot}22] \quad \supset \vdash . \text{Nc}`\alpha \geqslant \text{Nc}`\Lambda .$

$[*117{\cdot}107.*101{\cdot}1] \supset \vdash . \text{N}_0\text{c}`\alpha \geqslant 0 \qquad (1)$

$\vdash . (1) . *103{\cdot}2 . \quad \supset \vdash . \text{Prop}$

***117·501.** $\vdash : \mu \in \text{N}_0\text{C} . \equiv . \mu \geqslant 0 \quad [*117{\cdot}5{\cdot}104]$

***117·51.** $\vdash : \mu \in \text{N}_0\text{C} - \iota`0 . \supset . \mu > 0$

Dem.

$\vdash . *101{\cdot}15 . \supset \vdash : \text{Hp} . \supset . \mu \neq \text{sm}``0 \qquad (1)$

$\vdash . (1) . *117{\cdot}5{\cdot}104 . \supset \vdash . \text{Prop}$

***117·511.** $\vdash : \mu \in \text{N}_0\text{C} - \iota`0 . \equiv . \mu > 0 \quad [*117{\cdot}51{\cdot}15{\cdot}42]$

***117·52.** $\vdash : \exists ! \xi . \supset . \text{Nc}`\xi \geqslant 1$

Dem.

$\vdash . *51{\cdot}2 . \supset \vdash : \text{Hp} . \supset . (\exists x) . \iota`x \subset \xi .$

$[*117{\cdot}222] \quad \supset . (\exists x) . \text{Nc}`\xi \geqslant \text{Nc}`\iota`x .$

$[*101{\cdot}2] \quad \supset . \text{Nc}`\xi \geqslant 1 : \supset \vdash . \text{Prop}$

***117·53.** $\vdash : \mu \in N_0C - \iota`0 . \supset . \mu \geqslant 1$

Dem.

$\vdash . *101\cdot16 . *103\cdot2 . \supset \vdash : \mathrm{Hp} . \supset . (\exists\alpha) . N_0c`\alpha = \mu . \exists ! \alpha .$

$[*117\cdot52] \quad \supset . (\exists\alpha) . N_0c`\alpha = \mu . Nc`\alpha \geqslant 1 .$

$[*117\cdot107] \quad \supset . \mu \geqslant 1 : \supset \vdash . \mathrm{Prop}$

***117·531.** $\vdash : \mu \in N_0C - \iota`0 . \equiv . \mu \geqslant 1$

Dem.

$\vdash . *117\cdot104 . \quad \supset \vdash : \mu \geqslant 1 . \supset . \mu \in N_0C \quad (1)$

$\vdash . *117\cdot51 . *101\cdot22 . \supset \vdash . 1 > 0 .$

$[*117\cdot45] \quad \supset \vdash : \mu \geqslant 1 . \supset . \mu > 0 .$

$[*117\cdot42] \quad \supset . \mu \neq 0 \quad (2)$

$\vdash . (1) . (2) . *117\cdot53 . \supset \vdash . \mathrm{Prop}$

***117·54.** $\vdash :. 1 \geqslant \mu . \equiv : \mu = 0 . \vee . \mu = 1$

Dem.

$\vdash . *117\cdot241 . *101\cdot2 . *52\cdot22 . \supset$

$\vdash :. 1 \geqslant \mu . \equiv : (\exists\alpha, x) . \mu = N_0c`\alpha . \exists ! Nc`\alpha \cap Cl`\iota`x :$

$[*60\cdot362] \quad \equiv : (\exists\alpha, x) : \mu = N_0c`\alpha : \exists ! Nc`\alpha \cap \iota`\Lambda . \vee . \exists ! Nc`\alpha \cap \iota`\iota`x :$

$[*51\cdot31] \quad \equiv : (\exists\alpha, x) : \mu = N_0c`\alpha : \Lambda \in Nc`\alpha . \vee . \iota`x \in Nc`\alpha :$

$[*101\cdot17\cdot29] \quad \equiv : (\exists\alpha, x) : \mu = N_0c`\alpha : Nc`\alpha = Nc`\Lambda . \vee . Nc`\alpha = Nc`\iota`x :$

$[*103\cdot16] \quad \equiv : (\exists\alpha, x) . \mu = N_0c`\alpha : \mu = Nc`\Lambda . \vee . \mu = Nc`\iota`x :$

$[*101\cdot1\cdot2] \quad \equiv : (\exists\alpha) . \mu = N_0c`\alpha : \mu = 0 . \vee . \mu = 1 :$

$[*103\cdot2\cdot5\cdot51] \equiv : \mu = 0 . \vee . \mu = 1 :. \supset \vdash . \mathrm{Prop}$

***117·55.** $\vdash : \mu > 1 . \equiv . \mu \in N_0C - \iota`0 - \iota`1$

Dem.

$\vdash . *117\cdot281 . \supset \vdash : \mu > 1 . \equiv . \mu \geqslant 1 . \sim(1 \geqslant \mu) .$

$[*117\cdot531\cdot54] \quad \equiv . \mu \in N_0C - \iota`0 . \mu \neq 0 . \mu \neq 1 .$

$[*51\cdot15] \quad \equiv . \mu \in N_0C - \iota`0 - \iota`1 : \supset \vdash . \mathrm{Prop}$

***117·551.** $\vdash :. \mu \in N_0C - \iota`0 - \iota`1 . \equiv :$

$(\exists\alpha) : \mu = N_0c`\alpha : (\exists x, y) . x, y \in \alpha . x \neq y : \equiv . \mu \geqslant 2$

Dem.

$\vdash . *103\cdot2 . \supset \vdash :. \mu \in N_0C - \iota`0 - \iota`1 . \equiv :$

$(\exists\alpha) . \mu = N_0c`\alpha . N_0c`\alpha \neq 0 . N_0c`\alpha \neq 1 :$

$[*101\cdot14] \quad \equiv : (\exists\alpha) . \mu = N_0c`\alpha . \exists ! \alpha . N_0c`\alpha \neq 1 :$

$[*103\cdot26] \quad \equiv : (\exists\alpha) . \mu = N_0c`\alpha . \exists ! \alpha . \alpha \sim\in 1 :$

$[*52\cdot41] \quad \equiv : (\exists\alpha) : \mu = N_0c`\alpha : (\exists x, y) . x, y \in \alpha . x \neq y : \quad (1)$

$[*54\cdot26 . *51\cdot2] \equiv : (\exists\alpha) : \mu = N_0c`\alpha : (\exists x, y) . \iota`x \cup \iota`y \subset \alpha . \iota`x \cup \iota`y \in 2 :$

$[*13\cdot195] \quad \equiv : (\exists\alpha) : \mu = N_0c`\alpha : (\exists x, y, \beta) . \beta = \iota`x \cup \iota`y . \beta \subset \alpha . \beta \in 2 :$

$[*54\cdot101] \quad \equiv : (\exists\alpha) : \mu = N_0c`\alpha : (\exists\beta) . \beta \subset \alpha . \beta \in 2 :$

$[*117\cdot241] \quad \equiv : \mu \geqslant 2 \quad (2)$

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***117·56.** $\vdash : \mathrm{Nc}`\alpha \geqslant \mathrm{Nc}`\beta . \supset . \mathrm{Nc}`\alpha +_c \mathrm{Nc}`\gamma \geqslant \mathrm{Nc}`\beta +_c \mathrm{Nc}`\gamma$

Dem.

$\vdash . *110·12 . *117·221 . \supset$

$\vdash : \mathrm{Hp} . \supset . (\exists \delta) . \delta \subset \downarrow \Lambda_\gamma``\iota``\alpha . \delta \,\mathrm{sm} \downarrow \Lambda_\gamma``\iota``\beta .$

[*110·11.*73·71.(*110·01)] $\supset . (\exists \delta) . \delta \cup \Lambda_\alpha \downarrow ``\iota``\gamma \subset \alpha + \gamma .$

$\delta \cup \Lambda_\alpha \downarrow ``\iota``\gamma \,\mathrm{sm}\, (\beta + \gamma) .$

[*117·221] $\supset . \mathrm{Nc}`(\alpha + \gamma) \geqslant \mathrm{Nc}`(\beta + \gamma) .$

[*110·3] $\supset . \mathrm{Nc}`\alpha +_c \mathrm{Nc}`\gamma \geqslant \mathrm{Nc}`\beta +_c \mathrm{Nc}`\gamma : \supset \vdash . \mathrm{Prop}$

***117·561.** $\vdash : \mu \geqslant \nu . \varpi \in \mathrm{N_0C} . \supset . \mu +_c \varpi \geqslant \nu +_c \varpi$ [*117·56]

The proof of *117·561 follows from *117·56 in the same way as the proof of *117·31 follows from *117·3. In the remainder of this number we shall omit proofs of this kind.

***117·57.** $\vdash : \mathrm{Nc}`\alpha \geqslant \mathrm{Nc}`\beta . \supset . \mathrm{Nc}`\alpha \times_c \mathrm{Nc}`\gamma \geqslant \mathrm{Nc}`\beta \times_c \mathrm{Nc}`\gamma$

Dem.

$\vdash . *37·2 . \quad \supset \vdash : \rho \subset \alpha . \supset . \gamma \underset{''}{\downarrow} ``\rho \subset \gamma \underset{''}{\downarrow} ``\alpha .$

[*40·161.*113·1] $\supset . \rho \times \gamma \subset \alpha \times \gamma$ (1)

$\vdash . *113·13 . \supset \vdash : \rho \,\mathrm{sm}\, \beta . \supset . \rho \times \gamma \,\mathrm{sm}\, \beta \times \gamma$ (2)

$\vdash . (1) . (2) . \quad \supset \vdash : \rho \subset \alpha . \rho \,\mathrm{sm}\, \beta . \supset . \rho \times \gamma \subset \alpha \times \gamma . \rho \times \gamma \,\mathrm{sm}\, \beta \times \gamma .$

[*117·221] $\supset . \mathrm{Nc}`(\alpha \times \gamma) \geqslant \mathrm{Nc}`(\beta \times \gamma)$ (3)

$\vdash . (3) . *117·221 . \supset \vdash . \mathrm{Prop}$

***117·571.** $\vdash . \mu \geqslant \nu . \varpi \in \mathrm{N_0C} . \supset . \mu \times_c \varpi \geqslant \nu \times_c \varpi$ [*117·57]

***117·58.** $\vdash : \mathrm{Nc}`\alpha \geqslant \mathrm{Nc}`\beta . \supset . (\mathrm{Nc}`\alpha)^{\mathrm{Nc}`\gamma} \geqslant (\mathrm{Nc}`\beta)^{\mathrm{Nc}`\gamma}$

Dem.

$\vdash . *35·432·82 . \supset \vdash : \rho \subset \alpha . \supset . \rho \uparrow \gamma \subset \alpha \uparrow \gamma .$

[*80·15] $\supset . (\rho \uparrow \gamma)_\Delta `\gamma \subset (\alpha \uparrow \gamma)_\Delta `\gamma$ (1)

$\vdash . *116·15·19 . \supset \vdash : \rho \,\mathrm{sm}\, \beta . \supset . (\rho \uparrow \gamma)_\Delta `\gamma \,\mathrm{sm}\, (\beta \uparrow \gamma)_\Delta `\gamma$ (2)

$\vdash . (1) . (2) . *117·221 . \supset$

$\vdash : \rho \subset \alpha . \rho \,\mathrm{sm}\, \beta . \supset . \mathrm{Nc}`(\alpha \uparrow \gamma)_\Delta `\gamma \geqslant \mathrm{Nc}`(\beta \uparrow \gamma)_\Delta `\gamma .$

[*116·15·25] $\supset . (\mathrm{Nc}`\alpha)^{\mathrm{Nc}`\gamma} \geqslant (\mathrm{Nc}`\beta)^{\mathrm{Nc}`\gamma}$ (3)

$\vdash . (3) . *117·221 . \supset \vdash . \mathrm{Prop}$

***117·581.** $\vdash : \mu \geqslant \nu . \varpi \in \mathrm{N_0C} . \supset . \mu^\varpi \geqslant \nu^\varpi$ [*117·58]

The two following propositions are lemmas for *117·59.

***117·582.** $\vdash : \exists ! \gamma . \beta \subset \alpha . \sigma \in \gamma \exp (\alpha - \beta) . \supset . (\cup \sigma) \upharpoonright (\gamma \exp \beta) \in 1 \rightarrow 1 .$

$(\cup \sigma)``(\gamma \exp \beta) \subset \gamma \exp \alpha$

Dem.

$\vdash . *116·183 . \supset \vdash : \rho \in (\gamma \exp \beta) . \sigma \in \gamma \exp (\alpha - \beta) . \supset . \rho \subset \beta \times \gamma . \sigma \subset (\alpha - \beta) \times \gamma .$

[*113·19.*24·21] $\supset . \rho \cap \sigma = \Lambda$ (1)

$\vdash . (1) . *24·481 . \supset \vdash :: \mathrm{Hp} . \supset :. \rho, \rho' \in (\gamma \exp \beta) . \supset : \rho \cup \sigma = \rho' \cup \sigma . \equiv . \rho = \rho' :.$

[*71·58] $\supset :. (\cup \sigma) \upharpoonright (\gamma \exp \beta) \in 1 \rightarrow 1$ (2)

$\vdash . *113\cdot191 . \supset \vdash :. \text{Hp} . \supset : \gamma \downarrow_{,,} ``\beta \cap \gamma \downarrow_{,,} ``(\alpha - \beta) = \Lambda :$

$[*115\cdot14.(*116\cdot01)] \quad \supset : \rho \,\epsilon\, (\gamma \exp \beta) . \supset . \rho \cup \sigma \,\epsilon\, \text{Prod}`\{\gamma \downarrow_{,,} ``\beta \cup \gamma \downarrow_{,,} ``(\alpha - \beta)\} .$

$[*37\cdot22.*24\cdot411] \quad \supset . \rho \cup \sigma \,\epsilon\, (\gamma \exp \alpha) :$

$[*37\cdot61] \quad \supset : (\cup \sigma)``(\gamma \exp \beta) \subset \gamma \exp \alpha \qquad (3)$

$\vdash . (2) . (3) . \supset \vdash . \text{Prop}$

***117·583.** $\vdash : \beta \subset \alpha . \exists ! \gamma . \supset . (\exists \tau) . \tau \subset \gamma \exp \alpha . \tau \,\text{sm}\, (\gamma \exp \beta)$

Dem.

$\vdash . *116\cdot171 . \supset \vdash : \text{Hp} . \supset . \exists ! \gamma \exp (\alpha - \beta) \qquad (1)$

$\vdash . (1) . *117\cdot582 . *73\cdot15 . \supset \vdash . \text{Prop}$

***117·59.** $\vdash : \text{Nc}`\alpha \geqslant \text{Nc}`\beta . \exists ! \gamma . \supset . (\text{Nc}`\gamma)^{\text{Nc}`\alpha} \geqslant (\text{Nc}`\gamma)^{\text{Nc}`\beta}$

Dem.

$\vdash . *117\cdot221 . \supset \vdash :. \text{Hp} . \supset : (\exists \rho) . \rho \subset \alpha . \rho \,\text{sm}\, \beta : \exists ! \gamma :$

$[*117\cdot583] \quad \supset : (\exists \rho, \tau) . \rho \subset \alpha . \rho \,\text{sm}\, \beta . \tau \subset \gamma \exp \alpha . \tau \,\text{sm}\, (\gamma \exp \rho) :$

$[*116\cdot19] \quad \supset : (\exists \tau) . \tau \subset \gamma \exp \alpha . \tau \,\text{sm}\, (\gamma \exp \beta) :$

$[*117\cdot221] \quad \supset : \text{Nc}`(\gamma \exp \alpha) \geqslant \text{Nc}`(\gamma \exp \beta) \qquad (1)$

$\vdash . (1) . *116\cdot25 . \supset \vdash . \text{Prop}$

The hypothesis is essential in the above proposition, for $0^0 = 1$ while $0^1 = 0$, so that $0^0 > 0^1$.

***117·591.** $\vdash : \mu \geqslant \nu . \varpi \,\epsilon\, \text{N}_0\text{C} - \iota`0 . \supset . \varpi^\mu \geqslant \varpi^\nu \quad [*117\cdot59]$

***117·592.** $\vdash : \alpha^\delta = 1 . \alpha \neq 0 . \alpha \neq 1 . \supset . \delta = 0$

Dem.

$\vdash . *116\cdot203 . \supset \vdash :. \text{Hp} . \supset : \alpha, \delta \,\epsilon\, \text{N}_0\text{C} :$

$[*117\cdot551\cdot53] \quad \supset : \alpha \geqslant 2 : \delta \neq 0 . \supset . \delta \geqslant 1 :$

$[*117\cdot581\cdot591] \quad \supset : \delta \neq 0 . \supset . \alpha^\delta \geqslant 2^1 .$

$[*116\cdot321.*117\cdot244] \quad \supset . \alpha^\delta \geqslant 2 .$

$[*117\cdot551] \quad \supset . \alpha^\delta \neq 1 \qquad (1)$

$\vdash . (1) . \text{Transp} . \supset \vdash . \text{Prop}$

The above proposition is used in *120·53.

***117·6.** $\vdash : \mu, \nu \,\epsilon\, \text{N}_0\text{C} . \supset . \mu +_c \nu \geqslant \mu . \mu +_c \nu \geqslant \nu$

Dem.

$\vdash . *117\cdot561\cdot5 . \supset \vdash : \text{Hp} . \supset . \mu +_c \nu \geqslant \mu +_c 0 . \mu +_c \nu \geqslant 0 +_c \nu \qquad (1)$

$\vdash . (1) . *110\cdot6 . *117\cdot244 . \supset \vdash . \text{Prop}$

***117·61.** $\vdash : \nu > \mu . \supset . \mu +_c \nu > \mu \quad [*117\cdot6\cdot45]$

***117·62.** $\vdash : \mu, \nu \,\epsilon\, \text{N}_0\text{C} - \iota`0 . \supset . \mu \times_c \nu \geqslant \mu . \mu \times_c \nu \geqslant \nu$

Dem.

$\vdash . *117\cdot571\cdot53 . \supset \vdash : \text{Hp} . \supset . \mu \times_c \nu \geqslant \mu \times_c 1 . \mu \times_c \nu \geqslant 1 \times_c \nu \qquad (1)$

$\vdash . (1) . *113\cdot621 . *117\cdot244 . \supset \vdash . \text{Prop}$

***117·63.** $\vdash : \alpha, \beta \sim\epsilon\, 0 \cup 1 \,.\supset.\, \mathrm{Nc}\text{‘}\alpha \times_0 \mathrm{Nc}\text{‘}\beta \geqslant \mathrm{Nc}\text{‘}\alpha +_0 \mathrm{Nc}\text{‘}\beta$

Dem.

$\vdash .\, *52\cdot 4 \,.\, \mathrm{Transp} \,.\supset \vdash : \mathrm{Hp} \,.\supset.\, (\exists x, x', y, y') \,.\, x, x' \epsilon \alpha \,.\, y, y' \epsilon \beta \,.\, x \neq x' \,.\, y \neq y'$ (1)

$\vdash .\, *113\cdot 101 \,.\quad \supset \vdash : \mathrm{Hp} \,.\, x, x' \epsilon \alpha \,.\, y, y' \epsilon \beta \,.\, x \neq x' \,.\, y \neq y' \,.\, \rho = \downarrow y\text{“}\alpha \,.$
$\sigma = x \downarrow \text{“}(\beta - \iota\text{‘}y) \cup \iota\text{‘}x' \downarrow y' \,.\supset.\, \rho \cup \sigma \subset \beta \times \alpha$ (2)

$\vdash .\, *55\cdot 15 \,.\supset \vdash :. \mathrm{Hp}(2) \,.\supset : R \epsilon \rho \,.\supset_R .\, \mathrm{C\!\!\!I}\text{‘}R = \iota\text{‘}y :$
$S \epsilon x \downarrow \text{“}(\beta - \iota\text{‘}y) \,.\supset_S .\, \mathrm{C\!\!\!I}\text{‘}S \epsilon \beta - \iota\text{‘}y : \mathrm{C\!\!\!I}\text{‘}x' \downarrow y' = \iota\text{‘}y' :$

[$*51\cdot 23$] $\supset : R \epsilon \rho \,.\, S \epsilon \sigma \,.\supset_{R,S} .\, \mathrm{C\!\!\!I}\text{‘}R \neq \mathrm{C\!\!\!I}\text{‘}S :$

[$*24\cdot 37 . *30\cdot 37$] $\supset : \rho \cap \sigma = \Lambda$ (3)

$\vdash .\, *73\cdot 61\cdot 611 \,.\quad \supset \vdash : \mathrm{Hp}(2) \,.\supset.\, \rho \,\mathrm{sm}\, \alpha \,.\, x \downarrow \text{“}(\beta - \iota\text{‘}y) \,\mathrm{sm}\, (\beta - \iota\text{‘}y)$ (4)

$\vdash .\, *55\cdot 202 \,.\quad \supset \vdash : \mathrm{Hp}(2) \,.\supset.\, x' \downarrow y' \sim\epsilon\, x \downarrow \text{“}(\beta - \iota\text{‘}y)$ (5)

$\vdash .\, (4) . (5) .\, *73\cdot 71 \,.\quad \supset \vdash : \mathrm{Hp}(2) \,.\supset.\, \rho \,\mathrm{sm}\, \alpha \,.\, \sigma \,\mathrm{sm}\, \beta$ (6)

$\vdash .\, (3) . (6) .\, *110\cdot 13 \,.\quad \supset \vdash : \mathrm{Hp}(2) \,.\supset.\, \rho \cup \sigma \,\epsilon\, \mathrm{Nc}\text{‘}(\alpha + \beta)$ (7)

$\vdash .\, (2) . (7) .\, *117\cdot 221 \,.\supset \vdash : \mathrm{Hp}(2) \,.\supset.\, \mathrm{Nc}\text{‘}(\beta \times \alpha) \geqslant \mathrm{Nc}\text{‘}(\alpha + \beta)$ (8)

$\vdash .\, (1) . (8) .\, *113\cdot 141\cdot 25 \,.\, *110\cdot 3 \,.\supset \vdash .\, \mathrm{Prop}$

***117·631.** $\vdash : \mu, \nu \,\epsilon\, \mathrm{N_0C} - \iota\text{‘}0 - \iota\text{‘}1 \,.\supset.\, \mu \times_0 \nu \geqslant \mu +_0 \nu$ [$*117\cdot 63$]

The two following propositions are lemmas for *117·64.

***117·632.** $\vdash : \kappa \,\epsilon\, \mathrm{Cls}^2 \,\mathrm{excl} \,.\, \kappa \sim\epsilon\, 0 \cup 1 \,.\, \rho, \sigma \,\epsilon\, \mathrm{Prod}\text{‘}\kappa \,.\, \rho \cap \sigma = \Lambda \,.$
$T = \hat{\mu}\hat{x}\, \{(\exists \alpha, \beta) \,.\, \alpha, \beta \,\epsilon\, \kappa \,.\, \alpha \neq \beta \,.\, x \,\epsilon\, \beta \,.\, \mu = (\rho - \alpha - \beta) \cup (\sigma \cap \alpha) \cup \iota\text{‘}x\} \,.$
$\supset.\, T \,\epsilon\, 1 \to 1 \,.\, \mathrm{D}\text{‘}T \subset \mathrm{Prod}\text{‘}\kappa \,.\, \mathrm{C\!\!\!I}\text{‘}T = s\text{‘}\kappa$

Dem.

$\vdash .\, *115\cdot 11\cdot 145 \,.\supset \vdash :. \mathrm{Hp} \,.\, \alpha, \beta \,\epsilon\, \kappa \,.\, \alpha \neq \beta \,.\supset : \rho - \alpha - \beta \,\epsilon\, \mathrm{Prod}\text{‘}(\kappa - \iota\text{‘}\alpha - \iota\text{‘}\beta) :$

[$*115\cdot 11\cdot 145$] $\supset : (\rho - \alpha - \beta) \cup (\sigma \cap \alpha) \,\epsilon\, \mathrm{Prod}\text{‘}(\kappa - \iota\text{‘}\beta) :$

[$*115\cdot 145$] $\supset : x \,\epsilon\, \beta \,.\supset.\, (\rho - \alpha - \beta) \cup (\sigma \cap \alpha) \cup \iota\text{‘}x \,\epsilon\, \mathrm{Prod}\text{‘}\kappa$ (1)

$\vdash .\, (1) .\, *21\cdot 33 \,.\quad \supset \vdash : \mathrm{Hp} \,.\, \mu T x \,.\supset.\, \mu \,\epsilon\, \mathrm{Prod}\text{‘}\kappa$ (2)

$\vdash .\, *52\cdot 4 \,.\, \mathrm{Transp} \,.\quad \supset \vdash :. \mathrm{Hp} \,.\supset : \beta \,\epsilon\, \kappa \,.\, x \,\epsilon\, \beta \,.\supset.\, (\exists \alpha) \,.\, \alpha \,\epsilon\, \kappa \,.\, \alpha \neq \beta \,.$

[$*21\cdot 33 . *33\cdot 131$] $\supset.\, x \,\epsilon\, \mathrm{C\!\!\!I}\text{‘}T$ (3)

$\vdash .\, *21\cdot 33 \,.\, *33\cdot 131 \,.\supset \vdash :. \mathrm{Hp} \,.\supset : x \,\epsilon\, \mathrm{C\!\!\!I}\text{‘}T \,.\supset.\, (\exists \beta) \,.\, \beta \,\epsilon\, \kappa \,.\, x \,\epsilon\, \beta$ (4)

$\vdash .\, (3) . (4) .\quad \supset \vdash : \mathrm{Hp} \,.\supset.\, \mathrm{C\!\!\!I}\text{‘}T = s\text{‘}\kappa$ (5)

$\vdash .\, *21\cdot 33 \,.\, *13\cdot 172 \,.\supset \vdash :. \mathrm{Hp} \,.\supset : \mu T x \,.\, \nu T x \,.\supset.\, \mu = \nu$ (6)

$\vdash .\, *21\cdot 33 \,.\, *13\cdot 171 \,.\supset \vdash :. \mathrm{Hp} \,.\supset : \mu T x \,.\, \mu T x' \,.\supset.$
$(\exists \alpha, \alpha', \beta, \beta') \,.\, \alpha, \alpha' \,\epsilon\, \kappa \,.\, \beta, \beta' \,\epsilon\, \kappa \,.\, \alpha \neq \beta \,.\, \alpha' \neq \beta' \,.$
$(\rho - \alpha - \beta) \cup (\sigma \cap \alpha) \cup \iota\text{‘}x = (\rho - \alpha' - \beta') \cap (\sigma \cap \alpha') \cup \iota\text{‘}x' \,.$

[$*24\cdot 48 . \mathrm{Hp}$] $\supset.\, \iota\text{‘}x = \iota\text{‘}x'$ (7)

$\vdash .\, (2) . (5) . (6) . (7) .\supset \vdash .\, \mathrm{Prop}$

***117·633.** $\vdash :. \kappa \,\epsilon\, \mathrm{Cls}^2 \,\mathrm{excl} \,.\, \kappa \sim\epsilon\, 0 \cup 1 : (\exists \rho, \sigma) \,.\, \rho, \sigma \,\epsilon\, \mathrm{Prod}\text{‘}\kappa \,.\, \rho \cap \sigma = \Lambda : \supset.$
$\Pi \mathrm{Nc}\text{‘}\kappa \geqslant \Sigma \mathrm{Nc}\text{‘}\kappa$

Dem.

$\vdash .\, *117\cdot 632 \,.\supset \vdash : \mathrm{Hp} \,.\supset.\, (\exists \gamma) \,.\, \gamma \subset \mathrm{Prod}\text{‘}\kappa \,.\, \gamma \,\mathrm{sm}\, s\text{‘}\kappa \,.$

[$*117\cdot 221$] $\supset.\, \mathrm{Nc}\text{‘}\mathrm{Prod}\text{‘}\kappa \geqslant \mathrm{Nc}\text{‘}s\text{‘}\kappa$ (1)

$\vdash .\, (1) .\, *115\cdot 12 \,.\, *112\cdot 15 \,.\supset \vdash .\, \mathrm{Prop}$

***117·64.** $\vdash :. \kappa \epsilon \mathrm{Cls}^2 \mathrm{excl} : (\exists \rho, \sigma) . \rho, \sigma \epsilon \mathrm{Prod}`\kappa . \rho \cap \sigma = \Lambda : \supset .$
$\Pi \mathrm{Nc}`\kappa \geqslant \Sigma \mathrm{Nc}`\kappa$

Dem.

$\vdash . *112·321 . *114·21 . \supset \vdash : \kappa \epsilon 1 . \supset . \Pi\mathrm{Nc}`\kappa = \Sigma \mathrm{Nc}`\kappa$ (1)
$\vdash . *114·2 . *112·3 . \quad \supset \vdash : \kappa \epsilon 0 . \supset . \Pi\mathrm{Nc}`\kappa = 1 . \Sigma\mathrm{Nc}`\kappa = 0 .$
[*117·51] $\quad \supset . \Pi\mathrm{Nc}`\kappa > \Sigma\mathrm{Nc}`\kappa$ (2)
$\vdash . (1) . (2) . *117·633 . \supset \vdash . \mathrm{Prop}$

***117·651.** $\vdash : \alpha \sim \epsilon 0 \cup 1 . \supset . (\mathrm{Nc}`\alpha)^{\mathrm{Nc}`\beta} \geqslant \mathrm{Nc}`\alpha \times_0 \mathrm{Nc}`\beta$

Dem.

$\vdash . *52·4 . \mathrm{Transp} . \quad \supset \vdash : \mathrm{Hp} . \supset . (\exists x, y) . x, y \epsilon \alpha . x \neq y$ (1)
$\vdash . *116·152 . *55·23·202 . \supset \vdash : x, y \epsilon \alpha . x \neq y . \supset . x \downarrow {}``\beta, y \downarrow {}``\beta \epsilon (\alpha \exp \beta) .$
$x \downarrow {}``\beta \cap y \downarrow {}``\beta = \Lambda$ (2)
$\vdash . *113·111 . \quad \supset \vdash . \alpha \downarrow_y {}``\beta \epsilon \mathrm{Cls}^2 \mathrm{excl}$ (3)
$\vdash . (1) . (2) . (3) . *117·64 . *113·1·141·25 . *116·25 . (*116·01) . \supset \vdash . \mathrm{Prop}$

***117·652.** $\vdash : \mu \epsilon \mathrm{N_0C} - \iota`0 - \iota`1 . \nu \epsilon \mathrm{N_0C} . \supset . \mu^\nu \geqslant \mu \times_0 \nu$ [*117·651]

***117·66.** $\vdash . \mathrm{Nc}`\mathrm{Cl}`\alpha > \mathrm{Nc}`\alpha$

Dem.

$\vdash . *102·72 . \quad \supset \vdash . \sim(\exists \beta) . \beta \subset \alpha . \beta \,\mathrm{sm}\, \mathrm{Cl}`\alpha$ (1)
$\vdash . *100·6 . *60·61 . \quad \supset \vdash . \iota``\alpha \subset \mathrm{Cl}`\alpha . \iota``\alpha \,\mathrm{sm}\, \alpha$ (2)
$\vdash . (1) . (2) . *117·13 . \supset \vdash . \mathrm{Prop}$

***117·661.** $\vdash : \mu \epsilon \mathrm{N_0C} . \supset . 2^\mu > \mu$ [*117·66 . *116·72]

The above proposition is important. (See, however, the Introduction to the second edition.)

***117·67.** $\vdash : \kappa \epsilon \mathrm{Cls}^2 \mathrm{excl} . \exists ! \mathrm{Prod}`\kappa . \supset . \mathrm{Nc}`s`\kappa \geqslant \mathrm{Nc}`\kappa$

Dem.

$\vdash . *115·16·11 . \supset \vdash : \kappa \epsilon \mathrm{Cls}^2 \mathrm{excl} . \mu \epsilon \mathrm{Prod}`\kappa . \supset . \mu \,\mathrm{sm}\, \kappa . \mu \subset s`\kappa .$
[*117·22] $\quad \supset . \mathrm{Nc}`s`\kappa \geqslant \mathrm{Nc}`\kappa : \supset \vdash . \mathrm{Prop}$

***117·68.** $\vdash : R, S \epsilon \epsilon_\Delta`\kappa . R \dot\cap S = \dot\Lambda . T = \hat P \hat \rho \{\rho \epsilon \kappa . P = R \upharpoonright - \iota`\rho \cup S \upharpoonright \iota`\rho\}$
$\supset . T \epsilon 1 \to 1 . \mathrm{D}`T \subset \epsilon_\Delta`\kappa . \text{Ɑ}`T = \kappa$

Dem.

$\vdash . *21·33 . *13·172 . \supset \vdash :. \mathrm{Hp} . \supset : PT\rho . QT\rho . \supset . P = Q$ (1)
$\vdash . *23·631 . \quad \supset \vdash : \mathrm{Hp} . \rho \epsilon \kappa . \supset . (T`\rho) \dot\cap S = S \upharpoonright \iota`\rho :$
[*13·17] $\quad \supset \vdash : \mathrm{Hp} . \rho, \sigma \epsilon \kappa . T`\rho = T`\sigma . \supset . S \upharpoonright \iota`\rho = S \upharpoonright \iota`\sigma .$
[*35·65] $\quad \supset . \iota`\rho = \iota`\sigma .$
[*51·23] $\quad \supset . \rho = \sigma$ (2)
$\vdash . (1) . (2) . \quad \supset \vdash : \mathrm{Hp} . \supset . T \epsilon 1 \to 1$ (3)
$\vdash . *21·33 . *33·131 . \supset \vdash : \mathrm{Hp} . \supset . \text{Ɑ}`T = \kappa$ (4)
$\vdash . *80·36 . \quad \supset \vdash : \mathrm{Hp} . \supset . \mathrm{D}`T \subset \epsilon_\Delta`\kappa$ (5)
$\vdash . (3) . (4) . (5) . \quad \supset \vdash . \mathrm{Prop}$

***117·681.** $\vdash : (\exists R, S) . R, S \epsilon \epsilon_\Delta`\kappa . R \dot\cap S = \dot\Lambda . \supset . \mathrm{Nc}`\epsilon_\Delta`\kappa \geqslant \mathrm{Nc}`\kappa$ [*117·68·22]

∗117·682. $\vdash : \kappa \subset \lambda \,.\, \exists ! \, \epsilon_\Delta{}^{\boldsymbol{\prime}}(\lambda - \kappa) \,.\, \supset \,.\, \mathrm{Nc}{}^{\boldsymbol{\prime}}\epsilon_\Delta{}^{\boldsymbol{\prime}}\lambda \geqslant \mathrm{Nc}{}^{\boldsymbol{\prime}}\epsilon_\Delta{}^{\boldsymbol{\prime}}\kappa$

Dem.

$\vdash \,.\, *80{\cdot}65 \,.\, \supset \vdash :. \mathrm{Hp} \,.\, \supset : R \in \epsilon_\Delta{}^{\boldsymbol{\prime}}\kappa \,.\, S \in \epsilon_\Delta{}^{\boldsymbol{\prime}}(\lambda - \kappa) \,.\, \supset \,.\, R \cup S \in \epsilon_\Delta{}^{\boldsymbol{\prime}}\lambda$ (1)

$\vdash \,.\, *80{\cdot}14 \,.\, \supset \vdash : R \in \epsilon_\Delta{}^{\boldsymbol{\prime}}\lambda \,.\, S \in \epsilon_\Delta{}^{\boldsymbol{\prime}}(\lambda - \kappa) \,.\, \supset \,.\, \mathrm{C}{}^{\boldsymbol{\prime}}R \cap \mathrm{C}{}^{\boldsymbol{\prime}}S = \Lambda \,.$

[∗33·33] $\supset \,.\, R \mathbin{\dot\cap} S = \dot\Lambda \,.$

[∗25·4] $\supset \,.\, (R \cup S) \mathbin{\dot-} S = R$ (2)

$\vdash \,.\, (2) \,.\, *13{\cdot}171 \,.\, \supset \vdash : Q, R \in \epsilon_\Delta{}^{\boldsymbol{\prime}}\lambda \,.\, S \in \epsilon_\Delta{}^{\boldsymbol{\prime}}(\lambda - \kappa) \,.\, Q \cup S = R \cup S \,.\, \supset \,.\, Q = R$ (3)

$\vdash \,.\, (1) \,.\, (3) \,.\, \supset \vdash : \mathrm{Hp} \,.\, S \in \epsilon_\Delta{}^{\boldsymbol{\prime}}(\lambda - \kappa) \,.\, \supset \,.\, (\cup S) \upharpoonright \epsilon_\Delta{}^{\boldsymbol{\prime}}\kappa \in 1 \to 1 \,.\, (\cup S){}^{\boldsymbol{\prime\prime}}\epsilon_\Delta{}^{\boldsymbol{\prime}}\kappa \subset \epsilon_\Delta{}^{\boldsymbol{\prime}}\lambda \,.$

[∗117·22] $\supset \,.\, \mathrm{Nc}{}^{\boldsymbol{\prime}}\epsilon_\Delta{}^{\boldsymbol{\prime}}\lambda \geqslant \mathrm{Nc}{}^{\boldsymbol{\prime}}\epsilon_\Delta{}^{\boldsymbol{\prime}}\kappa : \supset \vdash \,.\, \mathrm{Prop}$

∗117·683. $\vdash :. \kappa \subset \lambda \,.\, \exists ! \, \epsilon_\Delta{}^{\boldsymbol{\prime}}(\lambda - \kappa) : (\exists R, S) \,.\, R, S \in \epsilon_\Delta{}^{\boldsymbol{\prime}}\kappa \,.\, R \mathbin{\dot\cap} S = \dot\Lambda : \supset \,.$
$\mathrm{Nc}{}^{\boldsymbol{\prime}}\epsilon_\Delta{}^{\boldsymbol{\prime}}\lambda \geqslant \mathrm{Nc}{}^{\boldsymbol{\prime}}\kappa$ [∗117·681·682]

∗117·684. $\vdash : \kappa \subset \lambda \,.\, \exists ! \, \epsilon_\Delta{}^{\boldsymbol{\prime}}\lambda : (\exists R, S) \,.\, R, S \in \epsilon_\Delta{}^{\boldsymbol{\prime}}\kappa \,.\, R \mathbin{\dot\cap} S = \dot\Lambda : \supset \,.$
$\mathrm{Nc}{}^{\boldsymbol{\prime}}\epsilon_\Delta{}^{\boldsymbol{\prime}}\lambda \geqslant \mathrm{Nc}{}^{\boldsymbol{\prime}}\kappa$ [∗117·683 . ∗88·22]

The above proposition is used in ∗120·765.

GENERAL NOTE ON CARDINAL CORRELATORS

The correlators established at various stages throughout Section B present certain analogies to each other, and they or others closely resembling them will be found to be the correlators required in relation-arithmetic (Part IV). We shall therefore here collect together the most important propositions hitherto proved on correlators.

When we have to deal with correlators of two different functions of a single class, as *e.g.* $\epsilon_\Delta{}^{\backprime}\kappa$ and $\text{Prod}{}^{\backprime}\kappa$, the correlator is usually D or $\dot{s}$ or $\dot{s} \,|\, \text{D}$, with a suitable limitation on the converse domain. Sometimes it is $\breve{\iota} \,|\, \text{D}$ or $\epsilon \,|\, \text{D}$. Thus for example the class $\epsilon \downarrow {}^{\backprime\backprime}\kappa$, by means of which $\Sigma{}^{\backprime}\kappa$ is defined (*112), has double similarity with κ if $\kappa \,\epsilon\, \text{Cls}^2 \,\text{excl}$ (*112·14); in this case, the double correlator is $\breve{\iota} \,|\, \text{D}$ with its converse domain limited, *i.e.*

$$\vdash : \kappa \,\epsilon\, \text{Cls}^2 \,\text{excl} \,.\, \supset .\, \breve{\iota} \,|\, \text{D} \upharpoonright \Sigma{}^{\backprime}\kappa \,\epsilon\, \kappa \,\overline{\text{sm}}\, \overline{\text{sm}}\, (\epsilon \downarrow {}^{\backprime\backprime}\kappa).$$

In the case of $\text{Prod}{}^{\backprime}\kappa$ and $\epsilon_\Delta{}^{\backprime}\kappa$, the correlator is D, *i.e.*

$$\vdash : \kappa \,\epsilon\, \text{Cls}^2 \,\text{excl} \,.\, \supset .\, \text{D} \upharpoonright \epsilon_\Delta{}^{\backprime}\kappa \,\epsilon\, (\text{Prod}{}^{\backprime}\kappa) \,\overline{\text{sm}}\, (\epsilon_\Delta{}^{\backprime}\kappa).$$

In the case of $\epsilon_\Delta{}^{\backprime}s{}^{\backprime}\kappa$ and $\epsilon_\Delta{}^{\backprime}\epsilon_\Delta{}^{\backprime\backprime}\kappa$, the correlator is $\dot{s} \,|\, \text{D}$, *i.e.*

$$\vdash : \kappa \,\epsilon\, \text{Cls}^2 \,\text{excl} \,.\, \supset .\, \dot{s} \,|\, \text{D} \upharpoonright \epsilon_\Delta{}^{\backprime}\epsilon_\Delta{}^{\backprime\backprime}\kappa \,\epsilon\, (\epsilon_\Delta{}^{\backprime}s{}^{\backprime}\kappa) \,\overline{\text{sm}}\, (\epsilon_\Delta{}^{\backprime}\epsilon_\Delta{}^{\backprime\backprime}\kappa).$$

$\dot{s} \,|\, \text{D}$ also correlates $\epsilon_\Delta{}^{\backprime}\kappa$ with $\epsilon_\Delta{}^{\backprime}\epsilon \downarrow {}^{\backprime\backprime}\kappa$ (*85·61) and $P_\Delta{}^{\backprime}\alpha$ with $\epsilon_\Delta{}^{\backprime}P \downarrow {}^{\backprime\backprime}\alpha$ (*85·53), and $P_\Delta{}^{\backprime}s{}^{\backprime}\kappa$ with $\epsilon_\Delta{}^{\backprime}P_\Delta{}^{\backprime\backprime}\kappa$ (*85·27·42) if $\kappa \,\epsilon\, \text{Cls}^2 \,\text{excl}$.

The correlator of $(\alpha \uparrow \beta)_\Delta{}^{\backprime}\beta$ with $(\alpha \exp \beta)$ is $\dot{s}$ (*116·131).

Another kind of correlator arises where we are given a correlator of κ and λ, and we wish to construct a correlator for some associated classes $W{}^{\backprime}\kappa$ and $W{}^{\backprime}\lambda$, or where we are given correlators of α with γ and of β with δ, and we wish to construct a correlator of $\alpha \,♀\, \beta$ with $\gamma \,♀\, \delta$, where ♀ is some double descriptive function in the sense of *38. In this case, the correlator will usually be of the form $R \,\|\, \breve{S}$ (with a limited converse domain). Sometimes R and S will be identical; sometimes S will be R_ϵ. Such correlators always depend upon

***55·61.** $\vdash : \text{E} \,!\, R{}^{\backprime}x \,.\, \text{E} \,!\, S{}^{\backprime}y \,.\, \supset .\, (R \,\|\, \breve{S}){}^{\backprime}(x \downarrow y) = (R{}^{\backprime}x) \downarrow (S{}^{\backprime}y)$

together with the propositions *74·77 *seq.* giving cases in which $(R \,\|\, \breve{S}) \upharpoonright \lambda$ is a one-one relation. It follows from *55·61 that if R and S are correlators whose converse domains include the domain and converse domain respectively of a relation P, then $(R \,\|\, \breve{S}){}^{\backprime}P$ will be a relation holding between $R{}^{\backprime}x$ and $S{}^{\backprime}y$ whenever P holds between x and y. Examples of such correlators as $R \,\|\, \breve{S}$ are

***112·153.** $\vdash : T \,\epsilon\, \kappa \,\overline{\text{sm}}\,\overline{\text{sm}}\, \lambda \,.\, \supset .\, (T \,\|\, \breve{T}_\epsilon) \upharpoonright s{}^{\backprime}\epsilon \downarrow {}^{\backprime\backprime}\lambda \,\epsilon\, (\epsilon \downarrow {}^{\backprime\backprime}\kappa) \,\overline{\text{sm}}\, \overline{\text{sm}}\, (\epsilon \downarrow {}^{\backprime\backprime}\lambda)$

$*113{\cdot}127.\ \vdash : R \restriction \gamma \in \alpha\, \overline{\text{sm}}\, \gamma \,.\, S \restriction \delta \in \beta\, \overline{\text{sm}}\, \delta \,.\, \supset .$

$$(R \parallel \breve{S}) \restriction (\delta \times \gamma) \in (\alpha \underset{;}{\downarrow} {}``\beta)\, \overline{\text{sm}}\, \overline{\text{sm}}\, (\gamma \underset{;}{\downarrow} {}``\delta)$$

$*113{\cdot}65.\ \vdash .\ \downarrow z``\alpha \times \downarrow z``\beta = (\downarrow z \parallel \text{Cnv}`\downarrow z)``(\alpha \times \beta)$

$*114{\cdot}51.\ \vdash : T \restriction s`\lambda \in \kappa\, \overline{\text{sm}}\, \overline{\text{sm}}\, \lambda \,.\, \supset .\, (T \parallel \breve{T}_\epsilon) \restriction \epsilon_\Delta`\lambda \in (\epsilon_\Delta`\kappa)\, \overline{\text{sm}}\, (\epsilon_\Delta`\lambda)$

$*116{\cdot}192.\ \vdash : R \restriction \gamma \in \alpha\, \overline{\text{sm}}\, \gamma \,.\, S \restriction \delta \in \beta\, \overline{\text{sm}}\, \delta \,.\, \supset .$

$$(R \parallel \breve{S}) \restriction (\delta \times \gamma) \in (\alpha \exp \beta)\, \overline{\text{sm}}\, \overline{\text{sm}}\, (\gamma \exp \delta) \,.$$

$$(R \parallel \breve{S})_\epsilon \restriction (\gamma \exp \delta) \in (\alpha \exp \beta)\, \overline{\text{sm}}\, (\gamma \exp \delta)$$

An exceptionally simple correlator is given by

$*115{\cdot}502.\ \vdash : T \restriction s`\lambda \in \kappa\, \overline{\text{sm}}\, \overline{\text{sm}}\, \lambda \,.\, \supset .\, T \restriction s`\text{Prod}`\lambda \in (\text{Prod}`\kappa)\, \overline{\text{sm}}\, \overline{\text{sm}}\, (\text{Prod}`\lambda)$

Another exceptionally simple case is

$*73{\cdot}63.\ \vdash : S \in \alpha\, \overline{\text{sm}}\, \beta \,.\, T \restriction \alpha, T \restriction \beta \in 1 \to 1 \,.\, \alpha \cup \beta \subset \text{Ɑ}`T \,.\, \supset .$

$$T \mid S \mid \breve{T} \in (T``\alpha)\, \overline{\text{sm}}\, (T``\beta)$$

By means of the above correlators, most correlators that are required can be calculated. Thus it will be seen that $*116{\cdot}192$ in the above list is an immediate consequence of $*113{\cdot}127$ and $*115{\cdot}502$, since

$$\alpha \exp \beta = \text{Prod}`\alpha \underset{;}{\downarrow} {}``\beta \text{ and } s`\text{Prod}`\gamma \underset{;}{\downarrow} {}``\delta = \delta \times \gamma.$$

In order to develop the subject, it is almost always necessary, not merely to prove that two classes are similar, but actually to construct a correlator of the two classes. This applies equally to relation-arithmetic, in which analogous correlators are used to prove ordinal similarity.

SECTION C

FINITE AND INFINITE

Summary of Section C.

The distinction of finite and infinite is not required, as appears from Section B, for the definition of the arithmetical operations or for the proof of their formal laws. There are, however, many important respects in which finite cardinals and classes differ respectively from infinite cardinals and classes, and these differences must now be investigated.

There are two different ways in which we may define the finite and the infinite, and these two ways cannot (so far as is known at present) be shown to be equivalent except by assuming the multiplicative axiom. As there seems no good reason for regarding one of these ways as giving more exactly than the other what is usually meant by the words "finite" and "infinite," we shall, to avoid confusion, give other names than these to each of the two ways of dividing classes and cardinals. The division effected by the first method of definition we shall call the division into *inductive* and *non-inductive*; that effected by the second method we shall call the division into *non-reflexive* and *reflexive*.

The division into inductive and non-inductive, which is treated in $*120$, is defined as follows. An inductive cardinal is one which can be reached from 0 by successive additions of 1; that is, an inductive cardinal is one which has to 0 the relation $(+_0 1)_*$, where (by $*38{\cdot}02$) $+_0 1$ is the relation of $\alpha +_0 1$ to α, and the subscript asterisk has the meaning defined in $*90$. Hence we put

$$\mathrm{NC\,induct} = \hat{\alpha}\,\{\alpha\,(+_0 1)_*\,\dot{0}\} \quad \mathrm{Df.}$$

By applying the definition of $*90$, this gives

$$\vdash :: \alpha \,\epsilon\, \mathrm{NC\,induct} \,.\equiv :.\, \xi \,\epsilon\, \mu \,.\supset_\xi .\, \xi +_0 1 \,\epsilon\, \mu : 0 \,\epsilon\, \mu : \supset_\mu .\, \alpha \,\epsilon\, \mu.$$

This proposition may be regarded as stating that an inductive cardinal is one which obeys mathematical induction starting from 0, *i.e.* it is one which possesses every property possessed by 0 and by the numbers obtained by adding 1 to numbers possessing the property. In elementary mathematics, it is customary to regard mathematical induction, as applied to the series of natural numbers, as a principle rather than a definition, but according to

the above procedure it becomes a definition rather than a principle. This procedure is unavoidable as soon as it is perceived that there are cardinals which do not obey mathematical induction starting from 0. (This only holds on the assumption that the total number of objects in any one type is not one of the inductive cardinals. This assumption, in a slightly different form, is introduced below as the "axiom of infinity.") Thus for example $0 \neq 1$, and $\xi \neq \xi +_c 1 . \supset . \xi +_c 1 \neq \xi +_c 2$. Hence if α is any inductive cardinal, $\alpha \neq \alpha +_c 1$. But we know that $\aleph_0$, the first of Cantor's transfinite cardinals*, satisfies $\aleph_0 = \aleph_0 +_c 1$. Thus mathematical induction starting from 0 cannot be validly applied to prove properties of $\aleph_0$. It follows that the inductive cardinals as above defined are only some among cardinals; nor does it appear that there is any way of defining them except as those that obey mathematical induction starting from 0. It follows that mathematical induction is not a principle, to be either proved or assumed as an axiom, but is merely a characteristic defining a certain class of cardinals, namely the class of inductive cardinals.

By a syllogism in Barbara, it is evident that 0 is an inductive cardinal; hence by the definition 1 is an inductive cardinal, and hence 2, 3, ... are inductive cardinals. Thus any given cardinal in the series of natural numbers can be shown to be an inductive cardinal. The usual elementary properties of inductive cardinals, such as the uniqueness of subtraction and division, are easily proved by mathematical induction.

We define an inductive class as a class the number of whose terms is an inductive cardinal. More simply, we put

$$\text{Cls induct} = s\text{‘NC induct} \quad \text{Df.}$$

It is then easily shown that an inductive class is one which can be reached from Λ by successive additions of single members. That is, if we put

$$M = \hat{\eta}\hat{\zeta}\{(\exists y) . \zeta = \eta \cup \iota\text{‘}y\},$$

then $$\text{Cls induct} = \overleftarrow{M_*}\text{‘}\Lambda.$$

Thus we have

$$\vdash :: \rho \in \text{Cls induct} . \equiv :. \eta \in \mu . \supset_{\eta, y} . \eta \cup \iota\text{‘}y \in \mu : \Lambda \in \mu : \supset_\mu . \rho \in \mu.$$

We might equally well have begun by defining inductive classes, and proceeded to define inductive cardinals as the cardinals of inductive classes; in that case, we should have used the above relation M to define inductive classes.

Some of the properties which we expect inductive cardinals to possess, such for example as $\alpha \neq \alpha +_c 1$, can only be proved by assuming that no inductive cardinal is null, *i.e.* that

$$\alpha \in \text{NC induct} . \supset_\alpha . \exists ! \alpha.$$

This amounts to the assumption that, in any fixed type, a class can be found

* For the definition of $\aleph_0$, cf. $*123 \cdot 01$ and p. 186 of this summary.

having any assigned inductive number of terms. If this were false, there would have to be some definite member of the series of natural numbers which gave the total number of objects of the type in question. Thus suppose there were exactly n individuals in the universe, and no more, where n is an inductive cardinal. We should then have 2^n classes, 2^{2^n} classes of classes, and so on. In that case, in the type of individuals we should have $n +_c 1 = \Lambda$, $n +_c 2 = \Lambda$, etc. Hence we should have

$$n +_c 1 = (n +_c 1) +_c 1, \text{ etc.}$$

In the type of classes, we should get similar results for 2^n, and so on. It is plain (though not demonstrable except in each particular case) that if the assumption $\alpha \epsilon \mathrm{NC\ induct} . \supset_\alpha . \exists ! \alpha$ fails in any one type, it fails in any other type in the same hierarchy, and if it holds in any one, it holds in any other; for if n be the total number of individuals, then if n is an inductive cardinal, the total number of any other type is an inductive cardinal, while if n is not an inductive cardinal, no more is the total number of any other type. Hence the assumption $\alpha \epsilon \mathrm{NC\ induct} . \supset_\alpha . \exists ! \alpha$ is either true in any type or false in any type in one hierarchy. We shall call it the "axiom of infinity," putting

$$\mathrm{Infin\ ax} . = : \alpha \epsilon \mathrm{NC\ induct} . \supset_\alpha . \exists ! \alpha \quad \mathrm{Df.}$$

This assumption, like the multiplicative axiom, will be adduced as a hypothesis whenever it is relevant. It seems plain that there is nothing in logic to necessitate its truth or falsehood, and that it can only be legitimately believed or disbelieved on empirical grounds. When we wish to use a typically definite form of the axiom, we shall employ the definition

$$\mathrm{Infin\ ax}\,(x) . = : \alpha \epsilon \mathrm{NC\ induct} . \supset_\alpha . \exists ! \alpha\,(x) \quad \mathrm{Df},$$

which asserts that, if α is any inductive cardinal, there are at least α terms of the same type as x.

It is important to observe that, although the axiom of infinity cannot (so far as appears) be proved a priori, we can prove that any given inductive cardinal exists in a sufficiently high type. For if the total number of individuals be n, the numbers of objects in succeeding types are 2^n, 2^{2^n}, etc., and these numbers grow beyond any assigned inductive cardinal. Owing, however, to the fact that we cannot add together an infinite number of classes whose types increase without limit, we cannot hence show that there is a type in which every inductive cardinal exists, though we can show of every inductive cardinal that there is a type in which it exists. *I.e.* if α is any inductive cardinal, there must be a type for x such that $\exists ! \alpha\,(x)$ is true; but there need not be a type for x such that if α is any inductive cardinal, $\exists ! \alpha\,(x)$ is true.

The axiom of infinity suffices to prove the existence, in appropriate types, of $\aleph_0$, $2^{\aleph_0}$, $2^{2^{\aleph_0}}$, ... $\aleph_1$, $\aleph_2$, ...*. It does not suffice, so far as we know, to prove

* For the definitions of $\aleph_1$, $\aleph_2$, etc., see *265.

the existence of $\aleph_\omega$ or any Aleph with a greater suffix than ω, because the existences of $\aleph_1$, $\aleph_2$, ... are proved in successively rising types, and no meaning can be found for a type whose order is infinite.

The other definition of finite and infinite is of less importance in practice than the definition by induction. It is dealt with in *124. According to this definition, we call a class *reflexive* when it contains a proper part similar to itself, *i.e.* we put

$$\text{Cls refl} = \hat{\alpha}\{(\exists R) . R \epsilon 1 \to 1 . \text{D}`R = \alpha . \text{G}`R \subset \alpha . \text{G}`R \neq \alpha\} \quad \text{Df},$$

or, what comes to the same thing,

$$\text{Cls refl} = \hat{\alpha}\{(\exists R) . R \epsilon 1 \to 1 . \text{G}`R \subset \text{D}`R . \exists ! \overrightarrow{B}`R . \alpha = \text{D}`R\} \quad \text{Df}.$$

We call a cardinal reflexive when it is the homogeneous cardinal of a reflexive class, *i.e.* we put

$$\text{NC refl} = \text{N}_0\text{c}``\text{Cls refl} \quad \text{Df}.$$

It is easy to show that

$$\text{NC refl} = \hat{\alpha}\{\exists ! \alpha . \alpha = \alpha +_{\text{c}} 1\}.$$

We find that inductive classes and cardinals are non-reflexive, and reflexive classes and cardinals are non-inductive. We find also that reflexive cardinals are those that are equal to or greater than $\aleph_0$, while inductive cardinals are those that are less than $\aleph_0$. By assuming the multiplicative axiom, we can show that every cardinal is equal to, greater than, or less than $\aleph_0$, whence it follows that every cardinal is either reflexive or inductive, thus identifying the two definitions of finite and infinite. But so long as we refrain from assuming either the multiplicative axiom or some special axiom *ad hoc*, it remains possible (so far as is known at present) that there may be cardinals neither greater than, nor equal to, nor less than $\aleph_0$. Such cardinals, if they exist, are neither inductive nor reflexive: they are infinite if we define infinity by the negation of induction, but finite if we define infinity by reflexiveness. It is possible that further investigation may either prove or disprove the existence of such cardinals; for the present, their existence must remain an open question, except for those who regard the multiplicative axiom as a self-evident truth.

In *121 we shall consider *intervals* in a discrete series; *i.e.* in a series generated by a one-one relation between consecutive terms. If P be the generating relation of such a series, and x and y be two members of the series, of which y is the later, the terms which lie between x and y are the terms z for which we have

$$xP_{\text{po}}z . zP_{\text{po}}y,$$

where P_{po} has the meaning defined in *91. Hence we put

$$P(x - y) = \overleftarrow{P_{\text{po}}}`x \cap \overrightarrow{P_{\text{po}}}`y \quad \text{Df},$$

where "$P(x-y)$" means "the P-interval between x and y." We want also

symbols for the interval together with one or both of its end-points. For these we put

$$P(x \dashv y) = \overleftarrow{P_{\text{po}}}`x \cap \overrightarrow{P_*}`y \quad \text{Df},$$
$$P(x \vdash y) = \overleftarrow{P_*}`x \cap \overrightarrow{P_{\text{po}}}`y \quad \text{Df},$$
$$P(x \mapsto y) = \overleftarrow{P_*}`x \cap \overrightarrow{P_*}`y \quad \text{Df*}.$$

Thus, for example, if x and y be inductive cardinals, and P be the relation of n to $n +_c 1$, and $x < y$, $P(x - y)$ will be the numbers greater than x and less than y, while $P(x \dashv y)$ will be these numbers together with y, $P(x \vdash y)$ will be these numbers together with x, and $P(x \mapsto y)$ will be these numbers together with both x and y. By means of intervals, we define a class of relations P_ν (where ν is any inductive cardinal), where "$xP_\nu z$" means that we can pass from x to z in ν steps. In order to fit the case in which x and z are identical, and to insure that no relation such as P_ν shall hold between terms which do not both belong to the field of P, we put

$$P_\nu = \hat{x}\hat{y}\{\text{Nc}`P(x \mapsto y) = \nu +_c 1\} \quad \text{Df.}$$

Then, provided $P_{\text{po}} \subset J$, $P_0 = I \upharpoonright C`P$, and if further $P \in 1 \rightarrow 1$, then $P_1 = P$, $P_2 = P^2$, etc. If P is a transitive serial relation, P_1 is the relation "immediately preceding," which has great importance in well-ordered series. In this case, $P_1 = P \mathbin{\dot{-}} P^2$. If P is a transitive serial relation generating a finite series or a progression or a series of the type of the negative and positive integers in order of magnitude, we have

$$P = (P_1)_{\text{po}}.$$

In $*121$ we shall only consider P_ν in the case where

$$P \in (1 \rightarrow \text{Cls}) \cup (\text{Cls} \rightarrow 1),$$

and generally we shall have the further hypothesis $P_{\text{po}} \subset J$. We can then prove that the interval between x and y is always an inductive class (it will be null unless xP_*y); this proposition is useful in its application to the number-series and to progressions generally.

When $P \in (1 \rightarrow \text{Cls}) \cup (\text{Cls} \rightarrow 1) \,.\, P_{\text{po}} \subset J$, the class of such relations as P_ν (where ν is an inductive cardinal) is identical with Potid$`P$, the class of powers of P (cf. $*91$ *seq.*). This identification (which does not hold in general without the above hypothesis) leads to many useful propositions. In $*91$ *seq.*, we treated powers of a relation without the use of numbers, *i.e.* without defining the νth power of P. When the powers of P are the class of such relations as P_ν, we can of course take P_ν as the νth power of P. The general definition of the νth power of P (where ν is an inductive cardinal) will be given later, in $*301$; we shall denote it by P^ν, thereby including the notation P^2 already defined.

* These symbols are suggested by those given in Peano's *Formulaire*, Vol. IV. p. 116. (*Algèbre*, § 46.)

In $*122$ we shall deal with progressions, *i.e.* with series of the type of the series of natural numbers. In this number, we shall deal with such series as generated by one-one relations; they will be dealt with at a later stage ($*263$) as generated by transitive relations. We define a progression as a one-one relation whose domain is the posterity of its first term, *i.e.*

$$\text{Prog} = (1 \to 1) \cap \hat{R}(\text{D}`R = \overleftarrow{R_*}`B`R) \quad \text{Df.}$$

According to this definition, there must be a first term $B`R$; $\text{Ꭵ}`R$ will be $\breve{R}``\overleftarrow{R_*}`B`R$, *i.e.* $\overleftarrow{R_{\text{po}}}`B`R$, which is contained in $\overleftarrow{R_*}`B`R$, *i.e.* in $\text{D}`R$; since $\text{Ꭵ}`R \subset \text{D}`R$, every term of the field of $R$ has a successor, so that there is no end to the series; since $C`R = \text{D}`R = \overleftarrow{R_*}`B`R$, every term of the series can be reached from the beginning by successive steps. These characteristics suffice to define progressions.

In $*123$ we proceed to the definition and discussion of $\aleph_0$, the smallest of reflexive cardinals. This is the cardinal number of any class whose terms can be arranged in a progression; hence it is the class of domains of progressions, *i.e.* we may put

$$\aleph_0 = \text{D}``\text{Prog} \quad \text{Df.}$$

With this definition, remembering that Λ is a cardinal, we can prove that $\aleph_0$ is a cardinal; but to prove that $\aleph_0$ is an *existent* cardinal, we need the axiom of infinity. The existence-theorem for $\aleph_0$ is then derived from the inductive cardinals, which, if no one of them is null, form a progression when arranged in order of magnitude. It should be observed that this existence-theorem is for a higher type than that for which the axiom of infinity is assumed. In order to get an existence-theorem for the same type, we need the multiplicative axiom as well.

After a number on reflexive classes and cardinals ($*124$) and a number on the axiom of infinity ($*125$), the Section ends with a number ($*126$) on "typically indefinite inductive cardinals." The constant inductive cardinals are the typically ambiguous symbols 0, 1, 2, ...; thus we want to define the class of inductive cardinals in such a way that a variable member of the class shall be typically ambiguous. This is not possible without a sacrifice of rigour, but in $*126$ it is shown how to minimize the sacrifice of rigour, and how to obviate the resulting logical dangers. A variable whose values are typically ambiguous is said to be "typically indefinite."

A proof that all inductive cardinals exist has often been derived from $*120{\cdot}57$ (below). But according to the doctrine of types, this proof is invalid, since "$\mu +_c 1$" in $*120{\cdot}57$ is necessarily of higher type than "μ."

$*118$. ARITHMETICAL SUBSTITUTION AND UNIFORM FORMAL NUMBERS

Summary of $*118$.

A difficulty arises respecting substitution in arithmetic. For if μ is a formal number and its occurrence in $f\mu$ is arithmetical, then by IIT μ is always to be taken in an existential type. Hence we can only substitute a real variable ξ for μ under the hypothesis $\exists ! \xi$, and we can only substitute another formal number σ for μ provided that the equation $\mu = \sigma$, which justifies the substitution, is arithmetical, *i.e.* provided that in this equation the type of μ is such that $\exists ! \mu$.

The result is that the application of $*20 \cdot 18$ is apt to lead to fallacies owing to the different meanings which a formal number may possess in different occurrences. Hitherto we have considered each case in detail, *e.g.* note on $*110 \cdot 61$, and proof of $*110 \cdot 56$.

The condition for the safe application of $*20 \cdot 18$ is given in $*118 \cdot 01$, namely

$*118 \cdot 01$. $\vdash :. \exists ! \mu . \mu = \sigma . \supset : f\mu . \equiv . f\sigma$ $[*20 \cdot 18]$

This question is more fully discussed in the prefatory statement of this volume. The first reference to $*118 \cdot 01$ is in $*120 \cdot 222$. Another way of evading the difficulty is to work with formal numbers which, together with all their components, are of the same type. This leads to the consideration of Uniform Formal Numbers, which with the exception of $*118 \cdot 01$ occupies the rest of the number.

The *dominant type* of a formal number as used in any context is the type of the formal number itself in that context, and the *subordinate types* of the formal number are the dominant types of its component formal numbers.

When the dominant types of some of the formal numbers are not expressly indicated by an explicit notation (cf. $*65$), the rules according to which the dominant types thus left ambiguous are to be related, so far as they are related, including the rules governing the relation of subordinate types, if left ambiguous, to dominant types, are given by conventions IT, IIT, and AT of the prefatory statement in this volume.

We have now to consider an important special case which arises when types are explicitly indicated by the use of $*65 \cdot 01 \cdot 03$. A formal number, whose subordinate types are the same as its dominant type, is called *uniform*; and if some of its subordinate types are the same as its dominant type, it is called *partially uniform*. A formal number can only be partially uniform, or at least so designated as to be necessarily partially uniform, when the dominant type and those subordinate types identical with it are expressly indicated by

$*65 \cdot 01 \cdot 03$. For otherwise the conventions IT, IIT, and perhaps also AT, apply; and these do not secure uniformity, and may perhaps in some contexts be inconsistent with it.

Common sense in its consideration of arithmetic habitually disregards the possibility of a formal number representing Λ. In other words, it always applies conventions IIT and AT. But also, owing to its disregard of types, it assumes that the formal numbers are all uniform. The assumption which is really essential to this common sense reasoning, so far as the form of its arithmetical conclusions are concerned, is the assumption that none of the numerical symbols represent Λ. This assumption is secured here, when no types are expressly indicated, by IIT and AT. We have now to consider the effect on arithmetical operations of the other assumption, that the formal numbers are uniform, or partially uniform. There is no difficulty arising from any change of convention for symbolism, since, as stated above, partial or complete uniformity is secured by express indication of type. Accordingly conventions IT, IIT continue, as always, to apply when the types of formal numbers are left ambiguous.

Convention AT will not be applied either in $*118$ or $*119$ or $*120$: in $*118$ the fact is entirely unimportant since the dominant types of equational occurrences are always indicated, so that no case arises when it could apply.

Apart from its intrinsic interest and its bearing on substitution, the arithmetic of uniform formal numbers is necessary for $*120$, where the fundamental arithmetical properties of inductive numbers are investigated.

The propositions of this number are proved by the use of the results of $*117$. The basis of the reasoning is

$*118 \cdot 13$. $\vdash :. \mu \leqslant \nu . \supset : \exists ! \, \mathrm{sm}_\xi `` \nu . \supset . \exists ! \, \mathrm{sm}_\xi `` \mu$

In $*118 \cdot 2 \cdot 3 \cdot 4$ the meaning of the symbolism for dominant types is stated, namely

$*118 \cdot 2$. $\vdash . (\mu +_c \nu)_\xi = \hat{\eta} \{(\exists \alpha, \beta) . \mu = \mathrm{N_0c}`\alpha . \nu = \mathrm{N_0c}`\beta . \eta \, \mathrm{sm}_\xi \, (\alpha + \beta)\}$

$*118 \cdot 3$. $\vdash . (\mu \times_c \nu)_\xi = \hat{\eta} \{(\exists \alpha, \beta) . \mu = \mathrm{N_0c}`\alpha . \nu = \mathrm{N_0c}`\beta . \eta \, \mathrm{sm}_\xi \, (\alpha \times \beta)\}$

$*118 \cdot 4$. $\vdash . (\mu^\nu)_\xi = \hat{\eta} \{(\exists \alpha, \beta) . \mu = \mathrm{N_0c}`\alpha . \nu = \mathrm{N_0c}`\beta . \eta \, \mathrm{sm}_\xi \, (\alpha \exp \beta)\}$

The important propositions which are finally reached for addition are

$*118 \cdot 23$. $\vdash : \mu, \nu \in \mathrm{NC} . \supset . (\mu +_c \nu)_\xi = (\mathrm{sm}_\xi `` \mu +_c \mathrm{sm}_\xi `` \nu)_\xi$

$*118 \cdot 24$. $\vdash : \nu \in \mathrm{NC} . \supset . (\mu +_c \nu)_\xi = (\mu +_c \mathrm{sm}_\xi `` \nu)_\xi$

$*118 \cdot 241$. $\vdash : \mu \in \mathrm{NC} . \supset . (\mu +_c \nu)_\xi = (\mathrm{sm}_\xi `` \mu +_c \nu)_\xi$

$*118 \cdot 25$. $\vdash . (\mu +_c \nu +_c \varpi)_\xi = \{(\mu +_c \nu)_\xi +_c \varpi\}_\xi = \{\mu +_c (\nu +_c \varpi)_\xi\}_\xi$

The important propositions for multiplication are

$*118 \cdot 33$. $\vdash : \mu, \nu \in \mathrm{NC} - \iota`0 . \supset . (\mu \times_c \nu)_\xi = (\mathrm{sm}_\xi `` \mu \times_c \mathrm{sm}_\xi `` \nu)_\xi$

$*118 \cdot 34$. $\vdash : \nu \in \mathrm{NC} . \mu \neq 0 . \supset . (\mu \times_c \nu)_\xi = (\mu \times_c \mathrm{sm}_\xi `` \nu)_\xi$

***118·341.** $\vdash : \mu \in \mathrm{NC} \,.\, \nu \neq 0 \,.\, \supset \,.\, (\mu \times_c \nu)_\xi = (\mathrm{sm}_\xi``\mu \times_c \nu)_\xi$

***118·35.** $\vdash : \varpi \neq 0 \,.\, \supset \,.\, (\mu \times_c \nu \times_c \varpi)_\xi = \{(\mu \times_c \nu)_\xi \times_c \varpi\}_\xi$

***118·351.** $\vdash : \mu \neq 0 \,.\, \supset \,.\, (\mu \times_c \nu \times_c \varpi)_\xi = \{\mu \times_c (\nu \times_c \varpi)_\xi\}_\xi$

The important propositions for exponentiation are

***118·43.** $\vdash : \mu, \nu \in \mathrm{NC} - \iota`0 \,.\, \mu \neq 1 \,.\, \supset \,.\, (\mu^\nu)_\xi = \{(\mathrm{sm}_\xi``\mu)^{\mathrm{sm}_\xi``\nu}\}_\xi$

***118·44.** $\vdash : \nu \in \mathrm{NC} \,.\, \mu \neq 0 \,.\, \mu \neq 1 \,.\, \supset \,.\, (\mu^\nu)_\xi = (\mu^{\mathrm{sm}_\xi``\nu})_\xi$

***118·441.** $\vdash : \mu \in \mathrm{NC} \,.\, \nu \neq 0 \,.\, \supset \,.\, (\mu^\nu)_\xi = \{(\mathrm{sm}_\xi``\mu)^\nu\}_\xi$

***118·45.** $\vdash : \mu \neq 0 \,.\, \mu \neq 1 \,.\, \supset \,.\, (\mu^{\nu \times_c \varpi})_\xi = \{\mu^{(\nu \times_c \varpi)_\xi}\}_\xi$

***118·451.** $\vdash : \varpi \neq 0 \,.\, \supset \,.\, (\mu^{\nu \times_c \varpi})_\xi = [\{(\mu^\nu)_\xi\}^\varpi]_\xi$

***118·46.** $\vdash : \mu \neq 0 \,.\, \mu \neq 1 \,.\, \supset \,.\, (\mu^{\nu +_c \varpi})_\xi = \{\mu^{(\nu +_c \varpi)_\xi}\}_\xi$

***118·461.** $\vdash \,.\, (\mu^{\nu +_c \varpi})_\xi = \{(\mu^\nu)_\xi \times_c (\mu^\varpi)_\xi\}_\xi$

with two analogous propositions *118·462·463,

***118·47.** $\vdash : \varpi \neq 0 \,.\, \supset \,.\, \{(\mu \times_c \nu)^\varpi\}_\xi = [\{(\mu \times_c \nu)_\xi\}^\varpi]_\xi$

***118·471.** $\vdash :. \mu \neq 0 \,.\, \nu \neq 0 \,.\, \mathrm{v} \,.\, \varpi = 0 \,.\, \mathrm{v} \,.\, \sim(\mu, \nu, \varpi \in \mathrm{N_0C}) : \supset \,.$

$$\{(\mu \times_c \nu)^\varpi\}_\xi = \{(\mu^\varpi)_\xi \times_c (\nu^\varpi)_\xi\}_\xi$$

with two analogous propositions *118·472·473.

It is thus seen that, apart from some exceptional cases connected with 0 and 1, in all arithmetical operations uniform, or partially uniform, formal numbers can replace those constructed in obedience to convention IIT.

***118·01.** $\vdash :. \exists ! \mu \,.\, \mu = \sigma \,.\, \supset : f\mu \,.\, \equiv \,.\, f\sigma$ [*20·18]

As far as the symbolism is concerned, this proposition with the omission of $\exists ! \mu$ from the hypothesis is a transcript of *20·18. But if μ or σ (not excluding both) is a formal number, $\exists ! \mu$ is required in case the occurrence of μ in $f\mu$ is arithmetical. In fact this proposition embodies the three fundamental propositions of the Principle of Arithmetical Substitution arrived at in the Prefatory Explanations on Types. Its necessity arises from the convention IIT which is explained there.

***118·11.** $\vdash : \exists ! \mathrm{Nc}(\xi)`\beta \,.\, \alpha \subset \beta \,.\, \supset \,.\, \exists ! \mathrm{Nc}(\xi)`\alpha$

Dem.

$\vdash \,.\, *100·31 \,.\, \supset \vdash :. \mathrm{Hp} \,.\, \supset :$

$\gamma \in \mathrm{Nc}(\xi)`\beta \,.\, \supset \,.\, \gamma \,\mathrm{sm}_\xi\, \beta \,.$

[*73·1] $\supset \,.\, (\exists R) \,.\, R \in 1(\xi) \to 1 \,.\, \gamma = \mathrm{D}`R \,.\, \beta = \mathrm{Cl}`R \,.$

[*22·55] $\supset \,.\, (\exists R) \,.\, R \in 1(\xi) \to 1 \,.\, \alpha \subset \mathrm{Cl}`R \,.\, R``\alpha = R``\alpha \,.$

[*73·12] $\supset \,.\, (\exists R) \,.\, R``\alpha \,\mathrm{sm}_\xi\, \alpha \,.$

[*100·31] $\supset \,.\, \exists ! \mathrm{Nc}(\xi)`\alpha : \supset \vdash \,.\, \mathrm{Prop}$

***118·12.** $\vdash :. \mathrm{Nc}`\alpha \leqslant \mathrm{Nc}`\beta . \supset : \exists ! \mathrm{Nc}(\xi)`\beta . \supset . \exists ! \mathrm{Nc}(\xi)`\alpha$ [*117·32·107 . *100·511]

***118·13.** $\vdash :. \mu \leqslant \nu . \supset : \exists ! \mathrm{sm}_\xi``\nu . \supset . \exists ! \mathrm{sm}_\xi``\mu$ [*117·32]

***118·2.** $\vdash . (\mu +_c \nu)_\xi = \hat{\eta} \{(\exists \alpha, \beta) . \mu = \mathrm{N_0c}`\alpha . \nu = \mathrm{N_0c}`\beta . \eta \, \mathrm{sm}_\xi (\alpha + \beta)\}$ [(*65·01·03) . *110·2]

***118·201.** $\vdash : \exists ! (\mu +_c \nu) . \supset . \mathrm{sm}_\xi``(\mu +_c \nu) = (\mu +_c \nu)_\xi$ [*110·44. Note change in enunciation]

***118·21.** $\vdash : \exists ! (\mu +_c \nu)_\xi . \supset . \exists ! \mathrm{sm}_\xi``\mu . \exists ! \mathrm{sm}_\xi``\nu$

Dem.

$\vdash . *110·4 . *118·2 . \supset \vdash : \mathrm{Hp} . \supset . \mu, \nu \in \mathrm{N_0C} .$

[*117·6] $\supset . \mu +_c \nu \geqslant \mu . \mu +_c \nu \geqslant \nu .$

[*118·13·201.(IIT)] $\supset . \exists ! \mathrm{sm}_\xi``\mu . \exists ! \mathrm{sm}_\xi``\nu : \supset \vdash . \mathrm{Prop}$

Here the reference (IIT) is to the convention IIT explained in the prefatory statement.

***118·22.** $\vdash :. \mu, \nu \in \mathrm{NC} . \supset : \exists ! (\mu +_c \nu)_\xi . \equiv . \exists ! (\mathrm{sm}_\xi``\mu +_c \mathrm{sm}_\xi``\nu)_\xi . \equiv .$
$\exists ! (\mu +_c \mathrm{sm}_\xi``\nu)_\xi . \equiv . \exists ! (\mathrm{sm}_\xi``\mu +_c \nu)_\xi$

Dem.

$\vdash . *118·21 . \supset \vdash :. \mathrm{Hp} . \supset : \exists ! (\mu +_c \nu)_\xi . \equiv . \exists ! (\mu +_c \nu)_\xi . \exists ! \mathrm{sm}_\xi``\mu . \exists ! \mathrm{sm}_\xi``\nu .$

[*110·25·4] $\equiv . \exists ! (\mathrm{sm}_\xi``\mu +_c \mathrm{sm}_\xi``\nu)_\xi$ (1)

$\vdash . *118·21 . *103·43 . *110·4 . \supset$

$\vdash :. \mathrm{Hp} . \supset : \exists ! (\mu +_c \nu)_\xi . \equiv . \exists ! (\mu +_c \nu)_\xi . \exists ! \mathrm{sm}``\mu \cap t_0`\mu . \exists ! \mathrm{sm}_\xi``\nu .$

[*103·43.*110·25·4] $\equiv . \exists ! (\mu +_c \mathrm{sm}_\xi``\nu)_\xi$ (2)

Similarly $\vdash :. \mathrm{Hp} . \supset : \exists ! (\mu +_c \nu)_\xi . \equiv . \exists ! (\mathrm{sm}_\xi``\mu +_c \nu)_\xi$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash . \mathrm{Prop}$

***118·23.** $\vdash : \mu, \nu \in \mathrm{NC} . \supset . (\mu +_c \nu)_\xi = (\mathrm{sm}_\xi``\mu +_c \mathrm{sm}_\xi``\nu)_\xi$

Dem.

$\vdash . *118·21 . *110·4·25 . \supset \vdash : \exists ! (\mu +_c \nu)_\xi . \supset . (\mu +_c \nu)_\xi = (\mathrm{sm}_\xi``\mu +_c \mathrm{sm}_\xi``\nu)_\xi$ (1)

$\vdash . *118·22 . \supset \vdash : \mathrm{Hp} . \sim \exists ! (\mu +_c \nu)_\xi . \supset . (\mu +_c \nu)_\xi = (\mathrm{sm}_\xi``\mu +_c \mathrm{sm}_\xi``\nu)_\xi$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***118·24.** $\vdash : \nu \in \mathrm{NC} . \supset . (\mu +_c \nu)_\xi = (\mu +_c \mathrm{sm}_\xi``\nu)_\xi$

Dem.

$\vdash . *118·21 . *110·4·25 . *103·43 . \supset$

$\vdash : \exists ! (\mu +_c \nu)_\xi . \supset . (\mu +_c \nu)_\xi = (\mu +_c \mathrm{sm}_\xi``\nu)_\xi$ (1)

$\vdash . *110·4 . \supset \vdash : \mu \sim\in \mathrm{NC} . \supset . (\mu +_c \nu)_\xi = (\mu +_c \mathrm{sm}_\xi``\nu)_\xi$ (2)

$\vdash . *118·22 . \supset \vdash : \mathrm{Hp} . \mu \in \mathrm{NC} . \sim \exists ! (\mu +_c \nu)_\xi . \supset . (\mu +_c \nu)_\xi = (\mu +_c \mathrm{sm}_\xi``\nu)_\xi$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash . \mathrm{Prop}$

***118·241.** $\vdash : \mu \in \mathrm{NC} . \supset . (\mu +_c \nu)_\xi = (\mathrm{sm}_\xi``\mu +_c \nu)_\xi$ [*118·24 . *110·51]

∗118·25. $\vdash . (\mu +_c \nu +_c \varpi)_\xi = \{(\mu +_c \nu)_\xi +_c \varpi\}_\xi = \{\mu +_c (\nu +_c \varpi)_\xi\}_\xi$

Dem.

$\vdash . *110{\cdot}42 . *118{\cdot}241{\cdot}201 . (\text{II T}) . \supset$

$\vdash : \mu, \nu \in \mathrm{N_0C} . \supset . (\mu +_c \nu +_c \varpi)_\xi = \{(\mu +_c \nu)_\xi +_c \varpi\}_\xi$ (1)

$\vdash . *110{\cdot}4 . \supset \vdash : \sim(\mu, \nu \in \mathrm{N_0C}) . \supset . \mu +_c \nu = \Lambda . (\mu +_c \nu)_\xi = \Lambda .$

[∗110·4] $\supset . (\mu +_c \nu +_c \varpi)_\xi = \{(\mu +_c \nu)_\xi +_c \varpi\}_\xi$ (2)

$\vdash . (1) . (2) . \supset \vdash . (\mu +_c \nu +_c \varpi)_\xi = \{(\mu +_c \nu)_\xi +_c \varpi\}_\xi$ (3)

Similarly $\vdash . (\mu +_c \nu +_c \varpi)_\xi = \{\mu +_c (\nu +_c \varpi)_\xi\}_\xi$ (4)

$\vdash . (3) . (4) . \supset \vdash . \mathrm{Prop}$

∗118·3. $\vdash . (\mu \times_c \nu)_\xi = \hat{\eta}\{(\exists \alpha, \beta) . \mu = \mathrm{N_0c}`\alpha . \nu = \mathrm{N_0c}`\beta . \eta \,\mathrm{sm}_\xi\, (\alpha \times \beta)\}$
[(∗65·01·03) . ∗113·2]

∗118·301. $\vdash : \exists ! (\mu \times_c \nu) . \supset . \mathrm{sm}_\xi``(\mu \times_c \nu) = (\mu \times_c \nu)_\xi$ [Proof as in ∗118·201]

∗118·31. $\vdash : \exists ! (\mu \times_c \nu)_\xi . \nu \neq 0 . \supset . \exists ! \mathrm{sm}_\xi``\mu$

Dem.

$\vdash . *101{\cdot}15{\cdot}12 . \supset \vdash : \mu = 0 . \supset . \exists ! \mathrm{sm}_\xi``\mu$ (1)

$\vdash . *113{\cdot}203 . *118{\cdot}3 . \supset \vdash : \mathrm{Hp} . \mu \neq 0 . \supset . \mu, \nu \in \mathrm{N_0C} - \iota`0 .$

[∗117·62] $\supset . \mu \times_c \nu \geqslant \mu .$

[∗118·13·301.(II T)] $\supset . \exists ! \mathrm{sm}_\xi``\mu$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

∗118·311. $\vdash : \exists ! (\mu \times_c \nu)_\xi . \mu \neq 0 . \supset . \exists ! \mathrm{sm}_\xi``\nu$ [∗118·31 . ∗113·27]

∗118·32. $\vdash :. \nu \in \mathrm{NC} . \mu \neq 0 . \supset : \exists ! (\mu \times_c \nu)_\xi . \equiv . \exists ! (\mu \times_c \mathrm{sm}_\xi``\nu)_\xi$

Dem.

$\vdash . *113{\cdot}203 . \supset \vdash : \exists ! (\mu \times_c \nu)_\xi . \supset . \mu \in \mathrm{NC}$ (1)

$\vdash . *113{\cdot}203 . \supset \vdash : \exists ! (\mu \times_c \mathrm{sm}_\xi``\nu)_\xi . \supset . \mu \in \mathrm{NC}$ (2)

$\vdash . *113{\cdot}203 . *118{\cdot}311 . \supset$

$\vdash :. \mathrm{Hp} . \supset : \exists ! (\mu \times_c \nu)_\xi . \supset . \exists ! \mu . \exists ! \mathrm{sm}_\xi``\nu .$

[∗103·43] $\supset . \exists ! \mathrm{sm}``\mu \cap t_0`\mu . \exists ! \mathrm{sm}_\xi``\nu .$

[(1).∗113·26.∗103·43] $\supset . \exists ! (\mu \times_c \mathrm{sm}_\xi``\nu)_\xi$ (3)

$\vdash . *113{\cdot}203 . *103{\cdot}43 . \supset$

$\vdash :. \mathrm{Hp} . \supset : \exists ! (\mu \times_c \mathrm{sm}_\xi``\nu)_\xi . \supset . \exists ! \mathrm{sm}``\mu \cap t_0`\mu . \exists ! \mathrm{sm}_\xi``\nu .$

[(2).∗113·26.∗103·43] $\supset . \exists ! (\mu \times_c \nu)_\xi$ (4)

$\vdash . (3) . (4) . \supset \vdash . \mathrm{Prop}$

∗118·33. $\vdash : \mu, \nu \in \mathrm{NC} - \iota`0 . \supset . (\mu \times_c \nu)_\xi = (\mathrm{sm}_\xi``\mu \times_c \mathrm{sm}_\xi``\nu)_\xi$
[Proof as in ∗118·23, using ∗118·31·311 . ∗113·203·26]

∗118·34. $\vdash : \nu \in \mathrm{NC} . \mu \neq 0 . \supset . (\mu \times_c \nu)_\xi = (\mu \times_c \mathrm{sm}_\xi``\nu)_\xi$

Dem.

$\vdash . *118{\cdot}311 . *113{\cdot}203{\cdot}26 . *103{\cdot}43 . \supset$

$\vdash : \exists ! (\mu \times_c \nu)_\xi . \mu \neq 0 . \supset . (\mu \times_c \nu)_\xi = (\mu \times_c \mathrm{sm}_\xi``\nu)_\xi$ (1)

$\vdash . *118{\cdot}32 . \supset \vdash : \mathrm{Hp} . \sim\exists ! (\mu \times_c \nu)_\xi . \supset . \sim\exists ! (\mu \times_c \mathrm{sm}_\xi``\nu)_\xi .$

[∗24·51] $\supset . (\mu \times_c \nu)_\xi = (\mu \times_c \mathrm{sm}_\xi``\nu)_\xi$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***118·341.** $\vdash : \mu \in \text{NC} \,.\, \nu \neq 0 \,.\, \supset \,.\, (\mu \times_c \nu)_\xi = (\text{sm}_\xi``\mu \times_c \nu)_\xi$ [*118·34 . *113·27]

***118·35.** $\vdash : \varpi \neq 0 \,.\, \supset \,.\, (\mu \times_c \nu \times_c \varpi)_\xi = \{(\mu \times_c \nu)_\xi \times_c \varpi\}_\xi$
[Proof similar to *118·25, using *118·341·301 . *113·203·23]

***118·351.** $\vdash : \mu \neq 0 \,.\, \supset \,.\, (\mu \times_c \nu \times_c \varpi)_\xi = \{\mu \times_c (\nu \times_c \varpi)_\xi\}_\xi$ [*118·35 . *113·27]

***118·352.** $\vdash : \mu \neq 0 \,.\, \varpi \neq 0 \,.\, \supset \,.\, \{\mu \times_c (\nu \times_c \varpi)_\xi\}_\xi = \{(\mu \times_c \nu)_\xi \times_c \varpi\}_\xi$
[*118·35·351]

***118·4.** $\vdash \,.\, (\mu^\nu)_\xi = \hat{\eta}\,\{(\exists \alpha, \beta) \,.\, \mu = \text{N}_0\text{c}`\alpha \,.\, \nu = \text{N}_0\text{c}`\beta \,.\, \eta \,\text{sm}_\xi\, (\alpha \exp \beta)\}$
[(*65·01·03) . *116·2]

***118·401.** $\vdash : \exists ! \mu^\nu \,.\, \supset \,.\, \text{sm}_\xi``\mu^\nu = (\mu^\nu)_\xi$ [Proof as in *118·201]

***118·402.** $\vdash :. \mu, \nu \in \text{N}_0\text{C} \,.\, \mu \neq 0 \,.\, \mu \neq 1 \,.\, \supset : \exists ! (\mu^\nu)_\xi \,.\, \supset \,.\, \exists ! (\mu \times_c \nu)_\xi$

Dem.

$\vdash \,.\, *103·2 \,.\, \supset \vdash :. \text{Hp} \,.\, \supset : (\exists \alpha, \beta) \,.\, \mu = \text{N}_0\text{c}`\alpha \,.\, \nu = \text{N}_0\text{c}`\beta \,.\, \alpha \sim\in 0 \cup 1 :$

[*117·651] $\supset : (\exists \alpha, \beta) \,.\, \mu = \text{N}_0\text{c}`\alpha \,.\, \nu = \text{N}_0\text{c}`\beta \,.$
$(\text{N}_0\text{c}`\alpha)^{\text{N}_0\text{c}`\beta} \geqslant \text{N}_0\text{c}`\alpha \times_c \text{N}_0\text{c}`\beta :$

[*118·13·301·401.(IIT)] $\supset : \exists ! (\mu^\nu)_\xi \,.\, \supset \,.\, \exists ! (\mu \times_c \nu)_\xi :. \supset \vdash \,.\, \text{Prop}$

***118·41.** $\vdash : \exists ! (\mu^\nu)_\xi \,.\, \nu \neq 0 \,.\, \supset \,.\, \exists ! \text{sm}_\xi``\mu$

Dem.

$\vdash \,.\, *118·402·31 \,.\, \supset \vdash : \text{Hp} \,.\, \mu \neq 1 \,.\, \mu \neq 0 \,.\, \supset \,.\, \exists ! \text{sm}_\xi``\mu$ (1)

$\vdash \,.\, *101·12·15·241·28 \,.\, \supset \vdash :. \mu = 0 \,.\, \text{v} \,.\, \mu = 1 : \supset \,.\, \exists ! \text{sm}_\xi``\mu$ (2)

$\vdash \,.\, (1) \,.\, (2) \,.\, \supset \vdash \,.\, \text{Prop}$

***118·411.** $\vdash : \exists ! (\mu^\nu)_\xi \,.\, \mu \neq 0 \,.\, \mu \neq 1 \,.\, \supset \,.\, \exists ! \text{sm}_\xi``\nu$ [*118·402·311]

***118·42.** $\vdash :. \nu \in \text{NC} \,.\, \mu \neq 0 \,.\, \mu \neq 1 \,.\, \supset : \exists ! (\mu^\nu)_\xi \,.\, \equiv \,.\, \exists ! (\mu^{\text{sm}_\xi``\nu})_\xi$
[Proof as in *118·32, using *116·203·26 . *118·411]

***118·421.** $\vdash :. \mu \in \text{NC} \,.\, \nu \neq 0 \,.\, \supset : \exists ! (\mu^\nu)_\xi \,.\, \equiv \,.\, \exists ! \{(\text{sm}_\xi``\mu)^\nu\}_\xi$
[Proof as in *118·32, using *116·203·26 . *118·41]

***118·43.** $\vdash : \mu, \nu \in \text{NC} - \iota`0 \,.\, \mu \neq 1 \,.\, \supset \,.\, (\mu^\nu)_\xi = \{(\text{sm}_\xi``\mu)^{\text{sm}_\xi``\nu}\}_\xi$
[Proof as in *118·23, using *118·41·411 . *116·203·26]

***118·44.** $\vdash : \nu \in \text{NC} \,.\, \mu \neq 0 \,.\, \mu \neq 1 \,.\, \supset \,.\, (\mu^\nu)_\xi = (\mu^{\text{sm}_\xi``\nu})_\xi$
[Proof as in *118·34, using *116·203·26 . *118·411·42]

***118·441.** $\vdash : \mu \in \text{NC} \,.\, \nu \neq 0 \,.\, \supset \,.\, (\mu^\nu)_\xi = \{(\text{sm}_\xi``\mu)^\nu\}_\xi$
[Proof as in *118·34, using *116·203·26 . *118·41·421]

***118·45.** $\vdash : \mu \neq 0 \,.\, \mu \neq 1 \,.\, \supset \,.\, (\mu^{\nu \times_c \varpi})_\xi = \{\mu^{(\nu \times_c \varpi)_\xi}\}_\xi$

Dem.

$\vdash \,.\, *113·23 \,.\, *118·44·301 \,.\, (\text{IIT}) \,.\, \supset$

$\vdash : \text{Hp} \,.\, \nu, \varpi \in \text{N}_0\text{C} \,.\, \supset \,.\, (\mu^{\nu \times_c \varpi})_\xi = \{\mu^{(\nu \times_c \varpi)_\xi}\}_\xi$ (1)

$\vdash \,.\, *113·203 \,.\, \supset \vdash : \sim(\nu, \varpi \in \text{N}_0\text{C}) \,.\, \supset \,.\, \nu \times_c \varpi = \Lambda \,.$

[*116·203] $\supset \,.\, (\mu^{\nu \times_c \varpi})_\xi = \{\mu^{(\nu \times_c \varpi)_\xi}\}_\xi$ (2)

$\vdash \,.\, (1) \,.\, (2) \,.\, \supset \vdash \,.\, \text{Prop}$

***118·451.** $\vdash : \varpi \neq 0 \,.\, \supset \,.\, (\mu^{\nu \times_0 \varpi})_\xi = [\{(\mu^\nu)_\xi\}^\varpi]_\xi$

Dem.

$\vdash \,.\, *116·63 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \mu \in \mathrm{NC} \,.\, \supset \,.\, (\mu^{\nu \times_0 \varpi})_\xi = \{(\mu^\nu)^\varpi\}_\xi$

$[*116·23 . *118·441·401 . (\mathrm{II\,T})] \qquad = [\{(\mu^\nu)_\xi\}^\varpi]_\xi$ (1)

$\vdash \,.\, *116·204 \,.\, \supset \vdash : \mu \sim \in \mathrm{NC} \,.\, \supset \,.\, (\mu^{\nu \times_0 \varpi})_\xi = [\{(\mu^\nu)_\xi\}^\varpi]_\xi$ (2)

$\vdash \,.\, (1) \,.\, (2) \,.\, \supset \vdash \,.\, \mathrm{Prop}$

***118·46.** $\vdash : \mu \neq 0 \,.\, \mu \neq 1 \,.\, \supset \,.\, (\mu^{\nu +_0 \varpi})_\xi = \{\mu^{(\nu +_0 \varpi)\xi}\}_\xi$

[Proof as in *118·45, using *118·44·201 . *116·203 . *110·4·42]

***118·461.** $\vdash \,.\, (\mu^{\nu +_0 \varpi})_\xi = \{(\mu^\nu)_\xi \times_0 (\mu^\varpi)_\xi\}_\xi$

Dem.

$\vdash \,.\, *116·52 \,.\, \supset \vdash : \mu \neq 0 \,.\, \supset \,.\, (\mu^{\nu +_0 \varpi})_\xi = (\mu^\nu \times_0 \mu^\varpi)_\xi$

$[*116·35·23 . *118·33·401 . (\mathrm{II\,T})] \qquad = \{(\mu^\nu)_\xi \times_0 (\mu^\varpi)_\xi\}_\xi$ (1)

$\vdash \,.\, *110·4 \,.\, *113·203 \,.\, *116·203 \,.\, \supset$

$\vdash : \sim(\nu, \varpi \in \mathrm{N_0C}) \,.\, \supset \,.\, (\mu^{\nu +_0 \varpi})_\xi = \{(\mu^\nu)_\xi \times_0 (\mu^\varpi)_\xi\}_\xi$ (2)

$\vdash \,.\, *116·311 \,.\, *113·601 \,.\, *110·62 \,.\, \supset$

$\vdash : \nu, \varpi \in \mathrm{N_0C} - \iota`0 \,.\, \mu = 0 \,.\, \supset \,.\, (\mu^{\nu +_0 \varpi})_\xi = \{(\mu^\nu)_\xi \times_0 (\mu^\varpi)_\xi\}_\xi$ (3)

$\vdash \,.\, *116·311·301 \,.\, *110·6 \,.\, *113·601 \,.\, \supset$

$\vdash : \nu \in \mathrm{N_0C} - \iota`0 \,.\, \varpi = 0 \,.\, \mu = 0 \,.\, \supset \,.\, (\mu^{\nu +_0 \varpi})_\xi = \{(\mu^\nu)_\xi \times_0 (\mu^\varpi)_\xi\}_\xi$ (4)

Similarly $\vdash : \varpi \in \mathrm{N_0C} - \iota`0 \,.\, \nu = 0 \,.\, \mu = 0 \,.\, \supset \,.\, (\mu^{\nu \times\ \varpi})_\xi = \{(\mu^\nu)_\xi \times_0 (\mu^\varpi)_\xi\}_\xi$ (5)

$\vdash \,.\, *116·301 \,.\, *113·621 \,.\, \supset$

$\vdash : \nu = 0 \,.\, \varpi = 0 \,.\, \mu = 0 \,.\, \supset \,.\, (\mu^{\nu +_0 \varpi})_\xi = \{(\mu^\nu)_\xi \times_0 (\mu^\varpi)_\xi\}_\xi$ (6)

$\vdash \,.\, (1) \,.\, (2) \,.\, (3) \,.\, (4) \,.\, (5) \,.\, (6) \,.\, \supset \vdash \,.\, \mathrm{Prop}$

***118·462.** $\vdash \,.\, (\mu^{\nu +_0 \varpi})_\xi = \{\mu^\nu \times_0 (\mu^\varpi)_\xi\}_\xi$ [Proof as in *118·461, using *118·34]

***118·463.** $\vdash \,.\, (\mu^{\nu +_0 \varpi})_\xi = \{(\mu^\nu)_\xi \times_0 \mu^\varpi\}_\xi$ [Proof as in *118·461, using *118·341]

***118·47.** $\vdash : \varpi \neq 0 \,.\, \supset \,.\, \{(\mu \times_0 \nu)^\varpi\}_\xi = [\{(\mu \times_0 \nu)_\xi\}^\varpi]_\xi$

[Proof as in *118·45, using *118·441]

***118·471.** $\vdash :. \mu \neq 0 \,.\, \nu \neq 0 \,.\, \mathrm{v} \,.\, \varpi = 0 \,.\, \mathrm{v} \,.\, \sim(\mu, \nu, \varpi \in \mathrm{N_0C}) : \supset \,.$

$\{(\mu \times_0 \nu)^\varpi\}_\xi = \{(\mu^\varpi)_\xi \times_0 (\nu^\varpi)_\xi\}_\xi$

Dem.

$\vdash \,.\, *116·55 \,.\, \supset \vdash : \mu \neq 0 \,.\, \nu \neq 0 \,.\, \supset \,.\, \{(\mu \times_0 \nu)^\varpi\}_\xi = \{\mu^\varpi \times_0 \nu^\varpi\}_\xi$

$[*116·35·23 . *118·33·401 . (\mathrm{II\,T})] \qquad = \{(\mu^\varpi)_\xi \times_0 (\nu^\varpi)_\xi\}_\xi$ (1)

$\vdash \,.\, *110·4 \,.\, *113·203 \,.\, *116·203 \,.\, \supset$

$\vdash : \sim(\mu, \nu, \varpi \in \mathrm{N_0C}) \,.\, \supset \,.\, \{(\mu \times_0 \nu)^\varpi\}_\xi = \{(\mu^\varpi)_\xi \times_0 (\nu^\varpi)_\xi\}_\xi$ (2)

$\vdash \,.\, *116·301 \,.\, *113·621 \,.\, \supset$

$\vdash : \mu, \nu \in \mathrm{N_0C} \,.\, \varpi = 0 \,.\, \supset \,.\, \{(\mu \times_0 \nu)^\varpi\}_\xi = \{(\mu^\varpi)_\xi \times_0 (\nu^\varpi)_\xi\}_\xi$ (3)

$\vdash \,.\, (1) \,.\, (2) \,.\, (3) \,.\, \supset \vdash \,.\, \mathrm{Prop}$

***118·472.** $\vdash :. \mu \neq 0 \,.\, \mathrm{v} \,.\, \varpi = 0 \,.\, \mathrm{v} \,.\, \sim(\mu, \nu, \varpi \in \mathrm{N_0C}) : \supset \,.\, \{(\mu \times_0 \nu)^\varpi\}_\xi = \{\mu^\varpi \times_0 (\nu^\varpi)_\xi\}_\xi$

[Proof as in *118·471, using *118·34]

***118·473.** $\vdash :. \nu \neq 0 \,.\, \mathrm{v} \,.\, \varpi = 0 \,.\, \mathrm{v} \,.\, \sim(\mu, \nu, \varpi \in \mathrm{N_0C}) : \supset \,.\, \{(\mu \times_0 \nu)^\varpi\}_\xi = \{(\mu^\varpi)_\xi \times_0 \nu^\varpi\}_\xi$

[Proof as in *118·471, using *118·341]

$*119$. SUBTRACTION

Summary of $*119$.

The treatment of subtraction follows the same general lines as that of addition, and is simplified by the results in $*110$. A difficulty arises from the fact that subtraction (in any ordinary sense of the term) is not always possible; and also from the fact that the result, when possible, is not always a cardinal number.

We put

$*119\cdot01$. $\gamma -_c \nu = \hat{\xi}\{\text{Nc}`\xi +_c \nu = \gamma \,.\, \exists ! \,\text{Nc}`\xi +_c \nu\}$ Df

Thus when subtraction (in the ordinary sense of the term) is not possible,

$$\gamma -_c \nu = \Lambda.$$

The question of existential adjustment of types is dealt with by IIT of the prefatory statement combined with the following definitions:

$*119\cdot02$. $\text{Nc}`\alpha -_c \nu = \text{N}_0\text{c}`\alpha -_c \nu$ Df

$*119\cdot03$. $\gamma -_c \text{Nc}`\beta = \gamma -_c \text{N}_0\text{c}`\beta$ Df

We then proceed to deduce the elementary properties derivable from these definitions.

$*119\cdot11$. $\vdash : \exists ! \,\gamma -_c \nu \,.\, \supset \,.\, \gamma, \nu \in \text{N}_0\text{C}$

$*119\cdot12$. $\vdash : \xi \in \text{Nc}`\alpha -_c \text{Nc}`\beta \,.\, \equiv \,.\, \alpha \,\text{sm}\, \xi + \beta$

$*119\cdot14$. $\vdash : \xi \in \gamma -_c \nu \,.\, \supset \,.\, \text{N}_0\text{c}`\xi \subset \gamma -_c \nu$

$*119\cdot25$. $\vdash : \gamma \geqslant \nu \,.\, \supset \,.\, \exists ! \,(\gamma -_c \nu) \cap t_0`\gamma$

$*119\cdot26$. $\vdash : \exists ! \,\gamma -_c \nu \,.\, \supset \,.\, \gamma \geqslant \nu$

The next group of propositions is concerned with some simple results of subtraction.

$*119\cdot32$. $\vdash : (\gamma +_c \nu) -_c \nu \in \text{N}_0\text{C} \,.\, \supset \,.\, \text{sm}``\gamma = (\gamma +_c \nu) -_c \nu$

$*119\cdot34$. $\vdash : \gamma -_c \nu \in \text{N}_0\text{C} \,.\, \supset \,.\, (\gamma -_c \nu) +_c \nu = \text{sm}``\gamma$

$*119\cdot35$. $\vdash : \gamma -_c \nu \in \text{N}_0\text{C} \,.\, \supset \,.\, \alpha +_c \gamma = (\alpha +_c \nu) +_c (\gamma -_c \nu)$

Associative laws are then considered.

$*119\cdot44$. $\vdash : \mu +_c (\nu -_c \varpi) \subset (\mu +_c \nu) -_c \varpi$

$*119\cdot45$. $\vdash : (\mu +_c \nu) -_c \varpi \in \text{NC} \,.\, \exists ! \{\mu +_c (\nu -_c \varpi)\} \,.\, \supset \,.\, \mu +_c (\nu -_c \varpi) = (\mu +_c \nu) -_c \varpi$

The question of types is then dealt with:

$*119\cdot52$. $\vdash : \text{sm}_{\delta, \gamma}``(\mu -_c \nu)_\gamma = (\mu -_c \nu)_\delta \cap \text{D}`\text{sm}_{\delta, \gamma}$

A difficulty arises from the fact that if τ_1 and τ_2 are two complete types whose members are classes, we cannot prove that, either $\tau_1 = \text{sm}``\tau_2$ or $\tau_2 = \text{sm}``\tau_1$. We put

$*119{\cdot}54.\quad \text{SM}(\delta,\gamma) \,.\!=:\, t`\delta = \text{D}`\text{sm}_{\delta,\gamma} \,.\vee.\, t`\gamma = \text{D}`\text{sm}_{\gamma,\delta} \quad \text{Df}$

Then we obtain

$*119{\cdot}541.\ \vdash : \text{SM}(\delta,\gamma) \,.\, (\mu -_c \nu)_\gamma \,\epsilon\, \text{N}_0\text{C} \,.\, (\mu -_c \nu)_\delta \,\epsilon\, \text{NC} \,.\supset.$

$$\text{sm}_{\delta,\gamma}``(\mu -_c \nu)_\gamma = (\mu -_c \nu)_\delta$$

Finally we show that any existential adjustment of types will suffice for the components:

$*119{\cdot}61.\quad \vdash : \mu \,\epsilon\, \text{N}_0\text{C} \,.\, \exists!\, \text{sm}_\xi``\mu \,.\supset.\, \mu -_c \nu = \text{sm}_\xi``\mu -_c \nu$

$*119{\cdot}62.\quad \vdash : \nu \,\epsilon\, \text{N}_0\text{C} \,.\, \exists!\, \text{sm}_\xi``\nu \,.\supset.\, \mu -_c \nu = \mu -_c \text{sm}_\xi``\nu$

Also $*119{\cdot}25{\cdot}26$ are now extended to

$*119{\cdot}64.\quad \vdash :.\, \exists!\, \text{sm}_\xi``\mu \,.\supset : \mu \geqslant \nu \,.\equiv.\, \exists!\, (\mu -_c \nu)_\xi$

The only applications of the propositions of this number are in connection with Inductive Cardinals (cf. $*120$).

$*119{\cdot}01.\quad \gamma -_c \nu = \hat{\xi}\{\text{Nc}`\xi +_c \nu = \gamma \,.\, \exists!\, \text{Nc}`\xi +_c \nu\} \quad \text{Df}$

Here the suffix to the sign of subtraction is introduced to show that we are concerned with *cardinal* subtraction. It will be found that $\gamma -_c \nu$ is not an NC except under hypotheses for γ and ν.

$*119{\cdot}02.\quad \text{Nc}`\alpha -_c \nu = \text{N}_0\text{c}`\alpha -_c \nu \quad \text{Df}$

$*119{\cdot}03.\quad \gamma -_c \text{Nc}`\beta = \gamma -_c \text{N}_0\text{c}`\beta \quad \text{Df}$

$*119{\cdot}04.\quad \vdash .\, \text{Nc}`\alpha -_c \text{Nc}`\beta = \text{N}_0\text{c}`\alpha -_c \text{N}_0\text{c}`\beta \quad [*119{\cdot}02{\cdot}03]$

Note that the occurrence of a formal number in the place of γ or ν in $\gamma -_c \nu$ is an arithmetic occurrence, and accordingly IIT applies to it.

$*119{\cdot}1.\quad \vdash : \xi \,\epsilon\, \gamma -_c \nu \,.\equiv.\, \text{Nc}`\xi +_c \nu = \gamma \,.\, \exists!\, \text{Nc}`\xi +_c \nu \quad [(*119{\cdot}01)]$

$*119{\cdot}101.\ \vdash : \xi \,\epsilon\, \text{Nc}`\alpha -_c \nu \,.\equiv.\, \text{Nc}`\xi +_c \nu = \text{N}_0\text{c}`\alpha \quad [(*119{\cdot}02) \,.\, *103{\cdot}13]$

$*119{\cdot}102.\ \vdash : \xi \,\epsilon\, \gamma -_c \text{Nc}`\beta \,.\equiv.\, \text{Nc}`\xi +_c \text{Nc}`\beta = \gamma \,.\, \exists!\, \text{Nc}`\xi +_c \text{Nc}`\beta$
$[(*119{\cdot}03) \,.\, *110{\cdot}3]$

$*119{\cdot}103.\ \vdash : \xi \,\epsilon\, \text{Nc}`\alpha -_c \text{Nc}`\beta \,.\equiv.\, \text{Nc}`\xi +_c \text{Nc}`\beta = \text{N}_0\text{c}`\alpha$
$[*119{\cdot}04 \,.\, *110{\cdot}3 \,.\, *103{\cdot}13]$

$*119{\cdot}11.\quad \vdash : \exists!\, \gamma -_c \nu \,.\supset.\, \gamma, \nu \,\epsilon\, \text{N}_0\text{C} \quad [*110{\cdot}4{\cdot}42 \,.\, *103{\cdot}34]$

$*119{\cdot}12.\quad \vdash : \xi \,\epsilon\, \text{Nc}`\alpha -_c \text{Nc}`\beta \,.\equiv.\, \alpha \,\text{sm}\, \xi + \beta$

Dem.

$$\begin{aligned}
&\vdash .\, *119{\cdot}103 \,.\supset \vdash : \xi \,\epsilon\, \text{Nc}`\alpha -_c \text{Nc}`\beta \,.\equiv.\, \text{Nc}`\xi +_c \text{Nc}`\beta = \text{N}_0\text{c}`\alpha \,. \\
&[*110{\cdot}3] \qquad\qquad \equiv.\, \text{Nc}`(\xi + \beta) = \text{N}_0\text{c}`\alpha \,. \\
&[*100{\cdot}35 . *103{\cdot}13] \qquad\qquad \equiv.\, \alpha \,\text{sm}\, \xi + \beta : \supset \vdash .\, \text{Prop}
\end{aligned}$$

Thus $\text{Nc}`\alpha -_c \text{Nc}`\beta$ is an NC when $\hat{\xi}(\alpha \,\text{sm}\, \xi + \beta)$ is an NC.

***119·13.** $\vdash : \mathrm{N_0c}`\gamma \subset \mathrm{Nc}`\alpha -_c \mathrm{Nc}`\beta \,.\equiv.\, \alpha \,\mathrm{sm}\, (\gamma + \beta)$

Dem.

$\vdash . *22·1 . \supset \vdash :. \mathrm{N_0c}`\gamma \subset \mathrm{Nc}`\alpha -_c \mathrm{Nc}`\beta . \equiv : \xi \in \mathrm{N_0c}`\gamma . \supset_\xi . \xi \in \mathrm{Nc}`\alpha -_c \mathrm{Nc}`\beta :$

$[*103·12.*119·12] \quad \supset : \alpha \,\mathrm{sm}\, (\gamma + \beta) \quad (1)$

$\vdash . *110·15 . *100·31 . \supset \vdash :. \alpha \,\mathrm{sm}\, (\gamma + \beta) . \supset : \xi \in \mathrm{N_0c}`\gamma . \supset . (\xi + \beta) \,\mathrm{sm}\, (\gamma + \beta) .$

$[*73·32] \quad \supset . \alpha \,\mathrm{sm}\, (\xi + \beta) .$

$[*119·12] \quad \supset . \xi \in \mathrm{Nc}`\alpha -_c \mathrm{Nc}`\beta \quad (2)$

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***119·14.** $\vdash : \xi \in \gamma -_c \nu . \supset . \mathrm{N_0c}`\xi \subset \gamma -_c \nu \quad [*119·1 . *100·31·321]$

***119·21.** $\vdash : \beta \subset \alpha . \supset . \exists ! (\mathrm{Nc}`\alpha -_c \mathrm{Nc}`\beta)_\alpha$

The notation is defined in *65·01.

Dem.

$\vdash . *24·411·21 . \supset \vdash : \mathrm{Hp} . \supset . \alpha = \beta \cup (\alpha - \beta) . \beta \cap (\alpha - \beta) = \Lambda .$

$[*110·32] \quad \supset . \mathrm{Nc}`\alpha = \mathrm{Nc}`\beta +_c \mathrm{Nc}`(\alpha - \beta) .$

$[*10·24] \quad \supset . (\exists \xi) . \xi \in t`\alpha . \mathrm{N_0c}`\alpha = \mathrm{Nc}`\beta +_c \mathrm{Nc}`\xi .$

$[*119·103] \quad \supset . \exists ! (\mathrm{Nc}`\alpha -_c \mathrm{Nc}`\beta)_\alpha : \supset \vdash . \mathrm{Prop}$

***119·22.** $\vdash : \mathrm{Nc}`\alpha \geqslant \mathrm{Nc}`\beta . \supset . \exists ! (\mathrm{Nc}`\alpha -_c \mathrm{Nc}`\beta)_\alpha$

Dem.

$\vdash . *117·221 . \supset \vdash : \mathrm{Hp} . \supset . (\exists \rho) . \rho \subset \alpha . \rho \,\mathrm{sm}\, \beta .$

$[*119·21] \quad \supset . (\exists \rho) . \exists ! (\mathrm{Nc}`\alpha -_c \mathrm{Nc}`\rho)_\alpha . \rho \,\mathrm{sm}\, \beta .$

$[*100·35.*119·04] \quad \supset . \exists ! (\mathrm{Nc}`\alpha -_c \mathrm{Nc}`\beta)_\alpha : \supset \vdash . \mathrm{Prop}$

***119·23.** $\vdash : \exists ! (\mathrm{Nc}`\alpha -_c \mathrm{Nc}`\beta) . \supset . (\exists \delta) . \delta \,\mathrm{sm}\, \beta . \delta \subset \alpha$

Dem.

$\vdash . *119·103 . \supset \vdash : \mathrm{Hp} . \supset . (\exists \xi) . \mathrm{N_0c}`\alpha = \mathrm{Nc}`\beta +_c \mathrm{Nc}`\xi .$

$[*110·71] \quad \supset . (\exists \delta) . \delta \,\mathrm{sm}\, \beta . \delta \subset \alpha : \supset \vdash . \mathrm{Prop}$

***119·24.** $\vdash : \exists ! (\mathrm{Nc}`\alpha -_c \mathrm{Nc}`\beta) . \supset . \mathrm{Nc}`\alpha \geqslant \mathrm{Nc}`\beta \quad [*119·23 . *117·221]$

***119·25.** $\vdash : \gamma \geqslant \nu . \supset . \exists ! (\gamma -_c \nu) \cap t_0`\gamma$

Dem.

$\vdash . *117·24 . \supset \vdash : \mathrm{Hp} . \supset . (\exists \alpha, \beta) . \gamma = \mathrm{N_0c}`\alpha . \nu = \mathrm{N_0c}`\beta . \mathrm{N_0c}`\alpha \geqslant \mathrm{N_0c}`\beta .$

$[*117·107] \quad \supset . (\exists \alpha, \beta) . \gamma = \mathrm{N_0c}`\alpha . \nu = \mathrm{N_0c}`\beta . \mathrm{Nc}`\alpha \geqslant \mathrm{Nc}`\beta .$

$[*119·22·04] \quad \supset . (\exists \alpha, \beta) . \gamma = \mathrm{N_0c}`\alpha . \nu = \mathrm{N_0c}`\beta . \exists ! (\mathrm{N_0c}`\alpha -_c \mathrm{N_0c}`\beta)_\alpha .$

$[(*63·02).*13·193] \quad \supset . \exists ! (\gamma -_c \nu) \cap t_0`\gamma : \supset \vdash . \mathrm{Prop}$

***119·26.** $\vdash : \exists ! \gamma -_c \nu . \supset . \gamma \geqslant \nu$

Dem.

$\vdash . *119·11 . \supset \vdash : \mathrm{Hp} . \supset . (\exists \alpha, \beta) . \gamma = \mathrm{N_0c}`\alpha . \nu = \mathrm{N_0c}`\beta . \exists ! (\mathrm{N_0c}`\alpha -_c \mathrm{N_0c}`\beta) .$

$[*119·04·24] \quad \supset . (\exists \alpha, \beta) . \gamma = \mathrm{N_0c}`\alpha . \nu = \mathrm{N_0c}`\beta . \mathrm{Nc}`\alpha \geqslant \mathrm{Nc}`\beta .$

$[*117·107.*13·193] \quad \supset . \gamma \geqslant \nu : \supset \vdash . \mathrm{Prop}$

***119·27.** $\vdash : \gamma \geqslant \nu . \equiv . \exists ! (\gamma -_c \nu) \cap t_0`\gamma \quad [*119·25·26]$

For the extension of this theorem cf. *119·64.

***119·31.** $\vdash : \gamma, \nu \in \mathrm{N_0C} . \supset . \mathrm{sm}``\gamma \subset (\gamma +_c \nu) -_c \nu$

Dem.

$\vdash . *119\cdot1 . (\mathrm{IIT}) . \supset \vdash : \xi \in (\gamma +_c \nu) -_c \nu . \equiv . \mathrm{Nc}`\xi +_c \nu = \gamma +_c \nu . \exists ! \gamma +_c \nu$ (1)

$\vdash . *100\cdot51\cdot521 . \quad \supset \vdash :. \mathrm{Hp} . \supset : \xi \in \mathrm{sm}``\gamma . \supset . \mathrm{Nc}`\xi = \gamma .$

$[*103\cdot22 . *118\cdot01] \quad \supset . \mathrm{Nc}`\xi +_c \nu = \gamma +_c \nu .$

$[*110\cdot22\cdot03 . *103\cdot13] \quad \supset . \mathrm{Nc}`\xi +_c \nu = \gamma +_c \nu . \exists ! \gamma +_c \nu .$

$[(1)] \quad \supset . \xi \in (\gamma +_c \nu) -_c \nu : \supset \vdash . \mathrm{Prop}$

The penultimate step in the proof employs the principle, explained in the prefatory statement, that, since in the previous line the equation

$$\mathrm{Nc}`\xi +_c \nu = \gamma +_c \nu$$

has its sides undetermined in type by the conventions IT and IIT, any convenient type can be chosen for them. The type chosen in this line is such that $\exists ! \gamma +_c \nu$, and the references indicate the existence of at least one such type.

***119·32.** $\vdash : (\gamma +_c \nu) -_c \nu \in \mathrm{N_0C} . \supset . \mathrm{sm}``\gamma = (\gamma +_c \nu) -_c \nu$
$[*119\cdot11\cdot31 . *103\cdot22 . *100\cdot52\cdot42]$

***119·33.** $\vdash : \mathrm{Nc}`\alpha -_c \mathrm{Nc}`\beta \in \mathrm{N_0C} . \supset . (\mathrm{Nc}`\alpha -_c \mathrm{Nc}`\beta) +_c \mathrm{Nc}`\beta = \mathrm{Nc}`\alpha$

Dem.

$\vdash . *119\cdot13 . \supset \vdash : \mathrm{N_0c}`\gamma = \mathrm{Nc}`\alpha -_c \mathrm{Nc}`\beta . \supset . \alpha \,\mathrm{sm}\, (\gamma + \beta)$ (1)

$\vdash . *20\cdot18 . *118\cdot01 . \supset \vdash :. \mathrm{Hp}(1) . \supset :$

$(\mathrm{Nc}`\alpha -_c \mathrm{Nc}`\beta) +_c \mathrm{Nc}`\beta = \mathrm{N_0c}`\xi . \equiv_\xi . \mathrm{Nc}`\gamma +_c \mathrm{Nc}`\beta = \mathrm{N_0c}`\xi .$

$[*110\cdot3 . *100\cdot35] \quad \equiv_\xi . \xi \,\mathrm{sm}\, (\gamma + \beta) .$

$[(1) . *103\cdot42] \quad \equiv_\xi . \mathrm{N_0c}`\xi = \mathrm{Nc}`\alpha$ (2)

$\vdash . *103\cdot2\cdot34 . \supset \vdash :. \mathrm{Hp} . \supset : \exists ! \mathrm{Nc}`\alpha . \supset . (\exists \xi) . \mathrm{N_0c}`\xi = \mathrm{Nc}`\alpha .$

$[(2) . *10\cdot1] \quad \supset . (\mathrm{Nc}`\alpha -_c \mathrm{Nc}`\beta) +_c \mathrm{Nc}`\beta = \mathrm{Nc}`\alpha$ (3)

$\vdash . *110\cdot42 . *103\cdot34\cdot2 . \supset \vdash :. \mathrm{Hp} . \supset :$

$\exists ! \{(\mathrm{Nc}`\alpha -_c \mathrm{Nc}`\beta) +_c \mathrm{Nc}`\beta\} . \supset . (\exists \xi) . \mathrm{N_0c}`\xi = (\mathrm{Nc}`\alpha -_c \mathrm{Nc}`\beta) +_c \mathrm{Nc}`\beta .$

$[(2) . *10\cdot1] \quad \supset . (\mathrm{Nc}`\alpha -_c \mathrm{Nc}`\beta) +_c \mathrm{Nc}`\beta = \mathrm{Nc}`\alpha$ (4)

$\vdash . (3) . (4) . \supset \vdash . \mathrm{Prop}$

***119·34.** $\vdash : \gamma -_c \nu \in \mathrm{N_0C} . \supset . (\gamma -_c \nu) +_c \nu = \mathrm{sm}``\gamma$
$[*119\cdot11\cdot33 . *103\cdot2 . *100\cdot51 . *118\cdot01]$

***119·35.** $\vdash : \gamma -_c \nu \in \mathrm{N_0C} . \supset . \alpha +_c \gamma = (\alpha +_c \nu) +_c (\gamma -_c \nu)$

Dem.

$\vdash . *110\cdot51\cdot56 . \supset \vdash : \mathrm{Hp} . \supset . (\alpha +_c \nu) +_c (\gamma -_c \nu) = \alpha +_c \{(\gamma -_c \nu) +_c \nu\}$

$[*119\cdot34] \quad = \alpha +_c \mathrm{sm}``\gamma$

$[*118\cdot24 . *119\cdot11] \quad = \alpha +_c \gamma : \supset \vdash . \mathrm{Prop}$

***119·41.** $\vdash :. \delta \in \mathrm{Nc}`\beta -_c \mathrm{Nc}`\gamma . \supset :$

$\xi \in (\mathrm{Nc}`\alpha +_c \mathrm{Nc}`\beta) -_c \mathrm{Nc}`\gamma . \equiv . \{(\alpha + \delta) + \gamma\} \,\mathrm{sm}\, (\xi + \gamma)$

Dem.

$\vdash . *119\cdot12 . *110\cdot3 . \supset \vdash : \xi \in (\mathrm{Nc}`\alpha +_c \mathrm{Nc}`\beta) -_c \mathrm{Nc}`\gamma . \equiv . (\alpha + \beta) \,\mathrm{sm}\, (\xi + \gamma)$ (1)

$\vdash . *119\cdot12 . \quad \supset \vdash : \mathrm{Hp} . \equiv . \beta \,\mathrm{sm}\, (\delta + \gamma)$ (2)

$\vdash . (1) . (2) . *110\cdot15\cdot53 . \supset \vdash . \mathrm{Prop}$

***119·42.** $\vdash :. \mathrm{Nc}`\beta -_0 \mathrm{Nc}`\gamma \in \mathrm{N_0C} . \eta \in \mathrm{Nc}`\alpha +_0 (\mathrm{Nc}`\beta -_0 \mathrm{Nc}`\gamma) . \supset :$

$$\xi \in (\mathrm{Nc}`\alpha +_0 \mathrm{Nc}`\beta) -_0 \mathrm{Nc}`\gamma . \equiv . (\eta + \gamma) \,\mathrm{sm}\, (\xi + \gamma)$$

Dem.

$\vdash . *118·01 . *110·3 . *103·2 . *100·31 . \supset \vdash :. \mathrm{N_0c}`\delta = \mathrm{Nc}`\beta -_0 \mathrm{Nc}`\gamma . \supset :$

$$\eta \in \mathrm{Nc}`\alpha +_0 (\mathrm{Nc}`\beta -_0 \mathrm{Nc}`\gamma) . \equiv . \eta \,\mathrm{sm}\, (\alpha + \delta) \quad (1)$$

$\vdash . *119·41 . (1) . *103·12 . *110·15 . \supset \vdash . \mathrm{Prop}$

Note that if γ be an infinite class, it does not follow from $(\eta + \gamma) \,\mathrm{sm}\, (\xi + \gamma)$ that $\eta \,\mathrm{sm}\, \xi$. This will be proved, however, when γ is an inductive class (cf. *120·41).

***119·43.** $\vdash : \mathrm{Nc}`\beta -_0 \mathrm{Nc}`\gamma \in \mathrm{N_0C} . \supset .$

$$\mathrm{Nc}`\alpha +_0 (\mathrm{Nc}`\beta -_0 \mathrm{Nc}`\gamma) \subset (\mathrm{Nc}`\alpha +_0 \mathrm{Nc}`\beta) -_0 \mathrm{Nc}`\gamma$$

Dem.

$\vdash . *119·42 . \supset \vdash :. \mathrm{Hp} . \eta \in \mathrm{Nc}`\alpha +_0 (\mathrm{Nc}`\beta -_0 \mathrm{Nc}`\gamma) . \supset :$

$$\eta \in (\mathrm{Nc}`\alpha +_0 \mathrm{Nc}`\beta) -_0 \mathrm{Nc}`\gamma . \equiv . (\eta + \gamma) \,\mathrm{sm}\, (\eta + \gamma) :$$

[*73·3] $\supset : \eta \in (\mathrm{Nc}`\alpha +_0 \mathrm{Nc}`\beta) -_0 \mathrm{Nc}`\gamma$ (1)

$\vdash . (1) . *22·1 . \supset \vdash . \mathrm{Prop}$

***119·44.** $\vdash : \mu +_0 (\nu -_0 \varpi) \subset (\mu +_0 \nu) -_0 \varpi$

Dem.

$\vdash . *119·11·43 . *103·2 . \supset$

$\vdash : \nu -_0 \varpi \in \mathrm{N_0C} . \mu \in \mathrm{N_0C} . \supset . \mu +_0 (\nu -_0 \varpi) \subset (\mu +_0 \nu) -_0 \varpi$ (1)

$\vdash . *110·4·42 . *119·11 . \supset$

$\vdash : \sim \{\nu -_0 \varpi \in \mathrm{N_0C} . \mu \in \mathrm{N_0C}\} . \supset . \mu +_0 (\nu -_0 \varpi) = \Lambda .$

[*24·12] $\supset . \mu +_0 (\nu -_0 \varpi) \subset (\mu +_0 \nu) -_0 \varpi$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***119·45.** $\vdash : (\mu +_0 \nu) -_0 \varpi \in \mathrm{NC} . \exists ! \{\mu +_0 (\nu -_0 \varpi)\} . \supset . \mu +_0 (\nu -_0 \varpi) = (\mu +_0 \nu) -_0 \varpi$
[*119·44 . *100·33·321 . *110·42]

***119·51.** $\vdash : \mathrm{sm}_{\delta,\gamma}``(\mathrm{Nc}`\alpha -_0 \mathrm{Nc}`\beta)_\gamma = (\mathrm{Nc}`\alpha -_0 \mathrm{Nc}`\beta)_\delta \cap \mathrm{D}`\mathrm{sm}_{\delta,\gamma}$

Dem.

$\vdash . *119·12 . \supset \vdash : \eta \in (\mathrm{Nc}`\alpha -_0 \mathrm{Nc}`\beta)_\gamma . \zeta \,\mathrm{sm}_{\delta,\gamma}\, \eta . \equiv . \alpha \,\mathrm{sm}\, \eta + \beta . \zeta \,\mathrm{sm}_{\delta,\gamma}\, \eta .$

[*110·15] $\equiv . \alpha \,\mathrm{sm}\, \zeta + \beta . \zeta \,\mathrm{sm}_{\delta,\gamma}\, \eta .$

[*119·12] $\equiv . \zeta \in (\mathrm{Nc}`\alpha -_0 \mathrm{Nc}`\beta)_\delta . \zeta \,\mathrm{sm}_{\delta,\gamma}\, \eta :$

[*37·1.*33·13] $\supset \vdash . \mathrm{sm}_{\delta,\gamma}``(\mathrm{Nc}`\alpha -_0 \mathrm{Nc}`\beta)_\gamma = (\mathrm{Nc}`\alpha -_0 \mathrm{Nc}`\beta)_\delta \cap \mathrm{D}`\mathrm{sm}_{\delta,\gamma} : \supset \vdash . \mathrm{Prop}$

***119·52.** $\vdash : \mathrm{sm}_{\delta,\gamma}``(\mu -_0 \nu)_\gamma = (\mu -_0 \nu)_\delta \cap \mathrm{D}`\mathrm{sm}_{\delta,\gamma}$ [*119·51·11]

The difficulty in respect to types, which arises from the fact that $\mathrm{sm}_{\delta,\gamma}``(\mu -_0 \nu)_\gamma$ and $(\mu -_0 \nu)_\delta$ have not been proved to be identical, does not exist when ν is an "inductive number"; cf. *120·413.

***119·53.** $\vdash :. t`\delta = \mathrm{D}`\mathrm{sm}_{\delta,\gamma} . \supset : \mathrm{sm}_{\delta,\gamma}``(\mu -_0 \nu)_\gamma = (\mu -_0 \nu)_\delta$ [*119·52 . (*65·01)]

***119·531** $\vdash : t`\delta = \mathrm{D}`\mathrm{sm}_{\delta,\gamma} . (\mu -_0 \nu)_\delta \in \mathrm{N_0C} . \supset . \mathrm{sm}_{\gamma,\delta}``(\mu -_0 \nu)_\delta \in \mathrm{N_0C}$

Dem.

$\vdash . *65·13 . \quad \supset \vdash : \mathrm{Hp} . \supset . (\mu -_0 \nu)_\delta \subset \mathrm{D}`\mathrm{sm}_{\delta,\gamma} .$

[*37·43.*103·22.(*65·1)] $\supset . \exists ! \mathrm{sm}_{\gamma,\delta}``(\mu -_0 \nu)_\delta .$

[*100·52.*103·34] $\supset . \mathrm{sm}_{\gamma,\delta}``(\mu -_0 \nu)_\delta \in \mathrm{N_0C} : \supset \vdash . \mathrm{Prop}$

***119·532.** $\vdash : t\text{‘}\delta = \mathrm{D}\text{‘}\mathrm{sm}_{\delta\,\gamma} \,.\, (\mu -_0 \nu)_\delta \in \mathrm{N}_0\mathrm{C} \,.\, (\mu -_0 \nu)_\gamma \in \mathrm{NC} \,.\supset .$
$\mathrm{sm}_{\gamma,\delta}\text{“}(\mu -_0 \nu)_\delta = (\mu -_0 \nu)_\gamma$

Dem.

$\vdash . *119\cdot52\cdot531 . \supset \vdash : \mathrm{Hp} . \supset . \exists ! (\mu -_0 \nu)_\gamma .$
$[*119\cdot52\cdot531 . *100\cdot34] \quad \supset . \mathrm{sm}_{\gamma,\delta}\text{“}(\mu -_0 \nu)_\delta = (\mu -_0 \nu)_\gamma : \supset \vdash . \mathrm{Prop}$

***119·54.** $\mathrm{SM}(\delta, \gamma) . = : t\text{‘}\delta = \mathrm{D}\text{‘}\mathrm{sm}_{\delta,\gamma} . \vee . t\text{‘}\gamma = \mathrm{D}\text{‘}\mathrm{sm}_{\gamma,\delta}$ Df

***119·541.** $\vdash : \mathrm{SM}(\delta, \gamma) . (\mu -_0 \nu)_\gamma \in \mathrm{N}_0\mathrm{C} . (\mu -_0 \nu)_\delta \in \mathrm{NC} . \supset .$
$\mathrm{sm}_{\delta,\gamma}\text{“}(\mu -_0 \nu)_\gamma = (\mu -_0 \nu)_\delta$ [*119·53·532]

***119·61.** $\vdash : \mu \in \mathrm{N}_0\mathrm{C} . \exists ! \mathrm{sm}_\xi\text{“}\mu . \supset . \mu -_c \nu = \mathrm{sm}_\xi\text{“}\mu -_0 \nu$

Dem.

$\vdash . *119\cdot1 . \supset \vdash :. \mathrm{Hp} . \supset : \eta \in \mu -_0 \nu . \equiv . \mathrm{Nc}\text{‘}\eta +_0 \nu = \mu . \exists ! \mu .$
$[*103\cdot16 . *118\cdot201 . *37\cdot29] \quad \equiv . (\mathrm{Nc}\text{‘}\eta +_0 \nu)_\xi = \mathrm{sm}_\xi\text{“}\mu .$
$[*119\cdot1] \quad \equiv . \eta \in \mathrm{sm}_\xi\text{“}\mu -_0 \nu :. \supset \vdash . \mathrm{Prop}$

***119·62.** $\vdash : \nu \in \mathrm{N}_0\mathrm{C} . \exists ! \mathrm{sm}_\xi\text{“}\nu . \supset . \mu -_0 \nu = \mu -_0 \mathrm{sm}_\xi\text{“}\nu$

Dem.

$\vdash . *119\cdot1 . \supset \vdash :. \mathrm{Hp} . \supset : \eta \in \mu -_0 \nu . \equiv . \mathrm{Nc}\text{‘}\eta +_0 \nu = \mu . \exists ! \mu .$
$[*110\cdot25] \quad \equiv . \mathrm{Nc}\text{‘}\eta +_0 \mathrm{sm}_\xi\text{“}\nu = \mu . \exists ! \mu .$
$[*119\cdot1] \quad \equiv . \eta \in \mu -_0 \mathrm{sm}_\xi\text{“}\nu :. \supset \vdash . \mathrm{Prop}$

***119·63.** $\vdash : \mu, \nu \in \mathrm{N}_0\mathrm{C} . \exists ! \mathrm{sm}_\xi\text{“}\mu . \supset . \mu -_0 \nu = \mathrm{sm}_\xi\text{“}\mu -_0 \mathrm{sm}_\xi\text{“}\nu$

Dem.

$\vdash . *119\cdot26 . \supset \vdash : \mathrm{Hp} . \exists ! \mu -_0 \nu . \supset . \mu \geqslant \nu .$
$[*118\cdot13] \quad \supset . \exists ! \mathrm{sm}_\xi\text{“}\nu .$
$[*119\cdot61\cdot62] \quad \supset . \mu -_0 \nu = \mathrm{sm}_\xi\text{“}\mu -_0 \mathrm{sm}_\xi\text{“}\nu$ (1)
$\vdash . *119\cdot11 . *103\cdot13 . \supset$
$\vdash : \mathrm{Hp} . \exists ! \mathrm{sm}_\xi\text{“}\mu -_0 \mathrm{sm}_\xi\text{“}\nu . \supset . \exists ! \mathrm{sm}_\xi\text{“}\nu .$
$[*119\cdot61\cdot62] \quad \supset . \mu -_0 \nu = \mathrm{sm}_\xi\text{“}\mu -_0 \mathrm{sm}_\xi\text{“}\nu$ (2)
$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***119·64.** $\vdash :. \exists ! \mathrm{sm}_\xi\text{“}\mu . \supset : \mu \geqslant \nu . \equiv . \exists ! (\mu -_0 \nu)_\xi$

Dem.

$\vdash . *117\cdot24 . \quad \supset \vdash :. \mathrm{Hp} . \supset : \mu \geqslant \nu . \supset . \mu, \nu \in \mathrm{N}_0\mathrm{C} . \exists ! \mathrm{sm}_\xi\text{“}\mu .$
$[*119\cdot61] \quad \supset . (\mu -_0 \nu)_\xi = (\mathrm{sm}_\xi\text{“}\mu -_0 \nu)_\xi$ (1)
$\vdash . *117\cdot24\cdot244 . \supset \vdash :. \mathrm{Hp} . \supset : \mu \geqslant \nu . \supset . \mathrm{sm}_\xi\text{“}\mu \geqslant \nu .$
$[*119\cdot27] \quad \supset . \exists ! (\mathrm{sm}_\xi\text{“}\mu -_0 \nu)_\xi .$
$[(1)] \quad \supset . \exists ! (\mu -_0 \nu)_\xi$ (2)
$\vdash . (2) . *119\cdot26 . \supset \vdash . \mathrm{Prop}$

$*120$. INDUCTIVE CARDINALS

Summary of $*120$.

Inductive Cardinals are those that obey mathematical induction starting from 0, *i.e.* in the language of Part II, Section E, they are the posterity of 0 with respect to the relation of ν to $\nu +_0 1$, or, in more popular language, they are those that can be reached from 0 by successive additions of 1. In former days, these were supposed to be all the cardinals, and mathematical induction was treated as a kind of self-evident axiom. We now know that only certain cardinals obey mathematical induction starting from 0. It is these cardinals which are to be considered in this number. They embrace 0, 1, 2, ... and generally all those cardinals which would be commonly called finite, all those which can be expressed in the usual Arabic system of numeration, and no others. The propositions to be proved concerning them in this number are elementary and familiar; the interest lies entirely in the definition and method of proof, not in the propositions themselves.

Put $\qquad \text{NC induct} = \hat{\alpha}\{\alpha(+_0 1)_* 0\} \quad \text{Df.}$

Since $(+_0 1)_*$ has necessarily its domain and converse domain of the same type, it is important to be careful in noting the relations of type. Accordingly we also put

$$\text{N}_\xi\text{C induct} = \hat{\alpha}\{\alpha(+_0 1)_* 0_\xi\} \quad \text{Df.}$$

We begin by applying the propositions of $*90$. Thus we have

$*120{\cdot}11$. $\vdash :. \alpha \epsilon \text{N}_\eta\text{C induct} : \phi\xi . \supset_\xi . \phi(\xi +_0 1) : \phi 0_\eta : \supset . \phi\alpha$

$*120{\cdot}12$. $\vdash . 0 \epsilon \text{NC induct}$

$*120{\cdot}121$. $\vdash : \alpha \epsilon \text{N}_\xi\text{C induct} . \supset . (\alpha +_0 1)_\xi \epsilon \text{N}_\xi\text{C induct}$

$*120{\cdot}13$. $\vdash :. \alpha \epsilon \text{N}_\eta\text{C induct} : \xi \epsilon \text{N}_\eta\text{C induct} . \phi\xi . \supset_\xi . \phi(\xi +_0 1) : \phi 0_\eta : \supset . \phi\alpha$

$*120{\cdot}15$. $\vdash : \alpha \epsilon \text{NC induct} . \exists ! \alpha . \supset . \text{sm}``\alpha \epsilon \text{NC induct}$

$*120{\cdot}151$. $\vdash : \alpha \epsilon \text{NC induct} . \exists ! \alpha . \supset . \alpha +_0 1 \epsilon \text{NC induct}$

$*120{\cdot}152$. $\vdash : \alpha \epsilon \text{NC} . \text{sm}``\alpha \epsilon \text{NC induct} - \iota`\Lambda . \supset . \alpha \epsilon \text{NC induct} - \iota`\Lambda$

We then proceed to deduce the elementary properties of inductive *classes*, putting

$$\text{Cls induct} = s`\text{NC induct}.$$

We have

$*120{\cdot}21$. $\vdash : \rho \epsilon \text{Cls induct} . \equiv . \text{N}_0\text{c}`\rho \epsilon \text{NC induct}$

$*120{\cdot}211$. $\vdash : \text{Nc}`\rho \epsilon \text{NC induct} - \iota`\Lambda . \supset . \rho \epsilon \text{Cls induct}$

(We do not have an equivalence here, because, for aught we know, it might be possible to determine the ambiguity of $\text{Nc}`\rho$ so that $\text{Nc}`\rho = \Lambda$, even when $\rho \epsilon \text{Cls induct}$. This will not be possible, however, if the axiom of infinity is assumed.)

$*120{\cdot}212{\cdot}213.$ $\vdash . \Lambda, \iota`x \in \text{Cls induct}$

$*120{\cdot}214.$ $\vdash :. \rho \,\text{sm}\, \sigma . \supset : \rho \in \text{Cls induct} . \equiv . \sigma \in \text{Cls induct}$

We have a set of propositions applying induction to classes *directly*, and not through the intermediary of cardinals. Thus we have

$*120{\cdot}251.$ $\vdash : \eta \in \text{Cls induct} . \supset . \eta \cup \iota`y \in \text{Cls induct}$

$*120{\cdot}26.$ $\vdash :. \rho \in \text{Cls induct} : \phi\eta . \supset_{\eta, x} . \phi(\eta \cup \iota`x) : \phi\Lambda : \supset . \phi\rho$

We then state the axiom of infinity, and prove ($*120{\cdot}33$) that it is equivalent to the assumption that if α is an inductive cardinal, $\alpha \neq \alpha +_c 1$. To prove this, we first prove various propositions about $\alpha +_c 1$, among others the following:

$*120{\cdot}311.$ $\vdash : \exists ! \alpha +_c 1 . \alpha +_c 1 = \beta +_c 1 . \supset . \alpha = \text{sm}``\beta . \exists ! \alpha$

$*120{\cdot}322.$ $\vdash :. \alpha \in \text{NC induct} . \supset : \exists ! \alpha . \equiv . \alpha \neq \alpha +_c 1$

We then proceed to consider subtraction ($*120{\cdot}41$—$\cdot418$), which only gives a cardinal number when the subtrahend is an inductive cardinal. We have

$*120{\cdot}41.$ $\vdash :. \nu \in \text{NC induct} . \exists ! \alpha +_c \nu . \supset : \alpha +_c \nu = \beta +_c \nu . \supset . \alpha = \text{sm}``\beta$

We might validly put $\alpha = \beta$ instead of $\alpha = \text{sm}``\beta$, since $\alpha = \beta$ will be true whenever it is significant.

We have

$*120{\cdot}411.$ $\vdash :. \nu \in \text{NC induct} . \supset :$

$$\exists ! \gamma -_c \nu . \supset . \gamma -_c \nu \in \text{N}_0\text{C} : \gamma \geqslant \nu . \equiv . (\gamma -_c \nu) \cap t_0`\gamma \in \text{N}_0\text{C}$$

$*120{\cdot}4111.$ $\vdash :. \nu \in \text{NC induct} . \exists ! \text{sm}_\xi``\gamma . \supset : \gamma \geqslant \nu . \equiv . (\gamma -_c \nu)_\xi \in \text{N}_0\text{C}$

Hence we arrive at the conditions requisite for the usual point of view of subtraction; namely,

$*120{\cdot}412.$ $\vdash : \nu \in \text{NC induct} . \gamma \geqslant \nu . \exists ! \text{sm}_\xi``\gamma . \supset . (\gamma -_c \nu)_\xi = \{(\imath\alpha)(\alpha +_c \nu = \gamma)\}_\xi$

Also from $*120{\cdot}4111$ we deduce

$*120{\cdot}414.$ $\vdash : \mu \in \text{N}_0\text{C} - \iota`0 . \exists ! \text{sm}_\xi``\mu . \supset . (\mu -_c 1)_\xi \in \text{N}_0\text{C}$

And from $*120{\cdot}411 . *119{\cdot}34$, we find

$*120{\cdot}416.$ $\vdash : \nu \in \text{NC induct} . \exists ! \gamma -_c \nu . \supset . (\gamma -_c \nu) +_c \nu = \text{sm}``\gamma$

We prove next that no proper part of an inductive class is similar to the whole ($*120{\cdot}426$), *i.e.* that inductive classes are non-reflexive, and various connected propositions, *e.g.*

$*120{\cdot}423.$ $\vdash : \alpha \in \text{N}_\eta\text{C induct} - \iota`0 . \equiv . (\exists\beta) . \beta \in \text{N}_\eta\text{C induct} . \alpha = (\beta +_c 1)_\eta$

$*120{\cdot}4232.$ $\vdash : \alpha \in \text{N}_\eta\text{C induct} - \iota`0 . \equiv . (\exists\beta) . \beta \in \text{N}_\eta\text{C induct} - \iota`\Lambda . \alpha = (\beta +_c 1)_\eta$

$*120{\cdot}428.$ $\vdash : \nu \in \text{NC induct} . \exists ! \alpha +_c \nu . \alpha \neq 0 . \supset . \alpha +_c \nu > \nu$

$*120{\cdot}429.$ $\vdash :. \nu \in \text{NC induct} . \supset : \mu > \nu . \equiv . \mu \geqslant \nu +_c 1$

The last two of the above propositions do not hold in general when ν is a cardinal which is not inductive.

We prove next that if α is an existent inductive cardinal, then any existent cardinal is greater than, equal to, or less than α (*120·441); that if α, β are inductive cardinals, so is $\alpha +_c \beta$ (*120·45·450), and if $\alpha +_c \beta$ is an inductive cardinal other than Λ, so are α and β (*120·452). We then have some propositions dealing with mathematical induction starting from 1 or 2, *e.g.*

***120·4622.** $\vdash :. \alpha \in \mathrm{NC} . \beta \in \mathrm{NC}(\eta) . \exists ! \, \mathrm{sm}_\xi ``\beta . \supset :$
$$\beta \, (+_c 1)_* \, \mathrm{sm}_\eta ``\alpha . \equiv . \mathrm{sm}_\xi ``\beta \, (+_c 1)_* \, \mathrm{sm}_\xi ``\alpha$$

***120·47.** $\vdash :: \beta \in \mathrm{N}_\eta \mathrm{C} \, \mathrm{induct} - \iota`0 . \equiv :. \xi \in \mu . \supset_\xi . (\xi +_c 1)_\eta \in \mu : 1_\eta \in \mu : \supset_\mu . \beta \in \mu$

From *120·452 we deduce

***120·48.** $\vdash : \beta \in \mathrm{NC} \, \mathrm{induct} . \beta \geqslant \alpha . \supset . \alpha \in \mathrm{NC} \, \mathrm{induct} - \iota`\Lambda$

so that any number less than an inductive number is inductive. Hence

***120·481.** $\vdash : \eta \in \mathrm{Cls} \, \mathrm{induct} . \xi \subset \eta . \supset . \xi \in \mathrm{Cls} \, \mathrm{induct}$

which is a proposition constantly used, and

***120·491.** $\vdash :. \xi \sim\in \mathrm{Cls} \, \mathrm{induct} . \equiv : \beta \in \mathrm{NC} \, \mathrm{induct} . \supset_\beta . \exists ! \, \beta \cap \mathrm{Cl}`\xi$

We then prove that if α, β are inductive cardinals, $\alpha \times_c \beta$ and α^β are either inductive cardinals or Λ (*120·5·52), while conversely if $\alpha \times_c \beta$ or α^β is an existent inductive cardinal, α and β are so also, with exceptions for 0 and 1 (*120·512·56·561). Hence we infer the uniqueness of division and the taking of roots (*120·51·53·55) so long as inductive numbers are concerned.

We have next a set of propositions on the axiom of infinity and the multiplicative axiom. We prove (*120·61) that if there is any existent cardinal which is not inductive, the axiom of infinity is true. From *83·9·904, we infer by induction that if κ is an inductive class of which Λ is not a number, $\epsilon_\Delta`\kappa$ exists (*120·62), whence it follows that either the multiplicative axiom or the axiom of infinity must be true (*120·64).

Finally, we have a set of propositions on inductive *classes*. We prove

***120·71.** $\vdash : \rho, \sigma \in \mathrm{Cls} \, \mathrm{induct} . \equiv . \rho \cup \sigma \in \mathrm{Cls} \, \mathrm{induct} . \equiv . \rho + \sigma \in \mathrm{Cls} \, \mathrm{induct}$

***120·74.** $\vdash : \rho \in \mathrm{Cls} \, \mathrm{induct} . \equiv . \mathrm{Cl}`\rho \in \mathrm{Cls} \, \mathrm{induct}$

***120·75.** $\vdash : s`\kappa \in \mathrm{Cls} \, \mathrm{induct} . \equiv . \kappa \in \mathrm{Cls} \, \mathrm{induct} . \kappa \subset \mathrm{Cls} \, \mathrm{induct}$

with analogous propositions (involving however a hypothesis as to κ) on the subject of $\epsilon_\Delta`\kappa$.

The propositions of the present number are essential to the ordinary arithmetic of finite numbers. In the present work, however, they are not much used after the present section until we reach Part V, Section E, where we deal with the ordinal theory of finite and infinite.

∗120·01. $\text{NC induct} = \hat{\alpha}\{\alpha\,(+_c 1)_* \,0\}$ Df

Note that in virtue of our general conventions for descriptive functions of two arguments (∗38),

$$+_c 1 = \hat{\alpha}\hat{\beta}\,(\alpha = \beta +_c 1).$$

That is, $+_c 1$ is the relation of a cardinal to its immediate predecessor. It is the number written in the usual mathematical notation as $+1$ in the series of positive and negative integers, just as its converse is the number -1. (It should be observed that if ν is any cardinal, $+\nu$ is not identical with ν, since $+\nu$ is a relation, while ν is a class of classes.)

∗120·011. $\text{N}_\xi\text{C induct} = \hat{\alpha}\{\alpha\,(+_c 1)_* \,0_\xi\}$ Df

All members of N_ξC induct belong to the same type as 0_ξ, so that, if α is any member of N_ξC induct, "$\xi \in \alpha$" is significant.

∗120·02. $\text{Cls induct} = s\text{‘NC induct}$ Df

∗120·021. $\text{Cls}_\xi\,\text{induct} = s\text{‘N}_\xi\text{C induct}$ Df

In virtue of these definitions an inductive class is one whose cardinal is an inductive cardinal.

∗120·03. $\text{Infin ax} \,.=:\, \alpha \in \text{NC induct} \,.\supset_\alpha .\, \exists!\,\alpha$ Df

"Infin ax," like "Mult ax," is an arithmetical hypothesis which some will consider self-evident, but which we prefer to keep as a hypothesis, and to adduce in that form whenever it is relevant. Like "Mult ax," it states an existence-theorem. In the above form, it states that, if α is any inductive cardinal, there is at least one class (of the type in question) which has α terms. An equivalent assumption would be that, if ρ is any inductive class, there are objects which are not members of ρ. For in that case, if x be such an object, $\text{Nc‘}(\rho \cup \iota\text{‘}x) = \text{Nc‘}\rho +_c 1$. Hence by induction, every inductive cardinal must exist. Another equivalent assumption would be that V (the class of all objects of the type in question) is not an inductive class. The assumption that $\aleph_0$ exists in the type in question is, as we shall see, a stronger assumption than the above, unless we assume the multiplicative axiom.

If the axiom of infinity is true, the inductive cardinals are all different one from another, *i.e.* $\alpha +_c \beta$, where α and β are inductive cardinals, is not equal to α unless $\beta = 0$. But if the axiom of infinity is false, then, in any assigned type, all the cardinals after a certain one are Λ. (Except in the lowest type, the last existent cardinal must be a power of 2.) That is, if (say) 8 were the largest existent cardinal in the type in question, we should have, in that type, $9 = \Lambda$, and the same would hold of 10, 11, ... This possibility has to be taken account of in what follows.

In order to give typical definiteness to the axiom of infinity, we write

***120·04.** Infin ax $(x) . = : \alpha \epsilon \text{NC induct} . \supset_\alpha . \exists ! \alpha (x)$ Df

Then "Infin ax (x)" states that, if α is any inductive cardinal, there are at least α objects of the same type as x.

***120·1.** $\vdash : \alpha \epsilon \text{NC induct} . \equiv . \alpha (+_c 1)_* 0$ [(*120·01)]

***120·101.** $\vdash :: \alpha \epsilon \text{NC induct} . \equiv :. \xi \epsilon \mu . \supset_\xi . \xi +_c 1 \epsilon \mu : 0 \epsilon \mu : \supset_\mu . \alpha \epsilon \mu$
[*120·1 . *90·131 . *38·12]

The right-hand side of the above equivalence gives the usual formula for mathematical induction. Observe that the conditions of significance require that $\xi +_c 1$ should be taken in the same type as ξ. This fact is specially relevant in the proof of *120·15.

The symbol "NC induct" is of ambiguous type not necessarily the same in different occurrences; also, according to the convention explained in the prefatory statement as holding for NC and NC induct, "$\alpha, \beta \epsilon \text{NC induct}$" will not imply that α and β are of the same type. Accordingly to avoid error in connection with *120·1·101 typical definiteness is required as in the three following propositions.

***120·102.** $\vdash : \alpha \epsilon \text{N}_\eta\text{C induct} . \equiv . \alpha (+_c 1)_* 0_\eta$ [(*120·011)]

***120·103.** $\vdash :: \alpha \epsilon \text{N}_\eta\text{C induct} . \equiv :. \xi \epsilon \mu . \supset_\xi . (\xi +_c 1)_\eta \epsilon \mu : 0_\eta \epsilon \mu : \supset_\mu . \alpha \epsilon \mu$
[*120·101]

***120·11.** $\vdash :. \alpha \epsilon \text{N}_\eta\text{C induct} : \phi\xi . \supset_\xi . \phi (\xi +_c 1) : \phi 0_\eta : \supset . \phi\alpha$
[*120·102 . *90·112]

***120·12.** $\vdash . 0 \epsilon \text{NC induct}$ $\left[*120{\cdot}101 \frac{0}{\alpha}\right]$

***120·121.** $\vdash : \alpha \epsilon \text{N}_\xi\text{C induct} . \supset . (\alpha +_c 1)_\xi \epsilon \text{N}_\xi\text{C induct}$ [*90·172 . *120·102]

By means of this proposition and *120·12, any assigned cardinal in the series of natural numbers can be shown to be an inductive cardinal; thus *e.g.* to show that 27 is an inductive cardinal, we shall only have to use *120·121 twenty-seven times in succession.

***120·122.** $\vdash . 1 \epsilon \text{NC induct}$ [*120·12·121 . *110·641]

***120·123.** $\vdash . 2 \epsilon \text{NC induct}$. etc. [*120·122·121 . *110·643]

***120·124.** $\vdash . \alpha +_c 1 \neq 0$

Dem.

$\vdash . *110{\cdot}4 . \text{Transp} . \supset \vdash : \alpha \sim\epsilon \text{NC} . \supset . \alpha +_c 1 = \Lambda .$
[*101·12] $\supset . \alpha +_c 1 \neq 0$ (1)
$\vdash . *110{\cdot}632 . \supset \vdash :. \alpha \epsilon \text{NC} . \supset : \xi \epsilon \alpha +_c 1 . \supset . \exists ! \xi :$
[*24·63] $\supset : \Lambda \sim\epsilon \alpha +_c 1 :$
[*54·102] $\supset : \alpha +_c 1 \neq 0$ (2)
$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

$*120\cdot13$. $\vdash :. \alpha \in N_\eta C \text{ induct} : \xi \in N_\eta C \text{ induct} . \phi\xi . \supset_\xi . \phi(\xi +_c 1) : \phi 0_\eta : \supset . \phi\alpha$

Dem.

$\vdash . *120\cdot121 . \supset \vdash :. \xi \in N_\eta C \text{ induct} . \phi\xi . \supset_\xi . \phi(\xi +_c 1) : \supset :$
$$\xi \in N_\eta C \text{ induct} . \phi\xi . \supset_\xi . (\xi +_c 1)_\eta \in N_\eta C \text{ induct} . \phi(\xi +_c 1) \quad (1)$$
$\vdash . *120\cdot12 . \supset \vdash : \phi 0_\eta . \supset . 0_\eta \in N_\eta C \text{ induct} . \phi 0_\eta \quad (2)$

$\vdash . (1) . (2) . \supset \vdash :. \text{Hp} . \supset :$

$\xi \in N_\eta C \text{induct} . \phi\xi . \supset_\xi . (\xi +_c 1)_\eta \in N_\eta C \text{induct} . \phi(\xi +_c 1) : 0_\eta \in N_\eta C \text{ induct} . \phi 0_\eta :$

$\left[*120\cdot11 \dfrac{\xi \in N_\eta C \text{ induct} . \phi\xi}{\phi\xi}\right] \supset : \alpha \in N_\eta C \text{ induct} . \phi\alpha :. \supset \vdash . \text{Prop}$

The above proposition is often convenient for inductive proofs.

$*120\cdot14$. $\vdash . \text{NC induct} \subset \text{NC}$

Dem.

$\vdash . *110\cdot42 . \text{Simp} . \supset \vdash : \alpha \in \text{NC} . \supset . \alpha +_c 1 \in \text{NC} \quad (1)$

$\vdash . (1) . *101\cdot11 . *120\cdot11 \dfrac{\alpha \in \text{NC}}{\phi\alpha} . \supset \vdash . \text{Prop}$

This proposition does not show that every inductive cardinal is an *existent* cardinal; to obtain this, we require the axiom of infinity.

$*120\cdot15$. $\vdash : \alpha \in \text{NC induct} . \exists ! \alpha . \supset . \text{sm}``\alpha \in \text{NC induct}$

I.e. a cardinal which is not null and is inductive in any one type is also inductive in any other type.

Dem.

$\vdash . *101\cdot15 . *120\cdot12 . \supset \vdash . \text{sm}_\eta``0_\xi \in N_\eta C \text{ induct} \quad (1)$

$\vdash . *110\cdot4 . \supset \vdash . \alpha = \Lambda_\xi . \supset . (\alpha +_c 1)_\xi = \Lambda_\xi \quad (2)$

$\vdash . *118\cdot201 . \supset \vdash : \exists ! (\alpha +_c 1)_\xi . \supset . \text{sm}_\eta``(\alpha +_c 1)_\xi = (\alpha +_c 1)_\eta$

$[*118\cdot241 . *110\cdot4] \quad = (\text{sm}_\eta``\alpha +_c 1)_\eta \quad (3)$

$\vdash . *120\cdot121 . \supset \vdash : \exists ! (\alpha +_c 1)_\xi . \text{sm}_\eta``\alpha \in N_\eta C \text{induct} . \supset . (\text{sm}_\eta``\alpha +_c 1)_\eta \in N_\eta C \text{ induct} .$

$[(3)] \quad \supset . \text{sm}_\eta``(\alpha +_c 1)_\xi \in N_\eta C \text{ induct} \quad (4)$

$\vdash . (4) . *2\cdot2 . \supset \vdash :. \text{sm}_\eta``\alpha \in N_\eta C \text{ induct} . \supset :$
$$(\alpha +_c 1)_\xi = \Lambda_\xi . \vee . \text{sm}_\eta``(\alpha +_c 1)_\xi \in N_\eta C \text{ induct} \quad (5)$$
$\vdash . (2) . (5) . *3\cdot48 . \supset \vdash :. \alpha = \Lambda_\xi . \vee . \text{sm}_\eta``\alpha \in N_\eta C \text{ induct} : \supset :$
$$(\alpha +_c 1)_\xi = \Lambda_\xi . \vee . \text{sm}_\eta``(\alpha +_c 1)_\xi \in N_\eta C \text{ induct} \quad (6)$$
$\vdash . (1) . (6) . *120\cdot11 . *4\cdot6 . \supset \vdash . \text{Prop}$

$*120\cdot151$. $\vdash : \alpha \in \text{NC induct} . \exists ! \alpha . \supset . \alpha +_c 1 \in \text{NC induct}$

Dem.

$\vdash . *120\cdot15 . \supset \vdash : \alpha \in N_\xi C \text{ induct} . \exists ! \alpha . \supset . \text{sm}_\eta``\alpha \in N_\eta C \text{ induct} .$

$[*120\cdot121] \quad \supset . (\text{sm}_\eta``\alpha +_c 1)_\eta \in N_\eta C \text{ induct} .$

$[*118\cdot241 . *120\cdot14] \quad \supset . (\alpha +_c 1)_\eta \in N_\eta C \text{ induct} : \supset \vdash . \text{Prop}$

$*120\cdot152$. $\vdash : \alpha \in \text{NC} . \text{sm}``\alpha \in \text{NC induct} - \iota`\Lambda . \supset . \alpha \in \text{NC induct} - \iota`\Lambda$

Dem.

$\vdash . *100\cdot521 . \supset \vdash : \text{Hp} . \supset . \text{sm}``\text{sm}``\alpha = \alpha .$

$[*120\cdot15] \quad \supset . \alpha \in \text{NC induct} \quad (1)$

$\vdash . *37\cdot29 . \supset \vdash : \text{Hp} . \supset . \exists ! \alpha \quad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

The following propositions, giving alternative forms for the definition of inductive *classes*, are inserted in order to show that the theory of inductive classes might be treated in a less arithmetical manner than we have adopted.

***120·2.** $\vdash : \rho \in \text{Cls induct} . \equiv . (\exists \alpha) . \alpha \in \text{NC induct} . \rho \in \alpha$ [(*120·02)]

***120·201.** $\vdash :. \rho \,\text{sm}\, \sigma . \supset : \text{N}_0\text{c}`\rho \in \text{NC induct} . \equiv . \text{N}_0\text{c}`\sigma \in \text{NC induct}$

Dem.

$\vdash . *100·35 . *103·13 . *100·511 . \supset$

$\vdash : \text{Hp} . \supset . \text{N}_0\text{c}`\rho = \text{sm}``\text{N}_0\text{c}`\sigma . \text{N}_0\text{c}`\sigma = \text{sm}``\text{N}_0\text{c}`\rho :$

[*120·152.*103·13] $\supset \vdash . \text{Prop}$

***120·21.** $\vdash : \rho \in \text{Cls induct} . \equiv . \text{N}_0\text{c}`\rho \in \text{NC induct}$

Dem.

$\vdash . *120·14·2 . \supset \vdash : \rho \in \text{Cls induct} . \equiv . (\exists \alpha) . \alpha \in \text{NC induct} . \alpha \in \text{NC} . \rho \in \alpha .$

[*103·27] $\equiv . (\exists \alpha) . \alpha \in \text{NC induct} . \text{N}_0\text{c}`\rho = \alpha .$

[*13·195] $\equiv . \text{N}_0\text{c}`\rho \in \text{NC induct} : \supset \vdash . \text{Prop}$

Note that "$\rho \in \text{Cls induct} . \equiv . \text{Nc}`\rho \in \text{NC induct}$" is not proved above. The proof encounters the difficulty that we may have $\text{Nc}`\rho = \Lambda$; in order to establish our proposition in this case, we have to show that if $\Lambda \in \text{NC induct}$, then *every* class is an inductive class. We can however prove the following implication.

***120·211.** $\vdash : \text{Nc}`\rho \in \text{NC induct} - \iota`\Lambda . \supset . \rho \in \text{Cls induct}$

Dem.

$\vdash . *100·511 . \supset \vdash : \text{Hp} . \supset . \text{sm}``\text{Nc}`\rho = \text{N}_0\text{c}`\rho .$

[*120·15] $\supset . \text{N}_0\text{c}`\rho \in \text{NC induct} .$

[*120·21] $\supset . \rho \in \text{Cls induct} : \supset \vdash . \text{Prop}$

***120·212.** $\vdash . \Lambda \in \text{Cls induct}$ [*120·211·12]

***120·213.** $\vdash . \iota`x \in \text{Cls induct}$ [*120·211·122]

***120·214.** $\vdash :. \rho \,\text{sm}\, \sigma . \supset : \rho \in \text{Cls induct} . \equiv . \sigma \in \text{Cls induct}$ [*120·201·21]

The following propositions are lemmas for *120·24.

***120·22.** $\vdash :: \eta \in \mu . \supset_{\eta, y} . \eta \cup \iota`y \in \mu : \Lambda \in \mu : \supset_\mu . \rho \in \mu :. \supset . \rho \in \text{Cls induct}$

Dem.

$\vdash . *120·212 . \supset \vdash . \Lambda \in \text{Cls induct}$ (1)

$\vdash . *51·2 . \supset \vdash :. y \in \eta . \supset : \eta \cup \iota`y = \eta :$

[*13·12] $\supset : \eta \in \text{Cls induct} . \supset . \eta \cup \iota`y \in \text{Cls induct}$ (2)

$\vdash . *110·63 . \supset \vdash :. y \sim\in \eta . \supset : \text{Nc}`(\eta \cup \iota`y) = \text{Nc}`\eta +_c 1$

[(*110·03)] $= \text{N}_0\text{c}`\eta +_c 1 :$

[*120·121] $\supset : \text{N}_0\text{c}`\eta \in \text{NC induct} . \supset . \text{N}_0\text{c}`(\eta \cup \iota`y) \in \text{NC induct} :$

[*120·21·211] $\supset : \eta \in \text{Cls induct} . \supset . \eta \cup \iota`y \in \text{Cls induct}$ (3)

$\vdash . (2) . (3) . \supset \vdash : \eta \in \text{Cls induct} . \supset . \eta \cup \iota`y \in \text{Cls induct}$ (4)

$\vdash . *10·1 . (1) . (4) . \supset \vdash . \text{Prop}$

***120·221.** $\vdash :. \eta \epsilon \mu . \supset_{\eta,y} . \eta \cup \iota{}^\backprime y \epsilon \mu : \mathrm{Nc}{}^\backprime\rho \subset \mu : \supset . \mathrm{Nc}{}^\backprime\rho +_0 1 \subset \mu$

Dem.

$\vdash . *110{\cdot}63 . *100{\cdot}31 . \supset$

$\vdash : \zeta \epsilon \mathrm{Nc}{}^\backprime\rho +_0 1 . \equiv . (\exists \eta, y) . \eta \epsilon \mathrm{Nc}{}^\backprime\rho . y \sim\epsilon \eta . \zeta = \eta \cup \iota{}^\backprime y$ (1)

$\vdash . *22{\cdot}1 . \quad \supset \vdash :. \mathrm{Hp} . \supset : \eta \epsilon \mathrm{Nc}{}^\backprime\rho . \supset . \eta \epsilon \mu .$

$[*10{\cdot}1] \quad \supset . \eta \cup \iota{}^\backprime y \epsilon \mu :$

$[*3{\cdot}41] \quad \supset : \eta \epsilon \mathrm{Nc}{}^\backprime\rho . y \sim\epsilon \eta . \supset . \eta \cup \iota{}^\backprime y \epsilon \mu :$

$[*13{\cdot}12] \quad \supset : \eta \epsilon \mathrm{Nc}{}^\backprime\rho . y \sim\epsilon \eta . \zeta = \eta \cup \iota{}^\backprime y . \supset . \zeta \epsilon \mu$ (2)

$\vdash . (1) . (2) . \supset \vdash :. \mathrm{Hp} . \supset : \zeta \epsilon \mathrm{Nc}{}^\backprime\rho +_0 1 . \supset . \zeta \epsilon \mu :. \supset \vdash . \mathrm{Prop}$

***120·222.** $\vdash :. \eta \epsilon \mu . \supset_{\eta,y} . \eta \cup \iota{}^\backprime y \epsilon \mu : \xi \epsilon \mathrm{NC} . \xi \subset \mu : \supset . \xi +_0 1 \subset \mu$

Dem.

$\vdash . *100{\cdot}4 . \supset \vdash : \mathrm{Hp} . \exists ! \xi . \supset . (\exists \alpha) . \xi = \mathrm{Nc}(\zeta){}^\backprime\alpha . \mathrm{Nc}(\zeta){}^\backprime\alpha \subset \mu .$

$[*120{\cdot}221] \quad \supset . (\exists \alpha) . \xi = \mathrm{Nc}(\zeta){}^\backprime\alpha . \mathrm{Nc}{}^\backprime\alpha +_0 1 \subset \mu .$

$[*118{\cdot}01] \quad \supset . \xi +_c 1 \subset \mu$ (1)

$\vdash . *110{\cdot}4 . \supset \vdash : \sim\exists ! \xi . \supset . \xi +_c 1 \subset \mu$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

The proof of this proposition might also proceed by the use of uniform formal numbers, employing *118·241.

***120·23.** $\vdash :. \eta \epsilon \mu . \supset_{\eta,y} . \eta \cup \iota{}^\backprime y \epsilon \mu : \Lambda \epsilon \mu : \supset . \mathrm{Cls\,induct} \subset \mu$

Dem.

$\vdash . *51{\cdot}2 . *54{\cdot}1 . \quad \supset \vdash : \mathrm{Hp} . \supset . 0 \subset \mu$ (1)

$\vdash . *120{\cdot}222{\cdot}14 . \quad \supset \vdash :. \mathrm{Hp} . \supset : \xi \epsilon \mathrm{NC\,induct} . \xi \subset \mu . \supset_\xi . \xi +_0 1 \subset \mu$ (2)

$\vdash . (1) . (2) . *120{\cdot}13 . \supset \vdash :. \mathrm{Hp} . \supset : \xi \epsilon \mathrm{NC\,induct} . \supset . \xi \subset \mu :$

$[*40{\cdot}151 . (*120{\cdot}02)] \quad \supset : \mathrm{Cls\,induct} \subset \mu :. \supset \vdash . \mathrm{Prop}$

***120·24.** $\vdash :: \rho \epsilon \mathrm{Cls\,induct} . \equiv :. \eta \epsilon \mu . \supset_{\eta,y} . \eta \cup \iota{}^\backprime y \epsilon \mu : \Lambda \epsilon \mu : \supset_\mu . \rho \epsilon \mu$

Dem.

$\vdash . *120{\cdot}23 . \supset \vdash :: \rho \epsilon \mathrm{Cls\,induct} . \supset :. \eta \epsilon \mu . \supset_{\eta,y} . \eta \cup \iota{}^\backprime y \epsilon \mu : \Lambda \epsilon \mu : \supset . \rho \epsilon \mu$ (1)

$\vdash . (1) . *120{\cdot}22 . \supset \vdash . \mathrm{Prop}$

This proposition might be used to define inductive classes. It gives a form of mathematical induction applicable to classes instead of to numbers. Virtually it states that an inductive class is one which can be formed by adding members one at a time, starting from Λ. This is made more explicit in *120·25. Instead of $\eta \epsilon \mu . \supset_{\eta,y} . \eta \cup \iota{}^\backprime y \epsilon \mu$, in the above propositions, as well as in those that follow, we may plainly substitute

$$\eta \epsilon \mu . y \sim\epsilon \eta . \supset_{\eta,y} . \eta \cup \iota{}^\backprime y \epsilon \mu .$$

***120·25.** $\vdash : M = \hat{\eta}\hat{\zeta}\{(\exists y) . \zeta = \eta \cup \iota{}^\backprime y\} . \supset . \mathrm{Cls\,induct} = \overleftarrow{M_*}{}^\backprime\Lambda$
$[*120{\cdot}24 . *90{\cdot}131]$

***120·251.** $\vdash : \eta \epsilon \mathrm{Cls\,induct} . \supset . \eta \cup \iota{}^\backprime y \epsilon \mathrm{Cls\,induct}$ $[*90{\cdot}172 . *120{\cdot}25]$

***120·26.** $\vdash :. \rho \epsilon \mathrm{Cls\,induct} : \phi\eta . \supset_{\eta,x} . \phi(\eta \cup \iota{}^\backprime x) : \phi\Lambda : \supset . \phi\rho$
$[*120{\cdot}25 . *90{\cdot}112]$

***120·261.** $\vdash :. \rho \in \text{Cls induct} : \eta \in \text{Cls induct} . \phi\eta . \supset_{\eta,x} . \phi(\eta \cup \iota`x) : \phi\Lambda : \supset . \phi\rho$
[*120·26·251·212]

***120·27.** $\vdash : \rho \in \text{Cls induct} . \supset . \text{Nc}`\rho \cap t`\gamma \in \text{NC induct}$

Dem.

$\vdash . *120{\cdot}12 . \supset \vdash . \text{Nc}`\Lambda \cap t`\gamma \in \text{NC induct}$ (1)

$\vdash . *13{\cdot}12 . \supset \vdash : \text{Nc}`\eta \cap t`\gamma \in \text{NC induct} . y \in \eta . \supset .$
$\text{Nc}`(\eta \cup \iota`y) \cap t`\gamma \in \text{NC induct}$ (2)

$\vdash . *110{\cdot}63 . *120{\cdot}121 . \supset$

$\vdash : \text{Nc}`\eta \cap t`\gamma \in \text{NC induct} . y \sim\in \eta . \supset . \text{Nc}`(\eta \cup \iota`y) \cap t`\gamma \in \text{NC induct}$ (3)

$\vdash . (1) . (2) . (3) . *120{\cdot}26 . \supset \vdash . \text{Prop}$

This proposition also follows immediately from *120·21·15.

***120·3.** $\vdash :. \text{Infin ax} . \equiv : \alpha \in \text{NC induct} . \supset_\alpha . \exists ! \alpha$ [(*120·03)]

***120·301.** $\vdash :. \text{Infin ax}(x) . \equiv : \alpha \in \text{NC induct} . \supset_\alpha . \exists ! \alpha(x)$ [(*120·04)]

***120·31.** $\vdash : \exists ! \text{Nc}`\alpha +_c 1 . \text{Nc}`\alpha +_c 1 = \text{Nc}`\beta +_c 1 . \supset . \text{Nc}`\alpha = \text{Nc}`\beta . \alpha \,\text{sm}\, \beta$

Dem.

$\vdash . *110{\cdot}63 . \supset \vdash :. \text{Nc}`\alpha +_c 1 = \text{Nc}`\beta +_c 1 . \equiv :$

$(\exists \gamma, y) . \gamma \,\text{sm}\, \alpha . y \sim\in \gamma . \xi = \gamma \cup \iota`y . \equiv_\xi . (\exists \delta, z) . \delta \,\text{sm}\, \beta . z \sim\in \delta . \xi = \delta \cup \iota`z :$

[*10·1] $\supset : \gamma \,\text{sm}\, \alpha . y \sim\in \gamma . \supset . (\exists \delta, z) . \delta \,\text{sm}\, \beta . z \sim\in \delta . \gamma \cup \iota`y = \delta \cup \iota`z .$

[*73·72·3] $\supset . (\exists \delta) . \delta \,\text{sm}\, \beta . \gamma \,\text{sm}\, \delta .$

[*73·32] $\supset . \gamma \,\text{sm}\, \beta .$

[*73·32] $\supset . \alpha \,\text{sm}\, \beta$ (1)

$\vdash . *110{\cdot}63 . \supset \vdash : \text{Hp} . \supset . (\exists \gamma, y) . \gamma \,\text{sm}\, \alpha . y \sim\in \gamma$ (2)

$\vdash . (1) . (2) . *100{\cdot}321 . \supset \vdash . \text{Prop}$

***120·311.** $\vdash : \exists ! \alpha +_c 1 . \alpha +_c 1 = \beta +_c 1 . \supset . \alpha = \text{sm}``\beta . \exists ! \alpha$
[*120·31 . *110·4 . *103·16·4·2]

***120·32.** $\vdash : \alpha \in \text{NC induct} . \exists ! \alpha . \supset . \alpha \neq \alpha +_c 1$

Dem.

$\vdash . *101{\cdot}22 . *110{\cdot}641 . \supset \vdash . 0_\xi \neq 0_\xi +_c 1$ (1)

$\vdash . *120{\cdot}311 . *110{\cdot}44 . \supset \vdash : \alpha \in \text{NC} . \exists ! \alpha +_c 1 . \alpha +_c 1 = \alpha +_c 1 +_c 1 . \supset . \alpha = \alpha +_c 1 :$

[Transp] $\supset \vdash : \alpha \in \text{NC} . \exists ! \alpha +_c 1 . \alpha \neq \alpha +_c 1 . \supset . \alpha +_c 1 \neq \alpha +_c 1 +_c 1 :$

[*118·2·25] $\supset \vdash : \alpha \in \text{NC}(\xi) . \exists ! (\alpha +_c 1)_\xi . \alpha \neq (\alpha +_c 1)_\xi . \supset . (\alpha +_c 1)_\xi \neq \{(\alpha +_c 1)_\xi +_c 1\}_\xi$ (2)

$\vdash . (2) . \supset \vdash :. \alpha \in \text{NC}(\xi) . \alpha \neq (\alpha +_c 1)_\xi . \supset : (\alpha +_c 1)_\xi = \Lambda . \vee . (\alpha +_c 1)_\xi \neq \{(\alpha +_c 1)_\xi +_c 1\}_\xi$ (3)

$\vdash . *110{\cdot}4 . \text{Transp} . \supset \vdash :. \alpha \sim\in \text{NC}(\xi) . \vee . \alpha = \Lambda_\xi : \supset . (\alpha +_c 1)_\xi = \Lambda_\xi$ (4)

$\vdash . (3) . (4) . \supset \vdash :. \alpha = \Lambda_\xi . \vee . \alpha \neq (\alpha +_c 1)_\xi : \supset :$
$(\alpha +_c 1)_\xi = \Lambda_\xi . \vee . (\alpha +_c 1)_\xi \neq \{(\alpha +_c 1)_\xi +_c 1\}_\xi$ (5)

$\vdash . (1) . (5) . *120{\cdot}11 . \supset \vdash :. \alpha \in \text{N}_\xi\text{C induct} . \supset : \alpha = \Lambda_\xi . \vee . \alpha \neq (\alpha +_c 1)_\xi :. \supset \vdash . \text{Prop}$

***120·321.** $\vdash : \alpha \neq \alpha +_c 1 . \supset . \exists ! \alpha$

Dem.

$\vdash . *110{\cdot}4 . \text{Transp} . \supset \vdash : \alpha = \Lambda . \supset . \alpha +_c 1 = \Lambda$ (1)

$\vdash . (1) . \text{Transp} . \quad \supset \vdash . \text{Prop}$

***120·322.** $\vdash :. \alpha \in \text{NC induct} . \supset : \exists ! \alpha . \equiv . \alpha \neq \alpha +_c 1$ [*120·32·321]

***120·33.** $\vdash :. \text{Infin ax} . \equiv : \alpha \in \text{NC induct} . \supset_\alpha . \alpha \neq \alpha +_c 1$ [*120·3·322]

***120·41.** $\vdash :. \nu \in \text{NC induct} . \exists ! \alpha +_c \nu . \supset : \alpha +_c \nu = \beta +_c \nu . \supset . \alpha = \mathrm{sm}``\beta$

Dem.

$\vdash . *110{\cdot}4 . \text{Transp} . *118{\cdot}25 . \supset \vdash : (\alpha +_c \nu)_\xi = \Lambda . \supset . \{\alpha +_c (\nu +_c 1)_\xi\}_\xi = \Lambda$ (1)

$\vdash . *118{\cdot}25 . \supset \vdash :: \exists ! \{\alpha +_c (\nu +_c 1)_\xi\}_\xi . \supset :. \exists ! \{(\alpha +_c \nu)_\xi +_c 1\}_\xi :.$

[*120·311.*110·4.*118·201]

$\supset :. \{(\alpha +_c \nu)_\xi +_c 1\}_\xi = \{(\beta +_c \nu)_\xi +_c 1\}_\xi . \supset . (\alpha +_c \nu)_\xi = (\beta +_c \nu)_\xi :.$

[Syll.*118·25] $\supset :. (\alpha +_c \nu)_\xi = (\beta +_c \nu)_\xi . \supset . \alpha = \mathrm{sm}``\beta : \supset :$

$\{\alpha +_c (\nu +_c 1)_\xi\}_\xi = \{\beta +_c (\nu +_c 1)_\xi\}_\xi . \supset . \alpha = \mathrm{sm}``\beta$ (2)

$\vdash . (2) . \text{Comm} . \supset \vdash :: (\alpha +_c \nu)_\xi = (\beta +_c \nu)_\xi . \supset . \alpha = \mathrm{sm}``\beta : \supset :.$

$\{\alpha +_c (\nu +_c 1)_\xi\}_\xi = \Lambda : \vee : \{\alpha +_c (\nu +_c 1)_\xi\}_\xi = \{\beta +_c (\nu +_c 1)_\xi\}_\xi . \supset . \alpha = \mathrm{sm}``\beta$ (3)

$\vdash . (1) . (3) . \supset \vdash :: (\alpha +_c \nu)_\xi = \Lambda : \vee : (\alpha +_c \nu)_\xi = (\beta +_c \nu)_\xi . \supset . \alpha = \mathrm{sm}``\beta :. \supset :.$

$\{\alpha +_c (\nu +_c 1)_\xi\}_\xi = \Lambda : \vee : \{\alpha +_c (\nu +_c 1)_\xi\}_\xi = \{\beta +_c (\nu +_c 1)_\xi\}_\xi . \supset . \alpha = \mathrm{sm}``\beta$ (4)

$\vdash . *110{\cdot}4 . *118{\cdot}21 . \supset \vdash :. \exists ! (\beta +_c 0)_\xi . \supset : \beta \in \text{NC} . \exists ! \mathrm{sm}_\xi``\beta :$

[*102·87.*100·51] $\supset : \mathrm{sm}_\xi``\alpha = \mathrm{sm}_\xi``\beta . \supset . \alpha = \mathrm{sm}``\beta$ (5)

$\vdash . *110{\cdot}64 . \supset \vdash : \exists ! (\alpha +_c 0)_\xi . (\alpha +_c 0)_\xi = (\beta +_c 0)_\xi . \supset . \mathrm{sm}_\xi``\alpha = \mathrm{sm}_\xi``\beta .$

[(5)] $\supset . \alpha = \mathrm{sm}``\beta$ (6)

$\vdash . (6) . \text{Exp} . *4{\cdot}6 . \supset \vdash :. (\alpha +_c 0)_\xi = \Lambda : \vee : (\alpha +_c 0)_\xi = (\beta +_c 0)_\xi . \supset . \alpha = \mathrm{sm}``\beta$ (7)

$\vdash . (4) . (7) . *120{\cdot}11 . \supset$

$\vdash :: \nu \in \text{N}_\xi\text{C induct} . \supset :. (\alpha +_c \nu)_\xi = \Lambda : \vee : (\alpha +_c \nu)_\xi = (\beta +_c \nu)_\xi . \supset . \alpha = \mathrm{sm}``\beta$ (8)

$\vdash . *110{\cdot}4 . \quad \supset \vdash : \nu = \Lambda_\eta . \supset . (\alpha +_c \nu)_\xi = \Lambda$ (9)

$\vdash . *120{\cdot}15 . \supset \vdash :: \nu \in \text{N}_\eta\text{C induct} - \iota`\Lambda . \supset :. \mathrm{sm}_\xi``\nu = \text{N}_\xi\text{C induct} :.$

[(8)] $\supset :. (\alpha +_c \mathrm{sm}_\xi``\nu)_\xi = \Lambda : \vee : (\alpha +_c \mathrm{sm}_\xi``\nu)_\xi = (\beta +_c \mathrm{sm}_\xi``\nu)_\xi . \supset . \alpha = \mathrm{sm}``\beta :.$

[*118·24] $\supset :. (\alpha +_c \nu)_\xi = \Lambda : \vee : (\alpha +_c \nu)_\xi = (\beta +_c \nu)_\xi . \supset . \alpha = \mathrm{sm}``\beta$ (10)

$\vdash . (9) . (10) . \supset \vdash . \text{Prop}$

The above proposition establishes (with the natural limitations) the uniqueness (within each type) of subtraction (conceived as in *120·412) when the subtrahend is an inductive cardinal. (When the subtrahend is a non-inductive cardinal, subtraction ceases to give a unique result.) Hence we are led to the following extensions of *118 for the case of inductive cardinals:

***120·411.** $\vdash :. \nu \in \mathrm{NC\ induct} . \supset :$

$$\exists ! \gamma -_0 \nu . \supset . \gamma -_0 \nu \in \mathrm{N_0C} : \gamma \geqslant \nu . \equiv . (\gamma -_0 \nu) \cap t_0{}^{\backprime}\gamma \in \mathrm{N_0C}$$

Dem.

$\vdash . *119·1 . \supset \vdash :. \nu \in \mathrm{NC\ induct} . \supset :$

$\xi, \eta \in \gamma -_c \nu . \supset . \mathrm{Nc}^{\backprime}\xi +_0 \nu = \gamma . \mathrm{Nc}^{\backprime}\eta +_0 \nu = \gamma . \exists ! \mathrm{Nc}^{\backprime}\xi +_c \nu .$

[*20·22] $\supset . \mathrm{Nc}^{\backprime}\xi +_0 \nu = \mathrm{Nc}^{\backprime}\eta +_0 \nu . \exists ! \mathrm{Nc}^{\backprime}\xi +_0 \nu .$

[*120·41.*100·511.(*110·03)] $\supset . \mathrm{Nc}^{\backprime}\xi = \mathrm{Nc}^{\backprime}\eta$ (1)

$\vdash . (1) . *119·14 .$ $\supset \vdash :. \mathrm{Hp} . \supset : \exists ! \gamma -_0 \nu . \supset . \gamma -_0 \nu \in \mathrm{N_0C}$ (2)

$\vdash . *119·27 . (2) .$ $\supset \vdash :. \mathrm{Hp} . \supset : \gamma \geqslant \nu . \supset . (\gamma -_0 \nu) \cap t_0{}^{\backprime}\gamma \in \mathrm{N_0C}$ (3)

$\vdash . *103·22 . *119·27 . \supset \vdash :. \mathrm{Hp} . \supset : (\gamma -_0 \nu) \cap t_0{}^{\backprime}\gamma \in \mathrm{N_0C} . \supset . \gamma \geqslant \nu$ (4)

$\vdash . (2) . (3) . (4) . \supset \vdash . \mathrm{Prop}$

***120·4111.** $\vdash :. \nu \in \mathrm{NC\ induct} . \exists ! \mathrm{sm}_\xi{}^{\backprime\backprime}\gamma . \supset : \gamma \geqslant \nu . \equiv . (\gamma -_0 \nu)_\xi \in \mathrm{N_0C}$

Dem.

$\vdash . *119·64 . \supset \vdash :. \mathrm{Hp} . \supset : \gamma \geqslant \nu . \supset . \exists ! (\gamma -_0 \nu)_\xi .$

[*120·411] $\supset . (\gamma -_0 \nu)_\xi \in \mathrm{N_0C}$ (1)

$\vdash . (1) . *119·26 . *103·13 . \supset \vdash . \mathrm{Prop}$

***120·412.** $\vdash : \nu \in \mathrm{NC\ induct} . \gamma \geqslant \nu . \exists ! \mathrm{sm}_\xi{}^{\backprime\backprime}\gamma . \supset . (\gamma -_0 \nu)_\xi = \{(\imath\alpha)(\alpha +_0 \nu = \gamma)\}_\xi$

Dem.

$\vdash . *120·4111 . \supset \vdash :. \mathrm{Hp} . \supset . (\gamma -_c \nu)_\xi \in \mathrm{N_0C} .$

[*119·34] $\supset . (\gamma -_c \nu)_\xi +_0 \nu = \gamma$ (1)

$\vdash . *120·41 . *103·43 . *37·29 . \supset \vdash :. \mathrm{Hp} . \supset : \alpha +_0 \nu = \gamma . \beta +_0 \nu = \gamma . \supset_{\alpha, \beta} . \alpha = \beta$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***120·413.** $\vdash : \mu \in \mathrm{N_0C} . \supset . \mu -_0 0 = \mathrm{sm}^{\backprime\backprime}\mu$

Dem.

$\vdash . *119·1 . \supset \vdash :. \mathrm{Hp} . \supset : \xi \in \mu -_0 0 . \equiv . \mathrm{N_0c}^{\backprime}\xi +_0 0 = \mu . \exists ! \mu .$

[*110·61.*103·13] $\equiv . \mathrm{Nc}^{\backprime}\xi = \mu .$

[*103·44·4] $\equiv . \mathrm{N_0c}^{\backprime}\xi = \mathrm{sm}^{\backprime\backprime}\mu .$

[*103·26] $\equiv . \xi \in \mathrm{sm}^{\backprime\backprime}\mu :. \supset \vdash . \mathrm{Prop}$

***120·414.** $\vdash : \mu \in \mathrm{N_0C} - \iota^{\backprime}0 . \exists ! \mathrm{sm}_\xi{}^{\backprime\backprime}\mu . \supset . (\mu -_c 1)_\xi \in \mathrm{N_0C}$

[*120·4111 . *117·53]

***120·415.** $\vdash : \mu \in \mathrm{N_0C} - \iota^{\backprime}0 - \iota^{\backprime}1 . \exists ! \mathrm{sm}_\xi{}^{\backprime\backprime}\mu . \supset . (\mu -_0 2)_\xi \in \mathrm{N_0C}$

[*120·4111 . *117·551]

***120·416.** $\vdash : \nu \in \mathrm{NC\ induct} . \exists ! \gamma -_c \nu . \supset . (\gamma -_0 \nu) +_0 \nu = \mathrm{sm}^{\backprime\backprime}\gamma$

[*120·411 . *119·34]

***120·417.** $\vdash : \mu \in \mathrm{N_0C} - \iota^{\backprime}0 . \exists ! \mathrm{sm}_\xi{}^{\backprime\backprime}\gamma . \supset . \alpha +_0 \gamma = (\alpha +_0 1) +_0 (\gamma -_0 1)_\xi$

[*120·414 . *119·35]

***120·418.** $\vdash : \nu \in \mathrm{NC\ induct} . \exists ! \mathrm{sm}_\xi{}^{\backprime\backprime}\gamma . \gamma \geqslant \nu . \supset . \alpha +_c \gamma = (\alpha +_c \nu) +_0 (\gamma -_0 \nu)_\xi$

[*120·4111 . *119·35]

***120·42.** $\vdash : \nu \in \text{NC induct} . \exists ! \nu . \alpha \neq 0 . \supset . \nu \neq \alpha +_c \nu$

Dem.

$\vdash . *110{\cdot}61 . *120{\cdot}14 . \supset \vdash : \nu \in \text{NC induct} . \supset . \nu = 0 +_c \nu$ (1)

$\vdash . *120{\cdot}41 . \supset \vdash : \nu \in \text{NC induct} . \exists ! 0 +_c \nu . 0 +_c \nu = \alpha +_c \nu . \supset . 0 = \alpha$ (2)

$\vdash . (1) . (2) . \supset \vdash : \nu \in \text{NC induct} . \exists ! \nu . \nu = \alpha +_c \nu . \supset . \alpha = 0 : \supset \vdash . \text{Prop}$

***120·422.** $\vdash : \alpha +_c 1 \in \text{NC induct} - \iota\text{‘}\Lambda . \supset . \alpha \in \text{NC induct} - \iota\text{‘}\Lambda$

Dem.

$\vdash . *120{\cdot}1{\cdot}124 . *91{\cdot}542 . \supset \vdash : \alpha +_c 1 \in \text{NC induct} . \supset . (\alpha +_c 1)(+_c 1)_{po} 0 .$

$[*91{\cdot}52] \quad \supset . (\exists \beta) . (\alpha +_c 1)(+_c 1) \beta . \beta (+_c 1)_* 0 .$

$[*120{\cdot}1] \quad \supset . (\exists \beta) . \alpha +_c 1 = \beta +_c 1 . \beta \in \text{NC induct}$ (1)

$\vdash . *120{\cdot}311 . \quad \supset \vdash :. \text{Hp} . \supset : \alpha +_c 1 = \beta +_c 1 . \supset . \alpha = \text{sm}\text{“}\beta . \exists ! \alpha$ (2)

$\vdash . (1) . (2) . *120{\cdot}15 . \supset \vdash : \text{Hp} . \supset . \alpha \in \text{NC induct}$ (3)

$\vdash . (3) . *110{\cdot}4 . \supset \vdash . \text{Prop}$

***120·423.** $\vdash : \alpha \in \text{N}_\eta\text{C induct} - \iota\text{‘}0 . \equiv . (\exists \beta) . \beta \in \text{N}_\eta\text{C induct} . \alpha = (\beta +_c 1)_\eta$

Dem.

$\vdash . *120{\cdot}121{\cdot}124 . \supset \vdash : \beta \in \text{N}_\eta\text{C induct} . \alpha = (\beta +_c 1)_\eta . \supset . \alpha \in \text{N}_\eta\text{C induct} - \iota\text{‘}0$ (1)

$\vdash . *120{\cdot}102 . *91{\cdot}542 . \supset \vdash : \alpha \in \text{N}_\eta\text{C induct} - \iota\text{‘}0 . \supset . \alpha (+_c 1)_{po} 0_\eta .$

$[*91{\cdot}52] \quad \supset . (\exists \beta) . \alpha (+_c 1) \beta . \beta (+_c 1)_* 0_\eta .$

$[*120{\cdot}102] \quad \supset . (\exists \beta) . \beta \in \text{N}_\eta\text{C induct} . \alpha = (\beta +_c 1)_\eta$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***120·4231.** $\vdash : \alpha \in \text{N}_\eta\text{C induct} . \supset . (\exists \beta) . \beta \in \text{N}_\eta\text{C induct} - \iota\text{‘}\Lambda . (\alpha +_c 1)_\eta = (\beta +_c 1)_\eta$

Dem.

$\vdash . *10{\cdot}24 . *101{\cdot}12 . *120{\cdot}12 . \supset$

$\vdash . (\exists \beta) . \beta \in \text{N}_\eta\text{C induct} - \iota\text{‘}\Lambda . (0 +_c 1)_\eta = (\beta +_c 1)_\eta$ (1)

$\vdash . *120{\cdot}121 . \supset \vdash :. \exists ! \xi . \supset :$

$\beta \in \text{N}_\eta\text{C induct} - \iota\text{‘}\Lambda . \xi = (\beta +_c 1)_\eta . \supset . \xi \in \text{N}_\eta\text{C induct} - \iota\text{‘}\Lambda . (\xi +_c 1)_\eta = (\xi +_c 1)_\eta :$

$[*10{\cdot}23{\cdot}24] \supset : (\exists \beta) . \beta \in \text{N}_\eta\text{C induct} - \iota\text{‘}\Lambda . \xi = (\beta +_c 1)_\eta . \supset .$

$\quad (\exists \gamma) . \gamma \in \text{N}_\eta\text{C induct} - \iota\text{‘}\Lambda . (\xi +_c 1)_\eta = (\gamma +_c 1)_\eta$ (2)

$\vdash . *110{\cdot}4 . *13{\cdot}17 . \supset$

$\vdash :. \sim \exists ! \xi . \supset : \beta \in \text{N}_\eta\text{C induct} - \iota\text{‘}\Lambda . \xi = (\beta +_c 1)_\eta . \supset . (\xi +_c 1)_\eta = (\beta +_c 1)_\eta :$

$[*10{\cdot}28] \quad \supset : (\exists \beta) . \beta \in \text{N}_\eta\text{C induct} - \iota\text{‘}\Lambda . \xi = (\beta +_c 1)_\eta . \supset .$

$\quad (\exists \beta) . \beta \in \text{N}_\eta\text{C induct} - \iota\text{‘}\Lambda . (\xi +_c 1)_\eta = (\beta +_c 1)_\eta$ (3)

$\vdash . (2) . (3) . \supset \vdash : (\exists \beta) . \beta \in \text{N}_\eta\text{C induct} - \iota\text{‘}\Lambda . \xi = (\beta +_c 1)_\eta . \supset .$

$\quad (\exists \beta) . \beta \in \text{N}_\eta\text{C induct} - \iota\text{‘}\Lambda . (\xi +_c 1)_\eta = (\beta +_c 1)_\eta$ (4)

$\vdash . (1) . (4) \dfrac{(\xi +_c 1)_\eta}{\xi} . *120{\cdot}11 . \supset$

$\vdash : \alpha \in \text{N}_\eta\text{C induct} . \supset . (\exists \beta) . \beta \in \text{N}_\eta\text{C induct} - \iota\text{‘}\Lambda . (\alpha +_c 1)_\eta = (\beta +_c 1)_\eta :$

$\supset \vdash . \text{Prop}$

***120·4232.** $\vdash : \alpha \in \text{N}_\eta\text{C induct} - \iota\text{‘}0 . \equiv . (\exists \beta) . \beta \in \text{N}_\eta\text{C induct} - \iota\text{‘}\Lambda . \alpha = (\beta +_c 1)_\eta$
[*120·423·4231]

***120·424.** $\vdash : \beta \neq 0 \,.\, \exists ! (\alpha +_c \beta)_\xi \,.\, \supset .\, (\alpha +_c \beta)_\xi -_c 1 = \alpha +_c (\beta -_c 1)_\xi$

Dem.

$\vdash .\, *110·42·62 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset .\, (\alpha +_c \beta)_\xi \in \mathrm{NC} - \iota^\prime\Lambda - \iota^\prime 0 \,.$

$[*120·414 . *103·13] \quad \supset .\, \exists ! (\alpha +_c \beta)_\xi \dot{-}_c 1$ (1)

$\vdash .\, *110·4 \,.\, *118·21 \,.\, *120·414 \,.\, *103·13 \,.\, \supset \vdash : \mathrm{Hp} . \supset .\, \exists ! (\beta -_c 1)_\xi$ (2)

$\vdash .\, (1) . (2) .\, *120·416 \,.\, \supset$

$\vdash : \mathrm{Hp} . \quad \supset .\, \{(\alpha +_c \beta)_\xi -_c 1\} +_c 1 = \alpha +_c \beta \,.\, (\beta -_c 1)_\xi +_c 1 = \beta \,.$ (3)

$[*110·56] \supset .\, \{(\alpha +_c \beta)_\xi -_c 1\} +_c 1 = \{\alpha +_c (\beta -_c 1)_\xi\} +_c 1$ (4)

$\vdash .\, (3) .\, \supset \vdash : \mathrm{Hp} \,.\, \supset .\, \exists ! [\{(\alpha +_c \beta)_\xi -_c 1\} +_c 1]_\xi$ (5)

$\vdash .\, (4) . (5) .\, *120·311 \,.\, *110·44 \,.\, \supset$

$\vdash : \mathrm{Hp} \,.\, \supset .\, (\alpha +_c \beta)_\xi -_c 1 = \alpha +_c (\beta -_c 1)_\xi : \supset \vdash .\, \mathrm{Prop}$

***120·425.** $\vdash :. (\alpha +_c \beta)_\xi \in \mathrm{N_0C} - \iota^\prime 0 \,.\, \supset :$

$(\alpha +_c \beta)_\xi -_c 1 = \alpha +_c (\beta -_c 1)_\xi \,.\, \mathrm{v} \,.\, (\alpha +_c \beta)_\xi -_c 1 = (\alpha -_c 1)_\xi +_c \beta$

Dem.

$\vdash .\, *110·62 \,.\, *103·22 \,.\, \supset \vdash :. \mathrm{Hp} \,.\, \supset : \alpha \neq 0 \,.\, \mathrm{v} \,.\, \beta \neq 0 : \exists ! (\alpha +_c \beta)_\xi$ (1)

$\vdash .\, (1) .\, *120·424 \,.\, \supset \vdash .\, \mathrm{Prop}$

***120·426.** $\vdash : \rho \in \mathrm{Cls\,induct} \,.\, \rho \subset \sigma \,.\, \exists ! \sigma - \rho \,.\, \supset .\, \sim(\rho \,\mathrm{sm}\, \sigma) \,.\, \mathrm{Nc}^\prime\rho < \mathrm{Nc}^\prime\sigma$

Dem.

$\vdash .\, *110·32 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset .\, \mathrm{Nc}^\prime\sigma = \mathrm{Nc}^\prime\rho +_c \mathrm{Nc}^\prime(\sigma - \rho)$ (1)

$\vdash .\, *101·14 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset .\, \mathrm{Nc}^\prime(\sigma - \rho) \neq 0$ (2)

$\vdash .\, (1) . (2) .\, *120·42 \,.\, *117·222·26 \,.\, \supset \vdash .\, \mathrm{Prop}$

***120·427.** $\vdash : R \in 1 \to 1 \,.\, \mathrm{C\!\!\!I}^\prime R \subset \mathrm{D}^\prime R \,.\, \exists ! \mathrm{D}^\prime R - \mathrm{C\!\!\!I}^\prime R \,.\, \supset .\, \mathrm{D}^\prime R \sim\in \mathrm{Cls\,induct}$

$[*120·426 \,.\, \mathrm{Transp}]$

The above proposition shows that no reflexive class is inductive.

***120·428.** $\vdash : \nu \in \mathrm{NC\,induct} \,.\, \exists ! \alpha +_c \nu \,.\, \alpha \neq 0 \,.\, \supset .\, \alpha +_c \nu > \nu$

Dem.

$\vdash .\, *117·511 \,.\, *110·4 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset .\, \alpha > 0 \,.\, \nu \in \mathrm{N_0C} \,.$

$[*117·561 . *110·6] \quad \supset .\, \alpha +_c \nu \geqslant \nu$ (1)

$\vdash .\, *120·42 \,.\, *110·4 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset .\, \alpha +_c \nu \neq \nu$ (2)

$\vdash .\, (1) . (2) .\, *117·26 \,.\, \supset \vdash .\, \mathrm{Prop}$

***120·429.** $\vdash :. \nu \in \mathrm{NC\,induct} \,.\, \supset : \mu > \nu \,.\, \equiv .\, \mu \geqslant \nu +_c 1$

Dem.

$\vdash .\, *120·428 \,.\, \supset \vdash :. \mathrm{Hp} \,.\, \supset : \mu \in \mathrm{N_0C} \,.\, \mu = \nu +_c 1 \,.\, \supset .\, \mu > \nu :$ (1)

$[*117·47·12] \quad \supset : \mu > \nu +_c 1 \,.\, \supset .\, \mu > \nu$ (2)

$\vdash .\, *117·31 \,.\, \supset \vdash : \mu > \nu \,.\, \supset .\, (\exists \varpi) \,.\, \varpi \in \mathrm{N_0C} \,.\, \mu = \nu +_c \varpi$ (3)

$\vdash .\, *117·26·12 \,.\, \supset \vdash : \mu > \nu \,.\, \supset .\, \mu \neq \nu +_c 0$ (4)

$\vdash .\, (3) . (4) . \quad \supset \vdash : \mu > \nu \,.\, \supset .\, (\exists \varpi) \,.\, \varpi \in \mathrm{N_0C} - \iota^\prime 0 \,.\, \mu = \nu +_c \varpi \,.$

$[*117·531] \quad \supset .\, (\exists \varpi) \,.\, \varpi \geqslant 1 \,.\, \mu = \nu +_c \varpi \,.$

$[*117·31] \quad \supset .\, (\exists \varpi, \rho) \,.\, \rho \in \mathrm{N_0C} \,.\, \varpi = \rho +_c 1 \,.\, \mu = \nu +_c \varpi \,.$

$[*13·195] \quad \supset .\, (\exists \rho) \,.\, \rho \in \mathrm{N_0C} \,.\, \mu = \nu +_c \rho +_c 1 \,.$

$[*117·31] \quad \supset .\, \mu \geqslant \nu +_c 1$ (5)

$\vdash .\, (1) . (2) . (5) .\, \supset \vdash .\, \mathrm{Prop}$

The following definition, in which "spec" stands for "species," defines the "species" of a cardinal β as all cardinals which are less than, equal to, or greater than β. We cannot prove, unless by assuming the multiplicative axiom, that all cardinals belong to the species of β, except in the case where β is an inductive cardinal. In all other cases there may, so far as is known at present, be other cardinals which are neither greater nor less than β.

***120·43.** $\text{spec}'\beta = \hat{\alpha}\{\alpha < \beta \,.\, \mathsf{v} \,.\, \alpha \geqslant \beta\}$ Df

***120·431.** $\vdash :. \alpha \in \text{spec}'\beta \,.\equiv : \alpha < \beta \,.\, \mathsf{v} \,.\, \alpha \geqslant \beta$ [(*120·43)]

***120·432.** $\vdash :. \alpha \in \text{spec}'\beta \,.\equiv : \alpha \leqslant \beta \,.\, \mathsf{v} \,.\, \alpha \geqslant \beta$ [*117·281 . *120·431]

***120·433.** $\vdash :. \text{Nc}'\rho \in \text{spec}'\text{Nc}'\sigma \,.\equiv : \exists ! \,\text{Cl}'\rho \cap \text{Nc}'\sigma \,.\, \mathsf{v} \,.\, \exists ! \,\text{Cl}'\sigma \cap \text{Nc}'\rho$
[*117·22 . *120·432]

***120·434.** $\vdash . \text{spec}'\beta \subset \text{N}_0\text{C}$ [*117·105·104·12 . *120·432]

***120·435.** $\vdash : \beta \in \text{N}_0\text{C} \,.\equiv .\, \beta \in \text{spec}'\beta \,.\equiv .\, \exists ! \,\text{spec}'\beta$ [*117·104 . *120·434]

***120·436.** $\vdash :. \alpha \in \text{spec}'\beta \,.\equiv : \alpha, \beta \in \text{N}_0\text{C} : (\exists\gamma) : \alpha +_c \gamma = \beta \,.\, \mathsf{v} \,.\, \beta +_c \gamma = \alpha$
[*120·432 . *117·31]

***120·437.** $\vdash : \beta \in \text{N}_0\text{C} \,.\supset .\, 0 \in \text{spec}'\beta$ [*117·5 . *120·432]

***120·438.** $\vdash : \alpha \in \text{spec}'\beta \,.\, \exists ! \,\alpha +_c 1 \,.\supset .\, \alpha +_c 1 \in \text{spec}'\beta$

Dem.

$\vdash . \text{*120·436} . \text{*110·4} . \supset \vdash :. \text{Hp} \,.\equiv : \alpha, \beta \in \text{N}_0\text{C} \,.\, \exists ! \,\alpha +_c 1 :$
$(\exists\gamma) : \gamma \in \text{N}_0\text{C} : \alpha +_c \gamma = \beta \,.\, \mathsf{v} \,.\, \beta +_c \gamma = \alpha$ (1)

$\vdash . \text{*110·61} . \supset \vdash : \alpha, \beta \in \text{N}_0\text{C} \,.\, \alpha +_c 0 = \beta \,.\supset .\, \alpha = \beta \,.$
[*13·12·15] $\supset .\, \alpha +_c 1 = \beta +_c 1 \,.$
[*120·436] $\supset .\, \alpha +_c 1 \in \text{spec}'\beta$ (2)

$\vdash . \text{*120·417} . \supset \vdash : \alpha, \beta, \gamma \in \text{N}_0\text{C} \,.\, \gamma \neq 0 \,.\, \alpha +_c \gamma = \beta \,.\supset .\, \alpha +_c 1 +_c (\gamma -_c 1) = \beta \,.$
[*120·436] $\supset .\, \alpha +_c 1 \in \text{spec}'\beta$ (3)

$\vdash . \text{*13·12·15} . \supset \vdash : \alpha, \beta, \gamma \in \text{N}_0\text{C} \,.\, \beta +_c \gamma = \alpha \,.\, \exists ! \,\alpha +_c 1 \,.\supset .\, \beta +_c \gamma +_c 1 = \alpha +_c 1 \,.$
[*120·436] $\supset .\, \alpha +_c 1 \in \text{spec}'\beta$ (4)

$\vdash . (1) . (2) . (3) . (4) . \supset \vdash . \text{Prop}$

***120·44.** $\vdash : \beta \in \text{N}_0\text{C} \,.\supset .\, \text{NC induct} - \iota'\Lambda \subset \text{spec}'\beta$

Dem.

$\vdash . \text{*120·437} . \supset \vdash : \text{Hp} \,.\supset .\, 0 \in \text{spec}'\beta$ (1)

$\vdash . \text{*120·438} . \text{*110·4} . \supset \vdash :: \text{Hp} \,.\supset :. \alpha = \Lambda \,.\, \mathsf{v} \,.\, \alpha \in \text{spec}'\beta : \supset :$
$\alpha +_c 1 = \Lambda \,.\, \mathsf{v} \,.\, \alpha +_c 1 \in \text{spec}'\beta$ (2)

$\vdash . (1) . (2) . \text{*120·11} . \supset \vdash :: \text{Hp} \,.\supset :. \alpha \in \text{NC induct} \,.\supset :$
$\alpha = \Lambda \,.\, \mathsf{v} \,.\, \alpha \in \text{spec}'\beta :: \supset \vdash . \text{Prop}$

***120·441.** $\vdash :. \alpha \in \text{NC induct} - \iota'\Lambda \,.\, \beta \in \text{NC} - \iota'\Lambda \,.\supset : \alpha < \beta \,.\, \mathsf{v} \,.\, \alpha = \text{sm}''\beta \,.\, \mathsf{v} \,.\, \alpha > \beta$
[*120·44 . *103·34]

***120·442.** $\vdash :. \alpha \in \text{NC induct} - \iota`\Lambda . \beta \in \text{NC} - \iota`\Lambda . \supset :$

$\alpha < \beta . \equiv . \sim(\alpha \geqslant \beta) : \alpha > \beta . \equiv . \sim(\alpha \leqslant \beta)$

Dem.

$\vdash . *117{\cdot}104 . *120{\cdot}441 . \supset \vdash :. \text{Hp} . \supset : \alpha < \beta . \text{v} . \alpha \geqslant \beta$ (1)

$\vdash . *117{\cdot}291 . \quad \supset \vdash : \alpha < \beta . \supset . \sim(\alpha \geqslant \beta)$ (2)

$\vdash . (1) . (2) . *5{\cdot}17 . \supset \vdash :. \text{Hp} . \supset : \alpha < \beta . \equiv . \sim(\alpha \geqslant \beta)$ (3)

Similarly $\vdash :. \text{Hp} . \supset : \alpha > \beta . \equiv . \sim(\alpha \leqslant \beta)$ (4)

$\vdash . (3) . (4) . \supset \vdash . \text{Prop}$

***120·45.** $\vdash : \alpha, \beta \in \text{N}_\xi\text{C induct} . \supset . (\alpha +_c \beta)_\xi \in \text{N}_\xi\text{C induct}$

Dem.

$\vdash . *110{\cdot}6 . \supset \vdash : \alpha \in \text{N}_\xi\text{C induct} . \supset . (\alpha +_c 0_\xi)_\xi \in \text{N}_\xi\text{C induct}$ (1)

$\vdash . *120{\cdot}121 . *118{\cdot}25 . \supset$

$\vdash : (\alpha +_c \beta)_\xi \in \text{N}_\xi\text{C induct} . \supset . \{\alpha +_c (\beta +_c 1)_\xi\}_\xi \in \text{N}_\xi\text{C induct}$ (2)

$\vdash . (1) . (2) . *120{\cdot}11 . \supset \vdash . \text{Prop}$

***120·4501.** $\vdash : \alpha, \beta \in \text{NC induct} - \iota`\Lambda . \supset . \alpha +_c \beta \in \text{NC induct}$

Dem.

$\vdash . *120{\cdot}15 . \supset \vdash : \text{Hp} . \supset . \text{sm}_\xi``\alpha, \text{sm}_\xi``\beta \in \text{N}_\xi\text{C induct} .$

$[*120{\cdot}45] \quad \supset . (\text{sm}_\xi``\alpha +_c \text{sm}_\xi``\beta)_\xi \in \text{N}_\xi\text{C induct} .$

$[*118{\cdot}23] \quad \supset . (\alpha +_c \beta)_\xi \in \text{N}_\xi\text{C induct} : \supset \vdash . \text{Prop}$

The following proposition is a lemma in the proof of *120·452.

***120·451.** $\vdash :. \gamma = (\alpha +_c \beta)_\xi . \supset_{\alpha, \beta} . \alpha, \beta \in \text{NC induct} - \iota`\Lambda :$

$\exists ! (\gamma +_c 1)_\xi . (\gamma +_c 1)_\xi = (\alpha' +_c \beta')_\xi : \supset . \alpha', \beta' \in \text{NC induct} - \iota`\Lambda$

Dem.

$\vdash . *120{\cdot}414{\cdot}124 . *110{\cdot}42 . \supset \vdash : \exists ! (\gamma +_c 1)_\xi . \supset . \{(\gamma +_c 1)_\xi -_c 1\}_\xi \in \text{N}_0\text{C} .$

$[*119{\cdot}32] \quad \supset . \gamma = \{(\gamma +_c 1)_\xi -_c 1\}_\xi$ (1)

$\vdash . (1) . *120{\cdot}124 . \supset \vdash :. \text{Hp} . \supset : \gamma = \{(\alpha' + \beta')_\xi -_c 1\}_\xi . (\alpha' + \beta')_\xi \neq 0 . \exists ! (\alpha' + \beta')_\xi :$

$[*120{\cdot}425] \supset : \gamma = \{\alpha' +_c (\beta' -_c 1)_\xi\}_\xi . \text{v} . \gamma = \{(\alpha' -_c 1)_\xi +_c \beta'\}_\xi :$

$[\text{Hp}] \quad \supset : \alpha', (\beta' -_c 1)_\xi \in \text{NC induct} - \iota`\Lambda . \text{v} . (\alpha' -_c 1)_\xi, \beta' \in \text{NC induct} - \iota`\Lambda :$

$[*119{\cdot}11] \quad \supset : \alpha', \beta' \in \text{NC induct} - \iota`\Lambda :. \supset \vdash . \text{Prop}$

This proposition could be extended to greater generality as regards types; but its sole use is as a lemma.

***120·452.** $\vdash : \alpha +_c \beta \in \text{NC induct} - \iota`\Lambda . \supset . \alpha, \beta \in \text{NC induct} - \iota`\Lambda$

Dem.

$\vdash . *110{\cdot}4 . \text{Transp} . \supset \vdash : \gamma = \Lambda . \supset . (\gamma +_c 1)_\eta = \Lambda$ (1)

$\vdash . *120{\cdot}451 . \supset \vdash :: \gamma = (\alpha +_c \beta)_\eta . \supset_{\alpha, \beta} . \alpha, \beta \in \text{NC induct} - \iota`\Lambda : \supset :.$

$(\gamma +_c 1)_\eta = \Lambda : \text{v} : (\gamma +_c 1)_\eta = (\alpha' +_c \beta')_\eta . \supset_{\alpha', \beta'} . \alpha', \beta' \in \text{NC induct} - \iota`\Lambda$ (2)

$\vdash . (1) . (2) . \quad \supset \vdash :: \gamma = \Lambda : \text{v} : \gamma = (\alpha +_c \beta)_\eta . \supset_{\alpha, \beta} . \alpha, \beta \in \text{NC induct} - \iota`\Lambda :. \supset :.$

$(\gamma +_c 1)_\eta = \Lambda : \text{v} : (\gamma +_c 1)_\eta = (\alpha' +_c \beta')_\eta . \supset_{\alpha', \beta'} . \alpha', \beta' \in \text{NC induct} - \iota`\Lambda$ (3)

$\vdash . *110{\cdot}62 . *120{\cdot}12 . \supset \vdash : 0 = (\alpha +_c \beta)_\eta . \supset_{\alpha, \beta} . \alpha, \beta \in \text{NC induct} - \iota`\Lambda$ (4)

$\vdash . (3) . (4) . *120{\cdot}11 . \supset \vdash :: \gamma \in \text{N}_\eta\text{C induct} . \supset :.$

$\gamma = \Lambda : \text{v} : \gamma = (\alpha +_c \beta)_\eta . \supset_{\alpha, \beta} . \alpha, \beta \in \text{NC induct} - \iota`\Lambda ::$

$[*13{\cdot}15] \quad \supset \vdash :. (\alpha +_c \beta)_\eta \in \text{N}_\eta\text{C induct} . \supset :$

$(\alpha +_c \beta)_\eta = \Lambda . \text{v} . \alpha, \beta \in \text{NC induct} - \iota`\Lambda :. \supset \vdash . \text{Prop}$

In the last line but one of the above proof, we substitute for the $\phi\xi$ of $*120\cdot11$ the function

$$\xi = \Lambda : \mathrm{v} : \xi = (\alpha +_c \beta)_\eta \,.\, \supset_{\alpha,\beta} \,.\, \alpha, \beta \,\epsilon\, \mathrm{NC\ induct} - \iota`\Lambda.$$

The following propositions are chiefly required as leading to $*120\cdot4621$ $\cdot4622\cdot47$, which are useful in proving propositions concerning all inductive cardinals other than zero.

$*120\cdot46$. $\vdash : \alpha \,\epsilon\, \mathrm{NC} \,.\, \gamma \,\epsilon\, \mathrm{N}_\eta\mathrm{C\ induct} \,.\, \supset \,.\, (\alpha +_c \gamma)_\eta \,(+_c 1)_*\, \mathrm{sm}_\eta``\alpha$

Dem.

$\vdash \,.\, *110\cdot6 \,.\, *118\cdot241 \,.\, \supset \vdash : \alpha \,\epsilon\, \mathrm{NC} \,.\, \supset \,.\, (\alpha +_c 0)_\eta \,(+_c 1)_*\, \mathrm{sm}_\eta``\alpha$ (1)

$\vdash \,.\, *90\cdot172 \,.\, *118\cdot25 \,.\, \supset \vdash : (\alpha +_c \gamma)_\eta \,(+_c 1)_*\, \mathrm{sm}_\eta``\alpha \,.\, \supset \,.$

$\{\alpha +_c (\gamma +_c 1)_\eta\}_\eta \,(+_c 1)_*\, \mathrm{sm}_\eta``\alpha$ (2)

$\vdash \,.\, (1) \,.\, (2) \,.\, *120\cdot11 \,.\, \supset \vdash \,.\, \mathrm{Prop}$

$*120\cdot461$. $\vdash : \alpha \,\epsilon\, \mathrm{NC} \,.\, \beta \,(+_c 1)_*\, \mathrm{sm}_\eta``\alpha \,.\, \supset \,.\, (\exists\gamma) \,.\, \gamma \,\epsilon\, \mathrm{N}_\eta\mathrm{C\ induct} \,.\, \beta = (\alpha +_c \gamma)_\eta$

Dem.

$\vdash \,.\, *110\cdot6 \,.\, *118\cdot23 \,.\quad \supset \vdash : \alpha \,\epsilon\, \mathrm{NC} \,.\, \beta = \mathrm{sm}_\eta``\alpha \,.\, \supset \,.\, \beta = (\alpha +_c 0)_\eta$ (1)

$\vdash \,.\, *120\cdot121 \,.\, *118\cdot25 \,.\, \supset \vdash : \beta = (\alpha +_c \gamma)_\eta \,.\, \gamma \,\epsilon\, \mathrm{N}_\eta\mathrm{C\ induct} \,.\, \supset \,.$

$(\beta +_c 1)_\eta = \{\alpha +_c (\gamma +_c 1)_\eta\}_\eta \,.\, (\gamma +_c 1)_\eta \,\epsilon\, \mathrm{N}_\eta\mathrm{C\ induct}$ (2)

$\vdash \,.\, (1) \,.\, (2) \,.\, *90\cdot112 \,.\, \supset \vdash \,.\, \mathrm{Prop}$

$*120\cdot462$. $\vdash :. \alpha \,\epsilon\, \mathrm{NC} \,.\, \supset : (\exists\gamma) \,.\, \gamma \,\epsilon\, \mathrm{N}_\eta\mathrm{C induct} \,.\, \beta = (\alpha +_c \gamma)_\eta \,.\, \equiv \,.\, \beta \,(+_c 1)_*\, \mathrm{sm}_\eta``\alpha$

[$*120\cdot46\cdot461$]

$*120\cdot4621$. $\vdash :. \alpha \,\epsilon\, \mathrm{NC} \,.\, \exists \,!\, \beta \,.\, \supset : \beta \,(+_c 1)_*\, \mathrm{sm}_\eta``\alpha \,.\, \supset \,.\, \mathrm{sm}_\xi``\beta \,(+_c 1)_*\, \mathrm{sm}_\xi``\alpha$

Dem.

$\vdash \,.\, *120\cdot461 \,.\, \supset \vdash :. \mathrm{Hp} \,.\, \supset :$

$\beta \,(+_c 1)_*\, \mathrm{sm}_\eta``\alpha \,.\, \supset \,.\, (\exists\gamma) \,.\, \gamma \,\epsilon\, \mathrm{N}_\eta\mathrm{C\ induct} \,.\, \beta = (\alpha +_c \gamma)_\eta \,.$

[$*110\cdot4$] $\supset \,.\, (\exists\gamma) \,.\, \gamma \,\epsilon\, \mathrm{N}_\eta\mathrm{C\ induct} - \iota`\Lambda \,.\, \beta = (\alpha +_c \gamma)_\eta \,.$

[$*120\cdot15 . *118\cdot201$] $\supset \,.\, (\exists\gamma) \,.\, \mathrm{sm}_\xi``\gamma \,\epsilon\, \mathrm{N}_\xi\mathrm{C\ induct} \,.\, \mathrm{sm}_\xi``\beta = (\alpha +_c \gamma)_\xi \,.$

[$*118\cdot24 . *120\cdot14$] $\supset \,.\, (\exists\gamma') \,.\, \gamma' \,\epsilon\, \mathrm{N}_\xi\mathrm{C\ induct} \,.\, \mathrm{sm}_\xi``\beta = (\alpha +_c \gamma')_\xi \,.$

[$*120\cdot462$] $\supset \,.\, \mathrm{sm}_\xi``\beta \,(+_c 1)_*\, \mathrm{sm}_\xi``\alpha :. \supset \vdash \,.\, \mathrm{Prop}$

$*120\cdot4622$. $\vdash :. \alpha \,\epsilon\, \mathrm{NC} \,.\, \beta \,\epsilon\, \mathrm{NC}\,(\eta) \,.\, \exists \,!\, \mathrm{sm}_\xi``\beta \,.\, \supset :$

$\beta \,(+_c 1)_*\, \mathrm{sm}_\eta``\alpha \,.\, \equiv \,.\, \mathrm{sm}_\xi``\beta \,(+_c 1)_*\, \mathrm{sm}_\xi``\alpha$

Dem.

$\vdash \,.\, *110\cdot4 \,.\, *37\cdot29 \,.\, *120\cdot461 \,.\, \supset$

$\vdash :. \mathrm{Hp} \,.\, \supset : \mathrm{sm}_\xi``\beta \,(+_c 1)_*\, \mathrm{sm}_\xi``\alpha \,.\, \supset \,.\, \exists \,!\, \mathrm{sm}_\xi``\alpha \,.\, \exists \,!\, \alpha \,.$ (1)

[$*100\cdot52$] $\supset \,.\, \mathrm{sm}_\xi``\alpha \,\epsilon\, \mathrm{NC}$ (2)

$\vdash \,.\, *120\cdot4621 \,.\, (2) \,.\, \supset \vdash :. \mathrm{Hp} \,.\, \supset :$

$\mathrm{sm}_\xi``\beta \,(+_c 1)_*\, \mathrm{sm}_\xi``\alpha \,.\, \supset \,.\, \mathrm{sm}_\eta``\mathrm{sm}_\xi``\beta \,(+_c 1)_*\, \mathrm{sm}_\eta``\mathrm{sm}_\xi``\alpha \,.$

[$*102\cdot87 . \mathrm{Hp} . (1)$] $\supset \,.\, \mathrm{sm}_\eta``\beta \,(+_c 1)_*\, \mathrm{sm}_\eta``\alpha \,.$

[$*103\cdot34$] $\supset \,.\, \beta \,(+_c 1)_*\, \mathrm{sm}_\eta``\alpha$ (3)

$\vdash . *37{\cdot}29 . *120{\cdot}4621 . \supset$

$\vdash :. \text{Hp} . \supset : \beta\,(+_c 1)_* \, \text{sm}_\eta{}^{\prime\prime}\alpha . \supset . \text{sm}_\xi{}^{\prime\prime}\beta\,(+_c 1)_* \, \text{sm}_\xi{}^{\prime\prime}\alpha$ (4)

$\vdash . (3) . (4) . \supset \vdash . \text{Prop}$

It is on this proposition that the irrelevance of types in the consideration of inductive cardinals depends.

$*120{\cdot}463$. $\vdash ::. \alpha \in \text{NC} . \supset :: (\exists\gamma) . \gamma \in \text{N}_\eta\text{C induct} . \beta = (\alpha +_c \gamma)_\eta . \equiv :.$
$\xi \in \mu . \supset_\xi . (\xi +_c 1)_\eta \in \mu : \text{sm}_\eta{}^{\prime\prime}\alpha \in \mu : \supset_\mu . \beta \in \mu$

$[*120{\cdot}462 . *90{\cdot}11]$

$*120{\cdot}47$. $\vdash :: \beta \in \text{N}_\eta\text{C induct} - \iota{}^\prime 0 . \equiv :. \xi \in \mu . \supset_\xi . (\xi +_c 1)_\eta \in \mu : 1_\eta \in \mu : \supset_\mu . \beta \in \mu$

$[*120{\cdot}423{\cdot}463]$

Thus mathematical induction starting from 1 will apply to all inductive cardinals except 0. Similar propositions can be similarly proved for 2, 3, ...

$*120{\cdot}471$. $\vdash : (\exists\alpha) . \alpha \in \text{NC induct} - \iota{}^\prime 0 . f\alpha . \equiv . (\exists\beta) . \beta \in \text{NC induct} . f(\beta +_c 1)$

Dem.

$\vdash . *120{\cdot}423 . \supset$

$\vdash : (\exists\alpha) . \alpha \in \text{NC induct} - \iota{}^\prime 0 . f\alpha . \equiv . (\exists\beta) . \beta \in \text{NC induct} . \alpha = \beta +_c 1 . f\alpha .$

$[*13{\cdot}195]$ $\equiv . (\exists\beta) . \beta \in \text{NC induct} . f(\beta +_c 1) : \supset \vdash . \text{Prop}$

$*120{\cdot}472$. $\vdash : (\exists\alpha) . \alpha \in \text{NC induct} - \iota{}^\prime 0 - \iota{}^\prime 1 . f\alpha . \equiv .$
$(\exists\beta) . \beta \in \text{NC induct} - \iota{}^\prime 0 . f(\beta +_c 1) . \equiv . (\exists\gamma) . \gamma \in \text{NC induct} . f(\gamma +_c 2)$

Dem.

$\vdash . *120{\cdot}471 . \supset$

$\vdash : (\exists\alpha) . \alpha \in \text{NC induct} - \iota{}^\prime 0 - \iota{}^\prime 1 . f\alpha . \equiv .$
$(\exists\beta) . \beta \in \text{NC induct} . \beta +_c 1 \neq 1 . f(\beta +_c 1) .$

$[*120{\cdot}42 . *110{\cdot}641]$ $\equiv . (\exists\beta) . \beta \in \text{NC induct} - \iota{}^\prime 0 . f(\beta +_c 1) .$ (1)

$[*120{\cdot}471]$ $\equiv . (\exists\gamma) . \gamma \in \text{NC induct} . f(\gamma +_c 1 +_c 1) .$

$[*110{\cdot}643]$ $\equiv . (\exists\gamma) . \gamma \in \text{NC induct} . f(\gamma +_c 2)$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

$*120{\cdot}473$. $\vdash :. \phi 1 : \xi \in \text{N}_\eta\text{C induct} - \iota{}^\prime 0 . \phi\xi . \supset_\xi . \phi(\xi +_c 1) : \supset :$
$\xi \in \text{N}_\eta\text{C induct} - \iota{}^\prime 0 . \supset . \phi\xi$

Dem.

$\vdash . *120{\cdot}122 . *101{\cdot}22 . \supset \vdash : \phi 1 . \supset . 1 \in \text{N}_\eta\text{C induct} - \iota{}^\prime 0 . \phi 1$ (1)

$\vdash . *120{\cdot}121{\cdot}124 . \supset \vdash : \xi \in \text{N}_\eta\text{C induct} - \iota{}^\prime 0 . \supset . \xi +_c 1 \in \text{N}_\eta\text{C induct} - \iota{}^\prime 0$ (2)

$\vdash . (1) . (2) . \supset$

$\vdash :. \text{Hp} . \supset : 1 \in \text{N}_\eta\text{C induct} - \iota{}^\prime 0 . \phi 1 : \xi \in \text{N}_\eta\text{C induct} - \iota{}^\prime 0 . \phi\xi . \supset_\xi .$
$\xi +_c 1 \in \text{N}_\eta\text{C induct} - \iota{}^\prime 0 . \phi(\xi +_c 1)$ (3)

$\vdash . (3) . *120{\cdot}47 \dfrac{\hat\xi(\xi \in \text{N}_\eta\text{C induct} - \iota{}^\prime 0 . \phi\xi)}{\mu} . \supset \vdash . \text{Prop}$

$*120 \cdot 48$. $\vdash : \beta \in \mathrm{NC\,induct} \,.\, \beta \geqslant \alpha \,.\, \supset \,.\, \alpha \in \mathrm{NC\,induct} - \iota`\Lambda$
[$*120 \cdot 452 \,.\, *117 \cdot 31$]

Thus every cardinal which is not greater than every inductive cardinal is an inductive cardinal.

$*120 \cdot 481$. $\vdash : \eta \in \mathrm{Cls\,induct} \,.\, \xi \subset \eta \,.\, \supset \,.\, \xi \in \mathrm{Cls\,induct}$ [$*117 \cdot 222 \,.\, *120 \cdot 21 \cdot 48$]

Thus if any inductive class can be found which contains a given class, the given class is also inductive.

$*120 \cdot 49$. $\vdash : \alpha \in \mathrm{NC} - \mathrm{NC\,induct} - \iota`\Lambda \,.\, \beta \in \mathrm{NC\,induct} - \iota`\Lambda \,.\, \supset \,.\, \alpha > \beta$

Dem.

$\vdash \,.\, *120 \cdot 48 \,.\, \mathrm{Transp} \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, \sim(\beta \geqslant \alpha)$ (1)

$\vdash \,.\, *120 \cdot 441 \,.\, \supset \vdash :. \mathrm{Hp} \,.\, \supset : \alpha > \beta \,.\, \mathrm{v} \,.\, \beta \geqslant \alpha$ (2)

$\vdash \,.\, (1) \,.\, (2) \,.\, \supset \vdash \,.\, \mathrm{Prop}$

Thus every non-inductive cardinal (except Λ) is greater than every inductive cardinal (except Λ).

$*120 \cdot 491$. $\vdash :. \xi \sim\in \mathrm{Cls\,induct} \,.\, \equiv : \beta \in \mathrm{NC\,induct} \,.\, \supset_\beta \,.\, \exists ! \beta \cap \mathrm{Cl}`\xi$

Dem.

$\vdash \,.\, *120 \cdot 49 \,.\, \supset \vdash : \xi \sim\in \mathrm{Cls\,induct} \,.\, \beta \in \mathrm{NC\,induct} - \iota`\Lambda \,.\, \supset \,.\, \mathrm{N_0c}`\xi > \beta \,.$

[$*120 \cdot 429 . *117 \cdot 12$] $\supset \,.\, \mathrm{N_0c}`\xi \geqslant \beta +_c 1 \,.\, \exists ! \beta \cap \mathrm{Cl}`\xi$ (1)

$\vdash \,.\, (1) \,.\, *117 \cdot 104 \cdot 12 \,.\, *103 \cdot 13 \,.\, \supset$

$\vdash : \xi \sim\in \mathrm{Cls\,induct} \,.\, \beta \in \mathrm{NC\,induct} - \iota`\Lambda \,.\, \supset \,.\, \beta +_c 1 \neq \Lambda$ (2)

$\vdash \,.\, (2) \,.\, *101 \cdot 12 \,.\, *120 \cdot 13 \,.\, \supset$

$\vdash :. \xi \sim\in \mathrm{Cls\,induct} \,.\, \supset : \beta \in \mathrm{NC\,induct} \,.\, \supset \,.\, \beta \neq \Lambda$ (3)

$\vdash \,.\, (1) \,.\, (3) \,.\, \supset \vdash : \xi \sim\in \mathrm{Cls\,induct} \,.\, \beta \in \mathrm{NC\,induct} \,.\, \supset \,.\, \exists ! \beta \cap \mathrm{Cl}`\xi$ (4)

$\vdash \,.\, *120 \cdot 121 \,.\, \supset$

$\vdash :. \beta \in \mathrm{NC\,induct} \,.\, \supset_\beta \,.\, \exists ! \beta \cap \mathrm{Cl}`\xi : \supset : \beta \in \mathrm{NC\,induct} \,.\, \supset_\beta \,.\, \exists ! (\beta +_c 1) \cap \mathrm{Cl}`\xi \,.$

[$*117 \cdot 242 . *120 \cdot 429$] $\supset_\beta \,.\, \mathrm{Nc}`\xi > \beta \,.$

[$*117 \cdot 42 . (*117 \cdot 03)$] $\supset_\beta \,.\, \mathrm{N_0c}`\xi \neq \beta :$

[$*13 \cdot 196$] $\supset : \mathrm{N_0c}`\xi \sim\in \mathrm{NC\,induct} :$

[$*120 \cdot 21$] $\supset : \xi \sim\in \mathrm{Cls\,induct}$ (5)

$\vdash \,.\, (4) \,.\, (5) \,.\, \supset \vdash \,.\, \mathrm{Prop}$

$*120 \cdot 492$. $\vdash : \alpha \in \mathrm{NC} - \mathrm{NC\,induct} \,.\, \beta \geqslant \alpha \,.\, \supset \,.\, \beta \in \mathrm{NC} - \mathrm{NC\,induct}$
[$*120 \cdot 48 \,.\, \mathrm{Transp}$]

In virtue of $*120 \cdot 491$, a class ξ which is not inductive contains sub-classes having 0, 1, 2, 3, ... terms. If we take the successive classes of sub-classes

$$0 \cap \mathrm{Cl}`\xi,\ 1 \cap \mathrm{Cl}`\xi,\ 2 \cap \mathrm{Cl}`\xi,\ \ldots,$$

these are mutually exclusive, and all exist provided Λ is not an inductive

cardinal, *i.e.* provided the axiom of infinity holds. Thus if the axiom of infinity holds, we get $\aleph_0$ classes of sub-classes contained in any non-inductive class. It follows, as we shall see later, that if ξ is a non-inductive class, $\text{Cl}`\text{Cl}`\xi$ is a reflexive class. This seems to be the nearest approach possible to identifying the two definitions of finite and infinite when the multiplicative axiom is not assumed. When the multiplicative axiom is assumed as well as the axiom of infinity, we pick out one class from $1 \cap \text{Cl}`\xi$, one from $2 \cap \text{Cl}`\xi$ and so on; then, forming the logical sum of all these classes, we get $\aleph_0$ terms which are members of ξ. Hence it follows that ξ is a reflexive class; for, as we shall see later, a reflexive class is one which contains sub-classes of $\aleph_0$ terms. Thus with the help of the multiplicative axiom, the two definitions of finite and infinite can be identified.

***120·493.** $\vdash :. \sigma \in \text{Cls induct} . \supset :$

$$\text{Nc}`\xi < \text{Nc}`\sigma . \equiv . (\exists \rho) . \rho \,\text{sm}\, \xi . \rho \subset \sigma . \exists ! \sigma - \rho . \equiv . \exists ! \text{Nc}`\xi \cap \text{Cl}`\sigma - \iota`\sigma$$

Dem.

$\vdash . *117·26·221 . \supset \vdash :. \text{Nc}`\xi < \text{Nc}`\sigma . \supset : \sim(\xi \,\text{sm}\, \sigma) : (\exists \rho) . \rho \,\text{sm}\, \xi . \rho \subset \sigma :$

$[*73·3·37] \qquad \supset : (\exists \rho) . \rho \,\text{sm}\, \xi . \rho \subset \sigma . \rho \neq \sigma \qquad (1)$

$\vdash . *120·481 . \supset \vdash :. \text{Hp} . \supset : \rho \subset \sigma . \exists ! \sigma - \rho . \supset . \rho \in \text{Cls induct} . \rho \subset \sigma . \exists ! \sigma - \rho .$

$[*120·426] \qquad \supset . \text{Nc}`\rho < \text{Nc}`\sigma :$

$[*100·321] \qquad \supset : \rho \,\text{sm}\, \xi . \rho \subset \sigma . \exists ! \sigma - \rho . \supset . \text{Nc}`\xi < \text{Nc}`\sigma \qquad (2)$

$\vdash . (1) . (2) . *24·6 . \supset \vdash . \text{Prop}$

***120·5.** $\vdash : \alpha, \beta \in \text{NC induct} . \exists ! \alpha \times_c \beta . \supset . \alpha \times_c \beta \in \text{NC induct}$

Dem.

$\vdash . *113·203 . \supset \vdash : \alpha \in \text{NC induct} . \exists ! \alpha \times_c 0 . \supset . \alpha \in \text{NC} - \iota`\Lambda .$

$[*113·601] \qquad \supset . \alpha \times_c 0 = 0 .$

$[*120·12] \qquad \supset . \alpha \times_c 0 \in \text{NC induct} \qquad (1)$

$\vdash . *113·671 . \qquad \supset \vdash . \alpha \times_c (\beta +_c 1) = (\alpha \times_c \beta) +_c \alpha .$

$[*120·4501 . *113·203] \supset \vdash : \alpha \in \text{NC induct} . \alpha \times_c \beta \in \text{NC induct} - \iota`\Lambda . \supset .$

$\qquad \alpha \times_c (\beta +_c 1) \in \text{NC induct} \qquad (2)$

$\vdash . (1) . (2) . *120·13 . \supset \vdash . \text{Prop}$

The restriction involved in $\exists ! \alpha \times_c \beta$ in the hypothesis of the above proposition is not necessary if we assume that the axiom of infinity must fail in any one type if it fails in any other, *i.e.*

$$\Lambda \cap t`\alpha \in \text{NC induct} . \supset . \Lambda \cap t`\beta \in \text{NC induct},$$

where α and β are any two objects of any two types. To prove this proposition would require assumptions, as to the interrelation of various types, which have not been made in our previous proofs.

∗120·51. $\vdash : \alpha, \beta, \gamma \in \mathrm{NC\,induct} \,.\, \alpha \neq 0 \,.\, \exists ! \, \alpha \times_c \beta \,.\, \alpha \times_c \beta = \alpha \times_c \gamma \,.\, \supset \,.\, \beta = \mathrm{sm}``\gamma$

This proposition establishes the uniqueness of division among inductive cardinals.

Dem.

$\vdash . *120·44·436 . \supset \vdash :. \mathrm{Hp} . \supset : (\exists \delta) : \beta = \gamma +_c \delta . \vee . \gamma = \beta +_c \delta$ (1)

$\vdash . *113·43 . \quad \supset \vdash : \mathrm{Hp} . \beta = \gamma +_c \delta . \supset . \alpha \times_c \gamma = (\alpha \times_c \gamma) +_c (\alpha \times_c \delta) .$

[∗120·42.Transp] $\supset . \alpha \times_c \delta = 0 .$

[∗113·602] $\supset . \delta = 0 .$

[∗110·6] $\supset . \beta = \mathrm{sm}``\gamma$ (2)

Similarly $\vdash : \mathrm{Hp} . \gamma = \beta +_c \delta . \supset . \gamma = \mathrm{sm}``\beta .$

[∗100·53.∗113·203] $\supset . \beta = \mathrm{sm}``\gamma$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash . \mathrm{Prop}$

If β, γ in the above are typically ambiguous symbols, such as

$$0, 1, 2, \ldots \mathrm{Nc}`\rho, \mathrm{Nc}`\sigma, \ldots,$$

we have $\beta = \gamma$; for in this case, $\beta = \mathrm{sm}``\beta . \gamma = \mathrm{sm}``\gamma$. Also if β and γ are of the same type, we have $\beta = \gamma$, in virtue of ∗103·43. Hence "$\beta = \gamma$" may, with truth, be substituted for "$\beta = \mathrm{sm}``\gamma$" in the above proposition, since the result is true whenever significant. But in this form the proposition gives less information, since it tells us nothing as to what happens when β and γ are not of the same type.

∗120·511. $\vdash : \alpha, \beta \in \mathrm{NC\,induct} \,.\, \alpha \neq 0 \,.\, \exists ! \, \alpha \,.\, \alpha \times_c \beta = \alpha \,.\, \supset \,.\, \beta = 1$

Dem.

$\vdash . *113·621 . \supset \vdash : \mathrm{Hp} . \supset . \alpha \times_c \beta = \alpha \times_c 1$ (1)

$\vdash . (1) . *120·51 . *101·28 . \supset \vdash . \mathrm{Prop}$

∗120·512. $\vdash : \alpha \times_c \beta \in \mathrm{NC\,induct} - \iota`0 - \iota`\Lambda \,.\, \supset \,.\, \alpha, \beta \in \mathrm{NC\,induct} - \iota`0 - \iota`\Lambda$

Dem.

$\vdash . *113·602·203 . \supset \vdash : \mathrm{Hp} . \supset . \alpha, \beta \in \mathrm{NC} - \iota`0 - \iota`\Lambda$ (1)

$\vdash . (1) . *117·62 . \supset \vdash : \mathrm{Hp} . \supset . \alpha \times_c \beta \geqslant \alpha . \alpha \times_c \beta \geqslant \beta .$

[∗120·48] $\supset . \alpha, \beta \in \mathrm{NC\,induct}$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

∗120·513. $\vdash : \alpha \in \mathrm{NC\,induct} - \iota`0 - \iota`\Lambda \,.\, \alpha \times_c \beta = \alpha \,.\, \supset \,.\, \beta = 1$ [∗120·511·512]

This proposition does not hold when α is a non-inductive cardinal.

∗120·52. $\vdash : \alpha, \beta \in \mathrm{NC\,induct} \,.\, \exists ! \, \alpha^\beta \,.\, \supset \,.\, \alpha^\beta \in \mathrm{NC\,induct}$

Dem.

$\vdash . *116·203·301 . \supset \vdash : \alpha \in \mathrm{NC\,induct} . \exists ! \, \alpha^0 . \supset . \alpha^0 = 1 .$

[∗120·122] $\supset . \alpha^0 \in \mathrm{NC\,induct}$ (1)

$\vdash . *116·321·52 . \supset \vdash : \exists ! \, \alpha^{\beta +_c 1} . \supset . \alpha^{\beta +_c 1} = \alpha^\beta \times_c \alpha .$

[∗120·5] $\supset \vdash : \alpha \in \mathrm{NC\,induct} . \alpha^\beta \in \mathrm{NC\,induct} . \exists ! \, \alpha^{\beta +_c 1} . \supset .$

$\alpha^{\beta +_c 1} \in \mathrm{NC\,induct}$ (2)

$\vdash . *116·52 . *113·204 . \supset \vdash : \alpha^\beta = \Lambda . \supset . \alpha^{\beta +_c 1} = \Lambda$ (3)

$\vdash . (1) . (2) . (3) . *120·11 . \supset \vdash . \mathrm{Prop}$

$*120{\cdot}53$. $\vdash : \alpha, \beta, \gamma \in \text{NC induct} . \alpha \neq 0 . \alpha \neq 1 . \exists ! \alpha^\beta . \alpha^\beta = \alpha^\gamma . \supset . \beta = \text{sm}``\gamma$

Dem.

$\vdash . *116{\cdot}203 . \quad \supset \vdash : \exists ! \alpha^\beta . \supset . \exists ! \beta \quad (1)$

$\vdash . *120{\cdot}44{\cdot}436 . \quad \supset \vdash :. \text{Hp} . \supset : (\exists \delta) : \beta = \gamma +_c \delta . \vee . \gamma = \beta +_c \delta \quad (2)$

$\vdash . *118{\cdot}01 . *116{\cdot}52 . \quad \supset \vdash : \beta = \gamma +_c \delta . \exists ! \beta . \supset . \alpha^\beta = \alpha^\gamma \times_c \alpha^\delta :$

$[*13{\cdot}171 . *118{\cdot}01 . (1)] \quad \supset \vdash : \alpha^\beta = \alpha^\gamma . \beta = \gamma +_c \delta . \exists ! \alpha^\beta . \supset . \alpha^\gamma = \alpha^\gamma \times_c \alpha^\delta \quad (3)$

$\vdash . *120{\cdot}52 . *116{\cdot}35 . (1) . \supset \vdash : \text{Hp} . \beta = \gamma +_c \delta . \supset . \alpha^\gamma \in \text{NC induct} - \iota`\Lambda - \iota`0 . \exists ! \beta .$

$[(3) . *120{\cdot}513] \quad \supset . \alpha^\delta = 1 .$

$[*117{\cdot}592] \quad \supset . \delta = 0 .$

$[*110{\cdot}6] \quad \supset . \beta = \text{sm}``\gamma \quad (4)$

Similarly $\quad \vdash : \text{Hp} . \gamma = \beta +_c \delta . \supset . \gamma = \text{sm}``\beta .$

$[*100{\cdot}53 . (1)] \quad \supset . \beta = \text{sm}``\gamma \quad (5)$

$\vdash . (2) . (4) . (5) . \supset \vdash . \text{Prop}$

If α, β, γ are typically ambiguous symbols, we have $\beta = \gamma$ in the conclusion of the above proposition, instead of $\beta = \text{sm}``\gamma$. Also if β and γ are of the same type, $\beta = \gamma$; thus $\beta = \gamma$ whenever "$\beta = \gamma$" is significant.

$*120{\cdot}54$. $\vdash : \xi, \rho \in \text{Cls induct} . \exists ! \xi . \rho \subset \sigma . \exists ! \sigma - \rho . \supset . (\text{Nc}`\rho)^{\text{Nc}`\xi} < (\text{Nc}`\sigma)^{\text{Nc}`\xi}$

For the proof, which is here given shortly, compare $*117{\cdot}58$.

Dem.

$\vdash . *35{\cdot}432{\cdot}82 . *80{\cdot}15 \,\vdots\, *116{\cdot}12 . \quad \supset \vdash : \text{Hp} . \supset . (\rho \uparrow \xi)_\Delta`\xi \subset (\sigma \uparrow \xi)_\Delta`\xi .$

$\exists ! (\sigma \uparrow \xi)_\Delta`\xi - (\rho \uparrow \xi)_\Delta`\xi \quad (1)$

$\vdash . *120{\cdot}52 . *116{\cdot}15{\cdot}251 . *120{\cdot}2 . \supset \vdash : \text{Hp} . \supset . (\rho \uparrow \xi)_\Delta`\xi \in \text{Cls induct} \quad (2)$

$\vdash . (1) . (2) . *120{\cdot}426 . \supset \vdash : \text{Hp} . \supset . \text{Nc}`(\rho \uparrow \xi)_\Delta`\xi < \text{Nc}`(\sigma \uparrow \xi)_\Delta`\xi : \supset \vdash . \text{Prop}$

$*120{\cdot}541$. $\vdash : \alpha, \beta \in \text{NC induct} - \iota`\Lambda . \alpha \neq 0 . \beta < \gamma . \supset . \beta^\alpha < \gamma^\alpha \quad [*120{\cdot}54{\cdot}493]$

$*120{\cdot}542$. $\vdash : \alpha, \gamma \in \text{NC induct} - \iota`\Lambda . \alpha \neq 0 . \beta > \gamma . \supset . \beta^\alpha > \gamma^\alpha \quad [*120{\cdot}541]$

$*120{\cdot}55$. $\vdash : \alpha, \beta, \gamma \in \text{NC induct} . \alpha \neq 0 . \exists ! \beta^\alpha . \beta^\alpha = \gamma^\alpha . \supset . \beta = \text{sm}``\gamma$

Dem.

$\vdash . *120{\cdot}541{\cdot}542 . \supset \vdash : \text{Hp} . \supset . \sim(\beta < \gamma) . \sim(\beta > \gamma) .$

$[*120{\cdot}441] \quad \supset . \beta = \text{sm}``\gamma : \supset \vdash . \text{Prop}$

$*120{\cdot}56$. $\vdash : \alpha \geqslant 2 . \alpha^\beta \in \text{NC induct} - \iota`\Lambda . \supset . \beta \in \text{NC induct}$

Dem.

$\vdash . *117{\cdot}581 . \supset \vdash : \text{Hp} . \supset . \alpha^\beta \geqslant 2^\beta .$

$[*117{\cdot}661] \quad \supset . \alpha^\beta > \beta \quad (1)$

$\vdash . (1) . *120{\cdot}48 . \supset \vdash . \text{Prop}$

$*120{\cdot}561$. $\vdash : \beta \geqslant 1 . \alpha^\beta \in \text{NC induct} - \iota`\Lambda . \supset . \alpha \in \text{NC induct}$

Dem.

$\vdash . *117{\cdot}591 . *116{\cdot}321 . \supset \vdash : \text{Hp} . \supset . \alpha^\beta \geqslant \alpha \quad (1)$

$\vdash . (1) . *120{\cdot}48 . \supset \vdash . \text{Prop}$

∗120·57. $\vdash : \mu \in \text{NC induct} - \iota`\Lambda \,.\, \supset . \text{Nc}`\hat{\nu}(\nu \leqslant \mu) = \mu +_c 1$

Here "$\mu +_c 1$" is necessarily in a higher type than "μ," because it applies to a class of which μ is a member.

Dem.

$\vdash . *117{\cdot}511 . \supset \vdash . \text{Nc}`\hat{\nu}(\nu \leqslant 0) \in 1$ (1)

$\vdash . *110{\cdot}4 . \quad \supset \vdash : \mu = \Lambda . \supset . \mu +_c 1 = \Lambda$ (2)

$\vdash . *120{\cdot}429{\cdot}442 . \supset$

$\vdash : \mu \in \text{NC induct} . \exists ! \mu +_c 1 . \supset . \hat{\nu}(\nu \leqslant \mu) = \hat{\nu}(\nu < \mu +_c 1) .$

$[*117{\cdot}104{\cdot}105] \qquad \supset . \hat{\nu}(\nu \leqslant \mu +_c 1) = \hat{\nu}(\nu \leqslant \mu) \cup \iota`(\mu +_c 1)$ (3)

$\vdash . *120{\cdot}428 . \supset \vdash : \text{Hp}(3) . \supset . \mu +_c 1 \sim\in \hat{\nu}(\nu \leqslant \mu)$ (4)

$\vdash . (3) . (4) . *110{\cdot}631 . \supset$

$\vdash : \text{Hp}(3) . \text{Nc}`\hat{\nu}(\nu \leqslant \mu) = \mu +_c 1 . \supset . \text{Nc}`\hat{\nu}(\nu \leqslant \mu +_c 1) = \mu +_c 2$ (5)

$\vdash . (2) . (5) . \supset \vdash :. \mu \in \text{NC induct} : \mu = \Lambda . \lor . \text{Nc}`\hat{\nu}(\nu \leqslant \mu) = \mu +_c 1 : \supset :$

$\mu +_c 1 = \Lambda . \lor . \text{Nc}`\hat{\nu}(\nu \leqslant \mu +_c 1) = \mu +_c 2$ (6)

$\vdash . (1) . (6) . *120{\cdot}13 . \supset \vdash . \text{Prop}$

∗120·6. $\vdash : (\exists \gamma) . \gamma > \alpha . \gamma \subset t`\eta . \supset . \exists ! (\alpha +_c 1) \cap t`\eta$

Dem.

$\vdash . *117{\cdot}1 . \supset$

$\vdash :. \text{Hp} . \supset : (\exists \gamma, \rho, \sigma) . \text{N}_0\text{c}`\rho = \alpha . \text{N}_0\text{c}`\sigma = \gamma . \exists ! \text{Nc}`\rho \cap \text{Cl}`\sigma . \sim\exists ! \text{Nc}`\sigma \cap \text{Cl}`\rho :$

$[*100{\cdot}1] \supset : (\exists \gamma, \rho, \sigma, \xi) . \text{N}_0\text{c}`\rho = \alpha . \text{N}_0\text{c}`\sigma = \gamma . \xi \,\text{sm}\, \rho . \xi \subset \sigma . \xi \neq \sigma :$

$[*24{\cdot}6] \quad \supset : (\exists \gamma, \rho, \sigma, \xi, x) . \text{N}_0\text{c}`\rho = \alpha . \text{N}_0\text{c}`\sigma = \gamma . \xi \,\text{sm}\, \rho . x \in \sigma - \xi :$

$[*110{\cdot}631] \supset : (\exists \xi, x) . \xi \cup \iota`x \in \alpha +_c 1 \cap t`\eta :. \supset \vdash . \text{Prop}$

∗120·61. $\vdash : \exists ! \text{N}_0\text{C} \cap t^3`x - \text{NC induct} . \supset . \text{Infin ax}\,(x)$

Dem.

$\vdash . *120{\cdot}49 . \supset \vdash :. \gamma \in \text{N}_0\text{C} \cap t^3`x - \text{NC induct} . \supset :$

$\alpha \in \text{NC induct} . \exists ! \alpha . \supset_\alpha . \gamma > \alpha . \gamma \subset t^2`x .$

$[*120{\cdot}6] \qquad \supset_\alpha . \exists ! \alpha +_c 1 \cap t^2`x$ (1)

$\vdash . (1) . *101{\cdot}12 . *120{\cdot}13 . \supset$

$\vdash :. \gamma \in \text{N}_0\text{C} - \text{NC induct} . \supset : \alpha \in \text{NC induct} . \supset_\alpha . \exists ! \alpha(x) :. \supset \vdash . \text{Prop}$

∗120·611. $\vdash : \beta \in \text{Cls induct} . \beta \subset \text{G}`P . \supset . \exists ! P_\Delta`\beta$

Dem.

$\vdash . *80{\cdot}26 . \supset \vdash . \exists ! P_\Delta`\Lambda .$

$[\text{Simp}] \quad \supset \vdash : \Lambda \subset \text{G}`P . \supset . \exists ! P_\Delta`\Lambda$ (1)

$\vdash . *80{\cdot}94 . \supset \vdash : \exists ! P_\Delta`\beta . z \in \text{G}`P . \supset . \exists ! P_\Delta`(\beta \cup \iota`z) :$

$[\text{Syll}] \quad \supset \vdash :. \beta \subset \text{G}`P . \supset . \exists ! P_\Delta`\beta : \supset : \beta \subset \text{G}`P . z \in \text{G}`P . \supset . \exists ! P_\Delta`(\beta \cup \iota`z) :$

$[*51{\cdot}238] \qquad \supset : \beta \cup \iota`z \subset \text{G}`P . \supset . \exists ! P_\Delta`(\beta \cup \iota`z)$ (2)

$\vdash . (1) . (2) . *120{\cdot}26 . \supset \vdash . \text{Prop}$

***120·62.** $\vdash : \kappa \in \mathrm{Cls\,induct} \,.\, \Lambda \sim\in \kappa \,.\, \supset \,.\, \exists \,!\, \epsilon_\Delta{}^{\backprime}\kappa$

Dem.

$\vdash \,.\, *83{\cdot}9 \,.\quad \supset \vdash \,.\, \exists \,!\, \epsilon_\Delta{}^{\backprime}\Lambda$ (1)

$\vdash \,.\, *83{\cdot}904 \,.\, \supset \vdash : \exists \,!\, \epsilon_\Delta{}^{\backprime}\kappa \,.\, \exists \,!\, \alpha \,.\, \supset \,.\, \exists \,!\, \epsilon_\Delta{}^{\backprime}(\kappa \cup \iota^{\backprime}\alpha) :$

[Syll] $\supset \vdash :. \Lambda \sim\in \kappa \,.\, \supset \,.\, \exists \,!\, \epsilon_\Delta{}^{\backprime}\kappa : \supset : \Lambda \sim\in \kappa \,.\, \exists \,!\, \alpha \,.\, \supset \,.\, \exists \,!\, \epsilon_\Delta{}^{\backprime}(\kappa \cup \iota^{\backprime}\alpha) :$

[*24·54] $\supset : \Lambda \sim\in (\kappa \cup \iota^{\backprime}\alpha) \,.\, \supset \,.\, \exists \,!\, \epsilon_\Delta{}^{\backprime}(\kappa \cup \iota^{\backprime}\alpha)$ (2)

$\vdash \,.\, (1) \,.\, (2) \,.\, *120{\cdot}26 \,.\, \supset \vdash \,.\, \mathrm{Prop}$

The above proposition may also be deduced from *120·611, by *62·231.

***120·63.** $\vdash \,.\, \mathrm{Cls\,induct} - \overleftarrow{\epsilon}{}^{\backprime}\Lambda \subset \mathrm{Cls^2\,mult}$ [*120·62 . *88·2]

In virtue of this proposition the multiplicative axiom is not required in dealing with a finite number of factors, even when some or all of the factors are themselves infinite.

***120·64.** $\vdash : \mathrm{Infin\,ax} \,.\, \mathrm{v} \,.\, \mathrm{Mult\,ax}$

Dem.

$\vdash \,.\, *120{\cdot}61 \,.\, \mathrm{Transp} \,.\, \supset \vdash :. \sim \mathrm{Infin\,ax} \,.\, \supset : \mathrm{N_0C} \subset \mathrm{NC\,induct} :$

[*120·21] $\supset : (\kappa) \,.\, \kappa \in \mathrm{Cls\,induct} :$

[*120·62] $\supset : (\kappa) : \Lambda \sim\in \kappa \,.\, \supset \,.\, \exists \,!\, \epsilon_\Delta{}^{\backprime}\kappa :$

[*88·37] $\supset : \mathrm{Mult\,ax} :. \supset \vdash \,.\, \mathrm{Prop}$

Thus of our two arithmetical axioms, the multiplicative axiom and the axiom of infinity, at least one must be true.

***120·7.** $\vdash : \alpha \in \mathrm{Cls\,induct} \,.\, \alpha \subset \beta \,.\, \alpha \neq \beta \,.\, \supset \,.\, \mathrm{Nc}^{\backprime}\alpha < \mathrm{Nc}^{\backprime}\beta$ [*120·426 . *24·6]

***120·71.** $\vdash : \rho, \sigma \in \mathrm{Cls\,induct} \,.\, \equiv \,.\, \rho \cup \sigma \in \mathrm{Cls\,induct} \,.\, \equiv \,.\, \rho + \sigma \in \mathrm{Cls\,induct}$

Dem.

$\vdash \,.\, *120{\cdot}481 \,.\, \supset \vdash : \rho \cup \sigma \in \mathrm{Cls\,induct} \,.\, \supset \,.\, \rho, \sigma \in \mathrm{Cls\,induct}$ (1)

$\vdash \,.\, *120{\cdot}481 \,.\, \supset \vdash : \rho, \sigma \in \mathrm{Cls\,induct} \,.\, \supset \,.\, \rho, \sigma - \rho \in \mathrm{Cls\,induct} \,.$

[*120·21] $\supset \,.\, \mathrm{N_0c}^{\backprime}\rho, \mathrm{N_0c}^{\backprime}(\sigma - \rho) \in \mathrm{NC\,induct} \,.$

[*120·45] $\supset \,.\, \mathrm{N_0c}^{\backprime}\rho +_c \mathrm{N_0c}^{\backprime}(\sigma - \rho) \in \mathrm{NC\,induct} \,.$

[*110·32] $\supset \,.\, \mathrm{Nc}^{\backprime}(\rho \cup \sigma) \in \mathrm{NC\,induct} \,.$

[*120·211] $\supset \,.\, \rho \cup \sigma \in \mathrm{Cls\,induct}$ (2)

$\vdash \,.\, (1) \,.\, (2) \,.\quad \supset \vdash : \rho, \sigma \in \mathrm{Cls\,induct} \,.\, \equiv \,.\, \rho \cup \sigma \in \mathrm{Cls\,induct}$ (3)

$\vdash \,.\, *110{\cdot}12 \,.\, *120{\cdot}214 \,.\, \supset \vdash : \rho, \sigma \in \mathrm{Cls\,induct} \,.\, \equiv \,.$

$\downarrow (\Lambda \cap \sigma)^{\backprime\backprime}\iota^{\backprime\backprime}\rho, (\Lambda \cap \rho) \downarrow {}^{\backprime\backprime}\iota^{\backprime\backprime}\sigma \in \mathrm{Cls\,induct} \,.$

[(3).(*110·01)] $\equiv \,.\, \rho + \sigma \in \mathrm{Cls\,induct}$ (4)

$\vdash \,.\, (3) \,.\, (4) \,.\, \supset \vdash \,.\, \mathrm{Prop}$

The above proposition is frequently used.

***120·72.** $\vdash : \rho, \sigma \in \text{Cls induct} . \supset . \rho \times \sigma \in \text{Cls induct}$

Dem.

$\vdash . *120\cdot21 . \supset \vdash : \text{Hp} . \supset . \text{N}_0\text{c}'\rho, \text{N}_0\text{c}'\sigma \in \text{NC induct} .$
$[*120\cdot5] \quad \supset . \text{Nc}'(\rho \times \sigma) \in \text{NC induct} .$
$[*120\cdot211] \quad \supset . \rho \times \sigma \in \text{Cls induct} : \supset \vdash . \text{Prop}$

***120·721.** $\vdash :. \exists ! \rho . \exists ! \sigma . \supset : \rho, \sigma \in \text{Cls induct} . \equiv . \rho \times \sigma \in \text{Cls induct}$

Dem.

$\vdash . *120\cdot512 . *113\cdot107 . \supset$
$\vdash :. \text{Hp} . \supset : \rho \times \sigma \in \text{Cls induct} . \supset . \text{Nc}'\rho, \text{Nc}'\sigma \in \text{NC induct} .$
$[*120\cdot211] \quad \supset . \rho, \sigma \in \text{Cls induct} \quad (1)$
$\vdash . (1) . *120\cdot72 . \supset \vdash . \text{Prop}$

***120·73.** $\vdash : \rho, \sigma \in \text{Cls induct} . \supset . (\rho \exp \sigma) \in \text{Cls induct} \quad [*120\cdot52 . *116\cdot251]$

***120·731.** $\vdash :. \exists ! \rho . \exists ! \sigma . \rho \sim\in 1 . \supset : \rho, \sigma \in \text{Cls induct} . \equiv . (\rho \exp \sigma) \in \text{Cls induct}$
$[*120\cdot56\cdot561\cdot73]$

***120·74.** $\vdash : \rho \in \text{Cls induct} . \equiv . \text{Cl}'\rho \in \text{Cls induct}$

Dem.

$\vdash . *116\cdot72 . *120\cdot21 . \supset \vdash : \text{Cl}'\rho \in \text{Cls induct} . \equiv . 2^{\text{Nc}'\rho} \cap t'\text{Cl}'\rho \in \text{NC induct} .$
$[*120\cdot123\cdot52\cdot56 . *116\cdot72 . (*116\cdot04)] \quad \equiv . \text{N}_0\text{c}'\rho \in \text{NC induct} .$
$[*120\cdot21] \quad \equiv . \rho \in \text{Cls induct} : \supset \vdash . \text{Prop}$

***120·741.** $\vdash : s'\kappa \in \text{Cls induct} . \supset . \kappa \in \text{Cls induct} . \kappa \subset \text{Cls induct}$

Dem.

$\vdash . *120\cdot74 . \quad \supset \vdash : \text{Hp} . \supset . \text{Cl}'s'\kappa \in \text{Cls induct} .$
$[*60\cdot57 . *120\cdot481] \quad \supset . \kappa \in \text{Cls induct} \quad (1)$
$\vdash . *40\cdot13 . *120\cdot481 . \supset \vdash :. \text{Hp} . \supset : \rho \in \kappa . \supset . \rho \in \text{Cls induct} \quad (2)$
$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***120·75.** $\vdash : s'\kappa \in \text{Cls induct} . \equiv . \kappa \in \text{Cls induct} . \kappa \subset \text{Cls induct}$

Dem.

$\vdash . *22\cdot58 . \quad \supset \vdash : \exists ! \kappa - \text{Cls induct} . \supset . \exists ! (\kappa \cup \iota'\alpha) - \text{Cls induct} \quad (1)$
$\vdash . *120\cdot71 . *53\cdot15 . \supset \vdash : s'\kappa \in \text{Cls induct} . \alpha \in \text{Cls induct} . \supset .$
$\quad s'(\kappa \cup \iota'\alpha) \in \text{Cls induct} :$
$[*5\cdot6] \quad \supset \vdash :. s'\kappa \in \text{Cls induct} . \supset :$
$\quad \alpha \sim\in \text{Cls induct} . \vee . s'(\kappa \cup \iota'\alpha) \in \text{Cls induct} :$
$[*51\cdot16] \supset : \exists ! (\kappa \cup \iota'\alpha) - \text{Cls induct} . \vee . s'(\kappa \cup \iota'\alpha) \in \text{Cls induct} \quad (2)$
$\vdash . (1) . (2) . \supset \vdash :. \exists ! \kappa - \text{Cls induct} . \vee . s'\kappa \in \text{Cls induct} : \supset :$
$\quad \exists ! (\kappa \cup \iota'\alpha) - \text{Cls induct} . \vee . s'(\kappa \cup \iota'\alpha) \in \text{Cls induct} \quad (3)$
$\vdash . *40\cdot21 . *120\cdot212 . \supset \vdash . s'\Lambda \in \text{Cls induct} \quad (4)$
$\vdash . (3) . (4) . *120\cdot26 . \supset \vdash :. \kappa \in \text{Cls induct} . \supset :$
$\quad \exists ! \kappa - \text{Cls induct} . \vee . s'\kappa \in \text{Cls induct} :.$
$[*5\cdot6] \supset \vdash : \kappa \in \text{Cls induct} . \kappa \subset \text{Cls induct} . \supset . s'\kappa \in \text{Cls induct} \quad (5)$
$\vdash . (5) . *120\cdot741 . \supset \vdash . \text{Prop}$

***120·76.** $\vdash : \kappa \in \text{Cls induct} . \kappa \subset \text{Cls induct} . \supset . \epsilon_\Delta‘\kappa \in \text{Cls induct}$

Dem.

$\vdash . *51·2 . \supset \vdash :. \alpha \in \kappa . \supset : \kappa = \kappa \cup \iota‘\alpha :$

$[*13·12] \quad \supset : \epsilon_\Delta‘\kappa \in \text{Cls induct} . \supset . \epsilon_\Delta‘(\kappa \cup \iota‘\alpha) \in \text{Cls induct} \quad (1)$

$\vdash . *83·41 . *114·301 . \supset \vdash : \alpha \sim\in \kappa . \supset . \epsilon_\Delta‘(\kappa \cup \iota‘\alpha) \text{ sm } \epsilon_\Delta‘\kappa \times \alpha \quad (2)$

$\vdash . (2) . *120·214 . \supset \vdash :. \alpha \sim\in \kappa . \supset :$

$\quad \epsilon_\Delta‘(\kappa \cup \iota‘\alpha) \in \text{Cls induct} . \equiv . \epsilon_\Delta‘\kappa \times \alpha \in \text{Cls induct} \quad (3)$

$\vdash . (3) . *120·72 . \supset \vdash :. \alpha \sim\in \kappa . \supset :$

$\quad \epsilon_\Delta‘\kappa, \alpha \in \text{Cls induct} . \supset . \epsilon_\Delta‘(\kappa \cup \iota‘\alpha) \in \text{Cls induct} \quad (4)$

$\vdash . (1) . (4) . \quad \supset \vdash : \epsilon_\Delta‘\kappa, \alpha \in \text{Cls induct} . \supset . \epsilon_\Delta‘(\kappa \cup \iota‘\alpha) \in \text{Cls induct} \quad (5)$

$\vdash . (5) . *51·2 . \text{Syll} . \quad \supset \vdash :. \kappa \subset \text{Cls induct} . \supset . \epsilon_\Delta‘\kappa \in \text{Cls induct} : \supset :$

$\quad \kappa \cup \iota‘\alpha \subset \text{Cls induct} . \supset . \epsilon_\Delta‘(\kappa \cup \iota‘\alpha) \in \text{Cls induct} \quad (6)$

$\vdash . *83·15 . *120·213 . \supset \vdash . \epsilon_\Delta‘\Lambda \in \text{Cls induct} .$

$[\text{Simp}] \quad \supset \vdash : \Lambda \subset \text{Cls induct} . \supset . \epsilon_\Delta‘\Lambda \in \text{Cls induct} \quad (7)$

$\vdash . (6) . (7) . *120·26 . \quad \supset \vdash :. \kappa \in \text{Cls induct} . \supset :$

$\quad \kappa \subset \text{Cls induct} . \supset . \epsilon_\Delta‘\kappa \in \text{Cls induct} :. \supset \vdash . \text{Prop}$

The following propositions are concerned in establishing the converse of *120·76 subject to a suitable hypothesis. The final outcome is given in *120·77.

***120·761.** $\vdash : \exists ! \epsilon_\Delta‘\kappa . \epsilon_\Delta‘\kappa \in \text{Cls induct} . \supset . \kappa \subset \text{Cls induct}$

Dem.

$\vdash . *83·41 . *114·301 . \supset \vdash :. \alpha \in \kappa . \supset : \epsilon_\Delta‘\kappa \text{ sm } \alpha \times \epsilon_\Delta‘(\kappa - \iota‘\alpha) : \quad (1)$

$[*120·214] \supset : \epsilon_\Delta‘\kappa \in \text{Cls induct} . \equiv . \alpha \times \epsilon_\Delta‘(\kappa - \iota‘\alpha) \in \text{Cls induct} \quad (2)$

$\vdash . (1) . *113·114 . \quad \supset \vdash : \exists ! \epsilon_\Delta‘\kappa . \alpha \in \kappa . \supset . \exists ! \alpha . \exists ! \epsilon_\Delta‘(\kappa - \iota‘\alpha) \quad (3)$

$\vdash . (2) . (3) . *120·721 . \supset \vdash :. \exists ! \epsilon_\Delta‘\kappa . \alpha \in \kappa . \supset :$

$\quad \epsilon_\Delta‘\kappa \in \text{Cls induct} . \supset . \alpha \in \text{Cls induct} \quad (4)$

$\vdash . (4) . \text{Comm} . \supset \vdash . \text{Prop}$

***120·762.** $\vdash : \kappa \in \text{Cls induct} . \Lambda \sim\in \kappa . \sim\exists ! 1 \cap \kappa . \supset . (\exists R, S) . R, S \in \epsilon_\Delta‘\kappa . R \dot{\cap} S = \dot{\Lambda}$

Dem.

$\vdash . *51·2 . \supset \vdash : R, S \in \epsilon_\Delta‘\kappa . R \dot{\cap} S = \dot{\Lambda} . \alpha \in \kappa . \supset . R, S \in \epsilon_\Delta‘(\kappa \cup \iota‘\alpha) . R \dot{\cap} S = \dot{\Lambda} \quad (1)$

$\vdash . *83·5 . *55·201 . \supset$

$\vdash : R, S \in \epsilon_\Delta‘\kappa . R \dot{\cap} S = \dot{\Lambda} . x, y \in \alpha . x \neq y . \alpha \sim\in \kappa . \supset .$

$\quad R \dot{\cup} x \downarrow \alpha, S \dot{\cup} y \downarrow \alpha \in \epsilon_\Delta‘(\kappa \cup \iota‘\alpha) . (R \dot{\cup} x \downarrow \alpha) \dot{\cap} (S \dot{\cup} y \downarrow \alpha) = \dot{\Lambda} \quad (2)$

$\vdash . (1) . (2) . *52·41 . \supset \vdash : R, S \in \epsilon_\Delta‘\kappa . R \dot{\cap} S = \dot{\Lambda} . \alpha \neq \Lambda . \alpha \sim\in 1 . \supset .$

$\quad (\exists P, Q) . P, Q \in \epsilon_\Delta‘(\kappa \cup \iota‘\alpha) . P \dot{\cap} Q = \dot{\Lambda} \quad (3)$

$\vdash . *51·16 . \supset \vdash :. \alpha = \Lambda . \vee . \alpha \in 1 : \supset : \Lambda \in (\kappa \cup \iota‘\alpha) . \vee . \exists ! 1 \cap (\kappa \cup \iota‘\alpha) \quad (4)$

$\vdash . *22·58 . \supset \vdash :. \Lambda \in \kappa . \vee . \exists ! 1 \cap \kappa : \supset : \Lambda \in (\kappa \cup \iota‘\alpha) . \vee . \exists ! 1 \cap (\kappa \cup \iota‘\alpha) \quad (5)$

$\vdash . (3) . (4) . (5) . \supset \vdash :. \Lambda \in \kappa . \vee . \exists ! 1 \cap \kappa . \vee . (\exists R, S) . R, S \in \epsilon_\Delta‘\kappa . R \dot{\cap} S = \dot{\Lambda} : \supset :$

$\quad \Lambda \in (\kappa \cup \iota‘\alpha) . \vee . \exists ! 1 \cap (\kappa \cup \iota‘\alpha) . \vee . (\exists R, S) . R, S \in \epsilon_\Delta‘(\kappa \cup \iota‘\alpha) . R \dot{\cap} S = \dot{\Lambda} \quad (6)$

$\vdash . *83·15 . \supset \vdash . (\exists R, S) . R, S \in \epsilon_\Delta‘\Lambda . R \dot{\cap} S = \dot{\Lambda} \quad (7)$

$\vdash . (6) . (7) . *120·26 . \supset \vdash . \text{Prop}$

∗120·764. $\vdash : \kappa \epsilon \text{Cls induct} . \Lambda \sim \epsilon \kappa . \sim \exists ! (1 \cap \kappa) . \supset . \text{Nc}^{\backprime}\epsilon_\Delta{}^{\backprime}\kappa \geqslant \text{Nc}^{\backprime}\kappa$
[∗120·762 . ∗117·681]

∗120·765. $\vdash : \kappa \epsilon \text{Cls induct} . \Lambda \sim \epsilon \kappa . \sim \exists ! (1 \cap \kappa) . \kappa \subset \lambda . \exists ! \epsilon_\Delta{}^{\backprime}\lambda . \supset .$
$\text{Nc}^{\backprime}\epsilon_\Delta{}^{\backprime}\lambda \geqslant \text{Nc}^{\backprime}\kappa$ [∗120·762 . ∗117·684]

∗120·766. $\vdash : \lambda \sim \epsilon \text{Cls induct} . \Lambda \sim \epsilon \lambda . \sim \exists ! (1 \cap \lambda) . \exists ! \epsilon_\Delta{}^{\backprime}\lambda . \supset .$
$\text{Nc}^{\backprime}\epsilon_\Delta{}^{\backprime}\lambda \sim \epsilon \text{NC induct}$

Dem.

$\vdash . ∗120·491 . \supset \vdash :. \text{Hp} . \supset : \nu \epsilon \text{NC induct} . \supset .$
$(\exists \kappa) . \kappa \subset \lambda . \text{Nc}^{\backprime}\kappa = \nu . \Lambda \sim \epsilon \kappa . \sim \exists ! (1 \cap \kappa) .$
[∗120·765] $\supset . \text{Nc}^{\backprime}\epsilon_\Delta{}^{\backprime}\lambda \geqslant \nu :$
[∗120·121] $\supset : \nu \epsilon \text{NC induct} . \supset . \text{Nc}^{\backprime}\epsilon_\Delta{}^{\backprime}\lambda \geqslant \nu +_c 1 .$
[∗120·429] $\supset . \text{Nc}^{\backprime}\epsilon_\Delta{}^{\backprime}\lambda > \nu :$
[∗117·42] $\supset : \text{Nc}^{\backprime}\epsilon_\Delta{}^{\backprime}\lambda \sim \epsilon \text{NC induct} :. \supset \vdash . \text{Prop}$

∗120·767. $\vdash : \epsilon_\Delta{}^{\backprime}\lambda \epsilon \text{Cls induct} . \Lambda \sim \epsilon \lambda . \sim \exists ! (1 \cap \lambda) . \exists ! \epsilon_\Delta{}^{\backprime}\lambda . \supset . \lambda \epsilon \text{Cls induct}$
[∗120·766 . Transp]

∗120·77. $\vdash :. \Lambda \sim \epsilon \kappa . \sim \exists ! (1 \cap \kappa) . \exists ! \epsilon_\Delta{}^{\backprime}\kappa . \supset :$
$\epsilon_\Delta{}^{\backprime}\kappa \epsilon \text{Cls induct} . \equiv . \kappa \epsilon \text{Cls induct} . \kappa \subset \text{Cls induct}$
[∗120·76·761·767]

$*121$. INTERVALS

Summary of $*121$.

The present number is concerned with the class of terms between x and y with respect to some relation P, *i.e.* those terms which lie on a road from x to y on which any two consecutive terms have the relation P. Such a road may be called a P-road, and if zPw, the step from z to w may be called a P-step. In order that a P-road from x to y should exist, it is necessary and sufficient that we should have $xP_{po}y$. When this condition is fulfilled, there will in general be many P-roads from x to y. But if $P \in \mathrm{Cls} \to 1 \,.\, \sim(yP_{po}y)$, or if $P \in 1 \to \mathrm{Cls} \,.\, \sim(xP_{po}x)$, then at most one road leads from x to y. This follows from the propositions of $*96$. In virtue of those propositions, if $P \in \mathrm{Cls} \to 1 \,.\, \sim(yP_{po}y) \,.\, xP_{po}y$, P is $1 \to 1$ throughout the road from x to y, and this road forms an open series. The two other possibilities with a $\mathrm{Cls} \to 1$ are (assuming $xP_{po}y$)

$$(1)\ xP_{po}x,$$
$$(2)\ yP_{po}y \,.\, \sim(xP_{po}x).$$

In the first case, there is a cyclic road from x to x, and there are two roads from x to y, one consisting of that part of the cycle which is required to reach y, the other consisting of this part together with the whole cycle required to travel from y back to y. Thus the class of terms which can be reached in some journey from x to y is the whole class of descendants of x, *i.e.* the class $\overleftarrow{R}_*{}'x$, which is the cycle composing the road from x to x.

In the second case, the descendants of x form a Q, and y is in the circular part of the Q. Here, as before, there are two roads from x to y, of which the first stops as soon as it reaches y, while the second proceeds to travel round the circle until it comes to y again. Thus here again, all the descendants of x lie on some road between x and y.

The *interval* between x and y is defined as the class of terms lying on some road from x to y. There will be four kinds of interval, according as we do or do not include the end-points as such. We denote the kind including both end-points by

$$P(x \vdash\!\dashv y),$$

that excluding both by

$$P(x - y),$$

and the other two respectively by

$$P(x -\!\!\dashv y),\ P(x \vdash\!\!- y).$$

The definitions are

$$\overleftarrow{P}_*{}'x \cap \overrightarrow{P}_*{}'y,\ \overleftarrow{P}_{po}{}'x \cap \overrightarrow{P}_{po}{}'y,\ \overleftarrow{P}_{po}{}'x \cap \overrightarrow{P}_*{}'y,\ \overleftarrow{P}_*{}'x \cap \overrightarrow{P}_{po}{}'y.$$

If P is either one-many or many-one, it will be one-one throughout the interval $P(x \mapsto y)$, except at most at one exceptional point, namely the junction of the tail and circle of the Q. If $xP_{po}x$ or $\sim(yP_{po}y)$, the interval between x and y cannot be Q-shaped, but must be either open or cyclic; in either case, P is $1 \rightarrow 1$ throughout $P(x \mapsto y)$, with no exceptions; for if $P \epsilon \text{Cls} \rightarrow 1$, P is $1 \rightarrow 1$ throughout the interval because the interval is contained in $\overleftarrow{P_*}\text{‘}x$, and if $P \epsilon 1 \rightarrow \text{Cls}$, because the interval is contained in $\overrightarrow{P_*}\text{‘}y$. Thus throughout this number we shall constantly have the hypothesis $P \epsilon (\text{Cls} \rightarrow 1) \cup (1 \rightarrow \text{Cls})$; if $P \epsilon \text{Cls} \rightarrow 1$, the interval is to be supposed traversed from x to y, while if $P \epsilon 1 \rightarrow \text{Cls}$, it is to be supposed traversed from y to x. In either case the interval between x and y must be an *inductive* class. This is proved in $*121{\cdot}47$. If, however, P is serial (cf. $*204$), and thus neither many-one nor one-many, the interval between x and y is the stretch of the series between x and y, with or without end-points according to the definition chosen, and need not be an inductive class.

If the interval between x and y (both included) has $\nu +_c 1$ members, we say that $xP_\nu y$. Thus if there is only one road from x to y, "$xP_\nu y$" means that it requires ν steps to get from x to y. Assuming $P \epsilon \text{Cls} \rightarrow 1$, if we also have $P_{po} \subset J$ (*i.e.* if none of the families of P are cyclic), then if $xP_\nu y$ and yPz, we shall have $xP_{\nu +_c 1} z$. On this basis an inductive theory of P_ν is built up, and it is shown that the class of such relations as P_ν for different inductive values of ν is the same as Potid‘P, the class of powers of P including $I \restriction C\text{‘}P$ ($*121{\cdot}5$). The definition of P_ν is

$$P_\nu = \hat{x}\hat{y}\,\{\text{N}_0\text{c‘}P(x \mapsto y) = \nu +_c 1\} \quad \text{Df.}$$

The whole class of such relations as P_ν for different inductive values of ν is called finid‘P, *i.e.* we put

$$\text{finid‘}P = \hat{R}\,\{(\exists \nu) \,.\, \nu \,\epsilon\, \text{NC induct} - \iota\text{‘}\Lambda \,.\, R = P_\nu\} \quad \text{Df.}$$

If $B\text{‘}P$ exists, and if $P \epsilon \text{Cls} \rightarrow 1$, then the descendants of $B\text{‘}P$, so long as we do not reach a term y for which $yP_{po}y$, may be unambiguously described as the 2nd, 3rd, ... νth, ... terms of the posterity of $B\text{‘}P$, $B\text{‘}P$ itself being the 1st term. The correlation thus effected with the inductive cardinals is the logical essence of the process of counting; the last cardinal used in the correlation is the cardinal number of terms counted. We will call these terms $1_P, 2_P, \ldots \nu_P, \ldots$, defining ν_P as follows:

$$\nu_P = \breve{P}_{\nu -_c 1}\text{‘}B\text{‘}P \quad \text{Df.}$$

This notation does not conflict with ν_ξ as defined in $*65{\cdot}01$. There ξ must be a class if ν is a cardinal, here ν must be a cardinal and P a relation.

Hence whenever ν_P exists, the number of terms from the beginning to ν_P (both included) is ν. This is the fact upon which counting relies. If P is a many-one and P_{po} is contained in diversity, and ν is any inductive cardinal

other than 0, then ν_P exists when and only when $\overleftarrow{P_*}{}^{\text{\textquoteleft}}B{}^{\text{\textquoteleft}}P$ has at least ν members; *i.e.* roughly speaking, ν_P exists whenever it could possibly be expected to exist. In this case the whole posterity of $B{}^{\text{\textquoteleft}}P$ is contained in the series $1_P, 2_P, \dots \nu_P, \dots$ (*121·62). If the posterity is an inductive class, this series stops; if not, it forms a *progression* (cf. *122).

The propositions of the present number are very useful, not only in this section, but in the ordinal theory of finite and infinite and in parts of the book subsequent to that theory.

After some propositions which merely repeat definitions and give immediate consequences, we proceed (*121·3 ff.) to the theory of P_ν. We have

***121·302.** $\vdash : P_{\mathrm{po}} \in J . \supset . P_0 = I \upharpoonright C{}^{\text{\textquoteleft}}P$

***121·305.** $\vdash : P_{\mathrm{po}} \in J . \supset . P_1 \in P$

***121·31.** $\vdash : P \in (1 \to \mathrm{Cls}) \cup (\mathrm{Cls} \to 1) . P_{\mathrm{po}} \in J . \supset . P_1 = P$

When P is a transitive serial relation, we shall have $P_1 = P \dot{-} P^2$.

***121·321.** $\vdash : \nu > 0 . \supset . P_\nu \in P_{\mathrm{po}}$

***121·333.** $\vdash : P \in \mathrm{Cls} \to 1 . P_{\mathrm{po}} \in J . \supset . P_{\nu +_c 1} = P \mid P_\nu$

***121·35·351·352.** $\vdash : P \in (1 \to \mathrm{Cls}) \cup (\mathrm{Cls} \to 1) . P_{\mathrm{po}} \in J . \mu, \nu \in \mathrm{NC\,induct} . \supset .$
$$P_\mu \mid P_\nu = P_\nu \mid P_\mu = P_{\mu +_c \nu}$$

A similar result holds for $(P_\mu)_\nu$, which $= P_{\mu \times_c \nu}$ in the same circumstances.

We next proceed to the proof that an interval (under a similar hypothesis) is always an inductive class. This occupies *121·4—·47, being summed up in the proposition

***121·47.** $\vdash : R \in (\mathrm{Cls} \to 1) \cup (1 \to \mathrm{Cls}) . \supset . R(x \mapsto\!\dashv z) \in \mathrm{Cls\,induct}$

This is an important proposition. It leads to

***121·481.** $\vdash :. R \in \mathrm{Cls} \to 1 . \supset : \mathrm{Nc}{}^{\text{\textquoteleft}}R(x \mapsto\!\dashv y) \leqslant \mathrm{Nc}{}^{\text{\textquoteleft}}R(x \mapsto\!\dashv z) . \equiv .$
$$R(x \mapsto\!\dashv y) \subset R(x \mapsto\!\dashv z)$$
with a similar proposition if $R \in 1 \to \mathrm{Cls}$.

The next set of propositions (*121·5—·52) is concerned with $\mathrm{finid}{}^{\text{\textquoteleft}}P$. Assuming $P \in (\mathrm{Cls} \to 1) \cup (1 \to \mathrm{Cls}) . P_{\mathrm{po}} \in J$, we prove that $\mathrm{finid}{}^{\text{\textquoteleft}}P = \mathrm{Potid}{}^{\text{\textquoteleft}}P$ and $\mathrm{finid}{}^{\text{\textquoteleft}}P - \iota{}^{\text{\textquoteleft}}P_0 \subset \mathrm{Pot}{}^{\text{\textquoteleft}}P$ (*121·5); that if P is not null, $\mathrm{finid}{}^{\text{\textquoteleft}}P - \iota{}^{\text{\textquoteleft}}P_0 = \mathrm{Pot}{}^{\text{\textquoteleft}}P$ (*121·501); that $\dot{s}{}^{\text{\textquoteleft}}\mathrm{finid}{}^{\text{\textquoteleft}}P = P_*$ (*121·52) and $\dot{s}{}^{\text{\textquoteleft}}(\mathrm{finid}{}^{\text{\textquoteleft}}P - \iota{}^{\text{\textquoteleft}}P_0) = P_{\mathrm{po}}$ (*121·502); and that $P_2 = P^2 . P_3 = P^3 .$ etc. (*121·51).

Our next set of propositions is concerned with ν_P (*121·6—·638). We have

***121·601.** $\vdash : \mathrm{E}\,!\, B{}^{\text{\textquoteleft}}P . \supset . B{}^{\text{\textquoteleft}}P = 1_P . \sim \{(B{}^{\text{\textquoteleft}}P)\, P_{\mathrm{po}}\, (B{}^{\text{\textquoteleft}}P)\}$

***121·602.** $\vdash : \mathrm{E}\,!\, B{}^{\text{\textquoteleft}}P . P \in 1 \to 1 . \supset . \breve{P}{}^{\text{\textquoteleft}}B{}^{\text{\textquoteleft}}P = 2_P$

***121·634.** $\vdash :. P \in \mathrm{Cls} \to 1 . P_{\mathrm{po}} \in J . \nu \in \mathrm{NC\,induct} - \iota{}^{\text{\textquoteleft}}0 . \supset :$
$$\nu_P \in \mathrm{D}{}^{\text{\textquoteleft}}P . \equiv . \mathrm{E}\,!\,(\nu +_c 1)_P$$

Finally we have three propositions (∗121·7—·72) on $\overrightarrow{R_*}\text{‘}x$, of which the most useful is

∗121·7. $\vdash : R \in 1 \to 1 \,.\, aBR \,.\, aR_* x \,.\, \supset \,.\, \overrightarrow{R_*}\text{‘}x = R\,(a \mapsto x) \,.\, \overrightarrow{R_*}\text{‘}x \in \text{Cls induct}$

∗121·01. $P\,(x - y) = \overleftarrow{P_{\text{po}}}\text{‘}x \cap \overrightarrow{P_{\text{po}}}\text{‘}y$ Df

∗121·011. $P\,(x \dashv y) = \overleftarrow{P_{\text{po}}}\text{‘}x \cap \overrightarrow{P_*}\text{‘}y$ Df

∗121·012. $P\,(x \vdash y) = \overleftarrow{P_*}\text{‘}x \cap \overrightarrow{P_{\text{po}}}\text{‘}y$ Df

∗121·013. $P\,(x \mapsto y) = \overleftarrow{P_*}\text{‘}x \cap \overrightarrow{P_*}\text{‘}y$ Df

∗121·02. $P_\nu = \hat{x}\hat{y}\,\{\text{N}_0\text{c‘}P\,(x \mapsto y) = \nu +_{\text{c}} 1\}$ Df

∗121·03. $\text{finid‘}P = \hat{R}\,\{(\exists \nu) \,.\, \nu \in \text{NC induct} - \iota\text{‘}\Lambda \,.\, R = P_\nu\}$ Df

∗121·031. $\text{fin‘}P = \hat{R}\,\{(\exists \nu) \,.\, \nu \in \text{NC induct} - \iota\text{‘}\Lambda - \iota\text{‘}0 \,.\, R = P_\nu\}$ Df

∗121·04. $\nu_P = \breve{P}_{\nu -_{\text{c}} 1}\text{‘}B\text{‘}P$ Df

∗121·1. $\vdash : z \in P\,(x - y) \,.\, \equiv \,.\, xP_{\text{po}}z \,.\, zP_{\text{po}}y$ [(∗121·01)]

∗121·101. $\vdash : z \in P\,(x \dashv y) \,.\, \equiv \,.\, xP_{\text{po}}z \,.\, zP_*y$

∗121·102. $\vdash : z \in P\,(x \vdash y) \,.\, \equiv \,.\, xP_*z \,.\, zP_{\text{po}}y$

∗121·103. $\vdash : z \in P\,(x \mapsto y) \,.\, \equiv \,.\, xP_*z \,.\, zP_*y$

∗121·11. $\vdash : xP_\nu y \,.\, \equiv \,.\, \text{N}_0\text{c‘}P\,(x \mapsto y) = \nu +_{\text{c}} 1$ [(∗121·02)]

∗121·12. $\vdash : R \in \text{finid‘}P \,.\, \equiv \,.\, (\exists \nu) \,.\, \nu \in \text{NC induct} - \iota\text{‘}\Lambda \,.\, R = P_\nu$ [(∗121·03)]

∗121·121. $\vdash : R \in \text{fin‘}P \,.\, \equiv .\, (\exists \nu) .\, \nu \in \text{NC induct} - \iota\text{‘}\Lambda - \iota\text{‘}0 .\, R = P_\nu$ [(∗121·031)]

∗121·13. $\vdash : f(\nu_P) \,.\, \equiv \,.\, f(\breve{P}_{\nu -_{\text{c}} 1}\text{‘}B\text{‘}P)$ [(∗121·04)]

∗121·131. $\vdash : \text{E}\,!\, \breve{P}_{\nu -_{\text{c}} 1}\text{‘}B\text{‘}P \,.\, \supset \,.\, \nu_P = \breve{P}_{\nu -_{\text{c}} 1}\text{‘}B\text{‘}P$ [∗121·13 . ∗14·28]

∗121·14. $\vdash \,.\, P\,(x - y) = \breve{P}\,(y - x)$ [∗121·1 . ∗91·53]

∗121·141. $\vdash \,.\, P\,(x \dashv y) = \breve{P}\,(y \vdash x)$

∗121·142. $\vdash \,.\, P\,(x \vdash y) = \breve{P}\,(y \dashv x)$

∗121·143. $\vdash \,.\, P\,(x \mapsto y) = \breve{P}\,(y \mapsto x)$

∗121·2. $\vdash : \sim(xP_{\text{po}}x) \,.\, \supset \,.\, x \sim\in P\,(x - y)$ [∗121·1]

∗121·201. $\vdash : \sim(yP_{\text{po}}y) \,.\, \supset \,.\, y \sim\in P\,(x - y)$

∗121·202. $\vdash : P_{\text{po}} \subseteq J \,.\, \supset \,.\, x, y \sim\in P\,(x - y)$ [∗121·2·201]

∗121·21. $\vdash : xP_{\text{po}}y \,.\, \equiv \,.\, y \in P\,(x \dashv y) \,.\, \equiv \,.\, \exists\,!\, P\,(x \dashv y)$

Dem.

$\vdash \,.\, ∗90·12 \,.\, ∗91·54 \,.\, \supset \vdash : xP_{\text{po}}y \,.\, \equiv \,.\, xP_{\text{po}}y \,.\, yP_*y \,.$

[∗121·101] $\equiv \,.\, y \in P\,(x \dashv y)$ (1)

$\vdash \,.\, ∗121·101 \,.\, \supset \vdash : \exists\,!\, P\,(x \dashv y) \,.\, \equiv \,.\, xP_{\text{po}} \,|\, P_*y \,.$

[∗91·574] $\equiv \,.\, xP_{\text{po}}y$ (2)

$\vdash \,.\, (1) \,.\, (2) \,.\, \supset \vdash \,.\, \text{Prop}$

∗121·22. $\vdash : xP_{\text{po}}y \,.\equiv.\, x \in P(x \vdash y) \,.\equiv.\, \exists ! P(x \vdash y)$

∗121·23. $\vdash : xP_{*}y \,.\equiv.\, x, y \in P(x \vdash\!\dashv y) \,.\equiv.\, \exists ! P(x \vdash\!\dashv y)$

∗121·231. $\vdash : x \in C\text{‘}P \,.\equiv.\, x \in P(x \vdash\!\dashv x) \,.\equiv.\, \exists ! P(x \vdash\!\dashv x)$ [∗121·23 . ∗90·12]

∗121·24. $\vdash : xP_{\text{po}}y \,.\supset.\, P(x \dashv y) = P(x - y) \cup \iota\text{‘}y$

Dem.

$\vdash . *91{\cdot}54 . *121{\cdot}101 . \supset$

$\vdash :. z \in P(x \dashv y) . \equiv : xP_{\text{po}}z : zP_{\text{po}}y . \vee . z = y . y \in C\text{‘}P :$

$[*13{\cdot}193 . *91{\cdot}504] \equiv : xP_{\text{po}}z . zP_{\text{po}}y . \vee . xP_{\text{po}}y . z = y$ (1)

$\vdash . (1) . *4{\cdot}73 . \supset \vdash :: \text{Hp} . \supset :. z \in P(x \dashv y) . \equiv : xP_{\text{po}}z . zP_{\text{po}}y . \vee . z = y :: \supset \vdash . \text{Prop}$

∗121·241. $\vdash : xP_{\text{po}}y \,.\supset.\, P(x \vdash y) = P(x - y) \cup \iota\text{‘}x$

∗121·242. $\vdash : xP_{*}y \,.\supset.\, P(x \vdash\!\dashv y) = P(x \dashv y) \cup \iota\text{‘}x = P(x \vdash y) \cup \iota\text{‘}y$

$= P(x - y) \cup \iota\text{‘}x \cup \iota\text{‘}y$

∗121·25. $\vdash . P_{\text{po}}(x - y) = P(x - y)$ [∗91·601 . ∗121·1]

∗121·251. $\vdash . P_{\text{po}}(x \dashv y) = P(x \dashv y)$

∗121·252. $\vdash . P_{\text{po}}(x \vdash y) = P(x \vdash y)$

∗121·253. $\vdash . P_{\text{po}}(x \vdash\!\dashv y) = P(x \vdash\!\dashv y)$

∗121·254. $\vdash . P_{\nu} = (P_{\text{po}})_{\nu}$ [∗121·253·11]

∗121·254 is frequently used in the theory of series.

∗121·26. $\vdash . \breve{P}_{\nu} = (\breve{P})_{\nu}$

Dem.

$\vdash . *121{\cdot}11{\cdot}143 . \supset \vdash : x\breve{P}_{\nu}y . \equiv . \text{N}_0\text{c}\text{‘}\breve{P}(x \vdash\!\dashv y) = \nu +_c 1 .$

$[*90{\cdot}132 . *121{\cdot}11] \quad \equiv . x(\breve{P})_{\nu}y : \supset \vdash . \text{Prop}$

∗121·27. $\vdash : xP_{\nu}y \,.\supset.\, \nu, \nu +_c 1 \in \text{NC} - \iota\text{‘}\Lambda$

Dem.

$\vdash . *121{\cdot}11 . *103{\cdot}12 . \supset \vdash : \text{Hp} . \supset . P(x \vdash\!\dashv y) \in \nu +_c 1$ (1)

$\vdash . (1) . *110{\cdot}4{\cdot}42 . \supset \vdash . \text{Prop}$

∗121·271. $\vdash : \sim(\nu, \nu +_c 1 \in \text{NC} - \iota\text{‘}\Lambda) \,.\supset.\, P_{\nu} = \dot{\Lambda}$ [∗121·27 . Transp]

∗121·272. $\vdash : \dot{\exists} ! P_{\nu} \,.\supset.\, \nu \geqslant 0 . \nu +_c 1 > 0 . \nu +_c 1 \geqslant 1$

Dem.

$\vdash . *117{\cdot}5 . *121{\cdot}27 . \supset \vdash : \text{Hp} . \supset . \nu \geqslant 0 .$ (1)

$[*117{\cdot}561 . *110{\cdot}641] \quad \supset . \nu +_c 1 \geqslant 1 .$ (2)

$[*117{\cdot}511{\cdot}531] \quad \supset . \nu +_c 1 > 0$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash . \text{Prop}$

∗121·273. $\vdash : \dot{\exists} ! P_{\nu +_c 1} \,.\supset.\, \nu +_c 1 > 0$

Dem.

$\vdash . *121{\cdot}27 . *110{\cdot}4 . \supset \vdash : \text{Hp} . \supset . \nu \in \text{NC} - \iota\text{‘}\Lambda .$

$[*117{\cdot}6] \quad \supset . \nu +_c 1 \geqslant 1 .$

$[*117{\cdot}511{\cdot}531] \quad \supset . \nu +_c 1 > 0 : \supset \vdash . \text{Prop}$

$*121\cdot3$. $\vdash . P_0 \subseteq I \upharpoonright C`P$

Dem.

$\vdash . *121\cdot11 . \supset \vdash : xP_0 y . \equiv . P(x \mapsto y) \in 1 .$
$[*121\cdot23] \quad \supset . xP_* y . x = y .$
$[*90\cdot12] \quad \supset . x(I \upharpoonright C`P) y : \supset \vdash . \text{Prop}$

$*121\cdot301$. $\vdash : \sim(xP_{po}x) . \supset : xP_0 y . \equiv . x \in C`P . x = y$

Dem.

$\vdash . *91\cdot542\cdot56 . \supset \vdash : xP_* z . zP_* x . x \neq z . \supset . xP_{po} x \quad (1)$
$\vdash . (1) . \text{Transp} . \supset \vdash :. \text{Hp} . \supset : xP_* z . zP_* x . \supset_{x,z} . x = z :$
$[*121\cdot231] \quad \supset : x \in C`P . \supset . P(x \mapsto x) = \iota`x :$
$[*13\cdot12 . *52\cdot22] \quad \supset : x \in C`P . x = y . \supset . P(x \mapsto y) \in 1 .$
$[*121\cdot11] \quad \supset . xP_0 y \quad (2)$
$\vdash . (2) . *121\cdot3 . \supset \vdash . \text{Prop}$

$*121\cdot302$. $\vdash : P_{po} \subseteq J . \supset . P_0 = I \upharpoonright C`P \quad [*121\cdot301]$

$*121\cdot303$. $\vdash : \text{Nc}`P(x \mapsto y) > 1 . \supset . xP_{po} y$

Dem.

$\vdash . *121\cdot23 . *52\cdot22 . *117\cdot42 . \supset \vdash :. \text{Hp} . \supset : x \in P(x \mapsto y) . P(x \mapsto y) \neq \iota`x :$
$[*51\cdot4 . \text{Transp}] \quad \supset : (\exists z) . z \neq x . z \in P(x \mapsto y) :$
$[*121\cdot103 . *91\cdot542] \quad \supset : (\exists z) . xP_{po} z . zP_* y :$
$[*91\cdot574] \quad \supset : xP_{po} y :. \supset \vdash . \text{Prop}$

$*121\cdot304$. $\vdash :. P_{po} \subseteq J . \supset : xP_1 y . \equiv . P(x \mapsto y) = \iota`x \cup \iota`y . x \neq y$

Dem.

$\vdash . *121\cdot303\cdot11 . \supset \vdash : \text{Hp} . xP_1 y . \supset . xP_{po} y .$
$[\text{Hp}] \quad \supset . x \neq y \quad (1)$
$\vdash . (1) . *54\cdot53\cdot101 . *121\cdot23\cdot11 . \supset \vdash . \text{Prop}$

$*121\cdot305$. $\vdash : P_{po} \subseteq J . \supset . P_1 \subseteq P$

Dem.

$\vdash . *121\cdot303 . \supset \vdash : \text{Hp} . xP_1 y . \supset . xP_{po} y .$
$[*91\cdot52] \quad \supset . (\exists z) . xPz . zP_* y \quad (1)$
$\vdash . *121\cdot304 . *91\cdot542 . \supset$
$\vdash :. \text{Hp} . xP_1 y . \supset : xP_{po} z . zP_* y . \supset . z = y :$
$[*91\cdot502] \quad \supset : xPz . zP_* y . \supset . z = y \quad (2)$
$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

$*121\cdot306$. $\vdash : P \in 1 \to \text{Cls} . \sim(xP_{po}x) . xPy . \supset . P(x \mapsto y) = \iota`x \cup \iota`y . x \neq y$

Dem.

$\vdash . *91\cdot542 . \supset \vdash : xP_* z . zP_* y . z \neq x . z \neq y . xPy . \supset : xP_{po} z . zP_{po} y . xPy :$
$[*34\cdot1] \quad \supset : xP_{po} z . zP_{po} | \breve{P} x :$
$[*92\cdot11] \quad \supset : P \in 1 \to \text{Cls} . \supset . xP_{po} z . zP_* x :$
$[*91\cdot574] \quad \supset : P \in 1 \to \text{Cls} . \supset . xP_{po} x \quad (1)$

$\vdash . (1) . \text{Transp} . \supset \vdash :: \text{Hp} . \supset :. xP_{*}z . zP_{*}y . \supset_z : z = x . \vee . z = y$ (2)

$\vdash . *121{\cdot}23 . \quad \supset \vdash : \text{Hp} . \supset . x, y \in P(x \mapsto y)$ (3)

$\vdash . *91{\cdot}502 . \quad \supset \vdash : \text{Hp} . \supset . x \neq y$ (4)

$\vdash . (2) . (3) . (4) . *121{\cdot}103 . \supset \vdash . \text{Prop}$

***121·307.** $\vdash : P \in \text{Cls} \to 1 . \sim(yP_{\text{po}}y) . xPy . \supset . P(x \mapsto y) = \iota'x \cup \iota'y . x \neq y$
[*121·306·143]

***121·308.** $\vdash : P \in (1 \to \text{Cls}) \cup (\text{Cls} \to 1) . P_{\text{po}} \in J . \supset . P \in P_1$
[*121·306·307·11 . *54·101]

***121·31.** $\vdash : P \in (1 \to \text{Cls}) \cup (\text{Cls} \to 1) . P_{\text{po}} \in J . \supset . P_1 = P$ [*121·305·308]

***121·32.** $\vdash . P_\nu \in P_{*}$

Dem.

$\vdash . *121{\cdot}11 . *120{\cdot}421 . *101{\cdot}14 . \text{Transp} . \supset \vdash : xP_\nu y . \supset . \exists ! P(x \mapsto y) .$

[*121·23] $\supset . xP_{*}y : \supset \vdash . \text{Prop}$

If ν is not a cardinal, or if $\nu +_c 1 = \Lambda$, $P_\nu = \dot{\Lambda}$.

***121·321.** $\vdash : \nu > 0 . \supset . P_\nu \in P_{\text{po}}$

Dem.

$\vdash . *120{\cdot}428 . *121{\cdot}11 . \supset \vdash : \text{Hp} . xP_\nu y . \supset . \text{Nc}'P(x \mapsto y) > 1 .$

[*117·55.*52·181.*121·23] $\supset . (\exists z) . z \in P(x \mapsto y) . z \neq x .$

[*121·103.*91·542] $\supset . (\exists z) . xP_{\text{po}}z . zR_{*}y .$

[*91·574] $\supset . xP_{\text{po}}y : \supset \vdash . \text{Prop}$

***121·322.** $\vdash . C'P_\nu \subset C'P$ [*121·32 . *90·14]

***121·323.** $\vdash : \nu > 0 . \supset . \text{D}'P_\nu \subset \text{D}'P . \text{Ꭰ}'P_\nu \subset \text{Ꭰ}'P$ [*121·321 . *91·504]

***121·324.** $\vdash . \text{D}'P_{\nu +_c 1} \subset \text{D}'P . \text{Ꭰ}'P_{\nu +_c 1} \subset \text{Ꭰ}'P$

Dem.

$\vdash . *121{\cdot}273{\cdot}323 . \supset \vdash : \dot{\exists} ! P_{\nu +_c 1} . \supset . \text{D}'P_{\nu +_c 1} \subset \text{D}'P . \text{Ꭰ}'P_{\nu +_c 1} \subset \text{Ꭰ}'P$ (1)

$\vdash . (1) . *33{\cdot}241 . \supset \vdash . \text{Prop}$

***121·325.** $\vdash : \dot{\exists} ! P_\mu \dot{\cap} P_\nu . \supset . \mu = \nu$

Dem.

$\vdash . *121{\cdot}11 . \supset \vdash : \text{Hp} . \supset . \exists ! (\mu +_c 1) \cap (\nu +_c 1) \cap t_0{}'\mu .$

[*100·42.*110·4] $\supset . \exists ! (\mu +_c 1) \cap t_0{}'\mu . (\mu +_c 1) \cap t_0{}'\mu = \nu +_c 1 .$

[*120·311] $\supset . \mu = \nu : \supset \vdash . \text{Prop}$

***121·326.** $\vdash . \text{fin}'P \subset \text{finid}'P . \text{finid}'P - \iota'P_0 \subset \text{fin}'P$ [*121·12·121]

***121·327.** $\vdash : \dot{\exists} ! P_0 . \supset . \text{fin}'P = \text{finid}'P - \iota'P_0$

Dem.

$\vdash . *121{\cdot}325 . \text{Transp} . *121{\cdot}121 . \supset \vdash :. \text{Hp} . \supset : R \in \text{fin}'P . \supset . R \neq P_0$ (1)

$\vdash . (1) . *121{\cdot}326 . \supset \vdash . \text{Prop}$

*121·33·331 are lemmas for *121·332, which is a very useful proposition.

***121·33.** $\vdash :. P \epsilon 1 \to \mathrm{Cls} . \supset : z \epsilon P(x - y) . \equiv . z \epsilon P(x \dashv P`y) :$
$z \epsilon P(x \vdash y) . \equiv . z \epsilon P(x \vdash\!\dashv P`y)$

Dem.

$\vdash . *71·7 . \supset \vdash :. \mathrm{Hp} . \supset : z P_* (P`y) . \equiv . z P_* | P y .$
$[*91·52] \qquad \equiv . z P_{\mathrm{po}} y \qquad (1)$
$\vdash . (1) . *121·1·101·102·103 . \supset \vdash . \mathrm{Prop}$

From the above proposition it follows that
$P \epsilon 1 \to \mathrm{Cls} . y \epsilon \text{ᗡ}`P . \supset . P(x - y) = P(x \dashv P`y) . P(x \vdash y) = P(x \vdash\!\dashv P`y).$

This does not follow unless $y \epsilon \text{ᗡ}`P$, because
$$P(x - y) = P(x \dashv P`y) . \supset . \mathrm{E}!\, P`y,$$
whereas
$$z \epsilon P(x - y) . \equiv_z . z \epsilon P(x \dashv P`y)$$
will always be true if $y \sim\epsilon \text{ᗡ}`P$, and therefore (when $P \epsilon 1 \to \mathrm{Cls}$) if $\sim \mathrm{E}!\, P`y$.

***121·331.** $\vdash :. P \epsilon 1 \to \mathrm{Cls} . P_{\mathrm{po}} \subset J . \supset : x P_\nu (P`y) . \equiv . x P_{\nu +_c 1} y$

Dem.

$\vdash . *121·324 . *71·16 . \supset \vdash :. \mathrm{Hp} . \supset : x P_{\nu +_c 1} y . \supset . \mathrm{E}!\, P`y \qquad (1)$
$\vdash . *121·33 . \qquad \supset \vdash : \mathrm{Hp} . \mathrm{E}!\, P`y . \supset . P(x \vdash y) = P(x \vdash\!\dashv P`y) \qquad (2)$
$\vdash . *121·242·32 . (2) . \quad \supset \vdash : \mathrm{Hp}(2) . x P_* y . \supset . P(x \vdash\!\dashv y) = P(x \vdash\!\dashv P`y) \cup \iota`y \quad (3)$
$\vdash . *91·52 . \qquad \supset \vdash : \mathrm{Hp} . \supset . \sim(y P_* | P y) .$
$[*71·7] \qquad \supset . \sim \{y P_* (P`y)\} .$
$[*121·103] \qquad \supset . \sim \{y \epsilon P(x \vdash\!\dashv P`y)\} \qquad (4)$
$\vdash . (3) . (4) . *110·63 . \supset \vdash : \mathrm{Hp}(3) . \supset . \mathrm{Nc}`P(x \vdash\!\dashv y) = \mathrm{Nc}`P(x \vdash\!\dashv P`y) +_c 1 \quad (5)$
$\vdash . (1) . (5) . *121·11·32 . \supset$
$\vdash : \mathrm{Hp} . x P_{\nu +_c 1} y . \supset . (\nu +_c 1) +_c 1 = \mathrm{Nc}`P(x \vdash\!\dashv P`y) +_c 1 .$
$[*120·311 . *121·27] \supset . \nu +_c 1 = \mathrm{Nc}`P(x \vdash\!\dashv P`y) .$
$[*121·11] \qquad \supset . x P_\nu (P`y) \qquad (6)$
$\vdash . (5) . *14·21 . *121·11·32 . \supset \vdash : \mathrm{Hp} . x P_\nu (P`y) . \supset . \mathrm{Nc}`P(x \vdash\!\dashv y) = (\nu +_c 1) +_c 1 .$
$[*121·11] \qquad \supset . x P_{\nu +_c 1} y \qquad (7)$
$\vdash . (6) . (7) . \supset \vdash . \mathrm{Prop}$

***121·332.** $\vdash : P \epsilon 1 \to \mathrm{Cls} . P_{\mathrm{po}} \subset J . \supset . P_{\nu +_c 1} = P_\nu | P \quad [*121·331]$

***121·333.** $\vdash : P \epsilon \mathrm{Cls} \to 1 . P_{\mathrm{po}} \subset J . \supset . P_{\nu +_c 1} = P | P_\nu$

***121·34.** $\vdash : P \epsilon 1 \to \mathrm{Cls} . P_{\mathrm{po}} \subset J . \nu \epsilon \mathrm{NC\ induct} . \supset . P_\nu \epsilon 1 \to \mathrm{Cls}$

Dem.

$\vdash . *121·3 . \quad \supset \vdash . P_0 \epsilon 1 \to \mathrm{Cls} \qquad (1)$
$\vdash . *121·332 . \supset \vdash :. \mathrm{Hp} . \supset : P_\nu \epsilon 1 \to \mathrm{Cls} . \supset . P_{\nu +_c 1} \epsilon 1 \to \mathrm{Cls} \qquad (2)$
$\vdash . (1) . (2) . *120·11 . \supset \vdash . \mathrm{Prop}$

***121·341.** $\vdash : P \epsilon \mathrm{Cls} \to 1 . P_{\mathrm{po}} \subset J . \nu \epsilon \mathrm{NC\ induct} . \supset . P_\nu \epsilon \mathrm{Cls} \to 1$

***121·342.** $\vdash : P \epsilon 1 \to 1 . P_{\mathrm{po}} \subset J . \nu \epsilon \mathrm{NC\ induct} . \supset . P_\nu \epsilon 1 \to 1 \quad [*121·34·341]$

∗121·35. $\vdash : P \in 1 \to \mathrm{Cls} \,.\, P_{\mathrm{po}} \in J \,.\, \mu, \nu \in \mathrm{NC\ induct} \,.\, \supset .\, P_\mu \,|\, P_\nu = P_{\mu +_c \nu}$

Dem.

$\vdash . *50\cdot62 \,.\, *121\cdot302\cdot322 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset .\, P_\mu \,|\, P_0 = P_{\mu +_c 0}$ (1)

$\vdash . *121\cdot332 \,.\quad \supset \vdash :. \mathrm{Hp} \,.\, \supset : \mu, \nu \in \mathrm{NC\ induct} \,.\, P_\mu \,|\, P_\nu = P_{\mu +_c \nu} \,.\, \supset .$

$P_\mu \,|\, P_{\nu +_c 1} = P_{\mu +_c \nu} \,|\, P$

[∗121·332] $\qquad = P_{\mu +_c \nu +_c 1}$ (2)

$\vdash . (1) . (2) . *120\cdot13 \,.\, \supset \vdash . \mathrm{Prop}$

∗121·351. $\vdash : P \in \mathrm{Cls} \to 1 \,.\, P_{\mathrm{po}} \in J \,.\, \mu, \nu \in \mathrm{NC\ induct} \,.\, \supset .\, P_\mu \,|\, P_\nu = P_{\mu +_c \nu}$

∗121·352. $\vdash : P \in (1 \to \mathrm{Cls}) \cup (\mathrm{Cls} \to 1) \,.\, P_{\mathrm{po}} \in J \,.\, \mu, \nu \in \mathrm{NC\ induct} \,.\, \supset .$

$P_\mu \,|\, P_\nu = P_\nu \,|\, P_\mu$ [∗121·35·351 . ∗110·51]

∗121·36. $\vdash : P \in (1 \to \mathrm{Cls}) \cup (\mathrm{Cls} \to 1) \,.\, P_{\mathrm{po}} \in J \,.\, \mu, \nu \in \mathrm{NC\ induct} - \iota\text{‘}0 \,.\, \supset .$

$(P_\mu)_\nu = P_{\mu \times_c \nu}$

Dem.

$\vdash . *121\cdot321 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset .\, P_\mu \in P_{\mathrm{po}} \,.$

[∗91·59·601] $\qquad \supset .\, (P_\mu)_{\mathrm{po}} \in J \,.$ (1)

[∗121·31·34·341] $\qquad \supset .\, (P_\mu)_1 = P_\mu$ (2)

$\vdash . *121\cdot332\cdot333\cdot352 \,.\, (1) \,.\, \supset$

$\vdash :. \mathrm{Hp} \,.\, \supset : (P_\mu)_{\nu +_c 1} = (P_\mu)_\nu \,|\, P_\mu :$

[∗34·27] $\supset : (P_\mu)_\nu = P_{\mu \times_c \nu} \,.\, \supset .\, (P_\mu)_{\nu +_c 1} = P_{\mu \times_c \nu} \,|\, P_\mu$

[∗121·35·351] $\qquad = P_{(\mu \times_c \nu) +_c \mu}$

[∗113·671] $\qquad = P_{\mu \times_c (\nu +_c 1)}$ (3)

$\vdash . (2) . (3) . *120\cdot47 \,.\, \supset \vdash . \mathrm{Prop}$

∗121·361. $\vdash : P \in (1 \to \mathrm{Cls}) \cup (\mathrm{Cls} \to 1) \,.\, P_{\mathrm{po}} \in J \,.\, \mu, \nu \in \mathrm{NC\ induct} - \iota\text{‘}0 \,.\, \supset .$

$(P_\mu)_\nu = (P_\nu)_\mu$ [∗121·36 . ∗113·27]

∗121·37. $\vdash : P \in \mathrm{Cls} \to 1 \,.\, y \in P(x \mapsto z) \,.\, \supset .\, P(x \mapsto z) = P(x \mapsto y) \cup P(y \mapsto z)$

Dem.

$\vdash . *121\cdot103 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset .\, xP_*y \,.\, yP_*z$ (1)

$\vdash . (1) . *121\cdot103 \,.\, \supset$

$\vdash :. \mathrm{Hp} \,.\, \supset : w \in P(x \mapsto z) \,.\, \equiv .\, xP_*w \,.\, wP_*z \,.\, xP_*y \,.\, yP_*z$ (2)

$\vdash . *96\cdot302 \,.\quad \supset \vdash :: \mathrm{Hp} \,.\, \supset :. xP_*w \,.\, xP_*y \,.\, \supset : wP_*y \,.\, \vee \,.\, yP_*w$ (3)

$\vdash . (2) . (3) . *4\cdot73 \,.\, \supset$

$\vdash :: \mathrm{Hp} \,.\, \supset :. w \in P(x \mapsto z) \,.\, \equiv : xP_*w \,.\, wP_*z \,.\, xP_*y \,.\, yP_*z \,.\, wP_*y \,.\, \vee \,.$

$xP_*w \,.\, wP_*z \,.\, xP_*y \,.\, yP_*z \,.\, yP_*w$ (4)

$\vdash . *90\cdot17 \,.\, *4\cdot73 \,.\, \supset \vdash : wP_*y \,.\, yP_*z \,.\, \equiv .\, wP_*z \,.\, wP_*y \,.\, yP_*z :$

$yP_*w \,.\, wP_*z \,.\, \equiv .\, yP_*z \,.\, wP_*z \,.\, yP_*w$ (5)

$\vdash . (4) . (5) . \supset \vdash :: \mathrm{Hp} \,.\, \supset :. w \in P(x \mapsto z) \,.\, \equiv : xP_*w \,.\, xP_*y \,.\, yP_*z \,.\, wP_*y \,.\, \vee \,.$

$xP_*w \,.\, wP_*z \,.\, xP_*y \,.\, yP_*w :$

[∗90·17.∗4·73] $\qquad \equiv : xP_*w \,.\, wP_*y \,.\, yP_*z \,.\, \vee \,.\, xP_*y \,.\, yP_*w \,.\, wP_*z :$

[(1).∗4·73] $\qquad \equiv : xP_*w \,.\, wP_*y \,.\, \vee \,.\, yP_*w \,.\, wP_*z :$

[∗121·103] $\qquad \equiv : w \in P(x \mapsto y) \cup P(y \mapsto z) :: \supset \vdash . \mathrm{Prop}$

$*121{\cdot}371$. $\vdash : P \in (\mathrm{Cls} \to 1) \cup (1 \to \mathrm{Cls}) \,.\, y \in P(x \vdash\!\dashv z) \,.\, \supset .$

$P(x \vdash\!\dashv z) = P(x \vdash\!\dashv y) \cup P(y \vdash\!\dashv z) = P(x \vdash\!\relbar y) \cup P(y \vdash\!\dashv z)$

$= P(x \vdash\!\dashv y) \cup P(y \relbar\!\dashv z)$ [Proof as in $*121{\cdot}37$]

$*121{\cdot}372$. $\vdash : P \in (\mathrm{Cls} \to 1) \cup (1 \to \mathrm{Cls}) \,.\, y \in P(x \relbar\!\dashv z) \,.\, \supset .$

$P(x \relbar\!\dashv z) = P(x \relbar\!\dashv y) \cup P(y \relbar\!\dashv z) = P(x \relbar\!\dashv y) \cup P(y \vdash\!\dashv z)$

$*121{\cdot}373$. $\vdash : P \in (\mathrm{Cls} \to 1) \cup (1 \to \mathrm{Cls}) \,.\, y \in P(x \vdash\!\relbar z) \,.\, \supset .$

$P(x \vdash\!\relbar z) = P(x \vdash\!\relbar y) \cup P(y \vdash\!\relbar z) = P(x \vdash\!\dashv y) \cup P(y \vdash\!\relbar z)$

$*121{\cdot}374$. $\vdash : P \in (\mathrm{Cls} \to 1) \cup (1 \to \mathrm{Cls}) \,.\, y \in P(x - z) \,.\, \supset .$

$P(x - z) = P(x \relbar\!\dashv y) \cup P(y - z) = P(x - y) \cup P(y \vdash\!\relbar z)$

$= P(x \relbar\!\dashv y) \cup P(y \vdash\!\relbar z)$

The proofs of these propositions are analogous to the proof of $*121{\cdot}37$.

$*121{\cdot}38$. $\vdash : R \in \mathrm{Cls} \to 1 \,.\, xR_{\mathrm{po}}x \,.\, \supset .\, R(x \vdash\!\dashv x) = \overleftarrow{R_*}{}^{\backprime}x$ [$*97{\cdot}5$]

$*121{\cdot}381$. $\vdash : R \in 1 \to \mathrm{Cls} \,.\, xR_{\mathrm{po}}x \,.\, \supset .\, R(x \vdash\!\dashv x) = \overrightarrow{R_*}{}^{\backprime}x$ [$*97{\cdot}501$]

$*121{\cdot}382$. $\vdash : R \in \mathrm{Cls} \to 1 \,.\, xR_{\mathrm{po}}x \,.\, xR_{\mathrm{po}}y \,.\, \supset .$

$R(x \vdash\!\dashv x) = R(x \vdash\!\dashv y) = \overleftarrow{R_*}{}^{\backprime}x = R(y \vdash\!\dashv y)$ [$*97{\cdot}5 \,.\, *91{\cdot}56$]

$*121{\cdot}383$. $\vdash : R \in 1 \to \mathrm{Cls} \,.\, xR_{\mathrm{po}}x \,.\, yR_{\mathrm{po}}x \,.\, \supset .$

$R(x \vdash\!\dashv x) = R(y \vdash\!\dashv x) = \overrightarrow{R_*}{}^{\backprime}x = R(y \vdash\!\dashv y)$

$*121{\cdot}384$. $\vdash : R \in (\mathrm{Cls} \to 1) \cup (1 \to \mathrm{Cls}) \,.\, xR_{\mathrm{po}}x \,.\, y \in R(x \vdash\!\dashv x) \,.\, \supset .$

$R(x \vdash\!\dashv x) = R(x \vdash\!\dashv y) = R(y \vdash\!\dashv x) = R(y \vdash\!\dashv y)$ [$*121{\cdot}382{\cdot}383$]

$*121{\cdot}39$. $\vdash :.\, R \in \mathrm{Cls} \to 1 \,.\, \supset : R(x \vdash\!\dashv y) \subset R(x \vdash\!\dashv z) \,.\, \vee \,.\, R(x \vdash\!\dashv z) \subset R(x \vdash\!\dashv y)$

Dem.

$\vdash . *96{\cdot}302 . \quad \supset \vdash :.\, \mathrm{Hp} \,.\, xR_*y \,.\, xR_*z \,.\, \supset : yR_*z \,.\, \vee \,.\, zR_*y$ (1)

$\vdash . *121{\cdot}37 . \quad \supset \vdash : \mathrm{Hp} \,.\, xR_*y \,.\, yR_*z \,.\, \supset .\, R(x \vdash\!\dashv y) \subset R(x \vdash\!\dashv z)$ (2)

$\vdash . *121{\cdot}37 . \quad \supset \vdash : \mathrm{Hp} \,.\, xR_*z \,.\, zR_*y \,.\, \supset .\, R(x \vdash\!\dashv z) \subset R(x \vdash\!\dashv y)$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash :.\, \mathrm{Hp} \,.\, xR_*y \,.\, xR_*z \,.\, \supset :$

$R(x \vdash\!\dashv y) \subset R(x \vdash\!\dashv z) \,.\, \vee \,.\, R(x \vdash\!\dashv z) \subset R(x \vdash\!\dashv y)$ (4)

$\vdash . *121{\cdot}23 . \quad \supset \vdash : \sim(xR_*y) \,.\, \supset .\, R(x \vdash\!\dashv y) = \Lambda \,.$

$[*24{\cdot}12] \quad \supset .\, R(x \vdash\!\dashv y) \subset R(x \vdash\!\dashv z)$ (5)

$\vdash . (5)\dfrac{z, y}{y, z} . \quad \supset \vdash : \sim(xR_*z) \,.\, \supset .\, R(x \vdash\!\dashv z) \subset R(x \vdash\!\dashv y)$ (6)

$\vdash . (4) . (5) . (6) . \supset \vdash . \mathrm{Prop}$

The following series of propositions are concerned with proving $*121{\cdot}47$, *i.e.*

$$R \in (\mathrm{Cls} \to 1) \cup (1 \to \mathrm{Cls}) \,.\, \supset .\, R(x \vdash\!\dashv z) \in \mathrm{Cls\ induct.}$$

The proof for $R \in 1 \to \mathrm{Cls}$ follows from that for $R \in \mathrm{Cls} \to 1$ by $*121{\cdot}143$. Confining ourselves, therefore, to $R \in \mathrm{Cls} \to 1$, we proceed as follows.

We prove first that, starting from z and going backwards, each new step adds only one term (which may not be distinct from all its predecessors); *i.e.* we have

$$R \in \mathrm{Cls} \to 1 \,.\, xRy \,.\, yR_*z \,.\, \supset .\, R(x \vdash\!\dashv z) = \iota^{\backprime}x \cup R(y \vdash\!\dashv z).$$

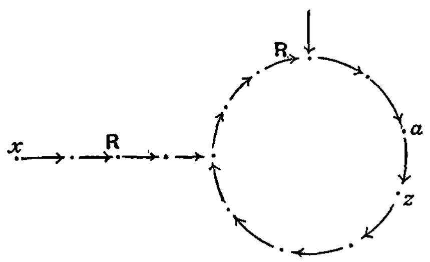
From this it follows by induction that if $R(z \vdash\!\dashv z)$ is an inductive class, so is $R(x \vdash\!\dashv z)$. Thus we only have to prove that $R(z \vdash\!\dashv z)$ is an inductive class. Here we must distinguish two cases, according as $\sim(zR_{\text{po}}z)$ or $zR_{\text{po}}z$. In the former case, we have

$$\exists ! R(z \vdash\!\dashv z) \,.\supset.\, R(z \vdash\!\dashv z) = \iota \text{`} z,$$

whence $R(z \vdash\!\dashv z)$ is an inductive class, and therefore so is $R(x \vdash\!\dashv z)$.

But in the latter case, when $zR_{\text{po}}z$, the matter is more difficult. In this case, z is a member of a cycle, the cycle being $R(z \vdash\!\dashv z)$. We have to prove that this cycle must be an inductive class. Given xR_*z, x will be a member of this cycle if $xR_{\text{po}}x$, and may be at the end of the tail of a Q, if $\sim(xR_{\text{po}}x)$. (Cf. $*96$.)

By $*96 \cdot 453$, we know that R is $1 \rightarrow 1$ when confined to $R(z \vdash\!\dashv z)$. Hence



in $R(z \vdash\!\dashv z)$, z has a unique predecessor, say a. Assume $a \neq z$. We then imagine a barrier placed between a and z, *i.e.* we construct a relation S which is to hold between any two consecutive members of $R(z \vdash\!\dashv z)$ except a and z. Putting $\alpha = R(z \vdash\!\dashv z) - \iota \text{`} a$, we have $S = \alpha \upharpoonleft R$. Then the relation S generates an *open* series consisting of all the terms of $R(z \vdash\!\dashv z)$; *i.e.* we have

$$\sim(aS_{\text{po}}a) \,.\, S(z \vdash\!\dashv a) = R(z \vdash\!\dashv z).$$

Hence, by our previous case, since $S(z \vdash\!\dashv a)$ is an inductive class, so is $R(z \vdash\!\dashv z)$.

If $a = z$, then by $*96 \cdot 33$ the cycle reduces to the single term z, and therefore $R(z \vdash\!\dashv z)$ is still an inductive class.

Hence $R(z \vdash\!\dashv z)$, and therefore $R(x \vdash\!\dashv z)$, is always an inductive class when $R \in \text{Cls} \rightarrow 1$, which was to be proved.

$*121 \cdot 4$. $\vdash : R \in \text{Cls} \rightarrow 1 \,.\, xRy \,.\, yR_*z \,.\supset.\, R(x \vdash\!\dashv z) = \iota \text{`} x \cup R(y \vdash\!\dashv z)$

Dem.

$\vdash . *90 \cdot 311 . \supset \vdash :: \text{Hp} . \supset :. xR_*w . \equiv : x = w . \vee . xR \,|\, R_*w :$

$[*71 \cdot 701 . \text{Hp}] \qquad \equiv : x = w . \vee . yR_*w$ (1)

$\vdash . *90 \cdot 172 . \supset \vdash :. \text{Hp} . \supset : x = w . \supset . wR_*z$ (2)

$\vdash . (1) . (2) . \supset \vdash :: \text{Hp} . \supset :. xR_*w . wR_*z . \equiv : x = w . \vee . yR_*w . wR_*z$ (3)

$\vdash . (3) . *121 \cdot 103 . \supset \vdash . \text{Prop}$

∗121·41. $\vdash : R \epsilon \mathrm{Cls} \to 1 \,.\, R(z \mapsto z) \epsilon \mathrm{Cls\ induct} \,.\, \supset \,.\, R(x \mapsto z) \epsilon \mathrm{Cls\ induct}$

Dem.

$\vdash \,.\, ∗121·4 \,.\, ∗120·251 \,.\, ∗90·172 \,.\, \supset \vdash :. \mathrm{Hp} \,.\, \supset :$

$yR_∗z \,.\, R(y \mapsto z) \epsilon \mathrm{Cls\ induct} \,.\, xRy \,.\, \supset \,.\, xR_∗z \,.\, R(x \mapsto z) \epsilon \mathrm{Cls\ induct}$ (1)

$\vdash \,.\, (1) \,.\, ∗90·112 \dfrac{\breve{R}}{R} \,.\quad \supset \vdash : \mathrm{Hp} \,.\, xR_∗z \,.\, \supset \,.\, R(x \mapsto z) \epsilon \mathrm{Cls\ induct}$ (2)

$\vdash \,.\, ∗121·23 \,.\, ∗120·212 \,.\, \supset \vdash : \sim(xR_∗z) \,.\, \supset \,.\, R(x \mapsto z) \epsilon \mathrm{Cls\ induct}$ (3)

$\vdash \,.\, (2) \,.\, (3) \,.\, \supset \vdash \,.\, \mathrm{Prop}$

In virtue of this proposition, we have only to prove $R(z \mapsto z) \epsilon \mathrm{Cls\ induct}$. This is obvious when $\sim(zR_{\mathrm{po}}z)$, for then either $R(z \mapsto z) = \iota`z$ or $R(z \mapsto z) = \Lambda$. But when $zR_{\mathrm{po}}z$, it is more difficult.

∗121·42. $\vdash : R \epsilon \mathrm{Cls} \to 1 \,.\, \sim(zR_{\mathrm{po}}z) \,.\, \supset \,.\, R(x \mapsto z) \epsilon \mathrm{Cls\ induct}$

Dem.

$\vdash \,.\, ∗121·303 \,.\, \mathrm{Transp} \,.\, ∗120·441 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, \mathrm{Nc}`R(z \mapsto z) \leqslant 1 \,.$

[∗120·48] $\supset \,.\, \mathrm{Nc}`R(z \mapsto z) \epsilon \mathrm{NC\ induct} \,.$

[∗120·211] $\supset \,.\, R(z \mapsto z) \epsilon \mathrm{Cls\ induct}$ (1)

$\vdash \,.\, (1) \,.\, ∗121·41 \,.\, \supset \vdash \,.\, \mathrm{Prop}$

∗121·43. $\vdash : R \epsilon \mathrm{Cls} \to 1 \,.\, zR_{\mathrm{po}}z \,.\, \supset \,.\, \mathrm{E}\,!\, \breve{\iota}`(\overrightarrow{R}`z \cap \overleftarrow{R_∗}`z)$

Dem.

$\vdash \,.\, ∗91·52 \,.\quad \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, (\exists a) \,.\, zR_∗a \,.\, aRz$ (1)

$\vdash \,.\, ∗96·453 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, (\overleftarrow{R_∗}`z) \upharpoonright R \epsilon 1 \to 1 \,.$

[∗71·122] $\supset \,.\, \hat{a}(zR_∗a \,.\, aRz) \epsilon 1 \cup \iota`\Lambda$ (2)

$\vdash \,.\, (1) \,.\, (2) \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, \hat{a}(zR_∗a \,.\, aRz) \epsilon 1 \,.$

[∗52·15] $\supset \,.\, \mathrm{E}\,!\, \breve{\iota}`(\overrightarrow{R}`z \cap \overleftarrow{R_∗}`z) : \supset \vdash \,.\, \mathrm{Prop}$

∗121·431. $\vdash : R \epsilon \mathrm{Cls} \to 1 \,.\, zR_{\mathrm{po}}z \,.\, a = \breve{\iota}`(\overrightarrow{R}`z \cap \overleftarrow{R_∗}`z) \,.\, \alpha = \overleftarrow{R_∗}`z - \iota`a \,.$
$S = \alpha \upharpoonright R \,.\, \supset \,.\, \sim(aS_{\mathrm{po}}a)$

Dem.

$\vdash \,.\, ∗35·61 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, a \sim\epsilon \mathrm{D}`S \,.$

[∗91·504] $\supset \,.\, a \sim\epsilon \mathrm{D}`S_{\mathrm{po}} \,.$

[∗33·14] $\supset \,.\, \sim(aS_{\mathrm{po}}a) : \supset \vdash \,.\, \mathrm{Prop}$

∗121·432. $\vdash : \mathrm{Hp}\ ∗121·431 \,.\, \supset \,.\, S(z \mapsto a) \epsilon \mathrm{Cls\ induct}$

Dem.

$\vdash \,.\, ∗71·261 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, S \epsilon \mathrm{Cls} \to 1$ (1)

$\vdash \,.\, (1) \,.\, ∗121·431·42 \,.\, \supset \vdash \,.\, \mathrm{Prop}$

***121·433.** $\vdash : \mathrm{Hp} *121{\cdot}431 \,.\, z \neq a \,.\, \supset .\, S(z \vdash\!\dashv a) = \overleftarrow{R_*}{}^{\backprime}z = R(z \vdash\!\dashv z)$

Dem.

$\vdash . *96{\cdot}11 . \quad \supset \vdash :. \mathrm{Hp} . \supset : zS_*w . \supset . zR_*w$ (1)

$\vdash . *51{\cdot}3 . *91{\cdot}504 . \supset \vdash :. \mathrm{Hp} . \supset : z \in \alpha . z \in \mathrm{D}{}^{\backprime}R :$

$[*35{\cdot}61] \quad \supset : z \in \mathrm{D}{}^{\backprime}S :$

$[*90{\cdot}12] \quad \supset : zS_*z$ (2)

$\vdash . (1) . *90{\cdot}16 . \quad \supset \vdash :: \mathrm{Hp} . \supset :. zS_*w . wRy . \supset : w \in \alpha \cup \iota{}^{\backprime}a . wRy :$

$[*35{\cdot}1] \quad \supset : wSy . \vee . w = a . wRy :$

$[\mathrm{Hp}.*71{\cdot}171] \quad \supset : wSy . \vee . y = z :$

$[*90{\cdot}16{\cdot}17.(2)] \quad \supset : zS_*y$ (3)

$\vdash . (2) . (3) . *90{\cdot}112 . \supset \vdash :. \mathrm{Hp} . \supset : zR_*w . \supset . zS_*w :$ (4)

$[\mathrm{Hp}] \quad \supset : zS_*a$ (5)

$\vdash . *71{\cdot}171 . \quad \supset \vdash : \mathrm{Hp} . aRy . \supset . y = z$ (6)

$\vdash . *91{\cdot}542{\cdot}504 . *35{\cdot}61 . \supset \vdash :. \mathrm{Hp} . \supset : wS_*a . w \neq a . wRy . \supset . wS_{\mathrm{po}}a . wSy .$

$[*92{\cdot}111] \quad \supset . yS_*a$ (7)

$\vdash . (5) . (6) . (7) . \quad \supset \vdash :. \mathrm{Hp} . \supset : wS_*a . wRy . \supset . yS_*a$ (8)

$\vdash . (5) . (8) . *90{\cdot}112 . \quad \supset \vdash :. \mathrm{Hp} . \supset : zR_*y . \supset . yS_*a$ (9)

$\vdash . (4) . (9) . \quad \supset \vdash :. \mathrm{Hp} . \supset : zR_*y . \supset . zS_*y . yS_*a$ (10)

$\vdash . (1) \frac{y}{w} . (10) . \quad \supset \vdash :. \mathrm{Hp} . \supset : zS_*y . yS_*a . \equiv . zR_*y :$

$[*121{\cdot}103] \quad \supset : S(z \vdash\!\dashv a) = \overleftarrow{R_*}{}^{\backprime}z$

$[*121{\cdot}38] \quad = R(z \vdash\!\dashv z) :. \supset \vdash . \mathrm{Prop}$

***121·434.** $\vdash : \mathrm{Hp} *121{\cdot}431 \,.\, z = a \,.\, \supset .\, \overleftarrow{R_*}{}^{\backprime}z = R(z \vdash\!\dashv z) = \iota{}^{\backprime}z$

Dem.

$\vdash . *32{\cdot}18 . \supset \vdash : \mathrm{Hp} . \supset . zRz .$

$[*96{\cdot}33] \quad \supset . \overleftarrow{R_*}{}^{\backprime}z = \iota{}^{\backprime}z .$ (1)

$[*121{\cdot}38] \quad \supset . R(z \vdash\!\dashv z) = \iota{}^{\backprime}z$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***121·44.** $\vdash : R \in \mathrm{Cls} \to 1 \,.\, zR_{\mathrm{po}}z \,.\, \supset .\, R(z \vdash\!\dashv z) \in \mathrm{Cls\ induct}$

Dem.

$\vdash . *121{\cdot}43{\cdot}432{\cdot}433 . \supset$

$\vdash : \mathrm{Hp} . z \neq \breve{\iota}{}^{\backprime}(\overrightarrow{R}{}^{\backprime}z \cap \overleftarrow{R_*}{}^{\backprime}z) . \supset . R(z \vdash\!\dashv z) \in \mathrm{Cls\ induct}$ (1)

$\vdash . *121{\cdot}434 . *120{\cdot}213 . \supset$

$\vdash : \mathrm{Hp} . z = \breve{\iota}{}^{\backprime}(\overrightarrow{R}{}^{\backprime}z \cap \overleftarrow{R_*}{}^{\backprime}z) . \supset . R(z \vdash\!\dashv z) \in \mathrm{Cls\ induct}$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***121·441.** $\vdash : R \in \mathrm{Cls} \to 1 \,.\, zR_{\mathrm{po}}z \,.\, \supset .\, R(x \vdash\!\dashv z) \in \mathrm{Cls\ induct}$ $[*121{\cdot}44{\cdot}41]$

***121·45.** $\vdash : R \in \mathrm{Cls} \to 1 \,.\, \supset .\, R(x \vdash\!\dashv z) \in \mathrm{Cls\ induct}$ $[*121{\cdot}42{\cdot}441]$

***121·46.** $\vdash : R \in 1 \to \mathrm{Cls} \,.\, \supset .\, R(x \vdash\!\dashv z) \in \mathrm{Cls\ induct}$ $[*121{\cdot}45{\cdot}143]$

***121·47.** $\vdash : R \in (\mathrm{Cls} \to 1) \cup (1 \to \mathrm{Cls}) \,.\, \supset .\, R(x \vdash\!\dashv z) \in \mathrm{Cls\ induct}$ $[*121{\cdot}45{\cdot}46]$

$*121{\cdot}48$. $\vdash :. R \in \mathrm{Cls} \to 1 . \supset :$

$$\mathrm{Nc}`R(x \vdash\!\dashv y) < \mathrm{Nc}`R(x \vdash\!\dashv z) . \equiv . \exists ! R(x \vdash\!\dashv z) - R(x \vdash\!\dashv y)$$

Dem.

$\vdash . *121{\cdot}39 . \supset \vdash :. \mathrm{Hp} . \supset : \exists ! R(x \vdash\!\dashv z) - R(x \vdash\!\dashv y) . \equiv .$
$\quad R(x \vdash\!\dashv y) \subset R(x \vdash\!\dashv z) . R(x \vdash\!\dashv y) \neq R(x \vdash\!\dashv z) .$

$[*120{\cdot}7 . *121{\cdot}45] \quad \supset . \mathrm{Nc}`R(x \vdash\!\dashv y) < \mathrm{Nc}`R(x \vdash\!\dashv z) \quad (1)$

$\vdash . *117{\cdot}222{\cdot}29 . \supset \vdash : \mathrm{Nc}`R(x \vdash\!\dashv y) < \mathrm{Nc}`R(x \vdash\!\dashv z) . \supset .$
$\quad \sim\{R(x \vdash\!\dashv z) \subset R(x \vdash\!\dashv y)\} .$

$[*24{\cdot}55] \quad \supset . \exists ! R(x \vdash\!\dashv z) - R(x \vdash\!\dashv y) \quad (2)$

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

$*121{\cdot}481$. $\vdash :. R \in \mathrm{Cls} \to 1 . \supset : \mathrm{Nc}`R(x \vdash\!\dashv y) \leqslant \mathrm{Nc}`R(x \vdash\!\dashv z) . \equiv .$
$\quad R(x \vdash\!\dashv y) \subset R(x \vdash\!\dashv z)$

Dem.

$\vdash . *121{\cdot}45 . *120{\cdot}441 . \supset$

$\vdash :. \mathrm{Hp} . \supset : \mathrm{Nc}`R(x \vdash\!\dashv y) \leqslant \mathrm{Nc}`R(x \vdash\!\dashv z) . \equiv . \sim\{\mathrm{Nc}`R(x \vdash\!\dashv z) < \mathrm{Nc}`R(x \vdash\!\dashv y)\} .$

$[*121{\cdot}48] \quad \equiv . \sim \exists ! R(x \vdash\!\dashv y) - R(x \vdash\!\dashv z) .$

$[*24{\cdot}55] \quad \equiv . R(x \vdash\!\dashv y) \subset R(x \vdash\!\dashv z) :. \supset \vdash . \mathrm{Prop}$

The above proposition is used in the proof of $*122{\cdot}35$, which is an important proposition in the theory of progressions.

The following propositions are concerned with the identification of such relations as P_ν with powers of P in the sense of $*91$.

$*121{\cdot}5$. $\vdash : P \in (\mathrm{Cls} \to 1) \cup (1 \to \mathrm{Cls}) . P_{\mathrm{po}} \subseteq J . \supset .$
$\quad \mathrm{finid}`P = \mathrm{Potid}`P . \mathrm{fin}`P = \mathrm{Pot}`P$

Dem.

$\vdash . *121{\cdot}302{\cdot}31 . \supset \vdash : \mathrm{Hp} . \supset . P_0 = I \upharpoonright C`P . P_1 = P \quad (1)$

$\vdash . (1) . *121{\cdot}332{\cdot}333{\cdot}352 . \supset \vdash :. \mathrm{Hp} . \nu \in \mathrm{NC\,induct} . \supset : P_{\nu +_c 1} = P_\nu | P : \quad (2)$

$[*91{\cdot}341] \quad \supset : P_\nu \in \mathrm{Potid}`P . \supset . P_{\nu +_c 1} \in \mathrm{Potid}`P : P_\nu \in \mathrm{Pot}`P . \supset . P_{\nu +_c 1} \in \mathrm{Pot}`P \quad (3)$

$\vdash . (1) . *91{\cdot}35 . \supset \vdash : \mathrm{Hp} . \supset . P_0 \in \mathrm{Potid}`P . P_1 \in \mathrm{Pot}`P \quad (4)$

$\vdash . (3) . (4) . *120{\cdot}13{\cdot}47 . \supset \vdash :. \mathrm{Hp} . \supset : \nu \in \mathrm{NC\,induct} . \supset . P_\nu \in \mathrm{Potid}`P :$
$\quad \nu \in \mathrm{NC\,induct} - \iota`0 . \supset . P_\nu \in \mathrm{Pot}`P :$

$[*121{\cdot}12{\cdot}121] \quad \supset : \mathrm{finid}`P \subset \mathrm{Potid}`P . \mathrm{fin}`P \subset \mathrm{Pot}`P \quad (5)$

$\vdash . (2) . *121{\cdot}121 . \supset \vdash :. \mathrm{Hp} . \supset : \nu \in \mathrm{NC\,induct} . \supset . P_\nu | P \in \mathrm{fin}`P :$

$[*121{\cdot}12] \quad \supset : Q \in \mathrm{finid}`P . \supset . Q | P \in \mathrm{fin}`P :$

$[(1) . *91{\cdot}17{\cdot}171] \quad \supset : \mathrm{Potid}`P \subset \mathrm{finid}`P . \mathrm{Pot}`P \subset \mathrm{fin}`P \quad (6)$

$\vdash . (5) . (6) . \supset \vdash . \mathrm{Prop}$

$*121{\cdot}501$. $\vdash : P \in (\mathrm{Cls} \to 1) \cup (1 \to \mathrm{Cls}) . P_{\mathrm{po}} \subseteq J . \dot{\exists} ! P . \supset .$
$\quad \mathrm{Pot}`P = \mathrm{finid}`P - \iota`P_0 = \mathrm{fin}`P$

Dem.

$\vdash . *121{\cdot}302 . \supset \vdash : \mathrm{Hp} . \supset . \dot{\exists} ! P_0 \quad (1)$

$\vdash . (1) . *121{\cdot}5{\cdot}327 . \supset \vdash . \mathrm{Prop}$

$*121\cdot502$. $\vdash : P \in (\text{Cls} \to 1) \cup (1 \to \text{Cls}) \,.\, \dot{P}_{\text{po}} \subset J \,.\, \supset .$
$$\dot{s}`(\text{finid}`P - \iota`P_0) = P_{\text{po}} = \dot{s}`\text{fin}`P$$

Dem.

$\vdash . *91\cdot504 . *33\cdot24 . *121\cdot5 . \supset \vdash : P = \dot{\Lambda} . \supset . \dot{s}`(\text{finid}`P - \iota`P_0) = \dot{\Lambda} = P_{\text{po}}$ (1)

$\vdash . (1) . *121\cdot501\cdot5 . \supset \vdash . \text{Prop}$

$*121\cdot51$. $\vdash : P \in (\text{Cls} \to 1) \cup (1 \to \text{Cls}) \,.\, P_{\text{po}} \subset J \,.\, \supset . P_2 = P^2 . P_3 = P^3 . \text{etc.}$

Dem.

$\vdash . *121\cdot31 . \quad \supset \vdash : \text{Hp} . \supset . P_1 = P$ (1)

$\vdash . *121\cdot332\cdot333 . \quad \supset \vdash : \text{Hp} . \supset . P_2 = P_1 | P_1$

$[(1)] \qquad = P^2$ (2)

$\vdash . *121\cdot332\cdot333\cdot352 . \supset \vdash : \text{Hp} . \supset . P_3 = P_2 | P_1$

$[(1).(2)] \qquad = P^3$ (3)

$\vdash . (2) . (3) . \text{etc.} . \supset \vdash . \text{Prop}$

$*121\cdot52$. $\vdash : P \in (\text{Cls} \to 1) \cup (1 \to \text{Cls}) \,.\, P_{\text{po}} \subset J \,.\, \supset . \dot{s}`\text{finid}`P = P_*$
$[*121\cdot5 . *91\cdot55]$

We shall at a later stage ($*301$) give a general definition of P^ν. When this definition has been introduced, we shall be able to prove, with the hypothesis of $*121\cdot51$,
$$\nu \in \text{NC induct} . \supset . P_\nu = P^\nu.$$
The definition of P^ν is postponed on account of various complications which render a general definition of P^ν difficult. The chief difficulty arises when $\exists ! P \dot{\wedge} I$. Thus suppose we have yPy; we shall also have yP^2y, yP^3y, etc. Hence if we have xPy, we have
$$\nu \in \text{NC induct} - \iota`0 . \supset_\nu . xP^\nu y.$$
Again, suppose this case excluded, but suppose
$$(\exists \mu, y) . \mu \in \text{NC induct} . y \in P(x \mapsto z) . yP^\mu y.$$
Then we shall have
$$\nu \in \text{NC induct} - \iota`0 - \iota`\Lambda . \supset . yP^{\mu \times_c \nu} y.$$
Thus the general definition of P^ν has to be complicated, except when $P_{\text{po}} \subset J$.

The following propositions are concerned with the series of relations P_ν and the series of terms ν_P. The relation P_ν holds between two terms (roughly speaking) when it requires ν steps to get from the first to the second; the term ν_P is the νth term starting from $B`P$, which, when it exists, is $1_P$. In order that $\nu_P$ should exist, it is necessary that $B`P$ should exist, and that there should be just one term x in the field of P such that the interval from $B`P$ to $x$ (both included) consists of $\nu$ terms. When this is the case for all inductive cardinals from 1 to $\nu$, we can say that $P$ generates a series starting from $B`P$ and having at least ν terms, each correlated with one of the cardinals in the interval from 1 to ν, both included; *i.e.* the series has a μth term, whenever $1 \leqslant \mu \leqslant \nu$. If this holds for all inductive values of ν, the family of

$B‘P$ is a progression*. (It will be observed that all such terms as ν_P belong to the family of $B‘P$, which need not form the whole field of P.)

$*121{\cdot}6$. $\vdash :. \nu \neq 0 . \supset . f(\nu_P) . \equiv . f[\breve{\iota}‘\hat{y}\,\{\mathrm{Nc}‘P\,(B‘P \mapsto y) = \nu\}]$

Dem.

$\vdash . *121{\cdot}11 . *120{\cdot}414{\cdot}416 . \supset \vdash :. \mathrm{Hp} . \supset :$

$f[\breve{\iota}‘\hat{y}\,\{\mathrm{Nc}‘P\,(B‘P \mapsto y) = \nu\}] . \equiv . f[\breve{\iota}‘\hat{y}\,\{(B‘P)\,P_{\nu \dot{-}_c 1}\,y\}] .$

$[*121{\cdot}13]$ $\equiv . f(\nu_P) : \supset \vdash . \mathrm{Prop}$

$*121{\cdot}601$. $\vdash : \mathrm{E} \,!\, B‘P . \supset . B‘P = 1_P . \sim\{(B‘P)\,P_{\mathrm{po}}\,(B‘P)\}$

Dem.

$\vdash . *91{\cdot}504 . *93{\cdot}1 . \supset \vdash . \sim\{(B‘P)\,P_{\mathrm{po}}\,(B‘P)\} .$ (1)

$[*121{\cdot}301]$ $\supset \vdash :. \mathrm{E} \,!\, B‘P . \supset : (B‘P)\,P_0\,y . \equiv_y . B‘P = y :$

$[*31{\cdot}17]$ $\supset : B‘P = \breve{P}_0‘B‘P :$

$[*121{\cdot}13]$ $\supset : B‘P = 1_P$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

$*121{\cdot}602$. $\vdash : \mathrm{E} \,!\, B‘P . P \,\epsilon\, 1 \to 1 . \supset . \breve{P}‘B‘P = 2_P$

Dem.

$\vdash . *121{\cdot}306{\cdot}601 . \supset \vdash : \mathrm{Hp} . \supset . P(B‘P \mapsto \breve{P}‘B‘P) \,\epsilon\, 2$ (1)

$\vdash . *121{\cdot}23{\cdot}601 . \supset \vdash :: \mathrm{Hp} . \supset :. (B‘P)\,P_{\mathrm{po}}\,y . \supset . B‘P, y \,\epsilon\, P(B‘P \mapsto y) . B‘P \neq y :.$

$[*54{\cdot}53 . *121{\cdot}303] \supset :. P(B‘P \mapsto y) \,\epsilon\, 2 . \supset : P(B‘P \mapsto y) = \iota‘B‘P \cup \iota‘y . (B‘P)\,P_{\mathrm{po}}\,y :$

$[*92{\cdot}111]$ $\supset : (\breve{P}‘B‘P)\,P_{*}\,y . P(B‘P \mapsto y) = \iota‘B‘P \cup \iota‘y :$

$[*121{\cdot}103{\cdot}601]$ $\supset : \breve{P}‘B‘P \,\epsilon\, \iota‘B‘P \cup \iota‘y . \breve{P}‘B‘P \neq B‘P :$

$[*51{\cdot}232]$ $\supset : y = \breve{P}‘B‘P$ (2)

$\vdash . (1) . (2) . *121{\cdot}6 . \supset \vdash . \mathrm{Prop}$

$*121{\cdot}61$. $\vdash : P \,\epsilon\, 1 \to \mathrm{Cls} . P_{\mathrm{po}} \subset J . x \,\epsilon\, s‘\mathrm{gen}‘P . \supset .$

$(\exists a, \nu) . aBP . \nu \,\epsilon\, \mathrm{NC\,induct} . aP_\nu x$

Dem.

$\vdash . *93{\cdot}36 . \supset \vdash :. P \,\epsilon\, 1 \to \mathrm{Cls} . x \,\epsilon\, s‘\mathrm{gen}‘P . \supset . (\exists a) . aBP . aP_{*}x$ (1)

$\vdash . *121{\cdot}52 . \supset \vdash :. P \,\epsilon\, 1 \to \mathrm{Cls} . P_{\mathrm{po}} \subset J . \supset : aP_{*}x . \equiv . a\,(\dot{s}‘\mathrm{finid}‘P)\,x$ (2)

$\vdash . (1) . (2) . \supset \vdash : \mathrm{Hp} . \supset . (\exists a) . aBP . a\,(\dot{s}‘\mathrm{finid}‘P)\,x .$

$[*121{\cdot}12]$ $\supset . (\exists a, \nu) . aBP . \nu \,\epsilon\, \mathrm{NC\,induct} . aP_\nu x : \supset \vdash . \mathrm{Prop}$

$*121{\cdot}62$. $\vdash : P \,\epsilon\, \mathrm{Cls} \to 1 . P_{\mathrm{po}} \subset J . (B‘P)\,P_{*}\,x . \supset .$

$(\exists \nu) . \nu \,\epsilon\, \mathrm{NC\,induct} - \iota‘0 . x = \nu_P$

Dem.

$\vdash . *121{\cdot}52 . \supset \vdash : \mathrm{Hp} . \supset . (B‘P)(\dot{s}‘\mathrm{finid}‘P)\,x .$

$[*121{\cdot}12]$ $\supset . (\exists \nu) . \nu \,\epsilon\, \mathrm{NC\,induct} . (B‘P)\,P_\nu\,x$ (1)

* Cf. $*122$, below.

$\vdash . *121{\cdot}341 . \supset \vdash : \mathrm{Hp} . \nu \in \mathrm{NC\,induct} . \supset . P_\nu \in \mathrm{Cls} \to 1$ (2)

$\vdash . (1) . (2) . \quad \supset \vdash : \mathrm{Hp} . \supset . (\exists \nu) . \nu \in \mathrm{NC\,induct} . x = \breve{P}_\nu {}^\prime B^\prime P .$

[$*121{\cdot}13$] $\supset . (\exists \nu) . \nu \in \mathrm{NC\,induct} . x = (\nu +_c 1)_P .$

[$*120{\cdot}471$] $\supset . (\exists \mu) . \mu \in \mathrm{NC\,induct} - \iota^\prime 0 . x = \mu_P : \supset \vdash . \mathrm{Prop}$

$*121{\cdot}63$. $\vdash : \mathrm{E} ! \, \nu_P . \supset . \mathrm{N_0c}^\prime P (B^\prime P \mapsto \nu_P) = \nu$

Dem.

$\vdash . *121{\cdot}13{\cdot}131 . \supset \vdash : \mathrm{Hp} . \supset . (B^\prime P) \, P_{\nu -_c 1} \, \nu_P .$

[$*121{\cdot}11$] $\supset . \mathrm{N_0c}^\prime P (B^\prime P \mapsto \nu_P) = \nu : \supset \vdash . \mathrm{Prop}$

$*121{\cdot}631$. $\vdash :. P \in \mathrm{Cls} \to 1 . P_{\mathrm{po}} \in J . \nu \in \mathrm{NC\,induct} - \iota^\prime 0 . \supset :$
$\mathrm{N_0c}^\prime P (B^\prime P \mapsto y) = \nu . \equiv . y = \nu_P . \equiv . (B^\prime P) \, P_{\nu -_c 1} \, y$

Dem.

$\vdash . *120{\cdot}414{\cdot}416 . *121{\cdot}11 . \supset$

$\vdash :. \mathrm{Hp} . \supset : \mathrm{N_0c}^\prime P (B^\prime P \mapsto y) = \nu . \equiv . (B^\prime P) \, P_{\nu -_c 1} \, y .$ (1)

[$*121{\cdot}341$] $\equiv . y = \breve{P}_{\nu -_c 1} {}^\prime B^\prime P .$

[$*121{\cdot}13$] $\equiv . y = \nu_P$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

$*121{\cdot}632{\cdot}633$ are required for proving $*121{\cdot}634$.

$*121{\cdot}632$. $\vdash : P \in \mathrm{Cls} \to 1 . P_{\mathrm{po}} \in J . \nu \in \mathrm{NC\,induct} - \iota^\prime 0 . y = \nu_P . y P z . \supset . z = (\nu +_c 1)_P$

Dem.

$\vdash . *121{\cdot}13 . \supset \vdash : \mathrm{Hp} . \supset . (B^\prime P) \, P_{\nu -_c 1} \, y . y P z .$

[$*121{\cdot}333{\cdot}352$] $\supset . (B^\prime P) \, P_\nu \, z .$

[$*121{\cdot}631$] $\supset . z = (\nu +_c 1)_P : \supset \vdash . \mathrm{Prop}$

$*121{\cdot}633$. $\vdash : P \in \mathrm{Cls} \to 1 . P_{\mathrm{po}} \in J . \nu \in \mathrm{NC\,induct} - \iota^\prime 0 . \nu_P \in \mathrm{D}^\prime P . \supset .$
$\mathrm{E} ! \, (\nu +_c 1)_P . (\nu +_c 1)_P = \breve{P}^\prime \nu_P$

[$*121{\cdot}632$]

$*121{\cdot}634$. $\vdash :. P \in \mathrm{Cls} \to 1 . P_{\mathrm{po}} \in J . \nu \in \mathrm{NC\,induct} - \iota^\prime 0 . \supset : \nu_P \in \mathrm{D}^\prime P . \equiv . \mathrm{E} ! \, (\nu +_c 1)_P$

[$*121{\cdot}633{\cdot}631{\cdot}333{\cdot}352$]

$*121{\cdot}635$. $\vdash : P \in \mathrm{Cls} \to 1 . P_{\mathrm{po}} \in J . \mathrm{E} ! \, \nu_P . \supset . \nu \in \mathrm{NC\,induct} - \iota^\prime 0$

Dem.

$\vdash . *121{\cdot}63{\cdot}45 . \supset \vdash : \mathrm{Hp} . \supset . \nu \in \mathrm{NC\,induct}$ (1)

$\vdash . *121{\cdot}13 . \quad \supset \vdash : \mathrm{E} ! \, \nu_P . \supset . \dot{\exists} ! \, P_{(\nu -_c 1)} .$

[$*121{\cdot}272$] $\supset . (\nu -_c 1) +_c 1 > 0 .$

[$*120{\cdot}416$] $\supset . \nu > 0$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

$*121{\cdot}636$. $\vdash : P \in \mathrm{Cls} \to 1 . P_{\mathrm{po}} \in J . \mathrm{E} ! \, \nu_P . \sim \mathrm{E} ! \, (\nu +_c 1)_P . \supset .$
$\overleftarrow{P_*} {}^\prime B^\prime P = P (B^\prime P \mapsto \nu_P) . \mathrm{N_0c}^\prime \overleftarrow{P_*} {}^\prime B^\prime P = \nu$

Dem.

$\vdash . *121{\cdot}635 . \supset \vdash : \mathrm{Hp} . \supset . \nu \in \mathrm{NC\,induct} - \iota^\prime 0 .$ (1)

[$*121{\cdot}634 . \mathrm{Hp}$] $\supset . \nu_P \sim \in \mathrm{D}^\prime P$ (2)

$\vdash . (1) . *121{\cdot}63 . \quad \supset \vdash :: \text{Hp} . \supset :. \exists ! P(B`P \mapsto \nu_P) :.$

$[*121{\cdot}23] \quad \supset :. (B`P) P_* \nu_P :.$

$[*96{\cdot}302 . *91{\cdot}542] \quad \supset :. (B`P) P_* z . \supset : z P_* \nu_P . \vee . \nu_P P_{\text{po}} z \quad (3)$

$\vdash . (2) . (3) . *91{\cdot}504 . \supset \vdash :. \text{Hp} . \supset : (B`P) P_* z . \supset . z P_* \nu_P :$

$[*4{\cdot}71] \quad \supset : (B`P) P_* z . \equiv . (B`P) P_* z . z P_* \nu_P :$

$[*121{\cdot}103] \quad \supset : \overleftarrow{P_*} `B`P = P(B`P \mapsto \nu_P) : \quad (4)$

$[*121{\cdot}63] \quad \supset : \text{N}_0\text{c}`\overleftarrow{P_*}`B`P = \nu \quad (5)$

$\vdash . (4) . (5) . \supset \vdash . \text{Prop}$

$*121{\cdot}637$. $\vdash : \text{E} ! \nu_P . \supset . \nu_P \in C`P$

Dem.

$\vdash . *121{\cdot}13 . *14{\cdot}28 . \supset \vdash : \text{E} ! \nu_P . \equiv . \nu_P = \breve{P}_{\nu -_0 1} `B`P .$

$[*121{\cdot}322] \quad \supset . \nu_P \in C`P : \supset \vdash . \text{Prop}$

$*121{\cdot}638$. $\vdash :. \text{E} ! (\nu +_0 1)_P . \supset : (B`P) P_\nu x . \equiv . x = (\nu +_0 1)_P : (\nu +_0 1) -_0 1 = \nu$

Dem.

$\vdash . *121{\cdot}13 . \supset \vdash : \text{E} ! (\nu +_0 1)_P . \equiv . \text{E} ! \breve{P}_{(\nu +_0 1) -_0 1} `B`P . \quad (1)$

$[*121{\cdot}272] \quad \supset . (\nu +_0 1) -_0 1 \geqslant 0 .$

$[*14{\cdot}21] \quad \supset . \text{E} ! (\nu +_0 1) -_0 1 .$

$[*14{\cdot}22 . (*120{\cdot}411)] \quad \supset . (\nu +_0 1) -_0 1 = \nu \quad (2)$

$\vdash . (2) . \supset \vdash :. \text{Hp} . \supset : (B`P) P_\nu x . \equiv . (B`P) P_{(\nu +_0 1) -_0 1} x .$

$[(1) . *30{\cdot}4] \quad \equiv . x = \breve{P}_{(\nu +_0 1) -_0 1} `B`P .$

$[*121{\cdot}13] \quad \equiv . x = (\nu +_0 1)_P \quad (3)$

$\vdash . (3) . (2) . \supset \vdash . \text{Prop}$

$*121{\cdot}64$. $\vdash : P \in \text{Cls} \to 1 . P_{\text{po}} \subseteq J . \nu \in \text{NC induct} - \iota`0 . \text{Nc}`\overleftarrow{P_*}`B`P \geqslant \nu . \supset . \text{E} ! \nu_P$

Dem.

$\vdash . *121{\cdot}636 . \supset \vdash :. \text{Hp} . \text{E} ! \nu_P . \supset : \sim \text{E} ! (\nu +_0 1)_P . \supset . \text{N}_0\text{c}`\overleftarrow{P_*}`B`P = \nu \quad (1)$

$\vdash . *120{\cdot}428 . \supset \vdash : \nu \in \text{NC induct} . \exists ! \nu +_0 1 . \supset . \nu +_0 1 > \nu .$

$[*117{\cdot}281] \quad \supset . \sim (\nu \geqslant \nu +_0 1) \quad (2)$

$\vdash . *117{\cdot}15 . \quad \supset \vdash : \sim \exists ! \nu +_0 1 . \supset . \sim (\nu \geqslant \nu +_0 1) \quad (3)$

$\vdash . (2) . (3) . \quad \supset \vdash : \nu \in \text{NC induct} . \supset . \sim (\nu \geqslant \nu +_0 1) \quad (4)$

$\vdash . (1) . (4) . \quad \supset \vdash :. \text{Hp} . \text{E} ! \nu_P . \supset :$

$\sim \text{E} ! (\nu +_0 1)_P . \supset . \sim (\text{Nc}`\overleftarrow{P_*}`B`P \geqslant \nu +_0 1) :$

$[\text{Transp}] \quad \supset : \text{Nc}`\overleftarrow{P_*}`B`P \geqslant \nu +_0 1 . \supset . \text{E} ! (\nu +_0 1)_P \quad (5)$

$\vdash . (5) . \text{Syll} . *117{\cdot}6 . \supset \vdash :. \text{Hp} : \text{Nc}`\overleftarrow{P_*}`B`P \geqslant \nu . \supset . \text{E} ! \nu_P : \supset :$

$\text{Nc}`\overleftarrow{P_*}`B`P \geqslant \nu +_0 1 . \supset . \text{E} ! (\nu +_0 1)_P \quad (6)$

$\vdash . *14{\cdot}21 . *121{\cdot}601 . \supset \vdash : \text{Nc}`\overleftarrow{P_*}`B`P \geqslant 1 . \supset . \text{E} ! 1_P \quad (7)$

$\vdash . (6) . (7) . *120{\cdot}473 . \supset$

$\vdash :. \text{Hp} . \supset : \text{Nc}`\overleftarrow{P_*}`B`P \geqslant \nu . \supset . \text{E} ! \nu_P :. \supset \vdash . \text{Prop}$

***121·641.** $\vdash :. P \in \text{Cls} \to 1 \,.\, P_{\text{po}} \subseteq J \,.\, \nu \in \text{NC induct} - \iota\text{'}0 \,.\, \supset :$

$\text{Nc'}\overleftarrow{P_*}\text{'}B\text{'}P \geqslant \nu \,.\, \equiv \,.\, \text{E!}\, \nu_P$

[*121·64·63·32]

***121·65.** $\vdash : P \in \text{Cls} \to 1 \,.\, P_{\text{po}} \subseteq J \,.\, \mu \neq 0 \,.\, \text{E!}\,(\mu +_c \nu)_P \,.\, \supset \,.\, \mu_P P_\nu (\mu +_c \nu)_P$

Dem.

$\vdash \,.\, *121\cdot631\cdot635\cdot64 \,.\, *120\cdot452 \,.\, \supset$

$\vdash : \text{Hp} \,.\, \supset \,.\, (B\text{'}P)\, P_{\mu -_c 1}\, \mu_P \,.\, (B\text{'}P)\, P_{\mu +_c \nu -_c 1}\, (\mu +_c \nu)_P \,.$

$[*121\cdot351 \,.\, *120\cdot424] \supset \,.\, (B\text{'}P)\, P_{\mu -_c 1}\, \mu_P \,.\, (B\text{'}P)\,(P_{\mu -_c 1} \mid P_\nu)\,(\mu +_c \nu)_P \,.$

$[*121\cdot341 \,.\, *72\cdot591] \supset \,.\, \mu_P P_\nu (\mu +_c \nu)_P : \supset \vdash \,.\, \text{Prop}$

***121·66.** $\vdash : P \in \text{Cls} \to 1 \,.\, P_{\text{po}} \subseteq J \,.\, \text{Nc'}P\,(B\text{'}P \vdash\!\dashv x) > \nu \,.\, \supset \,.\, x \in \text{Ɑ'}P_\nu$

Dem.

$\vdash \,.\, *121\cdot45 \,.\, *120\cdot48 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, \nu \in \text{NC induct} \,.$

$[*120\cdot429] \quad \supset \,.\, \text{Nc'}P\,(B\text{'}P \vdash\!\dashv x) \geqslant \nu +_c 1 \,.$

$[*117\cdot31] \quad \supset \,.\, (\exists \mu) \,.\, \text{Nc'}P\,(B\text{'}P \vdash\!\dashv x) = \nu +_c 1 +_c \mu \,.$

$[*121\cdot11] \quad \supset \,.\, (\exists \mu) \,.\, (B\text{'}P)\, P_{\nu +_c \mu}\, x \,.$

$[*121\cdot351\cdot352] \quad \supset \,.\, (\exists \mu) \,.\, (B\text{'}P)\,(P_\mu \mid P_\nu)\, x \,.$

$[*34\cdot36] \quad \supset \,.\, x \in \text{Ɑ'}P_\nu : \supset \vdash \,.\, \text{Prop}$

The following proposition is used in *122·38·381.

***121·7.** $\vdash : R \in 1 \to 1 \,.\, aBR \,.\, aR_* x \,.\, \supset \,.\, \overrightarrow{R_*}\text{'}x = R\,(a \vdash\!\dashv x) \,.\, \overrightarrow{R_*}\text{'}x \in \text{Cls induct}$

Dem.

$\vdash \,.\, *96\cdot25 \,.\, \supset \vdash :. \text{Hp} \,.\, \supset : yR_* x \,.\, \supset \,.\, aR_* y :$

$[*4\cdot71] \quad \supset : yR_* x \,.\, \equiv \,.\, aR_* y \,.\, yR_* x :$

$[*121\cdot103] \quad \supset : \overrightarrow{R_*}\text{'}x = R\,(a \vdash\!\dashv x) \qquad (1)$

$\vdash \,.\, (1) \,.\, *121\cdot45 \,.\, \supset \vdash \,.\, \text{Prop}$

***121·71.** $\vdash :. R \in 1 \to 1 : x \in s\text{'gen'}R \,.\, \vee \,.\, (\exists y) \,.\, y \in \overleftrightarrow{R_*}\text{'}x \,.\, yR_{\text{po}}y : \supset \,.$

$\overrightarrow{R_*}\text{'}x \in \text{Cls induct}$

Dem.

$\vdash \,.\, *121\cdot7 \,.\, *93\cdot36 \,.\, \supset \vdash : R \in 1 \to 1 \,.\, x \in s\text{'gen'}R \,.\, \supset \,.\, \overrightarrow{R_*}\text{'}x \in \text{Cls induct} \qquad (1)$

$\vdash \,.\, *97\cdot55\cdot111 \,.\, \supset \vdash :. R \in 1 \to 1 : (\exists y) \,.\, y \in \overleftrightarrow{R_*}\text{'}x \,.\, yR_{\text{po}}y : \supset :$

$y \in \overleftrightarrow{R_*}\text{'}x \,.\, \supset_y \,.\, yR_{\text{po}}y : x \in \overleftrightarrow{R_*}\text{'}x :$

$[*10\cdot26] \quad \supset : xR_{\text{po}}x :$

$[*121\cdot381] \quad \supset : \overrightarrow{R_*}\text{'}x = R\,(x \vdash\!\dashv x) :$

$[*121\cdot45] \quad \supset : \overrightarrow{R_*}\text{'}x \in \text{Cls induct} \qquad (2)$

$\vdash \,.\, (1) \,.\, (2) \,.\, \supset \vdash \,.\, \text{Prop}$

***121·72.** $\vdash : R \in 1 \to 1 \,.\, \overrightarrow{R_*}\text{'}x \sim\in \text{Cls induct} \,.\, \supset \,.\, x \in p\text{'Ɑ''Pot'}R \,.\, R_{\text{po}} \upharpoonright \overleftrightarrow{R_*}\text{'}x \subseteq J$

[*121·71 . Transp . *93·271 . *120·212 . *50·24]

$*122$. PROGRESSIONS

Summary of $*122$.

By a "progression" we mean a series which is like the series of the inductive cardinals in order of magnitude (assuming that all inductive cardinals exist), *i.e.* a series whose terms can be called

$$1_R, 2_R, 3_R, \ldots \nu_R, \ldots,$$

where every term of the series is correlated with some inductive cardinal, and every inductive cardinal is correlated with some term of the series. Such series belong to the relation-number (cf. $*152$ and $*263$) which Cantor calls ω. Their generating relation may be taken to be the transitive relation of earlier and later, or the one-one relation of immediate predecessor to immediate successor. We shall reserve the notation ω for the *transitive* generating relations of progressions; for the present, we are concerned with the one-one relations which generate progressions. The class of these relations we shall call "Prog."

It is not convenient to *define* a progression as a series which is ordinally similar to that of the inductive cardinals, both because this definition only applies if we assume the axiom of infinity, and because we have in any case to show that (assuming the axiom of infinity) the series of inductive cardinals has certain properties, which can be used to afford a direct definition of progressions. The existence of progressions, however, is only obtainable by means of the axiom of infinity, and is then most easily obtained from the fact that the inductive cardinals form a progression. We shall not consider the existence-theorem until the next number ($*123$).

From this number onwards convention Infin T of the Prefatory Statement is used when relevant.

The characteristics of the generating relation R of a progression, which we employ in the definition, are the following:

(1) R is a one-one relation;

(2) there is a first term, *i.e.* $\mathrm{E} \,!\, B\text{'}R$;

(3) the whole field is contained in the posterity of the first term, *i.e.* $C\text{'}R = \overleftarrow{R_*}\text{'}B\text{'}R$. (If this failed, $C\text{'}R$ would consist of two or more distinct families, of which, since we have $\mathrm{E} \,!\, B\text{'}R$, all but one would have to be cyclic families or infinite families with neither beginning nor end.)

(4) every term of the field has a successor, *i.e.* the series is endless. This is secured by $\text{Cl}\text{'}R \subset \mathrm{D}\text{'}R$, or (what is equivalent) $C\text{'}R = \mathrm{D}\text{'}R$.

These four properties suffice to define the one-one generating relations of progressions. It will be observed that (2), (3) and (4) are all secured by

$$\text{D}`R = \overleftarrow{R_*}`B`R.$$

This secures $\text{E} ! B`R$, by *14·21; it secures $\text{Ꭰ}`R \subset \text{D}`R$, by *37·25 and *90·163; hence, by *33·181, $\text{D}`R = C`R$, and therefore

$$C`R = \overleftarrow{R_*}`B`R.$$

Hence our definition of progressions is

$$\text{Prog} = (1 \to 1) \cap \hat{R}\,(\text{D}`R = \overleftarrow{R_*}`B`R) \quad \text{Df.}$$

Instead of stating in the definition that R is to be a one-one relation, it is sufficient to put $R \in \text{Cls} \to 1 \,.\, R_{\text{po}} \subseteq J$, which, with $\text{D}`R = \overleftarrow{R_*}`B`R$, implies $R \in 1 \to 1$, and may be substituted for $R \in 1 \to 1$ without altering the force of the definition (*122·17).

In the present number we shall prove, among other propositions, that every existent class contained in a progression has a first term (*122·23), *i.e.* that progressions are well-ordered series; that in a progression $R_{\text{po}} \subseteq J$ (*122·16), which makes the propositions of *121 available; that if ν is any inductive cardinal other than 0, ν_R exists (*122·33), *i.e.* the series has a νth term; that any class contained in $\text{D}`R$ and having a last term is an inductive class (*122·43), and that any class contained in $\text{D}`R$ and not having a last term is itself the domain of a progression (*122·45), so that every class contained in $\text{D}`R$ is either inductive or the domain of a progression (*122·46); that if P is a many-one, and x a member of its domain, and if the descendants of x have no last term and are none of them descendants of themselves, then P arranges these descendants in a progression (*122·51); and that the same holds if P is a one-one and $\sim(xPx)$ (*122·52); and that if $P \in 1 \to 1$ and x belongs to one of the generations of P, but not to one of the generations of $\breve{P}$, then P arranges the whole family of x in a progression (*122·54).

The following general observations on the families of one-one relations may serve to elucidate the bearing of the propositions of this section.

Given any relation P, we call $\overleftrightarrow{P_*}`x$, *i.e.* $\overrightarrow{P_*}`x \cup \overleftarrow{P_*}`x$, the *family* of x. If P is a one-one, this family may be of four different kinds. (1) It may be a closed series, like the angles of a polygon. This occurs if $xP_{\text{po}}x$. In this case the family forms an inductive class. (2) It may be an open series with a beginning and an end; this occurs if

$$\sim(xP_{\text{po}}x) \,.\, \text{E} ! \min_P`\overleftrightarrow{P_*}`x \,.\, \text{E} ! \max_P`\overleftrightarrow{P_*}`x.$$

In this case also the family forms an inductive class. (3) It may be an

open series with a beginning and no end, or an end and no beginning. This occurs if

$$\sim(xP_{po}x) \,.\, \text{E}\,!\,\text{min}_P`\overleftrightarrow{P}_*`x \,.\, \sim\text{E}\,!\,\text{max}_P`\overleftrightarrow{P}_*`x,$$

or if

$$\sim(xP_{po}x) \,.\, \sim\text{E}\,!\,\text{min}_P`\overleftrightarrow{P}_*`x \,.\, \text{E}\,!\,\text{max}_P`\overleftrightarrow{P}_*`x.$$

In this case, the series is of the type ω or $\text{Cnv}``\omega$, and is non-inductive and reflexive. (4) The series may be open and have neither beginning nor end. This occurs if

$$\sim(xP_{po}x) \,.\, \sim\text{E}\,!\,\text{min}_P`\overleftrightarrow{P}_*`x \,.\, \sim\text{E}\,!\,\text{max}_P`\overleftrightarrow{P}_*`x.$$

In this case we get a series whose relation-number is the sum (in the sense of $*180$) of $\text{Cnv}``\omega$ and ω, which again is non-inductive and reflexive. In all four cases, if y and z be any two members of the family of x, the interval between y ana z is an inductive class.

If x is a member of $\overrightarrow{B}`P$, or if the family of $x$ contains a member of $\overrightarrow{B}`P$, cases (1) and (4) are excluded, since the series has a beginning. In this case the number of predecessors of any term is an inductive number. It will be observed that every family is either wholly contained in $s`\text{gen}`P$ or wholly contained in $p`\text{Cl}``\text{Pot}`P$; families of kinds (2) and (3) (excluding, in (2), those which have an end but no beginning) are contained in $s`\text{gen}`P$, while families of kinds (1) and (4), and those of (2) which have an end but no beginning, are contained in $p`\text{Cl}``\text{Pot}`P$; families containing a member of $\overrightarrow{B}`P$ are contained in $s`\text{gen}`P$, while all others are contained in $p`\text{Cl}``\text{Pot}`P$.

Thus a one-one relation in general gives rise to a number of wholly disconnected series, some closed, others open and with or without a beginning or an end. The condition that all the series should be open is $P_{po} \subset J$.

The case of a Q-shaped family, considered in $*96$, cannot arise when $P \in 1 \rightarrow 1$, for in a Q-shaped family the term at the junction of the tail and the circle has two predecessors, one in the tail and one in the circle, so that the relation in question is not $1 \rightarrow 1$. It follows that, when $P \in 1 \rightarrow 1$, if α is a family containing a member of $\overrightarrow{B}`P$, $\alpha \rceil P_{po} \subset J$ (cf. $*96{\cdot}23$).

When $B`P$ exists, there is only one family which has a beginning. In this case, ignoring the other families (if any), we call the members of the family of $B`P$ respectively 1_P, 2_P, 3_P, ... If the family has ν members, where ν is an inductive cardinal, its last member will be ν_P. If on the other hand the number of members of the family is not an inductive cardinal, it must be $\aleph_0$; in this case, the family forms a progression, whose members are 1_P, 2_P, 3_P, ..., ν_P, ..., where ν_P always exists when ν is an inductive cardinal.

In addition to the propositions already mentioned, the following are important:

$*122\cdot21$. $\vdash :. R \in \text{Prog} . x, y \in C\text{'}R . \supset : x R_{\text{po}} y . \vee . x = y . \vee . y R_{\text{po}} x$

(Cf. note to $*122\cdot21$, below.)

$*122\cdot34$. $\vdash :. R \in \text{Prog} . \supset : \nu \in \text{NC induct} - \iota\text{'}0 . \equiv . \text{E} ! \nu_R$

$*122\cdot341$. $\vdash : R \in \text{Prog} . \supset . \text{D'}R = \hat{x}\{(\exists \nu) . \nu \in \text{NC induct} - \iota\text{'}0 . x = \nu_R\}$

In virtue of these two propositions, the terms of a progression are

$$1_R, 2_R, 3_R, \ldots \nu_R, \ldots,$$

where every inductive cardinal occurs. This is the same fact as is usually assumed when the terms are represented as

$$x_1, x_2, x_3, \ldots x_\nu, \ldots.$$

$*122\cdot35$. $\vdash : R \in \text{Prog} . \nu \in \text{NC induct} - \iota\text{'}0 . \supset . \overrightarrow{B}\text{'}R_\nu = R(1_R \mapsto \nu_R) . \overrightarrow{B}\text{'}R_\nu \in \nu$

$*122\cdot36$. $\vdash : \exists ! \text{Prog} \cap t^{11}\text{'}x . \supset . \text{Infin ax}(x)$

$*122\cdot37$. $\vdash : R \in \text{Prog} . \supset . \text{D'}R \sim\in \text{Cls induct} . \text{N}_0\text{c'D'}R \sim\in \text{NC induct}$

$*122\cdot38$. $\vdash : R \in \text{Prog} . \supset . \overrightarrow{R_*}\text{'}x \in \text{Cls induct}$

I.e. the number of terms up to any given point of a progression is inductive.

$*122\cdot01$. $\text{Prog} = (1 \rightarrow 1) \cap \hat{R}(\text{D'}R = \overleftarrow{R_*}\text{'}B\text{'}R)$ Df

$*122\cdot1$. $\vdash : R \in \text{Prog} . \equiv . R \in 1 \rightarrow 1 . \text{D'}R = \overleftarrow{R_*}\text{'}B\text{'}R$ [$(*122\cdot01)$]

$*122\cdot11$. $\vdash :. R \in \text{Prog} . \equiv : R \in 1 \rightarrow 1 . \text{E} ! B\text{'}R : x \in \text{D'}R . \equiv_x . x \in \overleftarrow{R_*}\text{'}B\text{'}R$

Dem.

$\vdash . *122\cdot1 . *14\cdot205 . \supset$

$\vdash :: R \in \text{Prog} . \equiv :. R \in 1 \rightarrow 1 : (\exists a) . a = B\text{'}R . \text{D'}R = \overleftarrow{R_*}\text{'}a :.$

$[*20\cdot43]$ $\equiv :. R \in 1 \rightarrow 1 :. (\exists a) : a = B\text{'}R : x \in \text{D'}R . \equiv_x . x \in \overleftarrow{R_*}\text{'}a :.$

$[*14\cdot15]$ $\equiv :. R \in 1 \rightarrow 1 :. (\exists a) : a = B\text{'}R : x \in \text{D'}R . \equiv_x . x \in \overleftarrow{R_*}\text{'}B\text{'}R :.$

$[*14\cdot204]$ $\equiv :. R \in 1 \rightarrow 1 . \text{E} ! B\text{'}R : x \in \text{D'}R . \equiv_x . x \in \overleftarrow{R_*}\text{'}B\text{'}R :: \supset \vdash . \text{Prop}$

Observe that, by the conventions as to descriptive symbols, $\text{D'}R = \overleftarrow{R_*}\text{'}B\text{'}R$ involves the existence of $B\text{'}R$, whereas $x \in \text{D'}R . \equiv_x . x \in \overleftarrow{R_*}\text{'}B\text{'}R$ does not, since, if $B\text{'}R$ does not exist, we have $(x) . x \sim\in \overleftarrow{R_*}\text{'}B\text{'}R$, and therefore $(x) . x \sim\in \text{D'}R$ will satisfy the equivalence, *i.e.* Λ will satisfy the equivalence although it has no first term. This is the reason why $\text{E} ! B\text{'}R$ appears explicitly in $*122\cdot11$, though it was only implicit in $*122\cdot1$.

$*122\cdot12$. $\vdash :: R \in \text{Prog} . \equiv :. R \in 1 \rightarrow 1 . \text{E} ! B\text{'}R :. x \in \text{D'}R . \equiv_x :$

$$B\text{'}R \in \alpha . \breve{R}\text{''}\alpha \subset \alpha . \supset_\alpha . x \in \alpha \quad [*122\cdot11 . *90\cdot1]$$

***122·14.** $\vdash : R \epsilon \mathrm{Prog} . \supset . \overleftarrow{R_{po}}`B`R = \text{ᗡ}`R$

Dem.

$\vdash . *122·1 . *37·25 . \supset \vdash : \mathrm{Hp} . \supset . \text{ᗡ}`R = \breve{R}``\overleftarrow{R_*}`B`R$

$[*91·52] \qquad = \overleftarrow{R_{po}}`B`R : \supset \vdash . \mathrm{Prop}$

***122·141.** $\vdash : R \epsilon \mathrm{Prog} . \supset . \text{ᗡ}`R \subset \mathrm{D}`R . C`R = \mathrm{D}`R$

Dem.

$\vdash . *122·1 . *37·25 . \supset \vdash : \mathrm{Hp} . \supset . \text{ᗡ}`R = \breve{R}``\overleftarrow{R_*}`B`R .$

$[*90·163] \qquad \supset . \text{ᗡ}`R \subset \overleftarrow{R_*}`B`R .$

$[*122·1 . *33·181] \qquad \supset . \text{ᗡ}`R \subset \mathrm{D}`R . C`R = \mathrm{D}`R : \supset \vdash . \mathrm{Prop}$

***122·142.** $\vdash : R \epsilon \mathrm{Prog} . P \epsilon \mathrm{Pot}`R . \supset . \mathrm{D}`P = \mathrm{D}`R \quad [*122·141 . *92·14]$

***122·143.** $\vdash : R \epsilon \mathrm{Prog} . P \epsilon \mathrm{Pot}`R . \supset . \text{ᗡ}`P \subset \mathrm{D}`P \quad [*122·142·141 . *91·271]$

***122·15.** $\vdash : R \epsilon \mathrm{Prog} . \supset . R = (\overleftarrow{R_*}`B`R) \rceil R = R \upharpoonright (\overleftarrow{R_{po}}`B`R) = R \upharpoonright (\overleftarrow{R_*}`B`R)$

Dem.

$\vdash . *122·1 . *35·63 . \supset \vdash : \mathrm{Hp} . \supset . R = (\overleftarrow{R_*}`B`R) \rceil R$

$[*96·2] \qquad = R \upharpoonright (\overleftarrow{R_{po}}`B`R)$

$[*96·21] \qquad = R \upharpoonright (\overleftarrow{R_*}`B`R) : \supset \vdash . \mathrm{Prop}$

***122·151.** $\vdash : R \epsilon \mathrm{Prog} . \supset . R_* = (\overleftarrow{R_*}`B`R) \rceil R_* = R_* \upharpoonright (\overleftarrow{R_*}`B`R)$
$[*35·63·66 . *90·14 . *122·141·1]$

***122·152.** $\vdash : R \epsilon \mathrm{Prog} . \supset . R_{po} = (\overleftarrow{R_*}`B`R) \rceil R_{po} = R_{po} \upharpoonright (\overleftarrow{R_{po}}`B`R)$
$= R_{po} \upharpoonright (\overleftarrow{R_*}`B`R)$
$[*35·63·66 . *91·504 . *121·1·14]$

***122·16.** $\vdash : R \epsilon \mathrm{Prog} . \supset . R_{po} \Subset J \quad [*96·23 . *122·152]$

This proposition enables us to apply to progressions all the propositions of $*121$ in which we have as hypothesis

$$R \epsilon \mathrm{Cls} \to 1 . R_{po} \Subset J, \text{ or } R \epsilon 1 \to \mathrm{Cls} . R_{po} \Subset J.$$

***122·17.** $\vdash : R \epsilon \mathrm{Prog} . \equiv . R \epsilon \mathrm{Cls} \to 1 . R_{po} \Subset J . \mathrm{D}`R = \overleftarrow{R_*}`B`R$

Dem.

$\vdash . *35·63 . \supset \vdash : \mathrm{D}`R = \overleftarrow{R_*}`B`R . \supset . R = (\overleftarrow{R_*}`B`R) \rceil R \qquad (1)$

$\vdash . *96·453 . \supset \vdash : R \epsilon \mathrm{Cls} \to 1 . (\overleftarrow{R_*}`B`R) \rceil R_{po} \Subset J . \supset . (\overleftarrow{R_*}`B`R) \rceil R \epsilon 1 \to 1 \quad (2)$

$\vdash . (1) . (2) . *122·1 . \supset \vdash : \mathrm{D}`R = \overleftarrow{R_*}`B`R . R \epsilon \mathrm{Cls} \to 1 . R_{po} \Subset J . \supset . R \epsilon \mathrm{Prog} \quad (3)$

$\vdash . (3) . *122·1·16 . \supset \vdash . \mathrm{Prop}$

To illustrate this proposition, consider its application to the inductive cardinals arranged in order of magnitude; *i.e.* take as a value of R the relation

$$\hat{\mu}\hat{\nu}(\mu \epsilon \mathrm{NC\,induct} \,.\, \nu = \mu +_c 1).$$

We then have $R \epsilon \mathrm{Cls} \to 1 \,.\, 0 = B'R$; also

$$\mathrm{NC\,induct} = \mathrm{D}'R = \overleftarrow{R_*}'B'R.$$

We have also

$$\exists \,!\, \mu +_c 1 \,.\, \mu +_c 1 = \nu +_c 1 \,.\, \supset .\, \mu = \nu,$$

so that $$R \upharpoonright (-\iota'\Lambda) \epsilon 1 \to \mathrm{Cls}.$$

Again

$$\mu R_{\mathrm{po}} \nu \,.\, \equiv .\, (\exists \varpi) \,.\, \varpi \epsilon \mathrm{NC\,induct} - \iota'0 \,.\, \nu = \mu +_c \varpi,$$

whence $$\mu R_{\mathrm{po}} \nu \,.\, \exists \,!\, \mu \,.\, \supset .\, \mu \neq \nu,$$

i.e. $$(-\iota'\Lambda) \upharpoonleft R_{\mathrm{po}} \subseteq J.$$

But we do not get $R \epsilon 1 \to \mathrm{Cls}$ or $R_{\mathrm{po}} \subseteq J$ unless we have

$$\Lambda \sim\epsilon \mathrm{NC\,induct},$$

which is the axiom of infinity. If this condition fails, we reach at last an inductive cardinal which $= \Lambda$, and we have

$$\Lambda = \Lambda +_c 1,$$

so that Λ has two immediate predecessors, namely itself and the last existent cardinal. The posterity of 0, in this case, is a Q in which the circle has narrowed to a single term, namely Λ.

Thus we need the axiom of infinity in order to prove

$$\hat{\mu}\hat{\nu}(\mu \epsilon \mathrm{NC\,induct} \,.\, \nu = \mu +_c 1) \epsilon \mathrm{Prog}.$$

***122·2.** $\vdash :. R \epsilon \mathrm{Prog} \,.\, x, y \epsilon C'R \,.\, \supset : xR_*y \,.\, \vee \,.\, yR_*x$ [*96·302 . *122·1·141]

***122·21.** $\vdash :. R \epsilon \mathrm{Prog} \,.\, x, y \epsilon C'R \,.\, \supset : xR_{\mathrm{po}}y \,.\, \vee \,.\, x = y \,.\, \vee \,.\, yR_{\mathrm{po}}x$
[*96·303 . *122·1·141]

This proposition, together with *122·16 and *91·56, shows that if $R \epsilon \mathrm{Prog}$, R_{po} has the three properties by which transitive serial relations are defined (cf. *204), namely it is (1) transitive, (2) contained in diversity, (3) connected, *i.e.* such that it relates any two distinct members of its field. We shall at a later stage define the ordinal number ω as the class of such relations as R_{po}, where $R \epsilon \mathrm{Prog}$.

***122·22.** $\vdash : R \epsilon \mathrm{Prog} \,.\, \alpha \subset \mathrm{D}'R \,.\, x, y \epsilon \alpha - \breve{R}_{\mathrm{po}}``\alpha \,.\, \supset .\, x = y$

Dem.

$\vdash .\, *122·21 \,.\, \supset \vdash :.\, \mathrm{Hp} \,.\, \supset : xR_{\mathrm{po}}y \,.\, \vee \,.\, x = y \,.\, \vee \,.\, yR_{\mathrm{po}}x$ (1)

$\vdash .\, *37·105 \,.\, \supset \vdash : x \epsilon \alpha \,.\, xR_{\mathrm{po}}y \,.\, \supset .\, y \epsilon \breve{R}_{\mathrm{po}}``\alpha :$

[Transp] $\supset \vdash : x \epsilon \alpha \,.\, y \sim\epsilon \breve{R}_{\mathrm{po}}``\alpha \,.\, \supset .\, \sim(xR_{\mathrm{po}}y)$ (2)

$\vdash .\, (2) \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset .\, \sim(xR_{\mathrm{po}}y) \,.\, \sim(yR_{\mathrm{po}}x)$ (3)

$\vdash .\, (1) \,.\, (3) \,.\, \supset \vdash .\, \mathrm{Prop}$

$*122\cdot23$. $\vdash : R \in \text{Prog} . \alpha \subset \text{D}`R . \exists ! \alpha . \supset .$

$$\text{E} ! \min (R_{\text{po}})`\alpha . \alpha - \breve{R}_{\text{po}}``\alpha = \iota`\min (R_{\text{po}})`\alpha$$

Dem.

$\vdash . *96\cdot52 . \quad \supset \vdash : \text{Hp} . \supset . \exists ! \overrightarrow{\min} (R_{\text{po}})`\alpha \quad (1)$

$\vdash . *93\cdot111 . *122\cdot22 . \supset \vdash :. \text{Hp} . \supset : x, y \in \overrightarrow{\min} (R_{\text{po}})`\alpha . \supset_{x,y} . x = y \quad (2)$

$\vdash . (1) . (2) . *32\cdot4 . *93\cdot111 . \supset \vdash . \text{Prop}$

This proposition shows that every existent class contained in a progression has a first term, *i.e.* that a progression is a well-ordered series (cf. $*250$).

$*122\cdot231$. $\vdash : R \in \text{Prog} . \alpha \subset \breve{R}_{\text{po}}``\alpha . \supset . \alpha = \Lambda$

Dem.

$\vdash . *91\cdot504 . \supset \vdash : \text{Hp} . \supset . \alpha \subset \text{Cl}`R \quad (1)$

$\vdash . *93\cdot11 . \quad \supset \vdash : \text{Hp} . \supset . \sim \text{E} ! \overrightarrow{\min} (R_{\text{po}})`\alpha \quad (2)$

$\vdash . (1) . (2) . *122\cdot23\cdot141 . \text{Transp} . \supset \vdash . \text{Prop}$

$*122\cdot24$. $\vdash : R \in \text{Prog} . P \in \text{Pot}`R . \supset . \text{D}`P = \breve{P}_*``\overrightarrow{B}`P = s`\text{gen}`P$

Dem.

$\vdash . *122\cdot1 . *92\cdot102 . \supset \vdash : \text{Hp} . \supset . P \in 1 \to 1 .$

$[*93\cdot42] \quad \supset . p`\text{Cl}``\text{Pot}`P = \breve{P}``p`\text{Cl}``\text{Pot}`P .$

$[*91\cdot581] \quad \supset . p`\text{Cl}``\text{Pot}`P \subset \breve{R}_{\text{po}}``p`\text{Cl}``\text{Pot}`P .$

$[*122\cdot231] \quad \supset . p`\text{Cl}``\text{Pot}`P = \Lambda \quad (1)$

$\vdash . (1) . *93\cdot37\cdot36 . \quad \supset \vdash : \text{Hp} . \supset . C`P = \breve{P}_*``\overrightarrow{B}`P = s`\text{gen}`P \quad (2)$

$\vdash . (2) . *122\cdot143 . \supset \vdash . \text{Prop}$

Except when $P = R$, $\overrightarrow{B}`P$ will not reduce to a single term. In fact, if $P = R_\nu$, $\overrightarrow{B}`P = R\,(1_R \mapsto \nu_R)$, *i.e.* $\overrightarrow{B}`P$ consists of the first ν terms of the progression.

$*122\cdot25$. $\vdash : R \in \text{Prog} . P \in \text{Pot}`R . x \in \text{D}`R . \supset .$

$$(\overleftarrow{P_*}`x) \upharpoonright P \in \text{Prog} . x = B`\{(\overleftarrow{P_*}`x) \upharpoonright P\}$$

Dem.

$\vdash . *122\cdot1 . *92\cdot102 . \supset \vdash : \text{Hp} . \supset . (\overleftarrow{P_*}`x) \upharpoonright P \in 1 \to 1 \quad (1)$

$\vdash . *122\cdot143 . \quad \supset \vdash : \text{Hp} . \supset . \overleftarrow{P_*}`x \subset \text{D}`P .$

$[*35\cdot62] \quad \supset . \text{D}`\{(\overleftarrow{P_*}`x) \upharpoonright P\} = \overleftarrow{P_*}`x \quad (2)$

$\vdash . *37\cdot4 . *91\cdot52 . \quad \supset \vdash . \text{Cl}`\{(\overleftarrow{P_*}`x) \upharpoonright P\} = \overleftarrow{P}_{\text{po}}`x \quad (3)$

$\vdash . *122\cdot16 . *91\cdot6 . \quad \supset \vdash : \text{Hp} . \supset . x \sim\in \overleftarrow{P}_{\text{po}}`x \quad (4)$

$\vdash . *91\cdot542 . \quad \supset \vdash : y \in \overleftarrow{P_*}`x . y \neq x . \supset . y \in \overleftarrow{P}_{\text{po}}`x \quad (5)$

$\vdash . (2) . (3) . (4) . (5) . \supset \vdash : \text{Hp} . \supset . x = B`\{(\overleftarrow{P_*}`x) \upharpoonright P\} \quad (6)$

$\vdash . (1) . (2) . (6) . *96\cdot131 . \supset \vdash . \text{Prop}$

The above proposition shows that what we may call an "arithmetical progression" in a progression is a progression, *i.e.* if, starting from any term of a progression, we take every other term, or every third term, or every νth term, we still have a progression.

***122·26.** $\vdash : R \epsilon \text{Prog} . \alpha \subset R_{\text{po}}``\alpha . \exists ! \alpha . \supset . \text{D}`R = R_{\text{po}}``\alpha$

Dem.

$\vdash . *22·1 . \quad \supset \vdash :. \text{Hp} . \supset : B`R \epsilon \alpha . \supset . B`R \epsilon R_{\text{po}}``\alpha \quad (1)$

$\vdash . *91·542 . *122·11 . \supset \vdash :. \text{Hp} . B`R \sim\epsilon \alpha . \supset : y \epsilon \alpha \cap \text{D}`R . \supset_y . (B`R) R_{\text{po}} y :$

$[*91·504 . *37·15] \quad \supset : y \epsilon \alpha . \supset_y . (B`R) R_{\text{po}} y :$

$[*10·55.\text{Hp}] \quad \supset : (\exists y) . y \epsilon \alpha . (B`R) R_{\text{po}} y :$

$[*37·1] \quad \supset : B`R \epsilon R_{\text{po}}``\alpha \quad (2)$

$\vdash . (1) . (2) . \supset \vdash : \text{Hp} . \supset . B`R \epsilon R_{\text{po}}``\alpha \quad (3)$

$\vdash . *92·111 . \supset \vdash :. \text{Hp} . \supset : x \epsilon R_{\text{po}}``\alpha . xRy . \supset . y \epsilon R_*``\alpha .$

$[*91·545] \quad \supset . y \epsilon \alpha \cup R_{\text{po}}``\alpha .$

$[\text{Hp}] \quad \supset . y \epsilon R_{\text{po}}``\alpha \quad (4)$

$\vdash . (3) . (4) . *90·112 . \supset \vdash :. \text{Hp} . \supset : (B`R) R_* y . \supset . y \epsilon R_{\text{po}}``\alpha \quad (5)$

$\vdash . (5) . *122·1 . \supset \vdash . \text{Prop}$

The above proposition shows that if an existent class contained in a progression has no maximum, then any assigned member of the progression is succeeded by members of the class.

The following proposition states that if α has members belonging to a progression, and there are members of the progression which do not precede any member of α, then there is in the progression a last member of α.

***122·27.** $\vdash : R \epsilon \text{Prog} . \exists ! \text{D}`R - R_{\text{po}}``\alpha . \exists ! \alpha \cap \text{D}`R . \supset .$
$\text{E} ! \max (R_{\text{po}})`\alpha . \exists ! \alpha \cap \text{D}`R - R_{\text{po}}``\alpha$

Dem.

$\vdash . *122·26 . \text{Transp} . *37·265 . \quad \supset \vdash : \text{Hp} . \supset . \exists ! \alpha \cap C`R - R_{\text{po}}``\alpha \quad (1)$

$\vdash . *122·21 . \quad \supset \vdash : \text{Hp} . x, y \epsilon \alpha \cap C`R - R_{\text{po}}``\alpha . \supset . x = y \quad (2)$

$\vdash . (1) . (2) . *93·115 . *122·141 . \supset \vdash . \text{Prop}$

***122·28.** $\vdash : R \epsilon \text{Prog} . \alpha \subset \overrightarrow{R_*}`x . \exists ! \alpha . \supset . \text{E} ! \max (R_{\text{po}})`\alpha . \exists ! \alpha \cap \text{D}`R - R_{\text{po}}``\alpha$

Dem.

$\vdash . *90·13 . *122·141 . \supset \vdash : \text{Hp} . \supset . \alpha \subset \text{D}`R \quad (1)$

$\vdash . *90·14 . *122·141 . \supset \vdash : \text{Hp} . \supset . x \epsilon \text{D}`R .$

$[*71·161 . *122·16] \quad \supset . \breve{R}`x \sim\epsilon R_{\text{po}}``\alpha .$

$[*122·1] \quad \supset . \exists ! \text{D}`R - R_{\text{po}}``\alpha \quad (2)$

$\vdash . (1) . (2) . *122·27 . \supset \vdash . \text{Prop}$

***122·3.** $\vdash : R \in \text{Prog} . \supset . \text{D}'R = \hat{x}\{(\exists \nu) . \nu \in \text{NC induct} . (B'R) R_\nu x\}$
[*121·52 . *122·1·16]

***122·31.** $\vdash : R \in \text{Prog} . \nu \in \text{NC induct} - \iota'0 . \supset . \text{ᗡ}'R_\nu = \hat{y}\{\text{Nc}'R(B'R \mapsto y) > \nu\}$

Dem.

$\vdash . *120{\cdot}429 . \supset \vdash :. \text{Hp} . \supset : \text{Nc}'R(B'R \mapsto y) > \nu . \equiv . \text{Nc}'R(B'R \mapsto y) \geqslant \nu +_c 1 .$

[*117·31] $\equiv . (\exists \mu) . \mu \in \text{NC} . \text{Nc}'R(B'R \mapsto y) = \mu +_c \nu +_c 1 .$
$\nu +_c 1, \mu +_c \nu +_c 1 \in \text{N}_0\text{C} .$

[*121·45.*120·452.*110·4] $\equiv . (\exists \mu) . \mu \in \text{NC induct} .$
$\text{Nc}'R(B'R \mapsto y) = \mu +_c \nu +_c 1 . \mu +_c \nu +_c 1 \in \text{N}_0\text{C} .$

[*121·11·35.*110·43.*100·3] $\equiv . (\exists \mu) . \mu \in \text{NC induct} . (B'R) R_\mu | R_\nu y .$

[*34·1] $\equiv . (\exists \mu, x) . \mu \in \text{NC induct} . (B'R) R_\mu x . x R_\nu y .$

[*122·3] $\equiv . (\exists x) . x \in \text{D}'R . x R_\nu y .$

[*121·323] $\equiv . (\exists x) . x R_\nu y .$

[*33·131] $\equiv . y \in \text{ᗡ}'R_\nu :. \supset \vdash . \text{Prop}$

***122·32.** $\vdash : R \in \text{Prog} . \nu \in \text{NC induct} - \iota'0 . \supset .$
$\overrightarrow{B}'R_\nu = \text{D}'R \cap \hat{x}\{\text{Nc}'R(B'R \mapsto x) \leqslant \nu\}$

Dem.

$\vdash . *122{\cdot}142 . *121{\cdot}501 . \supset \vdash : \text{Hp} . \supset . \text{D}'R_\nu = \text{D}'R$ (1)

$\vdash . *122{\cdot}31 . *120{\cdot}442 . \supset \vdash : \text{Hp} . \supset . -\text{ᗡ}'R_\nu = \hat{x}\{\text{Nc}'R(B'R \mapsto x) \leqslant \nu\}$ (2)

$\vdash . (1) . (2) . *93{\cdot}101 . \supset \vdash . \text{Prop}$

***122·33.** $\vdash : R \in \text{Prog} . \nu \in \text{NC induct} - \iota'0 . \supset . \text{E}! \, \nu_R$

Dem.

$\vdash . *121{\cdot}601 . *122{\cdot}11 . \supset \vdash : \text{Hp} . \supset . \text{E}! \, 1_R$ (1)

$\vdash . *121{\cdot}634{\cdot}637 . *122{\cdot}141 . \supset \vdash :. \text{Hp} . \supset : \text{E}! \, \nu_R . \supset . \text{E}! \, (\nu +_c 1)_R$ (2)

$\vdash . (1) . (2) . *120{\cdot}473 . \supset \vdash . \text{Prop}$

***122·34.** $\vdash :. R \in \text{Prog} . \supset : \nu \in \text{NC induct} - \iota'0 . \equiv . \text{E}! \, \nu_R$ [*122·33.*121·635]

***122·341.** $\vdash : R \in \text{Prog} . \supset . \text{D}'R = \hat{x}\{(\exists \nu) . \nu \in \text{NC induct} - \iota'0 . x = \nu_R\}$

Dem.

$\vdash . *122{\cdot}3{\cdot}34 . *121{\cdot}638 . \supset$

$\vdash : \text{Hp} . \supset . \text{D}'R = \hat{x}\{(\exists \nu) . \nu \in \text{NC induct} . x = (\nu +_c 1)_R\}$

[*120·471] $= \hat{x}\{(\exists \nu) . \nu \in \text{NC induct} - \iota'0 . x = \nu_R\} : \supset \vdash . \text{Prop}$

In virtue of *122·34·341, all the terms of a progression occur in the series $1_R, 2_R, \ldots \nu_R, \ldots$, and every inductive cardinal except 0 is used in forming this series.

***122·35.** $\vdash : R \in \text{Prog} . \nu \in \text{NC induct} - \iota`0 . \supset . \overrightarrow{B`R_\nu} = R(1_R \mapsto \nu_R) . \overrightarrow{B`R_\nu} \in \nu$

Dem.

$\vdash . *121·63 . *122·33 . \supset \vdash : \text{Hp} . \supset . \text{Nc}`R(B`R \mapsto \nu_R) = \nu .$ (1)

[*122·32] $\supset . \overrightarrow{B`R_\nu} = \text{D}`R \cap \hat{x}\{\text{Nc}`R(B`R \mapsto x) \leqslant \text{Nc}`R(B`R \mapsto \nu_R)\}$

[*121·481] $= \text{D}`R \cap \hat{x}\{R(B`R \mapsto x) \subset R(B`R \mapsto \nu_R)\}$

[*122·1.*121·103] $= \hat{x}\{(B`R) R_* x : y R_* x . \supset_y . y R_* \nu_R\}$

[*90·17·13.*10·1] $= \hat{x}\{(B`R) R_* x . x R_* \nu_R\}$

[*121·103] $= R(B`R \mapsto \nu_R)$ (2)

[*121·601.*122·11] $= R(1_R \mapsto \nu_R)$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash . \text{Prop}$

***122·36.** $\vdash : \exists ! \text{Prog} \cap t^{11}`x . \supset . \text{Infin ax}(x)$

Dem.

$\vdash . *122·35 . \quad \supset \vdash : R \in \text{Prog} \cap t^{11}`x . \nu \in \text{NC induct} - \iota`0 . \supset . \exists ! \nu(x)$ (1)

$\vdash . (1) . *101·12 . \supset \vdash :. R \in \text{Prog} \cap t^{11}`x . \supset : \nu \in \text{NC induct} . \supset_\nu . \exists ! \nu(x) :$

[*120·301] $\supset : \text{Infin ax}(x) :. \supset \vdash . \text{Prop}$

***122·37.** $\vdash : R \in \text{Prog} . \supset . \text{D}`R \sim\in \text{Cls induct} . \text{N}_0\text{c}`\text{D}`R \sim\in \text{NC induct}$

Dem.

$\vdash . *122·35 . \supset \vdash :. R \in \text{Prog} . \supset : \nu \in \text{NC induct} . \supset_\nu . \exists ! \text{Cl}`\text{D}`R \cap (\nu +_c 1) .$

[*117·22·107] $\supset_\nu . \text{N}_0\text{c}`\text{D}`R \geqslant \nu +_c 1 .$

[*120·429] $\supset_\nu . \text{N}_0\text{c}`\text{D}`R > \nu .$

[*117·42] $\supset_\nu . \text{N}_0\text{c}`\text{D}`R \neq \nu :$

[*13·196] $\supset : \text{N}_0\text{c}`\text{D}`R \sim\in \text{NC induct}$ (1)

$\vdash . (1) . *120·21 . \supset \vdash . \text{Prop}$

***122·38.** $\vdash : R \in \text{Prog} . \supset . \overrightarrow{R_*}`x \in \text{Cls induct}$ [*121·7 . *90·13 . *120·212]

***122·381.** $\vdash : R \in \text{Prog} . \nu \in \text{NC induct} - \iota`0 . \supset . \overrightarrow{R_*}`\nu_R = R(1_R \mapsto \nu_R) . \overrightarrow{R_*}`\nu_R \in \nu$
[*121·7 . *122·35]

The following series of propositions are concerned in proving that any class contained in a progression is inductive if it has a last term, and is a progression if it has no last term. In the latter case, it is supposed arranged in the same order as it had in the original progression. A certain complication is necessary in order to define its one-one generating relation. If R is the generating relation of the original progression, we proceed first to R_{po}, then to $R_{\text{po}} \upharpoonright \alpha$, where α is the class in question; this gives us a transitive generating relation for α. Calling this relation P, we then proceed to $P \dot{-} P^2$, *i.e.* the relation of consecutive members of the series generated by P. This relation turns out to be one-one, and to arrange α in a progression; hence our proposition is proved. The reason for the necessity of this detour is that consecutive members of α may not be consecutive members of the original progression.

∗122·41. $\vdash : R \in \text{Prog} . \alpha \subset \text{D}'R . y \in \alpha - R_{\text{po}}``\alpha . \supset . \alpha \subset R(B'R \mapsto y)$

Dem.

$\vdash . ∗37·1 . ∗10·51 . \supset \vdash :. \text{Hp} . \supset : z \in \alpha . \supset_z . \sim(yR_{\text{po}}z) .$

$[∗122·21] \qquad \supset_z . zR_*y \qquad (1)$

$\vdash . ∗122·1 . \qquad \supset \vdash :. \text{Hp} . \supset : z \in \alpha . \supset_z . (B'R) R_* z \qquad (2)$

$\vdash . (1) . (2) . ∗121·103 . \supset \vdash . \text{Prop}$

∗122·42. $\vdash : R \in \text{Prog} . \alpha \subset R(B'R \mapsto y) . y \in \alpha . \supset . y = \max_R{}'\alpha$

Dem.

$\vdash . ∗121·103 . \supset \vdash :. \text{Hp} . \supset : z \in \alpha . \supset_z . zR_*y .$

$[∗91·574.∗122·16] \qquad \supset_z . \sim(yR_{\text{po}}z) :$

$[∗37·1.∗10·51] \qquad \supset : y \sim\in R_{\text{po}}``\alpha : \qquad (1)$

$[∗96·303] \qquad \supset : z \in \alpha - R_{\text{po}}``\alpha . \supset_z . z = y \qquad (2)$

$\vdash . (1) . (2) . ∗93·115 . \supset \vdash . \text{Prop}$

∗122·43. $\vdash : R \in \text{Prog} . \alpha \subset \text{D}'R . \exists ! \alpha - R_{\text{po}}``\alpha . \supset . \alpha \in \text{Cls induct}$

$[∗122·41 . ∗121·45 . ∗120·481]$

Thus every class which is contained in a progression and has a last term is inductive. We have next to prove

$$R \in \text{Prog} . \alpha \subset \text{D}'R . \exists ! \alpha . \sim\exists ! \alpha - R_{\text{po}}``\alpha . \supset . \alpha \in \text{D}``\text{Prog}.$$

This is effected in the following propositions.

∗122·44. $\vdash : R \in \text{Prog} . \alpha \subset R_{\text{po}}``\alpha . \exists ! \alpha . P = R_{\text{po}} \upharpoonright \alpha . Q = P \dot{-} P^2 . \supset .$
$Q \in 1 \to 1 . Q \subseteq R_{\text{po}}$

Note. The hypothesis here exceeds what is necessary for the conclusion, but is the hypothesis required for ∗122·45, for which the present and the following propositions are lemmas.

Dem.

$\vdash . ∗23·43 . ∗35·442 . \supset \vdash : \text{Hp} . \supset . Q \subseteq R_{\text{po}} \qquad (1)$

$\vdash . ∗36·13 . \qquad \supset \vdash :. \text{Hp} . \supset : x, y, z \in \alpha . xR_{\text{po}}y . yR_{\text{po}}z . \supset . xP^2z :$

$[\text{Transp}] \qquad \supset : x, y, z \in \alpha . xR_{\text{po}}y . \sim(xP^2z) . \supset . \sim(yR_{\text{po}}z) :$

$[∗36·13] \qquad \supset : xPy . \sim(xP^2z) . \supset . \sim(yR_{\text{po}}z) :$

$[∗3·47] \qquad \supset : xQy . xQz . \supset . \sim(yR_{\text{po}}z) . \sim(zR_{\text{po}}y) .$

$[∗122·21.(1)] \qquad \supset . y = z \qquad (2)$

Similarly $\qquad \vdash :. \text{Hp} . \supset : xQz . yQz . \supset . x = y \qquad (3)$

$\vdash . (1) . (2) . (3) . \supset \vdash . \text{Prop}$

∗122·441. $\vdash : \text{Hp} ∗122·44 . \supset . \text{D}'Q = \alpha$

Dem.

$\vdash . ∗37·41 . \supset \vdash : \text{Hp} . \supset . \text{D}'Q \subset \alpha \qquad (1)$

$\vdash . ∗37·1 . \supset \vdash :. \text{Hp} . \supset : x \in \alpha . \supset . (\exists y) . y \in \alpha . xR_{\text{po}}y .$

$[∗36·13] \qquad \supset . \exists ! \overleftarrow{P}'x .$

$[∗122·23.∗93·11] \qquad \supset . \exists ! \overleftarrow{P}'x - \breve{R}_{\text{po}}``\overleftarrow{P}'x .$

$[∗35·442] \qquad \supset . \exists ! \overleftarrow{P}'x - \breve{P}``\overleftarrow{P}'x .$

$[∗37·311.∗32·31·35] \qquad \supset . \exists ! \overleftarrow{Q}'x \qquad (2)$

$\vdash . (1) . (2) . ∗33·4 . \supset \vdash . \text{Prop}$

$*122\cdot442$. $\vdash : \mathrm{Hp} *122\cdot44 . \supset . P = Q_{po}$

In proving $P \mathrel{\subseteq} Q_{po}$ below, we assume xPz and consider the maximum of $\overrightarrow{R_{po}}\text{‘}z \cap \overleftarrow{Q_*}\text{‘}x$, which is shown to exist and be $Q\text{‘}z$, whence $xQ_{po}z$.

Dem.

$\vdash . *23\cdot43 . \supset \vdash : \mathrm{Hp} . \supset . Q \mathrel{\subseteq} P$ (1)

$\vdash . *91\cdot56 . \supset \vdash :. \mathrm{Hp} . \supset : P^2 \mathrel{\subseteq} P :$

[(1)] $\supset : S \mathrel{\subseteq} P . \supset . S | Q \mathrel{\subseteq} P$ (2)

$\vdash . (1) . (2) . *91\cdot171 . *41\cdot151 . \supset \vdash : \mathrm{Hp} . \supset . Q_{po} \mathrel{\subseteq} P$ (3)

$\vdash . *36\cdot13 . *121\cdot1 . \supset \vdash :. \mathrm{Hp} . \supset : xP^2z . \equiv . x, z \in \alpha . \exists ! \alpha \cap R(x - z) :$

[Transp.Fact] $\supset : xQz . \equiv . x, z \in \alpha . xR_{po}z . \alpha \cap R(x - z) = \Lambda$ (4)

$\vdash . *122\cdot441 . \supset \vdash :. \mathrm{Hp} . xPz . \supset : x \in (\overrightarrow{R_{po}}\text{‘}z \cap \overleftarrow{Q_*}\text{‘}x) :$

[$*122\cdot27$] $\supset : \exists ! \overrightarrow{R_{po}}\text{‘}z \cap \overleftarrow{Q_*}\text{‘}x - R_{po}\text{“}(\overrightarrow{R_{po}}\text{‘}z \cap \overleftarrow{Q_*}\text{‘}x) :$

[$*37\cdot461$] $\supset : (\exists y) . y \in \overrightarrow{R_{po}}\text{‘}z \cap \overleftarrow{Q_*}\text{‘}x . \overleftarrow{R_{po}}\text{‘}y \cap \overrightarrow{R_{po}}\text{‘}z \cap \overleftarrow{Q_*}\text{‘}x = \Lambda :$

[$*90\cdot151$] $\supset : (\exists y) . y \in \overrightarrow{R_{po}}\text{‘}z \cap \overleftarrow{Q_*}\text{‘}x . \overleftarrow{R_{po}}\text{‘}y \cap \overrightarrow{R_{po}}\text{‘}z \cap \overleftarrow{Q_*}\text{‘}y = \Lambda :$

[(4)] $\supset : (\exists y) : y \in \overrightarrow{R_{po}}\text{‘}z \cap \overleftarrow{Q_*}\text{‘}x :$

$\sim(\exists w) . w \in \overleftarrow{R_{po}}\text{‘}y \cap \overrightarrow{R_{po}}\text{‘}z . \alpha \cap \overleftarrow{R_{po}}\text{‘}y \cap \overrightarrow{R_{po}}\text{‘}w = \Lambda :$

[$*22\cdot43 . *91\cdot56$] $\supset : (\exists y) : y \in \overrightarrow{R_{po}}\text{‘}z \cap \overleftarrow{Q_*}\text{‘}x :$

$\sim(\exists w) . w \in \alpha \cap \overleftarrow{R_{po}}\text{‘}y \cap \overrightarrow{R_{po}}\text{‘}z . \alpha \cap \overleftarrow{R_{po}}\text{‘}y \cap \overrightarrow{R_{po}}\text{‘}z \cap \overrightarrow{R_{po}}\text{‘}w = \Lambda :$

[$*37\cdot461$] $\supset : (\exists y) . y \in \overrightarrow{R_{po}}\text{‘}z \cap \overleftarrow{Q_*}\text{‘}x .$

$\sim \exists ! \alpha \cap \overleftarrow{R_{po}}\text{‘}y \cap \overrightarrow{R_{po}}\text{‘}z - \breve{R}_{po}\text{“}(\alpha \cap \overleftarrow{R_{po}}\text{‘}y \cap \overrightarrow{R_{po}}\text{‘}z) :$

[$*122\cdot28$.Transp] $\supset : (\exists y) . y \in \overrightarrow{R_{po}}\text{‘}z \cap \overleftarrow{Q_*}\text{‘}x . \alpha \cap \overleftarrow{R_{po}}\text{‘}y \cap \overrightarrow{R_{po}}\text{‘}z = \Lambda :$

[(4)] $\supset : (\exists y) . y \in \overleftarrow{Q_*}\text{‘}x . yQz :$

[$*91\cdot52$] $\supset : xQ_{po}z$ (5)

$\vdash . (3) . (5) . \supset \vdash . \mathrm{Prop}$

$*122\cdot443$. $\vdash : \mathrm{Hp} *122\cdot44 . \supset . \min(R_{po})\text{‘}\alpha = B\text{‘}Q . \mathrm{C\!\!\!I}\text{‘}Q = \alpha \cap \breve{R}_{po}\text{“}\alpha$

Dem.

$\vdash . *91\cdot504 . *122\cdot442 . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{C\!\!\!I}\text{‘}Q = \mathrm{C\!\!\!I}\text{‘}P$

[$*37\cdot41$] $= \alpha \cap \breve{R}_{po}\text{“}\alpha$ (1)

$\vdash . (1) . *122\cdot441 . \supset \vdash : \mathrm{Hp} . \supset . \overrightarrow{B}\text{‘}Q = \alpha - \breve{R}_{po}\text{“}\alpha$ (2)

$\vdash . (1) . (2) . *122\cdot23 . \supset \vdash . \mathrm{Prop}$

$*122\cdot444$. $\vdash : \mathrm{Hp} *122\cdot44 . \supset . \mathrm{D}\text{‘}Q = \overleftarrow{Q_*}\text{‘}B\text{‘}Q$

Dem.

$\vdash . *122\cdot443 . *14\cdot21 . \supset \vdash . \mathrm{Hp} . \supset . E ! B\text{‘}Q .$

[$*90\cdot13$] $\supset . \overleftarrow{Q_*}\text{‘}B\text{‘}Q \subset C\text{‘}Q .$

[$*122\cdot441\cdot443$] $\supset . \overleftarrow{Q_*}\text{‘}B\text{‘}Q \subset \alpha$ (1)

$\vdash . *122\cdot443 . *96\cdot303 . \supset$

$\vdash : \mathrm{Hp} . x \in \alpha . x \neq B\text{‘}Q . \supset . (B\text{‘}Q) R_{po} x . B\text{‘}Q, x \in \alpha .$

$[\mathrm{Hp}] \qquad \supset . (B`Q) P x .$

$[*122{\cdot}442] \qquad \supset . (B`Q) Q_{\mathrm{po}} x \qquad (2)$

$\vdash . (2) . *91{\cdot}54 . \supset \vdash : \mathrm{Hp} . x \in \alpha . \supset . (B`Q) Q_* x \qquad (3)$

$\vdash . (1) . (3) . \qquad \supset \vdash : \mathrm{Hp} . \supset . \overleftarrow{Q_*}`B`Q = \alpha$

$[*122{\cdot}441] \qquad = \mathrm{D}`Q : \supset \vdash . \mathrm{Prop}$

$*122{\cdot}45. \quad \vdash : R \in \mathrm{Prog} . \alpha \subset R_{\mathrm{po}}``\alpha . \exists ! \alpha . P = R_{\mathrm{po}} \upharpoonright \alpha . Q = P \dot{-} P^2 . \supset .$
$Q \in \mathrm{Prog} . \mathrm{D}`Q = \alpha \quad [*122{\cdot}44{\cdot}444{\cdot}441]$

This proposition shows that every series extracted from a progression and having no last term is a progression.

$*122{\cdot}46. \quad \vdash : R \in \mathrm{Prog} . \alpha \subset \mathrm{D}`R . \supset . \alpha \in \mathrm{Cls\,induct} \cup \mathrm{D}``\mathrm{Prog}$
$[*122{\cdot}43{\cdot}45 . *120{\cdot}212]$

This proposition shows that any number less than the number of terms in a progression is inductive. This result will be developed in the next number (*123).

$*122{\cdot}47. \quad \vdash :. R \in \mathrm{Prog} . \alpha \subset \mathrm{D}`R . \supset : \alpha \in \mathrm{Cls\,induct} - \iota`\Lambda . \equiv . \exists ! \alpha - R_{\mathrm{po}}``\alpha$

Dem.

$\vdash . *122{\cdot}45 . \supset \vdash : \mathrm{Hp} . \exists ! \alpha . \sim \exists ! \alpha - R_{\mathrm{po}}``\alpha . \supset . \alpha \in \mathrm{D}``\mathrm{Prog} .$

$[*122{\cdot}37] \qquad \supset . \alpha \sim\in \mathrm{Cls\,induct} \qquad (1)$

$\vdash . (1) . *122{\cdot}43 . \supset \vdash . \mathrm{Prop}$

$*122{\cdot}48. \quad \vdash : R \in \mathrm{Prog} . \alpha \subset \mathrm{D}`R . \alpha \in \mathrm{Cls\,induct} . \supset . \mathrm{D}`R - \alpha \sim\in \mathrm{Cls\,induct}$

Dem.

$\vdash . *120{\cdot}71 . \supset \vdash : \alpha \subset \mathrm{D}`R . \alpha, \mathrm{D}`R - \alpha \in \mathrm{Cls\,induct} . \supset . \mathrm{D}`R \in \mathrm{Cls\,induct} :$

$[\mathrm{Transp}] \qquad \supset \vdash : \alpha \subset \mathrm{D}`R . \alpha \in \mathrm{Cls\,induct} . \mathrm{D}`R \sim\in \mathrm{Cls\,induct} . \supset .$
$\mathrm{D}`R - \alpha \sim\in \mathrm{Cls\,induct} \qquad (1)$

$\vdash . (1) . *122{\cdot}37 . \supset \vdash . \mathrm{Prop}$

$*122{\cdot}49. \quad \vdash : R \in \mathrm{Prog} . \alpha \subset \mathrm{D}`R . \alpha \in \mathrm{Cls\,induct} . \supset . \mathrm{D}`R - \alpha \in \mathrm{D}``\mathrm{Prog}$
$[*122{\cdot}46{\cdot}48]$

The following propositions are concerned with circumstances under which the posterity or the family of a term forms a progression.

$*122{\cdot}51. \quad \vdash : P \in \mathrm{Cls} \to 1 . I_P`x = \Lambda . x \in \mathrm{D}`P . \overleftarrow{P_*}`x \subset \mathrm{D}`P . \supset . (\overleftarrow{P_*}`x) \upharpoonright P \in \mathrm{Prog}$

Here $I_P`x$ has the meaning defined in *96.

Dem.

$\vdash . *71{\cdot}261 . *96{\cdot}13 . \supset \vdash : \mathrm{Hp} . Q = (\overleftarrow{P_*}`x) \upharpoonright P . \supset .$
$Q \in \mathrm{Cls} \to 1 . Q_{\mathrm{po}} = (\overleftarrow{P_*}`x) \upharpoonright P_{\mathrm{po}} . \qquad (1)$

$[*96{\cdot}104] \qquad \supset . Q_{\mathrm{po}} \subseteq J \qquad (2)$

$\vdash . *35{\cdot}61 . *37{\cdot}4 . \quad \supset \vdash : \text{Hp}(1) . \supset . \text{D}'Q = \overleftarrow{P_*}'x . \text{ᗡ}'Q = \breve{P}``\overleftarrow{P_*}'x$ (3)

$[*91{\cdot}52] \quad = \overleftarrow{P_{\text{po}}}'x$

$[(1)] \quad = \overleftarrow{Q_{\text{po}}}'x$

$[(2).*91{\cdot}542] \quad = \overleftarrow{Q_*}'x - \iota'x$ (4)

$\vdash . (1) . (3) . (4) . \quad \supset \vdash : \text{Hp}(1) . \supset . \text{D}'Q = \overleftarrow{Q_*}'x . \text{ᗡ}'Q = \overleftarrow{Q_*}'x - \iota'x .$

$[*93{\cdot}101] \quad \supset . \overrightarrow{B}'Q = \iota'x . \text{D}'Q = \overleftarrow{Q_*}'x$ (5)

$\vdash . (1) . (2) . (5) . \supset$

$\vdash : \text{Hp} . Q = (\overleftarrow{P_*}'x) \upharpoonright P . \supset . Q \in \text{Cls} \rightarrow 1 . Q_{\text{po}} \subseteq J . \text{D}'Q = \overleftarrow{Q_*}'B'Q .$

$[*122{\cdot}17] \quad \supset . Q \in \text{Prog} : \supset \vdash . \text{Prop}$

The following proposition ($*122{\cdot}52$) is used in $*123{\cdot}191$, $*261{\cdot}4$ and $*264{\cdot}22$.

$*122{\cdot}52$. $\vdash : P \in 1 \rightarrow 1 . x \in \text{D}'P . \sim(xP_{\text{po}}x) . \overleftarrow{P_*}'x \subset \text{D}'P . \supset . (\overleftarrow{P_*}'x) \upharpoonright P \in \text{Prog}$

Dem.

$\vdash . *96{\cdot}492 . \supset \vdash : \text{Hp} . \supset . I_P{}'x = \Lambda$ (1)

$\vdash . (1) . *122{\cdot}51 . \supset \vdash . \text{Prop}$

The remaining propositions ($*122{\cdot}53{\cdot}54{\cdot}55$) are not used in the sequel.

$*122{\cdot}53$. $\vdash : P \in 1 \rightarrow 1 . x \in s'\text{gen}'P . \overleftarrow{P_*}'x \subset \text{D}'P . \supset . (\overleftrightarrow{P_*}'x) \upharpoonright P \in \text{Prog}$

Dem.

$\vdash . *97{\cdot}21 . \supset \vdash : \text{Hp} . \supset . (\exists y) . yBP . \overleftrightarrow{P_*}'x = \overleftarrow{P_*}'y .$

$[*96{\cdot}23.*93{\cdot}1] \quad \supset . (\exists y) . y \in \text{D}'P . \overleftrightarrow{P_*}'x = \overleftarrow{P_*}'y . I_P{}'y = \Lambda .$

$[*97{\cdot}17.*91{\cdot}504.\text{Hp}] \supset . (\exists y) . y \in \text{D}'P . \overleftarrow{P_*}'y \subset \text{D}'P . I_P{}'y = \Lambda . \overleftrightarrow{P_*}'x = \overleftarrow{P_*}'y .$

$[*122{\cdot}51] \quad \supset . (\overleftrightarrow{P_*}'x) \upharpoonright P \in \text{Prog} : \supset \vdash . \text{Prop}$

$*122{\cdot}54$. $\vdash : P \in 1 \rightarrow 1 . x \in s'\text{gen}'P - s'\text{gen}'\breve{P} . \supset . (\overleftrightarrow{P_*}'x) \upharpoonright P \in \text{Prog}$

Dem.

$\vdash . *93{\cdot}27{\cdot}272 . \supset \vdash : \text{Hp} . \supset . x \in s'\text{gen}'P \cap p'\text{ᗡ}``\text{Pot}'\breve{P} .$

$[*93{\cdot}381] \quad \supset . x \in s'\text{gen}'P . \overleftarrow{P_*}'x \subset \text{D}'P$ (1)

$\vdash . (1) . *122{\cdot}53 . \supset \vdash . \text{Prop}$

$*122{\cdot}55$. $\vdash :. P \in 1 \rightarrow 1 . \supset : x \in s'\text{gen}'P - s'\text{gen}'\breve{P} . \equiv . (\overleftrightarrow{P_*}'x) \upharpoonright P \in \text{Prog}$

Dem.

$\vdash . *35{\cdot}61 . \quad \supset \vdash : Q = (\overleftrightarrow{P_*}'x) \upharpoonright P . \supset . \text{D}'Q = \overleftrightarrow{P_*}'x \cap \text{D}'P$ (1)

$\vdash . *37{\cdot}4 . \quad \supset \vdash :. Q = (\overleftrightarrow{P_*}'x) \upharpoonright P . \supset : \text{ᗡ}'Q = \breve{P}``\overleftrightarrow{P_*}'x :$

$[*97{\cdot}17.*92{\cdot}111.*91{\cdot}54{\cdot}52] \supset : Q \in 1 \rightarrow 1 . \supset . \text{ᗡ}'Q = \overleftrightarrow{P_*}'x \cap \text{ᗡ}'P$ (2)

$\vdash . (1) . (2) . \supset \vdash :. \text{Hp} . \text{Hp}(1) . \supset : \exists ! \overrightarrow{B}\text{‘}Q . \supset . \exists ! \overleftrightarrow{P_*}\text{‘}x - \text{Cl‘}P .$

$[*97\cdot17 . *91\cdot504] \qquad \supset . \exists ! \overrightarrow{P_*}\text{‘}x - \text{Cl‘}P .$

$[*93\cdot38\cdot27] \qquad \supset . x \in s\text{‘gen‘}P \qquad (3)$

$\vdash . (1) . (2) . \supset \vdash :. \text{Hp}(3) . \supset : \text{D‘}Q = C\text{‘}Q . \supset . \overleftrightarrow{P_*}\text{‘}x \cap \text{D‘}P = \overleftrightarrow{P_*}\text{‘}x .$

$[*22\cdot621] \qquad \supset . \overleftrightarrow{P_*}\text{‘}x \subset \text{D‘}P .$

$[*97\cdot13] \qquad \supset . \overleftarrow{P_*}\text{‘}x \subset \text{D‘}P .$

$[*93\cdot381\cdot275] \qquad \supset . x \sim\in s\text{‘gen‘}\breve{P} \qquad (4)$

$\vdash . (3) . (4) . *122\cdot11\cdot141\cdot54 . \supset \vdash .$ Prop

$*123.\ \aleph_0$

Summary of $*123$.

In this number we are concerned with the arithmetical properties of $\aleph_0$, the smallest of Cantor's transfinite cardinals. Cantor defines $\aleph_0$ as the cardinal number of any class which can be put into one-one relation with the inductive cardinals. This definition assumes that $\nu \neq \nu +_c 1$, when ν is an inductive cardinal; in other words, it assumes the axiom of infinity; for without this, the inductive cardinals would form a finite series, with a last term, namely Λ. For this reason among others, we do not make similarity with the inductive cardinals our *definition*. We define $\aleph_0$ as the class of those classes which can be arranged in progressions, *i.e.* as $\mathrm{D}``\mathrm{Prog}$. We then have to prove that $\aleph_0$ so defined is a cardinal, and that if it is not null, it is the number of the inductive numbers.

For convenience we put for the moment N for the relation of μ to $\mu +_c 1$ when μ is an inductive cardinal. We then easily prove

$*123{\cdot}21{\cdot}23$. $\vdash . N \in \mathrm{Cls} \to 1 . \mathrm{D}`N = \mathrm{NC\,induct} . B`N = 0 . \overleftarrow{N_*}`0 = \mathrm{NC\,induct}$

The only thing further required to prove $N \in \mathrm{Prog}$ is $N \in 1 \to \mathrm{Cls}$, *i.e.*

$$\mu, \nu \in \mathrm{NC\,induct} . \mu +_c 1 = \nu +_c 1 . \supset . \mu = \nu.$$

By $*120{\cdot}311$, this holds if $\exists ! \mu +_c 1$, which holds if Infin ax holds. Hence

$*123{\cdot}25{\cdot}26$. $\vdash : \mathrm{Infin\,ax}(x) . \supset . N \upharpoonright t^3`x \in \mathrm{Prog} . \mathrm{NC\,induct} \cap t^3`x \in \aleph_0$

whence, by $*122{\cdot}36$,

$*123{\cdot}27$. $\vdash : \exists ! \aleph_0(x) . \supset . \mathrm{NC\,induct} \cap t^3`x \in \aleph_0$

Again it is obvious from $*122{\cdot}34{\cdot}341$ that if R is a progression, $\mathrm{D}`R$ can always be put into a $1 \to 1$ relation to the inductive cardinals ($*123{\cdot}3$) since $\mathrm{D}`R$ consists of the terms $1_R, 2_R, \ldots \nu_R, \ldots$, and all the inductive cardinals are used in putting $\mathrm{D}`R$ into this form. Hence

$*123{\cdot}31$. $\vdash : \alpha \in \aleph_0 . \supset . \alpha\ \mathrm{sm\ NC\,induct}$

whence also

$*123{\cdot}311$. $\vdash : \alpha, \beta \in \aleph_0 . \supset . \alpha\ \mathrm{sm}\ \beta$

It remains to prove that any class similar to the inductive cardinals is an $\aleph_0$; this can only be proved by assuming the axiom of infinity. We prove

first (*123·32) that if R is a progression, and S is a one-one whose converse domain is $\text{D}'R$, then $S \mid R \mid \breve{S}$ is a progression whose domain is $\text{D}'S$. Hence

***123·321.** $\vdash : \alpha \in \aleph_0 \,.\, \alpha \operatorname{sm} \beta \,.\, \supset .\, \beta \in \aleph_0$

From this and $\alpha, \beta \in \aleph_0 \,.\, \supset .\, \alpha \operatorname{sm} \beta$, we obtain

***123·322.** $\vdash : \alpha \in \aleph_0 \,.\, \supset .\, \aleph_0 = \text{Nc}'\alpha$

Hence by our previous results

***123·34.** $\vdash : \text{Infin ax}\,(x) \,.\, \supset .\, \aleph_0 = \text{Nc}'(\text{NC induct} \cap t^{3}{}'x)$

Also we have, by *123·322 above,

$$\exists ! \aleph_0 \,.\, \supset .\, \aleph_0 \in \text{NC},$$

whence, since $\Lambda \in \text{NC}$, we obtain at last

***123·36.** $\vdash .\, \aleph_0 \in \text{NC}$

As to the existence of $\aleph_0$ in various types, if $\text{Infin ax}\,(x)$ holds, *i.e.* if, given any inductive cardinal ν, there are classes having ν terms and composed of terms of the same type as x, then $\text{NC induct}\,(t'x) \in \aleph_0\,(t^2{}'x)$. Thus

***123·37.** $\vdash : \text{Infin ax}\,(x) \,.\, \supset .\, \exists ! \aleph_0\,(t^2{}'x) \,.\, \aleph_0\,(t^2{}'x) \in \text{N}_0\text{C}$

The arithmetical properties of $\aleph_0$ in regard to addition, multiplication and exponentiation by an inductive cardinal are easily proved. We have

***123·41.** $\vdash : \nu \in \text{NC induct} \,.\, \supset .\, \aleph_0 = \aleph_0 +_c \nu$

***123·421.** $\vdash .\, \aleph_0 = \aleph_0 +_c \aleph_0 = 2 \times_c \aleph_0$

***123·422.** $\vdash : \nu \in \text{NC induct} - \iota'0 \,.\, \supset .\, \nu \times_c \aleph_0 = \aleph_0$

***123·52.** $\vdash .\, \aleph_0 = \aleph_0 \times_c \aleph_0 = \aleph_0{}^2$

***123·53.** $\vdash : \nu \in \text{NC induct} - \iota'0 \,.\, \supset .\, \aleph_0{}^\nu = \aleph_0$

All these propositions are well known.

The early propositions of the present number are for the most part immediate consequences of propositions proved in *122.

***123·01.** $\aleph_0 = \text{D}''\text{Prog}$ Df

***123·02.** $N = \hat{\mu}\hat{\nu}\,\{\mu \in \text{NC induct} \,.\, \nu = (\mu +_c 1) \cap t_0{}'\mu\}$ Dft. [*123—4]

***123·1.** $\vdash : \alpha \in \aleph_0 \,.\, \equiv .\, (\exists R) \,.\, R \in \text{Prog} \,.\, \alpha = \text{D}'R$ [*37·1 . (*123·01)]

***123·101.** $\vdash : R \in \text{Prog} \,.\, \supset .\, \text{D}'R \in \aleph_0$ [*123·1]

***123·11.** $\vdash : R \in 1 \to 1 \,.\, \text{D}'R = \overleftarrow{R_*}{}'B'R \,.\, \supset .\, \text{D}'R \in \aleph_0$ [*123·101 . *122·1]

***123·12.** $\vdash : \alpha \in \aleph_0 \,.\, \supset .\, (\exists R) \,.\, \text{D}'R = \alpha \,.\, R \in 1 \to 1 \,.\, \text{C}'R \subset \text{D}'R \,.\, \overrightarrow{B}'R \in 1$
[*123·1 . *122·141·11]

***123·13.** $\vdash : \alpha \in \aleph_0 . \supset . \mathrm{Nc}`\alpha = \mathrm{Nc}`\alpha +_c 1$

Dem.

$\vdash . *123·12 . *110·32 . \supset$

$\vdash : \alpha \in \aleph_0 . \supset . (\exists R) . \mathrm{D}`R = \alpha . R \in 1 \to 1 . \mathrm{Nc}`\mathrm{D}`R = \mathrm{Nc}`\mathrm{G}`R +_c 1 .$

[*100·321] $\supset . (\exists R) . \mathrm{D}`R = \alpha . \mathrm{Nc}`\mathrm{D}`R = \mathrm{Nc}`\mathrm{D}`R +_c 1 .$

[*35·94.*13·195] $\supset . \mathrm{Nc}`\alpha = \mathrm{Nc}`\alpha +_c 1 : \supset \vdash . \mathrm{Prop}$

***123·14.** $\vdash : \alpha \in \aleph_0 . \nu \in \mathrm{NC\,induct} . \supset . \exists ! \nu \cap \mathrm{Cl}`\alpha$ [*122·35]

***123·15.** $\vdash : \alpha \in \aleph_0 . \supset . \alpha \sim\in \mathrm{Cls\,induct}$ [*122·37]

***123·16.** $\vdash : \alpha \in \aleph_0 . \supset . \mathrm{Cl}`\alpha \subset \mathrm{Cls\,induct} \cup \aleph_0$ [*122·46]

***123·17.** $\vdash : \alpha \in \aleph_0 . \beta \in \mathrm{Cls\,induct} . \supset . \alpha - \beta \in \aleph_0$

Dem.

$\vdash . *120·481 . \supset \vdash : \mathrm{Hp} . \supset . \alpha \cap \beta \in \mathrm{Cls\,induct} .$

[*122·49] $\supset . \alpha - (\alpha \cap \beta) \in \aleph_0 : \supset \vdash . \mathrm{Prop}$

***123·18.** $\vdash : \exists ! \aleph_0 (x) . \supset . \mathrm{Infin\,ax}\,(x)$ [*122·36]

***123·19.** $\vdash : R \in \mathrm{Prog} . \exists ! \alpha . \alpha \subset R_{\mathrm{po}}``\alpha . \supset . \alpha \in \aleph_0$ [*122·45]

***123·191.** $\vdash : R \in 1 \to 1 . x \in \mathrm{D}`R . \sim(x R_{\mathrm{po}} x) . \overleftarrow{R_*}`x \subset \mathrm{D}`R . \supset . \overleftarrow{R_*}`x \in \aleph_0$ [*122·52]

***123·192.** $\vdash : R \in 1 \to 1 . \mathrm{G}`R \subset \mathrm{D}`R . \supset . \overleftarrow{R_*}``\overrightarrow{B}`R \subset \aleph_0$

Dem.

$\vdash . *93·101 . \quad \supset \vdash : x \in \overrightarrow{B}`R . \supset . x \in \mathrm{D}`R$ (1)

$\vdash . *91·504 . *93·101 . \quad \supset \vdash : x \in \overrightarrow{B}`R . \supset . \sim(x R_{\mathrm{po}} x)$ (2)

$\vdash . *90·13 . \quad \supset \vdash : \mathrm{G}`R \subset \mathrm{D}`R . \supset . \overleftarrow{R_*}`x \subset \mathrm{D}`R$ (3)

$\vdash . (1) . (2) . (3) . *123·191 . \supset \vdash : \mathrm{Hp} . x \in \overrightarrow{B}`R . \supset . \overleftarrow{R_*}`x \in \aleph_0 : \supset \vdash . \mathrm{Prop}$

***123·2.** $\vdash : \mu N \nu . \equiv . \mu \in \mathrm{NC\,induct} . \nu = (\mu +_c 1) \cap t_0`\mu$ [(*123·02)]

***123·21.** $\vdash . N \in \mathrm{Cls} \to 1 . \mathrm{D}`N = \mathrm{NC\,induct} . \mathrm{G}`N = \mathrm{NC\,induct} - \iota`0 . B`N = 0$

Dem.

$\vdash . *123·2 . *13·172 . \quad \supset \vdash : \mu N \nu . \mu N \varpi . \supset . \nu = \varpi :$

[*71·171] $\supset \vdash . N \in \mathrm{Cls} \to 1$ (1)

$\vdash . *123·2 . \quad \supset \vdash . \mathrm{D}`N = \mathrm{NC\,induct}$ (2)

$\vdash . *123·2 . \quad \supset \vdash . \mathrm{G}`N = \hat{\nu} \{(\exists \mu) . \mu \in \mathrm{NC\,induct} . \nu = \mu +_c 1\}$

[*120·423] $= \mathrm{NC\,induct} - \iota`0$ (3)

$\vdash . (2) . (3) . *93·101 . \supset \vdash . B`N = 0$ (4)

$\vdash . (1) . (2) . (3) . (4) . \supset \vdash . \mathrm{Prop}$

***123·22.** $\vdash . \breve{N} = (+_c 1) \upharpoonright \mathrm{NC\,induct}$ [*123·2]

***123·23.** $\vdash . \overleftarrow{N_*}`0 = \text{NC induct} = \text{D}`N$

Dem.

$\vdash . *123·22 . \supset \vdash . \overleftarrow{N_*}`0 = \hat{\mu}[\mu \{(+_c 1) \upharpoonright \text{NC induct}\}_* 0]$

$[*120·1.*96·21·131] \quad = \hat{\mu}[\mu \{\text{NC induct} \upharpoonleft (+_c 1)_*\} 0]$

$[*120·1] \quad = \text{NC induct} \qquad (1)$

$\vdash . (1) . *123·21 . \supset \vdash . \text{Prop}$

***123·24.** $\vdash : \text{Infin ax}(x) . \supset . N \upharpoonright t^3`x \in 1 \rightarrow 1$

Dem.

$\vdash . *120·301·121 . \supset \vdash :: \text{Hp} . \supset :. \mu \in \text{NC induct} . \supset : \exists ! (\mu +_c 1) \cap t^2`x :$

$[*120·311] \quad \supset : (\mu +_c 1) \cap t^2`x = \nu +_c 1 . \supset . \mu = \nu :$

$[*123·2.*71·17] \quad \supset : N \upharpoonright t^3`x \in 1 \rightarrow \text{Cls} \qquad (1)$

$\vdash . (1) . *123·21 . \supset \vdash . \text{Prop}$

***123·25.** $\vdash : \text{Infin ax}(x) . \supset . N \upharpoonright t^3`x \in \text{Prog} \qquad [*123·21·23·24 . *122·1]$

***123·26.** $\vdash : \text{Infin ax}(x) . \supset . \text{NC induct} \cap t^3`x \in \aleph_0 \qquad [*123·25·21·101]$

***123·27.** $\vdash : \exists ! \aleph_0(x) . \supset . \text{NC induct} \cap t^3`x \in \aleph_0 \qquad [*123·26·18]$

***123·3.** $\vdash : R \in \text{Prog} . S = \hat{x}\hat{\nu}\{\nu \in \text{NC induct} . x = (\nu +_c 1)_R\} . \supset .$
$S \in 1 \rightarrow 1 . \text{D}`S = \text{D}`R . \text{Cl}`S = \text{NC induct}$

Dem.

$\vdash . *120·423 . \supset \vdash : \text{Hp} . \supset . \text{D}`S = \hat{x}\{(\exists\mu) . \mu \in \text{NC induct} - \iota`0 . x = \mu_R\}$

$[*122·341] \quad = \text{D}`R \qquad (1)$

$\vdash . *14·204 . *122·34 . \supset \vdash : \text{Hp} . \supset . \text{Cl}`S = \hat{\nu}\{\text{E} ! (\nu +_c 1)_R\}$

$[*122·34] \quad = \hat{\nu}\{\nu +_c 1 \in \text{NC induct} - \iota`0\} \qquad (2)$

$\vdash . *122·36 . *120·3 . \supset \vdash :. \text{Hp} . \supset : \nu +_c 1 \in \text{NC induct} . \supset . \exists ! \nu +_c 1 .$

$[*120·422] \quad \supset . \nu \in \text{NC induct} \qquad (3)$

$\vdash . (3) . *120·421·121 . \supset \vdash :. \text{Hp} . \supset : \nu +_c 1 \in \text{NC induct} - \iota`0 . \equiv .$
$\nu \in \text{NC induct} \qquad (4)$

$\vdash . (2) . (4) . \quad \supset \vdash : \text{Hp} . \supset . \text{Cl}`S = \text{NC induct} \qquad (5)$

$\vdash . *13·172 . *71·17 . \supset \vdash : \text{Hp} . \supset . S \in 1 \rightarrow \text{Cls} \qquad (6)$

$\vdash . *121·631 . \quad \supset \vdash :. \text{Hp} . \supset : xS\mu . xS\nu . \supset .$
$\text{Nc}`R(B`R \mapsto x) = \mu +_c 1 . \text{Nc}`R(B`R \mapsto x) = \nu +_c 1 .$

$[*13·171] \quad \supset . \mu +_c 1 = \nu +_c 1 \qquad (7)$

$\vdash . (5) . *122·36 . *120·3 . \supset \vdash :. \text{Hp} . \supset : xS\mu . \supset . \exists ! \mu +_c 1 :$

$[*120·41] \quad \supset : xS\mu . \mu +_c 1 = \nu +_c 1 . \supset . \mu = \nu :$

$[(7)] \quad \supset : xS\mu . xS\nu . \supset . \mu = \nu :$

$[*71·171] \quad \supset : S \in \text{Cls} \rightarrow 1 \qquad (8)$

$\vdash . (1) . (5) . (6) . (8) . \supset \vdash . \text{Prop}$

$*123\cdot31.\ \vdash : \alpha \in \aleph_0 . \supset . \alpha \,\mathrm{sm}\, \mathrm{NC\,induct}$ [$*123\cdot3$]

$*123\cdot311.\ \vdash : \alpha, \beta \in \aleph_0 . \supset . \alpha \,\mathrm{sm}\, \beta$ [$*123\cdot31 . *73\cdot31\cdot32$]

It is not assumed here that α and β are of the same type.

$*123\cdot312.\ \vdash : R \in \mathrm{Prog} . S \in 1 \to 1 . \text{Ɑ}\text{‘}S = \mathrm{D}\text{‘}R . \supset .$

$$S \mid R \mid \breve{S} \in 1 \to 1 . \mathrm{D}\text{‘}S = \mathrm{D}\text{‘}(S \mid R \mid \breve{S}) . S\text{‘}B\text{‘}R = B\text{‘}(S \mid R \mid \breve{S})$$

Dem.

$\vdash . *71\cdot252 . *122\cdot1 . \quad \supset \vdash : \mathrm{Hp} . \supset . S \mid R \mid \breve{S} \in 1 \to 1 \quad (1)$

$\vdash . *122\cdot141 . *37\cdot321 . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{D}\text{‘}(R \mid \breve{S}) = \mathrm{D}\text{‘}R = \text{Ɑ}\text{‘}S . \quad (2)$

$[*37\cdot323] \quad \supset . \mathrm{D}\text{‘}(S \mid R \mid \breve{S}) = \mathrm{D}\text{‘}S \quad (3)$

$\vdash . (2) . *37\cdot32 . \quad \supset \vdash : \mathrm{Hp} . \supset . \text{Ɑ}\text{‘}(S \mid R \mid \breve{S}) = S\text{‘‘}\text{Ɑ}\text{‘}R \quad (4)$

$\vdash . (3) . (4) . \quad \supset \vdash : \mathrm{Hp} . \supset . \overrightarrow{B}\text{‘}(S \mid R \mid \breve{S}) = \mathrm{D}\text{‘}S - S\text{‘‘}\text{Ɑ}\text{‘}R$

$[*37\cdot25.\mathrm{Hp}] \quad = S\text{‘‘}\mathrm{D}\text{‘}R - S\text{‘‘}\text{Ɑ}\text{‘}R$

$[*71\cdot381] \quad = S\text{‘‘}\overrightarrow{B}\text{‘}R$

$[*122\cdot11 . *53\cdot31] \quad = \iota\text{‘}S\text{‘}B\text{‘}R \quad (5)$

$\vdash . (1) . (3) . (5) . \supset \vdash . \mathrm{Prop}$

$*123\cdot313.\ \vdash : R \in \mathrm{Prog} . S \in 1 \to 1 . \text{Ɑ}\text{‘}S = \mathrm{D}\text{‘}R . P = S \mid R \mid \breve{S} . \supset . \mathrm{D}\text{‘}P = \overleftarrow{P_*}\text{‘}B\text{‘}P$

Dem.

$\vdash . *34\cdot36 . *123\cdot312 . \supset \vdash : \mathrm{Hp} . \supset . \text{Ɑ}\text{‘}P \subset \mathrm{D}\text{‘}P . \mathrm{E} ! \, B\text{‘}P .$

$[*90\cdot13] \quad \supset . \overleftarrow{P_*}\text{‘}B\text{‘}P \subset \mathrm{D}\text{‘}P \quad (1)$

$\vdash . *123\cdot312 . \quad \supset \vdash : \mathrm{Hp} . \supset . S\text{‘}B\text{‘}R \in \overleftarrow{P_*}\text{‘}B\text{‘}P \quad (2)$

$\vdash . *33\cdot14 . \quad \supset \vdash : \mathrm{Hp} . S\text{‘}x \in \overleftarrow{P_*}\text{‘}B\text{‘}P . xRy . \supset . y \in \text{Ɑ}\text{‘}R .$

$[*122\cdot141.\mathrm{Hp}] \quad \supset . y \in \text{Ɑ}\text{‘}S .$

$[*71\cdot16] \quad \supset . \mathrm{E} ! \, S\text{‘}y .$

$[*30\cdot32 . *34\cdot1] \quad \supset . S\text{‘}x \, (S \mid R \mid \breve{S}) \, S\text{‘}y .$

$[\mathrm{Hp}] \quad \supset . S\text{‘}x \, P \, S\text{‘}y .$

$[*90\cdot163] \quad \supset . S\text{‘}y \in \overleftarrow{P_*}\text{‘}B\text{‘}P \quad (3)$

$\vdash . (2) . (3) . *90\cdot112 . \supset \vdash :. \mathrm{Hp} . \supset : (B\text{‘}R) R_* x . \supset . S\text{‘}x \in \overleftarrow{P_*}\text{‘}B\text{‘}P :$

$[*37\cdot63] \quad \supset : S\text{‘‘}\overleftarrow{R_*}\text{‘}B\text{‘}R \subset \overleftarrow{P_*}\text{‘}B\text{‘}P :$

$[*122\cdot1] \quad \supset : S\text{‘‘}\mathrm{D}\text{‘}R \subset \overleftarrow{P_*}\text{‘}B\text{‘}P :$

$[*37\cdot25.\mathrm{Hp}] \quad \supset : \mathrm{D}\text{‘}S \subset \overleftarrow{P_*}\text{‘}B\text{‘}P :$

$[*123\cdot312] \quad \supset : \mathrm{D}\text{‘}P \subset \overleftarrow{P_*}\text{‘}B\text{‘}P \quad (4)$

$\vdash . (1) . (4) . \supset \vdash . \mathrm{Prop}$

$*123\cdot32.\ \vdash : R \in \mathrm{Prog} . S \in 1 \to 1 . \text{Ɑ}\text{‘}S = \mathrm{D}\text{‘}R . \supset .$

$$S \mid R \mid \breve{S} \in \mathrm{Prog} . \mathrm{D}\text{‘}S = \mathrm{D}\text{‘}(S \mid R \mid \breve{S}) . S\text{‘}B\text{‘}R = B\text{‘}(S \mid R \mid \breve{S}) \quad [*123\cdot312\cdot313]$$

***123·321.** $\vdash : \alpha \epsilon \aleph_0 . \alpha \operatorname{sm} \beta . \supset . \beta \epsilon \aleph_0$ [*123·32]

***123·322.** $\vdash : \alpha \epsilon \aleph_0 . \supset . \aleph_0 = \mathrm{Nc}`\alpha$

Dem.

$\vdash . *123·311·321 . \supset \vdash :. \alpha \epsilon \aleph_0 . \supset : \beta \epsilon \aleph_0 . \equiv . \beta \operatorname{sm} \alpha$ (1)

$\vdash . (1) . *100·1 . \quad \supset \vdash . \mathrm{Prop}$

***123·323.** $\vdash : R \epsilon \mathrm{Prog} . \supset . \aleph_0 = \mathrm{Nc}`\mathrm{D}`R$ [*123·322]

***123·33.** $\vdash :. \mathrm{Infin\,ax}(x) . \supset : \alpha \epsilon \aleph_0 . \equiv . \alpha \operatorname{sm} (\mathrm{NC\,induct} \cap t^3`x)$ [*123·26·321·31]

***123·34.** $\vdash : \mathrm{Infin\,ax}(x) . \supset . \aleph_0 = \mathrm{Nc}`(\mathrm{NC\,induct} \cap t^3`x)$ [*123·33]

***123·35.** $\vdash : \exists ! \aleph_0(x) . \supset . \aleph_0(x) = \mathrm{Nc}`(\mathrm{NC\,induct} \cap t^3`x)$ [*123·34·18]

***123·36.** $\vdash . \aleph_0 \epsilon \mathrm{NC}$ [*123·35 . *102·74]

***123·361.** $\vdash : \exists ! \aleph_0 . \supset . \aleph_0 \sim \epsilon \mathrm{NC\,induct}$ [*123·15·322 . *120·211]

***123·37.** $\vdash : \mathrm{Infin\,ax}(x) . \supset . \exists ! \aleph_0(t^2`x) . \aleph_0(t^2`x) \epsilon \mathrm{N_0C}$

Dem.

$\vdash . *120·301 . \supset \vdash :. \mathrm{Hp} . \supset : \nu \epsilon \mathrm{NC\,induct} . \supset_\nu . \exists ! \nu(x) :$

[*65·13] $\supset : \nu \epsilon \mathrm{NC\,induct} . \supset_\nu . \exists ! \nu . \nu = \nu(x) :$

[(*65·02)] $\supset : \nu \epsilon \mathrm{NC\,induct} . \supset_\nu . \exists ! \nu : \mathrm{NC\,induct} \subset t^3`x :$

[*123·34] $\supset : \mathrm{NC\,induct} \epsilon \aleph_0 . \mathrm{NC\,induct} \subset t^3`x :$

[(*65·02)] $\supset : \mathrm{NC\,induct} \epsilon \aleph_0(t^2`x)$ (1)

$\vdash . (1) . *103·34 . *123·36 . \supset \vdash . \mathrm{Prop}$

***123·39.** $\vdash . (\aleph_0)_\eta = (\aleph_0 +_c 1)_\eta$

Dem.

$\vdash . *118·12 . *117·6 . *123·322 . \supset \vdash : (\aleph_0)_\eta = \Lambda . \supset . (\aleph_0 +_c 1)_\eta = \Lambda$ (1)

$\vdash . *123·13·322 . \quad \supset \vdash : \exists ! (\aleph_0)_\eta . \supset . (\aleph_0)_\eta = (\aleph_0 +_c 1)_\eta$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***123·4.** $\vdash . \aleph_0 = \aleph_0 +_c 1$ [*123·39]

***123·401.** $\vdash : \exists ! \aleph_0 . \supset . \aleph_0 = \aleph_0 -_c 1$

Dem.

$\vdash . *120·124 . *123·36·4 . \supset \vdash : \exists ! \aleph_0 . \supset . \aleph_0 \epsilon \mathrm{NC} - \iota`0 .$

[*120·414·416] $\supset . (\aleph_0 -_c 1) +_c 1 = \aleph_0$

[*123·4] $= \aleph_0 +_c 1 .$

[*120·311] $\supset . \aleph_0 -_c 1 = \aleph_0$ (1)

$\vdash . *119·11 . \supset \vdash : (\aleph_0)_\eta = \Lambda . \supset . (\aleph_0)_\eta = (\aleph_0 -_c 1)_\eta$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***123·41.** $\vdash : \nu \epsilon \mathrm{NC\,induct} . \supset . \aleph_0 = \aleph_0 +_c \nu$ [*123·4 . *120·11]

***123·411.** $\vdash : \nu \epsilon \mathrm{NC\,induct} . \supset . \aleph_0 = \aleph_0 -_c \nu$ [*123·401 . *120·11]

***123·42.** $\vdash : P \in \text{Prog} \,.\, Q = P^2 \,.\, \supset \,.\, \overleftarrow{Q_*}‘1_P, \overleftarrow{Q_*}‘2_P \in \aleph_0 \,.\, \overleftarrow{Q_*}‘1_P \cap \overleftarrow{Q_*}‘2_P = \Lambda$

Note that $\overleftarrow{Q_*}‘1_P$ is the odd terms and $\overleftarrow{Q_*}‘2_P$ the even terms of $\text{D}‘P$.

Dem.

$\vdash \,.\, *91·6 \,.\, \supset \vdash :. \text{Hp} \,.\, \supset : \overleftarrow{Q_*}‘1_P \subset \overleftarrow{P_*}‘1_P \,.\, \overleftarrow{Q_*}‘2_P \subset \overleftarrow{P_*}‘2_P :$

$[*122·1] \quad \supset : \overleftarrow{Q_*}‘1_P \subset \text{D}‘P :$

$[*33·13] \quad \supset : y \in \overleftarrow{Q_*}‘1_P \,.\, \supset \,.\, (\exists z) \,.\, yPz \,.$

$[*122·141] \quad \supset \,.\, (\exists z, w) \,.\, yPz \,.\, zPw \,.$

$[\text{Hp}.*90·163.*91·503] \quad \supset \,.\, (\exists w) \,.\, yQw \,.\, w \in \overleftarrow{Q_*}‘1_P \,.\, yP_{\text{po}}w :$

$[*37·1] \quad \supset : \overleftarrow{Q_*}‘1_P \subset P_{\text{po}}“\overleftarrow{Q_*}‘1_P :$

$[*123·19] \quad \supset : \overleftarrow{Q_*}‘1_P \in \aleph_0$ (1)

Similarly $\vdash : \text{Hp} \,.\, \supset \,.\, \overleftarrow{Q_*}‘2_P \in \aleph_0$ (2)

$\vdash \,.\, *121·601·602 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, 1_P P\, 2_P \,.$

$[*122·16.*91·52·6] \quad \supset \,.\, \sim(2_P\, Q_*\, 1_P)$ (3)

$\vdash \,.\, *121·602 \,.\, *53·31 \,.\, *93·1 \,.\, \supset \vdash :. \text{Hp} \,.\, \supset : \overrightarrow{Q}‘2_P = \overrightarrow{P}‘1_P = \Lambda :$

$[*13·14] \quad \supset : yQz \,.\, \supset \,.\, z \neq 2_P :$

$[*91·542] \quad \supset : 2_P Q_* z \,.\, yQz \,.\, \supset \,.\, 2_P Q_{\text{po}} z \,.\, yQz \,.$

$[*92·11] \quad \supset \,.\, 2_P Q_* y :$

$[\text{Transp}] \quad \supset : \sim(2_P Q_* y) \,.\, yQz \,.\, \supset \,.\, \sim(2_P Q_* z)$ (4)

$\vdash \,.\, (3) \,.\, (4) \,.\, *90·112 \,.\, \supset \vdash :. \text{Hp} \,.\, \supset : 1_P Q_* z \,.\, \supset \,.\, \sim(2_P Q_* z)$ (5)

$\vdash \,.\, (1) \,.\, (2) \,.\, (5) \,.\, \supset \vdash \,.\, \text{Prop}$

***123·421.** $\vdash \,.\, \aleph_0 = \aleph_0 +_c \aleph_0 = 2 \times_c \aleph_0$

Dem.

$\vdash \,.\, *123·42 \,.\, \quad \supset \vdash : \alpha \in \aleph_0 \,.\, \supset \,.\, (\exists \beta, \gamma) \,.\, \beta, \gamma \in \aleph_0 \,.\, \beta \cap \gamma = \Lambda \,.\, \beta \cup \gamma \subset \alpha \,.$

$[*110·32.*117·22] \quad \supset \,.\, \text{Nc}‘\alpha \geqslant \aleph_0 +_c \aleph_0$ (1)

$\vdash \,.\, (1) \,.\, *117·6·23 \,.\, \supset \vdash : \exists ! \aleph_0 \,.\, \supset \,.\, \aleph_0 = \aleph_0 +_c \aleph_0$ (2)

$\vdash \,.\, (2) \,.\, *118·12 \,.\, *117·6 \,.\, \supset \vdash \,.\, \aleph_0 = \aleph_0 +_c \aleph_0$ (3)

$\vdash \,.\, (3) \,.\, *113·66 \,.\, \supset \vdash \,.\, \text{Prop}$

***123·422.** $\vdash : \nu \in \text{NC induct} - \iota‘0 \,.\, \supset \,.\, \nu \times_c \aleph_0 = \aleph_0$

Dem.

$\vdash \,.\, *113·671 \,.\, \supset \vdash : \nu \times_c \aleph_0 = \aleph_0 \,.\, \supset \,.\, (\nu +_c 1) \times_c \aleph_0 = \aleph_0 +_c \aleph_0$

$[*123·421] \quad = \aleph_0$ (1)

$\vdash \,.\, (1) \,.\, *120·47 \,.\, \supset \vdash \,.\, \text{Prop}$

***123·43.** $\vdash :. \exists ! \aleph_0 \,.\, \supset : \nu \in \text{NC induct} \,.\, \supset_\nu \,.\, \aleph_0 > \nu$

Dem.

$\vdash \,.\, *123·18·36·361 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, \aleph_0 \in \text{NC} - \text{NC induct} - \iota‘\Lambda \,.$

$\text{NC induct} \subset -\iota‘\Lambda$ (1)

$\vdash \,.\, (1) \,.\, *120·49 \,.\, \supset \vdash \,.\, \text{Prop}$

∗123·44. $\vdash :. \exists ! \aleph_0 . \supset : \nu \in \text{NC induct} \cup \iota`\aleph_0 . \equiv . \aleph_0 \geqslant \nu$

Dem.

$\vdash . *123·322 . \supset \vdash :. \alpha \in \aleph_0 . \supset : \aleph_0 \geqslant \nu . \supset . \text{Nc}`\alpha \geqslant \nu .$

$[*117·22·104·12] \quad \supset . \exists ! \nu \cap \text{Cl}`\alpha . \nu \in \text{N}_0\text{C} .$

$[*123·16] \quad \supset . \exists ! \nu \cap (\text{Cls induct} \cup \aleph_0) . \nu \in \text{N}_0\text{C} .$

$[*103·26] \quad \supset . (\exists \beta) . \nu = \text{N}_0\text{c}`\beta . \beta \in \text{Cls induct} \cup \aleph_0 .$

$[*120·21.*103·26] \quad \supset . \nu \in \text{NC induct} \cup \iota`\aleph_0 \quad (1)$

$\vdash . (1) . *123·43 . \supset \vdash . \text{Prop}$

∗123·45. $\vdash :. \exists ! \aleph_0 . \supset : \nu \in \text{NC induct} . \equiv . \aleph_0 > \nu . \equiv . \nu < \aleph_0$ [∗123·43·44]

∗123·46. $\vdash : \alpha \in \text{Cls induct} . \beta \in \aleph_0 . \supset . \alpha \cup \beta \in \aleph_0$

Dem.

$\vdash . *110·32 . *22·91 . \supset \vdash . \text{Nc}`(\alpha \cup \beta) = \text{Nc}`\beta +_c \text{Nc}`(\alpha - \beta) \quad (1)$

$\vdash . *120·481·21 . \quad \supset \vdash : \text{Hp} . \supset . \text{N}_0\text{c}`(\alpha - \beta) \in \text{NC induct} \quad (2)$

$\vdash . *123·322 . \quad \supset \vdash : \text{Hp} . \supset . \aleph_0 = \text{Nc}`\beta \quad (3)$

$\vdash . (2) . (3) . (*110·04) . *123·41 . \supset \vdash : \text{Hp} . \supset . \text{Nc}`\beta +_c \text{Nc}`(\alpha - \beta) = \aleph_0 \quad (4)$

$\vdash . (1) . (4) . *100·44 . \supset \vdash . \text{Prop}$

∗123·47. $\vdash :. \exists ! \aleph_0 . \supset : \alpha \in \text{Cls induct} \cup \aleph_0 . \equiv . (\exists \gamma) . \gamma \in \aleph_0 . \alpha \subset \gamma .$
$\equiv . \text{Nc}`\alpha \leqslant \aleph_0$

Dem.

$\vdash . *123·46 . \quad \supset \vdash :. \text{Hp} . \supset : \alpha \in \text{Cls induct} . \supset . (\exists \gamma) . \gamma \in \aleph_0 . \alpha \subset \gamma \quad (1)$

$\vdash . *22·42 . \quad \supset \vdash : \alpha \in \aleph_0 . \supset . (\exists \gamma) . \gamma \in \aleph_0 . \alpha \subset \gamma \quad (2)$

$\vdash . *123·16 . \quad \supset \vdash : (\exists \gamma) . \gamma \in \aleph_0 . \alpha \subset \gamma . \supset . \alpha \in \text{Cls induct} \cup \aleph_0 \quad (3)$

$\vdash . (1) . (2) . (3) . \supset \vdash :. \text{Hp} . \supset : \alpha \in \text{Cls induct} \cup \aleph_0 . \equiv . (\exists \gamma) . \gamma \in \aleph_0 . \alpha \subset \gamma \quad (4)$

$\vdash . *123·44·322 . \supset \vdash :. \beta \in \aleph_0 . \supset : \text{N}_0\text{c}`\alpha \in \text{NC induct} \cup \iota`\aleph_0 . \equiv . \text{N}_0\text{c}`\alpha \leqslant \text{N}_0\text{c}`\beta :$

$[*103·26.*120·21.*117·107] \quad \supset : \alpha \in \text{Cls induct} \cup \aleph_0 . \equiv . \text{Nc}`\alpha \leqslant \text{Nc}`\beta .$

$[*123·322] \quad \equiv . \text{Nc}`\alpha \leqslant \aleph_0 \quad (5)$

$\vdash . (5) . *10·11·23 . \supset \vdash :. \exists ! \aleph_0 . \supset : \alpha \in \text{Cls induct} \cup \aleph_0 . \equiv . \text{Nc}`\alpha \leqslant \aleph_0 \quad (6)$

$\vdash . (4) . (6) . \supset \vdash . \text{Prop}$

The following propositions are concerned in proving $\aleph_0^2 = \aleph_0$. The proof given is roughly Cantor's. It consists in showing that the relation R defined in the hypothesis of ∗123·5 is a progression.

∗123·5. $\vdash : P, Q \in \text{Prog} .$
$R = \hat{X} \hat{Y} [(\exists \mu, \nu) : X = \mu_P \downarrow \nu_Q . Y = (\mu +_c 1)_P \downarrow (\nu -_c 1)_Q . \mathbf{v} .$
$X = \mu_P \downarrow 1_Q . Y = 1_P \downarrow (\mu +_c 1)_Q] . \supset . R \in 1 \rightarrow 1$

Dem.

$\vdash . *122·34 . \supset \vdash :. \text{Hp} . \supset : X = \mu_P \downarrow \nu_Q . Y = (\mu +_c 1)_P \downarrow (\nu -_c 1)_Q . \supset .$
$\mu, \nu \in \text{NC induct} - \iota`0 . \nu \neq 1 \quad (1)$

$\vdash . (1) . \supset \vdash :. \text{Hp} . \supset : (\exists \mu, \nu) . X = \mu_P \downarrow \nu_Q . Y = (\mu +_c 1)_P \downarrow (\nu -_c 1)_Q . \supset .$
$\sim(\exists \mu) . X = \mu_P \downarrow 1_Q . Y = 1_P \downarrow (\mu +_c 1)_Q$ (2)

$\vdash . (2) . *123{\cdot}3 . \supset$
$\vdash :: \text{Hp} . \supset :. (\exists \mu, \nu) . X = \mu_P \downarrow \nu_Q . Y = (\mu +_c 1)_Q \downarrow (\nu -_c 1)_Q : XRY' . X'RY : \supset .$
$X = X' . Y = Y'$ (3)

$\vdash . (2) . \text{Transp} . *123{\cdot}3 . \supset$
$\vdash :: \text{Hp} . \supset :. (\exists \mu) . X = \mu_P \downarrow 1_Q . Y = 1_P \downarrow (\mu +_c 1)_Q : XRY' . X'RY : \supset .$
$X = X' . Y = Y'$ (4)

$\vdash . (3) . (4) . \supset \vdash : \text{Hp} . \supset . R \in 1 \to 1 : \supset \vdash . \text{Prop}$

***123·501.** $\vdash : \text{Hp} *123{\cdot}5 . \supset . \text{D}`R = \text{D}`P \times \text{D}`Q$

Dem.

$\vdash . *122{\cdot}34 . \supset \vdash :. \text{Hp} . \supset : \mu, \nu \in \text{NC induct} - \iota`0 . \nu \neq 1 . \supset .$
$(\mu_P \downarrow \nu_Q) R \{(\mu +_c 1)_P \downarrow (\nu -_c 1)_Q\}$ (1)

$\vdash . *122{\cdot}34 . \supset \vdash :. \text{Hp} . \supset : \mu \in \text{NC induct} - \iota`0 . \supset .$
$(\mu_P \downarrow 1_Q) R \{1_P \downarrow (\mu +_c 1)_Q\}$ (2)

$\vdash . (1) . (2) . \supset \vdash : \text{Hp} . \supset : \mu, \nu \in \text{NC induct} - \iota`0 . \supset . \mu_P \downarrow \nu_Q \in \text{D}`R :$
[*122·341] $\supset : x \in \text{D}`P . y \in \text{D}`Q . \supset . x \downarrow y \in \text{D}`R$ (3)

$\vdash . *21{\cdot}33 . \supset \vdash :. \text{Hp} . \supset : X \in \text{D}`R . \supset . (\exists \mu, \nu) . X = \mu_P \downarrow \nu_Q .$
[*122·341] $\supset . (\exists x, y) . x \in \text{D}`P . y \in \text{D}`Q . X = x \downarrow y$ (4)

$\vdash . (3) . (4) . *113{\cdot}101 . \supset \vdash . \text{Prop}$

***123·502.** $\vdash : \text{Hp} *123{\cdot}5 . \supset . \text{G}`R \subset \text{D}`R . \overleftarrow{R_*}`(1_P \downarrow 1_Q) \subset \text{D}`R$

Dem.

$\vdash . *21{\cdot}33 . \supset \vdash : \text{Hp} . Y = (\mu +_c 1)_P \downarrow (\nu -_c 1)_Q . \nu -_c 1 \neq 1 . \supset .$
$YR \{(\mu +_c 2)_P \downarrow (\nu -_c 2)_Q\}$ (1)

$\vdash . *21{\cdot}33 . \supset \vdash : \text{Hp} . Y = (\mu +_c 1)_P \downarrow 1_Q . \supset . YR \{1_P \downarrow (\mu +_c 2)_Q\}$ (2)

$\vdash . *21{\cdot}33 . \supset \vdash : \text{Hp} . Y = 1_P \downarrow (\mu +_c 1)_Q . \supset . YR\, 2_P \downarrow \mu_Q$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash : \text{Hp} . \supset \text{G}`R \subset \text{D}`R : \supset \vdash . \text{Prop}$

***123·503.** $\vdash : \text{Hp} *123{\cdot}5 . \supset . \text{D}`R \subset \overleftarrow{R_*}`(1_P \downarrow 1_Q)$

Dem.

$\vdash . *123{\cdot}501 . *122{\cdot}11 . \supset \vdash : \text{Hp} . \supset . 1_P \downarrow 1_Q \in \overleftarrow{R_*}`(1_P \downarrow 1_Q)$ (1)

$\vdash . *90{\cdot}16 . \supset \vdash : \text{Hp} . (1_P \downarrow 1_Q) R_* (\mu_P \downarrow \nu_Q) . \nu \neq 1 . \supset .$
$(1_P \downarrow 1_Q) R_* \{(\mu +_c 1)_P \downarrow (\nu -_c 1)_Q\}$ (2)

$\vdash . (2) . *120{\cdot}47 . \supset$
$\vdash : \text{Hp} . (1_P \downarrow 1_Q) R_* (\mu_P \downarrow \nu_Q) . \supset . (1_P \downarrow 1_Q) R_* \{(\mu +_c \nu -_c 1)_P \downarrow 1_Q\} .$
[*90·16] $\supset . (1_P \downarrow 1_Q) R_* \{1_P \downarrow (\mu +_c \nu)_Q\} .$
[(2).*120·47] $\supset . (1_P \downarrow 1_Q) R_* \{\mu_P \downarrow (\nu +_c 1)_Q\} .$ (3)
[*90·16] $\supset . (1_P \downarrow 1_Q) R_* \{(\mu +_c 1)_P \downarrow \nu_Q\}$ (4)

$\vdash . (1) . (3) . (4) . *120{\cdot}47 . \supset$
$\vdash :. \text{Hp} . \supset : \mu, \nu \in \text{NC induct} - \iota`0 . \supset . (1_P \downarrow 1_Q) R_* (\mu_P \downarrow \nu_Q)$ (5)

$\vdash . (5) . *122{\cdot}341 . \supset \vdash . \text{Prop}$

***123·504.** $\vdash : \text{Hp} *123{\cdot}5 . \supset . B{\text{‘}}R = 1_P \downarrow 1_Q$ [*123·34 . *120·414]

***123·51.** $\vdash : \text{Hp} *123{\cdot}5 . \supset . R \in \text{Prog} . \text{D}{\text{‘}}R = \text{D}{\text{‘}}P \times \text{D}{\text{‘}}Q$
[*123·5·501·502·503·504]

***123·52.** $\vdash . \aleph_0 = \aleph_0 \times_c \aleph_0 = \aleph_0{}^2$ [*123·51 . *116·34 . *113·25·204]

***123·53.** $\vdash : \nu \in \text{NC induct} - \iota{\text{‘}}0 . \supset . \aleph_0{}^\nu = \aleph_0$ [*123·52 . *116·52]

***123·7.** $\vdash : \text{Infin ax}\,(x) . \text{Mult ax} . \supset . \exists ! \aleph_0\,(t{\text{‘}}x)$

Dem.

$\vdash . *123{\cdot}34 . *120{\cdot}301 . \supset \vdash : \text{Hp} . \supset . \text{NC induct}\,(t{\text{‘}}x) \in \aleph_0$ (1)
$\vdash . *100{\cdot}43 . *120{\cdot}301 . \supset \vdash : \text{Hp} . \supset . \text{NC induct}\,(t{\text{‘}}x) \in \text{Cls ex}^2\,\text{excl}$ (2)
$\vdash . (1) . (2) . *88{\cdot}32 . \quad \supset \vdash : \text{Hp} . \supset . \exists ! \text{Prod}{\text{‘}}\text{NC induct}\,(t{\text{‘}}x)$ (3)
$\vdash . (1) . (2) . *115{\cdot}16 . \quad \supset \vdash : \text{Hp} . \supset . \text{Prod}{\text{‘}}\text{NC induct}\,(t{\text{‘}}x) \subset \aleph_0$ (4)
$\vdash . *115{\cdot}18 . (*65{\cdot}02) . \quad \supset \vdash : \kappa \in \text{Prod}{\text{‘}}\text{NC induct}\,(t{\text{‘}}x) . \supset . \kappa \in t{\text{‘}}t{\text{‘}}t{\text{‘}}x$ (5)
$\vdash . (3) . (4) . (5) . (*65{\cdot}02) . \supset \vdash . \text{Prop}$

$*124$. REFLEXIVE CLASSES AND CARDINALS

Summary of $*124$.

In this number, we have to take up the second definition of infinity mentioned in the introduction to this Section. A class which is infinite according to this definition we propose to call a reflexive class, because a class which is of this kind is capable of *reflexion* into a part of itself. A class is called *reflexive* when there is a one-one relation which correlates the class with a proper part of itself. (A *proper part* is a part not the whole.) A reflexive cardinal is the homogeneous cardinal of a reflexive class.

We prove easily that reflexive classes are not inductive ($*124{\cdot}271$), that reflexive cardinals are such as are greater than or equal to $\aleph_0$ ($*124{\cdot}23$), and such as are unchanged by adding 1 (excepting Λ) ($*124{\cdot}25$). To prove that classes which are not inductive must be reflexive has not hitherto been found possible without assuming the multiplicative axiom. We do not need, however, to assume the axiom generally, but only as applied to products of $\aleph_0$ factors. With this assumption, the result follows by a series of propositions explained below. Thus if a product of $\aleph_0$ factors, no one of which is zero, is never zero, then the two definitions of the finite and the infinite coincide ($*124{\cdot}56$).

We will call a cardinal ν a "multiplicative cardinal" if a product of ν factors none of which is zero is never zero. Thus all inductive cardinals are multiplicative cardinals; and the assumption needed for identifying the two definitions of finite and infinite is that $\aleph_0$ should be a multiplicative cardinal.

For a reflexive class we use the notation "Cls refl," and for a reflexive cardinal we use "NC refl." We define a reflexive cardinal as the *homogeneous* cardinal of a reflexive class, *i.e.* we put

$$\mathrm{NC\,refl} = \mathrm{N_0c}``\mathrm{Cls\,refl} \quad \mathrm{Df.}$$

The only effect of this is to exclude Λ from reflexive cardinals, which is convenient. We then need (on the analogy of $*110{\cdot}03{\cdot}04$) a definition of what is meant when an ambiguous symbol such as $\mathrm{Nc}`\alpha$ is said to be reflexive, and we therefore put

$$\mathrm{Nc}`\rho \in \mathrm{NC\,refl} \,.=.\, \mathrm{N_0c}`\rho \in \mathrm{NC\,refl} \quad \mathrm{Df.}$$

For the class of multiplicative cardinals we use the notation "NC mult." Thus we put

$$\mathrm{NC\,mult} = \mathrm{NC} \cap \hat{\alpha}\{\kappa \in \alpha \cap \mathrm{Cls\,ex^2\,excl} \,.\supset_\kappa .\, \exists ! \,\epsilon_\Delta`\kappa\} \quad \mathrm{Df,}$$

whence it follows that if $\alpha \in \mathrm{NC\,mult}$, a product of α factors, none of which is zero, will never be zero.

We begin, in this number, with the more obvious properties of Cls refl, proving that a Cls refl is one which contains sub-classes of $\aleph_0$ terms ($*124{\cdot}15$), that it is one whose number is unchanged when a single term is taken away ($*124{\cdot}17$), and that it remains reflexive if any inductive class is taken away from it ($*124{\cdot}182$).

We then give corresponding propositions concerning NC refl ($*124{\cdot}23{\cdot}25$ $\cdot 252$), proving, in addition to propositions already mentioned, that a reflexive cardinal is greater than every inductive cardinal ($*124{\cdot}26$), and that a class which is neither inductive nor reflexive (if there be such) is one which neither contains nor is contained in any progression ($*124{\cdot}34$). On such classes, see the remarks at the end of this number.

We then ($*124{\cdot}4{\cdot}41$) give a proposition merely embodying the definition of NC mult, and show that all inductive cardinals are multiplicative, which follows immediately from $*120{\cdot}62$.

The following series of propositions ($*124{\cdot}51$ ff.) are concerned with the proof that, if $\aleph_0$ is a multiplicative cardinal, then the two definitions of finite and infinite coalesce. The proof, which is somewhat complicated, proceeds as follows.

To begin with, we know that if ρ is a class which is not inductive, it contains classes having ν terms, if ν is any inductive cardinal. Thus we have

$$\exists ! 0 \cap \mathrm{Cl}`\rho, \ \exists ! 1 \cap \mathrm{Cl}`\rho, \ \ldots \ \exists ! \nu \cap \mathrm{Cl}`\rho, \ \ldots.$$

The classes of classes $0 \cap \mathrm{Cl}`\rho$, $1 \cap \mathrm{Cl}`\rho$, ... $\nu \cap \mathrm{Cl}`\rho$, ... thus form a progression, which is contained in $\mathrm{Cl}`\mathrm{Cl}`\rho$. Hence ($*124{\cdot}511$)

$$\vdash : \rho \sim \in \mathrm{Cls\,induct} \,.\supset .\, \mathrm{Cl}`\mathrm{Cl}`\rho \in \mathrm{Cls\,refl}.$$

So far, the multiplicative axiom is not required.

The above progression of classes of classes is

$$(\cap \mathrm{Cl}`\rho)``\mathrm{NC\,induct}.$$

If P is a selective relation for this class of classes, $\mathrm{D}`P$ is a progression contained in $\mathrm{Cl}`\rho$. Hence

$*124{\cdot}513$. $\vdash : \exists ! \epsilon_\Delta`(\cap \mathrm{Cl}`\rho)``\mathrm{NC\,induct} \,.\supset .\, \mathrm{Cl}`\rho \in \mathrm{Cls\,refl}$

whence

$*124{\cdot}514$. $\vdash :. \aleph_0 \in \mathrm{NC\,mult} \,.\supset :\, \rho \sim \in \mathrm{Cls\,induct} \,.\supset .\, \mathrm{Cl}`\rho \in \mathrm{Cls\,refl}$

To prove the next step, namely

$$\aleph_0 \in \mathrm{NC\,mult} \,.\, \exists ! \aleph_0 \cap \mathrm{Cl}`\mathrm{Cl}`\rho \,.\supset .\, \exists ! \aleph_0 \cap \mathrm{Cl}`\rho,$$

we make a fresh start. We have, by hypothesis, a progression R whose domain is contained in $\mathrm{Cl}`\rho$; hence $s`\mathrm{D}`R \subset \rho$. Thus it will suffice to prove

$$\aleph_0 \in \mathrm{NC\,mult} \,.\, R \in \mathrm{Prog} \,.\, \mathrm{D}`R \subset \mathrm{Cls\,induct} \,.\supset .\, \exists ! \aleph_0 \cap s`\mathrm{D}`R,$$

where the conditions of significance require that $\mathrm{D}`R$ should consist of classes.

For this purpose, we prove that no member of D‘R can be the last that has new members which have not occurred before. The proof proceeds by showing that if this were not so, s‘D‘R would be an inductive class, and therefore, by *120·75, D‘R would be an inductive class. Hence (*124·534) the members of D‘R which introduce new terms form an $\aleph_0$, by *123·19; and so therefore do the classes of new terms which they introduce (*124·535). Hence (*124·536) a selection from these classes of new terms, which is a subclass of s‘D‘R, is also an $\aleph_0$, and therefore (*124·54) there is a progression contained in s‘D‘R if the selection in question exists. This completes the proof.

In virtue of *124·511 and *120·74, we have, without the multiplicative axiom,

***124·6.** $\vdash : \rho \sim \epsilon \text{ Cls induct} . \equiv . \text{Cl‘Cl‘}\rho \,\epsilon \text{ Cls refl}$

Hence if it could be shown that Cl‘ρ cannot be reflexive unless ρ is reflexive, a double application of this would enable us, by means of *124·6, to identify the two definitions of the finite without the multiplicative axiom.

***124·01.** $\text{Cls refl} = \hat{\rho}\{(\exists R) . R \,\epsilon\, 1 \rightarrow 1 . \text{Cl‘}R \subset \text{D‘}R . \exists ! \overrightarrow{B}\text{‘}R . \rho = \text{D‘}R\}$ Df

An equivalent definition would be

$$\text{Cls refl} = \text{D“}\{(1 \rightarrow 1) \cap \text{Cl‘}B - \text{Cnv“Cl‘}B\} \quad \text{Df.}$$

***124·02.** $\text{NC refl} = \text{N}_0\text{c“Cls refl}$ Df

***124·021.** $\text{Nc‘}\rho \,\epsilon\, \text{NC refl} . = . \text{N}_0\text{c‘}\rho \,\epsilon\, \text{NC refl}$ Df

***124·03.** $\text{NC mult} = \text{NC} \cap \hat{\alpha}\{\kappa \,\epsilon\, \alpha \cap \text{Cls ex}^2 \text{ excl} . \supset_\kappa . \exists ! \epsilon_\Delta\text{‘}\kappa\}$ Df

***124·1.** $\vdash : \rho \,\epsilon\, \text{Cls refl} . \equiv . (\exists R) . R \,\epsilon\, 1 \rightarrow 1 . \text{Cl‘}R \subset \text{D‘}R . \exists ! \overrightarrow{B}\text{‘}R . \rho = \text{D‘}R$ [(*124·01)]

***124·11.** $\vdash : R \,\epsilon\, 1 \rightarrow 1 . \text{Cl‘}R \subset \text{D‘}R . \exists ! \overrightarrow{B}\text{‘}R . \supset . \text{D‘}R \,\epsilon\, \text{Cls refl}$ [*124·1]

***124·12.** $\vdash . \aleph_0 \subset \text{Cls refl}$ [*123·12 . *124·1]

***124·13.** $\vdash : \rho \,\epsilon\, \text{Cls refl} . \supset . \exists ! \aleph_0 \cap \text{Cl‘}\rho$ [*124·1 . *123·192]

***124·14.** $\vdash : \rho \,\epsilon\, \text{Cls refl} . \supset . \rho \cup \sigma \,\epsilon\, \text{Cls refl}$

Dem.

$\vdash . *71·242 . *50·5·52 . \supset$

$\vdash : R \,\epsilon\, 1 \rightarrow 1 . \text{Cl‘}R \subset \text{D‘}R . \exists ! \overrightarrow{B}\text{‘}R . \text{D‘}R = \rho . S = I \upharpoonright (\sigma - \rho) . \supset .$

$\quad R \cup S \,\epsilon\, 1 \rightarrow 1 . \text{D‘}(R \cup S) = \text{D‘}R \cup \sigma . \text{Cl‘}(R \cup S) = \text{Cl‘}R \cup (\sigma - \rho) .$

[Hp.*93·101] $\supset . R \cup S \,\epsilon\, 1 \rightarrow 1 . \text{D‘}(R \cup S) = \rho \cup \sigma . \overrightarrow{B}\text{‘}(R \cup S) = \overrightarrow{B}\text{‘}R .$

[Hp.*13·12] $\supset . R \cup S \,\epsilon\, 1 \rightarrow 1 . \text{D‘}(R \cup S) = \rho \cup \sigma . \exists ! \overrightarrow{B}\text{‘}(R \cup S) .$

[*124·11] $\supset . \rho \cup \sigma \,\epsilon\, \text{Cls refl}$ (1)

$\vdash . (1) . *124·1 . \supset \vdash . \text{Prop}$

***124·141.** $\vdash : \exists ! \mathrm{Cl}`\rho \cap \mathrm{Cls\,refl} . \supset . \rho \in \mathrm{Cls\,refl}$

Dem.

$\vdash . *124·14 . \supset \vdash : \mu \in \mathrm{Cls\,refl} . \supset . \mu \cup (\rho - \mu) \in \mathrm{Cls\,refl} .$

$[*24·411] \quad \supset \vdash : \mu \subset \rho . \mu \in \mathrm{Cls\,refl} . \supset . \rho \in \mathrm{Cls\,refl} : \supset \vdash . \mathrm{Prop}$

***124·15.** $\vdash : \rho \in \mathrm{Cls\,refl} . \equiv . \exists ! \aleph_0 \cap \mathrm{Cl}`\rho$

Dem.

$\vdash . *124·12 . \supset \vdash : \exists ! \aleph_0 \cap \mathrm{Cl}`\rho . \supset . \exists ! \mathrm{Cls\,refl} \cap \mathrm{Cl}`\rho .$

$[*124·141] \qquad \supset . \rho \in \mathrm{Cls\,refl} \qquad (1)$

$\vdash . (1) . *124·13 . \supset \vdash . \mathrm{Prop}$

***124·151.** $\vdash : \rho \in \mathrm{Cls\,refl} . \equiv . \mathrm{Nc}`\rho \geqslant \aleph_0 \quad [*124·15 . *117·22]$

***124·16.** $\vdash : \rho \in \mathrm{Cls\,refl} . \equiv . (\exists \sigma) . \sigma \subset \rho . \exists ! \rho - \sigma . \rho \,\mathrm{sm}\, \sigma .$

$\equiv . \exists ! \mathrm{Nc}`\rho \cap \mathrm{Cl}`\rho - \iota`\rho$

Dem.

$\vdash . *73·1 . \supset \vdash : (\exists \sigma) . \sigma \subset \rho . \exists ! \rho - \sigma . \rho \,\mathrm{sm}\, \sigma . \equiv .$

$(\exists R, \sigma) . \sigma \subset \rho . \exists ! \rho - \sigma . R \in 1 \to 1 . \mathrm{D}`R = \rho . \mathrm{Cl}`R = \sigma .$

$[*13·195] \qquad \equiv . (\exists R) . \mathrm{Cl}`R \subset \rho . \exists ! \rho - \mathrm{Cl}`R . R \in 1 \to 1 . \mathrm{D}`R = \rho .$

$[*13·193] \qquad \equiv . (\exists R) . \mathrm{Cl}`R \subset \mathrm{D}`R . \exists ! \mathrm{D}`R - \mathrm{Cl}`R . R \in 1 \to 1 . \mathrm{D}`R = \rho .$

$[*93·101 . *124·1] \quad \equiv . \rho \in \mathrm{Cls\,refl} : \supset \vdash . \mathrm{Prop}$

***124·17.** $\vdash : \rho \in \mathrm{Cls\,refl} . \equiv . (\exists x) . x \in \rho . \rho - \iota`x \,\mathrm{sm}\, \rho$

Dem.

$\vdash . *124·16 . \supset \vdash : (\exists x) . x \in \rho . \rho - \iota`x \,\mathrm{sm}\, \rho . \supset . \rho \in \mathrm{Cls\,refl} \qquad (1)$

$\vdash . *123·17·192·311 . \supset$

$\vdash : R \in 1 \to 1 . \mathrm{Cl}`R \subset \mathrm{D}`R . x \in \overrightarrow{B}`R . \supset . \overleftarrow{R_*}`x \,\mathrm{sm}\, \overleftarrow{R_*}`x - \iota`x .$

$[*73·7] \qquad \supset . (\mathrm{D}`R - \overleftarrow{R_*}`x) \cup \overleftarrow{R_*}`x \,\mathrm{sm}\, (\mathrm{D}`R - \overleftarrow{R_*}`x) \cup (\overleftarrow{R_*}`x - \iota`x) .$

$[*24·411·412] \qquad \supset . \mathrm{D}`R \,\mathrm{sm}\, \mathrm{D}`R - \iota`x \qquad (2)$

$\vdash . (2) . *124·1 . \supset \vdash : \rho \in \mathrm{Cls\,refl} . \supset . (\exists x) . x \in \rho . \rho \,\mathrm{sm}\, \rho - \iota`x \qquad (3)$

$\vdash . (1) . (3) . \supset \vdash . \mathrm{Prop}$

***124·18.** $\vdash : \rho \in \mathrm{Cls\,refl} . \rho \,\mathrm{sm}\, \sigma . \supset . \sigma \in \mathrm{Cls\,refl} \quad [*124·151 . *100·321]$

***124·181.** $\vdash : \rho \in \mathrm{Cls\,refl} . \supset . \rho - \iota`x \in \mathrm{Cls\,refl} . \rho - \iota`x \,\mathrm{sm}\, \rho$

Dem.

$\vdash . *124·17·18 . *73·72 . \supset$

$\vdash : \rho \in \mathrm{Cls\,refl} . x \in \rho . \supset . \rho - \iota`x \,\mathrm{sm}\, \rho . \rho - \iota`x \in \mathrm{Cls\,refl} \qquad (1)$

$\vdash . (1) . *51·222 . \supset \vdash . \mathrm{Prop}$

***124·182.** $\vdash : \rho \in \mathrm{Cls\,refl} . \sigma \in \mathrm{Cls\,induct} . \supset . \rho - \sigma \in \mathrm{Cls\,refl} . \rho - \sigma \,\mathrm{sm}\, \rho$

$[*124·181 . *120·26]$

***124·2.** $\vdash : \mu \in \mathrm{NC\,refl} . \equiv . (\exists \rho) . \rho \in \mathrm{Cls\,refl} . \mu = \mathrm{N_0c}`\rho \qquad [(*124·02)]$

***124·21.** $\vdash : \mu \in \mathrm{NC\,refl} . \equiv .$

$(\exists R) . R \in 1 \to 1 . \mathrm{Cl}`R \subset \mathrm{D}`R . \exists ! \overrightarrow{B}`R . \mu = \mathrm{N_0c}`\mathrm{D}`R \quad [*124·2·1]$

$*124\cdot23$. $\vdash : \mu \in \text{NC refl} . \equiv . \mu \geqslant \aleph_0$

Dem.

$\vdash . *117\cdot241 . \supset \vdash : \mu \geqslant \aleph_0 . \equiv . (\exists \alpha, \beta) . \mu = \text{N}_0\text{c}`\alpha . \aleph_0 = \text{N}_0\text{c}`\beta . \exists ! \text{Cl}`\alpha \cap \text{Nc}`\beta .$

$[*123\cdot36\cdot322 . *103\cdot26] \quad \equiv . (\exists \alpha, \beta) . \mu = \text{N}_0\text{c}`\alpha . \beta \in \aleph_0 . \exists ! \text{Cl}`\alpha \cap \aleph_0 .$

$[*10\cdot35] \quad \equiv . (\exists \alpha) . \mu = \text{N}_0\text{c}`\alpha . \exists ! \text{Cl}`\alpha \cap \aleph_0 .$

$[*124\cdot15] \quad \equiv . (\exists \alpha) . \mu = \text{N}_0\text{c}`\alpha . \alpha \in \text{Cls refl} .$

$[*124\cdot2] \quad \equiv . \mu \in \text{NC refl} : \supset \vdash . \text{Prop}$

$*124\cdot231$. $\vdash : \exists ! \text{NC refl} . \equiv . \exists ! \text{Cls refl} . \equiv . \exists ! \aleph_0 \quad [*124\cdot2\cdot12\cdot13]$

$*124\cdot232$. $\vdash : \exists ! \text{NC refl} . \supset . \text{Infin ax} \quad [*124\cdot231 . *123\cdot18]$

$*124\cdot24$. $\vdash :. \mu \in \text{NC refl} . \equiv : \mu \in \text{N}_0\text{C} : (\exists \nu) . \mu = \aleph_0 +_0 \nu . \nu \in \text{NC}$

Dem.

$\vdash . *124\cdot23 . *117\cdot31 . \supset$

$\vdash :. \mu \in \text{NC refl} . \equiv : \mu, \aleph_0 \in \text{N}_0\text{C} : (\exists \nu) . \nu \in \text{NC} . \mu = \aleph_0 +_0 \nu \quad (1)$

$\vdash . *110\cdot4 . \supset \vdash : \mu = \aleph_0 +_0 \nu . \mu \in \text{N}_0\text{C} . \supset . \aleph_0 \in \text{N}_0\text{C} \quad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

$*124\cdot25$. $\vdash : \mu \in \text{NC refl} . \equiv . \mu \in \text{N}_0\text{C} . \mu = \mu +_0 1 . \equiv . \exists ! \mu . \mu = \mu +_0 1$
$[*124\cdot17\cdot2]$

$*124\cdot251$. $\vdash : \mu \in \text{NC refl} . \supset . \mu = \mu +_0 1 \quad [*124\cdot25]$

$*124\cdot252$. $\vdash : \mu \in \text{NC refl} . \nu \in \text{NC induct} . \supset . \mu = \mu +_0 \nu$

Dem.

$\vdash . *124\cdot251 . \supset \vdash : \mu \in \text{NC refl} . \mu = \mu +_0 \nu . \supset . \mu = \mu +_0 \nu +_0 1 \quad (1)$

$\vdash . (1) . *120\cdot11 . \supset \vdash . \text{Prop}$

$*124\cdot253$. $\vdash : \mu \in \text{NC refl} . \supset . \mu = \mu +_0 \aleph_0$

Dem.

$\vdash . *124\cdot24 . \supset \vdash : \text{Hp} . \supset . (\exists \nu) . \mu = \aleph_0 +_0 \nu .$

$[*123\cdot421] \quad \supset . (\exists \nu) . \mu = \aleph_0 +_0 \aleph_0 +_0 \nu . \mu = \aleph_0 +_0 \nu .$

$[*13\cdot13] \quad \supset . \mu = \aleph_0 +_0 \mu : \supset \vdash . \text{Prop}$

$*124\cdot26$. $\vdash :. \mu \in \text{NC refl} . \supset : \nu \in \text{NC induct} . \supset_\nu . \mu > \nu$

Dem.

$\vdash . *124\cdot231 . \supset \vdash :. \text{Hp} . \supset : \exists ! \aleph_0 :$

$[*123\cdot43] \quad \supset : \nu \in \text{NC induct} . \supset_\nu . \aleph_0 > \nu \quad (1)$

$\vdash . (1) . *124\cdot23 . \supset \vdash . \text{Prop}$

$*124\cdot27$. $\vdash . \text{NC refl} \cap \text{NC induct} = \Lambda \quad [*124\cdot26 . *117\cdot42]$

$*124\cdot271$. $\vdash . \text{Cls refl} \cap \text{Cls induct} = \Lambda$

Dem.

$\vdash . *124\cdot2 . \supset \vdash : \rho \in \text{Cls refl} . \supset . \text{N}_0\text{c}`\rho \in \text{NC refl} .$

$[*124\cdot27] \quad \supset . \text{N}_0\text{c}`\rho \sim\in \text{NC induct} .$

$[*120\cdot21] \quad \supset . \rho \sim\in \text{Cls induct} : \supset \vdash . \text{Prop}$

$*124\cdot28$. $\vdash : \rho \in \mathrm{Cls\,refl} . \equiv . \mathrm{N_0c}`\rho \in \mathrm{NC\,refl} . \equiv . \mathrm{Nc}`\rho \in \mathrm{NC\,refl}$

Dem.

$\vdash . *4\cdot2 . (*124\cdot021) . \supset \vdash : \mathrm{Nc}`\rho \in \mathrm{NC\,refl} . \equiv . \mathrm{N_0c}`\rho \in \mathrm{NC\,refl} .$

$[*124\cdot2] \quad \equiv . (\exists\sigma) . \sigma \in \mathrm{Cls\,refl} . \mathrm{N_0c}`\rho = \mathrm{N_0c}`\sigma .$

$[*103\cdot14] \quad \equiv . (\exists\sigma) . \sigma \in \mathrm{Cls\,refl} . \rho \,\mathrm{sm}\, \sigma . \rho \in t`\sigma .$

$[*124\cdot18 . *73\cdot3 . *63\cdot103] \quad \equiv . \rho \in \mathrm{Cls\,refl} : \supset \vdash . \mathrm{Prop}$

$*124\cdot29$. $\vdash . s`\mathrm{NC\,refl} = \mathrm{Cls\,refl}$

Dem.

$\vdash . *40\cdot11 . \supset \vdash : \rho \in s`\mathrm{NC\,refl} . \equiv . (\exists\mu) . \mu \in \mathrm{NC\,refl} . \rho \in \mu .$

$[*103\cdot26] \quad \equiv . (\exists\mu) . \mu \in \mathrm{NC\,refl} . \mu = \mathrm{N_0c}`\rho .$

$[*13\cdot195] \quad \equiv . \mathrm{N_0c}`\rho \in \mathrm{NC\,refl} .$

$[*124\cdot28] \quad \equiv . \rho \in \mathrm{Cls\,refl} : \supset \vdash . \mathrm{Prop}$

$*124\cdot3$. $\vdash :: \exists ! \aleph_0 . \supset :. \mu < \aleph_0 . \vee . \mu \geqslant \aleph_0 : \equiv . \mu \in \mathrm{NC\,induct} \cup \mathrm{NC\,refl}$ $[*123\cdot45 . *124\cdot23]$

$*124\cdot31$. $\vdash : \exists ! \aleph_0 . \supset . \mathrm{spec}`\aleph_0 = \mathrm{NC\,induct} \cup \mathrm{NC\,refl}$ $[*124\cdot3 . *120\cdot431]$

In virtue of the above proposition, if there are any numbers which are neither inductive nor reflexive, they are such as are neither greater than, less than, nor equal to $\aleph_0$. (The existence of $\aleph_0$ in a suitable type can be deduced from the existence of numbers which are neither inductive nor reflexive; cf. $*124\cdot6$.) Two further propositions ($*124\cdot33\cdot34$) are given below on non-inductive non-reflexive classes and cardinals. The subject is resumed in the remarks at the end of the number.

$*124\cdot33$. $\vdash :. \exists ! \aleph_0 . \supset : \mu \in \mathrm{NC} - \mathrm{NC\,induct} - \mathrm{NC\,refl} . \equiv .$
$\mu \in \mathrm{NC} . \sim(\mu < \aleph_0) . \sim(\mu \geqslant \aleph_0)$ $[*124\cdot3 . \mathrm{Transp}]$

$*124\cdot34$. $\vdash :: \exists ! \aleph_0 . \supset :. \alpha \sim\in (\mathrm{Cls\,induct} \cup \mathrm{Cls\,refl}) . \equiv :$
$\sim(\exists\gamma) : \gamma \in \aleph_0 : \alpha \subset \gamma . \vee . \gamma \subset \alpha$

Dem.

$\vdash . *120\cdot21 . *124\cdot28 . \supset \vdash : \alpha \sim\in (\mathrm{Cls\,induct} \cup \mathrm{Cls\,refl}) . \equiv .$
$\mathrm{N_0c}`\alpha \sim\in (\mathrm{NC\,induct} \cup \mathrm{NC\,refl})$ (1)

$\vdash . *123\cdot36 . *103\cdot26 . \supset \vdash : \beta \in \aleph_0 . \supset . \aleph_0 = \mathrm{N_0c}`\beta$ (2)

$\vdash . (1) . (2) . *124\cdot31 . \supset \vdash :: \beta \in \aleph_0 . \supset :. \alpha \sim\in (\mathrm{Cls\,induct} \cup \mathrm{Cls\,refl}) . \equiv :$
$\mathrm{N_0c}`\alpha \sim\in \mathrm{spec}`\mathrm{N_0c}`\beta :$

$[*120\cdot432] \quad \equiv : \sim(\mathrm{N_0c}`\alpha \leqslant \mathrm{N_0c}`\beta) . \sim(\mathrm{N_0c}`\alpha \geqslant \mathrm{N_0c}`\beta) :$

$[*117\cdot107\cdot22] \quad \equiv : \sim(\mathrm{Nc}`\alpha \leqslant \mathrm{Nc}`\beta) : \sim(\exists\gamma) . \gamma \in \mathrm{Nc}`\beta . \gamma \subset \alpha :$

$[*123\cdot322] \quad \equiv : \sim(\mathrm{Nc}`\alpha \leqslant \aleph_0) : \sim(\exists\gamma) . \gamma \in \aleph_0 . \gamma \subset \alpha :$

$[*123\cdot47] \quad \equiv : \sim(\exists\gamma) . \gamma \in \aleph_0 . \alpha \subset \gamma : \sim(\exists\gamma) . \gamma \in \aleph_0 . \gamma \subset \alpha$ (3)

$\vdash . (3) . *10\cdot11\cdot21 . \supset \vdash . \mathrm{Prop}$

$*124\cdot4$. $\vdash :. \mu \in \mathrm{NC\,mult} . \equiv : \mu \in \mathrm{NC} : \kappa \in \mu \cap \mathrm{Cls\,ex^2\,excl} . \supset_\kappa . \exists ! \epsilon_\Delta`\kappa$
$[(*124\cdot03)]$

$*124{\cdot}41.$ $\vdash .\, \mathrm{NC\,induct} \subset \mathrm{NC\,mult}$ [$*120{\cdot}62 . *124{\cdot}4$]

The following propositions give the proof of $*124{\cdot}56$, which identifies the two definitions of the finite, on the assumption that $\aleph_0$ is a multiplicative cardinal. ($*124{\cdot}513$, however, is only used in proving $*124{\cdot}514$, and $*124{\cdot}514$ is not used in the proof. It is retained as marking a stage in the argument, although the actual propositions subsequently used are not it, but the lemmas which lead to it.)

$*124{\cdot}51.$ $\vdash : \rho \sim\epsilon \mathrm{Cls\,induct} . Q = (\cap \mathrm{Cl}`\rho) \,|\, N \,|\, \mathrm{Cnv}`(\cap \mathrm{Cl}`\rho) . \supset .$
$Q \epsilon \mathrm{Prog} . \mathrm{D}`Q \subset \mathrm{Cl}`\mathrm{Cl}`\rho . \mathrm{D}`Q = (\cap \mathrm{Cl}`\rho)``\mathrm{NC\,induct}$

N here has the meaning defined in $*123{\cdot}02$.

Dem.

$\vdash . *120{\cdot}61{\cdot}21 . *123{\cdot}25 . \supset \vdash : \mathrm{Hp} . \supset . N \epsilon \mathrm{Prog}$ (1)

$\vdash . *120{\cdot}491 . \supset \vdash :. \mathrm{Hp} . \supset : \mu, \nu \epsilon \mathrm{NC\,induct} . \supset_{\mu,\nu} . \exists ! \mu \cap \mathrm{Cl}`\rho . \exists ! \nu \cap \mathrm{Cl}`\rho :$

[$*22{\cdot}5$] $\supset : \mu, \nu \epsilon \mathrm{NC\,induct} . \mu \cap \mathrm{Cl}`\rho = \nu \cap \mathrm{Cl}`\rho . \supset_{\mu,\nu} .$
$\exists ! \mu \cap \nu \cap \mathrm{Cl}`\rho .$

[$*100{\cdot}43$] $\supset_{\mu,\nu} . \mu = \nu$

[$*71{\cdot}55$] $\supset : (\cap \mathrm{Cl}`\rho) \restriction \mathrm{NC\,induct} \epsilon 1 \rightarrow 1$ (2)

$\vdash . (1) . (2) . *123{\cdot}32 . \supset \vdash : \mathrm{Hp} . \supset . Q \epsilon \mathrm{Prog}$ (3)

$\vdash . *22{\cdot}43 . \supset \vdash : \alpha \epsilon \mathrm{D}`Q . \supset . \alpha \subset \mathrm{Cl}`\rho$ (4)

$\vdash . (3) . (4) . *37{\cdot}32{\cdot}321 . \supset \vdash . \mathrm{Prop}$

$*124{\cdot}511.$ $\vdash : \rho \sim\epsilon \mathrm{Cls\,induct} . \supset .$
$\mathrm{Cl}`\mathrm{Cl}`\rho \epsilon \mathrm{Cls\,refl} . (\cap \mathrm{Cl}`\rho)``\mathrm{NC\,induct} \epsilon \aleph_0 \cap \mathrm{Cls\,ex}^2\mathrm{excl}$
[$*124{\cdot}51{\cdot}15 . *120{\cdot}491 . *100{\cdot}43$]

$*124{\cdot}512.$ $\vdash : P \epsilon \epsilon_\Delta`(\cap \mathrm{Cl}`\rho)``\mathrm{NC\,induct} . \supset .$
$\mathrm{D}`P \epsilon \aleph_0 \cap \mathrm{Cl}`\mathrm{Cl}`\rho . \mathrm{D}`P \subset \mathrm{Cls\,induct}$

Dem.

$\vdash . *83{\cdot}11 . \mathrm{Transp} . \supset \vdash :. \mathrm{Hp} . \supset : \nu \epsilon \mathrm{NC\,induct} . \supset_\nu . \exists ! \nu \cap \mathrm{Cl}`\rho$ (1)

$\vdash . *115{\cdot}16 . (1) . *124{\cdot}511 . *120{\cdot}491 . \supset$
$\vdash : \mathrm{Hp} . \supset . \mathrm{D}`P \epsilon \mathrm{Nc}`(\cap \mathrm{Cl}`\rho)``\mathrm{NC\,induct} . \rho \sim\epsilon \mathrm{Cls\,induct} .$

[$*124{\cdot}511$] $\supset . \mathrm{D}`P \epsilon \aleph_0$ (2)

$\vdash . *83{\cdot}21 . \supset \vdash :. \mathrm{Hp} . \supset : \alpha \epsilon \mathrm{D}`P . \supset . (\exists \nu) . \nu \epsilon \mathrm{NC\,induct} . \alpha \epsilon \nu \cap \mathrm{Cl}`\rho .$

[$*10{\cdot}5 . *120{\cdot}2$] $\supset . \alpha \epsilon \mathrm{Cls\,induct} . \alpha \epsilon \mathrm{Cl}`\rho$ (3)

$\vdash . (2) . (3) . \supset \vdash . \mathrm{Prop}$

$*124{\cdot}513.$ $\vdash : \exists ! \epsilon_\Delta`(\cap \mathrm{Cl}`\rho)``\mathrm{NC\,induct} . \supset . \mathrm{Cl}`\rho \epsilon \mathrm{Cls\,refl}$ [$*124{\cdot}512{\cdot}15$]

$*124{\cdot}514.$ $\vdash :. \aleph_0 \epsilon \mathrm{NC\,mult} . \supset : \rho \sim\epsilon \mathrm{Cls\,induct} . \supset . \mathrm{Cl}`\rho \epsilon \mathrm{Cls\,refl}$
[$*124{\cdot}511{\cdot}513{\cdot}4$]

The following propositions are concerned in proving that, if $\aleph_0$ is a multiplicative cardinal, then a class such as $\mathrm{D}`P$ in $*124{\cdot}512$ must be such

that a progression is contained in s'D'P. The characteristics of D'P which are used in the proof are D'$P \epsilon \aleph_0$. D'$P \subset$ Cls induct. Since D'$P \epsilon \aleph_0$, we have $(\exists R) . R \epsilon \text{Prog} . \text{D}`P = \text{D}`R$. Hence the hypothesis with which the following series of propositions is concerned is

$$R \epsilon \text{Prog} . \text{D}`R \subset \text{Cls induct},$$

but the earlier propositions do not need the full hypothesis.

In what follows, note that if $\gamma \epsilon \text{D}`R$, $\gamma - s`\overrightarrow{R_{po}}`\gamma$ is the class of those terms which occur in γ and have never occurred before in any earlier member of D'R. We prove that, with our hypothesis, members of D'R for which this class of new terms is not null form a class which has no last member, and therefore form a progression.

***124·52.** $\vdash :. R \epsilon \text{Prog} . \sigma = \hat{\beta} \{(\exists\gamma) . \gamma \epsilon \text{D}`R . \beta = \gamma - s`\overrightarrow{R_{po}}`\gamma . \exists ! \beta\} . \supset :$
$\sigma \epsilon \text{Cls ex}^2 \text{ excl} : \gamma, \delta \epsilon \text{D}`R . \gamma \neq \delta . \supset . (\gamma - s`\overrightarrow{R_{po}}`\gamma) \cap (\delta - s`\overrightarrow{R_{po}}`\delta) = \Lambda$

Dem.

$\vdash . *20·33 . \quad \supset \vdash :. \text{Hp} . \supset : \beta \epsilon \sigma . \supset_\beta . \exists ! \beta \quad (1)$

$\vdash . *122·21 . \quad \supset \vdash :. \text{Hp} . \gamma, \delta \epsilon \text{D}`R . \gamma \neq \delta . \supset : \gamma R_{po} \delta . \vee . \delta R_{po} \gamma \quad (2)$

$\vdash . *40·13 . \quad \supset \vdash :. \text{Hp} . \gamma R_{po} \delta . \supset : \gamma \subset s`\overrightarrow{R_{po}}`\delta :$

[*24·3] $\quad \supset : (\gamma - s`\overrightarrow{R_{po}}`\gamma) \cap (\delta - s`\overrightarrow{R_{po}}`\delta) = \Lambda \quad (3)$

Similarly $\quad \vdash :. \text{Hp} . \delta R_{po} \gamma . \supset : (\gamma - s`\overrightarrow{R_{po}}`\gamma) \cap (\delta - s`\overrightarrow{R_{po}}`\delta) = \Lambda \quad (4)$

$\vdash . (2) . (3) . (4) . \supset \vdash : \text{Hp} . \gamma, \delta \epsilon \text{D}`R . \gamma \neq \delta . \supset .$
$(\gamma - s`\overrightarrow{R_{po}}`\gamma) \cap (\delta - s`\overrightarrow{R_{po}}`\delta) = \Lambda \quad (5)$

$\vdash . (5) . *20·33 . \supset \vdash : \text{Hp} . \beta, \beta' \epsilon \sigma . \beta \neq \beta' . \supset . \beta \cap \beta' = \Lambda \quad (6)$

$\vdash . (1) . (6) . (5) . \supset \vdash . \text{Prop}$

***124·521.** $\vdash : \text{Hp} *124·52 . \pi = \hat{\gamma} \{\gamma \epsilon \text{D}`R . \exists ! \gamma - s`\overrightarrow{R_{po}}`\gamma\} . \supset . \sigma \text{ sm } \pi$

Dem.

$\vdash . *124·52 . *24·57 . \supset$

$\vdash : \text{Hp} . \gamma, \delta \epsilon \pi . \gamma \neq \delta . \supset . \gamma - s`\overrightarrow{R_{po}}`\gamma \neq \delta - s`\overrightarrow{R_{po}}`\delta \quad (1)$

$\vdash . (1) . \supset \vdash : \text{Hp} . S = \hat{\beta}\hat{\gamma} \{\gamma \epsilon \text{D}`R . \beta = \gamma - s`\overrightarrow{R_{po}}`\gamma . \exists ! \beta\} . \supset .$
$S \epsilon 1 \rightarrow 1 . \text{D}`S = \sigma . \text{Q}`S = \pi :. \supset \vdash . \text{Prop}$

***124·53.** $\vdash : R \epsilon \text{Prog} . \supset . s`\text{D}`R \sim\epsilon \text{Cls induct}$ [*120·75 . *122·37]

***124·531.** $\vdash : R \epsilon \text{Prog} . \text{D}`R \subset \text{Cls induct} . \supset . s`\overrightarrow{R_*}`\gamma \epsilon \text{Cls induct}$

Dem.

$\vdash . *122·38 . \supset \vdash : \text{Hp} . \supset . \overrightarrow{R_*}`\gamma \epsilon \text{Cls induct} \quad (1)$

$\vdash . (1) . *120·75 . \supset \vdash . \text{Prop}$

***124·532.** $\vdash : R \epsilon \text{Prog} . \text{D}`R \subset \text{Cls induct} . \supset . \exists ! s`\text{D}`R - s`\overrightarrow{R_*}`\gamma$
[*124·53·531 . *120·481 . Transp]

$*124{\cdot}533.\ \vdash : R \in \text{Prog} . \text{D}'R \subset \text{Cls induct} . \gamma \in \text{D}'R . \supset .$

$$(\exists \beta) . \gamma R_{\text{po}} \beta . \exists ! \beta - s'\overrightarrow{R_{\text{po}}}'\beta$$

Dem.

$\vdash . *124{\cdot}532 . \supset \vdash : \text{Hp} . \supset . (\exists \beta) . \beta \in \text{D}'R . \exists ! \beta - s'\overrightarrow{R_*}'\gamma \qquad (1)$

$\vdash . *40{\cdot}13 . \quad \supset \vdash : \beta R_* \gamma . \supset . \beta \subset s'\overrightarrow{R_*}'\gamma :$

$[\text{Transp}] \quad \supset \vdash : \exists ! \beta - s'\overrightarrow{R_*}'\gamma . \supset . \sim(\beta R_* \gamma) :$

$[*122{\cdot}21] \quad \supset \vdash : \text{Hp} . \beta \in \text{D}'R . \exists ! \beta - s'\overrightarrow{R_*}'\gamma . \supset . \gamma R_{\text{po}} \beta \qquad (2)$

$\vdash . (1) . (2) . \supset$

$\vdash :. \text{Hp} . \supset : (\exists \beta) . \gamma R_{\text{po}} \beta . \exists ! \beta - s'\overrightarrow{R_*}'\gamma :$

$[*122{\cdot}23] \supset : \text{E} ! \min (R_{\text{po}})'\hat{\beta} \{\gamma R_{\text{po}} \beta . \exists ! \beta - s'\overrightarrow{R_{\text{po}}}'\beta\} :$

$[*93{\cdot}111] \supset : (\exists \beta) : \gamma R_{\text{po}} \beta . \exists ! \beta - s'\overrightarrow{R_*}'\gamma : \delta R_{\text{po}} \beta . \supset_\delta . \delta \subset s'\overrightarrow{R_*}'\gamma :$

$[*40{\cdot}151] \supset : (\exists \beta) . \gamma R_{\text{po}} \beta . \exists ! \beta - s'\overrightarrow{R_*}'\gamma . s'\overrightarrow{R_{\text{po}}}'\beta \subset s'\overrightarrow{R_*}'\gamma :$

$[*22{\cdot}81] \quad \supset : (\exists \beta) . \gamma R_{\text{po}} \beta . \exists ! \beta - s'\overrightarrow{R_{\text{po}}}'\beta :. \supset \vdash . \text{Prop}$

$*124{\cdot}534.\ \vdash : R \in \text{Prog} . \text{D}'R \subset \text{Cls induct} .$

$$\pi = \hat{\gamma} \{\gamma \in \text{D}'R . \exists ! \gamma - s'\overrightarrow{R_{\text{po}}}'\gamma\} . \supset . \pi \in \aleph_0$$

Dem.

$\vdash . *124{\cdot}533 . \supset \vdash : \text{Hp} . \supset . \exists ! \pi . \pi \subset R_{\text{po}}``\pi \qquad (1)$

$\vdash . (1) . *123{\cdot}19 . \supset \vdash . \text{Prop}$

$*124{\cdot}535.\ \vdash : R \in \text{Prog} . \text{D}'R \subset \text{Cls induct} .$

$$\sigma = \hat{\beta} \{(\exists \gamma) . \gamma \in \text{D}'R . \beta = \gamma - s'\overrightarrow{R_{\text{po}}}'\gamma . \exists ! \beta\} . \supset . \sigma \in \aleph_0$$

$[*124{\cdot}534{\cdot}521 . *123{\cdot}321]$

$*124{\cdot}536.\ \vdash : R \in \text{Prog} . \text{D}'R \subset \text{Cls induct} .$

$$\sigma = \hat{\beta} \{(\exists \gamma) . \gamma \in \text{D}'R . \beta = \gamma - s'\overrightarrow{R_{\text{po}}}'\gamma . \exists ! \beta\} .$$
$$S \in \epsilon_\Delta'\sigma . \supset . \text{D}'S \in \aleph_0 . \text{D}'S \subset s'\text{D}'R$$

Dem.

$\vdash . *115{\cdot}16 . *124{\cdot}52{\cdot}535 . \supset \vdash : \text{Hp} . \supset . \text{D}'S \in \aleph_0 \qquad (1)$

$\vdash . *83{\cdot}21 . \supset \vdash :. \text{Hp} . \supset : \text{D}'S \subset s'\sigma :$

$[*40{\cdot}11] \supset : x \in \text{D}'S . \supset . (\exists \beta, \gamma) . \gamma \in \text{D}'R . \beta = \gamma - s'\overrightarrow{R_{\text{po}}}'\gamma . \exists ! \beta . x \in \beta .$

$[*13{\cdot}195] \quad \supset . (\exists \gamma) . \gamma \in \text{D}'R . x \in \gamma - s'\overrightarrow{R_{\text{po}}}'\gamma .$

$[*22{\cdot}43] \quad \supset . (\exists \gamma) . \gamma \in \text{D}'R . x \in \gamma .$

$[*40{\cdot}11] \quad \supset . x \in s'\text{D}'R \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

$*124{\cdot}54.\ \vdash : \aleph_0 \in \text{NC mult} . R \in \text{Prog} . \text{D}'R \subset \text{Cls induct} . \supset . \exists ! \aleph_0 \cap \text{Cl}'s'\text{D}'R$

Dem.

$\vdash . *124{\cdot}52{\cdot}535{\cdot}4 . \supset$

$\vdash :. \text{Hp} . \supset : \sigma = \hat{\beta} \{(\exists \gamma) . \gamma \in \text{D}'R . \beta = \gamma - s'\overrightarrow{R_{\text{po}}}'\gamma . \exists ! \beta\} . \supset . \exists ! \epsilon_\Delta'\sigma .$

$[*124{\cdot}536] \quad \supset . \exists ! \aleph_0 \cap \text{Cl}'s'\text{D}'R :. \supset \vdash . \text{Prop}$

***124·541.** $\vdash : \aleph_0 \in \mathrm{NC\ mult} \,.\, P \in \epsilon_\Delta`(\cap \mathrm{Cl}`\rho)``\mathrm{NC\ induct} \,.\, \supset .$

$\exists ! \aleph_0 \cap \mathrm{Cl}`s`\mathrm{D}`P \,.\, s`\mathrm{D}`P \subset \rho$

Dem.

$\vdash . *124·512 . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{D}`P \in \aleph_0 . \mathrm{D}`P \subset \mathrm{Cls\ induct} .$

[*123·1] $\supset . (\exists R) . \mathrm{D}`P = \mathrm{D}`R . R \in \mathrm{Prog} . \mathrm{D}`R \subset \mathrm{Cls\ induct} .$

[*124·54] $\supset . (\exists R) . \mathrm{D}`P = \mathrm{D}`R . \exists ! \aleph_0 \cap \mathrm{Cl}`s`\mathrm{D}`R .$

[*13·193.*10·35] $\supset . \exists ! \aleph_0 \cap \mathrm{Cl}`s`\mathrm{D}`P$ (1)

$\vdash . *124·512 . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{D}`P \in \mathrm{Cl}`\mathrm{Cl}`\rho .$

[*60·2] $\supset . \mathrm{D}`P \subset \mathrm{Cl}`\rho .$

[*60·52] $\supset . s`\mathrm{D}`P \subset \rho$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***124·55.** $\vdash : \aleph_0 \in \mathrm{NC\ mult} . \rho \sim\in \mathrm{Cls\ induct} . \supset . \exists ! \aleph_0 \cap \mathrm{Cl}`\rho$

Dem.

$\vdash . *124·511·4 . \supset \vdash : \mathrm{Hp} . \supset . \exists ! \epsilon_\Delta`(\cap \mathrm{Cl}`\rho)``\mathrm{NC\ induct} .$

[*124·541.*60·4] $\supset . \exists ! \aleph_0 \cap \mathrm{Cl}`\rho : \supset \vdash . \mathrm{Prop}$

***124·56.** $\vdash : \aleph_0 \in \mathrm{NC\ mult} . \supset . -\mathrm{Cls\ induct} = \mathrm{Cls\ refl} . \mathrm{N_0C} - \mathrm{NC\ induct} = \mathrm{NC\ refl}$

Dem.

$\vdash . *124·55·15 . \supset \vdash : \mathrm{Hp} . \supset . -\mathrm{Cls\ induct} \subset \mathrm{Cls\ refl}$ (1)

$\vdash . *124·271 . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{Cls\ refl} \subset -\mathrm{Cls\ induct}$ (2)

$\vdash . (1) . (2) . \supset \vdash :. \mathrm{Hp} . \supset : -\mathrm{Cls\ induct} = \mathrm{Cls\ refl} :$ (3)

[*120·21.*124·28] $\supset : \mathrm{N_0c}`\rho \sim\in \mathrm{NC\ induct} . \equiv . \mathrm{N_0c}`\rho \in \mathrm{NC\ refl} :$

[*103·2.*124·2] $\supset : \alpha \in \mathrm{N_0C} - \mathrm{NC\ induct} . \equiv . \alpha \in \mathrm{NC\ refl}$ (4)

$\vdash . (3) . (4) . \supset \vdash . \mathrm{Prop}$

The above proposition identifies the two definitions of the finite, on the hypothesis $\aleph_0 \in \mathrm{NC\ mult}$.

***124·57.** $\vdash : \mu \in \mathrm{N_0C} - \mathrm{NC\ induct} . \supset . 2^{2^\mu} \in \mathrm{NC\ refl}$ [*124·511 . *116·72]

***124·58.** $\vdash :. 2^\mu \in \mathrm{NC\ refl} . \supset_\mu . \mu \in \mathrm{NC\ refl} : \supset . \mathrm{N_0C} - \mathrm{NC\ induct} = \mathrm{NC\ refl}$

Dem.

$\vdash . *124·57 . \supset \vdash :. \mathrm{Hp} . \supset : \mu \in \mathrm{N_0C} - \mathrm{NC\ induct} . \supset . 2^\mu \in \mathrm{NC\ refl} .$

[Hp] $\supset . \mu \in \mathrm{NC\ refl}$ (1)

$\vdash . (1) . *124·2·27 . \supset \vdash . \mathrm{Prop}$

The above proposition gives another hypothesis which would enable us to identify the two definitions of the finite if it could be proved, namely

$$2^\mu \in \mathrm{NC\ refl} . \supset_\mu . \mu \in \mathrm{NC\ refl},$$

or, what comes to the same thing,

$$\mathrm{Cl}`\rho \in \mathrm{Cls\ refl} . \supset . \rho \in \mathrm{Cls\ refl}.$$

$*124{\cdot}6$. $\vdash : \rho \sim \epsilon \text{ Cls induct} . \equiv . \text{Cl}`\text{Cl}`\rho \epsilon \text{ Cls refl}$

Dem.

$\vdash . *124{\cdot}511 . \supset \vdash : \rho \sim \epsilon \text{ Cls induct} . \supset . \text{Cl}`\text{Cl}`\rho \epsilon \text{ Cls refl}$ (1)

$\vdash . *120{\cdot}74 . \supset \vdash : \rho \epsilon \text{ Cls induct} . \supset . \text{Cl}`\text{Cl}`\rho \epsilon \text{ Cls induct} .$

$[*124{\cdot}271]$ $\supset . \text{Cl}`\text{Cl}`\rho \sim \epsilon \text{ Cls refl}$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

$*124{\cdot}61$. $\vdash :. \aleph_0 \epsilon \text{ NC mult} . \supset : \rho \epsilon \text{ Cls refl} . \equiv . \text{Cl}`\rho \epsilon \text{Cls refl} . \equiv . \text{Cl}`\text{Cl}`\rho \epsilon \text{Cls refl}$

Dem.

$\vdash . *124{\cdot}6{\cdot}271 . \supset \vdash : \rho \epsilon \text{ Cls refl} . \supset . \text{Cl}`\rho \epsilon \text{ Cls refl} . \supset . \text{Cl}`\text{Gl}`\rho \epsilon \text{ Cls refl}$ (1)

$\vdash . *124{\cdot}6{\cdot}56 . \supset \vdash :. \aleph_0 \epsilon \text{ NC mult} . \supset : \text{Cl}`\text{Cl}`\rho \epsilon \text{ Cls refl} . \supset . \rho \epsilon \text{ Cls refl} .$ (2)

$[(1)]$ $\supset . \text{Cl}`\rho \epsilon \text{ Cls refl}$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash . \text{Prop}$

The following properties of cardinals which are neither inductive nor reflexive (supposing there are such) are easily proved. Let us put

$$\text{NC med} = \text{N}_0\text{C} - \text{NC induct} - \text{NC refl} \quad \text{Df},$$

$$\text{Cls med} = - \text{Cls induct} - \text{Cls refl} \quad \text{Df},$$

where "med" stands for "mediate." Then

$$\mu \epsilon \text{ NC med} . \supset . \mu +_c 1 \epsilon \text{ NC med} . \mu -_c 1 \epsilon \text{ NC med} . \mu \neq \mu +_c 1 . \mu \neq \mu -_c 1.$$

Hence mediate cardinals have no maximum or minimum.

$$\mu, \nu \epsilon \text{ NC med} . \supset . \mu +_c \nu \epsilon \text{ NC med},$$

$$\mu \epsilon \text{ NC med} . \nu \epsilon \text{ NC med} \cup \text{NC induct} - \iota`0 . \supset . \mu \times_c \nu \epsilon \text{ NC med},$$

whence $\mu \epsilon \text{ NC med} . \supset . \mu^2, \mu^3, \ldots \epsilon \text{ NC med},$

$$\mu^\nu \epsilon \text{ NC med} . \supset : \mu \epsilon \text{ NC med} . \vee . \nu \epsilon \text{ NC med},$$

$$\mu \epsilon \text{ NC med} . \supset . 2^{2^\mu} \epsilon \text{ NC refl},$$

whence $\exists ! \text{ NC med} . \supset . (\exists \nu) . \nu \epsilon \text{ NC med} . 2^\nu \epsilon \text{ NC refl},$

since we have either $\mu \epsilon \text{ NC med} . 2^\mu \epsilon \text{ NC refl}$ or $2^\mu \epsilon \text{ NC med} . 2^{2^\mu} \epsilon \text{ NC refl}$.

$*125$. THE AXIOM OF INFINITY

Summary of $*125$.

The present number is merely concerned to give a few equivalent forms of the axiom of infinity, and of the kindred assumption of the existence of $\aleph_0$.

In virtue of $*125\cdot24\cdot25$ below, if the axiom of infinity holds in any one type, then it holds in any other type which can be derived from this one, or from any type from which this one can be derived. Hence if we assume, as it seems natural to do, that all extensional types are derived from a first type, namely that of individuals, then the axiom of infinity in any such type is equivalent to the assumption that the number of individuals is not inductive.

We deal, in this number, first with equivalent forms of Infin ax, then with equivalent forms of Infin ax (x), then with equivalent forms of $\exists ! \aleph_0$ or $\exists ! \aleph_0(x)$. When "Infin ax" or "$\exists ! \aleph_0$" occurs in this number without typical definition, it and all other typically ambiguous symbols are to be taken in the lowest logically possible types, or with the same *relative* types as if this had been done. The propositions of this number are often not referred to in the sequel, but are here collected together on account of their intrinsic interest.

$*125\cdot1$. $\vdash :. \text{Infin ax} . \equiv : \alpha \in \text{NC induct} . \supset_\alpha . \exists ! \alpha$ $\quad$ [$*120\cdot3$]

$*125\cdot11$. $\vdash :. \text{Infin ax} . \equiv : \alpha \in \text{NC induct} . \supset_\alpha . \alpha \neq \alpha +_c 1$ $\quad$ [$*120\cdot33$]

$*125\cdot12$. $\vdash :. \text{Infin ax} . \equiv : \alpha \in \text{NC induct} . \supset_\alpha . \exists ! \alpha +_c 1$

Dem.

$\vdash . *101\cdot12 . *125\cdot1 . \supset$

$\vdash :. \text{Infin ax} . \equiv : \alpha \in \text{NC induct} - \iota`0 . \supset_\alpha . \exists ! \alpha :$

[$*120\cdot423$] $\quad \equiv : \alpha \in \text{NC induct} . \supset_\alpha . \exists ! \alpha +_c 1 :. \supset \vdash . \text{Prop}$

$*125\cdot13$. $\vdash : \text{Infin ax} . \equiv . \Lambda \sim\in \text{NC induct}$ $\quad$ [$*125\cdot1 . *24\cdot63$]

$*125\cdot14$. $\vdash : \text{Infin ax} . \equiv . (+_c 1) \upharpoonright \text{NC induct} \in 1 \to 1$

Dem.

$\vdash . *123\cdot22\cdot24 . \supset \vdash : \text{Infin ax} . \supset . (+_c 1) \upharpoonright \text{NC induct} \in 1 \to 1$ $\quad$ (1)

$\vdash . *71\cdot55 . \quad \supset \vdash :. (+_c 1) \upharpoonright \text{NC induct} \in 1 \to 1 . \supset :$

$\alpha, \beta \in \text{NC induct} . \alpha +_c 1 = \beta +_c 1 . \supset_{\alpha,\beta} . \alpha = \beta :$

[Transp] $\quad \supset : \alpha, \beta \in \text{NC induct} . \alpha \neq \beta . \supset_{\alpha,\beta} . \alpha +_c 1 \neq \beta +_c 1 :$

[$*10\cdot1$] $\quad \supset : \Lambda, \beta \in \text{NC induct} . \Lambda \neq \beta . \supset_\beta . \Lambda +_c 1 \neq \beta +_c 1 :$

[$*110\cdot4$.Transp] $\supset : \Lambda \in \text{NC induct} . \beta \in \text{NC induct} . \exists ! \beta . \supset_\beta . \exists ! (\beta +_c 1)$ $\quad$ (2)

$\vdash . (2) . *101\cdot12 . *120\cdot13 . \supset$

$\vdash :. (+_c 1) \upharpoonright \text{NC induct} \epsilon 1 \to 1 . \Lambda \epsilon \text{NC induct} . \supset : \beta \epsilon \text{NC induct} . \supset_\beta . \exists ! \beta :$

$[*24\cdot63] \qquad \supset : \Lambda \sim\epsilon \text{NC induct} \qquad (3)$

$\vdash . (3) . *2\cdot01 . *125\cdot13 . \supset \vdash : (+_c 1) \upharpoonright \text{NC induct} \epsilon 1 \to 1 . \supset . \text{Infin ax} \qquad (4)$

$\vdash . (1) . (4) . \supset \vdash . \text{Prop}$

$*125\cdot15$. $\vdash :. \text{Infin ax} . \equiv : \rho \epsilon \text{Cls induct} . \supset_\rho . \exists ! - \rho$

Dem.

$\vdash . *110\cdot63 . \supset \vdash : x \sim\epsilon \rho . \supset_x . \rho \cup \iota`x \epsilon \text{Nc}`\rho +_c 1 :$

$[*10\cdot28] \qquad \supset \vdash : \exists ! - \rho . \supset . \exists ! \text{Nc}`\rho +_c 1 :$

$[\text{Syll}] \qquad \supset \vdash :. \rho \epsilon \text{Cls induct} . \supset_\rho . \exists ! - \rho : \supset :$

$\qquad \rho \epsilon \text{Cls induct} . \supset_\rho . \exists ! \text{Nc}`\rho +_c 1 :$

$[*120\cdot2] \qquad \supset : \alpha \epsilon \text{NC induct} . \rho \epsilon \alpha . \supset_{\alpha, \rho} . \exists ! \text{Nc}`\rho +_c 1 :$

$[*100\cdot45] \qquad \supset : \alpha \epsilon \text{NC induct} . \exists ! \alpha . \supset_\alpha . \exists ! \alpha +_c 1 :$

$[*120\cdot13 . *101\cdot12] \qquad \supset : \alpha \epsilon \text{NC induct} . \supset_\alpha . \exists ! \alpha \qquad (1)$

$\vdash . *13\cdot12 . \supset \vdash :. \alpha \epsilon \text{NC induct} . \supset_\alpha . \exists ! (\alpha +_c 1) : \supset :$

$\qquad \alpha \epsilon \text{NC induct} . \text{N}_0\text{c}`\rho = \alpha . \supset_{\alpha, \rho} . \exists ! (\text{N}_0\text{c}`\rho +_c 1) :$

$[*120\cdot21] \supset : \rho \epsilon \text{Cls induct} . \supset_\rho . \exists ! (\text{N}_0\text{c}`\rho +_c 1) .$

$[*103\cdot11 . *63\cdot101 . *110\cdot63] \quad \supset_\rho . (\exists \gamma, z) . \gamma \,\text{sm}\, \rho . z \sim\epsilon \gamma . \gamma \epsilon t`\rho \cup - \iota`\rho \qquad (2)$

$\vdash . *13\cdot12 . *10\cdot24 . \quad \supset \vdash : \gamma = \rho . z \sim\epsilon \gamma . \supset . \exists ! - \rho \qquad (3)$

$\vdash . *120\cdot426 . *24\cdot6 . \supset \vdash : \rho \epsilon \text{Cls induct} . \gamma \neq \rho . \gamma \subset \rho . \supset . \sim(\gamma \,\text{sm}\, \rho) :$

$[\text{Transp}] \qquad \supset \vdash : \rho \epsilon \text{Cls induct} . \gamma \,\text{sm}\, \rho . \gamma \neq \rho . \supset . \exists ! \gamma - \rho .$

$[*24\cdot561] \qquad \supset . \exists ! - \rho \qquad (4)$

$\vdash . (3) . (4) . \supset \vdash :. \rho \epsilon \text{Cls induct} : (\exists \gamma, z) . \gamma \,\text{sm}\, \rho . z \sim\epsilon \gamma . \gamma \epsilon t`\rho \cup - \iota`\rho : \supset .$

$\qquad \exists ! - \rho \qquad (5)$

$\vdash . (2) . (5) . \supset \vdash :. \alpha \epsilon \text{NC induct} . \supset_\alpha . \exists ! (\alpha +_c 1) : \supset :$

$\qquad \rho \epsilon \text{Cls induct} . \supset_\rho . \exists ! - \rho \qquad (6)$

$\vdash . (1) . (6) . *125\cdot12\cdot1 . \supset \vdash . \text{Prop}$

$*125\cdot16$. $\vdash : \text{Infin ax} . \equiv . \exists ! \text{Cls} - \text{Cls induct} . \equiv . \exists ! \text{N}_0\text{C} - \text{NC induct} . \equiv .$

$\qquad \text{V} \sim\epsilon \text{Cls induct}$

Dem.

$\vdash . *125\cdot15 . \qquad \supset \vdash :. \text{Infin ax} . \equiv : \rho \epsilon \text{Cls induct} . \supset_\rho . \rho \neq \text{V} :$

$[*13\cdot196] \qquad \equiv : \text{V} \sim\epsilon \text{Cls induct} \qquad (1)$

$\vdash . *120\cdot481 . \text{Transp} . \supset \vdash : \exists ! \text{Cls} - \text{Cls induct} . \supset . \text{V} \sim\epsilon \text{Cls induct} \qquad (2)$

$\vdash . (1) . (2) . *120\cdot21 . \supset \vdash . \text{Prop}$

$*125\cdot2$. $\vdash :. \text{Infin ax} (x) . \equiv : \alpha \epsilon \text{NC induct} . \supset_\alpha . \exists ! \alpha (x) \quad [*120\cdot301]$

$*125\cdot21$. $\vdash : \text{Infin ax} (x) . \equiv . t`x \sim\epsilon \text{Cls induct}$

Dem.

$\vdash . *125\cdot15 . \supset \vdash :. \text{Infin ax} (x) . \equiv : \rho \epsilon \text{Cls induct} \cap \text{Cl}`t`x . \supset_\rho . \exists ! - \rho :$

$[*63\cdot102] \qquad \equiv : \rho \epsilon \text{Cls induct} \cap \text{Cl}`t`x . \supset_\rho . \rho \neq t`x :$

$[*13\cdot196] \qquad \equiv : t`x \sim\epsilon \text{Cls induct} :. \supset \vdash . \text{Prop}$

∗125·22. $\vdash : \text{Infin ax}\,(x) . \equiv . t^{3}{}^{\backprime}x \in \text{Cls refl}$ [∗125·21 . ∗63·66 . ∗124·6]

∗125·23. $\vdash : \text{Infin ax}\,(x) . \equiv . \exists ! \aleph_0 (t^{2}{}^{\backprime}x)$ [∗125·22 . ∗124·15]

∗125·24. $\vdash : \text{Infin ax}\,(x) . \equiv . \text{Infin ax}\,(t^{\backprime}x) . \equiv . \text{Infin ax}\,(t^{2}{}^{\backprime}x) . \equiv . \text{etc.}$

Dem.

$\vdash . ∗125·21 . \supset \vdash : \text{Infin ax}\,(x) . \equiv . t^{\backprime}x \sim\in \text{Cls induct} .$
[∗120·74] $\equiv . \text{Cl}^{\backprime}t^{\backprime}x \sim\in \text{Cls induct} .$
[∗63·66] $\equiv . t^{2}{}^{\backprime}x \sim\in \text{Cls induct} .$
[∗125·21] $\equiv . \text{Infin ax}\,(t^{\backprime}x) : \supset \vdash . \text{Prop}$

∗125·25. $\vdash : \text{Infin ax}\,(\alpha) . \equiv . \text{Infin ax}\,(t_{00}{}^{\backprime}\alpha) . \equiv . \text{Infin ax}\,(t_0{}^{1}{}^{\backprime}\alpha) . \equiv .$
$\text{Infin ax}\,(t^{11}{}^{\backprime}\alpha) . \equiv . \text{etc.}$
[∗116·91·92 . ∗120·56·52 . ∗125·21]

∗125·3. $\vdash : \exists ! \aleph_0 . \equiv . \exists ! (1 \rightarrow 1) \cap \hat{R} (\exists ! \overrightarrow{B}{}^{\backprime}R . \sim \exists ! \overrightarrow{B}{}^{\backprime}\breve{R})$

Dem.

$\vdash . ∗123·1 . \supset \vdash : \exists ! \aleph_0 . \equiv . \exists ! \text{Prog} .$
[∗122·11·141] $\supset . \exists ! (1 \rightarrow 1) \cap \hat{R} (\exists ! \overrightarrow{B}{}^{\backprime}R . \sim \exists ! \overrightarrow{B}{}^{\backprime}\breve{R})$ (1)
$\vdash . ∗123·192 . \supset \vdash : \exists ! (1 \rightarrow 1) \cap \hat{R} (\exists ! \overrightarrow{B}{}^{\backprime}R . \sim \exists ! \overrightarrow{B}{}^{\backprime}\breve{R}) . \supset . \exists ! \aleph_0$ (2)
$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

∗125·31. $\vdash : \exists ! \aleph_0 (x) . \equiv . t^{\backprime}x \in \text{Cls refl}$ [∗124·15]

∗125·32. $\vdash : \exists ! \aleph_0 (x) . \equiv . \exists ! (1 \rightarrow 1) \cap \overleftarrow{\mathrm{D}}{}^{\backprime}t^{\backprime}x - \overleftarrow{\text{Ɑ}}{}^{\backprime}t^{\backprime}x$

Dem.

$\vdash . ∗63·102 . \supset \vdash : \exists ! (1 \rightarrow 1) \cap \overleftarrow{\mathrm{D}}{}^{\backprime}t^{\backprime}x - \overleftarrow{\text{Ɑ}}{}^{\backprime}t^{\backprime}x . \equiv .$
$(\exists R) . R \in 1 \rightarrow 1 . \mathrm{D}^{\backprime}R = t^{\backprime}x . \text{Ɑ}^{\backprime}R \subset t^{\backprime}x . \exists ! t^{\backprime}x - \text{Ɑ}^{\backprime}R .$
[∗124·1] $\equiv . t^{\backprime}x \in \text{Cls refl}$ (1)
$\vdash . (1) . ∗125·31 . \supset \vdash . \text{Prop}$

∗125·33. $\vdash :. \exists ! \aleph_0 (x) . \equiv : \alpha \subset t^{\backprime}x . \exists ! \alpha . \supset_\alpha . \exists ! (1 \rightarrow 1) \cap \overleftarrow{\mathrm{D}}{}^{\backprime}\alpha - \overleftarrow{\text{Ɑ}}{}^{\backprime}\alpha$

Dem.

$\vdash . ∗73·7 . ∗51·222 . \supset \vdash : \alpha \subset t^{\backprime}x . y \in \alpha . z \in t^{\backprime}x - \alpha . \supset . \alpha \,\text{sm}\, (\alpha - \iota^{\backprime}y) \cup \iota^{\backprime}z :$
[∗73·1] $\supset . (\exists R) . R \in 1 \rightarrow 1 . \mathrm{D}^{\backprime}R = \alpha . \text{Ɑ}^{\backprime}R = (\alpha - \iota^{\backprime}y) \cup \iota^{\backprime}z .$
[∗33·6·61] $\supset . \exists ! (1 \rightarrow 1) \cap \overleftarrow{\mathrm{D}}{}^{\backprime}\alpha - \overleftarrow{\text{Ɑ}}{}^{\backprime}\alpha$ (1)
$\vdash . (1) . \supset \vdash : \exists ! \alpha . \exists ! t^{\backprime}x - \alpha . \supset . \exists ! (1 \rightarrow 1) \cap \overleftarrow{\mathrm{D}}{}^{\backprime}\alpha - \overleftarrow{\text{Ɑ}}{}^{\backprime}\alpha :$
[∗63·102] $\supset \vdash : \exists ! \alpha . \alpha \subset t^{\backprime}x . \alpha \neq t^{\backprime}x . \supset . \exists ! (1 \rightarrow 1) \cap \overleftarrow{\mathrm{D}}{}^{\backprime}\alpha - \overleftarrow{\text{Ɑ}}{}^{\backprime}\alpha$ (2)
$\vdash . (2) . ∗125·32 . \supset \vdash . \text{Prop}$

∗125·34. $\vdash : \exists ! \aleph_0 (x) . \equiv . \iota^{\backprime}t^{\backprime}x \sim\in \text{NC}$

Dem.

$\vdash . ∗125·32 . \supset \vdash :. \sim \exists ! \aleph_0 (x) . \equiv : R \in 1 \rightarrow 1 . \mathrm{D}^{\backprime}R = t^{\backprime}x . \supset_R . \text{Ɑ}^{\backprime}R = t^{\backprime}x :$
[∗100·13] $\equiv : \text{Nc}^{\backprime}t^{\backprime}x = \iota^{\backprime}t^{\backprime}x .$
[∗100·41·45] $\equiv : \iota^{\backprime}t^{\backprime}x \in \text{NC}$ (1)
$\vdash . (1) . \text{Transp} . \supset \vdash . \text{Prop}$

$*125\cdot35$. $\vdash :. \aleph_0 \in \text{NC mult} . \supset : \exists ! \aleph_0(x) . \equiv . \text{Infin ax}(x)$

Dem.

$\vdash . *125\cdot21 . *124\cdot56 . \supset \vdash :. \text{Hp} . \supset : \text{Infin ax}(x) . \equiv . t`x \in \text{Cls refl} .$

$[*125\cdot31] \qquad \equiv . \exists ! \aleph_0(x) :. \supset \vdash . \text{Prop}$

$*125\cdot36$. $\vdash : \text{Infin ax}(\text{Cls}) . \equiv . \exists ! \aleph_0(\text{Cls})$

Dem.

$\vdash . *24\cdot14 . *63\cdot102\cdot66 . \supset \vdash . t^2`x = \text{Cl}`V$

$[*24\cdot11] \qquad = \text{Cls} \qquad (1)$

$\vdash . (1) . *125\cdot24 . \supset \vdash : \text{Infin ax}(\text{Cls}) . \equiv . \text{Infin ax}(x) .$

$[*125\cdot23.(1)] \qquad \equiv . \exists ! \aleph_0(\text{Cls}) : \supset \vdash . \text{Prop}$

$*126$. ON TYPICALLY INDEFINITE INDUCTIVE CARDINALS

Recapitulation of Conventions and Summary of $*126$.

We have now arrived at the stage where we can adopt the standpoint of ordinary arithmetic, and can for the future in arithmetical operations with cardinals ignore differences of type. In order to understand how this is so, it will be necessary briefly to recall the line of thought of some of the previous numbers and the conventions upon which the symbolism is based.

The symbolism of $*102$, though perfectly precise as to the typical relations of the various symbols, is in fact too complex for use, except in cases of absolute necessity. It is better to use the typically ambiguous symbols Nc and sm, combined with some simple rules of interpretation of the symbolism, so as to secure that the various occurrences of the same symbols are in their proper relationships of type. This is the course followed in $*100$, $*101$, and in every number from $*110$ onwards.

The important symbols which involve an explicit or implicit use of Nc or sm are called "formal numbers," and it is only necessary to make the rules of interpretation apply to them.

A constant formal number is any symbol representing a typically ambiguous constant such that there is a constant α such that, however the ambiguities of type may be determined, the former constant is identical with $\mathrm{Nc}`\alpha$. The variable formal numbers are defined by enumeration. They are divided into three Sets, the Primary Set, the Argumental Set, and the Arithmetical Set.

The Primary Set consists of $\mathrm{Nc}`\alpha$, $\Sigma\,\mathrm{Nc}`\kappa$, $\Pi\,\mathrm{Nc}`\kappa$, where α is a variable Cls of any type and κ is a variable Cls^2 of any type. Also α and κ may themselves be complex symbols which in some way involve variables.

The Argumental Set has only one member $\mathrm{sm}``\mu$, where $\mu$ is a variable $\mathrm{Cls}^2$ of any type. In its capacity of a formal number $\mathrm{sm}``\mu$ is only interesting when μ is an NC; then $\mathrm{sm}``\mu$ gives the corresponding NC in another type, provided that $\mu$ is not $\Lambda$. Also $\mu$ may be a complex symbol which in some way involves a variable, *e.g.* $\mathrm{sm}``\mathrm{Nc}`\alpha$ is a formal number of the Argumental Set: μ is called the *argument* of $\mathrm{sm}``\mu$.

The Arithmetical Set consists of $\mu +_0 \nu$, $\mu \times_0 \nu$, μ^ν, $\mu -_0 \nu$. These formal numbers are only interesting when μ and ν are also members of NC. Also μ and ν may be complex symbols, so long as one of them at least involves a variable. For example $2^{3+_0\nu}$ is a formal number, and so is $\alpha +_0 (3 +_0 \nu)$.

The Primary and Argumental and Arithmetical Sets of Formal Numbers are derived from the corresponding sets of *variable* formal numbers, by

adding to them the constant formal numbers obtained by substituting constants for the variables occurring in the expressions for the members of the variable set in question.

In the formal numbers of the arithmetical set as written above, μ and ν are called the *first components*. Thus every formal number of this set has two first components. The first components (if any) of the first components are also called *components* of the original formal number, and so on; so that components of components are components of the original symbol.

A formal number of the arithmetical set, whose components are all formal numbers, either constant or variable but *not* belonging to the argumental set, is called a *pure arithmetical* formal number. These are the formal numbers which it is important in arithmetic to secure from assuming the value Λ owing to lowness of type.

The logical investigation of $*100$ and $*101$, where typically ambiguous formal numbers are used, is directly concerned in investigating the premisses necessary to secure various propositions from fluctuating truth-values owing to the intrusion of null-values among the cardinals. The convention, necessary to avoid determinations of type which we never wish to consider, is as follows, where the terms used are explained fully in the prefatory statement:

IT. Argumental occurrences are bound to logical and attributive occurrences; and, if there are no argumental occurrences, equational occurrences are bound to logical occurrences. This rule only applies so far as meaning permits after the assignment of types to the real variables.

In $*110$, $*113$, $*116$, $*119$ we consider the arithmetical operations of addition, multiplication, exponentiation, and subtraction. Also in $*117$ we consider the comparison of cardinal numbers in respect to the relation of greater and less.

There is no interest in complicating our theorems by allowing for the cases when a pure arithmetical formal number, whose components are ambiguous as to type, becomes equal to Λ owing to the low type of one of its *components*. Also in the theory of greater and less the possibility of null-values in low types has no real interest. Accordingly these are excluded from any consideration by the definitions

$$*110{\cdot}03{\cdot}04,\ *113{\cdot}04{\cdot}05,\ *116{\cdot}03{\cdot}04,\ *117{\cdot}02{\cdot}03,$$

as far as members of the primary set of formal numbers are concerned; and for other formal numbers by the following convention:

IIT. Whenever a formal number σ occurs, so that, if it were replaced by $\text{Nc}`\alpha$, the dominant type of $\text{Nc}`\alpha$ would by definition have to be adequate, then the dominant type of σ is also to be adequate.

When σ is a pure arithmetical formal number, this convention secures that the type of every component is adequate.

But in arithmetic we also wish to avoid the intrusion of null-values into the consideration of equations, so far as this avoidance can be attained by the use of high types. Accordingly when we are concerned with the purely arithmetical point of view, we add also the following definition and convention (AT).

Definition. An *arithmetical equation* is an equation between pure arithmetical formal numbers whose dominant types are both determined adequately.

AT. All equations involving pure arithmetical formal numbers are to be arithmetical.

This convention is used in $*117$ and in some earlier propositions which are noted in the prefatory statement.

Its effect is to render the statement of hypotheses often unnecessary. Examples of its application to the numbers where it is not used in the symbolism are also considered in the prefatory statement.

In the case of the inductive numbers we cannot logically prove, apart from Infin ax, that one type exists which is adequate for all the formal numbers 0, 1, 2, 3, etc. But we can prove that for any particular inductive number, say 521, a type exists for which 521 is not equal to Λ. Accordingly for a given symbolic form, in which the symbolism necessarily has only finite complexity, when the types of variables which by hypothesis represent inductive classes or inductive numbers, not Λ, have been settled, it is always possible to fix on a type which will be adequate for all the pure arithmetical formal numbers produced by the symbolism of the form, and also at the same time (and here the peculiar properties of inductive numbers come in) to have chosen the original types of the variables so that any of the variables can assume the value of any assigned constant inductive number, say 521, without being null.

The result is that we may assume that the symbols representing inductive numbers are never null, and thereby obtain the stable truth-values of propositions about them.

Accordingly we proceed as follows: we put

$*$126·01. $\text{NC ind} = \text{NC induct} - \iota{}^{\backprime}\Lambda$ Df

We make the rule that when NC ind appears, convention AT is always applied. The result is that when a formal number is an NC ind we need never think about its type, and accordingly all the conventions vanish from the mind, as far as pure arithmetical indefinite inductive cardinals are concerned. We supersede all other conventions by the single one that, if it has been proved or assumed that a formal number represents an inductive cardinal,

the types are so arranged that that formal number is not equal to Λ. The proofs of propositions in this number consist largely of the production of a definite type in which this result is attained.

The important propositions are

***126·12.** $\vdash : \nu \in \mathrm{NC\,ind} \,.\, \supset \,.\, (\nu +_c 1) \cap t`\nu \in \mathrm{NC\,ind}$

***126·121.** $\vdash \,.\, 1, 2, 3, \ldots \in \mathrm{NC\,ind}$

***126·13·14·15.** $\vdash : \alpha, \beta \in \mathrm{NC\,ind} \,.\, \supset \,.\, \alpha +_c \beta,\ \alpha \times_c \beta,\ \alpha^\beta \in \mathrm{NC\,ind}$

***126·141.** $\vdash : \alpha, \beta \in \mathrm{NC\,ind} - \iota`0 \,.\, \equiv \,.\, \alpha \times_c \beta \in \mathrm{NC\,ind} - \iota`0$

***126·151.** $\vdash : \alpha, \beta \in \mathrm{NC\,ind} - \iota`0 \,.\, \alpha \neq 1 \,.\, \equiv \,.\, \alpha^\beta \in \mathrm{NC\,ind} - \iota`0 - \iota`1$

Also *126·4·42·43 give the fundamental propositions for subtraction, division, and "inverse exponentiation"; and *126·5·51·52·53 the fundamental propositions for the relations of greater and less.

***126·01.** $\mathrm{NC\,ind} = \mathrm{Nc\,induct} - \iota`\Lambda$ Df

Whenever the symbol NC ind is used the *Rule of Indefinite Numbers* is adhered to, so that all consideration of distinctions in type among inductive cardinals can be laid aside (cf. Prefatory Statement and also the Summary of this number).

***126·011.** $\vdash : \nu \in \mathrm{NC\,ind} \,.\, \equiv \,.\, \nu \in \mathrm{NC\,induct} - \iota`\Lambda$ [(*126·01)]

***126·1.** $\vdash : \nu \in \mathrm{NC\,ind} \,.\, \equiv \,.\, (\exists\alpha) \,.\, \alpha \in \mathrm{Cls\,induct} \,.\, \nu = \mathrm{Nc}`\alpha \,.\, \exists ! \,\nu$

Dem.

$\vdash \,.\, *120·14 \,.\, *100·4 \,.\, *126·011 \,.\, \supset$

$\vdash : \nu \in \mathrm{NC\,ind} \,.\, \supset \,.\, (\exists\alpha) \,.\, \nu = \mathrm{Nc}`\alpha \,.\, \nu \in \mathrm{NC\,induct} - \iota`\Lambda \,.$

[*118·01] $\supset \,.\, (\exists\alpha) \,.\, \nu = \mathrm{Nc}`\alpha \,.\, \mathrm{Nc}`\alpha \in \mathrm{NC\,induct} - \iota`\Lambda \,.\, \exists ! \,\nu \,.$

[*120·211] $\supset \,.\, (\exists\alpha) \,.\, \alpha \in \mathrm{Cls\,induct} \,.\, \nu = \mathrm{Nc}`\alpha \,.\, \exists ! \,\nu$ (1)

$\vdash \,.\, *120·21 \,.\, \supset \vdash : (\exists\alpha) \,.\, \alpha \in \mathrm{Cls\,induct} \,.\, \nu = \mathrm{Nc}`\alpha \,.\, \exists ! \,\nu \,.$

$\supset \,.\, (\exists\alpha) \,.\, \mathrm{N_0c}`\alpha \in \mathrm{NC\,induct} \,.\, \nu = \mathrm{Nc}`\alpha \,.\, \exists ! \,\nu \,.$

[*120·15.*100·511] $\supset \,.\, \nu \in \mathrm{NC\,induct} - \iota`\Lambda$ (2)

$\vdash \,.\, (1) \,.\, (2) \,.\, \supset \vdash \,.\, \mathrm{Prop}$

***126·101.** $\vdash :. \mu, \nu \in \mathrm{NC\,ind} \,.\, \exists ! \,\mu_\lambda \,.\, \supset : \mu_\lambda = \nu_\lambda \,.\, \equiv \,.\, \mu_\lambda = \nu \,.\, \equiv \,.\, \mu = \nu$
[*126·1 . *103·16]

***126·11.** $\vdash \,.\, 0 \in \mathrm{NC\,ind}$ [*120·12 . *101·12]

***126·12.** $\vdash : \nu \in \mathrm{NC\,ind} \,.\, \supset \,.\, (\nu +_c 1) \cap t`\nu \in \mathrm{NC\,ind}$

Dem.

$\vdash \,.\, *120·151 \,.\, \supset \vdash : \nu \in \mathrm{NC\,ind} \,.\, \supset \,.\, \nu +_c 1 \in \mathrm{NC\,induct}$ (1)

$\vdash \,.\, *117·66 \,.\, *118·01 \,.\, \supset \vdash : \alpha \in \mathrm{Cls\,induct} \,.\, \nu = \mathrm{N_0c}`\alpha \,.\, \exists ! \,\nu \,.\, \supset \,.\, \mathrm{Nc}`\mathrm{Cl}`\alpha > \nu \,.$

[*126·1.*120·429] $\supset \,.\, \mathrm{Nc}`\mathrm{Cl}`\alpha \geqslant \nu +_c 1 \,.$

[*103·13.*117·32] $\supset \,.\, \exists ! \,(\nu +_c 1) \cap t`\mathrm{Cl}`\alpha \,.$

[*103·12.*60·34] $\supset \,.\, \exists ! \,(\nu +_c 1) \cap t`\nu$ (2)

$\vdash \,.\, (1) \,.\, (2) \,.\, *126·1 \,.\, \supset \vdash \,.\, \mathrm{Prop}$

$*126{\cdot}121$. $\vdash . 1, 2, 3, \ldots \epsilon \mathrm{NC\,ind}$ [$*126{\cdot}11{\cdot}12$]

This proposition, taken in connection with $*120{\cdot}4232$, embodies the convention named the Rule of Indefinite Numbers and its justification. The convention is that 1, 2, 3, ... are always in future to be used in existential types. In other words whenever any particular inductive number is employed, it is determined in a type in which it is not Λ. The justification is that by $*126{\cdot}11{\cdot}12$ such a type can always be found for each particular inductive number.

The convention is also applied to arithmetical formal numbers in $*126{\cdot}13{\cdot}14{\cdot}15$.

For all arithmetical and equational occurrences this convention is really the outcome of IT, IIT, and AT.

$*126{\cdot}13$. $\vdash : \alpha, \beta \epsilon \mathrm{NC\,ind} . \equiv . \alpha +_0 \beta \epsilon \mathrm{NC\,ind}$
[$*120{\cdot}71 . *126{\cdot}1 . *110{\cdot}3 . *103{\cdot}13$]

$*126{\cdot}14$. $\vdash : \alpha, \beta \epsilon \mathrm{NC\,ind} . \supset . \alpha \times_0 \beta \epsilon \mathrm{NC\,ind}$
[$*120{\cdot}72 . *126{\cdot}1 . *113{\cdot}25 . *103{\cdot}13$]

$*126{\cdot}141$. $\vdash : \alpha, \beta \epsilon \mathrm{NC\,ind} - \iota`0 . \equiv . \alpha \times_0 \beta \epsilon \mathrm{NC\,ind} - \iota`0$
[$*120{\cdot}721 . *113{\cdot}114$]

$*126{\cdot}15$. $\vdash : \alpha, \beta \epsilon \mathrm{NC\,ind} . \supset . \alpha^\beta \epsilon \mathrm{NC\,ind}$ [$*120{\cdot}73 . *116{\cdot}25 . *103{\cdot}13$]

$*126{\cdot}151$. $\vdash : \alpha, \beta \epsilon \mathrm{NC\,ind} - \iota`0 . \alpha \neq 1 . \equiv . \alpha^\beta \epsilon \mathrm{NC\,ind} - \iota`0 - \iota`1$
[$*120{\cdot}731 . *116{\cdot}35 . *117{\cdot}592$]

$*126{\cdot}23$. $\vdash : \mu \epsilon \mathrm{NC} . \exists ! \mu \cap t`\alpha . \supset . \exists ! 2^\mu \cap t`t`\alpha . \exists ! (\mu +_0 1) \cap t`t`\alpha$

Dem.

$\vdash . *63{\cdot}661 . *116{\cdot}72 . \supset$

$\vdash : \mu \epsilon \mathrm{NC} . \beta \epsilon \mu \cap t`\alpha . \supset . \mathrm{Cl}`\beta \epsilon 2^\mu \cap t`t`\alpha$ (1)

$\vdash . (1) . *117{\cdot}32 . \supset$

$\vdash : \mathrm{Hp}(1) . 2^\mu \geqslant \nu . \supset . \exists ! \mathrm{sm}``\nu \cap t`t`\alpha$ (2)

$\vdash . *117{\cdot}661{\cdot}31 . \supset \vdash : \mathrm{Hp}(1) . \supset . 2^\mu \geqslant \mu +_0 1$ (3)

$\vdash . (1) . (2) . (3) . *100{\cdot}511 . \supset \vdash . \mathrm{Prop}$

$*126{\cdot}31$. $\vdash : \alpha +_0 1 \epsilon \mathrm{NC\,ind} . \equiv . \alpha \epsilon \mathrm{NC\,ind}$ [$*126{\cdot}12{\cdot}13{\cdot}121 . *120{\cdot}452$]

Note that the specification of the type of $\alpha +_0 1$ is omitted in accordance with the convention. The reference to $*126{\cdot}12$ shows that it is always possible to apply the convention.

$*126{\cdot}32$. $\vdash : \alpha \epsilon \mathrm{NC} - \iota`0 - \iota`\Lambda . \nu \epsilon \mathrm{NC\,ind} . \supset . \alpha +_0 \nu > \nu$ [$*120{\cdot}428 . *110{\cdot}3$]

$*126{\cdot}33$. $\vdash :. \alpha \epsilon \mathrm{NC\,ind} . \beta \epsilon \mathrm{NC} - \iota`\Lambda . \supset : \alpha < \beta . \vee . \alpha = \beta . \vee . \alpha > \beta$ [$*120{\cdot}441$]

$*126{\cdot}4$. $\vdash :. \mu, \nu, \varpi \epsilon \mathrm{NC\,ind} . \supset : \mu +_0 \varpi = \nu +_0 \varpi . \equiv . \mu = \nu$
[$*126{\cdot}13 . *120{\cdot}41$]

$*126{\cdot}41$. $\vdash :. \mu, \nu, \varpi \epsilon \mathrm{NC\,ind} . \varpi \neq 0 . \supset : \mu \times_0 \varpi = \nu \times_0 \varpi . \equiv . \mu = \nu$
[$*120{\cdot}51 . *126{\cdot}14$]

***126·42.** $\vdash :. \mu, \nu, \varpi \in \text{NC ind} . \varpi \neq 0 . \supset : \mu^{\varpi} = \nu^{\varpi} . \equiv . \mu = \nu$ [*120·55 . *126·15]

***126·43.** $\vdash :. \mu, \nu, \varpi \in \text{NC ind} . \varpi \neq 0 . \varpi \neq 1 . \supset : \varpi^{\mu} = \varpi^{\nu} . \equiv . \mu = \nu$ [*120·53 . *126·15]

***126·5.** $\vdash :. \mu, \nu, \varpi \in \text{NC ind} . \supset : \mu +_c \varpi > \nu +_c \varpi . \equiv . \mu > \nu$

Dem.

$\vdash . *117{\cdot}561 . \quad \supset \vdash : \text{Hp} . \mu > \nu . \supset . \mu +_c \varpi \geqslant \nu +_c \varpi \quad (1)$

$\vdash . *126{\cdot}4 . \quad \supset \vdash : \text{Hp} . \mu > \nu . \supset . \mu +_c \varpi \neq \nu +_c \varpi \quad (2)$

$\vdash . (1) . (2) . *117{\cdot}26 . \supset \vdash : \text{Hp} . \mu > \nu . \supset . \mu +_c \varpi > \nu +_c \varpi \quad (3)$

$\vdash . *117{\cdot}561 . \text{Transp} . *117{\cdot}281 . \supset$

$\vdash : \text{Hp} . \mu +_c \varpi > \nu +_c \varpi . \supset . \sim(\nu \geqslant \mu) .$

$[*126{\cdot}33] \quad \supset . \mu > \nu \quad (4)$

$\vdash . (3) . (4) . \supset \vdash . \text{Prop}$

***126·51.** $\vdash :. \mu, \nu, \varpi \in \text{NC ind} . \varpi \neq 0 . \supset : \mu \times_c \varpi > \nu \times_c \varpi . \equiv . \mu > \nu$ [*117·571 . *126·41]

The proof proceeds as in *126·5.

***126·52.** $\vdash :. \mu, \nu, \varpi \in \text{NC ind} . \varpi \neq 0 . \supset : \mu^{\varpi} > \nu^{\varpi} . \equiv . \mu > \nu$ [*117·581 . *126·42]

***126·53.** $\vdash :. \mu, \nu, \varpi \in \text{NC ind} . \varpi \neq 0 . \varpi \neq 1 . \supset : \varpi^{\mu} > \varpi^{\nu} . \equiv . \mu > \nu$ [*117·591 . *126·43]

PART IV

RELATION-ARITHMETIC

SUMMARY OF PART IV

THE subject to be treated in this Part is a general kind of arithmetic of which ordinal arithmetic is a particular application. The form of arithmetic to be treated in this Part is applicable to all relations, though its chief importance is in regard to such relations as generate series. The analogy with cardinal arithmetic is very close, and the reader will find that what follows is much facilitated by bearing the analogy in mind.

The outlines of relation-arithmetic are as follows. We first define a relation between relations, which we shall call *ordinal similarity* or *likeness*, and which plays the same part for relations as similarity plays for classes. Likeness between P and Q is constituted by the fact that the fields of P and Q can be so correlated by a one-one relation that if any two terms have the relation P, their correlates have the relation Q, and vice versa. If P and Q generate series, we may express this by saying that P and Q are like if their fields can be correlated without change of order. Having defined likeness, our next step is to define the *relation-number* of a relation P as the class of relations which are like P, just as the cardinal number of a class α is the class of classes which are similar to α. We then proceed to addition. The ordinal sum of two relations P and Q is defined as the relation which holds between x and y when x and y have the relation P or the relation Q, or when x is a member of $C\text{‘}P$ and y is a member of $C\text{‘}Q$. If P and Q generate series, it will be seen that this defines the sum of P and Q as the series resulting from adding the Q-series after the end of the P-series. The sum is thus not commutative. The sum of the relation-numbers of P and Q is of course the relation-number of their sum, provided $C\text{‘}P$ and $C\text{‘}Q$ have no common terms.

The ordinal product of two relations P and Q is the relation between two couples $z \downarrow x$, $w \downarrow y$, when x, y belong to $C\text{‘}P$ and z, w belong to $C\text{‘}Q$ and either xPy or $x = y \,.\, zQw$. Thus, for example, if the field of P consists of 1_P, 2_P, 3_P, and the field of Q consists of 1_Q, 2_Q, the relation $P \times Q$ will hold from any earlier to any later term of the following series:

$$1_Q \downarrow 1_P,\ 2_Q \downarrow 1_P,\ 1_Q \downarrow 2_P,\ 2_Q \downarrow 2_P,\ 1_Q \downarrow 3_P,\ 2_Q \downarrow 3_P.$$

It is plain that, denoting the ordinal product of P and Q by $P \times Q$, we have

$$C\text{‘}(P \times Q) = C\text{‘}P \times C\text{‘}Q,$$

where the second "$\times$" as standing between classes has the meaning defined in $*113{\cdot}01$.

Infinite ordinal sums and products will also be defined, but the definitions are somewhat complicated.

The arithmetic which results from the above definitions satisfies all those of the formal laws which are satisfied in ordinal arithmetic, when this is not confined to finite ordinals; that is to say, relation-numbers satisfy the associative law for addition and for multiplication*, they satisfy the distributive law in the shape (where the + and × are those appropriate to relation-numbers)

$$(\beta + \gamma) \times \alpha = (\beta \times \alpha) + (\gamma \times \alpha),$$

and they satisfy the exponential laws

$$\alpha^\beta \times \alpha^\gamma = \alpha^{\beta+\gamma},$$
$$(\alpha^\beta)^\gamma = \alpha^{\beta \times \gamma}.$$

They do not in general satisfy the commutative law either in addition or in multiplication, nor do they satisfy the distributive law in the form

$$\alpha \times (\beta + \gamma) = (\alpha \times \beta) + (\alpha \times \gamma),$$

nor the exponential law

$$\alpha^\gamma \times \beta^\gamma = (\alpha \times \beta)^\gamma.$$

But in the particular case in which the relations concerned are finite serial relations, the corresponding relation-numbers do satisfy these additional formal laws; hence the arithmetic of *finite* ordinals is exactly analogous to that of inductive cardinals (cf. Part V, Section E).

If the relations concerned are limited to well-ordered relations, relation-arithmetic becomes ordinal arithmetic as developed by Cantor; but many of Cantor's propositions, as we shall see in this Part, do not require the limitation to well-ordered relations.

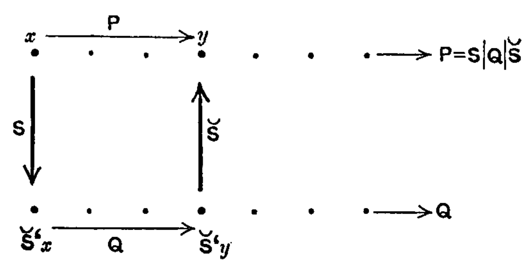
* For the associative law of multiplication, a hypothesis is required as to the kind of relation concerned. Cf. ∗174·241·25.

SECTION A

ORDINAL SIMILARITY AND RELATION-NUMBERS

Summary of Section A.

Two series generated by the relations P and Q respectively are said to be ordinally similar when their terms can be correlated as they stand, without



change of order. In the accompanying figure, the relation S correlates the members of $C\text{‘}P$ and $C\text{‘}Q$ in such a way that if xPy, then $(\breve{S}\text{‘}x)\,Q\,(\breve{S}\text{‘}y)$, and if zQw, then $(S\text{‘}z)\,P\,(S\text{‘}w)$. It is evident that the journey from x to y (where xPy) may, in such a case, be taken by going first to $\breve{S}\text{‘}x$, thence to $\breve{S}\text{‘}y$, and thence back to y, so that $xPy \,.\equiv.\, x\,(S\,|\,Q\,|\,\breve{S})\,y$, *i.e.* $P = S\,|\,Q\,|\,\breve{S}$. Hence to say that P and Q are ordinally similar is equivalent to saying that there is a one-one relation S which has $C\text{‘}Q$ for its converse domain and gives $P = S\,|\,Q\,|\,\breve{S}$. In this case we call S a *correlator* of Q and P.

We denote the relation of ordinal similarity by "smor," which is short for "similar ordinally." Thus

$$P \operatorname{smor} Q \,.\equiv.\, (\exists S) \,.\, S \in 1 \to 1 \,.\, C\text{‘}Q = \text{Ɑ}\text{‘}S \,.\, P = S\,|\,Q\,|\,\breve{S}.$$

It will be found that the relation $S\,|\,Q\,|\,\breve{S}$ plays the same part in relation to Q in relation-arithmetic as $S\text{‘‘}\beta$ plays in relation to β in cardinal arithmetic. It is therefore desirable to have a simpler notation for $S\,|\,Q\,|\,\breve{S}$. We put

$$S;Q = S\,|\,Q\,|\,\breve{S} \quad \text{Df.}$$

We shall find that the semi-colon so defined has the same kind of properties in relation-arithmetic as the two inverted commas have in cardinal arithmetic. Corresponding to the notation $S_\epsilon\text{‘}\beta$, we put

$$S\dagger Q = S\,|\,Q\,|\,\breve{S} \quad \text{Df.}$$

We shall thus have $S\dagger = S \| \breve{S}$. It will appear that $S\dagger$ has ordinal properties analogous to the cardinal properties of S_ϵ. Thus *e.g.* where $S \| \breve{S}_\epsilon$ appears as a cardinal correlator, $S \| \text{Cnv}'S\dagger$ will appear as an ordinal correlator (in each case with the converse domain suitably limited).

The elementary properties of $S;Q$ will be considered in $*150$. We shall then, in $*151$, be able to study ordinal similarity, taking as our definition of an ordinal correlator

$$P \,\overline{\text{smor}}\, Q = \hat{S}\{S \epsilon 1 \to 1 \,.\, C'Q = \text{Ꮧ}'S \,.\, P = S;Q\} \quad \text{Df},$$

and defining two relations as ordinally similar when they have at least one ordinal correlator, *i.e.* putting (on the analogy of $*73$)

$$\text{smor} = \hat{P}\hat{Q}\{\exists ! P \,\overline{\text{smor}}\, Q\} \quad \text{Df}.$$

There is no need to confine the notion of ordinal similarity (or likeness, as we shall also call it) to *serial* relations. When two relations have ordinal similarity, their internal structures are analogous, and they therefore have many common properties. Whenever similarity has been proved between two classes α and β, then if β is given as the field of some relation Q, and S is the correlating relation, $S;Q$ is like Q, and has α for its field. Hence similar classes are the fields of like relations. It must not be supposed, however, that like relations are coextensive with relations whose fields are similar. This does not hold even when we confine ourselves to serial relations, except in the special case of *finite* serial relations.

The definition of relation-numbers ($*152$) is as follows: The relation-number of P, which we call $\text{Nr}'P$, is the class of relations which are ordinally similar to P; and the class of relation-numbers, which we denote by NR, is the class of all classes of the form $\text{Nr}'P$. The elementary properties of relation-numbers, treated in $*152$, are closely analogous to those of cardinal numbers treated in $*100$.

After a few propositions about the ordinal 0 and the ordinal 2, which we call 0_r and 2_r ($*153$), we pass to the consideration of relation-numbers of various types. It will be observed that "smor," like "sm," is a relation which is ambiguous as to the type both of its domain and of its converse domain. Thus "$P \,\text{smor}\, Q$" only has an unambiguous meaning when the types of P and Q are determined. P and Q may or may not be of the same type; the only restriction upon the type of either is that both must be "homogeneous" relations, *i.e.* relations whose domain and converse domain are of the same type. This restriction results from the fact that $C'Q$ occurs in the definition of "$P \,\text{smor}\, Q$," and a relation does not have a field unless it is homogeneous; hence Q must be homogeneous, and therefore, whatever S may be, $S | Q | \breve{S}$ must be homogeneous, *i.e.* P must be homogeneous. Thus *e.g.* such relations as D, ι, or ϵ are not ordinally similar either to themselves

or to anything else. Whenever "$P \operatorname{smor} Q$" is *significant* for a suitable Q, we have $P \operatorname{smor} P$; but if P is not homogeneous, "$P \operatorname{smor} Q$" is never significant. Hence throughout the theory of ordinal similarity, the relations of which ordinal similarity is affirmed or denied must be homogeneous. The correlators, on the contrary, need not be homogeneous.

Owing to the homogeneity of our relations, the types of relation-numbers are much more easily dealt with than they otherwise would be; for the type of a homogeneous relation is determined by that of a single class, namely its field, whereas the type of a relation in general depends upon the types of *two* classes, namely its domain and its converse domain. Since, where likeness is concerned, the type of the field determines the type of the relation, propositions concerning the relations between different typical determinations of a given relation-number are, for the most part, exactly analogous to and deducible from those for cardinals. In fact, a relation ordinally similar to Q exists in the type of P when, and only when, a class similar to $C`Q$ exists in the type of $C`P$, *i.e.*

$$\exists ! \operatorname{Nr}(P)`Q \,.\equiv .\, \exists ! \operatorname{Nc}(C`P)`C`Q.$$

The half of this proposition follows from the fact that, if P is like Q, $C`P$ is similar to $C`Q$. The other half follows from the fact, mentioned above, that if $\beta = C`Q$ and $\alpha \operatorname{sm} \beta$, then there is a relation like $Q$ and having $\alpha$ for its field. Now if $\alpha$ belongs to the type of $C`P$, any relation having α for its field is contained in $t_0`C`P \uparrow t_0`C`P$. Hence in the case supposed there is a relation like Q and contained in $t_0`C`P \uparrow t_0`C`P$. But the relations contained in $t_0`C`P \uparrow t_0`C`P$ constitute $t`P$. Hence there is a relation which is like $Q$ and is a member of $t`P$, whence our proposition results. By means of this proposition and those of $*102$—6, the properties of relation-numbers with respect to types follow easily. The conventions IT, IIT and AT apply to relation-numbers as to cardinals; they are to be applied in the same way as in the analogous propositions of Part III, Section A.

*150. INTERNAL TRANSFORMATION OF A RELATION

Summary of *150.

In this number we introduce two notations which have uses in regard to relations closely analogous to the uses of $R``\alpha$ and R_ϵ in regard to classes. These two notations are defined as follows:

$$S \mathbin{;} Q = S \,|\, Q \,|\, \breve{S} \quad \text{Df},$$

$$S \dagger Q = S \,|\, Q \,|\, \breve{S} \quad \text{Df}.$$

We then have $\quad \vdash . \, S \dagger Q = S \mathbin{;} Q = S \,|\, Q \,|\, \breve{S} = (S \,\|\, \breve{S})`Q.$

$S \dagger Q$ is merely an alternative to $S \mathbin{;} Q$, just as $R_\epsilon`\alpha$ is an alternative to $R``\alpha$. Also $S\dagger = S \,\|\, \breve{S}$, in virtue of *38·01 and *43·01.

The uses of $S \mathbin{;} Q$ occur chiefly when S is a one-one relation and $C`Q \subset \text{Ɑ}`S$. This case is illustrated in the figure in the introduction to this section. Here if Q relates x and y, $S \mathbin{;} Q$ relates $S`x$ and $S`y$. Thus given a class α similar to $C`Q$, if S is the correlating relation, $S \mathbin{;} Q$ has α for its field, and has, in very many respects, properties analogous to those of Q.

$S \mathbin{;} Q$ is important for many special values of S. For example, let Q be a relation between relations; then $C \mathbin{;} Q$ will be the corresponding relation of the fields of these relations. If Q be any relation, $\downarrow x \mathbin{;} Q$ will be the corresponding relation between ordered couples of which x is the relatum; *i.e.* if yQz, the relation $\downarrow x \mathbin{;} Q$ will hold between $y \downarrow x$ and $z \downarrow x$. If Q is a relation between classes, and we have $\beta Q \gamma$, then the relation $\alpha \cup \mathbin{;} Q$ will hold between $\alpha \cup \beta$ and $\alpha \cup \gamma$. In short, whenever S is a one-many relation, and therefore gives rise to a descriptive function, then $S \mathbin{;} Q$ is the relation which holds between $S`y$ and $S`z$ whenever Q holds between y and z.

We introduce one other new notation in this number, corresponding to $\alpha \underset{,,}{♀} y$ in *38. This notation is thus defined:

$$Q \underset{\cdot ;}{♀} y = ♀ y \mathbin{;} Q \quad \text{Df}.$$

The purpose of this notation is to enable us to proceed to $Q \underset{\cdot ;}{♀} \mathbin{;} P$ and other similar notations; or, otherwise stated, to enable us to treat $♀ y \mathbin{;} Q$ as a function of y rather than of Q. Take for example the case of $x \downarrow \mathbin{;} Q$. We may wish to consider various relations $x \downarrow \mathbin{;} Q$, $y \downarrow \mathbin{;} Q$, where we are to have (say) xPy. To express the relation of $x \downarrow \mathbin{;} Q$ to $y \downarrow \mathbin{;} Q$ resulting from xPy, we need the above notation. By its help, we have

$$x \downarrow \mathbin{;} Q = Q \underset{\cdot ;}{\downarrow} `x \, . \, y \downarrow \mathbin{;} Q = Q \underset{\cdot ;}{\downarrow} `y.$$

Hence $\quad xPy \, . \equiv . \, (Q \underset{\cdot ;}{\downarrow} `x) (Q \underset{\cdot ;}{\downarrow} \mathbin{;} P) (Q \underset{\cdot ;}{\downarrow} `y).$

Thus $Q \underset{;}{\downarrow} {;}P$ is the relation between $x \downarrow {;}Q$ and $y \downarrow {;}Q$ corresponding to the relation P between x and y. $Q \underset{;}{\downarrow} {;}P$ plays the same part in relation-arithmetic as is played by $\alpha \underset{,,}{\downarrow} {``}\beta$ in cardinal arithmetic.

The notations of this number are capable of occasional uses in cardinal arithmetic*, but their chief utility is in relation-arithmetic, in which they are fundamental.

In order to minimize the use of brackets, we put

$$R`S{;}Q = R`(S{;}Q) \quad \text{Df},$$
$$R{;}S{;}Q = R{;}(S{;}Q) \quad \text{Df}.$$

As an immediate result of the definition of $S{;}Q$, we have

***150·11.** $\vdash : x\,(S{;}Q)\,y \,.\equiv .\, (\exists z, w) \,.\, xSz \,.\, ySw \,.\, zQw$

We have also

***150·12.** $\vdash . \text{Cnv}`S{;}Q = S{;}\breve{Q}$

***150·13.** $\vdash . R{;}S{;}Q = (R \mid S){;}Q$

This proposition, which is the analogue of $(P \mid Q)``\gamma = P``Q``\gamma$ (*37·33), is very often used. We have also

***150·3.** $\vdash . S{;}(Q \cup R) = S{;}Q \cup S{;}R$

***150·42.** $\vdash . S{;}\dot{\Lambda} = \dot{\Lambda}$

The remaining propositions of this number (with a few exceptions) may be thus classified:

(1) Propositions concerning the domain, converse domain, and field of $S{;}Q$ (*150·2—·23). Owing to the fact that the chief applications of this subject are to cases where Q and $S{;}Q$ are serial, the *field* of $S{;}Q$ is more important than its domain or converse domain. Thus the chief propositions here are

***150·22.** $\vdash : C`Q \subset \text{G}`S \,.\supset .\, C`S{;}Q = S``C`Q$

***150·23.** $\vdash : C`Q = \text{G}`S \,.\supset .\, C`S{;}Q = \text{D}`S$

The hypothesis $C`Q \subset \text{G}`S$ is verified in almost all applications of $S{;}Q$. When it is not verified, the part of $C`Q$ not contained in $\text{G}`S$ is irrelevant to the value of $S{;}Q$. The hypothesis $C`Q = \text{G}`S$ is very often verified in practice, since it is verified when S is a correlator of $S{;}Q$ and Q.

(2) Propositions concerning relations with limited domains, converse domains, or fields (*150·32—·38). Broadly speaking, a limitation on the *field* of Q is equivalent to a limitation on the *converse domain* of S, and both are equivalent to a corresponding limitation on the field of $S{;}Q$ provided

* *E.g.* in *116·53 and following propositions, where the notation $S\dagger$ was introduced by a temporary definition.

$S \in \mathrm{Cls} \rightarrow 1$. The limitations that occur in practice are limitations on the converse domain of S, with consequent limitations on the fields of Q and $S \mathbin{;} Q$.

The chief propositions on this subject are

***150·32.** $\vdash . (S \upharpoonright C'Q) \mathbin{;} Q = S \mathbin{;} Q$

***150·35.** $\vdash :. y \in C'Q . \supset_y . R'y = S'y : \supset . R \mathbin{;} Q = S \mathbin{;} Q$

(This follows from *150·32 and *35·71.)

***150·36.** $\vdash . (S \upharpoonright \beta) \mathbin{;} Q = S \mathbin{;} (Q \mathbin{\text{⫡}} \beta)$

***150·37.** $\vdash : S \in \mathrm{Cls} \rightarrow 1 . \supset . S \mathbin{;} (Q \mathbin{\text{⫡}} \beta) = (S \mathbin{;} Q) \mathbin{\text{⫡}} S``\beta = (S \upharpoonright \beta) \mathbin{;} Q = \{(S``\beta) \upharpoonleft S\} \mathbin{;} Q$

(3) Propositions on $S \mathbin{;} Q$ when S is one-many or many-one (*150·4—·56). We have

***150·4.** $\vdash :. S \in 1 \rightarrow \mathrm{Cls} . \supset : x (S \mathbin{;} Q) y . \equiv . (\exists z, w) . x = S'z . y = S'w . zQw$

This proposition is used constantly. Only slightly less useful is

***150·41.** $\vdash :. S \in \mathrm{Cls} \rightarrow 1 . \supset : x (S \mathbin{;} Q) y . \equiv . (\breve{S}'x) \, Q \, (\breve{S}'y)$

The remaining propositions of this set are chiefly applications of *150·4·41 to special cases.

(4) A few propositions on $Q \underset{\cdot\cdot}{♀} y$ (*150·6—·62). These are immediate consequences of the definition.

(5) A set of propositions on couples and matters connected with them (*150·7—·75). The chief of these is

***150·71.** $\vdash : S \in 1 \rightarrow \mathrm{Cls} . z, w \in \text{Ꭰ}'S . \supset . S \mathbin{;} (z \downarrow w) = (S'z) \downarrow (S'w)$

This proposition is very often used in relation-arithmetic. Useful also is

***150·73.** $\vdash . S \mathbin{;} (\alpha \uparrow \beta) = S``\alpha \uparrow S``\beta$

(6) We next have four propositions (*150·8—·83) on $S \mathbin{;} P$ when P is a power of Q. These belong with the propositions of *92; they are useful in the ordinal theory of finite and infinite. We have

***150·82·83.** $\vdash :. S \in \mathrm{Cls} \rightarrow 1 : \mathrm{D}'Q \subset \text{Ꭰ}'S . \mathrm{v} . \text{Ꭰ}'Q \subset \text{Ꭰ}'S : \supset .$
$\mathrm{Pot}'S \mathbin{;} Q = S\dagger``\mathrm{Pot}'Q . (S \mathbin{;} Q)_{\mathrm{po}} = S \mathbin{;} Q_{\mathrm{po}}$

It follows that, in the hypothesis supposed, if S is a correlator of P and Q, it is also a correlator of P_{po} and Q_{po}.

(7) Propositions concerning the relation $S\dagger$ (*150·14—·171 and *150·9—·94). These have uses analogous to those of propositions concerning S_ϵ. The most important are

***150·14.** $\vdash . R\dagger \mid S\dagger = (R \mid S)\dagger$

(This follows immediately from *150·13, above.)

***150·141.** $\vdash . S\dagger = S \,\|\, \breve{S}$

(This follows immediately from the definition.)

***150·16.** $\vdash . \dot{s}`R\dagger``\lambda = R\dagger(\dot{s}`\lambda) = R\mathbin{;}\dot{s}`\lambda$

This proposition is analogous to $s`R```\kappa = R``s`\kappa$ (*40·38), *i.e.* to

$$s`R_\epsilon``\kappa = R_\epsilon`s`\kappa = R``s`\kappa,$$

as appears on substituting $\dot{s}$ and $R\dagger$ for s and R_ϵ in this variant of *40·38.

The remaining propositions are mainly of the nature of lemmas, to be used once or twice each in relation-arithmetic.

***150·01.** $S\mathbin{;}Q = S \mid Q \mid \breve{S}$ Df

***150·02.** $S\dagger Q = S \mid Q \mid \breve{S}$ Df

***150·03.** $Q \underset{\bullet\,;}{♀} y = ♀ y \mathbin{;} Q$ Df

Here, as in *38, "♀" stands for any sign which, when placed between two letters, defines a descriptive function of the arguments represented by those letters. Thus for example "♀" may represent any of the following:

$$\cap,\ \cup,\ \dot{\cap},\ \dot{\cup},\ \mid,\ \upharpoonleft,\ \upharpoonright,\ \Lsh,\ \uparrow,\ \downarrow.$$

The two following definitions serve merely for the avoidance of brackets.

***150·04.** $R`S\mathbin{;}Q = R`(S\mathbin{;}Q)$ Df

***150·05.** $R\mathbin{;}S\mathbin{;}Q = R\mathbin{;}(S\mathbin{;}Q)$ Df

***150·1.** $\vdash . S\mathbin{;}Q = S \mid Q \mid \breve{S} = (S \parallel \breve{S})`Q = S\dagger Q = S\dagger`Q$
[*43·112 . *38·11 . (*150·01·02)]

***150·11.** $\vdash : x(S\mathbin{;}Q)y . \equiv . (\exists z, w) . xSz . ySw . zQw$ [*34·1 . *31·11]

***150·12.** $\vdash . \text{Cnv}`S\mathbin{;}Q = S\mathbin{;}\breve{Q}$ [*34·2 . *31·33]

***150·13.** $\vdash . R\mathbin{;}S\mathbin{;}Q = (R \mid S)\mathbin{;}Q$

Dem.

$\vdash . *150·1 . (*150·05) . \supset \vdash . R\mathbin{;}S\mathbin{;}Q = R\mathbin{;}(S \mid Q \mid \breve{S})$
[*150·1] $= R \mid S \mid Q \mid \breve{S} \mid \breve{R}$
[*34·2] $= R \mid S \mid Q \mid \text{Cnv}`(R \mid S)$
[*150·1] $= (R \mid S)\mathbin{;}Q . \supset \vdash . \text{Prop}$

***150·131.** $\vdash . (R\mathbin{;}S)\mathbin{;}Q = R\mathbin{;}(S \mid \breve{R})\mathbin{;}Q$

Dem.

$\vdash . *150·13 . \supset \vdash . R\mathbin{;}(S \mid \breve{R})\mathbin{;}Q = (R \mid S \mid \breve{R})\mathbin{;}Q$
[*150·1] $= (R\mathbin{;}S)\mathbin{;}Q . \supset \vdash . \text{Prop}$

Observe that we do not have $(R\mathbin{;}S)\mathbin{;}Q = R\mathbin{;}(S\mathbin{;}Q)$.

***150·14.** $\vdash . R\dagger \mid S\dagger = (R \mid S)\dagger$

Dem.

$\vdash . *150·1·13 . \supset \vdash . R\dagger`S\dagger`Q = (R \mid S)\dagger`Q$ (1)
$\vdash . (1) . *34·42 . \supset \vdash . \text{Prop}$

This proposition is the relational analogue of *37·34.

$*150{\cdot}141.\ \vdash . S\dagger = S \| \breve{S} \quad [*150{\cdot}1 . *30{\cdot}41]$

$*150{\cdot}15.\ \vdash . S\dagger \in 1 \to \mathrm{Cls} \quad [*72{\cdot}14]$

$*150{\cdot}151.\ \vdash :. (x) . E ! S\text{‘}x : S \in \mathrm{Cls} \to 1 : \supset . S\dagger \in 1 \to 1 \quad [*74{\cdot}772 . *150{\cdot}141]$

The following proposition is used in the theory of double ordinal similarity ($*164{\cdot}13$).

$*150{\cdot}152.\ \vdash : S \upharpoonright s\text{‘}C\text{“}\lambda \in \mathrm{Cls} \to 1 . s\text{‘}C\text{“}\lambda \subset \text{Ɑ}\text{‘}S . \supset . (S\dagger) \upharpoonright \lambda \in 1 \to 1$

Dem.

$\vdash . *74{\cdot}775 \dfrac{S, S}{Q, R} . \supset$

$\vdash : S \upharpoonright s\text{‘}C\text{“}\lambda \in \mathrm{Cls} \to 1 . s\text{‘}\mathrm{D}\text{“}\lambda \subset \text{Ɑ}\text{‘}S . s\text{‘}\text{Ɑ}\text{“}\lambda \subset \text{Ɑ}\text{‘}S . \supset .$

$$(S \| \breve{S}) \upharpoonright \lambda \in 1 \to 1 \qquad (1)$$

$\vdash . (1) . *40{\cdot}57 . *150{\cdot}141 . \supset \vdash . \mathrm{Prop}$

$*150{\cdot}153.\ \vdash :. S \upharpoonright s\text{‘}C\text{“}\lambda \in \mathrm{Cls} \to 1 . s\text{‘}C\text{“}\lambda \subset \text{Ɑ}\text{‘}S . Q, R \in \lambda . \supset :$
$S;Q = S;R . \supset . Q = R$

Dem.

$\vdash . *150{\cdot}152 . *71{\cdot}55 . \supset \vdash :. \mathrm{Hp} . \supset : S\dagger\text{‘}Q = S\dagger\text{‘}R . \supset . Q = R :$

$[*150{\cdot}1] \qquad \supset : S;Q = S;R . \supset . Q = R :. \supset \vdash . \mathrm{Prop}$

The above proposition is used in dealing with relations of relations of couples ($*165{\cdot}23$).

$*150{\cdot}16.\ \vdash . \dot{s}\text{‘}R\dagger\text{“}\lambda = R\dagger(\dot{s}\text{‘}\lambda) = R;\dot{s}\text{‘}\lambda \quad \left[*43{\cdot}43 \dfrac{\breve{R}}{S} . *150{\cdot}141{\cdot}1\right]$

The following proposition is a lemma for $*150{\cdot}171$.

$*150{\cdot}17.\ \vdash . (R \upharpoonright \lambda)\dagger = R\dagger \,|\, \lceil \lambda$

Dem.

$\vdash . *150{\cdot}1 . \supset \vdash . (R \upharpoonright \lambda)\dagger\text{‘}P = (R \upharpoonright \lambda) \,|\, P \,|\, (\lambda \upharpoonleft \breve{R})$

$[*35{\cdot}354] \qquad = R \,|\, \lambda \upharpoonleft P \upharpoonright \lambda \,|\, \breve{R}$

$[*150{\cdot}1 . *36{\cdot}11] \qquad = R\dagger(P \lceil \lambda)$

$[*38{\cdot}11] \qquad = R\dagger\text{‘}\lceil \lambda\text{‘}P \qquad (1)$

$\vdash . (1) . *34{\cdot}42 . \supset \vdash . \mathrm{Prop}$

$*150{\cdot}171.\ \vdash : s\text{‘}C\text{“}C\text{‘}Q \subset \lambda . \supset . (R \upharpoonright \lambda)\dagger;Q = R\dagger;Q . \lceil \lambda;Q = Q$

Dem.

$\vdash . *150{\cdot}17{\cdot}13 . \supset \vdash . (R \upharpoonright \lambda)\dagger;Q = R\dagger;\lceil \lambda;Q \qquad (1)$

$\vdash . *150{\cdot}11 . \quad \supset \vdash : M(\lceil \lambda;Q) N . \equiv . (\exists S, T) . M = S \lceil \lambda . N = T \lceil \lambda . SQT \qquad (2)$

$\vdash . *33{\cdot}17 . \quad \supset \vdash : SQT . \supset . S, T \in C\text{‘}Q .$

$[*37{\cdot}62] \qquad \supset . C\text{‘}S, C\text{‘}T \in C\text{“}C\text{‘}Q .$

$[*40{\cdot}13] \qquad \supset . C\text{‘}S \subset s\text{‘}C\text{“}C\text{‘}Q . C\text{‘}T \subset s\text{‘}C\text{“}C\text{‘}Q \qquad (3)$

$\vdash . (3) . \quad \supset \vdash :. \mathrm{Hp} . \supset : SQT . \supset . C\text{‘}S \subset \lambda . C\text{‘}T \subset \lambda .$

$[*36{\cdot}25] \qquad \supset . S \lceil \lambda = S . T \lceil \lambda = T \qquad (4)$

$\vdash . (2) . (4) . \supset \vdash :. \mathrm{Hp} . \supset : M(\lceil \lambda;Q) N . \equiv . (\exists S, T) . M = S . N = T . SQT .$

$[*13{\cdot}22] \qquad \equiv . MQT :$

$[*21{\cdot}43] \qquad \supset : \lceil \lambda;Q = Q \qquad (5)$

$\vdash . (1) . (5) . \supset \vdash . \mathrm{Prop}$

The above proposition is required in the theory of double ordinal similarity. It is used in proving $*164\cdot141$, which is used in $*164\cdot18$, which is a fundamental proposition in the theory of double ordinal similarity.

The following propositions, on the domain, converse domain and field of $S\mathbin{;}Q$, are much used, especially $*150\cdot202\cdot22\cdot23$. $*150\cdot201$ is hardly ever used, but is inserted in order that the general case may not remain unconsidered.

$*150\cdot2$. $\vdash . \mathrm{D}\text{‘}S\mathbin{;}Q = S\text{“}Q\text{“}\text{Ɑ}\text{‘}S . \text{Ɑ}\text{‘}S\mathbin{;}Q = S\text{“}\breve{Q}\text{“}\text{Ɑ}\text{‘}S$ [$*37\cdot32 . *150\cdot1$]

$*150\cdot201$. $\vdash . C\text{‘}S\mathbin{;}Q = S\text{“}(Q \cup \breve{Q})\text{“}\text{Ɑ}\text{‘}S = \mathrm{D}\text{‘}S\mathbin{;}(Q \cup \breve{Q})$

Dem.

$\vdash . *150\cdot2 . *37\cdot22 . \supset \vdash . C\text{‘}S\mathbin{;}Q = S\text{“}(Q\text{“}\text{Ɑ}\text{‘}S \cup \breve{Q}\text{“}\text{Ɑ}\text{‘}S)$

$[*37\cdot221] \qquad = S\text{“}(Q \cup \breve{Q})\text{“}\text{Ɑ}\text{‘}S$

$[*150\cdot2] \qquad = \mathrm{D}\text{‘}S\mathbin{;}(Q \cup \breve{Q}) . \supset \vdash . \text{Prop}$

$*150\cdot202$. $\vdash . \mathrm{D}\text{‘}S\mathbin{;}Q \subset S\text{“}\mathrm{D}\text{‘}Q . \text{Ɑ}\text{‘}S\mathbin{;}Q \subset S\text{“}\text{Ɑ}\text{‘}Q . C\text{‘}S\mathbin{;}Q \subset S\text{“}C\text{‘}Q$

Dem.

$\vdash . *37\cdot15\cdot16 . \supset \vdash . Q\text{“}\text{Ɑ}\text{‘}S \subset \mathrm{D}\text{‘}Q . \breve{Q}\text{“}\text{Ɑ}\text{‘}S \subset \text{Ɑ}\text{‘}Q .$

$[*37\cdot2 . *150\cdot2] \supset \vdash . \mathrm{D}\text{‘}S\mathbin{;}Q \subset S\text{“}\mathrm{D}\text{‘}Q . \text{Ɑ}\text{‘}S\mathbin{;}Q \subset S\text{“}\text{Ɑ}\text{‘}Q \qquad (1)$

$[*37\cdot22] \quad \supset \vdash . C\text{‘}S\mathbin{;}Q \subset S\text{“}C\text{‘}Q \qquad (2)$

$\vdash . (1) . (2) . \quad \supset \vdash . \text{Prop}$

$*150\cdot203$. $\vdash . C\text{‘}S\mathbin{;}Q \subset \mathrm{D}\text{‘}S$ [$*150\cdot202 . *37\cdot15$]

$*150\cdot21$. $\vdash : \text{Ɑ}\text{‘}Q \subset \text{Ɑ}\text{‘}S . \supset . \mathrm{D}\text{‘}S\mathbin{;}Q = S\text{“}\mathrm{D}\text{‘}Q = \mathrm{D}\text{‘}(S \mid Q)$ [$*150\cdot2 . *37\cdot27\cdot32$]

$*150\cdot211$. $\vdash : \mathrm{D}\text{‘}Q \subset \text{Ɑ}\text{‘}S . \supset . \text{Ɑ}\text{‘}S\mathbin{;}Q = S\text{“}\text{Ɑ}\text{‘}Q = \mathrm{D}\text{‘}(S \mid \breve{Q})$
[$*150\cdot2 . *37\cdot271\cdot32$]

$*150\cdot22$. $\vdash : C\text{‘}Q \subset \text{Ɑ}\text{‘}S . \supset . C\text{‘}S\mathbin{;}Q = S\text{“}C\text{‘}Q$ [$*150\cdot21\cdot211 . *37\cdot22$]

In practice, when $S\mathbin{;}Q$ is used, we almost always have $C\text{‘}Q \subset \text{Ɑ}\text{‘}S$. For the use of $S\mathbin{;}Q$ is to obtain a relation analogous to Q and having a different field; now $S\mathbin{;}Q$ is analogous to $Q \upharpoonright \text{Ɑ}\text{‘}S$, for the part of $C\text{‘}Q$ which lies outside $\text{Ɑ}\text{‘}S$ is unaffected by S. Hence if we have, to start with, a relation Q whose field is *not* contained in $\text{Ɑ}\text{‘}S$, we shall usually find it profitable to limit the field to $\text{Ɑ}\text{‘}S$, and consider the transformed relation rather as $S\mathbin{;}(Q \upharpoonright \text{Ɑ}\text{‘}S)$ than as $S\mathbin{;}Q$. Thus the hypothesis $C\text{‘}Q \subset \text{Ɑ}\text{‘}S$ will be verified in almost all useful applications of the notion of $S\mathbin{;}Q$.

$*150\cdot23$. $\vdash : C\text{‘}Q = \text{Ɑ}\text{‘}S . \supset . C\text{‘}S\mathbin{;}Q = \mathrm{D}\text{‘}S$ [$*150\cdot22 . *37\cdot25$]

$*150\cdot24$. $\vdash :. C\text{‘}Q \subset \text{Ɑ}\text{‘}S . \supset : \dot{\exists} ! S\mathbin{;}Q . \equiv . \dot{\exists} ! Q$

Dem.

$\vdash . *37\cdot43 . *150\cdot22 . \supset \vdash :. \text{Hp} . \supset : \exists ! C\text{‘}S\mathbin{;}Q . \equiv . \exists ! C\text{‘}Q :$

$[*33\cdot24] \qquad \supset : \dot{\exists} ! S\mathbin{;}Q . \equiv . \dot{\exists} ! Q :. \supset \vdash . \text{Prop}$

***150·25.** $\vdash :. (y) . E ! S`y . \supset : \dot{\exists} ! S;Q . \equiv . \dot{\exists} ! Q$ [*150·24 . *33·431]

***150·3.** $\vdash . S;(Q \mathbin{\dot{\cup}} R) = S;Q \mathbin{\dot{\cup}} S;R$ [*34·25·26 . *150·1]

***150·301.** $\vdash . S;(Q \mathbin{\dot{\cap}} R) \mathbin{\dot{\subset}} (S;Q) \mathbin{\dot{\cap}} (S;R)$ [*34·23·24 . *150·1]

***150·31.** $\vdash : P \mathbin{\dot{\subset}} Q . R \mathbin{\dot{\subset}} S . \supset . R;P \mathbin{\dot{\subset}} S;Q$ [*34·34 . *150·1]

The following propositions are frequently useful when we have to deal with correlators of the form $S \upharpoonright C`Q$, which often happens.

***150·32.** $\vdash . (S \upharpoonright C`Q);Q = S;Q$ $\left[ *43{\cdot}5 \dfrac{S, \breve{S}, C`Q, C`Q}{Q, R, \alpha, \beta} . *150{\cdot}141 \right]$

***150·33.** $\vdash : C`Q \subset \beta . \supset . (S \upharpoonright \beta);Q = S;Q$ $\left[*43{\cdot}5 \dfrac{S, \breve{S}}{Q, R} . *150{\cdot}141 \right]$

***150·34.** $\vdash : D`Q \subset \alpha . \mathrm{G}`Q \subset \beta . \supset . S \upharpoonright \alpha \,|\, Q \,|\, \beta \upharpoonleft \breve{S} = S;\alpha \upharpoonleft Q \upharpoonright \beta = S;Q$
[*43·5 . *35·354 . *150·141·1]

***150·35.** $\vdash :. y \in C`Q . \supset_y . R`y = S`y : \supset . R;Q = S;Q$

Dem.

$\vdash . *35{\cdot}71 . \supset \vdash : \mathrm{Hp} . \supset . R \upharpoonright C`Q = S \upharpoonright C`Q .$
$[*34{\cdot}27{\cdot}28 . *150{\cdot}1] \quad \supset . (R \upharpoonright C`Q);Q = (S \upharpoonright C`Q);Q .$
$[*150{\cdot}32] \quad \supset . R;Q = S;Q : \supset \vdash . \mathrm{Prop}$

The above proposition, which is the analogue of *37·69, is much used in relation-arithmetic.

The following proposition is much used after we reach the theory of well-ordered series, but not before (except in *150·37).

***150·36.** $\vdash . (S \upharpoonright \beta);Q = S;(Q \mathbin{\underline{\upharpoonright}} \beta)$

Dem.

$\vdash . *150{\cdot}11 . *35{\cdot}101 . \supset$
$\vdash : x \{(S \upharpoonright \beta);Q\} w . \equiv . (\exists y, z) . xSy . y \in \beta . yQz . z \in \beta . wSz .$
$[*36{\cdot}13] \quad \equiv . (\exists y, z) . xSy . y (Q \mathbin{\underline{\upharpoonright}} \beta) z . wSz .$
$[*150{\cdot}11] \quad \equiv . x \{S;(Q \mathbin{\underline{\upharpoonright}} \beta)\} w : \supset \vdash . \mathrm{Prop}$

***150·361.** $\vdash . (\alpha \upharpoonleft S);Q = (S;Q) \mathbin{\underline{\upharpoonright}} \alpha$ [Proof as in *150·36]

***150·37.** $\vdash : S \in \mathrm{Cls} \to 1 . \supset . S;(Q \mathbin{\underline{\upharpoonright}} \beta) = (S;Q) \mathbin{\underline{\upharpoonright}} S``\beta$
$= (S \upharpoonright \beta);Q = \{(S``\beta) \upharpoonleft S\};Q$

Dem.

$\vdash . *74{\cdot}141 . \supset \vdash : \mathrm{Hp} . \supset . (S \upharpoonright \beta);Q = \{(S``\beta) \upharpoonleft S\};Q \quad (1)$
$\vdash . (1) . *150{\cdot}36{\cdot}361 . \supset \vdash . \mathrm{Prop}$

The above proposition is not used until we reach the theory of series.

***150·38.** $\vdash : S \in 1 \to 1 . \supset . S ; \breve{S} ; Q = Q \upharpoonright \mathrm{D}'S$

Dem.

$$\begin{aligned} &\vdash . *150{\cdot}1 . &&\supset \vdash . S;\breve{S};Q = S \,|\, \breve{S} \,|\, Q \,|\, S \,|\, \breve{S} &(1)\\ &\vdash . (1) . *72{\cdot}59{\cdot}591 . \supset \vdash : \mathrm{Hp} . \supset . &&S;\breve{S};Q = (\mathrm{D}'S) \upharpoonleft Q \upharpoonright \mathrm{D}'S \\ &[*36{\cdot}11] && = Q \upharpoonright \mathrm{D}'S : \supset \vdash . \mathrm{Prop} \end{aligned}$$

The above proposition is used in dealing with the correlation of series (*208·2).

***150·4.** $\vdash :. S \in 1 \to \mathrm{Cls} . \supset : x (S;Q) y . \equiv . (\exists z, w) . x = S'z . y = S'w . zQw$
[*150·11 . *71·36]

This proposition is fundamental in the theory of $S;Q$, because in most of the uses of this notion S is one-many. The proposition states that when S is one-many, $S;Q$ is the relation between the S's of terms related by Q. Thus if S is the relation of wife to husband, and Q is the relation of brother to brother, $S;Q$ is the relation between wives of brothers. If Q is a relation between relations, $C;P$ will be the corresponding relation of their fields; and so on.

***150·41.** $\vdash :. S \in \mathrm{Cls} \to 1 . \supset : x (S;Q) y . \equiv . (\breve{S}'x) \, Q \, (\breve{S}'y)$ [*150·11 . *71·331]

***150·42.** $\vdash . S;\dot{\Lambda} = \dot{\Lambda}$ [*150·1 . *34·32]

The following propositions, down to *150·56, are, with the exception of *150·52—·535, all illustrations of *150·4·41.

***150·5.** $\vdash : \alpha (\overrightarrow{R};P) \beta . \equiv . (\exists x, y) . \alpha = \overrightarrow{R}'x . \beta = \overrightarrow{R}'y . xPy$

***150·51.** $\vdash : \alpha (\mathrm{D};R) \beta . \equiv . (\exists P, Q) . \alpha = \mathrm{D}'P . \beta = \mathrm{D}'Q . PRQ$

***150·511.** $\vdash : \alpha (\mathrm{G};R) \beta . \equiv . (\exists P, Q) . \alpha = \mathrm{G}'P . \beta = \mathrm{G}'Q . PRQ$

***150·512.** $\vdash : \alpha (C;R) \beta . \equiv . (\exists P, Q) . \alpha = C'R . \beta = C'Q . PRQ$

***150·52.** $\vdash : x (F;R) y . \equiv . (\exists P, Q) . x \in C'P . y \in C'Q . PRQ$
[*150·11 . *33·51]

$F;R$ is a relation which plays a great part in relation-arithmetic.

***150·53.** $\vdash . I;P = P$ [*50·4]

***150·531.** $\vdash . P;I = P \,|\, \breve{P}$ [*50·4]

***150·532.** $\vdash . P;I;Q = P;Q$ [*150·13 . *50·4]

***150·534.** $\vdash . (I \upharpoonright C'P);P = P$ [*150·53·32]

***150·535.** $\vdash : C'P \subset \alpha . \supset . (I \upharpoonright \alpha);P = P$ [*150·53·33]

***150·54.** $\vdash : \alpha (\iota;R) \beta . \equiv . (\breve{\iota}'\alpha) \, R \, (\breve{\iota}'\beta)$

***150·541.** $\vdash : x (\breve{\iota};R) y . \equiv . (\iota'x) \, R \, (\iota'y)$

***150·55.** $\vdash : Q (\downarrow z;P) R . \equiv . (\exists u, v) . Q = u \downarrow z . R = v \downarrow z . uPv$

$*150\cdot56$. $\vdash : M(S\dagger;Q)N . \equiv . (\exists X, Y) . XQY . M = S;X . N = S;Y$
$[*150\cdot4\cdot15\cdot1]$

$*150\cdot6$. $\vdash . P \underset{\cdot,}{♀} y = ♀ y;P$ $[(*150\cdot03)]$

$*150\cdot601$. $\vdash . P \underset{\cdot,}{♀} \epsilon 1 \to \mathrm{Cls}$ $[*150\cdot6 . *14\cdot21 . *71\cdot166]$

$*150\cdot61$. $\vdash : z(P \underset{\cdot,}{♀} y) w . \equiv . (\exists u, v) . z = u ♀ y . w = v ♀ y . uPv$
$[*150\cdot11 . *38\cdot101 . *150\cdot6]$

$*150\cdot62$. $\vdash : R(P \underset{\cdot,}{♀};Q) S . \equiv . (\exists z, w) . R = ♀ z;P . S = ♀ w;P . zQw$
$[*150\cdot4\cdot601\cdot6]$

Relations of the form $P \underset{\cdot,}{♀};Q$ are frequently useful in relation-arithmetic, especially in the particular case of $P \underset{\cdot,}{\downarrow};Q$, which takes the place taken by $\alpha \underset{,,}{\downarrow} {}``\beta$ in cardinal arithmetic. Relations of the form $P \underset{\cdot,}{\downarrow};Q$ will be considered in $*165$.

The following propositions are chiefly concerned with correlations of couples. They are of great utility in relation-arithmetic. $*150\cdot71$, in particular, is fundamental.

$*150\cdot7$. $\vdash . S;(z \downarrow w) = \overrightarrow{S}`z \uparrow \overrightarrow{S}`w$ $[*55\cdot6]$

$*150\cdot71$. $\vdash : S \epsilon 1 \to \mathrm{Cls} . z, w \epsilon \mathrm{C\!\!I}`S . \supset . S;(z \downarrow w) = (S`z) \downarrow (S`w)$ $[*55\cdot61]$

$*150\cdot72$. $\vdash : z \neq w . S = x \downarrow z \cup y \downarrow w . \supset . S;(z \downarrow w) = x \downarrow y$ $[*55\cdot62\cdot61]$

$*150\cdot73$. $\vdash . S;(\alpha \uparrow \beta) = S``\alpha \uparrow S``\beta$ $\left[*37\cdot82 \dfrac{S, \breve{S}}{R, S}\right]$

$*150\cdot74$. $\vdash . (S \cup T);Q = S;Q \cup T;Q \cup S | Q | \breve{T} \cup T | Q | \breve{S}$ $[*150\cdot1]$

$*150\cdot75$. $\vdash : \sim(yQy) . \supset . (S \cup x \downarrow y);Q = S;Q \cup S``\overrightarrow{Q}`y \uparrow \iota`x \cup \iota`x \uparrow S``\overleftarrow{Q}`y$

Dem.

$\vdash . *150\cdot1 . \supset \vdash : \mathrm{Hp} . \supset . (x \downarrow y);Q = \dot{\Lambda} .$

$[*150\cdot74]$ $\supset . (S \cup x \downarrow y);Q = S;Q \cup S | Q | y \downarrow x \cup x \downarrow y | Q | \breve{S}$

$[*55\cdot57\cdot571]$ $= S;Q \cup S``\overrightarrow{Q}`y \uparrow \iota`x \cup \iota`x \uparrow S``\overleftarrow{Q}`y : \supset \vdash . \mathrm{Prop}$

The four following propositions belong to the subject of $*92$, but could not be given in that number owing to the fact that they involve the notations of $*150$. They are required for proving that, if S is a correlator of P and Q, it is also a correlator of P_{po} and Q_{po} ($*151\cdot45$), and for one of the fundamental propositions in the ordinal theory of progressions ($*263\cdot17$).

$*150\cdot8$. $\vdash :. S \epsilon \mathrm{Cls} \to 1 : \mathrm{D}`Q \subset \mathrm{C\!\!I}`S . \vee . \mathrm{C\!\!I}`Q \subset \mathrm{C\!\!I}`S : P \epsilon \mathrm{Pot}`Q : \supset .$
$S;P \epsilon \mathrm{Pot}`(S;Q) . (S;P) | (S;Q) = S;(P | Q)$

Dem.

$\vdash . *91\cdot351 .$ $\supset \vdash . S;Q \epsilon \mathrm{Pot}`(S;Q)$ (1)

$\vdash . *150\cdot1 .$ $\supset \vdash . (S;P) | (S;Q) = S | P | \breve{S} | S | Q | \breve{S}$ (2)

$\vdash . (2) . *71\cdot191 . \quad \supset \vdash : \mathrm{Hp} . \supset . (S\,;P) \mid (S\,;Q) = S \mid P \mid I \upharpoonright \text{Ɑ}'S \mid Q \mid \breve{S}$ (3)

$\vdash . *50\cdot63 . \quad \supset \vdash : \mathrm{D}'Q \subset \text{Ɑ}'S . \supset . I \upharpoonright \text{Ɑ}'S \mid Q = Q$ (4)

$\vdash . *50\cdot62 . *91\cdot271 . \supset \vdash : P \in \mathrm{Pot}'Q . \text{Ɑ}'Q \subset \text{Ɑ}'S . \supset . P \mid I \upharpoonright \text{Ɑ}'S = P$ (5)

$\vdash . (3) . (4) . (5) . \quad \supset \vdash : \mathrm{Hp} . P \in \mathrm{Pot}'Q . \supset . (S\,;P) \mid (S\,;Q) = S \mid P \mid Q \mid \breve{S}$

[$*150\cdot1$] $\qquad = S\,;(P \mid Q)$ (6)

$\vdash . *91\cdot282 . \quad \supset \vdash : S\,;P \in \mathrm{Pot}'S\,;Q . \supset . (S\,;P) \mid (S\,;Q) \in \mathrm{Pot}'S\,;Q$ (7)

$\vdash . (6) . (7) . \quad \supset \vdash :. \mathrm{Hp} . P \in \mathrm{Pot}'Q . \supset : S\,;P \in \mathrm{Pot}'S\,;Q . \supset .$

$\qquad S\,;(P \mid Q) \in \mathrm{Pot}'S\,;Q$ (8)

$\vdash . (1) . (8) . *91\cdot373 \dfrac{S\,;P \in \mathrm{Pot}'S\,;Q}{\phi P} . \supset$

$\vdash :. \mathrm{Hp} . \supset : P \in \mathrm{Pot}'Q . \supset_P . S\,;P \in \mathrm{Pot}'(S\,;Q)$ (9)

$\vdash . (6) . (9) . \supset \vdash . \mathrm{Prop}$

$*150\cdot81$. $\vdash :. S \in \mathrm{Cls} \to 1 : \mathrm{D}'Q \subset \text{Ɑ}'S . \mathrm{v} . \text{Ɑ}'Q \subset \text{Ɑ}'S : T \in \mathrm{Pot}'S\,;Q : \supset .$

$\qquad (\exists P) . P \in \mathrm{Pot}'Q . T = S\,;P$

Dem.

$\vdash . *91\cdot351 . \supset \vdash . (\exists P) . P \in \mathrm{Pot}'Q . S\,;Q = S\,;P$ (1)

$\vdash . *150\cdot8 . \supset \vdash : \mathrm{Hp} . P \in \mathrm{Pot}'Q . T = S\,;P . \supset . T \mid (S\,;Q) = S\,;(P \mid Q) .$

[$*91\cdot282$] $\qquad \supset . (\exists R) . R \in \mathrm{Pot}'Q . T \mid (S\,;Q) = S\,;R$ (2)

$\vdash . (2) . *10\cdot23 . \supset$

$\vdash :. \mathrm{Hp} . \supset : (\exists P) . P \in \mathrm{Pot}'Q . T = S\,;P . \supset .$

$\qquad (\exists R) . R \in \mathrm{Pot}'Q . T \mid (S\,;Q) = S\,;R$ (3)

$\vdash . (1) . (3) . *91\cdot171 \dfrac{S\,;Q, \; T, \; (\exists P) . P \in \mathrm{Pot}'Q . T = S\,;P}{R, \quad S, \qquad \phi T} . \supset$

$\vdash : \mathrm{Hp} . T \in \mathrm{Pot}'S\,;Q . \supset . (\exists P) . P \in \mathrm{Pot}'Q . T = S\,;P : \supset \vdash . \mathrm{Prop}$

$*150\cdot82$. $\vdash :. S \in \mathrm{Cls} \to 1 : \mathrm{D}'Q \subset \text{Ɑ}'S . \mathrm{v} . \text{Ɑ}'Q \subset \text{Ɑ}'S : \supset . \mathrm{Pot}'S\,;Q = S\dagger``\mathrm{Pot}'Q$

Dem.

$\vdash . *150\cdot8\cdot81 . \supset$

$\vdash : \mathrm{Hp} . \supset . \mathrm{Pot}'S\,;Q = \hat{T}\{(\exists P) . P \in \mathrm{Pot}'Q . T = S\,;P\}$

[$*150\cdot1$] $\qquad = S\dagger``\mathrm{Pot}'Q : \supset \vdash . \mathrm{Prop}$

$*150\cdot83$. $\vdash :. S \in \mathrm{Cls} \to 1 : \mathrm{D}'Q \subset \text{Ɑ}'S . \mathrm{v} . \text{Ɑ}'Q \subset \text{Ɑ}'S : \supset . (S\,;Q)_{\mathrm{po}} = S\,;Q_{\mathrm{po}}$

Dem.

$\vdash . *150\cdot82 . (*91\cdot05) . \supset \vdash : \mathrm{Hp} . \supset . (S\,;Q)_{\mathrm{po}} = \dot{s}'S\dagger``\mathrm{Pot}'Q$

[$*150\cdot16 . (*91\cdot05)$] $\qquad = S\,;Q_{\mathrm{po}} : \supset \vdash . \mathrm{Prop}$

The following propositions, down to $*150\cdot94$ inclusive, resume the subject of the relation $S\dagger$, which has already been treated in $*150\cdot14$—$\cdot171$.

$*150\cdot9$. $\vdash . (I\dagger)\,;Q = Q$

Dem.

$\vdash . *150\cdot56 . \supset \vdash : M (I\dagger\,;Q) N . \equiv . (\exists X, Y) . XQY . M = I\,;X . N = I\,;Y .$

[$*150\cdot53$] $\qquad \equiv . (\exists X, Y) . XQY . M = X . N = Y .$

[$*13\cdot22$] $\qquad \equiv . MQY : \supset \vdash . \mathrm{Prop}$

The following propositions lead up to $*150\cdot931\cdot94$, which are used in the theory of double ordinal similarity ($*164\cdot3\cdot21$).

$*150\cdot91$. $\vdash : s\text{‘}C\text{“}C\text{‘}Q \subset \alpha . \supset . (I \upharpoonright \alpha)\dagger;Q = Q$

Dem.

$\vdash . *150\cdot535 . \quad \supset \vdash :. \mathrm{Hp} . \supset : X \in C\text{‘}Q . \supset . (I \upharpoonright \alpha);X = X \quad (1)$

$\vdash . (1) . *150\cdot56 . \supset \vdash :. \mathrm{Hp} . \supset :$

$M \{(I \upharpoonright \alpha)\dagger;Q\} N . \equiv . (\exists X, Y) . XQY . M = X . N = Y .$

$[*13\cdot22] \quad \equiv . MQN :. \supset \vdash . \mathrm{Prop}$

$*150\cdot92$. $\vdash : S \in \mathrm{Cls} \to 1 . s\text{‘}C\text{“}C\text{‘}Q \subset \text{\reflectbox{D}}\text{‘}S . \supset . \breve{S}\dagger;S\dagger;Q = Q$

Dem.

$\vdash . *150\cdot13\cdot14 . \quad \supset \vdash . \breve{S}\dagger;S\dagger;Q = (\breve{S} \,|\, S)\dagger;Q \quad (1)$

$\vdash . (1) . *71\cdot191 . \supset \vdash : \mathrm{Hp} . \supset . \breve{S}\dagger;S\dagger;Q = (I \upharpoonright \text{\reflectbox{D}}\text{‘}S)\dagger;Q$

$[*150\cdot91] \quad = Q : \supset \vdash . \mathrm{Prop}$

$*150\cdot921$. $\vdash : S \in 1 \to \mathrm{Cls} . s\text{‘}C\text{“}C\text{‘}P \subset \mathrm{D}\text{‘}S . \supset . S\dagger;\breve{S}\dagger;P = P$

$*150\cdot93$. $\vdash :. S \in 1 \to 1 . s\text{‘}C\text{“}C\text{‘}P \subset \mathrm{D}\text{‘}S . s\text{‘}C\text{“}C\text{‘}Q \subset \text{\reflectbox{D}}\text{‘}S . \supset :$

$P = S\dagger;Q . \equiv . Q = \breve{S}\dagger;P \quad [*150\cdot92\cdot921]$

$*150\cdot931$. $\vdash : s\text{‘}C\text{“}C\text{‘}Q \subset \text{\reflectbox{D}}\text{‘}S . \supset . C\text{“}C\text{‘}S\dagger;Q = S_\epsilon\text{“}C\text{“}C\text{‘}Q$

Dem.

$\vdash . *150\cdot22 . \quad \supset \vdash . C\text{‘}S\dagger;Q = S\dagger\text{“}C\text{‘}Q \quad (1)$

$\vdash . *150\cdot22\cdot1 . \supset \vdash :. \mathrm{Hp} . \supset : M \in C\text{‘}Q . \supset . C\text{‘}S\dagger\text{‘}M = S\text{“}C\text{‘}M :$

$[*37\cdot68\cdot11] \quad \supset : C\text{“}S\dagger\text{“}C\text{‘}Q = S_\epsilon\text{“}C\text{“}C\text{‘}Q \quad (2)$

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

$*150\cdot932$. $\vdash : s\text{‘}C\text{“}C\text{‘}Q \subset \text{\reflectbox{D}}\text{‘}S . \supset . s\text{‘}C\text{“}C\text{‘}S\dagger;Q = S\text{“}s\text{‘}C\text{“}C\text{‘}Q$

$[*150\cdot931 . *37\cdot11 . *40\cdot38]$

$*150\cdot933$. $\vdash : s\text{‘}C\text{“}C\text{‘}Q \subset \text{\reflectbox{D}}\text{‘}S . \supset . s\text{‘}C\text{“}C\text{‘}S\dagger;Q \subset \mathrm{D}\text{‘}S \quad [*150\cdot932 . *37\cdot15]$

$*150\cdot94$. $\vdash :. S \in 1 \to 1 . \supset : s\text{‘}C\text{“}C\text{‘}Q \subset \text{\reflectbox{D}}\text{‘}S . P = S\dagger;Q . \equiv .$

$s\text{‘}C\text{“}C\text{‘}P \subset \mathrm{D}\text{‘}S . Q = \breve{S}\dagger;P$

Dem.

$\vdash . *150\cdot933 . \quad \supset \vdash : s\text{‘}C\text{“}C\text{‘}Q \subset \text{\reflectbox{D}}\text{‘}S . P = S\dagger;Q . \equiv .$

$s\text{‘}C\text{“}C\text{‘}P \subset \mathrm{D}\text{‘}S . s\text{‘}C\text{“}C\text{‘}Q \subset \text{\reflectbox{D}}\text{‘}S . P = S\dagger;Q \quad (1)$

$\vdash . *150\cdot933 \dfrac{\breve{S}}{S} . \supset \vdash : s\text{‘}C\text{“}C\text{‘}P \subset \mathrm{D}\text{‘}S . Q = \breve{S}\dagger;P . \equiv .$

$s\text{‘}C\text{“}C\text{‘}P \subset \mathrm{D}\text{‘}S . s\text{‘}C\text{“}C\text{‘}Q \subset \text{\reflectbox{D}}\text{‘}S . Q = \breve{S}\dagger;P \quad (2)$

$\vdash . *150\cdot93 . *5\cdot32 . \supset$

$\vdash :. S \in 1 \to 1 . \supset : s\text{‘}C\text{“}C\text{‘}P \subset \mathrm{D}\text{‘}S . s\text{‘}C\text{“}C\text{‘}Q \subset \text{\reflectbox{D}}\text{‘}S . P = S\dagger;Q . \equiv .$

$s\text{‘}C\text{“}C\text{‘}P \subset \mathrm{D}\text{‘}S . s\text{‘}C\text{“}C\text{‘}Q \subset \text{\reflectbox{D}}\text{‘}S . Q = \breve{S}\dagger;P \quad (3)$

$\vdash . (1) . (2) . (3) . \supset \vdash . \mathrm{Prop}$

The above proposition is the analogue of $*74\cdot61$, which (with a few trivial transformations) may be written

$$\vdash :. S \in 1 \to 1 . \supset : s\text{‘}\lambda \subset \text{\reflectbox{D}}\text{‘}S . \kappa = S_\epsilon\text{“}\lambda . \equiv . s\text{‘}\kappa \subset \mathrm{D}\text{‘}S . \lambda = (\breve{S})_\epsilon\text{“}\kappa .$$

In obtaining ordinal analogues of such propositions, S_ϵ will be replaced by $S\dagger$, and the two inverted commas will be replaced by the semi-colon; a class of classes κ will be replaced, in most of its occurrences, by a relation of relations P, but will sometimes be replaced by $C``C`P$.

The above proposition (*150·94) is used in proving that the converse of a double correlator of P and Q is a double correlator of Q and P (*164·21). The corresponding cardinal proposition (*111·131) uses *74·6, which is practically the same proposition as *74·61, which is the analogue of *150·94.

***150·95.** $\vdash : C`R \subset \mathrm{Cl}`\alpha \,.\supset .\, (S \upharpoonright \alpha)_\epsilon ; R = S;R$

Dem.

$$\begin{array}{ll} \vdash .\, *37\cdot421 \,.\supset \vdash :.\, \mathrm{Hp} \,.\supset : \beta \in C`R \,.\supset .\, (S\upharpoonright\alpha)``\beta = S``\beta \,. & \\ [*37\cdot11] \qquad\qquad \supset .\, (S\upharpoonright\alpha)_\epsilon`\beta = S_\epsilon`\beta & (1) \\ \vdash .\, (1) \,.\, *150\cdot35 \,.\supset \vdash .\, \mathrm{Prop} & \end{array}$$

The above proposition is used in the theory of "first differences" (*170·41).

***150·96.** $\vdash : \mathrm{C}\!\mathrm{I}`\dot{s}`\lambda \subset \mathrm{C}\!\mathrm{I}`S \,.\supset .\, \mathrm{D};(T \,\|\, \breve{S}) \upharpoonright \lambda = T_\epsilon \upharpoonright \mathrm{D}``\lambda$

Dem.

$$\begin{array}{ll} \vdash .\, *150\cdot51 \,.\supset \vdash : \alpha \{\mathrm{D};(T\,\|\,\breve{S})\upharpoonright\lambda\}\, \beta \,.\equiv . & \\ \qquad (\exists M, N) \,.\, N \in \lambda \,.\, M = T \,|\, N \,|\, \breve{S} \,.\, \alpha = \mathrm{D}`M \,.\, \beta = \mathrm{D}`N & (1) \\ \vdash .\, *41\cdot13 \,.\supset \vdash :.\, \mathrm{Hp} \,.\supset : N \in \lambda \,.\supset .\, \mathrm{C}\!\mathrm{I}`N \subset \mathrm{C}\!\mathrm{I}`S \,. & \\ [*37\cdot321] \qquad\qquad \supset .\, \mathrm{D}`(N \,|\, \breve{S}) = \mathrm{D}`N \,. & \\ [*37\cdot32] \qquad\qquad \supset .\, \mathrm{D}`(T \,|\, N \,|\, \breve{S}) = T``\mathrm{D}`N & (2) \\ \vdash .\, (1) \,.\, (2) \,.\, *37\cdot6 \,.\supset & \\ \vdash :.\, \mathrm{Hp} \,.\supset : \alpha \{\mathrm{D};(T\,\|\,\breve{S})\upharpoonright\lambda\}\, \beta \,.\equiv .\, \beta \in \mathrm{D}``\lambda \,.\, \alpha = T``\beta \,. & \\ [*37\cdot101] \qquad\qquad \equiv .\, \alpha\, (T_\epsilon \upharpoonright \mathrm{D}``\lambda)\, \beta :.\supset \vdash .\, \mathrm{Prop} & \end{array}$$

***150·961.** $\vdash .\, \dot{s};(U \,\|\, \breve{W})_\epsilon \upharpoonright \lambda = (U \,\|\, \breve{W}) \upharpoonright \dot{s}``\lambda$

Dem.

$$\begin{array}{l} \vdash .\, *150\cdot4 \,.\supset \vdash : R \{\dot{s};(U\,\|\,\breve{W})_\epsilon \upharpoonright \lambda\}\, S \,.\equiv .\, (\exists\beta) \,.\, \beta \in \lambda \,.\, S = \dot{s}`\beta \,.\, R = \dot{s}`(U\,\|\,\breve{W})``\beta \,. \\ [*43\cdot43] \qquad\qquad \equiv .\, (\exists\beta) \,.\, \beta \in \lambda \,.\, S = \dot{s}`\beta \,.\, R = (U\,\|\,\breve{W})`\dot{s}`\beta \,. \\ [*13\cdot193 \,.\, *37\cdot6] \qquad\qquad = .\, S \in \dot{s}``\lambda \,.\, R = (U \,\|\, \breve{W})`S : \supset \vdash .\, \mathrm{Prop} \end{array}$$

The above proposition is used in the theory of ordinal exponentiation (*176·21).

$*151$. ORDINAL SIMILARITY

Summary of $*151$.

In this number, we give the definition of ordinal similarity, and various equivalent forms; we prove that ordinal similarity is reflexive ($*151\cdot13$), symmetrical ($*151\cdot14$) and transitive ($*151\cdot15$), and we give some particular cases of ordinal similarity ($*151\cdot6$ ff.). Propositions in this number should be compared with those in $*73$, to which they are analogous.

The class of ordinal correlators of P and Q is written $P \,\overline{\text{smor}}\, Q$, where "smor" stands for "similar ordinally." We put

$$P \,\overline{\text{smor}}\, Q = \hat{S}\{S \in 1 \rightarrow 1 \,.\, C\text{`}Q = \mathrm{G}\text{`}S \,.\, P = S;Q\} \quad \text{Df.}$$

(We might equally well put

$$P \,\overline{\text{smor}}\, Q = (1 \rightarrow 1) \cap \overleftarrow{\mathrm{G}}\text{`}C\text{`}Q \cap \overleftarrow{\dagger Q}\text{`}P \quad \text{Df,}$$

which is an equivalent but more condensed form of the definition.) We then define "P is ordinally similar to Q" as meaning that there is at least one ordinal correlator of P and Q, *i.e.*

$$\text{smor} = \hat{P}\hat{Q}(\exists \,!\, P \,\overline{\text{smor}}\, Q) \quad \text{Df.}$$

We shall find that if P and Q generate well-ordered series, they have at most one correlator ($*250\cdot6$), but this does not hold in general for other series.

After giving the elementary properties of ordinal similarity, we have three important propositions on its connection with cardinal similarity, namely: ($*151\cdot18$) if P is similar to Q, the field of P is similar to the field of Q (the converse does not hold in general, but holds if P and Q are finite serial relations); ($*151\cdot19$) if $C\text{`}P$ is similar to $C\text{`}Q$, there is a relation R similar to Q and having $C\text{`}P$ for its field, and vice versa; ($*151\cdot191$) $S$ is an ordinal correlator of $P$ and $Q$ when, and only when, it is a cardinal correlator of $C\text{`}P$ and $C\text{`}Q$ and $P = S;Q$.

We then have a set of propositions on correlators of the form $S \upharpoonright C\text{`}Q$ ($*151\cdot2$—$\cdot243$). Most of the correlators with which we shall be concerned are of this form. The most useful proposition here is

$*151\cdot22$. $\vdash : S \upharpoonright C\text{`}Q \in 1 \rightarrow 1 \,.\, C\text{`}Q \subset \mathrm{G}\text{`}S \,.\, P = S;Q \,.\equiv.\, S \upharpoonright C\text{`}Q \in P \,\overline{\text{smor}}\, Q$

A useful consequence of this proposition is

$*151\cdot231$. $\vdash :. (y) \,.\, \mathrm{E}\,!\, S\text{`}y : S \upharpoonright C\text{`}Q \in 1 \rightarrow 1 \,.\, P = S;Q : \supset .\, S \upharpoonright C\text{`}Q \in P \,\overline{\text{smor}}\, Q$

This consequence is useful because the hypothesis $(y) \,.\, \mathrm{E}\,!\, S\text{`}y$ is satisfied by most of the relations which occur as correlators.

We have next a number of propositions on the inferribility of $Q = \breve{S};P$ or $Q \subseteq \breve{S};P$ from $P = S;Q$ or $P \subseteq S;Q$, and connected matters (*151·25—·29). We have

***151·25.** $\vdash : S \in \mathrm{Cls} \to 1 \,.\, C`Q \subset \text{ᗡ}`S \,.\, P = S;Q \,.\supset .\, Q = \breve{S};P$

***151·26.** $\vdash :. S \in \mathrm{Cls} \to 1 \,.\, C`Q \subset \text{ᗡ}`S \,.\supset : P \subseteq S;Q \,.\supset .\, \breve{S};P \subseteq Q :$
$S;Q \subseteq P \,.\supset .\, Q \subseteq \breve{S};P$

***151·29.** $\vdash :. P \,\mathrm{smor}\, Q \,.\equiv : (\exists S) : xPy \,.\supset_{x,y} .\, (\breve{S}`x)\, Q\, (\breve{S}`y) :$
$zQw \,.\supset_{z,w} .\, (S`z)\, P\, (S`w)$

*151·29 is never used, but is inserted in order to show that our definition of "ordinal similarity" agrees with what is commonly understood by that term. If P and Q are regarded as serial, so that "xPy" means "x precedes y in the P-series," and "zQw" means "z precedes w in the Q-series," then our proposition states that two series are ordinally similar when their terms can be so correlated that predecessors in either are correlated with predecessors in the other, and successors with successors, *i.e.* when the two series can be correlated without change of order.

We have next (*151·31—·52) a set of miscellaneous propositions, of which the most useful are

***151·401.** $\vdash : T \upharpoonright C`P \in X \,\overline{\mathrm{smor}}\, P \,.\, T \upharpoonright C`Q \in Y \,\overline{\mathrm{smor}}\, Q \,.\, S \in P \,\overline{\mathrm{smor}}\, Q \,.\supset .$
$T;S \in X \,\overline{\mathrm{smor}}\, Y$

***151·5.** $\vdash : S \upharpoonright C`Q \in P \,\overline{\mathrm{smor}}\, Q \,.\supset .\, \mathrm{D}`P = S``\mathrm{D}`Q \,.\, \text{ᗡ}`P = S``\text{ᗡ}`Q \,.\, \overrightarrow{B}`P = S``\overrightarrow{B}`Q \,.$
$\overrightarrow{B}`\breve{P} = S``\overrightarrow{B}`\breve{Q}$

*151·401 will be useful in such cases as the following: Let P and Q be relations between relations, then $\mathrm{D};P$ and $\mathrm{D};Q$ will be the corresponding relations of their domains. Suppose $\mathrm{D} \upharpoonright C`P$, $\mathrm{D} \upharpoonright C`Q \in 1 \to 1$. Then, by *151·401, if S is a correlator of P and Q, $\mathrm{D};S$ is a correlator of $\mathrm{D};P$ and $\mathrm{D};Q$.

*151·5 shows that if S is a correlator of P and Q, it correlates $\mathrm{D}`P$ with $\mathrm{D}`Q$, $\text{ᗡ}`P$ with $\text{ᗡ}`Q$, $\overrightarrow{B}`P$ with $\overrightarrow{B}`Q$, and $\overrightarrow{B}`\breve{P}$ with $\overrightarrow{B}`\breve{Q}$.

Our next set of propositions (*151·53—·59) is concerned with the correlation of powers of P and Q and kindred matters. We show (*151·55) that a correlator of P and Q is also a correlator of P_{po} and Q_{po}, and therefore if P and Q are similar, so are P_{po} and Q_{po} (*151·56); we show also (*151·59) that if P and Q are similar, so are P_ν and Q_ν. These propositions are used in the theory of progressions (*263·17).

The remaining propositions (*151·6 to the end) are concerned with applications to particular cases. The most useful of these are

***151·61.** $\vdash .\, \iota;P \,\mathrm{smor}\, P$

which shows how to raise the type of a relation without changing its relation-number;

***151·64.** $\vdash . x \downarrow {}^{;}P \operatorname{smor} P . (x \downarrow) \upharpoonright C‘P \in (x \downarrow {}^{;}P) \overline{\operatorname{smor}} P$

***151·65.** $\vdash . \downarrow x {}^{;}P \operatorname{smor} P . (\downarrow x) \upharpoonright C‘P \in (\downarrow x {}^{;}P) \overline{\operatorname{smor}} P$

We prove also that all members of 2_r (*i.e.* all relations of the form $x \downarrow y$, where $x \neq y$) are similar (*151·63), and that all relations of the form $x \downarrow x$ are similar (*151·631).

***151·01.** $P \overline{\operatorname{smor}} Q = \hat{S}\{S \in 1 \to 1 . C‘Q = \text{ᗡ}‘S . P = S {}^{;} Q\}$ Df

***151·02.** $\operatorname{smor} = \hat{P}\hat{Q}\{\exists ! P \overline{\operatorname{smor}} Q\}$ Df

***151·1.** $\vdash : P \operatorname{smor} Q . \equiv . (\exists S) . S \in 1 \to 1 . C‘Q = \text{ᗡ}‘S . P = S {}^{;} Q$ [(*151·02)]

***151·11.** $\vdash : S \in P \overline{\operatorname{smor}} Q . \equiv . S \in 1 \to 1 . C‘Q = \text{ᗡ}‘S . P = S {}^{;} Q$ [(*151·01)]

***151·12.** $\vdash : P \operatorname{smor} Q . \equiv . \exists ! P \overline{\operatorname{smor}} Q$ [(*151·02)]

***151·121.** $\vdash . I \upharpoonright C‘Q \in (Q \overline{\operatorname{smor}} Q)$ [*72·17 . *50·5·52 . *150·534 . *151·11]

***151·13.** $\vdash . Q \operatorname{smor} Q$ [*151·121·12]

***151·131.** $\vdash : S \in P \overline{\operatorname{smor}} Q . \equiv . \breve{S} \in Q \overline{\operatorname{smor}} P$

Dem.

$\vdash . *71·212 . \supset \vdash : S \in 1 \to 1 . \equiv . \breve{S} \in 1 \to 1$ (1)

$\vdash . *150·13 . \supset \vdash : P = S {}^{;} Q . \supset . \breve{S} {}^{;} P = (\breve{S} \mid S) {}^{;} Q :$

$[*71·192] \supset \vdash : S \in 1 \to 1 . P = S {}^{;} Q . \supset . \breve{S} {}^{;} P = (I \upharpoonright \text{ᗡ}‘S) {}^{;} Q :$

$[*150·534] \supset \vdash : S \in 1 \to 1 . C‘Q = \text{ᗡ}‘S . P = S {}^{;} Q . \supset . \breve{S} {}^{;} P = Q$ (2)

$\vdash . *150·23 . \supset \vdash : C‘Q = \text{ᗡ}‘S . P = S {}^{;} Q . \supset . C‘P = \mathrm{D}‘S$ (3)

$\vdash . (1) . (2) . (3) . *33·21 . \supset$

$\vdash : S \in 1 \to 1 . C‘Q = \text{ᗡ}‘S . P = S {}^{;} Q . \supset . \breve{S} \in 1 \to 1 . C‘P = \text{ᗡ}‘\breve{S} . Q = \breve{S} {}^{;} P$ (4)

$\vdash . (4) \dfrac{\breve{S}, Q, P}{S, P, Q} . *31·33 . \supset$

$\vdash : \breve{S} \in 1 \to 1 . C‘P = \text{ᗡ}‘\breve{S} . Q = \breve{S} {}^{;} P . \supset . S \in 1 \to 1 . C‘Q = \text{ᗡ}‘S . P = S {}^{;} Q$ (5)

$\vdash . (4) . (5) . *151·11 . \supset \vdash . \text{Prop}$

***151·14.** $\vdash : P \operatorname{smor} Q . \equiv . Q \operatorname{smor} P$ [*151·131·12 . *31·52]

***151·141.** $\vdash : S \in P \overline{\operatorname{smor}} Q . T \in Q \overline{\operatorname{smor}} R . \supset . S \mid T \in P \overline{\operatorname{smor}} R$

Dem.

$\vdash . *151·11 . *71·252 . \supset \vdash : \text{Hp} . \supset . S \mid T \in 1 \to 1$ (1)

$\vdash . *151·11 . *150·23 . \supset \vdash : \text{Hp} . \supset . \text{ᗡ}‘S = C‘Q . \mathrm{D}‘T = C‘Q . \text{ᗡ}‘T = C‘R .$

$[*37·323] \qquad \supset . \text{ᗡ}‘(S \mid T) = \text{ᗡ}‘T . \text{ᗡ}‘T = C‘R .$

$[*13·17] \qquad \supset . \text{ᗡ}‘(S \mid T) = C‘R$ (2)

$\vdash . *151·11 . \qquad \supset \vdash : \text{Hp} . \supset . P = S {}^{;} T {}^{;} R$

$[*150·13] \qquad = (S \mid T) {}^{;} R$ (3)

$\vdash . (1) . (2) . (3) . *151·11 . \supset \vdash . \text{Prop}$

***151·15.** $\vdash : P \,\text{smor}\, Q \,.\, Q \,\text{smor}\, R \,.\supset .\, P \,\text{smor}\, R$ [*151·141]

***151·16.** $\vdash .\, I \Subset \text{smor}$ [*151·13]

***151·161.** $\vdash .\, \text{smor} = \text{Cnv}\text{‘}\text{smor}$ [*151·14]

***151·162.** $\vdash .\, (\text{smor})^2 = \text{smor}$ [*151·15·161 . *34·81]

***151·17.** $\vdash :.\, P \,\text{smor}\, Q \,.\supset :\, R \,\text{smor}\, P \,.\equiv .\, R \,\text{smor}\, Q$ [*151·14·15]

***151·18.** $\vdash : P \,\text{smor}\, Q \,.\supset .\, C\text{‘}P \,\text{sm}\, C\text{‘}Q$

Dem.

$\vdash .\, *151{\cdot}11 \,.\, *150{\cdot}23 \,.\supset \vdash : \text{Hp} \,.\supset .\, (\exists S) \,.\, S \in 1 \to 1 \,.\, \text{D}\text{‘}S = C\text{‘}P \,.\, \text{Ɑ}\text{‘}S = C\text{‘}Q \,.$

$[*73{\cdot}1] \qquad \supset .\, C\text{‘}P \,\text{sm}\, C\text{‘}Q : \supset \vdash .\, \text{Prop}$

***151·19.** $\vdash : C\text{‘}P \,\text{sm}\, C\text{‘}Q \,.\equiv .\, (\exists R) \,.\, C\text{‘}R = C\text{‘}P \,.\, R \,\text{smor}\, Q$

Dem.

$\vdash .\, *73{\cdot}1 \,.\supset \vdash : C\text{‘}P \,\text{sm}\, C\text{‘}Q \,.\equiv .\, (\exists S) \,.\, S \in 1 \to 1 \,.\, \text{D}\text{‘}S = C\text{‘}P \,.\, \text{Ɑ}\text{‘}S = C\text{‘}Q \,.$

$[*150{\cdot}23] \qquad \equiv .\, (\exists S) \,.\, S \in 1 \to 1 \,.\, \text{Ɑ}\text{‘}S = C\text{‘}Q \,.\, C\text{‘}S \,;\, Q = C\text{‘}P \,.$

$[*13{\cdot}195] \qquad \equiv .\, (\exists R, S) \,.\, S \in 1 \to 1 \,.\, \text{Ɑ}\text{‘}S = C\text{‘}Q \,.\, R = S \,;\, Q \,.\, C\text{‘}R = C\text{‘}P \,.$

$[*151{\cdot}1] \qquad \equiv .\, (\exists R) \,.\, C\text{‘}R = C\text{‘}P \,.\, R \,\text{smor}\, Q : \supset \vdash .\, \text{Prop}$

***151·191.** $\vdash : S \in P \,\overline{\text{smor}}\, Q \,.\equiv .\, S \in (C\text{‘}P) \,\overline{\text{sm}}\, (C\text{‘}Q) \,.\, P = S \,;\, Q$

Dem.

$\vdash .\, *151{\cdot}131{\cdot}11 \,.\supset \vdash : S \in P \,\overline{\text{smor}}\, Q \,.\supset .\, C\text{‘}P = \text{D}\text{‘}S :$

$[*4{\cdot}71 \,.\, *151{\cdot}11] \supset \vdash : S \in P \,\overline{\text{smor}}\, Q \,.\equiv .\, S \in 1 \to 1 \,.\, C\text{‘}Q = \text{Ɑ}\text{‘}S \,.\, P = S \,;\, Q \,.\, C\text{‘}P = \text{D}\text{‘}S \,.$

$[*73{\cdot}03] \qquad \equiv .\, S \in (C\text{‘}P) \,\overline{\text{sm}}\, (C\text{‘}Q) \,.\, P = S \,;\, Q : \supset \vdash .\, \text{Prop}$

***151·2.** $\vdash : S \in 1 \to 1 \,.\, C\text{‘}Q \subset \text{Ɑ}\text{‘}S \,.\, P = S \,;\, Q \,.\supset .\, S \upharpoonright C\text{‘}Q \in P \,\overline{\text{smor}}\, Q$

Dem.

$\vdash .\, *71{\cdot}29 \,.\supset \vdash : \text{Hp} \,.\supset .\, S \upharpoonright C\text{‘}Q \in 1 \to 1 \qquad (1)$

$\vdash .\, *35{\cdot}65 \,.\supset \vdash : \text{Hp} \,.\supset .\, \text{Ɑ}\text{‘}S \upharpoonright C\text{‘}Q = C\text{‘}Q \qquad (2)$

$\vdash .\, *150{\cdot}32 \,.\supset \vdash : \text{Hp} \,.\supset .\, P = S \upharpoonright C\text{‘}Q \,;\, Q \qquad (3)$

$\vdash .\, (1) \,.\, (2) \,.\, (3) \,.\, *151{\cdot}11 \,.\supset \vdash .\, \text{Prop}$

***151·21.** $\vdash : P \,\text{smor}\, Q \,.\equiv .\, (\exists S) \,.\, S \in 1 \to 1 \,.\, C\text{‘}Q \subset \text{Ɑ}\text{‘}S \,.\, P = S \,;\, Q$ [*151·2]

***151·22.** $\vdash : S \upharpoonright C\text{‘}Q \in 1 \to 1 \,.\, C\text{‘}Q \subset \text{Ɑ}\text{‘}S \,.\, P = S \,;\, Q \,.\equiv .\, S \upharpoonright C\text{‘}Q \in P \,\overline{\text{smor}}\, Q$

Dem.

$\vdash .\, *35{\cdot}65 \,.\, *150{\cdot}32 \,.\supset \vdash : S \upharpoonright C\text{‘}Q \in 1 \to 1 \,.\, C\text{‘}Q \subset \text{Ɑ}\text{‘}S \,.\, P = S \,;\, Q \,.\supset .$

$S \upharpoonright C\text{‘}Q \in P \,\overline{\text{smor}}\, Q \qquad (1)$

$\vdash .\, *151{\cdot}11 \,.\, *150{\cdot}32 \,.\supset \vdash : S \upharpoonright C\text{‘}Q \in P \,\overline{\text{smor}}\, Q \,.\supset .\, S \upharpoonright C\text{‘}Q \in 1 \to 1 \,.\, P = S \,;\, Q \qquad (2)$

$\vdash .\, *151{\cdot}11 \,. \qquad \supset \vdash : S \upharpoonright C\text{‘}Q \in P \,\overline{\text{smor}}\, Q \,.\supset .\, C\text{‘}Q = \text{Ɑ}\text{‘}(S \upharpoonright C\text{‘}Q)$

$[*35{\cdot}64] \qquad = C\text{‘}Q \cap \text{Ɑ}\text{‘}S \,.$

$[*22{\cdot}621] \qquad \supset .\, C\text{‘}Q \subset \text{Ɑ}\text{‘}S \qquad (3)$

$\vdash .\, (1) \,.\, (2) \,.\, (3) \,.\supset \vdash .\, \text{Prop}$

$*151\cdot23.\ \ \vdash : P \operatorname{smor} Q . \equiv . (\exists S) . S \upharpoonright C\text{‘}Q \in 1 \to 1 . C\text{‘}Q \subset \text{Ɑ}\text{‘}S . P = S;Q$ [$*151\cdot22$]

The above proposition ($*151\cdot23$) is very useful. It is the analogue of $*73\cdot15$. (It should be observed that, in all propositions concerning likeness, $S;Q$ plays the same part as $S\text{‘‘}\beta$ plays in propositions concerning similarity.) By means of $*151\cdot23$, we can establish likeness in all those numerous cases in which a relation which is not usually one-one becomes one-one when confined to a certain converse domain, as for example if we have to deal with $\mathrm{D} \upharpoonright \epsilon_\Delta\text{‘}\kappa$, where $\kappa \in \mathrm{Cls}^2 \operatorname{excl}$, or with $\mathrm{D} \upharpoonright P_\Delta\text{‘}\kappa$, where $P \upharpoonright \kappa \in \mathrm{Cls} \to 1$. Thus *e.g.* by the above proposition, if Q is any relation whose field is $P_\Delta\text{‘}\kappa$, where $P \upharpoonright \kappa \in \mathrm{Cls} \to 1$, $\mathrm{D};Q$ will be an ordinally similar relation whose field is $\mathrm{D}\text{‘‘}P_\Delta\text{‘}\kappa$.

$*151\cdot231.\ \ \vdash :. (y) . \mathrm{E} ! S\text{‘}y : S \upharpoonright C\text{‘}Q \in 1 \to 1 . P = S;Q : \supset . S \upharpoonright C\text{‘}Q \in P \,\overline{\operatorname{smor}}\, Q$
[$*151\cdot22 . *33\cdot431$]

$*151\cdot232.\ \ \vdash :. (\exists S) : (y) . \mathrm{E} ! S\text{‘}y : S \upharpoonright C\text{‘}Q \in 1 \to 1 . P = S;Q : \supset . P \operatorname{smor} Q$
[$*151\cdot231\cdot12$]

$*151\cdot24.\ \ \vdash :. (y) . \mathrm{E} ! S\text{‘}y : y, z \in C\text{‘}Q . S\text{‘}y = S\text{‘}z . \supset_{y,z} . y = z : P = S;Q : \supset .$
$S \upharpoonright C\text{‘}Q \in P \,\overline{\operatorname{smor}}\, Q . P \operatorname{smor} Q$ [$*71\cdot166\cdot55 . *33\cdot431 . *151\cdot22\cdot23$]

$*151\cdot241.\ \ \vdash :. S \in 1 \to \mathrm{Cls} . C\text{‘}Q \subset \text{Ɑ}\text{‘}S : y, z \in C\text{‘}Q . S\text{‘}y = S\text{‘}z . \supset_{y,z} . y = z : P = S;Q : \supset .$
$S \upharpoonright C\text{‘}Q \in P \,\overline{\operatorname{smor}}\, Q . P \operatorname{smor} Q$ [$*71\cdot55 . *151\cdot22\cdot23$]

$*151\cdot242.\ \ \vdash :: y, z \in C\text{‘}Q . \supset_{y,z} : S\text{‘}y = S\text{‘}z . \equiv . y = z :. P = S;Q :. \equiv . S \upharpoonright C\text{‘}Q \in P \,\overline{\operatorname{smor}}\, Q$
[$*71\cdot59 . *151\cdot22$]

$*151\cdot243.\ \ \vdash :: y, z \in C\text{‘}Q . \supset_{y,z} : S\text{‘}y = S\text{‘}z . \equiv . y = z :. P = S;Q :. \supset . P \operatorname{smor} Q$
[$*151\cdot242\cdot12$]

$*151\cdot25.\ \ \vdash : S \in \mathrm{Cls} \to 1 . C\text{‘}Q \subset \text{Ɑ}\text{‘}S . P = S;Q . \supset . Q = \breve{S};P$

Dem.

$\vdash . *150\cdot13 . \supset \vdash : \mathrm{Hp} . \supset . \breve{S};P = (\breve{S} \,|\, S);Q$
[$*71\cdot191$] $\quad = (I \upharpoonright \text{Ɑ}\text{‘}S);Q$
[$*150\cdot535$] $\quad = Q : \supset \vdash . \mathrm{Prop}$

$*151\cdot251.\ \ \vdash :. S \in 1 \to 1 . \supset : C\text{‘}Q \subset \text{Ɑ}\text{‘}S . P = S;Q . \equiv . C\text{‘}P \subset \mathrm{D}\text{‘}S . Q = \breve{S};P$
[$*151\cdot25 . *150\cdot22 . *37\cdot15$]

$*151\cdot252.\ \ \vdash : S \in \mathrm{Cls} \to 1 . C\text{‘}Q \subset \text{Ɑ}\text{‘}S . \supset . Q = \breve{S};S;Q$ [$*151\cdot25$]

$*151\cdot253.\ \ \vdash : S \in 1 \to \mathrm{Cls} . C\text{‘}P \subset \mathrm{D}\text{‘}S . \supset . P = S;\breve{S};P$ $\left[*151\cdot252 \dfrac{\breve{S}}{S}\right]$

$*151\cdot254.\ \ \vdash : S \in 1 \to 1 . \supset . S\dagger \upharpoonright \breve{C}\text{‘‘}\mathrm{Cl}\text{‘}\text{Ɑ}\text{‘}S = \mathrm{Cnv}\text{‘}\{\breve{S}\dagger \upharpoonright \breve{C}\text{‘‘}\mathrm{Cl}\text{‘}\mathrm{D}\text{‘}S\}$

Dem.

$\vdash . *151\cdot251 . \supset \vdash :. \mathrm{Hp} . \supset : C\text{‘}Q \in \mathrm{Cl}\text{‘}\text{Ɑ}\text{‘}S . P\,(S\dagger)\,Q . \equiv .$
$C\text{‘}P \in \mathrm{Cl}\text{‘}\mathrm{D}\text{‘}S . Q\,(\breve{S}\dagger)\,P :. \supset \vdash . \mathrm{Prop}$

This proposition is the analogue of $*72\cdot54$. "$\breve{S}\dagger$" means "$(\mathrm{Cnv}\text{‘}S)\dagger$," not "$\mathrm{Cnv}\text{‘}(S\dagger)$."

$*151{\cdot}26$. $\vdash :. S \in \mathrm{Cls} \to 1 \,.\, C'Q \subset \text{Ɑ}'S \,.\, \supset :$

$$P \Subset S;Q \,.\, \supset .\, \breve{S};P \Subset Q : S;Q \Subset P \,.\, \supset .\, Q \Subset \breve{S};P$$

Dem.

$\vdash .\, *150{\cdot}31 \,.\, \supset \vdash : P \Subset S;Q \,.\, \supset .\, \breve{S};P \Subset \breve{S};S;Q :$

$[*151{\cdot}252] \quad \supset \vdash :. \mathrm{Hp} \,.\, \supset : P \Subset S;Q \,.\, \supset .\, \breve{S};P \Subset Q \qquad (1)$

Similarly $\quad \vdash :. \mathrm{Hp} \,.\, \supset : S;Q \Subset P \,.\, \supset .\, Q \Subset \breve{S};P \qquad (2)$

$\vdash .\, (1) \,.\, (2) \,.\, \supset \vdash .\, \mathrm{Prop}$

$*151{\cdot}261$. $\vdash :. S \in 1 \to \mathrm{Cls} \,.\, C'P \subset \mathrm{D}'S \,.\, \supset :$

$$Q \Subset \breve{S};P \,.\, \supset .\, S;Q \Subset P : \breve{S};P \Subset Q \,.\, \supset .\, P \Subset S;Q \quad \left[*151{\cdot}26 \frac{\breve{S}, Q, P}{S, P, Q}\right]$$

$*151{\cdot}262$. $\vdash :. S \in 1 \to 1 \,.\, C'P \subset \mathrm{D}'S \,.\, C'Q \subset \text{Ɑ}'S \,.\, \supset :$

$$P \Subset S;Q \,.\, \equiv .\, \breve{S};P \Subset Q : Q \Subset \breve{S};P \,.\, \equiv .\, S;Q \Subset P \quad [*151{\cdot}26{\cdot}261]$$

$*151{\cdot}263$. $\vdash :. S \in 1 \to 1 \,.\, C'P \subset \mathrm{D}'S \,.\, C'Q \subset \text{Ɑ}'S \,.\, \supset :$

$$P \Subset S;Q \,.\, Q \Subset \breve{S};P \,.\, \equiv .\, \breve{S};P \Subset Q \,.\, S;Q \Subset P \,.\, \equiv .\, P = S;Q \,.\, \equiv .\, Q = \breve{S};P$$

$[*151{\cdot}262]$

$*151{\cdot}264$. $\vdash :. S \upharpoonright C'Q \in 1 \to 1 \,.\, \supset : P \Subset S;Q \,.\, Q \Subset \breve{S};P \,.\, \equiv .\, P = S;Q$

Dem.

$\vdash .\, *150{\cdot}202 \,.\, *37{\cdot}401 \,.\quad \supset \vdash : P \Subset S;Q \,.\, \supset .\, C'P \subset \mathrm{D}'S \upharpoonright C'Q \quad (1)$

$\vdash .\, (1) \,.\, *151{\cdot}262 \dfrac{S \upharpoonright C'Q}{Q} \,.\, \supset \vdash : \mathrm{Hp} \,.\, P \Subset S;Q \,.\, \supset :$

$$Q \Subset (C'Q \upharpoonleft \breve{S});P \,.\, \equiv .\, (S \upharpoonright C'Q);Q \Subset P :$$

$[*150{\cdot}361{\cdot}32] \quad \supset : Q \Subset (\breve{S};P) \upharpoonright C'Q \,.\, \equiv .\, S;Q \Subset P :$

$[*35{\cdot}9 \,.\, *36{\cdot}29] \supset : Q \Subset \breve{S};P \,.\, \equiv .\, S;Q \Subset P \qquad (2)$

$\vdash .\, (2) \,.\, *5{\cdot}32 \,.\, \supset \vdash .\, \mathrm{Prop}$

$*151{\cdot}27$. $\vdash : S \in 1 \to 1 \,.\, P \Subset S;Q \,.\, Q \Subset \breve{S};P \,.$

$\equiv .\, S \in 1 \to 1 \,.\, C'P \subset \mathrm{D}'S \,.\, C'Q \subset \text{Ɑ}'S \,.\, \breve{S};P \Subset Q \,.\, S;Q \Subset P \,.$

$\equiv .\, S \in 1 \to 1 \,.\, C'Q \subset \text{Ɑ}'S \,.\, P = S;Q \,.$

$\equiv .\, S \in 1 \to 1 \,.\, C'P \subset \mathrm{D}'S \,.\, Q = \breve{S};P$

$[*151{\cdot}263 \,.\, *5{\cdot}32 \,.\, *150{\cdot}203 \,.\, *4{\cdot}73]$

$*151{\cdot}271$. $\vdash : (\exists S) \,.\, S \in 1 \to 1 \,.\, P \Subset S;Q \,.\, Q \Subset \breve{S};P \,.$

$\equiv .\, (\exists S) \,.\, S \in 1 \to 1 \,.\, C'P \subset \mathrm{D}'S \,.\, C'Q \subset \text{Ɑ}'S \,.\, \breve{S};P \Subset Q \,.\, S;Q \Subset P \,.$

$\equiv .\, P \,\mathrm{smor}\, Q \quad [*151{\cdot}27{\cdot}21]$

***151·28.** $\vdash :. P \,\mathrm{smor}\, Q . \equiv : (\exists S) : S \in 1 \to 1 : xPy . \supset_{x,y} . (\breve{S}`x) \, Q \, (\breve{S}`y) :$
$zQw . \supset_{z,w} . (S`z) \, P \, (S`w)$

Dem.

$\vdash . *150·41 . \supset \vdash :: S \in 1 \to 1 . \supset :. (\breve{S}`x) Q (\breve{S}`y) . \equiv . x S ; Q y : (S`z) P (S`w) . \equiv . z \breve{S} ; P w :.$

$[*23·1] \quad \supset :. xPy . \supset_{x,y} . (\breve{S}`x) \, Q \, (\breve{S}`y) : \equiv . P \subseteq \breve{S} ; Q :$
$zQw . \supset_{z,w} . (S`z) \, P \, (S`w) : \equiv . Q \subseteq \breve{S} ; P :.$

$[*151·27] \supset :. xPy . \supset_{x,y} . (\breve{S}`x) \, Q \, (\breve{S}`y) : zQw . \supset_{z,w} . (S`z) \, P \, (S`w) : \equiv .$
$C`Q \subset \text{ɑ}`S . P = S ; Q \qquad (1)$

$\vdash . (1) . *5·32 . *151·21 . \supset \vdash . \mathrm{Prop}$

The above proposition shows that ordinal similarity as we have defined it has the properties which are commonly associated with the term "ordinal similarity," namely that P and Q are ordinally similar when their fields can be so correlated that two terms having the relation P are always correlated with two terms having the relation Q, and vice versa.

The hypothesis $S \in 1 \to 1$ is redundant in *151·28; this is shown in the following proposition.

***151·281.** $\vdash :. xPy . \supset_{x,y} . (\breve{S}`x) \, Q \, (\breve{S}`y) : zQw . \supset_{z,w} . (S`z) \, P \, (S`w) :$
$\supset . C`P \upharpoonleft S = S \upharpoonright C`Q . S \upharpoonright C`Q \in P \,\overline{\mathrm{smor}}\, Q$

Dem.

$\vdash . *14·21 . \supset \vdash :. \mathrm{Hp} . \supset : xPy . \supset . E ! \breve{S}`x . E ! \breve{S}`y :$

$[*33·352] \quad \supset : x \in C`P . \supset . E ! \breve{S}`x :$

$[*71·571] \quad \supset : (C`P) \upharpoonleft S \in \mathrm{Cls} \to 1 . C`P \subset D`S \qquad (1)$

Similarly $\vdash : \mathrm{Hp} . \supset . S \upharpoonright C`Q \in 1 \to \mathrm{Cls} . C`Q \subset \text{ɑ}`S \qquad (2)$

$\vdash . *33·17 . \supset \vdash :. \mathrm{Hp} . \supset : xPy . \supset . \breve{S}`x, \breve{S}`y \in C`Q :$

$[*33·352] \quad \supset : x \in C`P . \supset . \breve{S}`x \in C`Q :$

$[*14·21·26] \quad \supset : x \in C`P . xSz . \supset . z \in C`Q :$

$[*4·71] \quad \supset : x \in C`P . xSz . \equiv . x \in C`P . xSz . z \in C`Q :$

$[*35·1·102] \quad \supset : (C`P) \upharpoonleft S = (C`P) \upharpoonleft S \upharpoonright C`Q \qquad (3)$

Similarly $\vdash : \mathrm{Hp} . \supset . S \upharpoonright C`Q = (C`P) \upharpoonleft S \upharpoonright C`Q \qquad (4)$

$\vdash . (3) . (4) . \supset \vdash : \mathrm{Hp} . \supset . (C`P) \upharpoonleft S = S \upharpoonright C`Q . \qquad (5)$

$[(1).(2)] \quad \supset . S \upharpoonright C`Q \in 1 \to 1 . C`Q \subset \text{ɑ}`S \qquad (6)$

$\vdash . (6) . *35·7 . (1) . (2) . *150·4·41 . \supset \vdash : \mathrm{Hp} . \supset . P \subseteq S ; Q . Q \subseteq \breve{S} ; P .$

$[*151·264.(6)] \quad \supset . P = S ; Q \qquad (7)$

$\vdash . (5) . (6) . (7) . *151·22 . \supset \vdash . \mathrm{Prop}$

***151·29.** $\vdash :. P \,\mathrm{smor}\, Q . \equiv : (\exists S) : xPy . \supset_{x,y} . (\breve{S}`x) Q (\breve{S}`y) : zQw . \supset_{z,w} . (S`z) P (S`w)$
$[*151·28·281]$

***151·31.** $\vdash : S \in \mathrm{Cls} \to 1 . S ; Q = S ; R . C`Q \subset \text{ɑ}`S . C`R \subset \text{ɑ}`S . \supset . Q = R$

Dem.

$\vdash . *151·252 . \supset \vdash : \mathrm{Hp} . \supset . Q = \breve{S} ; S ; Q$

$[\mathrm{Hp}] \quad = \breve{S} ; S ; R$

$[*151·252] \quad = R : \supset \vdash . \mathrm{Prop}$

***151·32.** $\vdash :. P \operatorname{smor} Q . \supset : \dot{\exists} ! P . \equiv . \dot{\exists} ! Q$ [*151·18 . *73·36 . *33·24]

***151·33.** $\vdash : S \in P \,\overline{\operatorname{smor}}\, Q . \supset . P | S = S | Q . \breve{S} | P = Q | \breve{S}$

Dem.

$\vdash . *151{\cdot}11 . \supset \vdash : \mathrm{Hp} . \supset . P | S = S | Q | \breve{S} | S . S \in 1 \rightarrow 1 . C\text{'}Q = \text{Ɑ}\text{'}S .$

[*72·601] $\supset . P | S = S | Q$ (1)

Similarly $\vdash : \mathrm{Hp} . \supset . \breve{S} | P = Q | \breve{S}$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***151·4.** $\vdash : T \upharpoonright C\text{'}Q \in 1 \rightarrow 1 . C\text{'}P = T\text{''}C\text{'}Q . Q = \breve{T} ; P . \supset . T \upharpoonright C\text{'}Q \in P \,\overline{\operatorname{smor}}\, Q$

Dem.

$\vdash . *35{\cdot}52 . *37{\cdot}4 . \supset \vdash : \mathrm{Hp} . \supset . (C\text{'}Q) \rceil \breve{T} \in 1 \rightarrow 1 . \text{Ɑ}\text{'}\{(C\text{'}Q) \rceil \breve{T}\} = C\text{'}P$ (1)

$\vdash . *36{\cdot}33 .$ $\supset \vdash : \mathrm{Hp} . \supset . Q = (\breve{T} ; P) \upharpoonright C\text{'}Q$

[*150·361] $= \{(C\text{'}Q) \rceil \breve{T}\} ; P$ (2)

$\vdash . (1) . (2) . *151{\cdot}11 . \supset \vdash : \mathrm{Hp} . \supset . (C\text{'}Q) \rceil \breve{T} \in Q \,\overline{\operatorname{smor}}\, P .$

[*151·131] $\supset . T \upharpoonright C\text{'}Q \in P \,\overline{\operatorname{smor}}\, Q : \supset \vdash . \mathrm{Prop}$

***151·401.** $\vdash : T \upharpoonright C\text{'}P \in X \,\overline{\operatorname{smor}}\, P . T \upharpoonright C\text{'}Q \in Y \,\overline{\operatorname{smor}}\, Q . S \in P \,\overline{\operatorname{smor}}\, Q . \supset .$
$T ; S \in X \,\overline{\operatorname{smor}}\, Y$

Dem.

$\vdash . *151{\cdot}131{\cdot}141 . \supset \vdash : \mathrm{Hp} . \supset . T \upharpoonright C\text{'}P | S | (C\text{'}Q) \rceil \breve{T} \in X \,\overline{\operatorname{smor}}\, Y$ (1)

$\vdash . *151{\cdot}11{\cdot}131 .$ $\supset \vdash : \mathrm{Hp} . \supset . \mathrm{D}\text{'}S = C\text{'}P . \text{Ɑ}\text{'}S = C\text{'}Q .$

[*150·34] $\supset . T \upharpoonright C\text{'}P | S | (C\text{'}Q) \rceil \breve{T} = T ; S$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***151·41.** $\vdash : S \in P \,\overline{\operatorname{smor}}\, Q . T \upharpoonright C\text{'}P , T \upharpoonright C\text{'}Q \in 1 \rightarrow 1 . C\text{'}P \cup C\text{'}Q \subset \text{Ɑ}\text{'}T . \supset .$
$T ; S \in (T ; P) \,\overline{\operatorname{smor}}\, (T ; Q)$ [*151·401·22]

This proposition is the analogue of *73·63.

The following proposition is used frequently both in relation-arithmetic and in the theory of series.

***151·5.** $\vdash : S \upharpoonright C\text{'}Q \in P \,\overline{\operatorname{smor}}\, Q . \supset .$
$\mathrm{D}\text{'}P = S\text{''}\mathrm{D}\text{'}Q . \text{Ɑ}\text{'}P = S\text{''}\text{Ɑ}\text{'}Q . \overrightarrow{B}\text{'}P = S\text{''}\overrightarrow{B}\text{'}Q . \overrightarrow{B}\text{'}\breve{P} = S\text{''}\overrightarrow{B}\text{'}\breve{Q}$

Dem.

$\vdash . *151{\cdot}22 . *150{\cdot}21{\cdot}211 . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{D}\text{'}P = S\text{''}\mathrm{D}\text{'}Q . \text{Ɑ}\text{'}P = S\text{''}\text{Ɑ}\text{'}Q .$ (1)

[*93·101] $\supset . \overrightarrow{B}\text{'}P = S\text{''}\mathrm{D}\text{'}Q - S\text{''}\text{Ɑ}\text{'}Q .$

[*37·421.*151·22] $\supset . \overrightarrow{B}\text{'}P = (S \upharpoonright C\text{'}Q)\text{''}\mathrm{D}\text{'}Q - (S \upharpoonright C\text{'}Q)\text{''}\text{Ɑ}\text{'}Q$

[*71·381.*151·22] $= (S \upharpoonright C\text{'}Q)\text{''}(\mathrm{D}\text{'}Q - \text{Ɑ}\text{'}Q)$

[*93·101.*37·421] $= S\text{''}\overrightarrow{B}\text{'}Q$ (2)

Similarly $\vdash : \mathrm{Hp} . \supset . \overrightarrow{B}\text{'}\breve{P} = S\text{''}\overrightarrow{B}\text{'}\breve{Q}$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash . \mathrm{Prop}$

***151·51.** $\vdash : S \upharpoonright C\text{‘}Q \in P \,\overline{\text{smor}}\, Q \,.\, R \mathrel{\mathsf{G}} Q \,.\, \supset . \, S \upharpoonright C\text{‘}R \in (S ; R) \,\overline{\text{smor}}\, R \,.\, S ; R \mathrel{\mathsf{G}} P$

Dem.

$\vdash . *151\cdot22 . *33\cdot265 . \supset \vdash : \text{Hp} . \supset . C\text{‘}R \subset \text{Ɑ}\text{‘}S$ (1)

$\vdash . *151\cdot22 . *71\cdot222 . \supset \vdash : \text{Hp} . \supset . S \upharpoonright C\text{‘}R \in 1 \to 1$ (2)

$\vdash . *150\cdot31 . *151\cdot22 . \supset \vdash : \text{Hp} . \supset . S ; R \mathrel{\mathsf{G}} P$ (3)

$\vdash . (1) . (2) . (3) . *151\cdot22 . \supset \vdash . \text{Prop}$

***151·52.** $\vdash : P \,\text{smor}\, Q . \supset . \text{Rl}\text{‘}Q \subset \text{smor}\text{“}\text{Rl}\text{‘}P$ [*151·51·12·14]

***151·53.** $\vdash : S \upharpoonright C\text{‘}Q \in P \,\overline{\text{smor}}\, Q \,.\, T \in \text{Pot}\text{‘}Q . \supset .$

$S \upharpoonright C\text{‘}T \in (S ; T) \,\overline{\text{smor}}\, T \,.\, S ; T \in \text{Pot}\text{‘}P$

Dem.

$\vdash . *150\cdot8 . \supset \vdash : \text{Hp} . \supset . S ; T \in \text{Pot}\text{‘}P$ (1)

$\vdash . *91\cdot27 . \supset \vdash : \text{Hp} . \supset . C\text{‘}T \subset \text{Ɑ}\text{‘}S$ (2)

$\vdash . (1) . (2) . *151\cdot22 . \supset \vdash . \text{Prop}$

***151·54.** $\vdash : S \upharpoonright C\text{‘}Q \in P \,\overline{\text{smor}}\, Q . \supset . S \upharpoonright C\text{‘}Q \in P_{\text{po}} \,\overline{\text{smor}}\, Q_{\text{po}}$

Dem.

$\vdash . *91\cdot504 . *151\cdot22 . \supset \vdash : \text{Hp} . \supset . S \upharpoonright C\text{‘}Q = S \upharpoonright C\text{‘}Q_{\text{po}} \,.\, C\text{‘}Q_{\text{po}} \subset \text{Ɑ}\text{‘}S$ (1)

$\vdash . *150\cdot83 . *151\cdot22 . \supset \vdash : \text{Hp} . \supset . P_{\text{po}} = S ; Q_{\text{po}}$ (2)

$\vdash . (1) . (2) . *151\cdot22 . \supset \vdash . \text{Prop}$

***151·55.** $\vdash : S \in P \,\overline{\text{smor}}\, Q . \supset . S \in P_{\text{po}} \,\overline{\text{smor}}\, Q_{\text{po}}$ [*151·54]

***151·56.** $\vdash : P \,\text{smor}\, Q . \supset . P_{\text{po}} \,\text{smor}\, Q_{\text{po}}$ [*151·55]

*151·56 is used in *263·17.

The two following propositions are lemmas for *151·59, which is used in *263·17.

***151·57.** $\vdash : S \in P \,\overline{\text{smor}}\, Q \,.\, z, w \in C\text{‘}Q . \supset . P(S\text{‘}z \mapsto S\text{‘}w) = S\text{“}Q(z \mapsto w)$

Dem.

$\vdash . *151\cdot33\cdot55 . \supset \vdash : \text{Hp} . \supset . \overleftarrow{P}_{\text{po}}\text{‘}S\text{‘}z = S\text{“}\overleftarrow{Q}_{\text{po}}\text{‘}z \,.\, \overrightarrow{P}_{\text{po}}\text{‘}S\text{‘}w = S\text{“}\overrightarrow{Q}_{\text{po}}\text{‘}w \,.$

[*91·54] $\supset . \overleftarrow{P}_{*}\text{‘}S\text{‘}z = S\text{“}\overleftarrow{Q}_{\text{po}}\text{‘}z \cup \iota\text{‘}S\text{‘}z \,.$

$\overrightarrow{P}_{*}\text{‘}S\text{‘}w = S\text{“}\overrightarrow{Q}_{\text{po}}\text{‘}w \cup \iota\text{‘}S\text{‘}w \,.$

[*53·31.*91·54] $\supset . \overleftarrow{P}_{*}\text{‘}S\text{‘}z = S\text{“}\overleftarrow{Q}_{*}\text{‘}z \,.\, \overrightarrow{P}_{*}\text{‘}S\text{‘}w = S\text{“}\overrightarrow{Q}_{*}\text{‘}w \,.$

[(*121·103)] $\supset . P(S\text{‘}z \mapsto S\text{‘}w) = S\text{“}Q(z \mapsto w) : \supset \vdash . \text{Prop}$

***151·58.** $\vdash : S \in P \,\overline{\text{smor}}\, Q . \supset . S \upharpoonright C\text{‘}Q_\nu \in P_\nu \,\overline{\text{smor}}\, Q_\nu$

Dem.

$\vdash . *151\cdot57 . *73\cdot22 . \supset \vdash :. \text{Hp} . \supset : z, w \in C\text{‘}Q . \supset .$

$\text{Nc}\text{‘}P(S\text{‘}z \mapsto S\text{‘}w) = \text{Nc}\text{‘}Q(z \mapsto w)$ (1)

$\vdash . (1) . *121\cdot11 . \quad \supset \vdash :. \text{Hp} \,.\, z, w \in C\text{‘}Q . \supset : z Q_\nu w . \equiv . (S\text{‘}z) P_\nu (S\text{‘}w)$ (2)

$\vdash . (2) . *150\cdot41 . \quad \supset \vdash : \text{Hp} . \supset . Q_\nu = \breve{S} ; P_\nu \,.$

[*151·253.*121·322] $\supset . S ; Q_\nu = P_\nu \,.\, C\text{‘}Q_\nu \subset \text{Ɑ}\text{‘}S$ (3)

$\vdash . (3) . *151\cdot22 . \supset \vdash . \text{Prop}$

***151·59.** $\vdash : P \text{ smor } Q . \supset . P_\nu \text{ smor } Q_\nu$ [*151·58]

The remaining propositions of this number consist of applications to particular cases.

***151·6.** $\vdash . \text{Cnv}{}^{\text{;}}P \text{ smor } P . \text{Cnv} \upharpoonright C\text{‘}P \in (\text{Cnv}{}^{\text{;}}P) \overline{\text{smor}}\, P$
[*151·231 . *31·13 . *72·11]

This proposition is only significant when P is a relation between relations.

***151·61.** $\vdash . \iota{}^{\text{;}}P \text{ smor } P$ [*151·232 . *51·12 . *72·18]

***151·62.** $\vdash : C\text{‘}P \subset 1 . \supset . \breve{\iota}{}^{\text{;}}P \text{ smor } P$ [*52·62 . *151·243]

***151·63.** $\vdash : x \neq y . z \neq w . \supset . x \downarrow y \text{ smor } z \downarrow w . x \downarrow z \cup y \downarrow w \in (x \downarrow y) \overline{\text{smor}} (z \downarrow w)$

Dem.

$\vdash . *150{\cdot}72 . \supset \vdash : S = x \downarrow z \cup y \downarrow w . z \neq w . \supset . S{}^{\text{;}}(z \downarrow w) = x \downarrow y$ (1)

$\vdash . *72{\cdot}182 . *71{\cdot}242 . \supset \vdash : \text{Hp} . \text{Hp}(1) . \supset . S \in 1 \to 1$ (2)

$\vdash . *55{\cdot}15 . \quad \supset \vdash : \text{Hp}(1) . \supset . \text{G}\text{‘}S = C\text{‘}(z \downarrow w)$ (3)

$\vdash . (1) . (2) . (3) . *151{\cdot}1{\cdot}11 . \supset \vdash . \text{Prop}$

The above proposition shows that all ordinal couples (*i.e.* all members of 2_r) are ordinally similar. The following proposition shows the same for couples whose referent and relatum are identical.

***151·631.** $\vdash . x \downarrow x \text{ smor } z \downarrow z$

Dem.

$\vdash . *72{\cdot}182 . *55{\cdot}15 . \supset \vdash . x \downarrow z \in 1 \to 1 . \text{G}\text{‘}(x \downarrow z) = C\text{‘}(z \downarrow z)$ (1)

$\vdash . *55{\cdot}13 . \quad \supset \vdash : u \{x \downarrow z \,|\, z \downarrow z \,|\, \text{Cnv}\text{‘}(x \downarrow z)\} u' . \equiv .$

$u (x \downarrow z) z . u' (x \downarrow z)$

[*55·13] $\equiv . u = x . u' = x .$

[*55·13] $\equiv . u (x \downarrow x) u'$ (2)

$\vdash . (2) . *150{\cdot}1 . \quad \supset \vdash . (x \downarrow z){}^{\text{;}}(z \downarrow z) = x \downarrow x$ (3)

$\vdash . (1) . (3) . *151{\cdot}1 . \supset \vdash . \text{Prop}$

***151·64.** $\vdash . x \dot{\downarrow}{}^{\text{;}}P \text{ smor } P . (x \downarrow) \upharpoonright C\text{‘}P \in (x \downarrow{}^{\text{;}}P) \overline{\text{smor}}\, P$
[*72·184 . *55·12 . *151·231]

The following proposition is frequently used in relation-arithmetic.

***151·65.** $\vdash . \downarrow x{}^{\text{;}}P \text{ smor } P . (\downarrow x) \upharpoonright C\text{‘}P \in (\downarrow x{}^{\text{;}}P) \overline{\text{smor}}\, P$
[*72·184 . *55·121 . *151·231]

$*152$. DEFINITION AND ELEMENTARY PROPERTIES OF RELATION-NUMBERS

Summary of $*152$.

The relation-number of P, which we denote by $\text{Nr}'P$, is defined as the class of relations which are ordinally similar to P, *i.e.*

$$\text{Nr}'P = \overrightarrow{\text{smor}}'P.$$

Hence our definition is

$$\text{Nr} = \overrightarrow{\text{smor}} \quad \text{Df.}$$

The class of relation-numbers consists of all such classes as $\text{Nr}'P$, *i.e.*

$$\text{NR} = \text{D}'\text{Nr} \quad \text{Df.}$$

These two definitions are analogous to those of $*100$, merely substituting "smor" for "sm." They are justified by similar considerations, and lead to similar results. With the exception of $*152 \cdot 7 \cdot 71 \cdot 72$, the propositions of this number are the analogues of those of $*100$, and call for no remarks other than those in the introduction to $*100$ (*mutatis mutandis*).

$*152 \cdot 7 \cdot 71 \cdot 72$ give relations between relation-numbers and cardinals. $*152 \cdot 7$, which is constantly used, states that the cardinal number of $C'Q$ consists of the fields of the relation-number of Q, *i.e.* the classes similar to $C'Q$ are the fields of the relations similar to Q; in symbols,

$*152 \cdot 7$. $\vdash . \text{Nc}'C'Q = C``\text{Nr}'Q$

Hence it follows that the fields of a relation-number form a cardinal number, *i.e.*

$*152 \cdot 71$. $\vdash : \mu \in \text{NR} . \supset . C``\mu \in \text{NC}$

Hence also it follows that cardinals other than Λ consist of classes of the form $C``\mu$, where μ is a relation-number other than Λ, *i.e.*

$*152 \cdot 72$. $\vdash . \text{NC} - \iota'\Lambda = C```(\text{NR} - \iota'\Lambda)$

In $*154 \cdot 9$, we shall show how to remove the restriction to numbers other than Λ, thus arriving at

$$\vdash . \text{NC} = C```\text{NR}.$$

$*152 \cdot 01$. $\text{Nr} = \overrightarrow{\text{smor}}$ Df

$*152 \cdot 02$. $\text{NR} = \text{D}'\text{Nr}$ Df

$*152 \cdot 1$. $\vdash . \text{Nr}'P = \hat{Q}(Q \,\text{smor}\, P) = \hat{Q}(P \,\text{smor}\, Q)$
[$*32 \cdot 11 . (*152 \cdot 01) . *151 \cdot 14$]

$*152 \cdot 11$. $\vdash : Q \in \text{Nr}'P . \equiv . Q \,\text{smor}\, P . \equiv . P \,\text{smor}\, Q$ [$*152 \cdot 1$]

$*152 \cdot 2$. $\vdash . \text{E}!\, \text{Nr}'P$ [$*152 \cdot 1 . *14 \cdot 21$]

∗152·21. $\vdash . \text{C}'\text{Nr} = \text{Rel}$ [∗152·2 . ∗33·432]

∗152·22. $\vdash . \text{Nr} \epsilon 1 \to \text{Cls}$ [∗152·2 . ∗71·166]

∗152·3. $\vdash . P \epsilon \text{Nr}'P$ [∗151·13 . ∗152·11]

∗152·31. $\vdash : P \epsilon \text{Nr}'Q . \equiv . Q \epsilon \text{Nr}'P$ [∗152·11]

∗152·32. $\vdash : P \epsilon \text{Nr}'Q . Q \epsilon \text{Nr}'R . \supset . P \epsilon \text{Nr}'R$ [∗151·15 . ∗152·11]

∗152·321. $\vdash : P \text{ smor } Q . \supset . \text{Nr}'P = \text{Nr}'Q$ [∗151·17 . ∗152·1]

∗152·33. $\vdash : \exists ! \text{Nr}'P \cap \text{Nr}'Q . \supset . P \text{ smor } Q . \text{Nr}'P = \text{Nr}'Q$

Dem.

$\vdash . ∗152·11 . ∗151·14 . \supset \vdash : \text{Hp} . \supset . (\exists R) . P \text{ smor } R . R \text{ smor } Q .$

[∗151·15] $\supset . P \text{ smor } Q$ (1)

$\vdash . (1) . ∗152·321 . \supset \vdash . \text{Prop}$

∗152·35. $\vdash :. \exists ! \text{Nr}'P . \text{v} . \exists ! \text{Nr}'Q : \supset :$

$\text{Nr}'P = \text{Nr}'Q . \equiv . P \epsilon \text{Nr}'Q . \equiv . Q \epsilon \text{Nr}'P . \equiv . P \text{ smor } Q$

Dem.

$\vdash . ∗24·571 . \supset \vdash :. \text{Hp} . \supset : \text{Nr}'P = \text{Nr}'Q . \supset . \exists ! \text{Nr}'P \cap \text{Nr}'Q .$

[∗152·33] $\supset . P \text{ smor } Q$ (1)

$\vdash . (1) . ∗152·321 . \supset \vdash :. \text{Hp} . \supset : \text{Nr}'P = \text{Nr}'Q . \equiv . P \text{ smor } Q$ (2)

$\vdash . (2) . ∗152·11 . \supset \vdash . \text{Prop}$

In the above proposition, the same remarks as to types are to be made as in the case of ∗100·35. If in a certain type $\text{Nr}'P$ and $\text{Nr}'Q$ are both null, we have in that type $\text{Nr}'P = \text{Nr}'Q$, but we need not have $P \text{ smor } Q$. Thus for example we shall find that, in the type of $x \downarrow x$,

$$\text{Nr}'(t^{2}{}'x \uparrow t^{2}{}'x) = \Lambda = \text{Nr}'(t^{3}{}'x \uparrow t^{3}{}'x).$$

But we do not have

$$(t^{2}{}'x \uparrow t^{2}{}'x) \text{ smor } (t^{3}{}'x \uparrow t^{3}{}'x).$$

∗152·4. $\vdash : \mu \epsilon \text{NR} . \equiv . (\exists P) . \mu = \text{Nr}'P$ [∗37·78·79 . (∗152·02·01)]

Note that "$\text{Nr}'P$," like "$\text{Nc}'\alpha$," is a formal number, and may be subjected to the conventions I T, II T, A T.

∗152·41. $\vdash . \text{Nr}'P \epsilon \text{NR}$ [∗152·4·2]

∗152·42. $\vdash : \mu, \nu \epsilon \text{NR} . \exists ! \mu \cap \nu . \supset . \mu = \nu$ [∗152·33·4]

∗152·43. $\vdash . \text{NR} \epsilon \text{Cls}^2 \text{ excl}$ [∗152·42]

∗152·44. $\vdash :. \mu \epsilon \text{NR} : \exists ! \mu . \text{v} . \exists ! \text{Nr}'P . \supset : P \epsilon \mu . \equiv . \text{Nr}'P = \mu$
[∗152·35·4]

∗152·45. $\vdash : \mu \epsilon \text{NR} . P \epsilon \mu . \supset . \text{Nr}'P = \mu$ [∗152·44 . ∗10·24]

∗152·5. $\vdash : \mu \epsilon \text{NR} . P, Q \epsilon \mu . \supset . P \text{ smor } Q$ [∗152·31·32·4]

***152·51.** $\vdash : \mu \in \mathrm{NR} \,.\, P \in \mu \,.\, \supset \,.\, \mathrm{smor}``\mu = \mathrm{Nr}`P$

Dem.

$\vdash \,.\, *37\cdot1 \,.\, \supset \vdash : R \in \mathrm{smor}``\mu \,.\, \equiv \,.\, (\exists Q) \,.\, Q \in \mu \,.\, R \,\mathrm{smor}\, Q$ (1)

$\vdash \,.\, *152\cdot5 \,.\, \supset \vdash :. \mu \in \mathrm{NR} \,.\, P \in \mu \,.\, \supset : Q \in \mu \,.\, P \,\mathrm{smor}\, Q \,.\, \equiv \,.\, Q \in \mu$ (2)

$\vdash \,.\, (1) \,.\, (2) \,.\, \supset \vdash :. \mathrm{Hp} \,.\, \supset : R \in \mathrm{smor}``\mu \,.\, \equiv \,.\, (\exists Q) \,.\, Q \in \mu \,.\, P \,\mathrm{smor}\, Q \,.\, R \,\mathrm{smor}\, Q \,.$

[*151·17] $\equiv \,.\, (\exists Q) \,.\, Q \in \mu \,.\, P \,\mathrm{smor}\, Q \,.\, R \,\mathrm{smor}\, P \,.$

[(2)] $\equiv \,.\, (\exists Q) \,.\, Q \in \mu \,.\, R \,\mathrm{smor}\, P \,.$

[*10·35] $\equiv \,.\, \exists ! \mu \,.\, R \,\mathrm{smor}\, P$ (3)

$\vdash \,.\, *10\cdot24 \,.\, \supset \vdash :. \mathrm{Hp} \,.\, \supset : \exists ! \mu :$

[*4·73] $\supset : R \,\mathrm{smor}\, P \,.\, \equiv \,.\, \exists ! \mu \,.\, R \,\mathrm{smor}\, P$ (4)

$\vdash \,.\, (3) \,.\, (4) \,.\, \supset \vdash :. \mathrm{Hp} \,.\, \supset : R \in \mathrm{smor}``\mu \,.\, \equiv \,.\, R \,\mathrm{smor}\, P \,.$

[*152·11] $\equiv \,.\, R \in \mathrm{Nr}`P :. \supset \vdash \,.\, \mathrm{Prop}$

***152·52.** $\vdash : \mu \in \mathrm{NR} \,.\, \exists ! \mu \,.\, \supset \,.\, \mathrm{smor}``\mu \in \mathrm{NR}$ [*152·51·4]

The restriction involved in $\exists ! \mu$ is, as we shall see later, not necessary, since $\Lambda \in \mathrm{NR}$ in any assigned type.

***152·53.** $\vdash : \exists ! \mathrm{Nr}`Q \,.\, \supset \,.\, \mathrm{smor}``\mathrm{Nr}`Q = \mathrm{Nr}`Q$

Dem.

$\vdash \,.\, *152\cdot51 \,.\, \supset \vdash : P \in \mathrm{Nr}`Q \,.\, \supset \,.\, \mathrm{smor}``\mathrm{Nr}`Q = \mathrm{Nr}`P$ (1)

$\vdash \,.\, *152\cdot321 \,.\, \supset \vdash : P \in \mathrm{Nr}`Q \,.\, \supset \,.\, \mathrm{Nr}`P = \mathrm{Nr}`Q$ (2)

$\vdash \,.\, (1) \,.\, (2) \,.\, \supset \vdash \,.\, \mathrm{Prop}$

***152·54.** $\vdash :. \exists ! \mu \,.\, \exists ! \nu \,.\, \supset : \mu \in \mathrm{NR} \,.\, \nu = \mathrm{smor}``\mu \,.\, \equiv \,.\, \nu \in \mathrm{NR} \,.\, \mu = \mathrm{smor}``\nu$

[Proof as in *100·53]

***152·6.** $\vdash \,.\, \iota ; P \in \mathrm{Nr}`P$ [*151·61]

***152·62.** $\vdash \,.\, x \downarrow ; P \in \mathrm{Nr}`P$ [*151·64]

***152·63.** $\vdash \,.\, \downarrow x ; P \in \mathrm{Nr}`P$ [*151·65]

The utility of *152·6·62·63 is that they enable us to raise the type of a relation-number to any required extent. Thus $\iota ; P$ gives a relation whose field is a class of the next type above that of $C`P$, *i.e.* of the type $t^2`C`P$; while $x \downarrow ; P$ gives a relation whose field is $x \downarrow `` C`P$, which is of the type $t`t`(\iota`x \uparrow C`P)$. If $x \in C`P$, or, more generally, if $x \in t_0`C`P$, this is the type $t^2`P$. Thus if we put $Q = x \downarrow ; P$, we have

$$t`Q = t`(C`Q \uparrow C`Q) = t`(t`P \uparrow t`P) = t`(P \downarrow P).$$

Thus $x \downarrow ; P$ is a relation whose field consists of terms of the same type as P.

The following propositions on the relations of cardinals and relation-numbers are very important.

***152·7.** $\vdash \,.\, \mathrm{Nc}`C`Q = C``\mathrm{Nr}`Q$

Dem.

$\vdash \,.\, *151\cdot19 \,.\, *35\cdot942 \,.\, \supset \vdash : \alpha \in \mathrm{Nc}`C`Q \,.\, \supset \,.\, (\exists R) \,.\, C`R = \alpha \,.\, R \in \mathrm{Nr}`Q \,.$

[*37·6] $\supset \,.\, \alpha \in C``\mathrm{Nr}`Q$ (1)

$\vdash \,.\, *151\cdot18 \,.\, \supset \vdash : P \in \mathrm{Nr}`Q \,.\, \supset \,.\, C`P \in \mathrm{Nc}`C`Q$

[*37·61] $\supset \vdash \,.\, C``\mathrm{Nr}`Q \subset \mathrm{Nc}`C`Q$ (2)

$\vdash \,.\, (1) \,.\, (2) \,.\, \supset \vdash \,.\, \mathrm{Prop}$

***152·71.** $\vdash : \mu \in \mathrm{NR} . \supset . C``\mu \in \mathrm{NC}$ [*152·7]

***152·72.** $\vdash . \mathrm{NC} - \iota`\Lambda = C```(\mathrm{NR} - \iota`\Lambda)$

Dem.

$\vdash . *152{\cdot}71 . \quad \supset \vdash . C```\mathrm{NR} \subset \mathrm{NC} \quad (1)$

$\vdash . *152{\cdot}7 . *50{\cdot}5{\cdot}52 . \supset \vdash : \mu \in \mathrm{NC} . \alpha \in \mu . \supset . C``\mathrm{Nr}`(I \upharpoonright \alpha) = \mathrm{NC}`\alpha .$

$[*100{\cdot}45] \quad \supset . C``\mathrm{Nr}`(I \upharpoonright \alpha) = \mu .$

$[*37{\cdot}103] \quad \supset . \mu \in C```\mathrm{NR} \quad (2)$

$\vdash . (2) . *10{\cdot}11{\cdot}23{\cdot}35 . \supset \vdash : \mu \in \mathrm{NC} . \exists ! \mu . \supset . \mu \in C```\mathrm{NR} \quad (3)$

$\vdash . *37{\cdot}45 . \quad \supset \vdash : \mu = C``\nu . \exists ! \mu . \supset . \exists ! \nu :$

$[*37{\cdot}103] \quad \supset \vdash : \mu \in C```\mathrm{NR} . \exists ! \mu . \supset . \mu \in C```(\mathrm{NR} - \iota`\Lambda) \quad (4)$

$\vdash . (1) . (3) . (4) . \supset \vdash . \mathrm{Prop}$

We shall show in *154·9 that the exclusion of Λ in *152·72 is unnecessary.

*153. THE RELATION-NUMBERS 0_r, 2_r AND 1_s

Summary of *153.

The relation-numbers 0_r and 2_r have already been defined (in *56), though it remains for the present number to show that they are relation-numbers. They are the ordinal 0 and 2 respectively, *i.e.* they are the ordinal numbers of series of no terms and series of two terms respectively. But there is no means of introducing an ordinal 1 which shall be analogous to the cardinal 1 as completely as 0_r and 2_r are analogous to 0 and 2. The only relations whose fields are unit classes are relations of the form $x \downarrow x$. We therefore put

***153·01.** $1_s = \hat{R}\{(\exists x) . R = x \downarrow x\}$ Df

The above definition gives the nearest possible approach to an ordinal 1. 1_s so defined is a relation-number, and is the relation-number corresponding to 1 in the sense that it is the relation-number of all such relations as have a field consisting of one term. But 1_s is not what is called an "ordinal number," because this term is confined by usage to the relation-numbers of well-ordered series, and $x \downarrow x$ is not a serial relation. It is essential to a serial relation to be contained in diversity; and if, by definition, we include $x \downarrow x$ among series, we introduce more exceptions than we avoid. Moreover 1_s does not have the kind of properties which we wish 1 to have; *e.g.* $1_s \dot{+} 1_s$ is not 2_r.

We do not use 1_r, because we shall at a later stage define ν_r as the class of those *well-ordered series* whose fields have ν terms, so that $1_r = \Lambda$, while 0_r and 2_r have the values $\iota`\dot{\Lambda}$ and $\hat{R}\{(\exists x, y) . x \neq y . R = x \downarrow y\}$, as already defined. On account of this general definition of ν_r, we choose a different symbol for the relation-number 1, and 1_s has the merit of being as like 1_r as possible.

To illustrate, by anticipation, the way in which 1_s differs from proper ordinal numbers, we may point out that if 1_s is added to 2_r, we do not obtain 3_r. We shall define 3_r as the class of series which consist of three terms, *i.e.* the class of relations of the form

$$x \downarrow y \cup x \downarrow z \cup y \downarrow z,$$

where $x \neq y . x \neq z . y \neq z$. We shall define the sum of two ordinal numbers as the ordinal number of the sum of two relations having these ordinal numbers (cf. *180), and it will appear that if P and Q are relations whose fields have no members in common, then

$$P \cup Q \cup C`P \uparrow C`Q$$

has a relation-number which is the sum of those of P and Q. Suppose now

$P = x \downarrow y$ and $Q = z \downarrow z$, where $x \neq y \,.\, x \neq z \,.\, y \neq z$. Then

$$P \cup Q \cup C'P \uparrow C'Q = x \downarrow y \cup x \downarrow z \cup y \downarrow z \cup z \downarrow z.$$

This is not a member of 3_r, because of the additional term $z \downarrow z$. Thus the addition of one term to a series P does not give the same number as results from the addition of 1_s to Nr'P. Hence the addition of 1 to an ordinal number has to be separately treated*.

We prove in this number that $0_r = \mathrm{Nr}'\dot{\Lambda}$ (*153·11), that $2_r = \mathrm{Nr}'(\Lambda \downarrow \iota' x)$ (*153·24; observe that we have to take a couple of *classes* (or relations) in order to be sure of the existence of two different objects of the class in question), and that $1_s = \mathrm{Nr}'(y \downarrow y)$ (*153·32). We prove $C''0_r = 0$ (*153·18), $C''2_r = 2$ (*153·212), and $C''1_s = 1$ (*153·36). We have also $\breve{C}''0 = 0_r$ (not proved) and $\breve{C}''1 = 1_s$ (*153·301). But we do not have $\breve{C}''2 = 2_r$; *e.g.* $(x \downarrow y \cup y \downarrow x) \in \breve{C}''2$ if $x \neq y$, but $(x \downarrow y \cup y \downarrow x) \sim\in 2_r$. We have $\exists ! 0_r$ (*153·12) and $\exists ! 1_s$ (*153·34), but from our primitive propositions we cannot deduce $\exists ! 2_r$ unless we rise above the lowest type of relations. The case is exactly analogous to that of $\exists ! 2$ (cf. *101); we have

***153·26·262.** $\vdash . \exists ! 2_r \cap \mathrm{Rl}'(\mathrm{Cls} \uparrow \mathrm{Cls}) . \exists ! 2_r \cap \mathrm{Rel}^2$

But if, as monists aver, there is only one individual, we shall not have $\exists ! 2_r$ in the type of relations of individuals to individuals. Our primitive propositions do not suffice to disprove this supposition.

***153·01.** $1_s = \hat{R}\{(\exists x) . R = x \downarrow x\}$ Df

***153·1.** $\vdash : P \in 0_r . \equiv . P = \dot{\Lambda}$ [*56·104]

***153·101.** $\vdash : P \,\mathrm{smor}\, \dot{\Lambda} . \equiv . P = \dot{\Lambda}$

Dem.

$\vdash . *151·32 . \mathrm{Transp} . \supset \vdash : P \,\mathrm{smor}\, \dot{\Lambda} . \supset . \sim \dot{\exists} ! P$ (1)

$\vdash . *151·13 . \qquad \supset \vdash : P = \dot{\Lambda} . \supset . P \,\mathrm{smor}\, \dot{\Lambda}$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***153·11.** $\vdash . 0_r = \mathrm{Nr}'\dot{\Lambda}$ [*153·1·101 . *152·1]

***153·111.** $\vdash . 0_r \in \mathrm{NR}$ [*152·41 . *153·11]

***153·12.** $\vdash . \exists ! 0_r$ [*51·161]

***153·13.** $\vdash . \exists ! 0_r \cap \mathrm{Rl}'R . \Lambda \in 0_r \cap \mathrm{Rl}'R$ [*61·3]

***153·14.** $\vdash : \mathrm{Nr}'P = 0_r . \equiv . P = \Lambda$

Dem.

$\vdash . *152·44 . *153·111·12 . \supset \vdash : \mathrm{Nr}'P = 0_r . \equiv . P \in 0_r .$

[*152·1] $\equiv . P = \dot{\Lambda} : \supset \vdash . \mathrm{Prop}$

* Cf. *161 and *181, where this point is more fully elucidated.

***153·15.** $\vdash . \text{smor}``0_r = 0_r$

Dem.

$\vdash . *152·51 . *153·111·13 . \supset \vdash . \text{smor}``0_r = \text{Nr}`\dot{\Lambda}$

$[*153·11] \qquad\qquad = 0_r . \supset \vdash . \text{Prop}$

***153·16.** $\vdash :. \mu \epsilon \text{NR} - \iota`0_r . \supset : P \epsilon \mu . \supset_P . \dot{\exists} ! P$

Dem.

$\vdash . *153·13 . *152·42 . \supset \vdash :. \mu \epsilon \text{NR} . \supset : \dot{\Lambda} \epsilon \mu . \supset . \mu = 0_r :$

$[\text{Transp}] \qquad\qquad \supset : \mu \neq 0_r . \supset . \dot{\Lambda} \sim\epsilon \mu \qquad (1)$

$\vdash . (1) . *25·63 . \supset \vdash . \text{Prop}$

***153·17.** $\vdash : \dot{\Lambda} \epsilon \text{Nr}`P . \equiv . \text{Nr}`P = 0_r . \equiv . \text{Nr}`P = \text{Nr}`\dot{\Lambda} . \equiv . P = \dot{\Lambda}$

$[*152·35 . *153·11·14]$

***153·18.** $\vdash . C``0_r = 0$

Dem.

$\vdash . *53·31 . \supset \vdash . C``\iota`\dot{\Lambda} = \iota`C`\dot{\Lambda} \qquad (1)$

$\vdash . (1) . *33·241 . (*56·03 . *54·01) . \supset \vdash . \text{Prop}$

***153·2.** $\vdash : P \epsilon 2_r . \equiv . (\exists x, y) . x \neq y . P = x \downarrow y \quad [*56·11]$

***153·201.** $\vdash : x \neq y . \equiv . x \downarrow y \epsilon 2_r \quad [*56·17]$

***153·202.** $\vdash : P, Q \epsilon 2_r . \supset . P \,\text{smor}\, Q \quad [*151·63 . *153·2]$

***153·203.** $\vdash : Q \epsilon 2_r . P \,\text{smor}\, Q . \supset . P \epsilon 2_r$

Dem.

$\vdash . *113·123 . \supset \vdash : S \epsilon 1 \rightarrow \text{Cls} . z, w \epsilon \text{ɑ}`S . \supset . S;(z \downarrow w) = (S`z) \downarrow (S`w) :$

$[*55·15] \qquad \supset \vdash : S \epsilon 1 \rightarrow \text{Cls} . C`(z \downarrow w) = \text{ɑ}`S . \supset .$

$S;(z \downarrow w) = (S`z) \downarrow (S`w) \qquad (1)$

$\vdash . *71·56 . \quad \supset \vdash :. S \epsilon 1 \rightarrow 1 . C`(z \downarrow w) = \text{ɑ}`S . \supset : z = w . \equiv . S`z = S`w :$

$[\text{Transp}] \qquad\qquad \supset : z \neq w . \equiv . S`z \neq S`w \qquad (2)$

$\vdash . (1) . (2) . *153·201 . \supset$

$\vdash : S \epsilon 1 \rightarrow 1 . z \neq w . C`(z \downarrow w) = \text{ɑ}`S . P = S;(z \downarrow w) . \supset . P \epsilon 2_r :$

$[*151·1] \qquad \supset \vdash : z \neq w . P \,\text{smor}\, (z \downarrow w) . \supset . P \epsilon 2_r :$

$[*153·2] \qquad \supset \vdash : Q \epsilon 2_r . P \,\text{smor}\, Q . \supset . P \epsilon 2_r : \supset \vdash . \text{Prop}$

***153·21.** $\vdash : P \epsilon 2_r . \supset . 2_r = \text{Nr}`P \quad [*153·202·203]$

***153·211.** $\vdash : x \neq y . \supset . 2_r = \text{Nr}`(x \downarrow y) \quad [*153·21·201]$

***153·212.** $\vdash . C``2_r = 2 \quad [*55·15 . *56·11 . *54·101]$

***153·22.** $\vdash : \exists ! 2_r \cap t''`z . \equiv . \exists ! 2(z) . \equiv . (\exists x, y) . x \neq y . x \epsilon t`z \quad [*153·211 . *101·4]$

***153·23.** $\vdash : P \epsilon 2_r . \supset . \text{Rl}`P \subset 0_r \cup 2_r \quad [*56·261]$

This proposition illustrates the reasons for not putting

$$1_r = \hat{P} \{(\exists x) . P = x \downarrow x\} \quad \text{Df.}$$

We want the inductive ordinals, like the inductive cardinals, to form a series in order of magnitude; but, as the above proposition illustrates, the relation-number of such relations as $x \downarrow x$ is not in the same series with 0_r and 2_r. The above proposition should be contrasted with *54·411.

***153·24.** $\vdash . 2_r = \text{Nr}'(\Lambda \downarrow \iota' x)$ [*153·211 . *51·161]

***153·25.** $\vdash . 2_r \in \text{NR}$ [*153·24 . *152·41]

***153·251.** $\vdash . 2_r \neq 0_r . 2_r \cap 0_r = \Lambda$

Dem.

$\vdash . *153·212·18 . *101·34·35 . \supset \vdash . C``2_r \neq C``0_r . C``2_r \cap C``0_r = \Lambda .$

[*13·12.Transp.*37·21] $\supset \vdash . 2_r \neq 0_r . C``(2_r \cap 0_r) = \Lambda .$

[*37·45] $\supset \vdash . 2_r \neq 0_r . 2_r \cap 0_r = \Lambda$

***153·26.** $\vdash . \exists ! 2_r \cap \text{Rl}'(\text{Cls} \uparrow \text{Cls})$ [*153·24 . *152·3]

***153·261.** $\vdash . \dot{\Lambda} \downarrow (x \downarrow x) \in 2_r$ [*55·134 . *56·11]

***153·262.** $\vdash . \exists ! 2_r \cap \text{Rel}^2$ [*153·261 . (*61·03)]

***153·27.** $\vdash . 2_r = \text{smor}``(2_r \cap \text{Rl}'\text{Cls}) = \text{smor}``(2_r \cap \text{Rel}^2)$
[*152·53 . *153·26·262·24]

***153·28.** $\vdash : x \neq y . \supset . B'(x \downarrow y) = x . B'\text{Cnv}'(x \downarrow y) = y$

Dem.

$\vdash . *93·101 . *55·15 . \supset \vdash : \text{Hp} . \supset . \overrightarrow{B}'(x \downarrow y) = \iota' x . \overrightarrow{B}'\text{Cnv}'(x \downarrow y) = \iota' y : \supset \vdash . \text{Prop}$

***153·281.** $\vdash : P \in 2_r . \supset . B'P = \iota'\text{D}'P . B'\breve{P} = \iota'\text{G}'P$ [*153·28 . *55·15]

The above proposition is used in the theory of series (*204·48).

***153·3.** $\vdash . 1_s = \dot{2} - 2_r = \hat{R}\{(\exists x) . R = x \downarrow x\}$ [*56·13 . (*153·01)]

***153·301.** $\vdash . 1_s = \breve{C}``1$ [*153·3 . *56·39]

***153·31.** $\vdash . x \downarrow y \in (x \downarrow x) \,\overline{\text{smor}}\, (y \downarrow y)$

Dem.

$\vdash . *72·182 . *55·15 . \supset \vdash . x \downarrow y \in 1 \to 1 . \text{G}'(x \downarrow y) = C'(y \downarrow y)$ (1)

$\vdash . *35·89 . *55·1 . \supset \vdash . x \downarrow y \,|\, y \downarrow y = x \downarrow y .$

[*150·1.*55·14] $\supset \vdash . (x \downarrow y) ; (y \downarrow y) = x \downarrow y \,|\, y \downarrow x$

[*35·89.*55·1] $= x \downarrow x$ (2)

$\vdash . (1) . (2) . *151·11 . \supset \vdash . \text{Prop}$

***153·311.** $\vdash : Q \in 1_s . P \,\text{smor}\, Q . \supset . P \in 1_s$

Dem.

$\vdash . *153·3 . *151·1 . \supset \vdash : \text{Hp} . \supset .$

$(\exists S, y) . Q = y \downarrow y . S \in 1 \to 1 . \text{G}'S = \iota' y . P = S ; Q .$

[*150·71] $\supset . (\exists S, y) . P = (S'y) \downarrow (S'y) .$

[*153·3] $\supset . P \in 1_s : \supset \vdash . \text{Prop}$

$*153\cdot32$. $\vdash . 1_s = \mathrm{Nr}`(y \downarrow y)$ $[*153\cdot31\cdot311]$

$*153\cdot33$. $\vdash . 1_s \in \mathrm{NR}$ $[*153\cdot32]$

$*153\cdot34$. $\vdash . \exists ! 1_s . 1_s \neq 0_r . 1_s \neq 2_r . 1_s \cap 0_r = \Lambda . 1_s \cap 2_r = \Lambda$

Dem.

$\vdash . *153\cdot3 . \qquad \supset \vdash . x \downarrow x \in 1_s .$

$[*10\cdot24] \qquad \supset \vdash . \exists ! 1_s$ (1)

$\vdash . *56\cdot103\cdot104 . \qquad \supset \vdash . 1_s \cap 0_r = \Lambda$ (2)

$\vdash . (1) . (2) . \qquad \supset \vdash . 1_s \neq 0_r$ (3)

$\vdash . *153\cdot301 . *56\cdot113 . \supset \vdash . 1_s \cap 2_r \subset \breve{C}``1 \cap \breve{C}``2 .$

$[*72\cdot41 . *101\cdot35] \qquad \supset \vdash . 1_s \cap 2_r = \Lambda$ (4)

$[(1)] \qquad \supset \vdash . 1_s \neq 2_r$ (5)

$\vdash . (1) . (2) . (3) . (4) . (5) . \supset \vdash . \mathrm{Prop}$

$*153\cdot341$. $\vdash : R \in 1_s . \equiv . \mathrm{Nr}`R = 1_s$ $[*153\cdot33\cdot34 . *152\cdot44]$

$*153\cdot35$. $\vdash : R \in 1_s . \supset . \mathrm{Nc}`C`R = C``\mathrm{Nr}`R = 1$

Dem.

$\vdash . *55\cdot15 . *153\cdot3 . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{Nc}`C`R = 1$ (1)

$\vdash . (1) . *152\cdot7 . \supset \vdash . \mathrm{Prop}$

$*153\cdot36$. $\vdash . C``1_s = 1$

Dem.

$\vdash . *153\cdot301 . \supset \vdash . C``1_s = C``\breve{C}``1$

$[*72\cdot502] \qquad = 1 . \supset \vdash . \mathrm{Prop}$

$*154.$ RELATION-NUMBERS OF ASSIGNED TYPES

Summary of $*154$.

This number gives propositions analogous to those of $*102$. In accordance with our general notations for typical definiteness, "$\mathrm{Nr}(P)\text{‘}Q$" means "the class of relations like Q and of the same type as P," "$\mathrm{Nr}(P_Q)$" means "the relation to a relation of the type of Q of the class of relations like it and of the type of P." By a special definition, "$\mathrm{NR}^Q(P)$" is to mean all typically definite relation-numbers of the form "$\mathrm{Nr}(P_Q)\text{‘}R$," *i.e.* all relation-numbers generated by the relation $\mathrm{Nr}(P_Q)$, *i.e.* the domain of $\mathrm{Nr}(P_Q)$.

Existence-theorems in this subject can be proved by means of $*154 \cdot 14$, which states that relations like Q exist in the type of P when, and only when, classes similar to $C\text{‘}Q$ exist in the type of $C\text{‘}P$. In virtue of this proposition, the existence-theorems of our present topic are deducible from those for cardinals. In symbols, this proposition is

$*154 \cdot 14.$ $\vdash : \exists ! \, \mathrm{Nr}(P)\text{‘}Q \, . \equiv . \, \exists ! \, \mathrm{Nc}(C\text{‘}P)\text{‘}C\text{‘}Q$

Hence by $*102 \cdot 73$ we deduce

$*154 \cdot 242.$ $\vdash . \, \Lambda \in \mathrm{NR}^{\iota\text{‘}P}(P)$

whence, by $*152 \cdot 72$,

$*154 \cdot 9.$ $\vdash . \, \mathrm{NC} = C\text{“}\mathrm{NR}$

The remaining propositions are chiefly analogues of those in $*102$. Very few of them are subsequently referred to.

$*154 \cdot 01.$ $\mathrm{NR}^Y(X) = \mathrm{D}\text{‘}\mathrm{Nr}(X_Y)$ Df

$*154 \cdot 1.$ $\vdash : \exists ! \, \mathrm{Rl}\text{‘}P \cap \mathrm{Nr}\text{‘}Q \, . \supset . \, \exists ! \, \mathrm{Cl}\text{‘}C\text{‘}P \cap \mathrm{Nc}\text{‘}C\text{‘}Q$

Dem.

$\vdash . *152 \cdot 1 . \supset \vdash : \mathrm{Hp} . \supset . (\exists R) . R \Subset P . R \, \mathrm{smor} \, Q .$

$[*151 \cdot 18] \quad \supset . (\exists R) . R \Subset P . C\text{‘}R \, \mathrm{sm} \, C\text{‘}Q .$

$[*33 \cdot 265] \quad \supset . (\exists R) . C\text{‘}R \subset C\text{‘}P . C\text{‘}R \, \mathrm{sm} \, C\text{‘}Q .$

$[*100 \cdot 1] \quad \supset . \exists ! \, \mathrm{Cl}\text{‘}C\text{‘}P \cap \mathrm{Nc}\text{‘}C\text{‘}Q : \supset \vdash . \mathrm{Prop}$

$*154 \cdot 11.$ $\vdash : \exists ! \, \mathrm{Cl}\text{‘}C\text{‘}P \cap \mathrm{Nc}\text{‘}C\text{‘}Q \, . \supset . \, (\exists R) . R \, \mathrm{smor} \, Q \, . \, C\text{‘}R \subset C\text{‘}P$

Dem.

$\vdash . *100 \cdot 1 . *73 \cdot 1 . \supset \vdash : \mathrm{Hp} . \supset . (\exists S) . S \in 1 \to 1 . \mathrm{D}\text{‘}S \subset C\text{‘}P . \text{Ɑ}\text{‘}S = C\text{‘}Q .$

$[*151 \cdot 1] \quad \supset . (\exists S) . \mathrm{D}\text{‘}S \subset C\text{‘}P . S;Q \, \mathrm{smor} \, Q .$

$[*150 \cdot 203] \quad \supset . (\exists S) . C\text{‘}S;Q \subset C\text{‘}P . S;Q \, \mathrm{smor} \, Q : \supset \vdash . \mathrm{Prop}$

***154·12.** $\vdash : \exists ! \, \mathrm{Rl}`(\alpha \uparrow \alpha) \cap \mathrm{Nr}`Q \,.\equiv.\, \exists ! \, \mathrm{Cl}`\alpha \cap \mathrm{Nc}`C`Q \,.\equiv.\, \mathrm{Nc}`\alpha \geqslant \mathrm{Nc}`C`Q$

Dem.

$\vdash . *154·1 . *35·9 . \quad \supset \vdash : \exists ! \, \mathrm{Rl}`(\alpha \uparrow \alpha) \cap \mathrm{Nr}`Q \,.\supset.\, \exists ! \, \mathrm{Cl}`\alpha \cap \mathrm{Nc}`C`Q \quad (1)$

$\vdash . *154·11 . *35·92 . \supset \vdash : \exists ! \, \mathrm{Cl}`\alpha \cap \mathrm{Nc}`C`Q \,.\supset.\, \exists ! \, \mathrm{Rl}`(\alpha \uparrow \alpha) \cap \mathrm{Nr}`Q \quad (2)$

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***154·121.** $\vdash . \mathrm{Rl}`(t_0`C`P \uparrow t_0`C`P) = t`P = t_{00}`C`P$

Dem.

$\vdash . *64·5 . \quad \supset \vdash . \mathrm{Rl}`(t_0`C`P \uparrow t_0`C`P) = t`(C`P \uparrow C`P) \quad (1)$

$\vdash . *64·201 . \supset \vdash . t`(C`P \uparrow C`P) = t`P \quad (2)$

$\vdash . (1) . (2) . *64·54 . \supset \vdash . \mathrm{Prop}$

***154·13.** $\vdash : \exists ! \, t`P \cap \mathrm{Nr}`Q \,.\equiv.\, \exists ! \, t`C`P \cap \mathrm{Nc}`C`Q \,.\equiv.\, \mathrm{Nc}`t_0`C`P \geqslant \mathrm{Nc}`C`Q$

Dem.

$\vdash . *154·12 \dfrac{t_0`C`P}{\alpha} . *154·121 . \supset$

$\vdash : \exists ! \, t`P \cap \mathrm{Nr}`Q \,.\equiv.\, \exists ! \, \mathrm{Cl}`t_0`C`P \cap \mathrm{Nc}`C`Q \quad (1)$

$\vdash . (1) . *63·65 . *117·22 . \supset \vdash . \mathrm{Prop}$

***154·14.** $\vdash : \exists ! \, \mathrm{Nr}(P)`Q \,.\equiv.\, \exists ! \, \mathrm{Nc}(C`P)`C`Q$ [*154·13 . (*65·04)]

In virtue of *154·14 and the propositions of *102, *103, *104, *105, *106, we see that all homogeneous or ascending relation-numbers exist, while Λ is a member of every descending type of relation-numbers. Remembering that the relations concerned must be homogeneous, we see that there are two kinds of steps by which their types may be raised, namely (1) from P to relations of the type of $t`C`P \uparrow t`C`P$, *i.e.* from P to relations of the type of $C`P \downarrow C`P$, or of $\iota ; P$; (2) from P to relations of the type of $t`P \uparrow t`P$, *i.e.* from P to relations of the type of $P \downarrow P$, or of $\downarrow x ; P$ if $x \in t_0`C`P$. Thus repetitions of the two steps from P to $\iota ; P$, and from P to $\downarrow x ; P$, where $x \in t_0`C`P$, will enable us, without changing the relation-number, to raise its type indefinitely. It will be observed that, in accordance with our general definitions for relative types, the type of $\iota ; P$ is $t^{11}`C`P$, and the type of $\downarrow x ; P$ (where $x \in t_0`C`P$) is $t^{11}`P$.

***154·2.** $\vdash . \mathrm{Nr}(X_Y)`Q = \hat{P}\{P \, \mathrm{smor}_{(X,Y)} Q\}$ [*65·2 . (*152·01)]

***154·201.** $\vdash . \mathrm{Nr}(X)`Q = \mathrm{Nr}`Q \cap t`X$ [Proof as in *102·6]

***154·202.** $\vdash : P \in \mathrm{Nr}(X_Y)`Q \,.\equiv.\, P \in \mathrm{Nr}(X)`Q \,.\, Q \in t`Y \,.\equiv.$
$P \in \mathrm{Nr}`Q \,.\, P \in t`X \,.\, Q \in t`Y$ [*152·2·201 . (*65·1)]

***154·203.** $\vdash : Q \in t`Y \,.\supset.\, \mathrm{Nr}(X_Y)`Q = \mathrm{Nr}(X)`Q$ [*154·202]

When Q belongs to any other type than $t`Y$, $\mathrm{Nr}(X_Y)`Q$ is meaningless.

***154·21.** $\vdash . \mathrm{NR}^Y(X) = \hat{\lambda}\{(\exists Q) . \lambda = \mathrm{Nr}(X_Y)`Q\}$ [(*154·01)]

***154·22.** $\vdash . \mathrm{NR}^Y(X) = \mathrm{Nr}(X)``t`Y = (\cap t`X)``\mathrm{Nr}``t`Y$

Dem.

$\vdash . *154\cdot21\cdot202 . \supset$

$\vdash :. \lambda \in \mathrm{NR}^Y(X) . \equiv : (\exists Q) : P \in \lambda . \equiv_P . P \in \mathrm{Nr}(X)`Q . Q \in t`Y :$

[*63·108.*4·73] $\equiv : (\exists Q) : Q \in t`Y : P \in \lambda . \equiv_P . P \in \mathrm{Nr}(X)`Q :$

[*20·43] $\equiv : (\exists Q) . Q \in t`Y . \lambda = \mathrm{Nr}(X)`Q :$

[*37·6] $\equiv : \lambda \in \mathrm{Nr}(X)``t`Y$ (1)

$\vdash . (1) . *154\cdot201 . \supset \vdash .$ Prop

***154·23.** $\vdash : \Lambda \in \mathrm{NR}^Q(P) . \equiv . \Lambda \in \mathrm{NC}^{C`Q}(C`P) . \equiv . \Lambda \in \mathrm{NC}(C`P)``t`C`Q$

Dem.

$\vdash . *154\cdot22 . \supset \vdash : \Lambda \in \mathrm{NR}^Q(P) . \equiv . \Lambda \in \mathrm{Nr}(P)``t`Q .$

[*37·6] $\equiv . (\exists R) . R \in t`Q . \Lambda = \mathrm{Nr}(P)`R .$

[*154·14.Transp] $\equiv . (\exists R) . R \in t`Q . \Lambda = \mathrm{Nc}(C`P)`C`R .$

[*64·24] $\equiv . (\exists R) . C`R \in t`C`Q . \Lambda = \mathrm{Nc}(C`P)`C`R .$

[*35·942] $\equiv . (\exists \alpha) . \alpha \in t`C`Q . \Lambda = \mathrm{Nc}(C`P)`\alpha .$

[*37·6] $\equiv . \Lambda \in \mathrm{Nc}(C`P)``t`C`Q .$ (1)

[*102·62] $\equiv . \Lambda \in \mathrm{NC}^{C`Q}(C`P)$ (2)

$\vdash . (1) . (2) . \supset \vdash .$ Prop

***154·24.** $\vdash : C`Q = t`C`P . \supset . \mathrm{Nr}(P)`Q = \Lambda$ [*102·73 . *154·14]

***154·241.** $\vdash . \mathrm{Nr}(P)`I \upharpoonright t`C`P = \Lambda$ [*154·24]

***154·242.** $\vdash . \Lambda \in \mathrm{NR}^{\iota`P}(P)$

Dem.

$\vdash . *35\cdot91 . \supset \vdash . I \upharpoonright t`C`P \subset t`C`P \uparrow t`C`P$

[*63·64] $\subset t_0`\iota``C`P \uparrow t_0`\iota``C`P$

[*150·22] $\subset t_0`C`\iota;P \uparrow t_0`C`\iota;P$ (1)

$\vdash . (1) . *154\cdot121 . \supset \vdash . I \upharpoonright t`C`P \in t`\iota;P$ (2)

$\vdash . (2) . *154\cdot22\cdot241 . \supset \vdash .$ Prop

***154·25.** $\vdash : C`Q = t_{00}`C`P . \supset . \mathrm{Nr}(P)`Q = \Lambda$ [*106·53 . *154·14]

***154·251.** $\vdash . \Lambda \in \mathrm{NR}^{P \downarrow P}(P)$

Dem.

$\vdash . *154\cdot23 . \supset \vdash : \Lambda \in \mathrm{NR}^{P \downarrow P}(P) . \equiv . \Lambda \in \mathrm{Nc}(C`P)``t`C`(P \downarrow P) .$

[*55·15] $\equiv . \Lambda \in \mathrm{Nc}(C`P)``t`\iota`P .$

[*63·61] $\equiv . \Lambda \in \mathrm{Nc}(C`P)``t`t`P .$

[*154·121] $\equiv . \Lambda \in \mathrm{Nc}(C`P)``t`t_{00}`C`P$ (1)

$\vdash . (1) . *106\cdot53 . *104\cdot264 . \supset \vdash .$ Prop

***154·26.** $\vdash : P \in t`Q . \supset . \exists ! \mathrm{Nr}(P)`Q$ [*64·231 . *103·3·13 . *154·14]

***154·261.** $\vdash : C`P \in t^{2}`C`Q . \supset . \exists ! \mathrm{Nr}(P)`Q$ [*104·21·1 . *154·14]

***154·262.** $\vdash : C`P \in t_{00}`C`Q . \supset . \exists ! \mathrm{Nr}(P)`Q$ [*106·21·1 . *154·14]

The following propositions are concerned with the two particular transformations from P to $\iota ; P$ and from P to $x \downarrow ; P$, which are useful in raising the type of a relation-number.

***154·31.** $\vdash . t`\iota ; P = t^{11}`C`P$

Dem.

$\vdash . *154·121 . *150·22 . \supset \vdash . t`\iota ; P = \mathrm{Rl}`(t_0`\iota``C`P \uparrow t_0`\iota``C`P)$
$[*63·64] \quad = \mathrm{Rl}`(t`C`P \uparrow t`C`P)$
$[*64·56] \quad = t^{11}`C`P . \supset \vdash . \mathrm{Prop}$

***154·311.** $\vdash . \exists ! \mathrm{Nr}(t^{11}`C`P)`P \quad [*154·31 . *152·6]$

***154·32.** $\vdash : x \in t_0`C`P . \supset . t`x \downarrow ; P = t^{11}`P . t_0`x \downarrow ``C`P = t`P$

Dem.

$\vdash . *154·121 . *150·22 . \supset \vdash . t`x \downarrow ; P = \mathrm{Rl}`\{(t_0`x \downarrow ``C`P) \uparrow (t_0`x \downarrow ``C`P)\}$ (1)
$\vdash . *64·52 . \quad \supset \vdash : x, y \in t_0`C`P . \supset . x \downarrow y \in t`(t_0`C`P \uparrow t_0`C`P) .$
$[*154·121] \quad \supset . x \downarrow y \in t`P$ (2)
$\vdash . (2) . \quad \supset \vdash : x \in t_0`C`P . \supset . x \downarrow ``C`P \subset t`P .$
$[*63·21] \quad \supset . t_0`x \downarrow ``C`P = t`P$ (3)
$\vdash . (1) . (3) . \quad \supset \vdash : \mathrm{Hp} . \supset . t`x \downarrow ; P = \mathrm{Rl}`(t`P \uparrow t`P)$
$[*64·56] \quad = t^{11}`P$ (4)
$\vdash . (3) . (4) . \supset \vdash . \mathrm{Prop}$

***154·321.** $\vdash . \exists ! \mathrm{Nr}(t^{11}`P)`P \quad [*154·32 . *152·62 . *63·18]$

***154·322.** $\vdash : x \in t_0`C`P . \supset . t` \downarrow x ; P = t^{11}`P$ [Proof as in *154·32]

***154·33.** $\vdash : x \in t_0`C`P . \supset . t`P \downarrow ; x \downarrow ; P = t^{11}`s`t^{11}`P$

Dem.

$\vdash . *154·32 . \supset \vdash : \mathrm{Hp} . \supset . P \in t_0`x \downarrow ``C`P .$
$[*150·22] \quad \supset . P \in t_0`C`x \downarrow ; P .$
$[*154·32] \quad \supset . t`P \downarrow ; x \downarrow ; P = t^{11}`x \downarrow ; P$
$[*64·23] \quad = t^{11}`s`t`x \downarrow ; P$
$[*154·32] \quad = t^{11}`s`t^{11}`P : \supset \vdash . \mathrm{Prop}$

***154·331.** $\vdash . \exists ! \mathrm{Nr}(t^{11}`s`t^{11}`P) \quad [*154·33 . *152·62 . *63·18]$

***154·4.** $\vdash . \mathrm{Nr}(X_Y)`Q = \hat{P}\{(\exists S) . S \in 1 \to 1 . \text{Ꮯ}`S = C`Q . P = S ; Q .$
$\mathrm{D}`S \in t`C`X . \text{Ꮯ}`S \in t`C`Y\}$

Dem.

$\vdash . *154·202 . *152·1 . \supset$
$\vdash :. P \in \mathrm{Nr}(X_Y)`Q . \equiv : (\exists S) . S \in 1 \to 1 . \text{Ꮯ}`S = C`Q . P = S ; Q : P \in t`X . Q \in t`Y :$
$[*64·24] \quad \equiv : (\exists S) . S \in 1 \to 1 . \text{Ꮯ}`S = C`Q . P = S ; Q . C`P \in t`C`X .$
$C`Q \in t`C`Y :$
$[*13·193 . *150·23] \quad \equiv : (\exists S) . S \in 1 \to 1 . \text{Ꮯ}`S = C`Q . P = S ; Q .$
$\mathrm{D}`S \in t`C`X . \text{Ꮯ}`S \in t`C`Y :. \supset \vdash . \mathrm{Prop}$

*154·401. $\vdash . \mathrm{Nr}(X_Y)`Q = \hat{P}\{\exists ! (P \,\overline{\mathrm{smor}}\, Q) \cap t`(C`X \uparrow C`Y)\}$
[*154·4 . *151·11 . *64·63]

The remaining propositions of this number (except *154·9) are the analogues of those whose numbers have the same decimal part in *102. They are here given without proof, because the proofs are, step by step, analogous to the proofs of the corresponding propositions in *102.

*154·41. $\vdash : P \epsilon \mathrm{Nr}(X_Z)`R . Q \epsilon \mathrm{Nr}(Y_Z)`R . \supset . P \epsilon \mathrm{Nr}(X_Y)`Q . Q \epsilon \mathrm{Nr}(Y_X)`P$

*154·42. $\vdash . P \epsilon \mathrm{Nr}(P_P)`P$

*154·43. $\vdash . \exists ! \mathrm{Nr}(P_P)`P$

*154·46. $\vdash : P \epsilon \mathrm{Nr}(X_Y)`Q . \equiv . Q \epsilon \mathrm{Nr}(Y_X)`P . \equiv . P \,\mathrm{smor}\, Q . P \epsilon t`X . Q \epsilon t`Y$

*154·52. $\vdash : \exists ! \mathrm{Nr}(X_Y)`Q . \supset . \mathrm{Nr}(X_Y)`Q \epsilon \mathrm{NR}^X(X)$

*154·53. $\vdash . \mathrm{NR}^Y(X) - \iota`\Lambda \subset \mathrm{NR}^X(X)$

*154·55. $\vdash : \Lambda \sim\epsilon \mathrm{NR}^X(Y) . \supset . \mathrm{NR}^Y(X) - \iota`\Lambda = \mathrm{NR}^X(X)$

*154·64. $\vdash : \mu \epsilon \mathrm{NR} . \exists ! \mu . \supset . (\exists P, Q) . \mu = \mathrm{Nr}(P)`Q$

*154·641. $\vdash : \mu \epsilon \mathrm{NR} . \supset . (\exists P, Q) . \mu = \mathrm{Nr}(P)`Q$ [*154·64·241]

*154·8. $\vdash : P \epsilon \mathrm{Nr}(X_Y)`Q . P \,\mathrm{smor}\, R . R \epsilon t`S . \supset . R \epsilon \mathrm{Nr}(S_Y)`Q . R \epsilon \mathrm{Nr}(S_X)`P$

*154·81. $\vdash : P \epsilon \mathrm{Nr}(X_Y)`Q . \supset . \mathrm{smor}``\mathrm{Nr}(X_Y)`Q \cap t`S = \mathrm{Nr}(S_Y)`Q = \mathrm{Nr}(S_X)`P$

*154·82. $\vdash : \mu \epsilon \mathrm{NR}^Y(X) . \exists ! \mu . \supset . \mathrm{smor}``\mu \cap t`S \epsilon \mathrm{NR}^Y(S)$

*154·83. $\vdash : \mu \epsilon \mathrm{NR}^Y(X) . \nu = \mathrm{smor}``\mu \cap t`S . \exists ! \nu . \supset .$
$\mathrm{smor}``\mu \cap t`S = \mathrm{smor}``\nu \cap t`S . \mu = \mathrm{smor}``\nu \cap t`X$

*154·84. $\vdash : (\exists P) . P \,\mathrm{smor}\, X . P \epsilon t`X . Q \,\mathrm{smor}\, P . \equiv . Q \,\mathrm{smor}\, X$

*154·85. $\vdash . \mathrm{smor}``\mu \cap t`Y = \mathrm{smor}_Y``\mu$

*154·86. $\vdash : \mu = \mathrm{Nr}(X)`Q . \exists ! \mu . \supset . \mathrm{smor}_Y``\mu = \mathrm{Nr}(Y)`Q$

*154·861. $\vdash . \mathrm{smor}_X``\mathrm{smor}_Y``\mu \subset \mathrm{smor}_X``\mu$

*154·87. $\vdash : \mu = \mathrm{Nr}(Y)`Q . \exists ! \mathrm{Nr}(X)`Q . \supset . \mathrm{smor}_P``\mu = \mathrm{smor}_P``\mathrm{smor}_X``\mu$

*154·88. $\vdash : \mu = \mathrm{Nr}(Y)`Q . \exists ! \mathrm{smor}_P``\mu . \supset .$
$\mathrm{smor}_P``\mu = \mathrm{Nr}(P)`Q . \mathrm{smor}_X``\mu = \mathrm{Nr}(X)`Q .$
$\mathrm{smor}_X``\mu = \mathrm{smor}_X``\mathrm{smor}_P``\mu = \mathrm{smor}_X``\mathrm{Nr}(P)`Q$

*154·9. $\vdash . \mathrm{NC} = C```\mathrm{NR}$

Dem.

$\vdash . *37·29 . \supset \vdash : \mu = \Lambda . \supset . \mu = C``\Lambda .$
[*154·241] $\supset . \mu \epsilon C```\mathrm{NR}$ (1)
$\vdash . *37·29 . \supset \vdash : \nu = \Lambda . \supset . C``\nu = \Lambda .$
[*102·73] $\supset . C``\nu \epsilon \mathrm{NC}$ (2)
$\vdash . (1) . (2) . *152·72 . \supset \vdash . \mathrm{Prop}$

$*155$. HOMOGENEOUS RELATION-NUMBERS

Summary of $*155$.

A relation-number is called *homogeneous* when it is generated by a homogeneous relation of likeness, *i.e.* when it consists of all relations which are like a given relation P and of the same type as P. For the homogeneous relation-number of P we write "$\mathrm{N_0r}`P$"; thus $\mathrm{N_0r}`P = \mathrm{Nr}`P \cap t`P$. When P is given, $\mathrm{N_0r}`P$ is typically definite. We have always $P \epsilon \mathrm{N_0r}`P$, hence $\exists ! \mathrm{N_0r}`P$. Conversely, if a typically definite relation-number is not null, it is a homogeneous relation-number; in fact, if $P$ is a member of it, it is $\mathrm{N_0r}`P$. Thus the homogeneous relation-numbers are all the relation-numbers except Λ.

Homogeneous relation-numbers play the same part in relation-arithmetic as homogeneous cardinals play in cardinal arithmetic. The propositions of this number (except $*155{\cdot}6{\cdot}61$) are the analogues of those with the same decimal part in $*103$. Their proofs are exactly analogous to the proofs of their analogues in $*103$, and are therefore omitted.

The following propositions are the most useful in this number.

$*155{\cdot}11$. $\vdash : Q \epsilon \mathrm{N_0r}`P . \equiv . Q \,\mathrm{smor}\, P . Q \epsilon t`P . \equiv . Q \epsilon \mathrm{Nr}`P . Q \epsilon t`P$

This merely embodies the definition.

$*155{\cdot}12$. $\vdash . P \epsilon \mathrm{N_0r}`P$

whence

$*155{\cdot}13$. $\vdash . \exists ! \mathrm{N_0r}`P$

$*155{\cdot}16$. $\vdash : \mathrm{N_0r}`P = \mathrm{Nr}`Q . \equiv . \mathrm{Nr}`P = \mathrm{Nr}`Q$

This proposition is used in the theory of well-ordered series ($*253$ and $*255$). It requires that the equation "$\mathrm{Nr}`P = \mathrm{Nr}`Q$" on the right-hand side should be subject to the convention AT. Otherwise, the typical ambiguities might be so determined as to give $\mathrm{Nr}`P = \mathrm{Nr}`Q = \Lambda$, which would not imply $\mathrm{N_0r}`P = \mathrm{Nr}`Q$.

$*155{\cdot}2$. $\vdash : \mu \epsilon \mathrm{N_0R} . \equiv . (\exists P) . \mu = \mathrm{Nr}`P \cap t`P . \equiv . (\exists P) . \mu = \mathrm{N_0r}`P$

This merely embodies the definition of $\mathrm{N_0R}$.

$*155{\cdot}22$. $\vdash : \mu \epsilon \mathrm{N_0R} . \supset . \exists ! \mu$

$*155{\cdot}26$. $\vdash :. \mu \epsilon \mathrm{NR} . \supset : P \epsilon \mu . \equiv . \mathrm{N_0r}`P = \mu$

$*155{\cdot}27$. $\vdash : \mu = \mathrm{N_0r}`P . \equiv . \mu \epsilon \mathrm{NR} . P \epsilon \mu$

$*155{\cdot}34$. $\vdash . \mathrm{NR} - \iota`\Lambda \subset \mathrm{N_0R}$

*155·4. $\vdash . \text{smor}``\text{N}_0\text{r}`P = \text{Nr}`P$

*155·5. $\vdash . 0_r \in \text{N}_0\text{R}$

*155·6. $\vdash . C``\text{N}_0\text{r}`P = \text{N}_0\text{c}`C`P$

This last proposition connects homogeneous relation-numbers with homogeneous cardinals.

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*155·01. $\text{N}_0\text{r}`P = \text{Nr}`P \cap t`P$ Df

*155·02. $\text{N}_0\text{R} = \text{D}`\text{N}_0\text{r}$ Df

*155·1. $\vdash . \text{N}_0\text{r}`P = (\text{Nr}`P)_P = \text{Nr}(P)`P = \text{Nr}(P_P)`P$

*155·11. $\vdash : Q \in \text{N}_0\text{r}`P . \equiv . Q \,\text{smor}\, P . Q \in t`P . \equiv . Q \in \text{Nr}`P . Q \in t`P$

*155·12. $\vdash . P \in \text{N}_0\text{r}`P$

*155·13. $\vdash . \exists ! \text{N}_0\text{r}`P$

*155·14. $\vdash : \text{N}_0\text{r}`P = \text{N}_0\text{r}`Q . \equiv . P \in \text{N}_0\text{r}`Q . \equiv . Q \in \text{N}_0\text{r}`P . \equiv . P \,\text{smor}\, Q . Q \in t`P$

*155·15. $\vdash : \exists ! \text{N}_0\text{r}`P \cap \text{N}_0\text{r}`Q . \equiv . \text{N}_0\text{r}`P = \text{N}_0\text{r}`Q$

*155·16. $\vdash : \text{N}_0\text{r}`P = \text{Nr}`Q . \equiv . \text{Nr}`P = \text{Nr}`Q$

*155·2. $\vdash : \mu \in \text{N}_0\text{R} . \equiv . (\exists P) . \mu = \text{Nr}`P \cap t`P . \equiv . (\exists P) . \mu = \text{N}_0\text{r}`P$

*155·21. $\vdash . \text{N}_0\text{r}`P \in \text{N}_0\text{R} . \text{N}_0\text{r}`P \in \text{NR}$

*155·22. $\vdash : \mu \in \text{N}_0\text{R} . \supset . \exists ! \mu$

*155·23. $\vdash . \Lambda \sim\in \text{N}_0\text{R}$

*155·24. $\vdash . \text{N}_0\text{R} \in \text{Cls ex}^2 \text{excl}$

*155·25. $\vdash :. \mu, \nu \in \text{N}_0\text{R} . \supset : \exists ! \mu \cap \nu . \equiv . \mu = \nu$

*155·26. $\vdash :. \mu \in \text{NR} . \supset : P \in \mu . \equiv . \text{N}_0\text{r}`P = \mu$

*155·27. $\vdash : \mu = \text{N}_0\text{r}`P . \equiv . \mu \in \text{NR} . P \in \mu$

*155·28. $\vdash : (\exists R) . R \,\text{smor}\, P . \mu = \text{N}_0\text{r}`R . \equiv . \exists ! \mu . \mu = \text{Nr}`P$

*155·3. $\vdash : Q \in t`P . \supset . \text{N}_0\text{r}`Q = \text{Nr}(P)`Q = \text{Nr}(P_P)`Q = \text{Nr}`Q \cap t`P$

*155·301. $\vdash . \text{NR}^P(P) = \text{N}_0\text{R}(P)$

*155·31. $\vdash : \exists ! \text{Nr}(X_Y)`Q . \supset . \text{Nr}(X_Y)`Q \in \text{N}_0\text{R}(X)$

*155·32. $\vdash . \text{NR}^Y(X) - \iota`\Lambda \subset \text{N}_0\text{R}(X)$

*155·33. $\vdash . \text{NR}(X) - \iota`\Lambda \subset \text{N}_0\text{R}(X)$

*155·34. $\vdash . \text{NR} - \iota`\Lambda \subset \text{N}_0\text{R}$

*155·35. $\vdash : \Lambda \sim\in \text{NR}^X(Y) . \supset . \text{NR}^Y(X) - \iota`\Lambda = \text{N}_0\text{R}(X)$

*155·4. $\vdash . \text{smor}``\text{N}_0\text{r}`P = \text{Nr}`P$

*155·41. $\vdash . \text{smor}``\text{N}_0\text{r}`P \cap t`Q = \text{Nr}(Q)`P$

*155·42. $\vdash : Q \,\text{smor}\, P . \equiv . \text{Nr}(Q)`P = \text{N}_0\text{r}`Q$

*155·43. $\vdash : \mu \in \text{NR} . \supset . \text{smor}``\mu \cap t_0`\mu = \mu$

*155·44. $\vdash :. \mu, \nu \in \mathrm{N_0R} . \supset : \mu = \mathrm{smor}``\nu . \equiv . \nu = \mathrm{smor}``\mu$

*155·5. $\vdash . 0_r \in \mathrm{N_0R}$

*155·51. $\vdash . 2_r \cap \mathrm{Rl}`\mathrm{Cls} \in \mathrm{N_0R}$

*155·52. $\vdash . 2_r \cap \mathrm{Rel}^2 \in \mathrm{N_0R}$

The following propositions have no analogue in *103.

*155·6. $\vdash . C``\mathrm{N_0r}`P = \mathrm{N_0c}`C`P$

Dem.

$\vdash . *100·11 . *103·11 . \supset \vdash :. \alpha \in \mathrm{N_0c}`C`P . \equiv :$

$\alpha \in t`C`P : (\exists S) . S \in 1 \rightarrow 1 . \mathrm{D}`S = \alpha . \mathrm{Cl}`S = C`P :$

[*150·23] $\equiv : \alpha \in t`C`P : (\exists S) . S \in 1 \rightarrow 1 . C`S ; P = \alpha . \mathrm{Cl}`S = C`P :$

[*151·11] $\equiv : \alpha \in t`C`P : (\exists Q) . Q \,\mathrm{smor}\, P . \alpha = C`Q :$

[*64·24] $\equiv : (\exists Q) . Q \,\mathrm{smor}\, P . Q \in t`P . \alpha = C`Q :$

[*152·11.*155·11] $\equiv : \alpha \in C``\mathrm{N_0r}`P :. \supset \vdash .$ Prop

*155·61. $\vdash . C```\mathrm{N_0R} = \mathrm{N_0C}$ [*155·6]

On ascending and descending relation-numbers, propositions analogous to those of *104, *105, and *106 might be proved by proofs analogous to those given in those numbers. It is, however, scarcely necessary to add anything to the propositions already proved, namely *154·24·241·242·25·251 on descending relation-numbers, *154·26·261·262·31·311·32·321·322·33·331 on ascending relation-numbers, and *155·23·34 giving the relations of non-homogeneous to homogeneous relation-numbers. Ascending relation-numbers all exist, and those that start from the type of P, wherever they end*, are the correspondents† of the homogeneous relation-numbers of the type of P, and are only some of the homogeneous relation-numbers of the type in which they end. Descending relation-numbers consist of Λ together with the homogeneous relation-numbers of the type in which they end: they are the correspondents of only some of the type in which they begin, or rather, Λ is the common correspondent of all those relation-numbers in the initial type which are not correspondents of any homogeneous relation-number in the end-type. These properties are exactly the same as in the case of cardinals, as might be foreseen by *154·14.

* We say that $\mathrm{Nr}\,(P)`Q$ starts from the type of Q and ends in the type of P.

† We call two typically definite relation-numbers *correspondents* when they only differ as to the typical determination, *i.e.* $\mathrm{Nr}\,(X)`P$ and $\mathrm{Nr}\,(Y)`P$ are correspondents.

SECTION B

ADDITION OF RELATIONS, AND THE PRODUCT OF TWO RELATIONS

Summary of Section B.

In the present section, we have to consider the kind of addition of relations which is required in ordinal arithmetic. In cardinal arithmetic, if κ is a class of mutually exclusive classes, $s'\kappa$ has the properties required of their sum, and thus we do not require a new kind of logical addition before dealing with arithmetical addition. But in ordinal arithmetic this is not so. Suppose P and Q are the generating relations of two series, and we wish to add the Q-series at the end of the P-series. Then we wish every term of the P-series to precede every term of the Q-series; thus $P \mathbin{\dot\cup} Q$ is not the generating relation of the new series, since $P \mathbin{\dot\cup} Q$ gives no relation between the terms of the P-series and the terms of the Q-series. The relation we want is

$$P \mathbin{\dot\cup} Q \mathbin{\dot\cup} C'P \uparrow C'Q,$$

since this makes every term of the P-series precede every term of the Q-series. Hence we put

$$P \mathbin{\updownarrow} Q = P \mathbin{\dot\cup} Q \mathbin{\dot\cup} C'P \uparrow C'Q \quad \text{Df.}$$

It will be seen that $P \mathbin{\updownarrow} Q$ is in general different from $Q \mathbin{\updownarrow} P$.

If $C'P$ and $C'Q$ have no common terms, the sum of the relation-numbers of P and Q is the relation-number of $P \mathbin{\updownarrow} Q$ (cf. $*180$).

The addition of a single term to a series requires a new definition, and cannot be dealt with as a particular case of the addition of two relations. It might be thought that, just as $\alpha \cup \iota'x$ gives the result of adding the one term x to the class α, so $P \mathbin{\updownarrow} (x \downarrow x)$ would give the result of adding the one term x to the series P. But this is not the case, since, when we add a term to a series, we do not want this term to precede itself, whereas $P \mathbin{\updownarrow} (x \downarrow x)$ is a relation which x has to itself. What we want is a relation which every member of $C'P$ has to x but which x does not have to itself; thus we take $P \mathbin{\dot\cup} C'P \uparrow \iota'x$ as our relation, and put

$$P \mathbin{+\!\!\!\rightarrow} x = P \mathbin{\dot\cup} C'P \uparrow \iota'x \quad \text{Df.}$$

This definition defines the generating relation of the series obtained by adding x at the *end* of the P-series; similarly for adding x at the *beginning* we put

$$x \mathbin{\leftarrow\!\!\!+} P = \iota'x \uparrow C'P \mathbin{\dot\cup} P \quad \text{Df.}$$

If x is not a member of $C'P$, the relation-number of $P \rightarrow\!\!\!\!+ x$ is the sum of the relation-number of P and the ordinal 1, which we represent by $\dot{1}$. (The ordinal 1 has no meaning by itself, but only as a summand.)

The sum of a series of series is defined in the same way as the sum of two series was defined. Let P be a serial relation whose field consists of serial relations. Then the sum of all the series generated by members of $C'P$, when these series are taken in the order generated by P, must be a relation which holds between x and y whenever either (1) x and y both belong to the field of one of the series, and x precedes y in this series, or (2) x belongs to the field of an earlier series than that to which y belongs. In the first case, we have $(\exists Q) . Q \epsilon C'P . xQy$, *i.e.* $x (\dot{s}'C'P) y$. In the second case, we have $(\exists Q, R) . QPR . x \epsilon C'Q . y \epsilon C'R$, *i.e.* $(\exists Q, R) . QPR . xFQ . yFR$, *i.e.* $x (F ; P) y$. Hence the generating relation of the sum of all the series is $\dot{s}'C'P \cup F ; P$. Hence we put

$$\Sigma'P = \dot{s}'C'P \cup F ; P \quad \text{Df.}$$

The relation $\Sigma'P$ has all the properties which we should expect of the sum of a series of series.

If a series is to result from the addition of a series of series, it is necessary that no two of the series should have any common terms. For if we have

$$QPR . x \epsilon C'Q \cap C'R,$$

we shall also have $$x (\Sigma'P) x.$$

Hence instead of a series, we shall have cycles; for it is essential to a series that no term should precede itself. (What seem to be series in which there is repetition are always the result of a one-many correlation with series in which there is no repetition, so that a term can be counted once as the correlate of one term, and again as the correlate of a later term.) For this reason, as well as for many others, it is important to consider relations between mutually exclusive relations, *i.e.* between relations whose fields have no common terms. We put

$$\text{Rel}^2 \text{excl} = \hat{P} \{Q, R \epsilon C'P . Q \neq R . \supset_{Q,R} . C'Q \cap C'R = \Lambda\} \quad \text{Df.}$$

Then Rel^2 excl has much the same utility in relation-arithmetic as Cls^2 excl has in cardinal arithmetic. We have

$$\vdash : P \epsilon \text{Rel}^2 \text{excl} . \equiv . F \upharpoonright C'P \epsilon \text{Cls} \rightarrow 1,$$

which is analogous to the proposition (∗84·14)

$$\vdash : \kappa \epsilon \text{Cls}^2 \text{excl} . \equiv . \epsilon \upharpoonright \kappa \epsilon \text{Cls} \rightarrow 1.$$

It will be found that in relation-arithmetic the relation F often appears where ϵ appears in the analogous proposition of cardinal arithmetic.

Analogous to "sm sm" is the relation of double ordinal similarity. This holds between two relations P and Q when they are ordinally similar relations between ordinally similar relations with known correlators, *i.e.* when, if T is

an ordinal correlator of P and Q, so that $P = T ; Q$, then if X is a member of $C`P$, and $Y$ is the corresponding member of $C`Q$, so that XTY, we shall have X smor Y, and shall be able to specify a member of $X\,\overline{\text{smor}}\,Y$. But as in cardinals, so here, we have to frame our definition of double ordinal similarity in such a way as to minimize the use of the multiplicative axiom. We therefore take as our definition the following: P and Q are said to have donble ordinal similarity when there is a one-one relation S which has $C`\Sigma`Q$ for its converse domain, and is such that $P = S\dagger ; Q$. A relation S which has these properties is called a *double correlator* of P and Q, *i.e.* we put

$$P\,\overline{\text{smor}}\,\overline{\text{smor}}\,Q = (1 \to 1) \cap \overleftarrow{\text{Cl}}`C`\Sigma`Q \cap \hat{S}(P = S\dagger ; Q) \quad \text{Df},$$

a definition which, as will be perceived, is closely analogous to that of $\kappa\,\overline{\text{sm}}\,\overline{\text{sm}}\,\lambda$ in $*111$. Two relations have double similarity when they have a double correlator, *i.e.*

$$\text{smor smor} = \hat{P}\hat{Q}\,\{\exists\,!\,P\,\overline{\text{smor}}\,\overline{\text{smor}}\,Q\} \quad \text{Df.}$$

S is a double correlator of P and Q when S is a correlator of $\Sigma`P$ and $\Sigma`Q$ and $S\dagger \restriction C`Q$ is a correlator of P and Q. This might be taken as the definition of a double correlator, since it is equivalent to the above definition.

If we assume the multiplicative axiom, we can prove that double similarity holds between similar relations of mutually exclusive similar relations, *i.e.* between two relations of mutually exclusive relations P and Q which have a correlator S such that, if $Y \in C`Q$, then $Y$ and $S`Y$ are always similar. In this case, $S \subset$ smor. Thus if we assume the multiplicative axiom we have, if $P, Q \in \text{Rel}^2\text{excl}$,

$$P\,\text{smor smor}\,Q \,.\equiv.\, \exists\,!\,P\,\overline{\text{smor}}\,Q \cap \text{Rl}`\text{smor}.$$

In the particular case in which the fields of P and Q consist of *well-ordered* relations (*i.e.* relations generating well-ordered series), this equivalence can be proved without the use of the multiplicative axiom, because two similar well-ordered relations have only one correlator, so that the difficulty of selecting among correlators does not arise.

Double ordinal correlators have the same importance in proving the formal laws of relation-arithmetic that double cardinal correlators have in cardinal arithmetic. The construction of double correlators in various cases constitutes a large part of relation-arithmetic.

In defining the ordinal product of two relation-numbers, and in defining exponentiation, we use a relation which has properties analogous to those of $\alpha \underset{\prime\prime}{\downarrow} ``\beta$. This relation is $P \underset{;}{\downarrow} ; Q$, of which the structure is as follows: Let z, w be two terms having the relation Q; then form the two relations $\downarrow z ; P$, $\downarrow w ; P$. The relation $\downarrow z ; P$ holds between two couples $x \downarrow z$ and $y \downarrow z$ whenever xPy; thus it arranges couples whose referents are members of $C`P$, and whose relata are z, in an order similar to P. The relations $\downarrow z ; P$ and $\downarrow w ; P$

are (by $*150{\cdot}03$) the same as $P \underset{\cdot\cdot}{\downarrow} z$ and $P \underset{\cdot\cdot}{\downarrow} w$. Thus $P \underset{\cdot\cdot}{\downarrow} {}^{;}Q$ arranges such relations as $\downarrow z{}^{;}P$ in an order similar to Q. Thus $P \underset{\cdot\cdot}{\downarrow} {}^{;}Q$ is similar to Q, and every member of its field is similar to P. Thus the relation-number of $P \underset{\cdot\cdot}{\downarrow} {}^{;}Q$ is $\mathrm{Nr}{}^{\backprime}Q$, and every member of its field has the relation-number $\mathrm{Nr}{}^{\backprime}P$. Moreover $P \underset{\cdot\cdot}{\downarrow} {}^{;}Q$, as it is easy to see, is a relation of mutually exclusive relations. Hence it is suitable for defining the product of Q and P, and we put

$$Q \times P = \Sigma{}^{\backprime}P \underset{\cdot\cdot}{\downarrow} {}^{;}Q \quad \text{Df.}$$

In the next section, after we have defined the product of a relation of relations, we shall use the same relation $P \underset{\cdot\cdot}{\downarrow} {}^{;}Q$ for the definition of exponentiation, putting

$$P \exp Q = \mathrm{Prod}{}^{\backprime}P \underset{\cdot\cdot}{\downarrow} {}^{;}Q \quad \text{Df.}$$

These two definitions should be compared with those in $*113$ and $*116$.

In virtue of the definition of Σ, the relation $\Sigma{}^{\backprime}P \underset{\cdot\cdot}{\downarrow} {}^{;}Q$ holds between terms which either have one of the relations of the form $P \underset{\cdot\cdot}{\downarrow} z$, or belong respectively to the fields of two relations $P \underset{\cdot\cdot}{\downarrow} z$, $P \underset{\cdot\cdot}{\downarrow} w$, where zQw. Thus the relation $\Sigma{}^{\backprime}P \underset{\cdot\cdot}{\downarrow} {}^{;}Q$ holds between $x \downarrow z$ and $y \downarrow z$ whenever xPy and $z \in C{}^{\backprime}Q$, and also between $x \downarrow z$ and $y \downarrow w$ whenever $x, y \in C{}^{\backprime}P \,.\, zQw$. Thus if, for the sake of illustration, P and Q generate finite series, so that their fields are

$$1_P, 2_P, \ldots, \mu_P,$$
$$1_Q, 2_Q, \ldots, \nu_Q,$$

then the field of $\Sigma{}^{\backprime}P \underset{\cdot\cdot}{\downarrow} {}^{;}Q$ will consist of the couples

$$1_P \downarrow 1_Q,\ 2_P \downarrow 1_Q,\ \ldots,\ \mu_P \downarrow 1_Q;$$
$$1_P \downarrow 2_Q,\ 2_P \downarrow 2_Q,\ \ldots,\ \mu_P \downarrow 2_Q;$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$1_P \downarrow \nu_Q,\ 2_P \downarrow \nu_Q,\ \ldots,\ \mu_P \downarrow \nu_Q;$$

and their order as arranged by $\Sigma{}^{\backprime}P \underset{\cdot\cdot}{\downarrow} {}^{;}Q$ is that in which they are written above. Thus the above couples in the above order constitute the series $Q \times P$, and it is evident that this series has $\nu \times \mu$ terms.

When the factors of a product are not enumerated, but are given as the field of a relation, a new definition of multiplication is required. This definition, which has the advantage of being applicable to infinite products, will be dealt with in the following section.

$*160$. THE SUM OF TWO RELATIONS

Summary of $*160$.

In this number, we introduce the definition

$$P \mathbin{\stackrel{\uparrow}{+}} Q = P \cup Q \cup C\text{‘}P \uparrow C\text{‘}Q \quad \text{Df},$$

which was explained in the introduction to this section. Although the propositions of this and other numbers in this Part do not require that P and Q should be such as to generate series, yet the reader will find it convenient to imagine them to be such, since the important applications of the ideas of this Part are to series. Thus we may regard the sum of P and Q as a relation which holds between x and y when either x precedes y in the P-series, or x precedes y in the Q-series, or x belongs to the P-series and y belongs to the Q-series.

The most important propositions of this number are:

$*160{\cdot}14$. $\vdash . C\text{‘}(P \mathbin{\stackrel{\uparrow}{+}} Q) = C\text{‘}P \cup C\text{‘}Q$

$*160{\cdot}21$. $\vdash . P \mathbin{\stackrel{\uparrow}{+}} \dot{\Lambda} = P$

$*160{\cdot}22$. $\vdash . \dot{\Lambda} \mathbin{\stackrel{\uparrow}{+}} Q = Q$

$*160{\cdot}31$. $\vdash . (P \mathbin{\stackrel{\uparrow}{+}} Q) \mathbin{\stackrel{\uparrow}{+}} R = P \mathbin{\stackrel{\uparrow}{+}} (Q \mathbin{\stackrel{\uparrow}{+}} R)$

which is the associative law, and

$*160{\cdot}4$. $\vdash . (P \cup Q) \mathbin{\stackrel{\uparrow}{+}} R = (P \mathbin{\stackrel{\uparrow}{+}} R) \cup (Q \mathbin{\stackrel{\uparrow}{+}} R)$

which is the distributive law for logical and arithmetical addition;

$*160{\cdot}44$. $\vdash : C\text{‘}P \subset \text{Ɑ}\text{‘}S . C\text{‘}Q \subset \text{Ɑ}\text{‘}S . \supset . S\text{⁏}(P \mathbin{\stackrel{\uparrow}{+}} Q) = S\text{⁏}P \mathbin{\stackrel{\uparrow}{+}} S\text{⁏}Q$

which is also a kind of distributive law;

$*160{\cdot}47$. $\vdash : C\text{‘}P \cap C\text{‘}Q = \Lambda . C\text{‘}P' \cap C\text{‘}Q' = \Lambda . S \in P \,\overline{\text{smor}}\, P' . T \in Q \,\overline{\text{smor}}\, Q' . \supset .$
$S \cup T \in (P \mathbin{\stackrel{\uparrow}{+}} Q) \,\overline{\text{smor}}\, (P' \mathbin{\stackrel{\uparrow}{+}} Q')$

whence

$*160{\cdot}48$. $\vdash : C\text{‘}P \cap C\text{‘}Q = \Lambda . C\text{‘}P' \cap C\text{‘}Q' = \Lambda . P \,\text{smor}\, P' . Q \,\text{smor}\, Q' . \supset .$
$P \mathbin{\stackrel{\uparrow}{+}} Q \,\text{smor}\, P' \mathbin{\stackrel{\uparrow}{+}} Q'$

whence it follows that if P and Q are mutually exclusive, the relation-number of their sum depends only upon the relation-numbers of P and Q;

$*160{\cdot}5$. $\vdash : C\text{‘}P \cap C\text{‘}Q = \Lambda . \supset . (P \mathbin{\stackrel{\uparrow}{+}} Q) \upharpoonright C\text{‘}P = P . (P \mathbin{\stackrel{\uparrow}{+}} Q) \upharpoonright C\text{‘}Q = Q$

$*160{\cdot}52$. $\vdash : C\text{‘}P \cap C\text{‘}Q = \Lambda . C\text{‘}P \cap C\text{‘}R = \Lambda . P \mathbin{\stackrel{\uparrow}{+}} Q = P \mathbin{\stackrel{\uparrow}{+}} R . \supset . Q = R$

$*160{\cdot}01$. $P \mathbin{\stackrel{\uparrow}{+}} Q = P \cup Q \cup C\text{‘}P \uparrow C\text{‘}Q \quad \text{Df}$

$*160{\cdot}1$. $\vdash . P \mathbin{\stackrel{\uparrow}{+}} Q = P \cup Q \cup C\text{‘}P \uparrow C\text{‘}Q$ $\quad$ [($*160{\cdot}01$)]

$*160{\cdot}11$. $\vdash :. x (P \mathbin{\stackrel{\uparrow}{+}} Q) y . \equiv : xPy . \vee . xQy . \vee . x \in C\text{‘}P . y \in C\text{‘}Q$ $\quad$ [$*160{\cdot}1$]

$*160{\cdot}111$. $\vdash :. x (P \mathbin{\stackrel{\uparrow}{+}} Q) y . \equiv : xPy . \vee . xQy . \vee . xFP . yFQ$ $\quad$ [$*160{\cdot}11 . *33{\cdot}51$]

$*160{\cdot}12.\ \vdash : \dot{\exists} ! Q . \supset . \mathrm{D}`(P \dotplus Q) = C`P \cup \mathrm{D}`Q \quad [*33{\cdot}26 . *35{\cdot}85 . *160{\cdot}1]$

$*160{\cdot}13.\ \vdash : \dot{\exists} ! P . \supset . \mathrm{G}`(P \dotplus Q) = \mathrm{G}`P \cup C`Q$

$*160{\cdot}14.\ \vdash . C`(P \dotplus Q) = C`P \cup C`Q$

Dem.

$\vdash . *33{\cdot}262 . *160{\cdot}1 . \supset \vdash . C`(P \dotplus Q) = C`P \cup C`Q \cup C`(C`P \uparrow C`Q) \quad (1)$

$\vdash . *35{\cdot}85{\cdot}86{\cdot}88 . \quad \supset \vdash . C`(C`P \uparrow C`Q) \subset C`P \cup C`Q \quad (2)$

$\vdash . (1) . (2) . \supset \vdash .$ Prop

The above proposition is constantly used. The following propositions ($*160{\cdot}15$—$\cdot161$) are not used, but are inserted to show that $P \dotplus Q$ has the kind of structure that we should expect of a sum.

$*160{\cdot}15.\ \vdash : \dot{\exists} ! P . \supset . \overrightarrow{B}`(P \dotplus Q) = \overrightarrow{B}`P - C`Q$

Dem.

$\vdash . *160{\cdot}12{\cdot}13 . \supset \vdash : \dot{\exists} ! P . \dot{\exists} ! Q . \supset . \overrightarrow{B}`(P \dotplus Q) = (C`P \cup \mathrm{D}`Q) - (\mathrm{G}`P \cup C`Q)$

$[*93{\cdot}101 . *33{\cdot}161] \quad = \overrightarrow{B}`P - C`Q \quad (1)$

$\vdash . *160{\cdot}1 . \supset \vdash : Q = \dot{\Lambda} . \supset . P \dotplus Q = P .$

$[*30{\cdot}37] \quad \supset . \overrightarrow{B}`(P \dotplus Q) = \overrightarrow{B}`P$

$[*33{\cdot}241] \quad = \overrightarrow{B}`P - C`Q \quad (2)$

$\vdash . (1) . (2) . \supset \vdash .$ Prop

$*160{\cdot}151.\ \vdash : \dot{\exists} ! Q . \supset . \overrightarrow{B}`\mathrm{Cnv}`(P \dotplus Q) = \overrightarrow{B}`\breve{Q} - C`P$

$*160{\cdot}16.\ \vdash : \dot{\exists} ! P . \overrightarrow{B}`P \cap C`Q = \Lambda . \supset . \overrightarrow{B}`(P \dotplus Q) = \overrightarrow{B}`P \quad [*160{\cdot}15]$

$*160{\cdot}161.\ \vdash : \dot{\exists} ! Q . \overrightarrow{B}`\breve{Q} \cap C`P = \Lambda . \supset . \overrightarrow{B}`\mathrm{Cnv}`(P \dotplus Q) = \overrightarrow{B}`\breve{Q}$

$*160{\cdot}2.\ \vdash . \mathrm{Cnv}`(P \dotplus Q) = \breve{Q} \dotplus \breve{P} \quad [*31{\cdot}15 . *35{\cdot}84]$

$*160{\cdot}21.\ \vdash . P \dotplus \dot{\Lambda} = P \quad [*35{\cdot}88 . *25{\cdot}24]$

$*160{\cdot}22.\ \vdash . \dot{\Lambda} \dotplus Q = Q$

$*160{\cdot}3.\ \vdash . (P \dotplus Q) \dotplus R = P \mathbin{\dot{\cup}} Q \mathbin{\dot{\cup}} R \mathbin{\dot{\cup}} C`P \uparrow C`Q \mathbin{\dot{\cup}} C`P \uparrow C`R \mathbin{\dot{\cup}} C`Q \uparrow C`R$

Dem.

$\vdash . *160{\cdot}14{\cdot}1 . \supset$

$\vdash . (P \dotplus Q) \dotplus R = (P \dotplus Q) \mathbin{\dot{\cup}} R \mathbin{\dot{\cup}} (C`P \cup C`Q) \uparrow C`R$

$[*160{\cdot}1 . *35{\cdot}41{\cdot}82] = P \mathbin{\dot{\cup}} Q \mathbin{\dot{\cup}} C`P \uparrow C`Q \mathbin{\dot{\cup}} R \mathbin{\dot{\cup}} C`P \uparrow C`R \mathbin{\dot{\cup}} C`Q \uparrow C`R . \supset \vdash .$ Prop

$*160{\cdot}31.\ \vdash . (P \dotplus Q) \dotplus R = P \dotplus (Q \dotplus R)$

Dem.

$\vdash . *160{\cdot}14{\cdot}1 . \supset$

$\vdash . P \dotplus (Q \dotplus R) = P \mathbin{\dot{\cup}} Q \mathbin{\dot{\cup}} R \mathbin{\dot{\cup}} C`P \uparrow C`Q \mathbin{\dot{\cup}} C`P \uparrow C`R \mathbin{\dot{\cup}} C`Q \uparrow C`R \quad (1)$

$\vdash . (1) . *160{\cdot}3 . \supset \vdash .$ Prop

***160·32.** $P \dotplus Q \dotplus R = (P \dotplus Q) \dotplus R$ Df

This definition serves merely for the avoidance of brackets.

***160·33.** $\vdash : P \mathrel{\dot\subset} Q . \supset . P \dotplus R \mathrel{\dot\subset} Q \dotplus R$ [*33·265 . *160·1]

***160·34.** $\vdash : R \mathrel{\dot\subset} S . \supset . Q \dotplus R \mathrel{\dot\subset} Q \dotplus S$ [*33·265 . *160·1]

***160·35.** $\vdash : P \mathrel{\dot\subset} Q . R \mathrel{\dot\subset} S . \supset . P \dotplus Q \mathrel{\dot\subset} R \dotplus S$ [*160·33·34]

***160·4.** $\vdash . (P \mathbin{\dot\cup} Q) \dotplus R = (P \dotplus R) \mathbin{\dot\cup} (Q \dotplus R)$

Dem.

$\vdash . *160{\cdot}1 . \supset \vdash . (P \mathbin{\dot\cup} Q) \dotplus R = P \mathbin{\dot\cup} Q \mathbin{\dot\cup} R \mathbin{\dot\cup} C\text{‘}(P \mathbin{\dot\cup} Q) \uparrow C\text{‘}R$

[*33·262.*23·56] $= P \mathbin{\dot\cup} R \mathbin{\dot\cup} Q \mathbin{\dot\cup} R \mathbin{\dot\cup} (C\text{‘}P \cup C\text{‘}Q) \uparrow C\text{‘}R$

[*35·41·82] $= P \mathbin{\dot\cup} R \mathbin{\dot\cup} Q \mathbin{\dot\cup} R \mathbin{\dot\cup} C\text{‘}P \uparrow C\text{‘}R \mathbin{\dot\cup} C\text{‘}Q \uparrow C\text{‘}R$

[*160·1] $= (P \dotplus R) \mathbin{\dot\cup} (Q \dotplus R) . \supset \vdash . \text{Prop}$

***160·401.** $\vdash . P \dotplus (Q \mathbin{\dot\cup} R) = (P \dotplus Q) \mathbin{\dot\cup} (P \dotplus R)$

The above two propositions state the distributive law for logical and arithmetical addition. The three following propositions give the generalized form of this law, when $\dot{s}\text{‘}\lambda$ replaces $P \mathbin{\dot\cup} Q$; these propositions are not subsequently used but are inserted for the sake of their intrinsic interest.

***160·41.** $\vdash : \exists ! \lambda . \supset . \dot{s}\text{‘}\lambda \dotplus R = \dot{s}\text{‘} \dotplus R\text{“}\lambda = \dot{s}\text{‘}(\lambda \underset{,,}{\dotplus} R)$

Dem.

$\vdash . *41{\cdot}11 . \supset \vdash :. x (\dot{s}\text{‘} \dotplus R\text{“}\lambda) y . \equiv : (\exists P) . P \in \lambda . x (P \dotplus R) y :$

[*160·11] $\equiv : (\exists P) : P \in \lambda : xPy . \vee . xRy . \vee . x \in C\text{‘}P . y \in C\text{‘}R :$

[*10·42] $\equiv : (\exists P) . P \in \lambda . xPy . \vee . (\exists P) . P \in \lambda . xRy . \vee .$

$(\exists P) . P \in \lambda . x \in C\text{‘}P . y \in C\text{‘}R :$

[*41·11.*10·35.*41·45] $\equiv : x (\dot{s}\text{‘}\lambda) y . \vee . \exists ! \lambda . xRy . \vee . x \in C\text{‘}\dot{s}\text{‘}\lambda . y \in C\text{‘}R$ (1)

$\vdash . (1) . \supset \vdash :: \text{Hp} . \supset :. x (\dot{s}\text{‘} \dotplus R\text{“}\lambda) . \equiv : x (\dot{s}\text{‘}\lambda) y . \vee . xRy . \vee . x \in C\text{‘}\dot{s}\text{‘}\lambda . y \in C\text{‘}R :$

[*160·11] $\equiv : x (\dot{s}\text{‘}\lambda \dotplus R) y :: \supset \vdash . \text{Prop}$

***160·411.** $\vdash : \exists ! \lambda . \supset . P \dotplus \dot{s}\text{‘}\lambda = \dot{s}\text{‘}P \dotplus \text{“}\lambda$ [Proof as in *160·41]

***160·412.** $\vdash : \exists ! \lambda . \exists ! \mu . \supset . \dot{s}\text{‘}\lambda \dotplus \dot{s}\text{‘}\mu = \dot{s}\text{‘}s\text{‘}\lambda \underset{,,}{\dotplus} \text{“}\mu$

Dem.

$\vdash . *160{\cdot}411 . \supset \vdash : \exists ! \mu . \supset . \dot{s}\text{‘}\lambda \dotplus \dot{s}\text{‘}\mu = \dot{s}\text{‘}(\dot{s}\text{‘}\lambda) \dotplus \text{“}\mu$ (1)

$\vdash . *160{\cdot}41 . \supset \vdash : \exists ! \lambda . \supset . (\dot{s}\text{‘}\lambda) \dotplus \text{“}\mu = \dot{s}\text{“}\lambda \underset{,,}{\dotplus} \text{“}\mu$ (2)

$\vdash . (1) . (2) . \supset \vdash : \exists ! \lambda . \exists ! \mu . \supset . \dot{s}\text{‘}\lambda \dotplus \dot{s}\text{‘}\mu = \dot{s}\text{‘}\dot{s}\text{“}\lambda \underset{,,}{\dotplus} \text{“}\mu$

[*42·12] $= \dot{s}\text{‘}s\text{‘}\lambda \underset{,,}{\dotplus} \text{“}\mu : \supset \vdash . \text{Prop}$

The following propositions lead up to *160·44, which is frequently used.

***160·42.** $\vdash . (P \ddagger Q) | S = P | S \cup Q | S \cup C\text{‘}P \uparrow \breve{S}\text{“}C\text{‘}Q$

Dem.

$\vdash . *160\cdot1 . \supset \vdash . (P \ddagger Q) | S = P | S \cup Q | S \cup (C\text{‘}P \uparrow C\text{‘}Q) | S$

$[*37\cdot8] \qquad = P | S \cup Q | S \cup C\text{‘}P \uparrow \breve{S}\text{“}C\text{‘}Q . \supset \vdash . \text{Prop}$

***160·421.** $\vdash . S | (P \ddagger Q) = S | P \cup S | Q \cup S\text{“}C\text{‘}P \uparrow C\text{‘}Q$

***160·43.** $\vdash . S;(P \ddagger Q) = S;P \cup S;Q \cup S\text{“}C\text{‘}P \uparrow S\text{“}C\text{‘}Q$

Dem.

$\vdash . *150\cdot1 . *160\cdot421 . \supset$

$\vdash . S;(P \ddagger Q) \quad = (S | P \cup S | Q \cup S\text{“}C\text{‘}P \uparrow C\text{‘}Q) | \breve{S}$

$[*150\cdot1 . *37\cdot8] = S;P \cup S;Q \cup S\text{“}C\text{‘}P \uparrow S\text{“}C\text{‘}Q . \supset \vdash . \text{Prop}$

***160·44.** $\vdash : C\text{‘}P \subset \text{Ɑ}\text{‘}S . C\text{‘}Q \subset \text{Ɑ}\text{‘}S . \supset . S;(P \ddagger Q) = S;P \ddagger S;Q$

Dem.

$\vdash . *160\cdot43 . *150\cdot22 . \supset$

$\vdash : \text{Hp} . \supset . S;(P \ddagger Q) = S;P \cup S;Q \cup (C\text{‘}S;P) \uparrow (C\text{‘}S;Q)$

$[*160\cdot1] \qquad = S;P \ddagger S;Q : \supset \vdash . \text{Prop}$

***160·45.** $\vdash : S \upharpoonright (C\text{‘}P' \cup C\text{‘}Q') \in 1 \to 1 . S \upharpoonright C\text{‘}P' \in P \,\overline{\text{smor}}\, P' . S \upharpoonright C\text{‘}Q' \in Q \,\overline{\text{smor}}\, Q' . \supset .$
$S \upharpoonright C\text{‘}(P' \ddagger Q') \in (P \ddagger Q) \,\overline{\text{smor}}\, (P' \ddagger Q')$

Dem.

$\vdash . *151\cdot22 . \supset \vdash : \text{Hp} . \supset . C\text{‘}P' \subset \text{Ɑ}\text{‘}S . C\text{‘}Q' \subset \text{Ɑ}\text{‘}S . P = S;P' . Q = S;Q' . \quad (1)$

$[*160\cdot44] \qquad \supset . P \ddagger Q = S;(P' \ddagger Q') \quad (2)$

$\vdash . (1) . *160\cdot14 . \supset \vdash : \text{Hp} . \supset . C\text{‘}(P' \ddagger Q') \subset \text{Ɑ}\text{‘}S \quad (3)$

$\vdash . *160\cdot14 . \qquad \supset \vdash : \text{Hp} . \supset . S \upharpoonright C\text{‘}(P' \ddagger Q') \in 1 \to 1 \quad (4)$

$\vdash . (2) . (3) . (4) . *151\cdot22 . \supset \vdash . \text{Prop}$

***160·451.** $\vdash : S \upharpoonright C\text{‘}P' \in P \,\overline{\text{smor}}\, P' . S \upharpoonright C\text{‘}Q' \in Q \,\overline{\text{smor}}\, Q' . S\text{“}(C\text{‘}P' - C\text{‘}Q') \cap C\text{‘}Q = \Lambda .$
$\supset . S \upharpoonright C\text{‘}(P' \ddagger Q') \in (P \ddagger Q) \,\overline{\text{smor}}\, (P' \ddagger Q')$

Dem.

$\vdash . *151\cdot22 . *150\cdot22 . \supset \vdash : \text{Hp} . \supset . C\text{‘}Q = S\text{“}C\text{‘}Q' .$

$[*71\cdot381 . *37\cdot421] \qquad \supset . S\text{“}(C\text{‘}P' - C\text{‘}Q') \cap S\text{“}C\text{‘}Q' = \Lambda .$

$[*74\cdot823] \qquad \supset . S \upharpoonright (C\text{‘}P' \cup C\text{‘}Q') \in 1 \to 1 \quad (1)$

$\vdash . (1) . *160\cdot45 . \supset \vdash . \text{Prop}$

***160·452.** $\vdash : S \upharpoonright C\text{‘}P' \in P \,\overline{\text{smor}}\, P' . S \upharpoonright C\text{‘}Q' \in Q \,\overline{\text{smor}}\, Q' . C\text{‘}P \cap C\text{‘}Q = \Lambda . \supset .$
$S \upharpoonright C\text{‘}(P' \ddagger Q') \in (P \ddagger Q) \,\overline{\text{smor}}\, (P' \ddagger Q')$

Dem.

$\vdash . *151\cdot22 . *150\cdot22 . \supset \vdash : \text{Hp} . \supset . C\text{‘}P = S\text{“}C\text{‘}P' . C\text{‘}Q = S\text{“}C\text{‘}Q' .$

$[\text{Hp}] \qquad \supset . S\text{“}C\text{‘}P' \cap S\text{“}C\text{‘}Q' = \Lambda .$

$[*74\cdot833] \qquad \supset . S \upharpoonright C\text{‘}(P' \ddagger Q') \in 1 \to 1 \quad (1)$

$\vdash . (1) . *160\cdot45 . \supset \vdash . \text{Prop}$

***160·46.** $\vdash : C'P = \text{Ɑ}'S \,.\, C'Q = \text{Ɑ}'T \,.\, C'P \cap C'Q = \Lambda \,.\supset .$
$$(S \dot\cup T) ; (P \ddagger Q) = S ; P \ddagger T ; Q$$

Dem.

$\vdash . *160·44 . \supset \vdash : \text{Hp} . \supset . (S \dot\cup T) ; (P \ddagger Q) = (S \dot\cup T) ; P \ddagger (S \dot\cup T) ; Q$

$[*150·32] \qquad = \{(S \dot\cup T) \restriction C'P\} ; P \ddagger \{(S \dot\cup T) \restriction C'Q\} ; Q$

$[*35·644 . \text{Hp}] \qquad = (S \restriction C'P) ; P \ddagger (T \restriction C'Q) ; Q$

$[*150·32] \qquad = S ; P \ddagger T ; Q : \supset \vdash . \text{Prop}$

***160·47.** $\vdash : C'P \cap C'Q = \Lambda \,.\, C'P' \cap C'Q' = \Lambda \,.\, S \in P \,\overline{\text{smor}}\, P' \,.\, T \in Q \,\overline{\text{smor}}\, Q' \,.\supset .$
$$S \dot\cup T \in (P \ddagger Q) \,\overline{\text{smor}}\, (P' \ddagger Q')$$

Dem.

$\vdash . *151·11·131 . \supset \vdash : \text{Hp} . \supset . \text{D}'S = C'P \,.\, \text{D}'T = C'Q \,.\, \text{Ɑ}'S = C'P' \,.$
$\text{Ɑ}'T = C'Q' \,.$ (1)

$[\text{Hp}] \qquad \supset . \text{D}'S \cap \text{D}'T = \Lambda \,.\, \text{Ɑ}'S \cap \text{Ɑ}'T = \Lambda \,.$

$[*151·11 . *71·242] \qquad \supset . S \dot\cup T \in 1 \to 1$ (2)

$\vdash . (1) . *160·14 . \supset \vdash : \text{Hp} . \supset . C'(P' \ddagger Q') = \text{Ɑ}'S \cup \text{Ɑ}'T$

$[*33·261] \qquad = \text{Ɑ}'(S \dot\cup T)$ (3)

$\vdash . *160·46 . *151·11 . \supset \vdash : \text{Hp} . \supset . P \ddagger Q = (S \dot\cup T) ; (P' \ddagger Q')$ (4)

$\vdash . (2) . (3) . (4) . *151·11 . \supset \vdash . \text{Prop}$

***160·48.** $\vdash : C'P \cap C'Q = \Lambda \,.\, C'P' \cap C'Q' = \Lambda \,.\, P \,\text{smor}\, P' \,.\, Q \,\text{smor}\, Q' \,.\supset .$
$$P \ddagger Q \,\text{smor}\, P' \ddagger Q' \quad [*160·47 . *151·12]$$

***160·5.** $\vdash : C'P \cap C'Q = \Lambda \,.\supset . (P \ddagger Q) \lceil C'P = P \,.\, (P \ddagger Q) \lceil C'Q = Q$

Dem.

$\vdash . *160·1 . *36·23 . \supset$

$\vdash . (P \ddagger Q) \lceil C'P = P \lceil C'P \dot\cup (C'P \uparrow C'Q) \lceil C'P \dot\cup Q \lceil C'P$

$[*36·29·33] \qquad = P \dot\cup \{(C'P \uparrow C'Q) \dot\cap (C'P \uparrow C'P)\} \dot\cup Q \lceil C'P$ (1)

$\vdash . *36·31 . \qquad \supset \vdash : \text{Hp} . \supset . Q \lceil C'P = \dot\Lambda$ (2)

$\vdash . *35·834·88 . \supset \vdash : \text{Hp} . \supset . \{(C'P \uparrow C'Q) \dot\cap (C'P \uparrow C'P)\} = \dot\Lambda$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash : \text{Hp} . \supset . (P \ddagger Q) \lceil C'P = P$ (4)

Similarly $\qquad \vdash : \text{Hp} . \supset . (P \ddagger Q) \lceil C'Q = Q$ (5)

$\vdash . (4) . (5) . \supset \vdash . \text{Prop}$

***160·51.** $\vdash : C'P \cap C'Q = \Lambda \,.\supset . (P \ddagger Q)^2 = P^2 \dot\cup Q^2 \dot\cup \text{D}'P \uparrow C'Q \dot\cup C'P \uparrow \text{Ɑ}'Q$

Dem.

$\vdash . *34·73 . \quad \supset \vdash : \text{Hp} . \supset . (P \dot\cup Q)^2 = P^2 \dot\cup Q^2$ (1)

$\vdash . *35·895 . \supset \vdash : \text{Hp} . \supset . (C'P \uparrow C'Q)^2 = \dot\Lambda$ (2)

$\vdash . *34·62 . \quad \supset \vdash . (P \ddagger Q)^2 = (P \dot\cup Q)^2 \dot\cup (C'P \uparrow C'Q)^2$
$\dot\cup (P \dot\cup Q) \,|\, (C'P \uparrow C'Q) \dot\cup (C'P \uparrow C'Q) \,|\, (P \dot\cup Q)$

$[(1) . (2)] \qquad = P^2 \dot\cup Q^2 \dot\cup (P \dot\cup Q) \,|\, (C'P \uparrow C'Q) \dot\cup (C'P \uparrow C'Q) \,|\, (P \dot\cup Q)$ (3)

$\vdash . *37·81 . \quad \supset \vdash : \text{Hp} . \supset . (P \dot\cup Q) \,|\, (C'P \uparrow C'Q) = \text{D}'P \uparrow C'Q$ (4)

$\vdash . *37·8 . \quad \supset \vdash : \text{Hp} . \supset . (C'P \uparrow C'Q) \,|\, (P \dot\cup Q) = C'P \uparrow \text{Ɑ}'Q$ (5)

$\vdash . (3) . (4) . (5) . \supset \vdash . \text{Prop}$

The above proposition is useful in proving that, if $C^{\backprime}P \cap C^{\backprime}Q = \Lambda$, $P \ddagger Q$ is transitive when P and Q are transitive (cf. $*201{\cdot}4$).

$*160{\cdot}52$. $\vdash : C^{\backprime}P \cap C^{\backprime}Q = \Lambda \,.\, C^{\backprime}P \cap C^{\backprime}R = \Lambda \,.\, P \ddagger Q = P \ddagger R \,.\, \supset \,.\, Q = R$

Dem.

$\vdash \,.\, *160{\cdot}14 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, (P \ddagger Q) \upharpoonright (-C^{\backprime}P) = (P \ddagger Q) \upharpoonright C^{\backprime}Q \,.$

$\qquad (P \ddagger Q) \upharpoonright (-C^{\backprime}P) = (P \ddagger R) \upharpoonright C^{\backprime}R \,.$

$[*160{\cdot}5] \qquad \supset \,.\, (P \ddagger Q) \upharpoonright (-C^{\backprime}P) = Q \,.\, (P \ddagger R) \upharpoonright (-C^{\backprime}P) = R \,.$

$[\mathrm{Hp}] \qquad \supset \,.\, Q = R : \supset \vdash \,.\, \mathrm{Prop}$

The above proposition is used in dealing with the series of segments of a series ($*213{\cdot}561$).

$*161$. ADDITION OF A TERM TO A RELATION

Summary of $*161$.

The addition of a term has two forms, according as it occurs at the beginning or end of the field of the relation in question. If we add first x and then y at the end, the result is the same as if we added $x \downarrow y$ ($*161\cdot22$); if at the beginning, it is the same as if we added $y \downarrow x$ ($*161\cdot221$). The propositions of the present number are all obvious, and offer no difficulties of any kind. As explained in the introduction to this section, we put

$$P \rightarrow\!\!\!\!\!+\, x = P \cup C\text{‘}P \uparrow \iota\text{‘}x \quad \text{Df},$$

$$x \,+\!\!\!\!\!\leftarrow P = \iota\text{‘}x \uparrow C\text{‘}P \cup P \quad \text{Df}.$$

Most of the propositions of this number require the hypothesis $\dot{\exists}\,!\,P$, because if $P = \dot{\Lambda}$, $P \rightarrow\!\!\!\!\!+\, x = x \,+\!\!\!\!\!\leftarrow P = \dot{\Lambda}$ ($*161\cdot2\cdot201$). This is connected with the fact that there is no ordinal number 1. Apart from propositions already mentioned, the chief propositions of this number are the following (we omit propositions about $x \,+\!\!\!\!\!\leftarrow P$ when they are merely analogues of propositions about $P \rightarrow\!\!\!\!\!+\, x$):

$*161\cdot12$. $\vdash . x \,+\!\!\!\!\!\leftarrow P = \text{Cnv}\text{‘}(\breve{P} \rightarrow\!\!\!\!\!+\, x)$

$*161\cdot14$. $\vdash :\dot{\exists}\,!\,P . \supset . C\text{‘}(P \rightarrow\!\!\!\!\!+\, x) = C\text{‘}P \cup \iota\text{‘}x = C\text{‘}(x \,+\!\!\!\!\!\leftarrow P)$

$*161\cdot15$. $\vdash :\dot{\exists}\,!\,P . x \sim\in C\text{‘}P . \supset .$

$$\overrightarrow{B}\text{‘}(P \rightarrow\!\!\!\!\!+\, x) = \overrightarrow{B}\text{‘}P . \overrightarrow{B}\text{‘}\text{Cnv}\text{‘}(P \rightarrow\!\!\!\!\!+\, x) = \iota\text{‘}x . B\text{‘}(x \,+\!\!\!\!\!\leftarrow \breve{P}) = x$$

$*161\cdot211$. $\vdash . x \,+\!\!\!\!\!\leftarrow (y \downarrow z) = x \downarrow y \cup x \downarrow z \cup y \downarrow z = (x \downarrow y) \rightarrow\!\!\!\!\!+\, z$

$*161\cdot31$. $\vdash : P \,\text{smor}\, Q . x \sim\in C\text{‘}P . y \sim\in C\text{‘}Q . \supset .$

$$P \rightarrow\!\!\!\!\!+\, x \,\text{smor}\, Q \rightarrow\!\!\!\!\!+\, y . x \,+\!\!\!\!\!\leftarrow P \,\text{smor}\, y \,+\!\!\!\!\!\leftarrow Q$$

$*161\cdot4$. $\vdash : C\text{‘}Q \subset \text{Ɑ}\text{‘}S . x \in \text{Ɑ}\text{‘}S . S \in 1 \rightarrow \text{Cls} . \supset . S;(Q \rightarrow\!\!\!\!\!+\, x) = S;Q \rightarrow\!\!\!\!\!+\, S\text{‘}x$

$*161\cdot01$. $P \rightarrow\!\!\!\!\!+\, x = P \cup C\text{‘}P \uparrow \iota\text{‘}x \quad \text{Df}$

$*161\cdot02$. $x \,+\!\!\!\!\!\leftarrow P = \iota\text{‘}x \uparrow C\text{‘}P \cup P \quad \text{Df}$

$*161\cdot1$. $\vdash . P \rightarrow\!\!\!\!\!+\, x = P \cup C\text{‘}P \uparrow \iota\text{‘}x$ [($*161\cdot01$)]

$*161\cdot101$. $\vdash . x \,+\!\!\!\!\!\leftarrow P = \iota\text{‘}x \uparrow C\text{‘}P \cup P$ [($*161\cdot02$)]

$*161\cdot11$. $\vdash :. y (P \rightarrow\!\!\!\!\!+\, x) z . \equiv : yPz . \vee . y \in C\text{‘}P . z = x$ [$*161\cdot1$]

$*161\cdot111$. $\vdash :. y (x \,+\!\!\!\!\!\leftarrow P) z . \equiv : y = x . z \in C\text{‘}P . \vee . yPz$ [$*161\cdot101$]

$*161\cdot12$. $\vdash . x \,+\!\!\!\!\!\leftarrow P = \text{Cnv}\text{‘}(\breve{P} \rightarrow\!\!\!\!\!+\, x)$ [$*161\cdot1\cdot101 . *35\cdot84 . *33\cdot22$]

***161·13.** $\vdash . \mathrm{D}'(P \mathbin{+\!\!\!\rightarrow} x) = C'P . \text{Ɑ}'(x \mathbin{\leftarrow\!\!\!+} P) = C'P$

Dem.

$\vdash . *161·1 . \supset \vdash . \mathrm{D}'(P \mathbin{+\!\!\!\rightarrow} x) = \mathrm{D}'P \cup \mathrm{D}'(C'P \uparrow \iota'x)$

$[*35·85] \qquad = \mathrm{D}'P \cup C'P$

$[*33·161] \qquad = C'P \qquad (1)$

Similarly $\quad \vdash . \text{Ɑ}'(x \mathbin{\leftarrow\!\!\!+} P) = C'P \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash .$ Prop

***161·131.** $\vdash : \dot{\exists} ! P . \supset . \text{Ɑ}'(P \mathbin{+\!\!\!\rightarrow} x) = \text{Ɑ}'P \cup \iota'x . \mathrm{D}'(x \mathbin{\leftarrow\!\!\!+} P) = \mathrm{D}'P \cup \iota'x$

[*35·86 . *161·1]

***161·14.** $\vdash : \dot{\exists} ! P . \supset . C'(P \mathbin{+\!\!\!\rightarrow} x) = C'P \cup \iota'x = C'(x \mathbin{\leftarrow\!\!\!+} P)$ [*161·13·131]

The hypothesis $\dot{\exists} ! P$ is necessary in this proposition, since without it we have $P \mathbin{+\!\!\!\rightarrow} x = \dot{\Lambda}$.

***161·141.** $\vdash : \dot{\exists} ! P . \supset . \overrightarrow{B}'(P \mathbin{+\!\!\!\rightarrow} x) = \overrightarrow{B}'P - \iota'x . \overrightarrow{B}'\mathrm{Cnv}'(P \mathbin{+\!\!\!\rightarrow} x) = \iota'x - C'P$

[*161·13·131 . *93·101]

***161·15.** $\vdash : \dot{\exists} ! P . x \sim \epsilon\, C'P . \supset .$

$\overrightarrow{B}'(P \mathbin{+\!\!\!\rightarrow} x) = \overrightarrow{B}'P . \overrightarrow{B}'\mathrm{Cnv}'(P \mathbin{+\!\!\!\rightarrow} x) = \iota'x . B'(x \mathbin{\leftarrow\!\!\!+} \breve{P}) = x$ [*161·141]

***161·16.** $\vdash : x \sim \epsilon\, C'P . \supset . (P \mathbin{+\!\!\!\rightarrow} x) \upharpoonright C'P = (P \mathbin{+\!\!\!\rightarrow} x) \upharpoonright (-\iota'x) = P$ [*161·1]

The above proposition is used in the theory of connected relations (*202·412).

***161·161.** $\vdash : x \sim \epsilon\, C'P . \supset . (x \mathbin{\leftarrow\!\!\!+} P) \upharpoonright C'P = (x \mathbin{\leftarrow\!\!\!+} P) \upharpoonright (-\iota'x) = P$

The two following propositions are frequently used.

***161·2.** $\vdash . \dot{\Lambda} \mathbin{+\!\!\!\rightarrow} x = \dot{\Lambda}$ [*35·75·82 . *161·1]

***161·201.** $\vdash . x \mathbin{\leftarrow\!\!\!+} \dot{\Lambda} = \dot{\Lambda}$

***161·21.** $\vdash . (x \downarrow y) \mathbin{+\!\!\!\rightarrow} z = x \downarrow y \mathbin{\dot{\cup}} x \downarrow z \mathbin{\dot{\cup}} y \downarrow z$

Dem.

$\vdash . *161·1 . *55·15 . \supset \vdash . (x \downarrow y) \mathbin{+\!\!\!\rightarrow} z = x \downarrow y \mathbin{\dot{\cup}} (\iota'x \cup \iota'y) \uparrow \iota'z$

$[*35·82·41 . *55·1] \qquad = x \downarrow y \mathbin{\dot{\cup}} x \downarrow z \mathbin{\dot{\cup}} y \downarrow z . \supset \vdash .$ Prop

Note that $x \downarrow y \mathbin{\dot{\cup}} x \downarrow z \mathbin{\dot{\cup}} y \downarrow z$ is the relation which orders x and y and z in the order x, y, z.

***161·211.** $\vdash . x \mathbin{\leftarrow\!\!\!+} (y \downarrow z) = x \downarrow y \mathbin{\dot{\cup}} x \downarrow z \mathbin{\dot{\cup}} y \downarrow z = (x \downarrow y) \mathbin{+\!\!\!\rightarrow} z$

[Proof as in *161·21]

***161·212.** $P \mathbin{+\!\!\!\rightarrow} x \mathbin{+\!\!\!\rightarrow} y = (P \mathbin{+\!\!\!\rightarrow} x) \mathbin{+\!\!\!\rightarrow} y$ Df

***161·213.** $x \mathbin{\leftarrow\!\!\!+} y \mathbin{\leftarrow\!\!\!+} P = x \mathbin{\leftarrow\!\!\!+} (y \mathbin{\leftarrow\!\!\!+} P)$ Df

These definitions serve merely for the avoidance of brackets.

***161·22.** $\vdash : \dot{\exists} ! P . \supset . (P \mathbin{+\!\!\!\rightarrow} x) \mathbin{+\!\!\!\rightarrow} y = P \mathbin{\text{⇞}} (x \downarrow y)$

Dem.

$\vdash . *161·14·1 . \supset \vdash : \mathrm{Hp} . \supset . (P \mathbin{+\!\!\!\rightarrow} x) \mathbin{+\!\!\!\rightarrow} y = P \mathbin{\dot{\cup}} C'P \uparrow \iota'x \mathbin{\dot{\cup}} (C'P \cup \iota'x) \uparrow \iota'y$

$[*35·82·41] \qquad = P \mathbin{\dot{\cup}} C'P \uparrow \iota'x \mathbin{\dot{\cup}} C'P \uparrow \iota'y \mathbin{\dot{\cup}} \iota'x \uparrow \iota'y$

$[*35·82·412] \qquad = P \mathbin{\dot{\cup}} C'P \uparrow (\iota'x \cup \iota'y) \mathbin{\dot{\cup}} \iota'x \uparrow \iota'y$

$[*55·1·15] \qquad = P \mathbin{\dot{\cup}} C'P \uparrow C'(x \downarrow y) \mathbin{\dot{\cup}} x \downarrow y$

$[*160·1] \qquad = P \mathbin{\text{⇞}} (x \downarrow y) : \supset \vdash .$ Prop

***161·221.** $\vdash : \dot{\exists} ! P . \supset . x \mathbin{\leftarrow\!\!+} (y \mathbin{\leftarrow\!\!+} P) = (x \downarrow y) \mathbin{\Uparrow} P$

***161·23.** $\vdash : \dot{\exists} ! Q . \supset . (P \mathbin{\Uparrow} Q) \mathbin{+\!\!\rightarrow} y = P \mathbin{\Uparrow} (Q \mathbin{+\!\!\rightarrow} y)$

Dem.

$\vdash . *161\cdot14\cdot1 . *160\cdot1 . \supset \vdash : \mathrm{Hp} . \supset .$

$P \mathbin{\Uparrow} (Q \mathbin{+\!\!\rightarrow} y) = P \mathbin{\dot{\cup}} Q \mathbin{\dot{\cup}} C\text{‘}Q \uparrow \iota\text{‘}y \mathbin{\dot{\cup}} C\text{‘}P \uparrow (C\text{‘}Q \cup \iota\text{‘}y)$

[$*35\cdot82\cdot412$] $= P \mathbin{\dot{\cup}} Q \mathbin{\dot{\cup}} C\text{‘}P \uparrow C\text{‘}Q \mathbin{\dot{\cup}} C\text{‘}P \uparrow \iota\text{‘}y \mathbin{\dot{\cup}} C\text{‘}Q \uparrow \iota\text{‘}y$

[$*160\cdot1$] $= P \mathbin{\Uparrow} Q \mathbin{\dot{\cup}} C\text{‘}P \uparrow \iota\text{‘}y \mathbin{\dot{\cup}} C\text{‘}Q \uparrow \iota\text{‘}y$

[$*35\cdot82\cdot41 . *160\cdot14$] $= P \mathbin{\Uparrow} Q \mathbin{\dot{\cup}} C\text{‘}(P \mathbin{\Uparrow} Q) \uparrow \iota\text{‘}y$

[$*161\cdot1$] $= (P \mathbin{\Uparrow} Q) \mathbin{+\!\!\rightarrow} y : \supset \vdash .$ Prop

***161·231.** $\vdash : \dot{\exists} ! P . \supset . x \mathbin{\leftarrow\!\!+} (P \mathbin{\Uparrow} Q) = (x \mathbin{\leftarrow\!\!+} P) \mathbin{\Uparrow} Q$

***161·232.** $\vdash : \dot{\exists} ! P . \dot{\exists} ! Q . \supset . P \mathbin{\Uparrow} (x \mathbin{\leftarrow\!\!+} Q) = (P \mathbin{+\!\!\rightarrow} x) \mathbin{\Uparrow} Q$

Dem.

$\vdash . *161\cdot14\cdot101 . *160\cdot1 . \supset \vdash : \mathrm{Hp} . \supset .$

$P \mathbin{\Uparrow} (x \mathbin{\leftarrow\!\!+} Q) = P \mathbin{\dot{\cup}} \iota\text{‘}x \uparrow C\text{‘}Q \mathbin{\dot{\cup}} Q \mathbin{\dot{\cup}} C\text{‘}P \uparrow (\iota\text{‘}x \cup C\text{‘}Q)$

[$*35\cdot82\cdot412$] $= P \mathbin{\dot{\cup}} C\text{‘}P \uparrow \iota\text{‘}x \mathbin{\dot{\cup}} Q \mathbin{\dot{\cup}} C\text{‘}P \uparrow C\text{‘}Q \mathbin{\dot{\cup}} \iota\text{‘}x \uparrow C\text{‘}Q$

[$*161\cdot1\cdot14 . *35\cdot82\cdot41$] $= (P \mathbin{+\!\!\rightarrow} x) \mathbin{\dot{\cup}} Q \mathbin{\dot{\cup}} C\text{‘}(P \mathbin{+\!\!\rightarrow} x) \uparrow C\text{‘}Q$

[$*160\cdot1$] $= (P \mathbin{+\!\!\rightarrow} x) \mathbin{\Uparrow} Q : \supset \vdash .$ Prop

***161·24.** $\vdash . x \mathbin{\leftarrow\!\!+} (P \mathbin{+\!\!\rightarrow} y) = (x \mathbin{\leftarrow\!\!+} P) \mathbin{+\!\!\rightarrow} y$

Dem.

$\vdash . *161\cdot101\cdot14 . \supset \vdash : \dot{\exists} ! P . \supset .$

$x \mathbin{\leftarrow\!\!+} (P \mathbin{+\!\!\rightarrow} y) = \iota\text{‘}x \uparrow (C\text{‘}P \cup \iota\text{‘}y) \mathbin{\dot{\cup}} P \mathbin{\dot{\cup}} C\text{‘}P \uparrow \iota\text{‘}y$

[$*35\cdot82\cdot412$] $= \iota\text{‘}x \uparrow C\text{‘}P \mathbin{\dot{\cup}} P \mathbin{\dot{\cup}} \iota\text{‘}x \uparrow \iota\text{‘}y \mathbin{\dot{\cup}} C\text{‘}P \uparrow \iota\text{‘}y$

[$*35\cdot82\cdot41 . *161\cdot101\cdot14$] $= (x \mathbin{\leftarrow\!\!+} P) \mathbin{\dot{\cup}} C\text{‘}(x \mathbin{\leftarrow\!\!+} P) \uparrow \iota\text{‘}y$

[$*161\cdot1$] $= (x \mathbin{\leftarrow\!\!+} P) \mathbin{+\!\!\rightarrow} y$ (1)

$\vdash . *161\cdot2\cdot201 . \supset \vdash : P = \dot{\Lambda} . \supset . x \mathbin{\leftarrow\!\!+} (P \mathbin{+\!\!\rightarrow} y) = \dot{\Lambda} . (x \mathbin{\leftarrow\!\!+} P) \mathbin{+\!\!\rightarrow} y = \dot{\Lambda}$ (2)

$\vdash . (1) . (2) . \supset \vdash .$ Prop

***161·25.** $\vdash : \dot{\exists} ! P . \dot{\exists} ! Q . \supset . (P \mathbin{+\!\!\rightarrow} x) \mathbin{\Uparrow} (y \mathbin{\leftarrow\!\!+} Q) = P \mathbin{\Uparrow} (x \downarrow y) \mathbin{\Uparrow} Q$

Dem.

$\vdash . *161\cdot14 . *160\cdot1 . \supset$

$\vdash : \mathrm{Hp} . \supset . (P \mathbin{+\!\!\rightarrow} x) \mathbin{\Uparrow} (y \mathbin{\leftarrow\!\!+} Q) = (P \mathbin{+\!\!\rightarrow} x) \mathbin{\dot{\cup}} (y \mathbin{\leftarrow\!\!+} Q) \mathbin{\dot{\cup}} (C\text{‘}P \cup \iota\text{‘}x) \uparrow (C\text{‘}Q \cup \iota\text{‘}y)$

[$*161\cdot1\cdot101$] $= P \mathbin{\dot{\cup}} C\text{‘}P \uparrow \iota\text{‘}x \mathbin{\dot{\cup}} \iota\text{‘}y \uparrow C\text{‘}Q \mathbin{\dot{\cup}} Q$

$\mathbin{\dot{\cup}} (C\text{‘}P \cup \iota\text{‘}x) \uparrow (C\text{‘}Q \cup \iota\text{‘}y)$

[$*35\cdot82\cdot41\cdot412$] $= P \mathbin{\dot{\cup}} C\text{‘}P \uparrow (\iota\text{‘}x \cup \iota\text{‘}y) \mathbin{\dot{\cup}} \iota\text{‘}x \uparrow \iota\text{‘}y \mathbin{\dot{\cup}} Q$

$\mathbin{\dot{\cup}} (C\text{‘}P \cup \iota\text{‘}x \cup \iota\text{‘}y) \uparrow C\text{‘}Q$

[$*55\cdot15\cdot1 . *160\cdot14\cdot1$] $= \{P \mathbin{\Uparrow} (x \downarrow y)\} \mathbin{\dot{\cup}} Q \mathbin{\dot{\cup}} C\text{‘}\{P \mathbin{\Uparrow} (x \downarrow y)\} \uparrow C\text{‘}Q$

[$*160\cdot1 . (*160\cdot32)$] $= P \mathbin{\Uparrow} (x \downarrow y) \mathbin{\Uparrow} Q : \supset \vdash .$ Prop

$*161{\cdot}26$. $\vdash . x \leftarrow\!\!\!+ \{y \leftarrow\!\!\!+ (z \downarrow w)\} = (x \downarrow y) \ddagger (z \downarrow w) = \{(x \downarrow y) +\!\!\!\rightarrow z\} +\!\!\!\rightarrow w$
$= \{x \leftarrow\!\!\!+ (y \downarrow z)\} +\!\!\!\rightarrow w$

Dem.

$\vdash . *161{\cdot}221 . *55{\cdot}134 . \supset \vdash . x \leftarrow\!\!\!+ \{y \leftarrow\!\!\!+ (z \downarrow w)\} = (x \downarrow y) \ddagger (z \downarrow w)$
$[*161{\cdot}22 . *55{\cdot}134] \qquad = \{(x \downarrow y) +\!\!\!\rightarrow z\} +\!\!\!\rightarrow w$
$[*161{\cdot}211] \qquad = \{x \leftarrow\!\!\!+ (y \downarrow z)\} +\!\!\!\rightarrow w . \supset \vdash . \text{Prop}$

The following propositions lead up to $*161{\cdot}33$.

$*161{\cdot}3$. $\vdash : \dot{\exists} ! Q . S \in P \,\overline{\text{smor}}\, Q . x \sim\in C'P . y \sim\in C'Q . \supset .$
$S \cup x \downarrow y \in (P +\!\!\!\rightarrow x) \,\overline{\text{smor}}\, (Q +\!\!\!\rightarrow y)$

Dem.

$\vdash . *151{\cdot}11{\cdot}131 . \supset \vdash : \text{Hp} . \supset . S \in 1 \rightarrow 1 . C'Q = \text{Ɑ}'S . P = S;Q . C'P = \text{D}'S \quad (1)$
$\vdash . (1) . *55{\cdot}15 . \supset \vdash : \text{Hp} . \supset . \text{D}'S \cap \text{D}'(x \downarrow y) = \Lambda . \text{Ɑ}'S \cup \text{Ɑ}'(x \downarrow y) = \Lambda . \quad (2)$
$[*72{\cdot}182 . *71{\cdot}242] \qquad \supset . S \cup x \downarrow y \in 1 \rightarrow 1 \quad (3)$
$\vdash . *55{\cdot}15 . *151{\cdot}11 . \supset \vdash : \text{Hp} . \supset . \text{Ɑ}'(S \cup x \downarrow y) = C'Q \cup \iota'y$
$[*161{\cdot}14] \qquad = C'(Q +\!\!\!\rightarrow y) \quad (4)$
$\vdash . (1) . (2) . *34{\cdot}301 . \supset \vdash : \text{Hp} . \supset . (x \downarrow y) | Q = \dot{\Lambda} . Q | (y \downarrow x) = \dot{\Lambda} .$
$(x \downarrow y) | (C'Q \uparrow \iota'y) = \dot{\Lambda} . (C'Q \uparrow \iota'y) | \breve{S} = \dot{\Lambda} .$
$[*34{\cdot}25{\cdot}26] \qquad \supset . (S \cup x \downarrow y) | (Q \cup C'Q \uparrow \iota'y) = S | (Q \cup C'Q \uparrow \iota'y) .$
$(Q \cup C'Q \uparrow \iota'y) | (y \downarrow x \cup \breve{S}) = Q | \breve{S} \cup (C'Q \uparrow \iota'y) | (y \downarrow x)$
$[*35{\cdot}89 . *55{\cdot}1] \qquad = Q | \breve{S} \cup C'Q \uparrow \iota'x \quad (5)$
$\vdash . (5) . *150{\cdot}1 . \supset \vdash : \text{Hp} . \supset . (S \cup x \downarrow y);(Q \cup C'Q \uparrow \iota'y) = S | \{Q | \breve{S} \cup C'Q \uparrow \iota'x\}$
$[*150{\cdot}1] \qquad = S;Q \cup S | C'Q \uparrow \iota'x$
$[*37{\cdot}81 . (1) . *150{\cdot}23] \qquad = P \cup C'P \uparrow \iota'x \quad (6)$
$\vdash . (6) . *161{\cdot}1 . \supset \vdash : \text{Hp} . \supset . (S \cup x \downarrow y);(Q +\!\!\!\rightarrow y) = P +\!\!\!\rightarrow x \quad (7)$
$\vdash . (3) . (4) . (7) . *151{\cdot}11 . \supset \vdash . \text{Prop}$

$*161{\cdot}301$. $\vdash : \dot{\exists} ! Q . S \in P \,\overline{\text{smor}}\, Q . x \sim\in C'P . y \sim\in C'Q . \supset .$
$x \downarrow y \cup S \in (x \leftarrow\!\!\!+ P) \,\overline{\text{smor}}\, (y \leftarrow\!\!\!+ Q)$

$*161{\cdot}31$. $\vdash : P \,\text{smor}\, Q . x \sim\in C'P . y \sim\in C'Q . \supset .$
$P +\!\!\!\rightarrow x \,\text{smor}\, Q +\!\!\!\rightarrow y . x \leftarrow\!\!\!+ P \,\text{smor}\, y \leftarrow\!\!\!+ Q$

Dem.

$\vdash . *161{\cdot}3{\cdot}301 . *151{\cdot}12 . \supset$
$\vdash : \text{Hp} . \dot{\exists} ! Q . \supset . P +\!\!\!\rightarrow x \,\text{smor}\, Q +\!\!\!\rightarrow y . x \leftarrow\!\!\!+ P \,\text{smor}\, y \leftarrow\!\!\!+ Q \quad (1)$
$\vdash . *151{\cdot}32 . *161{\cdot}2{\cdot}201 . \supset$
$\vdash : \text{Hp} . Q = \dot{\Lambda} . \supset . P +\!\!\!\rightarrow x = \dot{\Lambda} . Q +\!\!\!\rightarrow y = \dot{\Lambda} . x \leftarrow\!\!\!+ P = \dot{\Lambda} . y \leftarrow\!\!\!+ Q = \dot{\Lambda} .$
$[*153{\cdot}101] \qquad \supset . P +\!\!\!\rightarrow x \,\text{smor}\, Q +\!\!\!\rightarrow y . x \leftarrow\!\!\!+ P \,\text{smor}\, y \leftarrow\!\!\!+ Q \quad (2)$
$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***161·32.** $\vdash : \dot{\exists} ! Q . x \sim\in C'P . y \sim\in C'Q . S \in (P \dashrightarrow x) \,\overline{\text{smor}}\, (Q \dashrightarrow y) . \supset .$
$S \upharpoonright (- \iota'y) \in P \,\overline{\text{smor}}\, Q . xSy$

Dem.

$\vdash . *151·5 . *161·15 . \supset \vdash : \text{Hp} . \supset . xSy$ (1)

$\vdash . (1) . *150·1 . \supset \vdash :. \text{Hp} . \supset : u \{S \upharpoonright (- \iota'y) ; Q\} v .$

$\equiv . (\exists z, w) . z (Q \dashrightarrow y) w . u \neq x . v \neq x . uSz . vSw .$

[*151·11] $\equiv . u \neq x . v \neq x . u (P \dashrightarrow x) v .$

[*161·11] $\equiv . uPv$ (2)

$\vdash . *35·64 . \supset \vdash : \text{Hp} . \supset . \text{D}'S \upharpoonright (- \iota'y) = C'(Q \dashrightarrow y) - \iota'y$

[*161·14·2] $= C'Q$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash . \text{Prop}$

***161·321.** $\vdash : \dot{\exists} ! Q . x \sim\in C'P . y \sim\in C'Q . S \in (x \leftarrow P) \,\overline{\text{smor}}\, (y \leftarrow Q) . \supset .$
$S \upharpoonright (- \iota'y) \in P \,\overline{\text{smor}}\, Q . xSy$

***161·33.** $\vdash :. x \sim\in C'P . y \sim\in C'Q . \supset :$
$P \,\text{smor}\, Q . \equiv . (P \dashrightarrow x) \,\text{smor}\, (Q \dashrightarrow y) . \equiv . (x \leftarrow P) \,\text{smor}\, (y \leftarrow Q)$
[*161·31·32·321·2·201 . *153·101]

The above proposition justifies addition of 1 or subtraction of 1 in ordinal arithmetic.

The following proposition (*161·4) is much used.

***161·4.** $\vdash : C'Q \subset \text{D}'S . x \in \text{D}'S . S \in 1 \rightarrow \text{Cls} . \supset . S;(Q \dashrightarrow x) = S;Q \dashrightarrow S'x$

Dem.

$\vdash . *161·1 . *150·3 . \quad \supset \vdash . S;(Q \dashrightarrow x) = S;Q \cup S;(C'Q \uparrow \iota'x)$

[*150·73] $= S;Q \cup (S``C'Q) \uparrow (S``\iota'x)$ (1)

$\vdash . (1) . *150·22 . *53·31 . \supset \vdash : \text{Hp} . \supset . S;(Q \dashrightarrow x) = S;Q \cup (C'S;Q) \uparrow (\iota'S'x)$

[*161·1] $= S;Q \dashrightarrow S'x : \supset \vdash . \text{Prop}$

***161·41.** $\vdash : C'Q \subset \text{D}'S . x \in \text{D}'S . S \in 1 \rightarrow \text{Cls} . \supset . S;(x \leftarrow Q) = S'x \leftarrow S;Q$

***161·42.** $\vdash . \downarrow y;(Q \dashrightarrow x) = \downarrow y;Q \dashrightarrow (x \downarrow y)$ [*161·4 . *55·21 . *72·184]

***161·43.** $\vdash . \downarrow y;(x \leftarrow Q) = (x \downarrow y) \leftarrow \downarrow y;Q$

$*$162. THE SUM OF THE RELATIONS OF A FIELD

Summary of $*$162.

The form of summation defined in $*$160 cannot be extended beyond a finite number of summands, since it involves explicit mention of all the summands. In the present number, we shall be concerned with a form of summation which is not subject to this restriction. It will be observed that, since relational summation is not permutative, we cannot define the sum of a *class* of relations, for this would not determine the order in which the summation is to be effected. Our relations must be given as the field of some relation which orders them; thus the sum appears not as the sum of a class, but as the sum of a relation, namely of a relation whose field is the relations to be summed. In the case of two relations Q and R, the sum of $Q \downarrow R$, as defined in the present number, will be equal to $Q \mathbin{\text{⤉}} R$; similarly for three, the sum of $Q \downarrow R \cup Q \downarrow S \cup R \downarrow S$ will be equal to $Q \mathbin{\text{⤉}} R \mathbin{\text{⤉}} S$, and so on for any finite number of summands.

As explained in the introduction to this Section, if P is a relation between relations, we put

$$\Sigma\text{‘}P = \dot{s}\text{‘}C\text{‘}P \cup F\,;P \quad \text{Df.}$$

It is convenient to suppose that P is serial, and that every member of $C\text{‘}P$ is also serial. Then $\Sigma\text{‘}P$ holds between x and y if either (1) there is a series, in the field of P, in which x precedes y, or (2) x belongs to a series which is earlier, in the P-series, than the series to which y belongs. The following are the chief propositions of this number:

$*$162·22·23. $\vdash . C\text{‘}\Sigma\text{‘}P = s\text{‘}C\text{‘‘}C\text{‘}P = C\text{‘}\dot{s}\text{‘}C\text{‘}P = F\text{‘‘}C\text{‘}P = \overrightarrow{F^2}\text{‘}P$

$*$162·26. $\vdash . \Sigma\text{‘}(P \cup Q) = \Sigma\text{‘}P \cup \Sigma\text{‘}Q$

$*$162·3. $\vdash . \Sigma\text{‘}(Q \downarrow R) = Q \mathbin{\text{⤉}} R$

$*$162·31. $\vdash . \Sigma\text{‘}Q \mathbin{\text{⤉}} \Sigma\text{‘}R = \Sigma\text{‘}(Q \mathbin{\text{⤉}} R)$

$*$162·34. $\vdash . \Sigma\text{‘}\Sigma\,;P = \Sigma\text{‘}\Sigma\text{‘}P$ [Associative Law. Cf. $*$42·1.]

$*$162·35. $\vdash : C\text{‘}\Sigma\text{‘}Q \subset \text{Ꮖ}\text{‘}R \,.\supset .\, \Sigma\text{‘}R\dagger\,;Q = R\,;\Sigma\text{‘}Q$

This is the analogue of $*$40·38. (Cf. note to $*$162·35, below.)

$*$162·4. $\vdash . \Sigma\text{‘}\dot{\Lambda} = \dot{\Lambda}$

$*$162·42. $\vdash : \dot{\exists}\,!\,\Sigma\text{‘}P \,.\equiv .\, \dot{\exists}\,!\,\dot{s}\text{‘}C\text{‘}P \,.\equiv .\, \exists\,!\,C\text{‘}P - \iota\text{‘}\dot{\Lambda}$

$*$162·43. $\vdash : \dot{\exists}\,!\,P \,.\supset .\, \Sigma\text{‘}(P \mathbin{+\!\!\!\rightarrow} R) = \Sigma\text{‘}P \mathbin{\text{⤉}} R$

It should be observed that the ordinal analogues of propositions about classes of classes often involve the substitution of Σ (not $\dot{s}$) for s. Examples are afforded by $*$162·34·35, quoted above.

∗162·01. $\Sigma`P = \dot{s}`C`P \cup F;P$ Df

∗162·1. $\vdash . \Sigma`P = \dot{s}`C`P \cup F;P$ [(∗162·01)]

∗162·11. $\vdash :. x(\Sigma`P)y . \equiv : x(\dot{s}`C`P)y . \vee . x(F;P)y$ [∗162·1]

∗162·12. $\vdash :. x(\Sigma`P)y . \equiv : (\exists Q) . Q \in C`P . xQy . \vee . (\exists Q, R) . xFQ . yFR . QPR$
[∗162·1 . ∗41·11 . ∗150·11]

∗162·13. $\vdash :. x(\Sigma`P)y . \equiv : (\exists Q) . Q \in C`P . xQy .$
$\vee . (\exists Q, R) . x \in C`Q . y \in C`R . QPR$ [∗161·12 . ∗33·51]

∗162·14. $\vdash :. x(\Sigma`P)y . \equiv : (\exists Q) . QFP . xQy . \vee . (\exists Q, R) . xFQ . yFR . QPR$
[∗161·12 . ∗33·51]

∗162·2. $\vdash . \mathrm{Cnv}`\Sigma`P = \Sigma`\mathrm{Cnv};\breve{P}$

Dem.

$\vdash . ∗162·13 . \supset \vdash :. x(\Sigma`\mathrm{Cnv};\breve{P})y . \equiv : (\exists Q) . Q \in C`\mathrm{Cnv};\breve{P} . xQy .$
$\vee . (\exists Q, R) . Q(\mathrm{Cnv};\breve{P})R . x \in C`Q . y \in C`R :$
[∗150·22·41] $\equiv : (\exists Q) . Q \in \mathrm{Cnv}``C`P . xQy .$
$\vee . (\exists Q, R) . \breve{Q}\breve{P}\breve{R} . x \in C`Q . y \in C`R :$
[∗37·64 . ∗33·22] $\equiv : (\exists Q) . Q \in C`P . yQx . \vee . (\exists Q, R) . RPQ . x \in C`Q . y \in C`R :$
[∗162·13] $\equiv : y(\Sigma`P)x :. \supset \vdash . \mathrm{Prop}$

∗162·21. $\vdash . \mathrm{D}`\Sigma`P = s`\mathrm{D}``C`P \cup s`C``\mathrm{D}`(P \upharpoonright - \iota`\Lambda)$

Dem.

$\vdash . ∗162·13 . \supset \vdash :. x \in \mathrm{D}`\Sigma`P . \equiv : (\exists Q, y) . Q \in C`P . xQy .$
$\vee . (\exists Q, R, y) . QPR . x \in C`Q . y \in C`R :$
[∗33·13·24] $\equiv : (\exists Q) . Q \in C`P . x \in \mathrm{D}`Q . \vee . (\exists Q, R) . QPR . x \in C`Q . \dot{\exists} ! R :$
[∗40·4 . ∗35·101] $\equiv : x \in s`\mathrm{D}``C`P . \vee . x \in s`C``\mathrm{D}`(P \upharpoonright - \iota`\Lambda) :. \supset \vdash . \mathrm{Prop}$

∗162·211. $\vdash . \text{Ꮖ}`\Sigma`P = s`\text{Ꮖ}``C`P \cup s`C``\text{Ꮖ}`(- \iota`\Lambda) \upharpoonleft P$

∗162·212. $\vdash : \Lambda \sim\in \text{Ꮖ}`P . \supset . \mathrm{D}`\Sigma`P = s`C``\mathrm{D}`P \cup s`\mathrm{D}``\overrightarrow{B}`\breve{P}$

Dem.

$\vdash . ∗162·21 . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{D}`\Sigma`P = s`\mathrm{D}``C`P \cup s`C``\mathrm{D}`P$
[∗40·31 . ∗93·12] $= s`\mathrm{D}``\mathrm{D}`P \cup s`\mathrm{D}``\overrightarrow{B}`\breve{P} \cup s`C``\mathrm{D}`P$
[∗40·57] $= s`\mathrm{D}``\overrightarrow{B}`\breve{P} \cup s`C``\mathrm{D}`P : \supset \vdash . \mathrm{Prop}$

∗162·213. $\vdash : \Lambda \sim\in \mathrm{D}`P . \supset . \text{Ꮖ}`\Sigma`P = s`C``\text{Ꮖ}`P \cup s`\text{Ꮖ}``\overrightarrow{B}`P$

The above proposition is used in ∗163·22.

The two following propositions are used very often.

∗162·22. $\vdash . C`\Sigma`P = s`C``C`P$

Dem.

$\vdash . ∗162·21·211 . ∗40·57 . \supset$
$\vdash . C`\Sigma`P = s`C``C`P \cup s`C``\mathrm{D}`(P \upharpoonright - \iota`\Lambda) \cup s`C``\text{Ꮖ}`(- \iota`\Lambda) \upharpoonleft P$
[∗40·161] $= s`C``C`P . \supset \vdash . \mathrm{Prop}$

***162·23.** $\vdash . C`\Sigma`P = C`\dot{s}`C`P = F``C`P = \overrightarrow{F^2}`P$ [*162·22 . *42·2]

***162·26.** $\vdash . \Sigma`(P \cup Q) = \Sigma`P \cup \Sigma`Q$

Dem. $\vdash . *162\cdot1 . \supset \vdash . \Sigma`(P \cup Q) = \dot{s}`C`(P \cup Q) \cup F;(P \cup Q)$

[*33·262.*41·171.*150·3] $= \dot{s}`C`P \cup \dot{s}`C`Q \cup F;P \cup F;Q$

[*162·1] $= \Sigma`P \cup \Sigma`Q . \supset \vdash .$ Prop

***162·27.** $\vdash . \Sigma`S;(P \cup Q) = \Sigma`S;P \cup \Sigma`S;Q$ [*162·26 . *150·3]

***162·3.** $\vdash . \Sigma`(Q \downarrow R) = Q \ddagger R$

Dem. $\vdash . *160\cdot1 . \supset \vdash . \Sigma`(Q \downarrow R) = \dot{s}`C`(Q \downarrow R) \cup F;(Q \downarrow R)$

[*55·15.*150·7] $= \dot{s}`(\iota`Q \cup \iota`R) \cup \overrightarrow{F}`Q \uparrow \overrightarrow{F}`R$

[*53·13.*33·5] $= Q \cup R \cup C`Q \uparrow C`R$

[*160·1] $= Q \ddagger R . \supset \vdash .$ Prop

This proposition establishes the connection between the two kinds of arithmetical addition of relations.

***162·31.** $\vdash . \Sigma`Q \ddagger \Sigma`R = \Sigma`(Q \ddagger R)$

Dem.

$\vdash . *160\cdot1 . \supset \vdash . \Sigma`Q \ddagger \Sigma`R = \Sigma`Q \cup \Sigma`R \cup C`\Sigma`Q \uparrow C`\Sigma`R$

[*162·123] $= \dot{s}`C`Q \cup F;Q \cup \dot{s}`C`R \cup F;R \cup (F``C`Q) \uparrow (F``C`R)$

[*150·73] $= \dot{s}`C`Q \cup \dot{s}`C`R \cup F;Q \cup F;R \cup F;(C`Q \uparrow C`R)$

[*41·171.*160·14.*150·3.*160·1] $= \dot{s}`C`(Q \ddagger R) \cup F;(Q \ddagger R)$

[*162·1] $= \Sigma`(Q \ddagger R) . \supset \vdash .$ Prop

The following propositions lead up to *162·34.

***162·32.** $\vdash . \Sigma`\dot{s}`\kappa = \dot{s}`\Sigma``\kappa$

Dem. $\vdash . *41\cdot6 . *162\cdot1 . *150\cdot1 . \supset \vdash . \dot{s}`\Sigma``\kappa = \dot{s}`\dot{s}``C``\kappa \cup \dot{s}`F\dagger``\kappa$

[*42·12.*150·16] $= \dot{s}`s`C``\kappa \cup F;\dot{s}`\kappa$

[*41·45] $= \dot{s}`C`\dot{s}`\kappa \cup F;\dot{s}`\kappa$

[*162·1] $= \Sigma`\dot{s}`\kappa . \supset \vdash .$ Prop

***162·33.** $\vdash . \Sigma`\Sigma`P = \dot{s}`C`\dot{s}`C`P \cup F;\dot{s}`C`P \cup F^2;P$

Dem. $\vdash . *162\cdot1 . \supset \vdash . \Sigma`\Sigma`P = \dot{s}`C`\Sigma`P \cup F;\Sigma`P$

[*162·23] $= \dot{s}`C`\dot{s}`C`P \cup F;(\dot{s}`C`P \cup F;P)$

[*150·3·13] $= \dot{s}`C`\dot{s}`C`P \cup F;\dot{s}`C`P \cup F^2;P . \supset \vdash .$ Prop

***162·331.** $\vdash . F | \Sigma = F | \dot{s} | C = F^2$

Dem. $\vdash . *71\cdot7 . \supset \vdash : x (F | \Sigma) P . \equiv . x F (\Sigma`P) .$

[*33·51] $\equiv . x \in C`\Sigma`P .$

[*162·23] $\equiv . x F^2 P$ (1)

$\vdash . *71\cdot7 . \supset \vdash : x (F | \dot{s} | C) P . \equiv . x F (\dot{s}`C`P) .$

[*33·51] $\equiv . x \in C`\dot{s}`C`P .$

[*42·2] $\equiv . x F^2 P$ (2)

$\vdash . (1) . (2) . \supset \vdash .$ Prop

$*162{\cdot}332$. $\vdash . \Sigma'\Sigma;P = \dot{s}'C'\dot{s}'C'P \cup F;\dot{s}'C'P \cup F^2;P$

Dem.

$\vdash . *162{\cdot}1 . \supset \vdash . \Sigma'\Sigma;P = \dot{s}'C'\Sigma;P \cup F;\Sigma;P$

$[*150{\cdot}22{\cdot}13] \qquad = \dot{s}'\Sigma``C'P \cup (F \,|\, \Sigma);P$

$[*162{\cdot}32{\cdot}331] \qquad = \Sigma'\dot{s}'C'P \cup F^2;P$

$[*162{\cdot}1] \qquad = \dot{s}'C'\dot{s}'C'P \cup F;\dot{s}'C'P \cup F^2;P . \supset \vdash . \text{Prop}$

$*162{\cdot}34$. $\vdash . \Sigma'\Sigma;P = \Sigma'\Sigma'P$ $[*162{\cdot}33{\cdot}332]$

This is the associative law for arithmetical sums of relations.

The following propositions lead up to $*162{\cdot}35$.

$*162{\cdot}341$. $\vdash :. C'Q \subset \text{Ɑ}'R . \supset : x(F \,|\, R\dagger)\, Q . \equiv . x(R \,|\, F)\, Q$

Dem.

$\vdash . *71{\cdot}7 . *150{\cdot}1 . \supset \vdash : x(F \,|\, R\dagger)\, Q . \equiv . xF(R;Q) .$

$[*33{\cdot}51] \qquad \equiv . x \in C'R;Q \qquad (1)$

$\vdash . (1) . *150{\cdot}22 . \supset \vdash :. \text{Hp} . \supset : x(F \,|\, R\dagger)\, Q . \equiv . x \in R``C'Q .$

$[*33{\cdot}5] \qquad \equiv . x \in R``\overrightarrow{F}'Q .$

$[*37{\cdot}3 . *32{\cdot}18] \qquad \equiv . x(R \,|\, F)\, Q :. \supset \vdash . \text{Prop}$

$*162{\cdot}342$. $\vdash : C'\dot{s}'\lambda \subset \text{Ɑ}'R . \supset . (F \,|\, R\dagger) \upharpoonright \lambda = (R \,|\, F) \upharpoonright \lambda$

Dem.

$\vdash . *41{\cdot}13 . \supset \vdash :. \text{Hp} . \supset : Q \in \lambda . \supset . C'Q \subset \text{Ɑ}'R :$

$[*162{\cdot}341] \qquad \supset : Q \in \lambda . x(F \,|\, R\dagger)\, Q . \equiv . Q \in \lambda . x(R \,|\, F)\, Q :. \supset \vdash . \text{Prop}$

$*162{\cdot}343$. $\vdash : C'\Sigma'P \subset \text{Ɑ}'R . \supset . F;R\dagger;P = R;F;P$

Dem.

$\vdash . *162{\cdot}23 . \supset \vdash : \text{Hp} . \supset . C'\dot{s}'C'P \subset \text{Ɑ}'R .$

$[*162{\cdot}342] \qquad \supset . (F \,|\, R\dagger) \upharpoonright (C'P);P = (R \,|\, F) \upharpoonright (C'P);P .$

$[*150{\cdot}32] \qquad \supset . (F \,|\, R\dagger);P = (R \,|\, F);P .$

$[*150{\cdot}13] \qquad \supset . F;R\dagger;P = R;F;P : \supset \vdash . \text{Prop}$

$*162{\cdot}35$. $\vdash : C'\Sigma'Q \subset \text{Ɑ}'R . \supset . \Sigma'R\dagger;Q = R;\Sigma'Q$

Dem.

$\vdash . *162{\cdot}1 . *150{\cdot}22 . \supset \vdash . \Sigma'R\dagger;Q = \dot{s}'R\dagger``C'Q \cup F;R\dagger;Q$

$[*150{\cdot}16] \qquad = R;\dot{s}'C'Q \cup F;R\dagger;Q \qquad (1)$

$\vdash . (1) . *162{\cdot}343 . \supset \vdash : \text{Hp} . \supset . \Sigma'R\dagger;Q = R;\dot{s}'C'Q \cup R;F;Q$

$[*150{\cdot}3 . *162{\cdot}1] \qquad = R;\Sigma'Q : \supset \vdash . \text{Prop}$

This proposition is important, since it enables us to infer (with a suitable hypothesis) that if $R;M$ is always like M when $M \in C'Q$, then the arithmetical sum of all such relations as $R;M$ is like $\Sigma'Q$, being in fact $R;\Sigma'Q$. In other words, if, whenever $M \in C'Q$, $R \upharpoonright C'M$ is a correlator of $R;M$ and M, then $R \upharpoonright \Sigma'Q$ is a correlator of $\Sigma'R\dagger;Q$ and $\Sigma'Q$. This proposition is analogous in its uses to the proposition

$$s'R_\epsilon``\kappa = R``s'\kappa,$$

which is $*40\cdot38$. In general, in obtaining relational analogues of cardinal propositions, $R``\kappa$ is to be replaced by $R;Q$, $R_\epsilon$ by $R\dagger$, and $s$ by $\Sigma$. When these substitutions are made in $s`R_\epsilon``\kappa = R``s`\kappa$, $*162\cdot35$ results, except for its hypothesis.

If we regard $R;Q$ as a kind of product of R and Q, $*162\cdot35$ becomes a distributive law. For it asserts that if we multiply each member of $C`Q$ by $R$, and then sum the resulting products, we get the same relation as if we first sum $C`Q$, and then multiply by R. The following application of $*162\cdot35$ to the sum of two relations makes its distributive character more evident.

$*162\cdot36$. $\vdash : C`P \cup C`Q \subset \text{Cl}`R . \supset . R;P \Downarrow R;Q = R;(P \Downarrow Q)$

Dem.

$\vdash . *162\cdot3 . \quad \supset \vdash . R;P \Downarrow R;Q = \Sigma`\{(R;P) \downarrow (R;Q)\}$

$[*150\cdot1\cdot71] \qquad = \Sigma`R\dagger;(P \downarrow Q) \qquad (1)$

$\vdash . (1) . *162\cdot35 . \supset \vdash : \text{Hp} . \supset . R;P \Downarrow R;Q = R;\Sigma`(P \downarrow Q)$

$[*162\cdot3] \qquad = R;(P \Downarrow Q) : \supset \vdash . \text{Prop}$

This proposition can be extended to any finite number of summands.

$*162\cdot37$. $\vdash : \exists ! \lambda . \exists ! \mu . \supset . \Sigma`(\lambda \uparrow \mu) = \dot{s}`\lambda \Downarrow \dot{s}`\mu$

Dem.

$\vdash . *35\cdot85\cdot86 . \supset \vdash : \text{Hp} . \supset . C`(\lambda \uparrow \mu) = \lambda \cup \mu .$

$[*162\cdot1] \qquad \supset . \Sigma`(\lambda \uparrow \mu) = \dot{s}`(\lambda \cup \mu) \dot{\cup} F;(\lambda \uparrow \mu)$

$[*41\cdot171 . *150\cdot73] \qquad = \dot{s}`\lambda \dot{\cup} \dot{s}`\mu \dot{\cup} (F``\lambda) \uparrow (F``\mu)$

$[*41\cdot45 . *40\cdot56] \qquad = \dot{s}`\lambda \dot{\cup} \dot{s}`\mu \dot{\cup} (C`\dot{s}`\lambda) \uparrow (C`\dot{s}`\mu)$

$[*160\cdot1] \qquad = \dot{s}`\lambda \Downarrow \dot{s}`\mu : \supset \vdash . \text{Prop}$

$*162\cdot371$. $\vdash : \exists ! \alpha . \supset . \Sigma`(\alpha \uparrow \iota`Q) = \dot{s}`\alpha \Downarrow Q \quad [*162\cdot37 . *53\cdot04]$

$*162\cdot372$. $\vdash : \exists ! \beta . \supset . \Sigma`(\iota`P) \uparrow \beta = P \Downarrow \dot{s}`\beta$

$*162\cdot4$. $\vdash . \Sigma`\dot{\Lambda} = \dot{\Lambda}$

Dem.

$\vdash . *33\cdot241 . *41\cdot21 . \supset \vdash . \dot{s}`C`\dot{\Lambda} = \dot{\Lambda} \qquad (1)$

$\vdash . *150\cdot42 . \quad \supset \vdash . F;\dot{\Lambda} = \dot{\Lambda} \qquad (2)$

$\vdash . (1) . (2) . *162\cdot1 . \supset \vdash . \text{Prop}$

$*162\cdot41$. $\vdash . \Sigma`(\dot{\Lambda} \downarrow \dot{\Lambda}) = \dot{\Lambda}$

Dem.

$\vdash . *162\cdot3 . \supset \vdash . \Sigma`(\dot{\Lambda} \downarrow \dot{\Lambda}) = \dot{\Lambda} \Downarrow \dot{\Lambda}$

$[*160\cdot21] \qquad = \dot{\Lambda} . \supset \vdash . \text{Prop}$

$*162\cdot42$. $\vdash : \dot{\exists} ! \Sigma`P . \equiv . \dot{\exists} ! \dot{s}`C`P . \equiv . \exists ! C`P - \iota`\dot{\Lambda}$

Dem.

$\vdash . *162\cdot23 . *33\cdot24 . \supset \vdash : \dot{\exists} ! \Sigma`P . \equiv . \dot{\exists} ! \dot{s}`C`P .$

$[*41\cdot26] \qquad \equiv . \exists ! C`P - \iota`\dot{\Lambda}$

***162·43.** $\vdash : \dot{\exists} ! P . \supset . \Sigma\text{‘}(P \mathbin{+\!\!\!\rightarrow} R) = \Sigma\text{‘}P \mathbin{\uparrow\!\!\!\!+} R$

Dem.

$\vdash . *162{\cdot}26 . *161{\cdot}1 . \supset \vdash . \Sigma\text{‘}(P \mathbin{+\!\!\!\rightarrow} R) = \Sigma\text{‘}P \cup \Sigma\text{‘}(C\text{‘}P \uparrow \iota\text{‘}R)$ (1)

$\vdash . *162{\cdot}371 . *33{\cdot}24 . \supset \vdash : \dot{\exists} ! P . \supset . \Sigma\text{‘}(C\text{‘}P \uparrow \iota\text{‘}R) = \dot{s}\text{‘}C\text{‘}P \mathbin{\uparrow\!\!\!\!+} R$ (2)

$\vdash . (1) . (2) . *160{\cdot}1 . \supset$

$\vdash : \mathrm{Hp} . \supset . \Sigma\text{‘}(P \mathbin{+\!\!\!\rightarrow} R) = \Sigma\text{‘}P \cup \dot{s}\text{‘}C\text{‘}P \cup R \cup (C\text{‘}\dot{s}\text{‘}C\text{‘}P) \uparrow C\text{‘}R$

$[*162{\cdot}1{\cdot}23] \quad = \Sigma\text{‘}P \cup R \cup (C\text{‘}\Sigma\text{‘}P) \uparrow C\text{‘}R$

$[*160{\cdot}1] \quad = \Sigma\text{‘}P \mathbin{\uparrow\!\!\!\!+} R : \supset \vdash . \mathrm{Prop}$

***162·431.** $\vdash : \dot{\exists} ! P . \supset . \Sigma\text{‘}(R \mathbin{\leftarrow\!\!\!+} P) = R \mathbin{\uparrow\!\!\!\!+} \Sigma\text{‘}P$ [Proof as in *162·43]

Observe that in *162·43·431, P and R must be of different types, in fact R must be of the type to which members of $C\text{‘}P$ belong. *162·43·431 are often useful.

***162·44.** $\vdash . \Sigma\text{‘}(P \mathbin{+\!\!\!\rightarrow} \dot{\Lambda}) = \Sigma\text{‘}(\dot{\Lambda} \mathbin{\leftarrow\!\!\!+} P) = \Sigma\text{‘}P$

Dem.

$\vdash . *162{\cdot}43 . \quad \supset \vdash : \dot{\exists} ! P . \supset . \Sigma\text{‘}(P \mathbin{+\!\!\!\rightarrow} \dot{\Lambda}) = \Sigma\text{‘}P \mathbin{\uparrow\!\!\!\!+} \dot{\Lambda}$

$[*160{\cdot}21] \quad = \Sigma\text{‘}P$ (1)

$\vdash . *33{\cdot}241 . *35{\cdot}88 . \supset \vdash : P = \dot{\Lambda} . \supset . C\text{‘}P \uparrow \iota\text{‘}\dot{\Lambda} = \dot{\Lambda} .$

$[*162{\cdot}4] \quad \supset . \Sigma\text{‘}(C\text{‘}P \uparrow \iota\text{‘}\dot{\Lambda}) = \dot{\Lambda} .$

$[*25{\cdot}24] \quad \supset . \Sigma\text{‘}P = \Sigma\text{‘}P \cup \Sigma\text{‘}(C\text{‘}P \uparrow \iota\text{‘}\dot{\Lambda})$

$[*162{\cdot}26] \quad = \Sigma\text{‘}(P \cup C\text{‘}P \uparrow \iota\text{‘}\dot{\Lambda})$

$[*161{\cdot}1] \quad = \Sigma\text{‘}(P \mathbin{+\!\!\!\rightarrow} \dot{\Lambda})$ (2)

$\vdash . (1) . (2) . \supset \vdash . \Sigma\text{‘}(P \mathbin{+\!\!\!\rightarrow} \dot{\Lambda}) = \Sigma\text{‘}P$ (3)

Similarly $\vdash . \Sigma\text{‘}(\dot{\Lambda} \mathbin{\leftarrow\!\!\!+} P) = \Sigma\text{‘}P$ (4)

$\vdash . (3) . (4) . \supset \vdash . \mathrm{Prop}$

***162·45.** $\vdash : \dot{\exists} ! P . \Sigma\text{‘}P = \dot{\Lambda} . \equiv . P = \dot{\Lambda} \downarrow \dot{\Lambda}$

Dem.

$\vdash . *162{\cdot}42 . \supset \vdash : \Sigma\text{‘}P = \dot{\Lambda} . \equiv . C\text{‘}P \subset \iota\text{‘}\dot{\Lambda} .$

$[*33{\cdot}16] \quad \equiv . \mathrm{D}\text{‘}P \subset \iota\text{‘}\dot{\Lambda} . \mathrm{G}\text{‘}P \subset \iota\text{‘}\dot{\Lambda}$ (1)

$\vdash . *33{\cdot}24 . \supset \vdash : \dot{\exists} ! P . \equiv . \exists ! \mathrm{D}\text{‘}P . \exists ! \mathrm{G}\text{‘}P$ (2)

$\vdash . (1) . (2) . *51{\cdot}4 . \supset$

$\vdash : \dot{\exists} ! P . \Sigma\text{‘}P = \dot{\Lambda} . \equiv . \mathrm{D}\text{‘}P = \iota\text{‘}\dot{\Lambda} . \mathrm{G}\text{‘}P = \iota\text{‘}\dot{\Lambda} .$

$[*55{\cdot}16] \quad \equiv . P = \dot{\Lambda} \downarrow \dot{\Lambda} : \supset \vdash . \mathrm{Prop}$

The above proposition is used in *174·162.

$*163$. RELATIONS OF MUTUALLY EXCLUSIVE RELATIONS

Summary of $*163$.

In the present number we have to define mutually exclusive relations, and to give a few of their properties. Mutually exclusive relations play much the same part in relation-arithmetic as mutually exclusive classes play in cardinal arithmetic. *Prima facie,* there are various ways in which we might define them. We might define P as a relation of mutually exclusive relations when

$$QPR \,.\, Q \neq R \,.\, \supset_{Q,R} \,.\, Q \dot{\cap} R = \dot{\Lambda},$$

or when $$Q, R \in C\text{`}P \,.\, Q \neq R \,.\, \supset_{Q,R} \,.\, Q \dot{\cap} R = \dot{\Lambda},$$

or when

$$Q, R \in C\text{`}P \,.\, Q \neq R \,.\, \supset_{Q,R} \,.\, \text{D}\text{`}Q \cap \text{D}\text{`}R = \Lambda \,.\, \text{G}\text{`}Q \cap \text{G}\text{`}R = \Lambda,$$

or in several other ways. But in fact the most useful property to choose is the property that any two members of the field have mutually exclusive fields, *i.e.*

$$Q, R \in C\text{`}P \,.\, Q \neq R \,.\, \supset_{Q,R} \,.\, C\text{`}Q \cap C\text{`}R = \Lambda.$$

The principal applications of the subjects studied in this Part are to series, and in series it is always the *fields* of the relations that are important. We want, for instance, to define relations of mutually exclusive relations in such a way that, if P is a serial relation, and every member of $C\text{`}P$ is a serial relation, then $\Sigma\text{`}P$ is a serial relation. For this purpose it is necessary that $\Sigma\text{`}P$ should be contained in diversity, which requires that $F;P$ should be contained in diversity, *i.e.* that

$$QPR \,.\, \supset_{Q,R} \,.\, C\text{`}Q \cap C\text{`}R = \Lambda.$$

If P is a serial relation, as we are supposing, this is equivalent to

$$Q, R \in C\text{`}P \,.\, Q \neq P \,.\, \supset_{Q,R} \,.\, C\text{`}Q \cap C\text{`}R = \Lambda.$$

Again we want to define relations of mutually exclusive relations in such a way that, if P and Q are two such relations, and P and Q have double likeness (cf. $*164$), then $\Sigma\text{`}P$ is like $\Sigma\text{`}Q$; *i.e.* if we are given a correlator S of P and Q, and for every M and N which S correlates, we are again given a correlator, then $\Sigma\text{`}P$ is to be like $\Sigma\text{`}Q$. That is, if λ is the class of relations which correlate pairs of relations M and N, where $N \in C\text{`}Q \,.\, MSN$, we want $\dot{s}\text{`}\lambda$ to be a correlator of P and Q. Now this requires that $\dot{s}\text{`}\lambda$ should be a one-one relation, which requires

$$M, M' \in C\text{`}P \,.\, M \neq M' \,.\, \supset_{M,M'} \,.\, \text{D}\text{`}M \cap \text{D}\text{`}M' = \Lambda \,.\, \text{G}\text{`}M \cap \text{G}\text{`}M' = \Lambda.$$

This is secured by

$$M, M' \in C\text{`}P \,.\, M \neq M' \,.\, \supset_{M,M'} \,.\, C\text{`}M \cap C\text{`}M' = \Lambda,$$

but except for special classes of relations it is not secured by

$$MPM' \,.\, \supset_{M,M'} \,.\, C\text{`}M \cap C\text{`}M' = \Lambda,$$

since there may be two relations M and M' which both belong to the field of P, but of which neither has the relation P to the other. Again, the analogy with cardinal arithmetic fails at many points unless, when P is a relation of mutually exclusive relations, $C``C`P$ is a class of mutually exclusive classes. But this is not secured by any of the other possible definitions we have been considering. There are further reasons, connected with the arithmetical product of a relation of relations, for choosing as the definition

$$Q, R \in C`P \,.\, Q \neq R \,.\, \supset_{Q,R} \,.\, C`Q \cap C`R = \Lambda.$$

From a technical point of view, the properties of a Cls^2 excl depend mainly upon the fact that when κ is such a class, $\epsilon \upharpoonright \kappa \in \text{Cls} \rightarrow 1$ (*84·14); in like manner the properties of a Rel^2 excl depend upon

$$F \upharpoonright C`P \in \text{Cls} \rightarrow 1,$$

which requires our definition, and is equivalent to it (*163·12). We thus become able to use the propositions of *81 on selections from many-one relations, which would not otherwise be the case.

It should be observed that

$$Q, R \in C`P \,.\, Q \neq R \,.\, \supset_{Q,R} \,.\, C`Q \cap C`R = \Lambda$$

is not equivalent to

$$C``C`P \in \text{Cls}^2\,\text{excl},$$

though it implies this. The converse implication will fail if $C`P$ contains two different relations with the same field. *E.g.* take a relation $P$ whose field consists of the four relations $S$, $\breve{S}$, $T$, $\breve{T}$, and suppose $C`S \cap C`T = \Lambda$. Then $C``C`P = \iota`C`S \cup \iota`C`T$, and $C``C`P \in \text{Cls}^2\,\text{excl}$. But unless $S = \breve{S}$ and $T = \breve{T}$ we shall not have

$$Q, R \in C`P \,.\, Q \neq R \,.\, \supset_{Q,R} \,.\, C`Q \cap C`R = \Lambda.$$

The property by which we define relations of mutually exclusive relations is a property which only depends on the field, so that we might equally well put

$$(\text{Cl}`\text{Rel})\,\text{excl} = \hat{\lambda}\,\{Q, R \in \lambda \,.\, Q \neq R \,.\, \supset_{Q,R} \,.\, C`Q \cap C`R = \Lambda\} \quad \text{Df.}$$

But for our purposes this would be less convenient than the definition of Rel^2 excl.

We thus put

***163·01.** $\text{Rel}^2\,\text{excl} = \hat{P}\,\{Q, R \in C`P \,.\, Q \neq R \,.\, \supset_{Q,R} \,.\, C`Q \cap C`R = \Lambda\}$ Df

We have

***163·11.** $\vdash :.\, P \in \text{Rel}^2\,\text{excl} \,.\, \equiv :\, Q, R \in C`P \,.\, \exists !\, C`Q \cap C`R \,.\, \supset_{Q,R} \,.\, Q = R$

***163·12.** $\vdash :\, P \in \text{Rel}^2\,\text{excl} \,.\, \equiv \,.\, F \upharpoonright C`P \in \text{Cls} \rightarrow 1$

***163·17.** $\vdash :\, P \in \text{Rel}^2\,\text{excl} \,.\, \equiv \,.\, C \upharpoonright C`P \in 1 \rightarrow 1 \,.\, C``C`P \in \text{Cls}^2\,\text{excl}$

Any of the above might have been used to define Rel^2 excl. The following propositions are important.

*163·3. $\vdash : Q \epsilon \text{Rel}^2\,\text{excl} \,.\, S \epsilon \text{Cls} \to 1 \,.\, \supset \,.\, S \dagger \dot{;} Q \epsilon \text{Rel}^2\,\text{excl}$

This is the analogue of *84·53.

*163·4·41. $\vdash \,.\, \dot{\Lambda}, P \downarrow P \epsilon \text{Rel}^2\,\text{excl}$

*163·441. $\vdash : P, Q \epsilon \text{Rel}^2\,\text{excl} \,.\, C\text{‘}\Sigma\text{‘}P \cap C\text{‘}\Sigma\text{‘}Q = \Lambda \,.\, \supset \,.\, P \ddagger Q \epsilon \text{Rel}^2\,\text{excl}$

*163·451. $\vdash : P \epsilon \text{Rel}^2\,\text{excl} \,.\, C\text{‘}\Sigma\text{‘}P \cap C\text{‘}R = \Lambda \,.\, \supset \,.\, P \dashrightarrow R \epsilon \text{Rel}^2\,\text{excl}$

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*163·01. $\text{Rel}^2\,\text{excl} = \hat{P}\{Q, R \epsilon C\text{‘}P \,.\, Q \neq R \,.\, \supset_{Q,R} \,.\, C\text{‘}Q \cap C\text{‘}R = \Lambda\}$ Df

*163·1. $\vdash :. P \epsilon \text{Rel}^2\,\text{excl} \,.\, \equiv : Q, R \epsilon C\text{‘}P \,.\, Q \neq R \,.\, \supset_{Q,R} \,.\, C\text{‘}Q \cap C\text{‘}R = \Lambda$
[(*163·01)]

*163·11. $\vdash :. P \epsilon \text{Rel}^2\,\text{excl} \,.\, \equiv : Q, R \epsilon C\text{‘}P \,.\, \exists ! \, C\text{‘}Q \cap C\text{‘}R \,.\, \supset_{Q,R} \,.\, Q = R$
[*163·1 . Transp]

*163·12. $\vdash : P \epsilon \text{Rel}^2\,\text{excl} \,.\, \equiv \,.\, F \restriction C\text{‘}P \epsilon \text{Cls} \to 1$ [*163·1 . *74·632]

For many purposes, this proposition gives the most useful equivalent of $P \epsilon \text{Rel}^2\,\text{excl}$.

Instead of the above proof, we may use *74·62, which gives us the result in virtue of *33·5.

*163·13. $\vdash :. P \epsilon \text{Rel}^2\,\text{excl} \,.\, \supset :$

$$Q, R \epsilon C\text{‘}P \,.\, Q \neq R \,.\, \supset_{Q,R} \,.\, \text{D}\text{‘}Q \cap \text{D}\text{‘}R = \Lambda \,.\, \text{Cl}\text{‘}Q \cap \text{Cl}\text{‘}R = \Lambda$$

[*24·402 . *163·1]

*163·14. $\vdash : P \epsilon \text{Rel}^2\,\text{excl} \,.\, \supset \,.\, C \restriction C\text{‘}P \epsilon 1 \to 1$ [*163·12 . *74·32 . *33·5]

*163·15. $\vdash : P \epsilon \text{Rel}^2\,\text{excl} \,.\, \supset \,.\, \text{D} \restriction C\text{‘}P, \text{Cl} \restriction C\text{‘}P \epsilon 1 \to 1$

Dem.

$\vdash \,.\, *74·63 \,.\, *163·13 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, (\epsilon \,|\, \text{D}) \restriction C\text{‘}P \epsilon \text{Cls} \to 1 \,.$

[*74·32] $\supset \,.\, \overrightarrow{\epsilon \,|\, \text{D}} \restriction C\text{‘}P \epsilon 1 \to 1 \,.$

[*72·27] $\supset \,.\, \text{D} \restriction C\text{‘}P \epsilon 1 \to 1$ (1)

Similarly $\vdash : \text{Hp} \,.\, \supset \,.\, \text{Cl} \restriction C\text{‘}P \epsilon 1 \to 1$ (2)

$\vdash \,.\, (1) \,.\, (2) \,.\, \supset \vdash \,.\, \text{Prop}$

*163·16. $\vdash : P \epsilon \text{Rel}^2\,\text{excl} \,.\, \supset \,.\, C\text{“}C\text{‘}P \epsilon \text{Cls}^2\,\text{excl}$ [*84·51 . *33·5 . *163·12]

*163·17. $\vdash : P \epsilon \text{Rel}^2\,\text{excl} \,.\, \equiv \,.\, C \restriction C\text{‘}P \epsilon 1 \to 1 \,.\, C\text{“}C\text{‘}P \epsilon \text{Cls}^2\,\text{excl}$
[*163·12 . *84·522 . *33·5]

*163·2. $\vdash : P \epsilon \text{Rel}^2\,\text{excl} \,.\, \supset \,.\, \text{D} \restriction F_\Delta\text{‘}C\text{‘}P \epsilon 1 \to 1 \,.\, F_\Delta\text{‘}C\text{‘}P \subset 1 \to 1$
[*81·21·1 . *163·12]

*163·21. $\vdash : P \epsilon \text{Rel}^2\,\text{excl} \,.\, \supset \,.\, \text{D}\text{“}F_\Delta\text{‘}C\text{‘}P = \text{Prod}\text{‘}C\text{“}C\text{‘}P$

Dem.

$\vdash \,.\, *85·1 \dfrac{F, C\text{‘}P}{Q, \lambda} \,.\, *163·12 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, \text{D}\text{“}F_\Delta\text{‘}C\text{‘}P = \text{D}\text{“}\epsilon_\Delta\text{‘}\overrightarrow{F}\text{“}C\text{‘}P$

[*115·1.*33·5] $= \text{Prod}\text{‘}C\text{“}C\text{‘}P : \supset \vdash \,.\, \text{Prop}$

This proposition is important in connection with the multiplication of relations, for we shall define as the product of a relation P (whose field consists of relations) a relation whose field is $\text{D}``F_\Delta`C`P$. Thus by the above proposition, whenever P is a $\text{Rel}^2\,\text{excl}$, the field of its product is the product (in the cardinal sense) of the fields of its field, just as the field of its sum is (by $*162\cdot22$) the sum of the fields of its field.

$*163\cdot22$. $\vdash : P \epsilon \text{Rel}^2\,\text{excl} . \Lambda \sim\epsilon C`P . \supset .$
$$\overrightarrow{B}`\Sigma`P = B``\overrightarrow{B}`P . \overrightarrow{B}`\text{Cnv}`\Sigma`P = B``\text{Cnv}``\overrightarrow{B}`\breve{P}$$

Dem.

$\vdash . *162\cdot23\cdot213 . *93\cdot103 . \supset \vdash : \text{Hp} . \supset . \overrightarrow{B}`\Sigma`P = F``C`P - s`C``\text{Ꮯ}`P - s`\text{Ꮯ}``\overrightarrow{B}`P$

$[*40\cdot56]$ $\qquad = F``C`P - F``\text{Ꮯ}`P - s`\text{Ꮯ}``\overrightarrow{B}`P$

$[*71\cdot381 . *37\cdot421 . *163\cdot12]$ $\qquad = F``(C`P - \text{Ꮯ}`P) - s`\text{Ꮯ}``\overrightarrow{B}`P$

$[*40\cdot56 . *93\cdot103]$ $\qquad = s`C``\overrightarrow{B}`P - s`\text{Ꮯ}``\overrightarrow{B}`P \qquad (1)$

$\vdash . *163\cdot11 . \supset \vdash :: \text{Hp} . \supset :. Q \epsilon \overrightarrow{B}`P . x \epsilon C`Q . \supset : R \epsilon \overrightarrow{B}`P . x \epsilon \text{Ꮯ}`R . \supset . R = Q .$

$[*13\cdot12]$ $\qquad \supset . x \epsilon \text{Ꮯ}`Q :$

$[*40\cdot4]$ $\qquad \supset : x \epsilon s`\text{Ꮯ}``\overrightarrow{B}`P . \supset . x \epsilon \text{Ꮯ}`Q :$

$[*40\cdot13]$ $\qquad \supset : x \epsilon s`\text{Ꮯ}``\overrightarrow{B}`P . \equiv . x \epsilon \text{Ꮯ}`Q \qquad (2)$

$\vdash . (2) . *5\cdot32 . \supset \vdash :. \text{Hp} . \supset : Q \epsilon \overrightarrow{B}`P . x \epsilon C`Q . x \sim\epsilon s`\text{Ꮯ}``\overrightarrow{B}`P . \equiv .$
$$Q \epsilon \overrightarrow{B}`P . x \epsilon C`Q . x \sim\epsilon \text{Ꮯ}`Q :$$

$[*10\cdot281 . *40\cdot4 . *93\cdot103] \supset : x \epsilon s`C``\overrightarrow{B}`P - s`\text{Ꮯ}``\overrightarrow{B}`P . \equiv . (\exists Q) . Q \epsilon \overrightarrow{B}`P . xBQ .$

$[*37\cdot1]$ $\qquad \equiv . x \epsilon B``\overrightarrow{B}`P \qquad (3)$

$\vdash . (1) . (3) . \supset \vdash : \text{Hp} . \supset . \overrightarrow{B}`\Sigma`P = B``\overrightarrow{B}`P \qquad (4)$

$\vdash . (4) . *162\cdot2 . *33\cdot22 . *163\cdot1 . \supset \vdash :\text{Hp} . \supset . \overrightarrow{B}`\text{Cnv}`\Sigma`P = B``\overrightarrow{B}`\text{Cnv};\breve{P}$

$[*151\cdot6\cdot5]$ $\qquad = B``\text{Cnv}``\overrightarrow{B}`\breve{P} \qquad (5)$

$\vdash . (4) . (5) . \supset \vdash . \text{Prop}$

$*163\cdot3$. $\vdash : Q \epsilon \text{Rel}^2\,\text{excl} . S \epsilon \text{Cls} \to 1 . \supset . S\dagger;Q \epsilon \text{Rel}^2\,\text{excl}$

Dem.

$\vdash . *72\cdot421 . \supset \vdash :. \text{Hp} . \supset : M, N \epsilon C`Q . \exists ! S``C`M \cap S``C`N . \supset . \exists ! C`M \cap C`N .$

$[*163\cdot11]$ $\qquad \supset . M = N .$

$[*30\cdot37]$ $\qquad \supset . S\dagger M = S\dagger N \qquad (1)$

$\vdash . (1) . *150\cdot202 . \supset \vdash :. \text{Hp} . \supset :$
$$M, N \epsilon C`Q . \exists ! C`(S\dagger M) \cap C`(S\dagger N) . \supset . S\dagger M = S\dagger N \qquad (2)$$

$\vdash . (2) . *163\cdot11 . \supset \vdash . \text{Prop}$

$*163\cdot31$. $\vdash :. C`P = C`Q . \supset : P \epsilon \text{Rel}^2\,\text{excl} . \equiv . Q \epsilon \text{Rel}^2\,\text{excl} \quad [*163\cdot1 . *13\cdot12]$

∗163·311. $\vdash :. C\text{‘}Q = \mathrm{Cnv}\text{‘‘}C\text{‘}P . \supset : P \in \mathrm{Rel}^2\,\mathrm{excl} . \equiv . Q \in \mathrm{Rel}^2\,\mathrm{excl}$

Dem.

$\vdash . {*}72{\cdot}513 . \supset \vdash :: \mathrm{Hp} . \supset :. M, N \in C\text{‘}P . \equiv . \breve{M}, \breve{N} \in C\text{‘}Q :.$

[∗31·32] $\supset :. M, N \in C\text{‘}P . M \neq N . \equiv . \breve{M}, \breve{N} \in C\text{‘}Q . \breve{M} \neq \breve{N} :.$

[∗33·22] $\supset :. M, N \in C\text{‘}P . M \neq N . \supset . C\text{‘}M \cap C\text{‘}N = \Lambda : \equiv :$
$\breve{M}, \breve{N} \in C\text{‘}Q . \breve{M} \neq \breve{N} . \supset . C\text{‘}\breve{M} \cap C\text{‘}\breve{N} = \Lambda :.$

[∗11·33.∗163·1] $\supset :. P \in \mathrm{Rel}^2\,\mathrm{excl} . \equiv :$
$\breve{M}, \breve{N} \in C\text{‘}Q . \breve{M} \neq \breve{N} . \supset_{M, N} . C\text{‘}\breve{M} \cap C\text{‘}\breve{N} = \Lambda :$

[∗31·51] $\equiv : M, N \in C\text{‘}Q . M \neq N . \supset_{M, N} . C\text{‘}M \cap C\text{‘}N = \Lambda :$

[∗163·1] $\equiv : Q \in \mathrm{Rel}^2\,\mathrm{excl} :: \supset \vdash . \mathrm{Prop}$

∗163·32. $\vdash : P \in \mathrm{Rel}^2\,\mathrm{excl} . \equiv . \breve{P} \in \mathrm{Rel}^2\,\mathrm{excl} . \equiv . \mathrm{Cnv};P \in \mathrm{Rel}^2\,\mathrm{excl} . \equiv .$
$\mathrm{Cnv};\breve{P} \in \mathrm{Rel}^2\,\mathrm{excl}$ [∗163·31·311 . ∗33·22 . ∗150·22·12]

∗163·33. $\vdash : P \Uparrow Q \in \mathrm{Rel}^2\,\mathrm{excl} . \equiv . Q \Uparrow P \in \mathrm{Rel}^2\,\mathrm{excl}$ [∗163·31 . ∗160·14]

∗163·331. $\vdash : P \nrightarrow R \in \mathrm{Rel}^2\,\mathrm{excl} . \equiv . R \nleftarrow P \in \mathrm{Rel}^2\,\mathrm{excl}$
[∗163·31 . ∗161·14·2·201]

∗163·4. $\vdash . \dot{\Lambda} \in \mathrm{Rel}^2\,\mathrm{excl}$

Dem.

$\vdash . {*}33{\cdot}241 . {*}24{\cdot}105 . \supset \vdash . (Q) . Q \sim\in C\text{‘}\dot{\Lambda} .$

[∗11·57] $\supset \vdash . (Q, R) . Q, R \sim\in C\text{‘}\dot{\Lambda} .$

[∗11·63] $\supset \vdash : Q, R \in C\text{‘}\dot{\Lambda} . Q \neq R . \supset_{Q, R} . C\text{‘}Q \cap C\text{‘}R = \Lambda$ (1)

$\vdash . (1) . {*}163{\cdot}1 . \supset \vdash . \mathrm{Prop}$

∗163·41. $\vdash . P \downarrow P \in \mathrm{Rel}^2\,\mathrm{excl}$

Dem.

$\vdash . {*}54{\cdot}25 . {*}55{\cdot}15 . \supset \vdash . C\text{‘}(P \downarrow P) \in 1 .$

[∗52·41.Transp] $\supset \vdash . \sim(\exists Q, R) . Q, R \in C\text{‘}(P \downarrow P) . Q \neq R .$

[∗11·63] $\supset \vdash : Q, R \in C\text{‘}(P \downarrow P) . Q \neq R . \supset_{Q, R} . C\text{‘}Q \cap C\text{‘}R = \Lambda$ (1)

$\vdash . (1) . {*}163{\cdot}1 . \supset \vdash . \mathrm{Prop}$

∗163·42. $\vdash :. P \downarrow Q \in \mathrm{Rel}^2\,\mathrm{excl} . \equiv : P = Q . \vee . C\text{‘}P \cap C\text{‘}Q = \Lambda$

Dem.

$\vdash . {*}163{\cdot}1 . {*}55{\cdot}15 . \supset$

$\vdash :. P \downarrow Q \in \mathrm{Rel}^2\,\mathrm{excl} . \equiv : M, N \in \iota\text{‘}P \cup \iota\text{‘}Q . M \neq N . \supset_{M, N} . C\text{‘}M \cap C\text{‘}N = \Lambda :$

[∗54·441] $\equiv : P = Q . \vee . C\text{‘}P \cap C\text{‘}Q = \Lambda . C\text{‘}Q \cap C\text{‘}P = \Lambda :$

[∗22·51] $\equiv : P = Q . \vee . C\text{‘}P \cap C\text{‘}Q = \Lambda :. \supset \vdash . \mathrm{Prop}$

The above proposition is used in ∗251·22.

***163·43.** $\vdash : P \in \text{Rel}^2\,\text{excl} \,.\, Q \Subset P \,.\, \supset .\, Q \in \text{Rel}^2\,\text{excl}$

Dem.

$\vdash . *33{\cdot}265 . \supset \vdash :. \text{Hp} . \supset : M, N \in C`Q . \supset . M, N \in C`P :$

[Fact] $\supset : M, N \in C`Q \,.\, M \neq N . \supset . M, N \in C`P \,.\, M \neq N :$

[*163·1.Hp] $\supset . C`M \cap C`N = \Lambda$ (1)

$\vdash . (1) . *163{\cdot}1 . \supset \vdash . \text{Prop}$

***163·431.** $\vdash : P \in \text{Rel}^2\,\text{excl} . \supset . \text{Rl}`P \subset \text{Rel}^2\,\text{excl}$ [*163·43]

***163·44.** $\vdash : P \ddagger Q \in \text{Rel}^2\,\text{excl} . \equiv .$

$P, Q \in \text{Rel}^2\,\text{excl} \,.\, s`C``C`P \cap s`C``(C`Q - C`P) = \Lambda$

Dem.

$\vdash . *163{\cdot}12 . *160{\cdot}14 . \supset \vdash : P \ddagger Q \in \text{Rel}^2\,\text{excl} . \equiv . F \upharpoonright (C`P \cup C`Q) \in \text{Cls} \to 1 .$

[*74·821] $\equiv . F \upharpoonright C`P, F \upharpoonright C`Q \in \text{Cls} \to 1 \,.\, F``C`P \cap F``(C`Q - C`P) = \Lambda .$

[*163·12.*40·56] $\equiv . P, Q \in \text{Rel}^2\text{excl} \,.\, s`C``C`P \cap s`C``(C`Q - C`P) = \Lambda : \supset \vdash . \text{Prop}$

***163·441.** $\vdash : P, Q \in \text{Rel}^2\,\text{excl} \,.\, C`\Sigma`P \cap C`\Sigma`Q = \Lambda . \supset . P \ddagger Q \in \text{Rel}^2\,\text{excl}$
[*163·44 . *162·22]

The above proposition is used in *173·26.

***163·442.** $\vdash :. C`P \cap C`Q = \Lambda . \supset :$

$P \ddagger Q \in \text{Rel}^2\,\text{excl} . \equiv . P, Q \in \text{Rel}^2\,\text{excl} \,.\, C`\Sigma`P \cap C`\Sigma`Q = \Lambda$

Dem.

$\vdash . *24{\cdot}313 . \supset \vdash : \text{Hp} . \supset . C`Q - C`P = C`Q$ (1)

$\vdash . (1) . *163{\cdot}44 . *162{\cdot}22 . \supset \vdash . \text{Prop}$

***163·45.** $\vdash : P \dashv\!\!\!\rightarrow R \in \text{Rel}^2\,\text{excl} . \equiv . P \in \text{Rel}^2\,\text{excl} \,.\, s`C``(C`P - \iota`R) \cap C`R = \Lambda$

Dem.

$\vdash . *161{\cdot}14 . *163{\cdot}12 . \supset$

$\vdash :. \exists ! P . \supset : P \dashv\!\!\!\rightarrow R \in \text{Rel}^2\,\text{excl} . \equiv . F \upharpoonright (C`P \cup \iota`R) \in \text{Cls} \to 1 .$

[*74·821.*53·301.*33·5]

$\equiv . F \upharpoonright C`P, F \upharpoonright \iota`R \in \text{Cls} \to 1 \,.\, F``(C`P - \iota`R) \cap C`R = \Lambda .$

[*35·101.*71·171] $\equiv . F \upharpoonright C`P \in \text{Cls} \to 1 \,.\, F``(C`P - \iota`R) \cap C`R = \Lambda .$

[*163·12.*40·56] $\equiv : P \in \text{Rel}^2\,\text{excl} \,.\, s`C``(C`P - \iota`R) \cap C`R = \Lambda$ (1)

$\vdash . *161{\cdot}2 . *163{\cdot}4 . \supset \vdash : P = \dot{\Lambda} . \supset . P \dashv\!\!\!\rightarrow R \in \text{Rel}^2\,\text{excl} \,.\, P \in \text{Rel}^2\,\text{excl}$ (2)

$\vdash . *33{\cdot}241 . *37{\cdot}29 . *40{\cdot}21 . \supset \vdash : P = \dot{\Lambda} . \supset . s`C``(C`P - \iota`R) \cap C`R = \Lambda$ (3)

$\vdash . (2) . (3) . \text{Comp} . *5{\cdot}1 . \supset \vdash :. P = \dot{\Lambda} . \supset :$

$P \dashv\!\!\!\rightarrow R \in \text{Rel}^2\,\text{excl} . \equiv . P \in \text{Rel}^2\,\text{excl} \,.\, s`C``(C`P - \iota`R) \cap C`R = \Lambda$ (4)

$\vdash . (1) . (4) . \supset \vdash . \text{Prop}$

***163·451.** $\vdash : P \in \text{Rel}^2\,\text{excl} \,.\, C`\Sigma`P \cap C`R = \Lambda . \supset . P \dashv\!\!\!\rightarrow R \in \text{Rel}^2\,\text{excl}$
[*163·45 . *162·22]

The above proposition is used in *173·25.

***163·452.** $\vdash :. R \sim \epsilon C\text{‘}P . \supset : P \dotplus\!\!\rightarrow R \,\epsilon\, \mathrm{Rel}^2 \mathrm{excl} . \equiv . P \,\epsilon\, \mathrm{Rel}^2 \mathrm{excl} . C\text{‘}\Sigma\text{‘}P \cap C\text{‘}R = \Lambda$
[*51·222 . *163·45 . *162·22]

***163·46.** $\vdash : R \leftarrow\!\!\dotplus P \,\epsilon\, \mathrm{Rel}^2 \mathrm{excl} . \equiv . P \,\epsilon\, \mathrm{Rel}^2 \mathrm{excl} . s\text{‘}C\text{“}(C\text{‘}P - \iota\text{‘}R) \cap C\text{‘}R = \Lambda$
[*163·45·331]

***163·461.** $\vdash : P \,\epsilon\, \mathrm{Rel}^2 \mathrm{excl} . C\text{‘}\Sigma\text{‘}P \cap C\text{‘}R = \Lambda . \supset . R \leftarrow\!\!\dotplus P \,\epsilon\, \mathrm{Rel}^2 \mathrm{excl}$
[*163·451·331]

***163·462.** $\vdash :. R \sim \epsilon C\text{‘}P . \supset : R \leftarrow\!\!\dotplus P \,\epsilon\, \mathrm{Rel}^2 \mathrm{excl} . \equiv . P \,\epsilon\, \mathrm{Rel}^2 \mathrm{excl} . C\text{‘}\Sigma\text{‘}P \cap C\text{‘}R = \Lambda$
[*163·452·331]

$*164$. DOUBLE LIKENESS

Summary of $*164$.

The subject of this number is of great importance throughout relation-arithmetic and its applications. Double likeness, or double ordinal similarity, is a relation which is to hold between P and Q when (1) P and Q are like, (2) correlated members of the fields of P and Q are like, with a specific given correlator in each case. (It is necessary, in general, to have a given correlator in each case, to avoid the necessity of the multiplicative axiom for selecting among correlators.) This definition can be somewhat simplified by starting from a relation correlating $\Sigma\text{‘}P$ and $\Sigma\text{‘}Q$. If S is such a correlator, so that

$$S \epsilon 1 \to 1 \,.\, \text{Ɑ‘}S = C\text{‘}\Sigma\text{‘}Q \,.\, \Sigma\text{‘}P = S;\Sigma\text{‘}Q,$$

we want S to be such that it not only correlates the whole of $\Sigma\text{‘}P$ with the whole of $\Sigma\text{‘}Q$, but also correlates each member of $C\text{‘}P$ with the corresponding member of $C\text{‘}Q$, *i.e.* such that, if N is any member of $C\text{‘}Q$, $S;N$ is the corresponding member of $C\text{‘}P$. This requires

$$NQN' \,.\equiv .\, (S;N)\,P\,(S;N'),$$

i.e. writing $S\dagger\text{‘}N$, $S\dagger\text{‘}N'$ in place of $S;N$, $S;N'$, it requires

$$P = S\dagger;Q.$$

When $P = S\dagger;Q$ and $\text{Ɑ‘}S = C\text{‘}\Sigma\text{‘}Q$, we have $\Sigma\text{‘}P = S;\Sigma\text{‘}Q$ by $*162\cdot35$. Hence double likeness will subsist if there is a relation S such that

$$S \epsilon 1 \to 1 \,.\, \text{Ɑ‘}S = C\text{‘}\Sigma\text{‘}Q \,.\, P = S\dagger;Q.$$

A relation S fulfilling this condition will be called a *double correlator* of P and Q. Thus two relations P and Q have double likeness when there exists a double correlator of P and Q, *i.e.* when

$$(\exists S) \,.\, S \epsilon 1 \to 1 \,.\, \text{Ɑ‘}S = C\text{‘}\Sigma\text{‘}Q \,.\, P = S\dagger;Q.$$

A double correlator of P and Q is a relation S which is a correlator of $\Sigma\text{‘}P$ and $\Sigma\text{‘}Q$ and is such that $S\dagger \upharpoonright C\text{‘}Q$ is a correlator of P and Q.

It will be seen that this definition has the usual analogy to the corresponding definition in cardinals ($*111\cdot01$). The two inverted commas of the cardinal definition are replaced by the semi-colon, and S_ϵ is replaced by $S\dagger$, and $s\text{‘}\lambda$ is replaced by $\Sigma\text{‘}Q$ or $C\text{‘}\Sigma\text{‘}Q$. The propositions of the present number consist largely of analogues of the propositions of $*111$, in accordance with the above substitutions.

If it were not for the difficulty of choice among correlators, we could define two relations as having double likeness when they are like relations of like relations, *i.e.* when, if P and Q are the two relations, they have a correlator S such that, if MSN, then M smor N. In this case, $S \epsilon P \,\overline{\text{smor}}\, Q \cap \text{Rl‘smor}$.

Thus we have to consider the relations of the class $P \,\overline{\mathrm{smor}}\, Q \cap \mathrm{Rl}'\mathrm{smor}$ to the class of double correlators, and we have to consider the relation of the relation "$\exists ! P \,\overline{\mathrm{smor}}\, Q \cap \mathrm{Rl}'\mathrm{smor}$" to the relation of double likeness. The propositions to be proved on this subject in the present number are analogous to the propositions of $*111$. But at a later stage ($*251{\cdot}61$) we shall show that if the field of P consists entirely of relations which generate *well-ordered* series, then the use of the multiplicative axiom ceases to be necessary in identifying double likeness with the relation $\exists ! P \,\overline{\mathrm{smor}}\, Q \cap \mathrm{Rl}'\mathrm{smor}$, the reason being that two well-ordered series can never be correlated in more than one way.

Our definitions are

$*164{\cdot}01$. $P \,\overline{\mathrm{smor}}\,\overline{\mathrm{smor}}\, Q = (1 \rightarrow 1) \cap \overleftarrow{\mathrm{G}}'C'\Sigma'Q \cap \hat{S}(P = S\dagger{};Q)$ Df

$*164{\cdot}02$. $\mathrm{smor\,smor} = \hat{P}\hat{Q}(\exists ! P \,\overline{\mathrm{smor}}\,\overline{\mathrm{smor}}\, Q)$ Df

The principal propositions of this number are

$*164{\cdot}15$. $\vdash : S \in P \,\overline{\mathrm{smor}}\,\overline{\mathrm{smor}}\, Q \, . \equiv . \, S \in \Sigma'P \,\overline{\mathrm{smor}}\, \Sigma'Q \, . \, (S\dagger) \upharpoonright C'Q \in P \,\overline{\mathrm{smor}}\, Q$

whence

$*164{\cdot}151$. $\vdash : P \,\mathrm{smor\,smor}\, Q \, . \supset . \, \Sigma'P \,\mathrm{smor}\, \Sigma'Q \, . \, P \,\mathrm{smor}\, Q$

$*164{\cdot}18$. $\vdash : S \upharpoonright C'\Sigma'Q \in P \,\overline{\mathrm{smor}}\,\overline{\mathrm{smor}}\, Q \, . \equiv .$

$$S \upharpoonright C'\Sigma'Q \in 1 \rightarrow 1 \, . \, C'\Sigma'Q \subset \mathrm{G}'S \, . \, P = S\dagger{};Q$$

This is usually the most convenient proposition when a double correlation has to be proved.

$*164{\cdot}201{\cdot}211{\cdot}221$. Double likeness is reflexive, symmetrical and transitive.

$*164{\cdot}31$. $\vdash : S \in P \,\overline{\mathrm{smor}}\,\overline{\mathrm{smor}}\, Q \, . \equiv . \, S \in (C''C'P) \,\overline{\mathrm{sm}}\,\overline{\mathrm{sm}}\, (C''C'Q) \, . \, P = S\dagger{};Q$

(Cf. note to $*164{\cdot}31$, below.)

We then have a set of propositions ($*164{\cdot}4$ to the end) on the identification of $\exists ! P \,\overline{\mathrm{smor}}\, Q \cap \mathrm{Rl}'\mathrm{smor}$ with double likeness by means of the multiplicative axiom. We have

$*164{\cdot}43$. $\vdash :. \, P, Q \in \mathrm{Rel}^2\,\mathrm{excl} \, . \, S \in P \,\overline{\mathrm{smor}}\, Q \, .$

$$\mu = \hat{\lambda}\{(\exists N) \, . \, N \in C'Q \, . \, \lambda = (S'N) \,\overline{\mathrm{smor}}\, N\} \, . \supset :$$
$$R \in \epsilon_{\Delta}'\mu \, . \supset . \, \dot{s}'\mathrm{D}'R \in P \,\overline{\mathrm{smor}}\,\overline{\mathrm{smor}}\, Q \, . \, S = (\dot{s}'\mathrm{D}'R)\dagger \upharpoonright C'Q$$

That is to say, given that P and Q are like relations of like mutually exclusive relations, if we can pick out one correlator for each pair of correlated members of $C'P$ and $C'Q$, then the sum ($\dot{s}$) of such selected correlators is a double correlator of P and Q. Hence, observing that if S is a double correlator of P and Q, $(S\dagger) \upharpoonright C'Q \in P \,\overline{\mathrm{smor}}\, Q \cap \mathrm{Rl}'\mathrm{smor}$ ($*164{\cdot}15{\cdot}16$), we arrive at

$*164{\cdot}45$. $\vdash :: \mathrm{Mult\,ax} \, . \supset :.$

$$P, Q \in \mathrm{Rel}^2\,\mathrm{excl} \, . \supset : \exists ! P \,\overline{\mathrm{smor}}\, Q \cap \mathrm{Rl}'\mathrm{smor} \, . \equiv . \, P \,\mathrm{smor\,smor}\, Q$$

From $*164{\cdot}43$ we deduce also

***164·46.** $\vdash :.\ \text{Mult ax} \,.\supset:$

$$P, Q \,\epsilon\, \text{Rel}^2\,\text{excl} \,.\, \exists !\, P \,\overline{\text{smor}}\, Q \cap \text{Rl‘smor} \,.\supset.\, \Sigma\text{‘}P \,\text{smor}\, \Sigma\text{‘}Q$$

***164·48.** $\vdash :.\ \text{Mult ax} \,.\supset:\, R, S \,\epsilon\, \text{Rel}^2\,\text{excl} \cap \text{Nr‘}Q \,.\, C\text{‘}R, C\text{‘}S \,\epsilon\, \text{Cl‘Nr‘}P \,.\supset.$

$$R \,\text{smor smor}\, S \,.\, \Sigma\text{‘}R \,\text{smor}\, \Sigma\text{‘}S$$

I.e. in effect, assuming the multiplicative axiom, if two series (Σ‘R and Σ‘S) can each be divided into β sets of α terms (α, β being relation-numbers), then the two series are ordinally similar, and the β sets in the one case have double similarity with the β sets in the other. (Here we have written α, β in place of the Nr‘P and Nr‘Q of the enunciation.)

It is by means of the above propositions that ordinal addition and multiplication are connected, as will appear in *166.

***164·01.** $P \,\overline{\text{smor}}\,\overline{\text{smor}}\, Q = (1 \rightarrow 1) \cap \overleftarrow{\text{Ɑ}}\text{‘}C\text{‘}\Sigma\text{‘}Q \cap \hat{S}(P = S\dagger ; Q)$ Df

***164·02.** $\text{smor smor} = \hat{P}\hat{Q}\,(\exists !\, P \,\overline{\text{smor}}\,\overline{\text{smor}}\, Q)$ Df

***164·1.** $\vdash : S \,\epsilon\, P \,\overline{\text{smor}}\,\overline{\text{smor}}\, Q \,.\equiv.\, S \,\epsilon\, 1 \rightarrow 1 \,.\, \text{Ɑ‘}S = C\text{‘}\Sigma\text{‘}Q \,.\, P = S\dagger ; Q$
[(*164·01)]

***164·11.** $\vdash : P \,\text{smor smor}\, Q \,.\equiv.\, \exists !\, P \,\overline{\text{smor}}\,\overline{\text{smor}}\, Q$ [(*164·02)]

***164·12.** $\vdash : P \,\text{smor smor}\, Q \,.\equiv.\, (\exists S) \,.\, S \,\epsilon\, 1 \rightarrow 1 \,.\, \text{Ɑ‘}S = C\text{‘}\Sigma\text{‘}Q \,.\, P = S\dagger ; Q$
[*164·1·11]

***164·13.** $\vdash : S \restriction C\text{‘}\Sigma\text{‘}Q \,\epsilon\, 1 \rightarrow 1 \,.\, C\text{‘}\Sigma\text{‘}Q \subset \text{Ɑ‘}S \,.\supset.\, (S\dagger) \restriction C\text{‘}Q \,\epsilon\, 1 \rightarrow 1$
[*150·152 . *162·22]

***164·131.** $\vdash : \text{Ɑ‘}S = C\text{‘}\Sigma\text{‘}Q \,.\, P = S\dagger ; Q \,.\supset.\, \text{D‘}S = C\text{‘}\Sigma\text{‘}P \,.\, \Sigma\text{‘}P = S ; \Sigma\text{‘}Q$

Dem.

$\vdash .\, *162·35 \,.\supset \vdash : \text{Hp} \,.\supset.\, \Sigma\text{‘}P = S ; \Sigma\text{‘}Q$ (1)

[*150·23.Hp] $\supset.\, C\text{‘}\Sigma\text{‘}P = \text{D‘}S$ (2)

$\vdash .\, (1) \,.\, (2) \,.\supset \vdash .\, \text{Prop}$

***164·14.** $\vdash : S \,\epsilon\, P \,\overline{\text{smor}}\,\overline{\text{smor}}\, Q \,.\supset.\, S \,\epsilon\, \Sigma\text{‘}P \,\overline{\text{smor}}\, \Sigma\text{‘}Q$ [*164·1·131 . *151·11]

The two following propositions are required for proving *164·18.

***164·141.** $\vdash : C\text{‘}\Sigma\text{‘}Q \subset \alpha \,.\supset.\, (T \restriction \alpha)\dagger ; Q = T\dagger ; Q$ [*150·171 . *162·22]

***164·142.** $\vdash .\, (T \restriction C\text{‘}\Sigma\text{‘}Q)\dagger ; Q = T\dagger ; Q = \{(T\dagger) \restriction C\text{‘}Q\} ; Q$ [*164·141 . *150·32]

***164·143.** $\vdash : S \,\epsilon\, P \,\overline{\text{smor}}\,\overline{\text{smor}}\, Q \,.\supset.\, (S\dagger) \restriction C\text{‘}Q \,\epsilon\, P \,\overline{\text{smor}}\, Q$

Dem.

$\vdash .\, *164·1·13 \,.\quad \supset \vdash : \text{Hp} \,.\supset.\, (S\dagger) \restriction C\text{‘}Q \,\epsilon\, 1 \rightarrow 1$ (1)

$\vdash .\, *35·65 \,.\quad \supset \vdash .\, \text{Ɑ‘}(S\dagger) \restriction C\text{‘}Q = C\text{‘}Q$ (2)

$\vdash .\, *164·1 \,.\, *150·32 \,.\supset \vdash : \text{Hp} \,.\supset.\, P = \{(S\dagger) \restriction C\text{‘}Q\} ; Q$ (3)

$\vdash .\, (1) \,.\, (2) \,.\, (3) \,.\, *151·11 \,.\supset \vdash .\, \text{Prop}$

***164·15.** $\vdash : S \in P \,\overline{\text{smor}}\, \overline{\text{smor}}\, Q . \equiv . S \in \Sigma\text{‘}P \,\overline{\text{smor}}\, \Sigma\text{‘}Q . (S\dagger) \upharpoonright C\text{‘}Q \in P \,\overline{\text{smor}}\, Q$

Dem.

$\vdash . *164{\cdot}14{\cdot}143 . \supset$

$\vdash : S \in P \,\overline{\text{smor}}\, \overline{\text{smor}}\, Q . \supset . S \in \Sigma\text{‘}P \,\overline{\text{smor}}\, \Sigma\text{‘}Q . (S\dagger) \upharpoonright C\text{‘}Q \in P \,\overline{\text{smor}}\, Q \quad (1)$

$\vdash . *151{\cdot}11 . \supset$

$\vdash : S \in \Sigma\text{‘}P \,\overline{\text{smor}}\, \Sigma\text{‘}Q . (S\dagger) \upharpoonright C\text{‘}Q \in P \,\overline{\text{smor}}\, Q . \supset .$

$S \in 1 \to 1 . \text{Ɑ‘}S = C\text{‘}\Sigma\text{‘}Q . P = \{(S\dagger) \upharpoonright C\text{‘}Q\} ; Q \quad (2)$

$\vdash . (2) . *150{\cdot}32 . *164{\cdot}1 . \supset$

$\vdash : S \in \Sigma\text{‘}P \,\overline{\text{smor}}\, \Sigma\text{‘}Q . (S\dagger) \upharpoonright C\text{‘}Q \in P \,\overline{\text{smor}}\, Q . \supset . S \in P \,\overline{\text{smor}}\, \overline{\text{smor}}\, Q \quad (3)$

$\vdash . (1) . (3) . \supset \vdash .$ Prop

***164·151.** $\vdash : P \,\text{smor}\, \text{smor}\, Q . \supset . \Sigma\text{‘}P \,\text{smor}\, \Sigma\text{‘}Q . P \,\text{smor}\, Q$ [*164·15·11]

***164·16.** $\vdash : S \in P \,\overline{\text{smor}}\, \overline{\text{smor}}\, Q . \supset . (S\dagger) \upharpoonright C\text{‘}Q \subset \text{smor}$

Dem.

$\vdash . *35{\cdot}101 . *150{\cdot}1 . \supset \vdash : M \{(S\dagger) \upharpoonright C\text{‘}Q\} N . \equiv . N \in C\text{‘}Q . M = S ; N \quad (1)$

$\vdash . *164{\cdot}1 . *162{\cdot}22 . \supset \vdash :. \text{Hp} . \supset : S \in 1 \to 1 : N \in C\text{‘}Q . \supset_N . C\text{‘}N \subset \text{Ɑ‘}S :$

[*151·23] $\supset : N \in C\text{‘}Q . M = S ; N . \supset_{M,N} . M \,\text{smor}\, N$

[(1)] $\supset : M \{(S\dagger) \upharpoonright C\text{‘}Q\} N . \supset_{M,N} . M \,\text{smor}\, N :. \supset \vdash .$ Prop

***164·17.** $\vdash : P \,\text{smor}\, \text{smor}\, Q . \supset . \exists ! \, P \,\overline{\text{smor}}\, Q \cap \text{Rl‘smor}$ [*164·143·16]

This proposition states that when P and Q have double likeness, there is a correlator of P and Q which couples like with like relations; *i.e.* if S is the correlator, then, if MSN, M and N are ordinally similar. The converse of this proposition, namely, that if P and Q have a correlator which couples ordinally similar relations, then P and Q have double likeness, can be proved if the multiplicative axiom is assumed, but not otherwise, except in special cases, such as that of well-ordered series.

The following proposition is used frequently, owing to the fact that, in the cases we are concerned with, double correlators generally have the form $S \upharpoonright C\text{‘}\Sigma\text{‘}Q$, where S is some relation for which we have $(y) . E ! S\text{‘}y$.

***164·18.** $\vdash : S \upharpoonright C\text{‘}\Sigma\text{‘}Q \in P \,\overline{\text{smor}}\, \overline{\text{smor}}\, Q . \equiv .$

$S \upharpoonright C\text{‘}\Sigma\text{‘}Q \in 1 \to 1 . C\text{‘}\Sigma\text{‘}Q \subset \text{Ɑ‘}S . P = S\dagger ; Q$

Dem.

$\vdash . *35{\cdot}64 . *22{\cdot}621 . \supset \vdash : \text{Ɑ‘}(S \upharpoonright C\text{‘}\Sigma\text{‘}Q) = C\text{‘}\Sigma\text{‘}Q . \equiv . C\text{‘}\Sigma\text{‘}Q \subset \text{Ɑ‘}S \quad (1)$

$\vdash . *164{\cdot}142 . \quad \supset \vdash : P = (S \upharpoonright C\text{‘}\Sigma\text{‘}Q)\dagger ; Q . \equiv . P = S\dagger ; Q \quad (2)$

$\vdash . *164{\cdot}1 . \quad \supset \vdash : S \upharpoonright C\text{‘}\Sigma\text{‘}Q \in P \,\overline{\text{smor}}\, \overline{\text{smor}}\, Q . \equiv .$

$S \upharpoonright C\text{‘}\Sigma\text{‘}Q \in 1 \to 1 . \text{Ɑ‘}(S \upharpoonright C\text{‘}\Sigma\text{‘}Q) = C\text{‘}\Sigma\text{‘}Q . P = (S \upharpoonright C\text{‘}\Sigma\text{‘}Q)\dagger ; Q \quad (3)$

$\vdash . (1) . (2) . (3) . \supset \vdash .$ Prop

***164·181.** $\vdash : P \,\mathrm{smor}\,\mathrm{smor}\, Q \,.\equiv .\, (\exists S) \,.\, S \upharpoonright C`\Sigma`Q \,\epsilon\, 1 \to 1 \,.\, C`\Sigma`Q \subset \text{Ɑ}`S \,.\, P = S\dagger ; Q$

Dem.

$\vdash . *35{\cdot}66 . *164{\cdot}1 . \supset$

$\vdash : S \,\epsilon\, P \,\overline{\mathrm{smor}}\,\overline{\mathrm{smor}}\, Q \,.\supset .\, S \upharpoonright C`\Sigma`Q \,\epsilon\, 1 \to 1 \,.\, C`\Sigma`Q \subset \text{Ɑ}`S \,.\, P = S\dagger ; Q \qquad (1)$

$\vdash . (1) . *164{\cdot}11 . \supset$

$\vdash : P \,\mathrm{smor}\,\mathrm{smor}\, Q \,.\supset .\, (\exists S) \,.\, S \upharpoonright C`\Sigma`Q \,\epsilon\, 1 \to 1 \,.\, C`\Sigma`Q \subset \text{Ɑ}`S \,.\, P = S\dagger ; Q \qquad (2)$

$\vdash . *164{\cdot}18{\cdot}11 . \supset$

$\vdash : (\exists S) \,.\, S \upharpoonright C`\Sigma`Q \,\epsilon\, 1 \to 1 \,.\, C`\Sigma`Q \subset \text{Ɑ}`S \,.\, P = S\dagger ; Q : \supset . P \,\mathrm{smor}\,\mathrm{smor}\, Q \qquad (3)$

$\vdash . (2) . (3) . \supset \vdash . \mathrm{Prop}$

The following propositions are concerned in proving that double likeness is reflexive, symmetrical, and transitive.

***164·2.** $\vdash . I \upharpoonright C`\Sigma`P \,\epsilon\, P \,\overline{\mathrm{smor}}\,\overline{\mathrm{smor}}\, P$

Dem.

$\vdash . *151{\cdot}121 . \supset \vdash . I \upharpoonright C`\Sigma`P \,\epsilon\, \Sigma`P \,\overline{\mathrm{smor}}\, \Sigma`P \,.\, I \upharpoonright C`P \,\epsilon\, P \,\overline{\mathrm{smor}}\, P \qquad (1)$

$\vdash . *35{\cdot}101 . *150{\cdot}1 . \supset$

$\vdash : M \{(I \upharpoonright C`\Sigma`P)\dagger \upharpoonright C`P\} N \,.\equiv .\, N \,\epsilon\, C`P \,.\, M = (I \upharpoonright C`\Sigma`P) ; N \,.$

$[*150{\cdot}33 . *162{\cdot}22] \qquad \equiv . N \,\epsilon\, C`P \,.\, M = I ; N \,.$

$[*150{\cdot}53] \qquad \equiv . M (I \upharpoonright C`P) N \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . I \upharpoonright C`\Sigma`P \,\epsilon\, \Sigma`P \,\overline{\mathrm{smor}}\, \Sigma`P \,.\, (I \upharpoonright C`\Sigma`P)\dagger \upharpoonright C`P \,\epsilon\, P \,\overline{\mathrm{smor}}\, P \,.$

$[*164{\cdot}15] \quad \supset \vdash . \mathrm{Prop}$

***164·201.** $\vdash . P \,\mathrm{smor}\,\mathrm{smor}\, P \quad [*164{\cdot}2{\cdot}11]$

***164·21.** $\vdash : S \,\epsilon\, P \,\overline{\mathrm{smor}}\,\overline{\mathrm{smor}}\, Q \,.\equiv .\, \breve{S} \,\epsilon\, Q \,\overline{\mathrm{smor}}\,\overline{\mathrm{smor}}\, P$

Dem.

$\vdash . *164{\cdot}1 . *71{\cdot}212 . \supset \vdash : S \,\epsilon\, P \,\overline{\mathrm{smor}}\,\overline{\mathrm{smor}}\, Q \,.\supset .\, \breve{S} \,\epsilon\, 1 \to 1 \qquad (1)$

$\vdash . *164{\cdot}131{\cdot}1 . \qquad \supset \vdash : S \,\epsilon\, P \,\overline{\mathrm{smor}}\,\overline{\mathrm{smor}}\, Q \,.\supset .\, \text{Ɑ}`\breve{S} = C`\Sigma`P \qquad (2)$

$\vdash . *150{\cdot}94 . *164{\cdot}1 . *162{\cdot}22 . \supset \vdash : S \,\epsilon\, P \,\overline{\mathrm{smor}}\,\overline{\mathrm{smor}}\, Q \,.\supset .\, Q = \breve{S}\dagger ; P \qquad (3)$

$\vdash . (1) . (2) . (3) . *164{\cdot}1 . \supset \vdash : S \,\epsilon\, P \,\overline{\mathrm{smor}}\,\overline{\mathrm{smor}}\, Q \,.\supset .\, \breve{S} \,\epsilon\, Q \,\overline{\mathrm{smor}}\,\overline{\mathrm{smor}}\, P \qquad (4)$

$\vdash . (4) \dfrac{\breve{S}, Q, P}{S, P, Q} . \qquad \supset \vdash : \breve{S} \,\epsilon\, Q \,\overline{\mathrm{smor}}\,\overline{\mathrm{smor}}\, P \,.\supset .\, S \,\epsilon\, P \,\overline{\mathrm{smor}}\,\overline{\mathrm{smor}}\, Q \qquad (5)$

$\vdash . (4) . (5) . \supset \vdash . \mathrm{Prop}$

***164·211.** $\vdash : P \,\mathrm{smor}\,\mathrm{smor}\, Q \,.\equiv .\, Q \,\mathrm{smor}\,\mathrm{smor}\, P \quad [*164{\cdot}21{\cdot}11]$

***164·22.** $\vdash : S \,\epsilon\, P \,\overline{\mathrm{smor}}\,\overline{\mathrm{smor}}\, Q \,.\, T \,\epsilon\, Q \,\overline{\mathrm{smor}}\,\overline{\mathrm{smor}}\, R \,.\supset .\, S \,|\, T \,\epsilon\, P \,\overline{\mathrm{smor}}\,\overline{\mathrm{smor}}\, R$

Dem.

$\vdash . *164{\cdot}1 . \qquad \supset \vdash : \mathrm{Hp} . \supset . S, T \,\epsilon\, 1 \to 1 \,.$

$[*71{\cdot}252] \qquad \supset . S \,|\, T \,\epsilon\, 1 \to 1 \qquad (1)$

$\vdash . *164{\cdot}1{\cdot}131 . \supset \vdash : \mathrm{Hp} . \supset . \text{Ɑ}`S = C`\Sigma`Q \,.\, \mathrm{D}`T = C`\Sigma`Q \,.$

$[*37{\cdot}323] \qquad \supset . \text{Ɑ}`(S \,|\, T) = \text{Ɑ}`T \,.$

$[*164{\cdot}1] \qquad \supset \therefore \text{Ɑ}`(S \,|\, T) = C`\Sigma`R \qquad (2)$

$\vdash . *150\cdot13\cdot14 . \supset \vdash . (S \mid T)\dagger \,;\, R = S\dagger \,;\, T\dagger \,;\, R$ (3)

$\vdash . *164\cdot1 . \quad \supset \vdash : \text{Hp} . \supset . T\dagger \,;\, R = Q . S\dagger \,;\, Q = P$ (4)

$\vdash . (3) . (4) . \quad \supset \vdash : \text{Hp} . \supset . (S \mid T)\dagger \,;\, R = P$ (5)

$\vdash . (1) . (2) . (5) . *164\cdot1 . \supset \vdash . \text{Prop}$

$*164\cdot221$. $\vdash : P \text{ smor smor } Q . Q \text{ smor smor } R . \supset . P \text{ smor smor } R$ [$*164\cdot22\cdot11$]

$*164\cdot23$. $\vdash :. P \text{ smor smor } Q . \supset : P \in \text{Rel}^2 \text{ excl} . \equiv . Q \in \text{Rel}^2 \text{ excl}$

Dem.

$\vdash . *164\cdot12 . \quad \supset \vdash :. \text{Hp} . \supset : (\exists T) . T \in 1 \to 1 . \text{ᗡ}`T = C`\Sigma`Q . P = T\dagger \,;\, Q :$

[$*163\cdot3$] $\quad \supset : Q \in \text{Rel}^2 \text{ excl} . \supset . P \in \text{Rel}^2 \text{ excl}$ (1)

$\vdash . (1) . *164\cdot211 . \supset \vdash :. \text{Hp} . \supset : P \in \text{Rel}^2 \text{ excl} . \supset . Q \in \text{Rel}^2 \text{ excl}$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

$*164\cdot3$. $\vdash : S \in P \,\overline{\text{smor}}\,\overline{\text{smor}}\, Q . \supset . S \in (C``C`P) \,\overline{\text{sm}}\,\overline{\text{sm}}\, (C``C`Q)$

Dem.

$\vdash . *164\cdot1 . *162\cdot22 . \supset \vdash : \text{Hp} . \supset . S \in 1 \to 1 . \text{ᗡ}`S = s`C``C`Q . P = S\dagger \,;\, Q .$ (1)

[$*150\cdot931$] $\quad \supset . C``C`P = S_\epsilon``C``C`Q$ (2)

$\vdash . (1) . (2) . *111\cdot1 . \supset \vdash . \text{Prop}$

$*164\cdot301$. $\vdash : P \text{ smor smor } Q . \supset . C``C`P \text{ sm sm } C``C`Q$ [$*164\cdot3\cdot11 . *111\cdot4$]

$*164\cdot31$. $\vdash : S \in P \,\overline{\text{smor}}\,\overline{\text{smor}}\, Q . \equiv . S \in (C``C`P) \,\overline{\text{sm}}\,\overline{\text{sm}}\, (C``C`Q) . P = S\dagger \,;\, Q$

Dem.

$\vdash . *164\cdot3\cdot1 . \supset$

$\vdash : S \in P \,\overline{\text{smor}}\,\overline{\text{smor}}\, Q . \supset . S \in (C``C`P) \,\overline{\text{sm}}\,\overline{\text{sm}}\, (C``C`Q) . P = S\dagger \,;\, Q$ (1)

$\vdash . *111\cdot1 . *162\cdot22 . \supset$

$\vdash : S \in (C``C`P) \,\overline{\text{sm}}\,\overline{\text{sm}}\, (C``C`Q) . \supset . S \in 1 \to 1 . \text{ᗡ}`S = C`\Sigma`Q$ (2)

$\vdash . (2) . \text{Fact} . *164\cdot1 . \supset$

$\vdash : S \in (C``C`P) \,\overline{\text{sm}}\,\overline{\text{sm}}\, (C``C`Q) . P = S\dagger \,;\, Q . \supset . S \in P \,\overline{\text{smor}}\,\overline{\text{smor}}\, Q$ (3)

$\vdash . (1) . (3) . \supset \vdash . \text{Prop}$

This proposition has the merit of reducing the ordinal element in double likeness to a minimum. The proof of

$$S \in (C``C`P) \,\overline{\text{sm}}\,\overline{\text{sm}}\, (C``C`Q)$$

is a cardinal problem, and what has to be added for ordinal purposes is merely

$$P = S\dagger \,;\, Q.$$

$*164\cdot32$. $\vdash . \dot\Lambda \in (\dot\Lambda \,\overline{\text{smor}}\,\overline{\text{smor}}\, \dot\Lambda) . \dot\Lambda \text{ smor smor } \dot\Lambda$

In this proposition, the various $\dot\Lambda$'s need not be of the same type. Hence "$\dot\Lambda$ smor smor $\dot\Lambda$" is not an immediate consequence of $*164\cdot201$.

Dem.

$\vdash . *72\cdot1 . *162\cdot4 . \quad \supset \vdash . \dot\Lambda \in 1 \to 1 . \text{ᗡ}`\dot\Lambda = C`\Sigma`\dot\Lambda$ (1)

$\vdash . *150\cdot42 . \quad \supset \vdash . \dot\Lambda = \dot\Lambda \,;\, \dot\Lambda$ (2)

$\vdash . (1) . (2) . *164\cdot1 . \supset \vdash . \dot\Lambda \in (\dot\Lambda \,\overline{\text{smor}}\,\overline{\text{smor}}\, \dot\Lambda) .$ (3)

[$*164\cdot11$] $\quad \supset . \dot\Lambda \text{ smor smor } \dot\Lambda$ (4)

$\vdash . (3) . (4) . \supset \vdash . \text{Prop}$

***164·33.** $\vdash : M \epsilon P \,\overline{\text{smor}}\, R \,.\, N \epsilon Q \,\overline{\text{smor}}\, S \,.\, C'P \cap C'Q = \Lambda \,.\, C'R \cap C'S = \Lambda \,.\supset .$
$M \cup N \epsilon (P \downarrow Q) \,\overline{\text{smor}}\,\overline{\text{smor}}\, (R \downarrow S)$

Dem.

$\vdash .*160·47 . \supset \vdash : \text{Hp} . \supset . M \cup N \epsilon (P \ddagger Q) \,\overline{\text{smor}}\, (R \ddagger S) .$

$[*162·3.*151·11] \quad \supset . M \cup N \epsilon 1 \to 1 . \text{Ɑ}'(M \cup N) = C'\Sigma'(R \downarrow S) \qquad (1)$

$\vdash .*150·32 . \supset \vdash : \text{Hp} . \supset . (M \cup N) ; R = \{(M \cup N) \upharpoonright C'R\} ; R$

$[*35·644.*150·32] \qquad = M ; R$

$[*151·11] \qquad = P \qquad (2)$

Similarly $\quad \vdash : \text{Hp} . \supset . (M \cup N) ; S = Q \qquad (3)$

$\vdash .*150·71·1 . \supset \vdash : \text{Hp} . \supset . (M \cup N) \dagger ;(R \downarrow S) = \{(M \cup N) ; R\} \downarrow \{(M \cup N) ; S\}$

$[(2).(3)] \qquad = P \downarrow Q \qquad (4)$

$\vdash . (1) . (4) . *164·1 . \supset \vdash . \text{Prop}$

***164·34.** $\vdash : P \,\text{smor}\, R \,.\, Q \,\text{smor}\, S \,.\, C'P \cap C'Q = \Lambda \,.\, C'R \cap C'S = \Lambda \,.\supset .$
$P \downarrow Q \,\text{smor smor}\, R \downarrow S$

$[*164·33·11 . *151·12]$

The following propositions are concerned in showing that, if P and Q are like relations, and the correlator of P and Q is contained in likeness (*i.e.* correlates relations which have the relation of likeness), a correlator being given for each pair of relations coupled by the correlator of P and Q, then the logical sum of such correlators is a double correlator of P and Q, provided P and Q are relations of mutually exclusive relations. That is, assuming S to be the correlator of P and Q, and assuming that $S'N \,\text{smor}\, N$ whenever $N \epsilon C'Q$, let it be possible to choose one correlator out of the class of correlators $(S'N) \,\overline{\text{smor}}\, N$, for every N which belongs to $C'Q$. That is, assume that it is possible to make a selection from the class of classes of correlators. If μ is such a selection, then $\dot{s}'\mu$ will be a double correlator of P and Q, if $P, Q \epsilon \text{Rel}^2 \,\text{excl}$.

The following propositions, down to *164·421, are lemmas for *164·43.

***164·4.** $\vdash :. N \epsilon C'Q . \supset_N . R'N \epsilon (S'N) \,\overline{\text{smor}}\, N : \supset . \text{Ɑ}'\dot{s}'R``C'Q = C'\Sigma'Q$

Dem.

$\vdash .*41·44 . \supset \vdash . \text{Ɑ}'\dot{s}'R``C'Q = s'\text{Ɑ}``R``C'Q \qquad (1)$

$\vdash .*151·11 . \supset \vdash :. \text{Hp} . \supset : N \epsilon C'Q . \supset . \text{Ɑ}'R'N = C'N :$

$[*37·68] \qquad \supset : \text{Ɑ}``R``C'Q = C``C'Q \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash :. \text{Hp} . \supset : \text{Ɑ}'\dot{s}'R``C'Q = s'C``C'Q$

$[*162·22] \qquad = C'\Sigma'Q : \supset \vdash . \text{Prop}$

***164·41.** $\vdash :. Q \epsilon \text{Rel}^2 \,\text{excl} : N \epsilon C'Q . \supset_N . R'N \epsilon (S'N) \,\overline{\text{smor}}\, N : \supset .$
$\dot{s}'R``C'Q \epsilon 1 \to \text{Cls}$

Dem.

$\vdash .*151·11 . \supset \vdash :. \text{Hp} . \supset : M, N \epsilon C'Q . \exists ! \text{Ɑ}'R'M \cap \text{Ɑ}'R'N . \supset .$
$\exists ! C'\dot{M} \cap C'N .$

$[*163\cdot11] \qquad \supset . M = N .$

$[*30\cdot37] \qquad \supset . R`M = R`N \qquad (1)$

$\vdash . *151\cdot11 . \supset \vdash :. \mathrm{Hp} . \supset : M \,\epsilon\, C`Q . \supset . R`M \,\epsilon\, 1 \rightarrow 1 \qquad (2)$

$\vdash . (1) . (2) . *72\cdot32 . \supset \vdash . \mathrm{Prop}$

$*164\cdot411$. $\vdash : S;Q \,\epsilon\, \mathrm{Rel}^2\,\mathrm{excl} . S \upharpoonright C`Q \,\epsilon\, 1 \rightarrow 1 . \mathrm{Hp}\,*164\cdot4 . \supset . \dot{s}`R``C`Q \,\epsilon\, \mathrm{Cls} \rightarrow 1$

Dem.

$\vdash . *151\cdot11 . \supset \vdash :. \mathrm{Hp} . \supset : M, N \,\epsilon\, C`Q . \exists ! \mathrm{D}`R`M \cap \mathrm{D}`R`N . \supset .$

$\exists ! C`S`M \cap C`S`N .$

$[*163\cdot11 . *150\cdot22] \qquad \supset . S`M = S`N .$

$[*71\cdot532] \qquad \supset . M = N .$

$[*30\cdot37] \qquad \supset . R`M = R`N \qquad (1)$

$\vdash . *151\cdot11 . \supset \vdash :. \mathrm{Hp} . \supset : M \,\epsilon\, C`Q . \supset . R`M \,\epsilon\, 1 \rightarrow 1 \qquad (2)$

$\vdash . (1) . (2) . *72\cdot321 . \supset \vdash . \mathrm{Prop}$

$*164\cdot412$. $\vdash :. S;Q, Q \,\epsilon\, \mathrm{Rel}^2\,\mathrm{excl} . S \upharpoonright C`Q \,\epsilon\, 1 \rightarrow 1 :$

$N \,\epsilon\, C`Q . \supset_N . R`N \,\epsilon\, (S`N)\, \overline{\mathrm{smor}}\, N : \supset . \dot{s}`R``C`Q \,\epsilon\, 1 \rightarrow 1$

$[*164\cdot41\cdot411]$

$*164\cdot413$. $\vdash :. \mathrm{Hp}\,*164\cdot41 . \supset :$

$N \,\epsilon\, C`Q . \supset . R`N = (\dot{s}`R``C`Q) \upharpoonright C`N . S`N = (\dot{s}`R``C`Q);N$

Dem.

$\vdash . *41\cdot13 . \supset \vdash : \mathrm{Hp} . N \,\epsilon\, C`Q . \supset . R`N \subset \dot{s}`R``C`Q .$

$[*72\cdot92 . *164\cdot41] \qquad \supset . R`N = (\dot{s}`R``C`Q) \upharpoonright \mathrm{G}`R`N$

$[*151\cdot11 . \mathrm{Hp}] \qquad = (\dot{s}`R``C`Q) \upharpoonright C`N \qquad (1)$

$\vdash . *151\cdot11 . \supset \vdash : \mathrm{Hp} . N \,\epsilon\, C`Q . \supset . S`N = (R`N);N$

$[(1) . *150\cdot32] \qquad = (\dot{s}`R``C`Q);N \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

$*164\cdot414$. $\vdash : \mathrm{Hp}\,*164\cdot41 . \supset . S;Q = (\dot{s}`R``C`Q) \dagger ;Q \quad [*164\cdot413 . *150\cdot1\cdot35]$

$*164\cdot42$. $\vdash :. Q, S;Q \,\epsilon\, \mathrm{Rel}^2\,\mathrm{excl} . S \upharpoonright C`Q \,\epsilon\, 1 \rightarrow 1 :$

$N \,\epsilon\, C`Q . \supset_N . R`N \,\epsilon\, (S`N)\, \overline{\mathrm{smor}}\, N : \supset .$

$\dot{s}`R``C`Q \,\epsilon\, (S;Q)\, \overline{\mathrm{smor}}\, \overline{\mathrm{smor}}\, Q \quad [*164\cdot4\cdot412\cdot414\cdot1]$

$*164\cdot421$. $\vdash :. P, Q \,\epsilon\, \mathrm{Rel}^2\,\mathrm{excl} . S \upharpoonright C`Q \,\epsilon\, P\, \overline{\mathrm{smor}}\, Q :$

$N \,\epsilon\, C`Q . \supset_N . R`N \,\epsilon\, (S`N)\, \overline{\mathrm{smor}}\, N : \supset .$

$\dot{s}`R``C`Q \,\epsilon\, P\, \overline{\mathrm{smor}}\, \overline{\mathrm{smor}}\, Q \quad [*164\cdot42]$

The following proposition, besides being used in proving all subsequent propositions of this number (except $*164\cdot432\cdot433$, which are mere lemmas for $*164\cdot44$), is used in $*251\cdot6$, in the theory of ordinal numbers.

***164·43.** $\vdash :. P, Q \epsilon \text{Rel}^2 \text{excl} . S \epsilon P \,\overline{\text{smor}}\, Q .$

$$\mu = \hat{\lambda} \{(\exists N) . N \epsilon C'Q . \lambda = (S'N) \,\overline{\text{smor}}\, N\} . \supset :$$
$$R \epsilon \epsilon_\Delta{}'\mu . \supset . \dot{s}'\text{D}'R \epsilon P \,\overline{\text{smor}}\,\overline{\text{smor}}\, Q . S = (\dot{s}'\text{D}'R)\dagger \upharpoonright C'Q$$

Dem.

$\vdash . *83{\cdot}2{\cdot}22 . \supset \vdash :. \text{Hp} . R \epsilon \epsilon_\Delta{}'\mu . \supset :$

$$N \epsilon C'Q . \supset . R'\{(S'N) \,\overline{\text{smor}}\, N\} \epsilon (S'N) \,\overline{\text{smor}}\, N : \dot{s}'\text{D}'R = R``\mu \quad (1)$$

$\vdash . (1) . \supset \vdash :. \text{Hp}(1) . T = \hat{\lambda}\hat{N}\{N \epsilon C'Q . \lambda = (S'N) \,\overline{\text{smor}}\, N\} . \supset :$

$$N \epsilon C'Q . \supset . R'T'N \epsilon (S'N) \,\overline{\text{smor}}\, N : \dot{s}'\text{D}'R = R``T``C'Q : \quad (2)$$

$\left[*164{\cdot}42 \dfrac{R|T}{R}\right] \supset : \dot{s}'\text{D}'R \epsilon P \,\overline{\text{smor}}\,\overline{\text{smor}}\, Q \quad (3)$

$\vdash . (2) . *164{\cdot}413 \dfrac{R|T}{R} . *151{\cdot}11 . *35{\cdot}71 . \supset \vdash : \text{Hp}(2) . \supset . S = (\dot{s}'\text{D}'R)\dagger \upharpoonright C'Q \quad (4)$

$\vdash . (3) . (4) . \supset \vdash . \text{Prop}$

***164·431.** $\vdash :. P, Q \epsilon \text{Rel}^2 \text{excl} : (\exists S) . S \epsilon P \,\overline{\text{smor}}\, Q .$

$$\exists ! \epsilon_\Delta{}'\hat{\lambda} \{(\exists N) . N \epsilon C'Q . \lambda = (S'N) \,\overline{\text{smor}}\, N\} : \supset . P \text{ smor smor } Q$$

[*163·43·11]

***164·432.** $\vdash : S \epsilon P \,\overline{\text{smor}}\, Q \cap \text{Rl}'\text{smor} . \supset .$

$$\Lambda \sim\epsilon \hat{\lambda} \{(\exists N) . N \epsilon C'Q . \lambda = (S'N) \,\overline{\text{smor}}\, N\}$$

Dem.

$\vdash . *151{\cdot}11 . \supset \vdash :. \text{Hp} . \supset : N \epsilon C'Q . \supset . N \epsilon \text{Cl}'S .$

[*71·31] $\supset . (S'N) S N .$

[Hp] $\supset . (S'N) \text{ smor } N .$

[*151·12] $\supset . \exists ! (S'N) \,\overline{\text{smor}}\, N :. \supset \vdash . \text{Prop}$

***164·433.** $\vdash :. \text{Mult ax} . \supset : S \epsilon P \,\overline{\text{smor}}\, Q \cap \text{Rl}'\text{smor} . \supset .$

$$\exists ! \epsilon_\Delta{}'\hat{\lambda} \{(\exists N) . N \epsilon C'Q . \lambda = (S'N) \,\overline{\text{smor}}\, N\}$$

[*164·432 . *88·37]

All the remaining propositions of the number are important.

***164·44.** $\vdash :. \text{Mult ax} . \supset : P, Q \epsilon \text{Rel}^2 \text{excl} . \exists ! P \,\overline{\text{smor}}\, Q \cap \text{Rl}'\text{smor} . \supset .$

$$P \text{ smor smor } Q \quad [*164{\cdot}433{\cdot}431]$$

***164·45.** $\vdash :: \text{Mult ax} . \supset :. P, Q \epsilon \text{Rel}^2 \text{excl} . \supset :$

$$\exists ! P \,\overline{\text{smor}}\, Q \cap \text{Rl}'\text{smor} . \equiv . P \text{ smor smor } Q \quad [*164{\cdot}44{\cdot}17]$$

***164·46.** $\vdash :. \text{Mult ax} . \supset : P, Q \epsilon \text{Rel}^2 \text{excl} . \exists ! P \,\overline{\text{smor}}\, Q \cap \text{Rl}'\text{smor} . \supset .$

$$\Sigma'P \text{ smor } \Sigma'Q \quad [*164{\cdot}44{\cdot}151]$$

***164·47.** $\vdash : R, S \in \mathrm{Nr}`Q \,.\, C`R, C`S \in \mathrm{Cl}`\mathrm{Nr}`P \,.\, \supset \,.\, \exists ! R \,\overline{\mathrm{smor}}\, S \cap \mathrm{Rl}`\mathrm{smor}$

Dem.

$\vdash \,.\, *152·5·4 \,.\quad \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, R \,\mathrm{smor}\, S \,.$

$[*151·12] \qquad \supset \,.\, \exists ! R \,\overline{\mathrm{smor}}\, S \qquad (1)$

$\vdash \,.\, *60·2 \,.\quad \supset \vdash :. \mathrm{Hp} \,.\, \supset : M \in C`R \,.\, N \in C`S \,.\, \supset \,.\, M, N \in \mathrm{Nr}`P \,.$

$[*152·5·4] \qquad \supset \,.\, M \,\mathrm{smor}\, N \qquad (2)$

$\vdash \,.\, *151·1·131 \,.\, \supset \vdash :. T \in R \,\overline{\mathrm{smor}}\, S \,.\, \supset : MTN \,.\, \supset \,.\, M \in C`R \,.\, N \in C`S \qquad (3)$

$\vdash \,.\, (2) \,.\, (3) \,.\quad \supset \vdash :. \mathrm{Hp} \,.\, \supset : T \in R \,\overline{\mathrm{smor}}\, S \,.\, \supset \,.\, T \subseteq \mathrm{smor} \qquad (4)$

$\vdash \,.\, (1) \,.\, (4) \,.\, \supset \vdash \,.\, \mathrm{Prop}$

***164·48.** $\vdash :. \mathrm{Mult\,ax} \,.\, \supset : R, S \in \mathrm{Rel}^2\,\mathrm{excl} \cap \mathrm{Nr}`Q \,.\, C`R, C`S \in \mathrm{Cl}`\mathrm{Nr}`P \,.\, \supset \,.$
$R \,\mathrm{smor\,smor}\, S \,.\, \Sigma`R \,\mathrm{smor}\, \Sigma`S \quad [*164·47·44·46]$

$*165$. RELATIONS OF RELATIONS OF COUPLES

Summary of $*165$.

In the present number, we shall give various propositions concerning the relation $P \underset{\text{;}}{\downarrow} \text{;}Q$, which has the same uses in relation-arithmetic as $\alpha \underset{\text{''}}{\downarrow} \text{``}\beta$ has in cardinal arithmetic. The propositions of this number will be used in the next number to establish the properties of the arithmetical product of two relations Q and P, which is defined as $\Sigma\text{‘}P \underset{\text{;}}{\downarrow} \text{;}Q$. Again in connection with exponentiation the propositions of the present number will be useful, since, after the product of a relation of relations has been defined ($*172$), we shall define exponentiation by means of the definition

$$P \exp Q = \text{Prod}\text{‘}P \underset{\text{;}}{\downarrow} \text{;}Q \quad \text{Df.} \quad (\text{Cf. } *176.)$$

There will also be occasional uses of the propositions of this number throughout the theory of series. The relation $P \underset{\text{;}}{\downarrow} \text{;}Q$ is important because its structure is thoroughly known. It is a $\text{Rel}^2\,\text{excl}$ which consists of $\text{Nr}\text{‘}Q$ relations, each like P ($*165{\cdot}27$); and if $P \,\text{smor}\, P' \,.\, Q \,\text{smor}\, Q'$, we can construct a double correlator of $P \underset{\text{;}}{\downarrow} \text{;}Q$ and $P' \underset{\text{;}}{\downarrow} \text{;}Q'$ without invoking the multiplicative axiom. In fact we have

$*165{\cdot}362$. $\vdash : R \upharpoonright C\text{‘}P' \in P \,\overline{\text{smor}}\, P' \,.\, S \upharpoonright C\text{‘}Q' \in Q \,\overline{\text{smor}}\, Q' \,.\, \supset .$

$$(R \,\|\, \breve{S}) \upharpoonright C\text{‘}\Sigma\text{‘}P' \underset{\text{;}}{\downarrow} \text{;}Q' \in (P \underset{\text{;}}{\downarrow} \text{;}Q) \,\overline{\text{smor}}\, \overline{\text{smor}}\, (P' \underset{\text{;}}{\downarrow} \text{;}Q')$$

This proposition should be compared with $*113{\cdot}127$. In virtue of $*164{\cdot}31$, together with various propositions of $*165$ and $*166$, it will appear that $*165{\cdot}362$ includes $*113{\cdot}127$ as part of what it asserts.

In the present number, we begin with a set of propositions on fields. We have

$*165{\cdot}12$. $\vdash . C\text{‘}P \underset{\text{;}}{\downarrow} \text{;}Q = P \underset{\text{;}}{\downarrow} \text{``}C\text{‘}Q$

$*165{\cdot}13$. $\vdash . C\text{‘}P \underset{\text{;}}{\downarrow} z = \downarrow z\text{``}C\text{‘}P = (C\text{‘}P) \underset{\text{''}}{\downarrow} z$

whence

$*165{\cdot}14$. $\vdash . C\text{``}C\text{‘}P \underset{\text{;}}{\downarrow} \text{;}Q = (C\text{‘}P) \underset{\text{''}}{\downarrow} \text{``}C\text{‘}Q$

which connects the theory of $P \underset{\text{;}}{\downarrow} \text{;}Q$ with that of $\alpha \underset{\text{''}}{\downarrow} \text{``}\beta$ ($*113$ and $*116$). Hence

$*165{\cdot}16$. $\vdash . C\text{‘}\Sigma\text{‘}P \underset{\text{;}}{\downarrow} \text{;}Q = C\text{‘}Q \times C\text{‘}P$

In $*166$, we shall define $Q \times P$ as $\Sigma\text{‘}P \underset{\text{;}}{\downarrow} \text{;}Q$; thus the above will become

$$\vdash . C\text{‘}(Q \times P) = C\text{‘}Q \times C\text{‘}P.$$

We next have a set of propositions concerned with $P \underset{\cdot,}{\downarrow}$ as a relation, and with the circumstances under which we can infer $x = y$ or $P = Q$ from data as to $P \underset{\cdot,}{\downarrow} x$ and $Q \underset{\cdot,}{\downarrow} y$. We have

***165·21.** $\vdash . P \underset{\cdot,}{\downarrow} ; Q \in \mathrm{Rel}^2 \,\mathrm{excl}$

***165·211.** $\vdash : \exists ! \, C`P \underset{\cdot,}{\downarrow} x \cap C`P \underset{\cdot,}{\downarrow} y . \supset . x = y$

***165·22.** $\vdash : \dot{\exists} ! \, P . \supset . P \underset{\cdot,}{\downarrow} \in 1 \rightarrow 1$

We then have various propositions concerning $\dot{\Lambda}$, of which the chief are

***165·241.** $\vdash : Q = \dot{\Lambda} . \supset . P \underset{\cdot,}{\downarrow} ; Q = \dot{\Lambda}$

***165·242.** $\vdash : P = \dot{\Lambda} . \dot{\exists} ! \, Q . \supset . P \underset{\cdot,}{\downarrow} ; Q = \dot{\Lambda} \downarrow \dot{\Lambda}$

We have next four propositions which are constantly used, proving that $P \underset{\cdot,}{\downarrow} ; Q$ consists of $\mathrm{Nr}`Q$ relations each like P. These propositions are

***165·25.** $\vdash : \dot{\exists} ! \, P . \supset . P \underset{\cdot,}{\downarrow} ; Q \,\mathrm{smor}\, Q . (P \underset{\cdot,}{\downarrow}) \upharpoonright C`Q \in (P \underset{\cdot,}{\downarrow} ; Q) \,\overline{\mathrm{smor}}\, Q$

***165·251.** $\vdash . P \underset{\cdot,}{\downarrow} x \,\mathrm{smor}\, P . (\downarrow x) \upharpoonright C`P \in (P \underset{\cdot,}{\downarrow} x) \,\overline{\mathrm{smor}}\, P$

***165·26.** $\vdash . C`P \underset{\cdot,}{\downarrow} ; Q \subset \mathrm{Nr}`P$

***165·27.** $\vdash : \dot{\exists} ! \, P . \supset . P \underset{\cdot,}{\downarrow} ; Q \in \mathrm{Rel}^2 \,\mathrm{excl} \cap \mathrm{Nr}`Q . C`P \underset{\cdot,}{\downarrow} ; Q \in \mathrm{Cl}`\mathrm{Nr}`P$

From *165·3 to *165·372, we are concerned with constructing a double correlator of $P \underset{\cdot,}{\downarrow} ; Q$ and $P' \underset{\cdot,}{\downarrow} ; Q'$ when we are given simple correlators of P with P' and of Q with Q'. The result (*165·362) has already been given. Hence we have

***165·37.** $\vdash : P \,\mathrm{smor}\, P' . Q \,\mathrm{smor}\, Q' . \supset . P \underset{\cdot,}{\downarrow} ; Q \,\mathrm{smor \, smor}\, P' \underset{\cdot,}{\downarrow} ; Q'$

and by *164·48 and *165·27 we have

***165·38.** $\vdash :. \mathrm{Mult\,ax} . \supset :$

$$R \in \mathrm{Rel}^2 \,\mathrm{excl} \cap \mathrm{Nr}`Q . C`R \subset \mathrm{Nr}`P . \supset . R \,\mathrm{smor\,smor}\, P \underset{\cdot,}{\downarrow} ; Q$$

Hence propositions concerning a series of β series, each containing α terms (where α and β are relation-numbers), which in general require the multiplicative axiom, can be deduced, assuming that axiom, from propositions (not requiring the axiom) concerning $P \underset{\cdot,}{\downarrow} ; Q$, where $\mathrm{Nr}`P = \alpha$ and $\mathrm{Nr}`Q = \beta$. Thus the use of $P \underset{\cdot,}{\downarrow} ; Q$ enables us to minimize the use of the multiplicative axiom.

***165·01.** $\vdash . P \underset{\cdot,}{\downarrow} z = \downarrow z ; P$ [*150·6]

***165·1.** $\vdash : R \, (P \underset{\cdot,}{\downarrow} ; Q) \, S . \equiv . (\exists z, w) . zQw . R = \downarrow z ; P . S = \downarrow w ; P$ [*150·62]

***165·11.** $\vdash : X \, (\downarrow z ; P) \, Y . \equiv . (\exists x, y) . xPy . X = x \downarrow z . Y = y \downarrow z$ [*150·55]

***165·12.** $\vdash . C`P \underset{\cdot,}{\downarrow} ; Q = P \underset{\cdot,}{\downarrow} `` C`Q$ [*150·22]

***165·13.** $\vdash . C\text{‘}P \underset{\cdot,}{\downarrow} z = \downarrow z\text{“}C\text{‘}P = (C\text{‘}P) \underset{,,}{\downarrow} z$ [*165·01 . *150·22 . *38·2]

***165·131.** $\vdash . C\text{“}P \underset{\cdot,}{\downarrow} \text{“}\beta = (C\text{‘}P) \underset{,,}{\downarrow} \text{“}\beta$ [*165·13 . *38·11 . *37·68]

***165·14.** $\vdash . C\text{“}C\text{‘}P \underset{\cdot,}{\downarrow} ;Q = (C\text{‘}P) \underset{,,}{\downarrow} \text{“}C\text{‘}Q$ [*165·12·131]

***165·15.** $\vdash . s\text{‘}C\text{“}C\text{‘}P \underset{\cdot,}{\downarrow} ;Q = C\text{‘}Q \times C\text{‘}P$ [*165·14 . *113·1]

***165·16.** $\vdash . C\text{‘}\Sigma\text{‘}P \underset{\cdot,}{\downarrow} ;Q = C\text{‘}Q \times C\text{‘}P$ [*165·15 . *162·22]

***165·161.** $\vdash : M(F;P \underset{\cdot,}{\downarrow} ;Q)\, N . \equiv .$

$$(\exists x, y, z, w) . x, y \in C\text{‘}P . zQw . M = x \downarrow z . N = y \downarrow w$$

Dem.

$\vdash . *150{\cdot}52 . \supset$

$\vdash :. M(F;P \underset{\cdot,}{\downarrow} ;Q)\, N . \equiv : (\exists R, S) . R\,(P \underset{\cdot,}{\downarrow} ;Q)\, S . M \in C\text{‘}R . N \in C\text{‘}S .$

[*165·1] $\equiv : (\exists R, S, z, w) . zQw . R = \downarrow z ; P . S = \downarrow w ; P . M \in C\text{‘}R . N \in C\text{‘}S .$

[*165·01·13] $\equiv : (\exists R, S, z, w) . zQw . R = \downarrow z ; P . S = \downarrow w ; P .$

$$M \in \downarrow z\text{“}C\text{‘}P . N \in \downarrow w\text{“}C\text{‘}P .$$

[*21·151] $\equiv : (\exists z, w) . zQw . M \in \downarrow z\text{“}C\text{‘}P . N \in \downarrow w\text{“}C\text{‘}P .$

[*38·131] $\equiv : (\exists x, y, z, w) . zQw . x, y \in C\text{‘}P . M = x \downarrow z . N = y \downarrow w :. \supset \vdash .$ Prop

***165·162.** $\vdash : M(\dot{s}\text{‘}C\text{‘}P \underset{\cdot,}{\downarrow} ;Q) N . \equiv . (\exists x, y, z) . xPy . z \in C\text{‘}Q . M = x \downarrow z . N = y \downarrow z$

Dem.

$\vdash . *165{\cdot}12 . *41{\cdot}11 . \supset$

$\vdash : M(\dot{s}\text{‘}C\text{‘}P \underset{\cdot,}{\downarrow} ;Q)\, N . \equiv . (\exists R) . R \in P \underset{\cdot,}{\downarrow} \text{“}C\text{‘}Q . MRN .$

[*38·13] $\equiv . (\exists z) . z \in C\text{‘}Q . M(P \underset{\cdot,}{\downarrow} z)\, N .$

[*165·01·11] $\equiv . (\exists x, y, z) . xPy . z \in C\text{‘}Q . M = x \downarrow z . N = y \downarrow z : \supset \vdash .$ Prop

***165·17.** $\vdash :. M(\Sigma\text{‘}P \underset{\cdot,}{\downarrow} ;Q)\, N . \equiv : (\exists x, y, z, w) :$

$$x, y \in C\text{‘}P . z, w \in C\text{‘}Q : zQw . \vee . z = w . xPy : M = x \downarrow z . N = y \downarrow w$$

Dem.

$\vdash . *165{\cdot}161{\cdot}162 . *162{\cdot}11 . \supset$

$\vdash :. M(\Sigma\text{‘}P \underset{\cdot,}{\downarrow} ;Q)\, N . \equiv : (\exists x, y, z, w) . x, y \in C\text{‘}P . zQw . M = x \downarrow z . N = y \downarrow w . \vee .$

$$(\exists x, y, z) . xPy . z \in C\text{‘}Q . M = x \downarrow z . N = y \downarrow w :$$

[*13·195] $\equiv : (\exists x, y, z, w) . x, y \in C\text{‘}P . zQw . M = x \downarrow z . N = y \downarrow w . \vee .$

$$(\exists x, y, z, w) . xPy . z, w \in C\text{‘}Q . z = w . M = x \downarrow z . N = y \downarrow w :$$

[*33·17 . *4·71] $\equiv : (\exists x, y, z, w) . x, y \in C\text{‘}P . z, w \in C\text{‘}Q . zQw . M = x \downarrow z . N = y \downarrow w . \vee .$

$$(\exists x, y, z, w) . x, y \in C\text{‘}P . z, w \in C\text{‘}Q . xPy . z = w . M = x \downarrow z . N = y \downarrow w :$$

[*11·41 . *4·4] $\equiv : (\exists x, y, z, w) : x, y \in C\text{‘}P . z, w \in C\text{‘}Q : zQw . \vee . z = w . xPy :$

$$M = x \downarrow z . N = y \downarrow w :. \supset \vdash . \text{Prop}$$

***165·18.** $\vdash . \mathrm{Cnv}\text{‘}P \underset{\cdot,}{\downarrow} ;Q = P \underset{\cdot,}{\downarrow} ;\breve{Q}$ [*150·12]

***165·181.** $\vdash . \mathrm{Cnv}\text{‘}P \underset{\cdot,}{\downarrow} z = \breve{P} \underset{\cdot,}{\downarrow} z$ [*165·01 . *150·12]

***165·182.** $\vdash . \text{Cnv};P \underset{;}{\downarrow} ;Q = \breve{P} \underset{;}{\downarrow} ;Q$ [*165·181 . *150·35]

***165·19.** $\vdash . \text{Cnv}\text{‘}\text{Cnv};P \underset{;}{\downarrow} ;Q = \breve{P} \underset{;}{\downarrow} ;\breve{Q} = \text{Cnv};\text{Cnv}\text{‘}P \underset{;}{\downarrow} ;Q$ [*165·18·182]

***165·2.** $\vdash . P \underset{;}{\downarrow} \epsilon 1 \rightarrow \text{Cls}$ [*72·14]

***165·201.** $\vdash . C\text{‘}(P \underset{;}{\downarrow} z) = (C\text{‘}P \uparrow \iota\text{‘}z)_{\Delta}\text{‘}\iota\text{‘}z$

Dem. $\vdash . *35·103 . \supset \vdash : y\,(C\text{‘}P \uparrow \iota\text{‘}z)\,z . \equiv . y \,\epsilon\, C\text{‘}P :$

[*85·51] $\supset \vdash . (C\text{‘}P \uparrow \iota\text{‘}z)_{\Delta}\text{‘}\iota\text{‘}z = \downarrow z\text{‘‘}C\text{‘}P$

[*165·13] $= C\text{‘}(P \underset{;}{\downarrow} z) . \supset \vdash . \text{Prop}$

***165·202.** $\vdash . C\text{‘‘}C\text{‘}P \underset{;}{\downarrow} ;Q = (C\text{‘}P \uparrow C\text{‘}Q)_{\Delta}\text{‘‘}\iota\text{‘‘}C\text{‘}Q$ [*165·14 . *113·103]

***165·203.** $\vdash . C\text{‘‘}C\text{‘}P \underset{;}{\downarrow} ;Q \,\epsilon\, \text{Cls}^2\,\text{excl}$ [*84·55 . *165·202]

***165·204.** $\vdash : C\text{‘}P \underset{;}{\downarrow} x = C\text{‘}P \underset{;}{\downarrow} y . \equiv . P \underset{;}{\downarrow} x = P \underset{;}{\downarrow} y$

Dem.

$\vdash . *165·13 . *55·232 . \supset$

$\vdash : C\text{‘}P \underset{;}{\downarrow} x = C\text{‘}P \underset{;}{\downarrow} y . \exists ! C\text{‘}P \underset{;}{\downarrow} x . \supset . x = y .$

[*30·37] $\supset . P \underset{;}{\downarrow} x = P \underset{;}{\downarrow} y$ (1)

$\vdash . *33·241 . \supset \vdash : C\text{‘}P \underset{;}{\downarrow} x = C\text{‘}P \underset{;}{\downarrow} y . C\text{‘}P \underset{;}{\downarrow} x = \Lambda . \supset . P \underset{;}{\downarrow} x = \dot{\Lambda} . P \underset{;}{\downarrow} y = \dot{\Lambda}$ (2)

$\vdash . (1) . (2) . \supset \vdash : C\text{‘}P \underset{;}{\downarrow} x = C\text{‘}P \underset{;}{\downarrow} y . \supset . P \underset{;}{\downarrow} x = P \underset{;}{\downarrow} y$ (3)

$\vdash . (3) . *30·37 . \supset \vdash . \text{Prop}$

***165·205.** $\vdash . C \upharpoonright \text{D}\text{‘}P \underset{;}{\downarrow} \epsilon 1 \rightarrow 1$ [*165·204 . *71·58]

***165·206.** $\vdash : (x) . \text{E} ! P \underset{;}{\downarrow} \text{‘}x : (\alpha) . \alpha \subset \text{Ɑ}\text{‘}P \underset{;}{\downarrow}$ [*38·12 . *33·431]

***165·21.** $\vdash . P \underset{;}{\downarrow} ;Q \,\epsilon\, \text{Rel}^2\,\text{excl}$

Dem. $\vdash . *165·205 . *150·203 . \supset \vdash . C \upharpoonright C\text{‘}P \underset{;}{\downarrow} ;Q \,\epsilon\, 1 \rightarrow 1$ (1)

$\vdash . (1) . *165·203 . *163·17 . \supset \vdash . \text{Prop}$

***165·211.** $\vdash : \exists ! C\text{‘}P \underset{;}{\downarrow} x \cap C\text{‘}P \underset{;}{\downarrow} y . \supset . x = y$ [*165·13 . *55·232]

***165·212.** $\vdash : \dot{\exists} ! P . \equiv . \dot{\exists} ! P \underset{;}{\downarrow} x$

Dem.

$\vdash . *165·11·01 . \supset \vdash : \dot{\exists} ! P \underset{;}{\downarrow} x . \equiv . (\exists X, Y, x, y) . xPy . X = x \downarrow z . Y = y \downarrow z .$

[*13·19] $\equiv . (\exists x, y) . xPy : \supset \vdash . \text{Prop}$

***165·22.** $\vdash : \dot{\exists} ! P . \supset . P \underset{;}{\downarrow} \epsilon 1 \rightarrow 1$

Dem.

$\vdash . *165·212 . \supset \vdash :. \text{Hp} . \supset : \dot{\exists} ! P \underset{;}{\downarrow} x :$

[*30·37.*24·571.*33·24] $\supset : P \underset{;}{\downarrow} x = P \underset{;}{\downarrow} y . \supset . \exists ! C\text{‘}P \underset{;}{\downarrow} x \cap C\text{‘}P \underset{;}{\downarrow} y .$

[*165·211] $\supset . x = y$ (1)

$\vdash . (1) . *71·54 . *165·2 . \supset \vdash . \text{Prop}$

∗165·221. $\vdash :. \dot{\exists} ! P . \supset : \dot{\exists} ! P \underset{\cdot,}{\downarrow} x \mathbin{\dot{\cap}} P \underset{\cdot,}{\downarrow} y . \equiv . P \underset{\cdot,}{\downarrow} x = P \underset{\cdot,}{\downarrow} y . \equiv . x = y$

Dem.

$\vdash . *33 \cdot 252 . \supset \vdash : \dot{\exists} ! P \underset{\cdot,}{\downarrow} x \mathbin{\dot{\cap}} P \underset{\cdot,}{\downarrow} y . \supset . \exists ! C`P \underset{\cdot,}{\downarrow} x \cap C`P \underset{\cdot,}{\downarrow} y .$

$[*165 \cdot 211] \qquad \supset . x = y \qquad (1)$

$\vdash . *165 \cdot 212 . *25 \cdot 571 . \supset \vdash :. \dot{\exists} ! P . \supset : x = y . \supset . \dot{\exists} ! P \underset{\cdot,}{\downarrow} x \mathbin{\dot{\cap}} P \underset{\cdot,}{\downarrow} y \qquad (2)$

$\vdash . (1) . (2) . *165 \cdot 212 . *30 \cdot 37 . \supset \vdash . \text{Prop}$

∗165·222. $\vdash :. \dot{\exists} ! P . \supset : \exists ! C`P \underset{\cdot,}{\downarrow} x \cap C`P \underset{\cdot,}{\downarrow} y . \equiv . C`P \underset{\cdot,}{\downarrow} x = C`P \underset{\cdot,}{\downarrow} y . \equiv . x = y$

[Proof as in ∗165·221]

∗165·223. $\vdash :. \dot{\exists} ! P . \supset : P \underset{\cdot,}{\downarrow} ; Q = P \underset{\cdot,}{\downarrow} ; R . \equiv . Q = R$

Dem.

$\vdash . *151 \cdot 31 . *165 \cdot 22 . \supset \vdash :. \text{Hp} . \supset : P \underset{\cdot,}{\downarrow} ; Q = P \underset{\cdot,}{\downarrow} ; R . \supset . Q = R \qquad (1)$

$\vdash . *34 \cdot 29 . *150 \cdot 1 . \quad \supset \vdash : Q = R . \supset . P \underset{\cdot,}{\downarrow} ; Q = P \underset{\cdot,}{\downarrow} ; R \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

∗165·23. $\vdash : P \underset{\cdot,}{\downarrow} x = Q \underset{\cdot,}{\downarrow} y . \supset . P = Q$

Dem.

$\vdash . *72 \cdot 184 . *150 \cdot 153 . \supset \vdash : \downarrow x ; P = \downarrow x ; Q . \supset . P = Q \qquad (1)$

$\vdash . (1) . *165 \cdot 01 . \supset \vdash . \text{Prop}$

∗165·231. $\vdash : P \underset{\cdot,}{\downarrow} x = Q \underset{\cdot,}{\downarrow} x . \equiv . P = Q \quad [*165 \cdot 23 . *30 \cdot 37]$

∗165·232. $\vdash :. \dot{\exists} ! P . \vee . \dot{\exists} ! Q : \supset : P \underset{\cdot,}{\downarrow} x = Q \underset{\cdot,}{\downarrow} y . \equiv . P = Q . x = y$

Dem.

$\vdash . *165 \cdot 23 . \supset \vdash :. P \underset{\cdot,}{\downarrow} x = Q \underset{\cdot,}{\downarrow} y . \supset : P = Q : \qquad (1)$

$[*13 \cdot 12 . \text{Hp} (1)] \qquad \supset : P \underset{\cdot,}{\downarrow} x = P \underset{\cdot,}{\downarrow} y . Q \underset{\cdot,}{\downarrow} x = Q \underset{\cdot,}{\downarrow} y :$

$[*165 \cdot 221] \qquad \supset : \dot{\exists} ! P . \supset . x = y : \dot{\exists} ! Q . \supset . x = y \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash :. \dot{\exists} ! P . \vee . \dot{\exists} ! Q : \supset : P \underset{\cdot,}{\downarrow} x = Q \underset{\cdot,}{\downarrow} y . \supset . P = Q . x = y \qquad (3)$

$\vdash . (3) . *13 \cdot 12 \cdot 15 . \supset \vdash . \text{Prop}$

∗165·233. $\vdash : \exists ! C`P \underset{\cdot,}{\downarrow} x \cap C`Q \underset{\cdot,}{\downarrow} y . \equiv . x = y . \exists ! C`P \cap C`Q$

[∗55·232 . ∗165·13]

∗165·24. $\vdash : P = \dot{\Lambda} . \supset . P \underset{\cdot,}{\downarrow} x = \dot{\Lambda} . P \underset{\cdot,}{\downarrow} = \iota`\dot{\Lambda} \uparrow V$

Dem.

$\vdash . *165 \cdot 212 . \text{Transp} . \supset \vdash : P = \dot{\Lambda} . \supset . P \underset{\cdot,}{\downarrow} x = \dot{\Lambda} \qquad (1)$

$\vdash . (1) . *38 \cdot 1 . \qquad \supset \vdash :. P = \dot{\Lambda} . \supset : R (P \underset{\cdot,}{\downarrow}) x . \equiv . R = \dot{\Lambda} .$

$[*51 \cdot 15 . *24 \cdot 104] \qquad \equiv . R \in \iota`\dot{\Lambda} . x \in V .$

$[*35 \cdot 103] \qquad \equiv . R (\iota`\dot{\Lambda} \uparrow V) x \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

$*165\cdot241$. $\vdash : Q = \dot{\Lambda} \,.\supset .\, P \underset{;}{\downarrow} ;Q = \dot{\Lambda}$ $[*150\cdot42]$

$*165\cdot242$. $\vdash : P = \dot{\Lambda} \,.\, \dot{\exists} ! Q \,.\supset .\, P \underset{;}{\downarrow} ;Q = \dot{\Lambda} \downarrow \dot{\Lambda}$

Dem.

$\vdash . *165\cdot1\cdot24 \,.\supset \vdash :. P = \dot{\Lambda} \,.\supset : R (P \underset{;}{\downarrow} ;Q) S \,.\equiv .\, (\exists z, w) \,.\, zQw \,.\, R = \dot{\Lambda} \,.\, S = \dot{\Lambda} \,.$

$[*10\cdot35]$ $\equiv .\, \dot{\exists} ! Q \,.\, R = \dot{\Lambda} \,.\, S = \dot{\Lambda}$ (1)

$\vdash . (1) . *55\cdot13 \,.\supset \vdash .$ Prop

$*165\cdot243$. $\vdash : \dot{\exists} ! Q \,.\equiv .\, \dot{\exists} ! P \underset{;}{\downarrow} ;Q$

Dem.

$\vdash . *165\cdot1 \,.\supset \vdash : \dot{\exists} ! P \underset{;}{\downarrow} ;Q \,.\equiv .\, (\exists x, y, R, S) \,.\, xQy \,.\, R = P \underset{;}{\downarrow} x \,.\, S = P \underset{;}{\downarrow} y \,.$

$[*13\cdot19]$ $\equiv .\, (\exists x, y) \,.\, xQy : \supset \vdash .$ Prop

$*165\cdot244$. $\vdash : \dot{\Lambda} \in C\text{‘}P \underset{;}{\downarrow} ;Q \,.\equiv .\, P = \dot{\Lambda} \,.\, \dot{\exists} ! Q \,.\equiv .\, P \underset{;}{\downarrow} ;Q = \dot{\Lambda} \downarrow \dot{\Lambda}$

Dem.

$\vdash . *165\cdot212\cdot12 \,.$ $\supset \vdash : \dot{\Lambda} \in C\text{‘}P \underset{;}{\downarrow} ;Q \,.\supset .\, P = \dot{\Lambda}$ (1)

$\vdash . *10\cdot24 . *33\cdot24 \,.\supset \vdash : \dot{\Lambda} \in C\text{‘}P \underset{;}{\downarrow} ;Q \,.\supset .\, \dot{\exists} ! P \underset{;}{\downarrow} ;Q \,.$

$[*165\cdot243]$ $\supset .\, \dot{\exists} ! Q$ (2)

$\vdash . *165\cdot242 . *55\cdot15 \,.\supset \vdash : P = \dot{\Lambda} \,.\, \dot{\exists} ! Q \,.\supset .\, \dot{\Lambda} \in C\text{‘}P \underset{;}{\downarrow} ;Q$ (3)

$\vdash . (1) . (2) . (3) \,.$ $\supset \vdash : \dot{\Lambda} \in C\text{‘}P \underset{;}{\downarrow} ;Q \,.\equiv .\, P = \dot{\Lambda} \,.\, \dot{\exists} ! Q$ (4)

$\vdash . *55\cdot15 \,.$ $\supset \vdash : P \underset{;}{\downarrow} ;Q = \dot{\Lambda} \downarrow \dot{\Lambda} \,.\supset .\, \dot{\Lambda} \in C\text{‘}P \underset{;}{\downarrow} ;Q \,.$

$[(4)]$ $\supset .\, P = \dot{\Lambda} \,.\, \dot{\exists} ! Q$ (5)

$\vdash . (5) . *165\cdot242 \,.$ $\supset \vdash : P = \dot{\Lambda} \,.\, \dot{\exists} ! Q \,.\equiv .\, P \underset{;}{\downarrow} ;Q = \dot{\Lambda} \downarrow \dot{\Lambda}$ (6)

$\vdash . (4) . (6) \,.\supset \vdash .$ Prop

$*165\cdot245$. $\vdash :. \dot{\exists} ! P \,.\vee .\, Q = \dot{\Lambda} : \equiv .\, \dot{\Lambda} \sim\in C\text{‘}P \underset{;}{\downarrow} ;Q \,.\equiv .\, \dot{\Lambda} \sim\in (C\text{‘}P) \underset{;}{\downarrow} \text{“}C\text{‘}Q$

$[*165\cdot244 \,.$ Transp $. *33\cdot241 . *165\cdot14]$

$*165\cdot25$. $\vdash : \dot{\exists} ! P \,.\supset .\, P \underset{;}{\downarrow} ;Q \,\text{smor}\, Q \,.\, (P \underset{;}{\downarrow}) \upharpoonright C\text{‘}Q \in (P \underset{;}{\downarrow} ;Q) \,\overline{\text{smor}}\, Q$

$[*165\cdot22\cdot206 . *151\cdot231]$

$*165\cdot251$. $\vdash . P \underset{;}{\downarrow} x \,\text{smor}\, P \,.\, (\downarrow x) \upharpoonright C\text{‘}P \in (P \underset{;}{\downarrow} x) \,\overline{\text{smor}}\, P$

$[*72\cdot184 . *55\cdot21 . *151\cdot22]$

$*165\cdot26$. $\vdash . C\text{‘}P \underset{;}{\downarrow} ;Q \subset \text{Nr}\text{‘}P$ $[*165\cdot251\cdot12 . *152\cdot11]$

$*165\cdot27$. $\vdash : \dot{\exists} ! P \,.\supset .\, P \underset{;}{\downarrow} ;Q \in \text{Rel}^2 \,\text{excl} \cap \text{Nr}\text{‘}Q \,.\, C\text{‘}P \underset{;}{\downarrow} ;Q \in \text{Cl}\text{‘}\text{Nr}\text{‘}P$

$[*165\cdot21\cdot25 . *152\cdot11 . *165\cdot26]$

The following propositions are concerned in proving that, if R is a correlator of P and P', and S is a correlator of Q and Q', then $R \| \breve{S}$ (with its converse domain limited) is a double correlator of $P \underset{;}{\downarrow} ;Q$ and $P' \underset{;}{\downarrow} ;Q'$. This proposition is required subsequently in establishing likenesses.

***165·3.** $\vdash : \mathrm{E} \,!\, R`y \,.\, \supset \,.\, \downarrow z`R`y = R|`\downarrow z`y$

Dem.

$\vdash . *34\cdot1 . *38\cdot11 . \supset \vdash : u\,\{R|`\downarrow z`y\}\,w \,.\equiv .\, (\exists v) \,.\, uRv \,.\, v\,(y \downarrow z)\,w \,.$

$[*55\cdot13] \qquad \equiv .\, uRy \,.\, w = z \qquad (1)$

$\vdash . (1) . *30\cdot4 . \quad \supset \vdash :.\, \mathrm{Hp} \,.\, \supset : u\,\{R|`\downarrow z`y\}\,w \,.\equiv .\, u = R`y \,.\, w = z \,.$

$[*55\cdot13 . *38\cdot11] \qquad \equiv .\, u\,(\downarrow z`R`y)\,w :.\, \supset \vdash .\, \mathrm{Prop}$

***165·301.** $\vdash : R \in 1 \to \mathrm{Cls} \,.\, \supset .\, \downarrow z \,|\, R = (R|)\,|\,(\downarrow z) \upharpoonright \mathrm{C\!\!\!I}`R$

Dem.

$\vdash . *165\cdot3 . \quad \supset \vdash :.\, \mathrm{E}\,!\,R`y \,.\, \supset : M\,\{(\downarrow z)\,|\,R\}\,y \,.\equiv .\, M\,\{(R|)\,|\downarrow z\}\,y :.$

$[*71\cdot16 . *34\cdot36] \supset \vdash :.\, \mathrm{Hp} \,.\, \supset : M\,\{(\downarrow z)\,|\,R\}\,y \,.\equiv .$

$M\,\{(R|)\,|\downarrow z\}\,y \,.\, y \in \mathrm{C\!\!\!I}`R :.\, \supset \vdash .\, \mathrm{Prop}$

***165·302.** $\vdash : \mathrm{E}\,!!\, R``C`P \,.\, \supset .\, \downarrow z \,;\, R \,;\, P = R|\,;\downarrow z \,;\, P$

Dem.

$\vdash . *165\cdot3 . \supset \vdash :.\, \mathrm{Hp} \,.\, \supset : y \in C`P \,.\, \supset .\, \downarrow z`R`y = R|`\downarrow z`y \qquad (1)$

$\vdash . (1) . *150\cdot35\cdot13 . \supset \vdash .\, \mathrm{Prop}$

***165·31.** $\vdash : \mathrm{E}\,!!\, R``C`P \,.\, \supset .\, (R\,;P) \underset{;}{\downarrow} z = R|\,;P \underset{;}{\downarrow} z \,.\, (R\,;P) \underset{;}{\downarrow} \,;Q = (R|)\dagger\,;P \underset{;}{\downarrow} \,;Q$

Dem.

$\vdash . *165\cdot302\cdot01 . \supset \vdash : \mathrm{Hp} \,.\, \supset .\, (R\,;P) \underset{;}{\downarrow} z = R|\,;P \underset{;}{\downarrow} z \qquad (1)$

$[*150\cdot1] \qquad = (R|)\dagger`P \underset{;}{\downarrow} z \qquad (2)$

$\vdash . (1) . (2) . *150\cdot35 . \supset \vdash .\, \mathrm{Prop}$

***165·311.** $\vdash : R \upharpoonright C`P \in 1 \to \mathrm{Cls} \,.\, C`P \subset \mathrm{C\!\!\!I}`R \,.\, \supset .$

$(R\,;P) \underset{;}{\downarrow} z = R|\,;P \underset{;}{\downarrow} z \,.\, (R\,;P) \underset{;}{\downarrow} \,;Q = (R|)\dagger\,;P \underset{;}{\downarrow} \,;Q$

$[*165\cdot31 . *71\cdot571]$

***165·32.** $\vdash : \mathrm{E}\,!\,S`z \,.\, \supset .\, \downarrow (S`z) = (|\breve{S})\,|\downarrow z \,.\, \downarrow (S`z)\,;P = |\breve{S}\,;\downarrow z\,;P$

Dem.

$\vdash . *34\cdot1 . *43\cdot101 . *38\cdot101 . \supset$

$\vdash : M\,\{(|\breve{S})\,|\downarrow z\}\,x \,.\equiv .\, (\exists N) \,.\, M = N\,|\,\breve{S} \,.\, N = x \downarrow z \,.$

$[*13\cdot195] \qquad \equiv .\, M = (x \downarrow z)\,|\,\breve{S} \qquad (1)$

$\vdash . (1) . *55\cdot581 . \supset$

$\vdash :.\, \mathrm{Hp} \,.\, \supset : M\,\{(|\breve{S})\,|\downarrow z\}\,x \,.\equiv .\, M = x \downarrow (S`z) \,.$

$[*38\cdot101] \qquad \equiv .\, M\,\{\downarrow (S`z)\}\,x \qquad (2)$

$\vdash . (2) . *21\cdot43 . \supset \vdash : \mathrm{Hp} \,.\, \supset .\, \downarrow (S`z) = (|\breve{S})\,|\downarrow z \,.$

$[*150\cdot13] \qquad \supset .\, \downarrow (S`z)\,;P = |\breve{S}\,;\downarrow z\,;P : \supset \vdash .\, \mathrm{Prop}$

***165·321.** $\vdash : \mathrm{E}\,!\,S`z \,.\, \supset .\, P \underset{;}{\downarrow} (S`z) = |\breve{S}\,;P \underset{;}{\downarrow} z \quad [*165\cdot32\cdot01]$

$*165\cdot33$. $\vdash : \mathrm{E}\,!\,!\, S``C`Q \,.\supset .\, P\underset{\smile}{\downarrow} ; S ; Q = (|\breve{S})\dagger ; P\underset{\smile}{\downarrow} ; Q$

Dem.

$\vdash . *165\cdot321 . *38\cdot11 . *150\cdot1 . \supset$

$\vdash :. \mathrm{Hp} . \supset : z \in C`Q . \supset . P\underset{\smile}{\downarrow} `S`z = (|\breve{S})\dagger`P\underset{\smile}{\downarrow}`z$ (1)

$\vdash . (1) . *150\cdot35 . \supset \vdash . \mathrm{Prop}$

$*165\cdot331$. $\vdash : S \upharpoonright C`Q \in 1 \to \mathrm{Cls} . C`Q \subset \text{Ɑ}`S . \supset . P\underset{\smile}{\downarrow} ; S ; Q = (|\breve{S})\dagger ; P\underset{\smile}{\downarrow} ; Q$
[$*165\cdot33 . *71\cdot571$]

$*165\cdot34$. $\vdash : \mathrm{E}\,!\,!\, R``C`P . \mathrm{E}\,!\,!\, S``C`Q . \supset . (R;P)\underset{\smile}{\downarrow} ; (S;Q) = (R \| \breve{S})\dagger ; (P\underset{\smile}{\downarrow} ; Q)$

Dem.

$\vdash . *165\cdot31 . \supset \vdash : \mathrm{Hp} . \supset . (R;P)\underset{\smile}{\downarrow} ; (S;Q) = (R|)\dagger ; P\underset{\smile}{\downarrow} ; (S;Q)$

[$*165\cdot33$] $= (R|)\dagger ; (|\breve{S})\dagger ; P\underset{\smile}{\downarrow} ; Q$

[$*150\cdot13\cdot14.(*43\cdot01)$] $= (R \| \breve{S})\dagger ; (P\underset{\smile}{\downarrow} ; Q) : \supset \vdash . \mathrm{Prop}$

$*165\cdot341$. $\vdash : R \upharpoonright C`P , S \upharpoonright C`Q \in 1 \to \mathrm{Cls} . C`P \subset \text{Ɑ}`R . C`Q \subset \text{Ɑ}`S . \supset .$
$(R;P)\underset{\smile}{\downarrow} ; (S;Q) = (R \| \breve{S})\dagger ; P\underset{\smile}{\downarrow} ; Q$ [$*165\cdot34 . *71\cdot571$]

$*165\cdot35$. $\vdash : R \upharpoonright C`P \in \mathrm{Cls} \to 1 . C`P \subset \text{Ɑ}`R . \supset . (R|) \upharpoonright C`\Sigma`P\underset{\smile}{\downarrow} ; Q \in 1 \to 1$

Dem.

$\vdash . *113\cdot118 . *165\cdot16 . \supset \vdash . s`\mathrm{D}``C`\Sigma`P\underset{\smile}{\downarrow} ; Q \subset C`P$ (1)

$\vdash . (1) . *74\cdot751 \dfrac{C`\Sigma`P\underset{\smile}{\downarrow} ; Q ,\; C`P}{\lambda , \quad \alpha} . \supset \vdash . \mathrm{Prop}$

$*165\cdot351$. $\vdash : S \upharpoonright C`Q \in \mathrm{Cls} \to 1 . C`Q \subset \text{Ɑ}`S . \supset . (|\breve{S}) \upharpoonright C`\Sigma`P\underset{\smile}{\downarrow} ; Q \in 1 \to 1$

Dem.

$\vdash . *113\cdot118 . *165\cdot16 . \supset \vdash . s`\text{Ɑ}``C`\Sigma`P\underset{\smile}{\downarrow} ; Q \subset C`Q .$

$\left[*74\cdot75 \dfrac{\breve{S}}{Q}\right]$ $\supset \vdash : \mathrm{Hp} . \supset . (|\breve{S}) \upharpoonright C`\Sigma`P\underset{\smile}{\downarrow} ; Q \in 1 \to 1 : \supset \vdash . \mathrm{Prop}$

$*165\cdot352$. $\vdash : R \upharpoonright C`P , S \upharpoonright C`Q \in \mathrm{Cls} \to 1 . C`P \subset \text{Ɑ}`R . C`Q \subset \text{Ɑ}`S . \supset .$
$(R \| \breve{S}) \upharpoonright C`\Sigma`P\underset{\smile}{\downarrow} ; Q \in 1 \to 1$

Dem.

$\vdash . *113\cdot118 . *165\cdot16 . \supset$

$\vdash : \mathrm{Hp} . \supset . s`\mathrm{D}``C`\Sigma`P\underset{\smile}{\downarrow} ; Q \subset C`P . s`\text{Ɑ}``C`\Sigma`P\underset{\smile}{\downarrow} ; Q \subset C`Q .$

[$*74\cdot773$] $\supset . (R \| \breve{S}) \upharpoonright C`\Sigma`P\underset{\smile}{\downarrow} ; Q \in 1 \to 1 : \supset \vdash . \mathrm{Prop}$

∗165·36. $\vdash : R \upharpoonright C`P' \in P \,\overline{\text{smor}}\, P' . \supset .$

$$(R|) \upharpoonright C`\Sigma`P' \downarrow ; Q \in (P \downarrow ; Q) \,\overline{\text{smor}}\,\overline{\text{smor}}\, (P' \downarrow ; Q)$$

Dem.

$\vdash . ∗151·22 . ∗165·35 . \supset \vdash : \text{Hp} . \supset . (R|) \upharpoonright C`\Sigma`P' \downarrow ; Q \in 1 \to 1 \quad (1)$

$\vdash . ∗43·3 . \qquad \supset \vdash . C`\Sigma`P' \downarrow ; Q \subset \text{Cl}`R| \quad (2)$

$\vdash . ∗151·22 . ∗165·311 . \supset \vdash : \text{Hp} . \supset . P \downarrow ; Q = (\| R)\dagger ; P' \downarrow ; Q \quad (3)$

$\vdash . (1) . (2) . (3) . ∗164·18 . \supset \vdash . \text{Prop}$

∗165·361. $\vdash : S \upharpoonright C`Q' \in Q \,\overline{\text{smor}}\, Q' . \supset .$

$$(|\breve{S}) \upharpoonright C`\Sigma`P \downarrow ; Q' \in (P \downarrow ; Q) \,\overline{\text{smor}}\,\overline{\text{smor}}\, (P \downarrow ; Q')$$

[∗165·351·331]

The proof proceeds as in ∗165·36.

∗165·362. $\vdash : R \upharpoonright C`P' \in P \,\overline{\text{smor}}\, P' . S \upharpoonright C`Q' \in Q \,\overline{\text{smor}}\, Q' . \supset .$

$$(R \| \breve{S}) \upharpoonright C`\Sigma`P' \downarrow ; Q' \in (P \downarrow ; Q) \,\overline{\text{smor}}\,\overline{\text{smor}}\, (P' \downarrow ; Q')$$

[∗165·352·341]

The above three propositions are of great utility in relation-arithmetic.

∗165·37. $\vdash : P \,\text{smor}\, P' . Q \,\text{smor}\, Q' . \supset . P \downarrow ; Q \,\text{smor}\,\text{smor}\, P' \downarrow ; Q'$

[∗165·362 . ∗164·11 . ∗151·12]

∗165·38. $\vdash :. \text{Mult ax} . \supset : R \in \text{Rel}^2\text{excl} \cap \text{Nr}`Q . C`R \subset \text{Nr}`P . \supset .$

$$R \,\text{smor}\,\text{smor}\, P \downarrow ; Q$$

Dem.

$\vdash . ∗164·48 . ∗165·27 . \supset \vdash : \text{Hp} . \dot{\exists} ! P . \supset . R \,\text{smor}\,\text{smor}\, P \downarrow ; Q \quad (1)$

$\vdash . ∗153·17 . ∗165·241 . \supset$

$\vdash : Q = \dot{\Lambda} . R \in \text{Rel}^2\text{excl} \cap \text{Nr}`Q . \supset . R = \dot{\Lambda} . P \downarrow ; Q = \dot{\Lambda} .$

$[∗164·32] \qquad \supset . R \,\text{smor}\,\text{smor}\, P \downarrow ; Q \quad (2)$

$\vdash . ∗165·242 . \supset \vdash : P = \dot{\Lambda} . \dot{\exists} ! Q . \supset . P \downarrow ; Q = \dot{\Lambda} \downarrow \dot{\Lambda} \quad (3)$

$\vdash . ∗153·17 . ∗51·4 . ∗151·32 . \supset$

$\vdash : R \in \text{Nr}`Q . C`R \subset \text{Nr}`P . P = \dot{\Lambda} . \dot{\exists} ! Q . \supset . C`R = \iota`\dot{\Lambda} .$

$[∗56·381] \qquad \supset . R = \dot{\Lambda} \downarrow \dot{\Lambda} \quad (4)$

$\vdash . (3) . (4) . ∗153·101 . ∗164·34 . \supset$

$\vdash : R \in \text{Nr}`Q . C`R \subset \text{Nr}`P . P = \dot{\Lambda} . \dot{\exists} ! Q . \supset . R \,\text{smor}\,\text{smor}\, P \downarrow ; Q \quad (5)$

$\vdash . (1) . (2) . (5) . \supset \vdash . \text{Prop}$

$*166$. THE PRODUCT OF TWO RELATIONS

Summary of $*166$.

The product $Q \times P$ is defined as $\Sigma`P \underset{\bullet}{\downarrow} ;Q$. This is a relation which has for its field all the couples that can be formed by choosing the referent in $C`P$ and the relatum in $C`Q$. These couples are arranged by $Q \times P$ on the following principle: If the relatum of the one couple has the relation Q to the relatum of the other, we put the one before the other, and if the relata of the two couples are equal while the referent of the one has the relation P to the referent of the other, we put the one before the other. Thus in advancing from any term $x \downarrow y$ in the field of $Q \times P$, we first keep y fixed and alter x into later terms as long as possible; then we alter y into a later term, move x back to the beginning, and so on. Thus with a given y, we get a series which is like P, and this series is wholly followed or wholly preceded by the series with the referent y', where y' follows or precedes y.

The propositions of this number are for the most part immediate consequences of those of $*165$. The most important of them are:

$*166{\cdot}12$. $\vdash . C`(P \times Q) = C`P \times C`Q$

$*166{\cdot}13$. $\vdash :. P \times Q = \dot{\Lambda} . \equiv : P = \dot{\Lambda} . \vee . Q = \dot{\Lambda}$

Hence it follows that an ordinal product of a finite number of factors vanishes when, and only when, one of its factors vanishes.

$*166{\cdot}16$. $\vdash . \overrightarrow{B}`(P \times Q) = \overrightarrow{B}`P \times \overrightarrow{B}`Q . \overrightarrow{B}`\mathrm{Cnv}`(P \times Q) = \overrightarrow{B}`\breve{P} \times \overrightarrow{B}`\breve{Q}$

$*166{\cdot}23$. $\vdash : P \,\mathrm{smor}\, P' . Q \,\mathrm{smor}\, Q' . \supset . Q \times P \,\mathrm{smor}\, Q' \times P'$

This proposition shows that the relation-number of a product $Q \times P$ depends only upon the relation-numbers of its factors.

$*166{\cdot}24$. $\vdash :. \mathrm{Mult\,ax} . \supset : R \in \mathrm{Rel}^2\,\mathrm{excl} \cap \mathrm{Nr}`Q . C`R \subset \mathrm{Nr}`P . \supset .$
$$\Sigma`R \,\mathrm{smor}\, Q \times P$$

This proposition connects addition and multiplication (cf. note to $*166{\cdot}24$, below).

$*166{\cdot}42$. $\vdash . (P \times Q) \times R \,\mathrm{smor}\, P \times (Q \times R)$

This is the associative law. The distributive law has two forms:

$*166{\cdot}44$. $\vdash . \Sigma` \times P ;Q = (\Sigma`Q) \times P$

$*166{\cdot}45$. $\vdash . (Q \dotplus R) \times P = (Q \times P) \dotplus (R \times P)$

We do not have in general (cf. note before $*166{\cdot}44$, below)
$$P \times (Q \dotplus R) = (P \times Q) \dotplus (P \times R).$$

We have also a distributive law for the addition of a single term, *i.e.*

$*166{\cdot}53.\ \vdash : \dot{\exists} ! Q . \supset . (Q \dashv\!\!\rightarrow y) \times P = (Q \times P) \ddagger (P \underset{\bullet}{\downarrow} y)$

$*166{\cdot}531.\ \vdash : \dot{\exists} ! Q . \supset . (y \leftarrow\!\!\vdash Q) \times P = (P \underset{\bullet}{\downarrow} y) \ddagger (Q \times P)$

Here again the law does not hold in general for $P \times (Q \dashv\!\!\rightarrow y)$ or $P \times (y \leftarrow\!\!\vdash Q)$.

$*166{\cdot}01.\ Q \times P = \Sigma\text{`}P \underset{\bullet}{\downarrow} ;Q$ Df

$*166{\cdot}1.\ \vdash . Q \times P = \Sigma\text{`}P \underset{\bullet}{\downarrow} ;Q$ [(*166·01)]

$*166{\cdot}11.\ \vdash :. M (Q \times P) N . \equiv : (\exists x, y, z, w) : x, y \in C\text{`}P . z, w \in C\text{`}Q : zQw . \vee .$
$z = w . xPy : M = x \downarrow z . N = y \downarrow w$ [*165·17 . *166·1]

$*166{\cdot}111.\ \vdash :. M (P \times Q) N . \equiv : (\exists x, y, z, w) : x, y \in C\text{`}P . z, w \in C\text{`}Q : xPy . \vee .$
$x = y . zQw : M = z \downarrow x . N = w \downarrow y$ [*165·17 . *166·1]

$*166{\cdot}112.\ \vdash :. (x \downarrow z) (Q \times P) (y \downarrow w) . \equiv : x, y \in C\text{`}P . z, w \in C\text{`}Q : zQw . \vee .$
$z = w . xPy$ [*166·11 . *55·202 . *13·22]

$*166{\cdot}113.\ \vdash :: x, y \in C\text{`}P . z, w \in C\text{`}Q . \supset :.$
$(x \downarrow z) (Q \times P) (y \downarrow w) . \equiv : zQw . \vee . z = w . xPy$ [*166·112]

$*166{\cdot}12.\ \vdash . C\text{`}(P \times Q) = C\text{`}P \times C\text{`}Q$ [*165·16 . *166·1]

$*166{\cdot}13.\ \vdash :. P \times Q = \dot{\Lambda} . \equiv : P = \dot{\Lambda} . \vee . Q = \dot{\Lambda}$ [*166·12 . *113·114 . *33·241]

$*166{\cdot}14.\ \vdash : \dot{\exists} ! P \times Q . \equiv . \dot{\exists} ! P . \dot{\exists} ! Q$ [*166·13]

$*166{\cdot}15.\ \vdash . \mathrm{Cnv}\text{`}(P \times Q) = \breve{P} \times \breve{Q}$ [*165·19 . *162·2]

$*166{\cdot}16.\ \vdash . \overrightarrow{B}\text{`}(P \times Q) = \overrightarrow{B}\text{`}P \times \overrightarrow{B}\text{`}Q . \overrightarrow{B}\text{`}\mathrm{Cnv}\text{`}(P \times Q) = \overrightarrow{B}\text{`}\breve{P} \times \overrightarrow{B}\text{`}\breve{Q}$

Dem.

$\vdash . *166{\cdot}111 . *93{\cdot}103 . \supset$

$\vdash :. M \in \overrightarrow{B}\text{`}(P \times Q) . \equiv : (\exists x, z) : x \in C\text{`}P . z \in C\text{`}Q . M = z \downarrow x :$
$\sim(\exists y) . yPx : \sim(\exists w) . wQz :$

[*93·103] $\equiv : (\exists x, z) . x \in \overrightarrow{B}\text{`}P . y \in \overrightarrow{B}\text{`}Q . M = z \downarrow x :$

[*113·101] $\equiv : M \in \overrightarrow{B}\text{`}P \times \overrightarrow{B}\text{`}Q$ (1)

$\vdash . (1) . *166{\cdot}15 . \supset \vdash . \overrightarrow{B}\text{`}\mathrm{Cnv}\text{`}(P \times Q) = \overrightarrow{B}\text{`}\breve{P} \times \overrightarrow{B}\text{`}\breve{Q}$ (2)

$\vdash . (1) . (2) . \supset \vdash .$ Prop

The above proposition is used in the ordinal theory of progressions (*263·62·65).

$*166{\cdot}2.\ \vdash : R \upharpoonright C\text{`}P' \in P \,\overline{\mathrm{smor}}\, P' . \supset . (R \| ) \upharpoonright C\text{`}(Q \times P') \in (Q \times P) \,\overline{\mathrm{smor}}\, (Q \times P')$
[*165·36 . *166·1 . *164·14]

$*166\cdot21.\ \vdash : S \upharpoonright C`Q' \in Q \,\overline{\text{smor}}\, Q' . \supset . (|\breve{S}) \upharpoonright C`(Q' \times P) \in (Q \times P) \,\overline{\text{smor}}\, (Q' \times P)$
[$*165\cdot361 . *166\cdot1 . *164\cdot14$]

$*166\cdot22.\ \vdash : R \upharpoonright C`P' \in P \,\overline{\text{smor}}\, P' . S \upharpoonright C`Q' \in Q \,\overline{\text{smor}}\, Q' . \supset .$
$(R \,\|\, \breve{S}) \upharpoonright C`(Q' \times P') \in (Q \times P) \,\overline{\text{smor}}\, (Q' \times P')$
[$*165\cdot362 . *166\cdot1 . *164\cdot14$]

This proposition gives the correlator for the product when correlators are given for the factors.

$*166\cdot23.\ \vdash : P \text{ smor } P' . Q \text{ smor } Q' . \supset . Q \times P \text{ smor } Q' \times P'$
[$*166\cdot22 . *151\cdot12$]

This proposition enables us to use $Q \times P$ to define the product of the relation-numbers of Q and P, for it shows that the relation-number of $Q \times P$ is determinate when the relation-numbers of Q and P are given. We shall therefore (in Section D of this part) define the product of two relation-numbers ν and μ as the relation-number of $Q \times P$ when $N_0r`Q = \nu$ and $N_0r`P = \mu$.

$*166\cdot24.\ \vdash :. \text{Mult ax} . \supset : R \in \text{Rel}^2 \text{ excl} \cap \text{Nr}`Q . C`R \subset \text{Nr}`P . \supset .$
$\Sigma`R \text{ smor } Q \times P$ [$*165\cdot38 . *164\cdot151 . *166\cdot1$]

This proposition exhibits the connection of addition and multiplication. If we put $\text{Nr}`P = \mu$ and $\text{Nr}`Q = \nu$, then $\Sigma`R$ in the above proposition is the sum of $\nu$ relations of which each is a $\mu$. In virtue of the above proposition, it follows that (if the multiplicative axiom is assumed) $\text{Nr}`\Sigma`R = \nu \times \mu$. In other words, assuming the multiplicative axiom, the sum of ν series (or other relations), each of which has μ terms, has $\nu \times \mu$ terms.

$*166\cdot3.\ \vdash : \exists ! C`(P \times Q) \cap C`(P' \times Q') . \equiv . \exists ! C`P \cap C`P' . \exists ! C`Q \cap C`Q'$
[$*166\cdot12 . *113\cdot19$]

The analogous proposition
$\dot{\exists} ! (P \times Q) \mathbin{\dot{\cap}} (P' \times Q') . \equiv :$
$\dot{\exists} ! (P \mathbin{\dot{\cap}} P') . \exists ! C`Q \cap C`Q' . \lor . \dot{\exists} ! (Q \mathbin{\dot{\cap}} Q') . \exists ! C`P \cap C`P'$
is only true in general if $P \mathrel{\dot{\subset}} J . P' \mathrel{\dot{\subset}} J$

$*166\cdot31.\ \vdash . s`C`(Q \times P) = C`P \uparrow C`Q$ [$*113\cdot115 . *166\cdot12$]

$*166\cdot311.\ \vdash : \dot{\exists} ! Q . \supset . s`D``C`(Q \times P) = C`P : \dot{\exists} ! P . \supset . s`\text{Ǝ}``C`(Q \times P) = C`Q$
[$*113\cdot116 . *166\cdot12 . *33\cdot24$]

$*166\cdot312.\ \vdash . s`D``C`(Q \times P) \subset C`P . s`\text{Ǝ}``C`(Q \times P) \subset C`Q$
[$*113\cdot118 . *166\cdot12$]

The following propositions are lemmas for the associative law ($*166\cdot42$).

$*166\cdot4.\ \vdash :. M \{(P \times Q) \times R\} M' . \equiv : (\exists x, y, z, x', y', z') :$
$x, x' \in C`P . y, y' \in C`Q . z, z' \in C`R :$
$xPx' . \lor . x = x' . yQy' . \lor . x = x' . y = y' . zRz' :$
$M = z \downarrow (y \downarrow x) . M' = z' \downarrow (y' \downarrow x')$

Dem.

$\vdash . *116\cdot111 . \supset \vdash :. M\{(P \times Q) \times R\} M' . \equiv : (\exists N, N', z, z') :$
$N, N' \in C`(P \times Q) . z, z' \in C`R : N(P \times Q) N' . \vee . N = N' . zRz' :$
$M = z \downarrow N . M' = z' \downarrow N' :$

$[*116\cdot12 . *113\cdot101] \equiv : (\exists N, N', x, x', y, y', z, z') :$
$x, x' \in C`P . y, y' \in C`Q . z, z' \in C`R . N = y \downarrow x . N' = y' \downarrow x' :$
$N(P \times Q) N' . \vee . N = N' . zRz' : M = z \downarrow N . M' = z' \downarrow N' :$

$[*13\cdot22 . *116\cdot113] \equiv : (\exists x, x', y, y', z, z') : x, x' \in C`P . y, y' \in C`Q . z, z' \in C`R :$
$xPx' . \vee . x = x' . yQy' . \vee . y \downarrow x = y' \downarrow x' . zRz' :$
$M = z \downarrow (y \downarrow x) . M' = z' \downarrow (y' \downarrow x') :$

$[*55\cdot202] \quad \equiv : (\exists x, x', y, y', z, z') : x, x' \in C`P . y, y' \in C`Q . z, z' \in C`R :$
$xPx' . \vee . x = x' . yQy' . \vee . x = x' . y = y' . zRz' :$
$M = z \downarrow (y \downarrow x) . M' = z' \downarrow (y' \downarrow x') :. \supset \vdash . \text{Prop}$

$*166\cdot401$. $\vdash :. N\{P \times (Q \times R)\} N' . \equiv : (\exists x, x', y, y', z, z') :$
$x, x' \in C`P . y, y' \in C`Q . z, z' \in C`R :$
$xPx' . \vee . x = x' . yQy' . \vee . x = x' . y = y' . zRz' :$
$N = (z \downarrow y) \downarrow x . N' = (z' \downarrow y') \downarrow x'$

[Proof as in $*166\cdot4$]

$*166\cdot41$. $\vdash : T = \hat{M}\hat{N}\{(\exists x, y, z) . x \in C`P . y \in C`Q . z \in C`R . M = z \downarrow (y \downarrow x) .$
$N = (z \downarrow y) \downarrow x\} . \supset . T \in \{(P \times Q) \times R\} \overline{\text{smor}} \{P \times (Q \times R)\}$

Dem.

$\vdash . *21\cdot33 . \supset \vdash :: \text{Hp} . \supset :. MTN . M'TN . \supset :$
$(\exists x, x', y, y', z, z') : x, x' \in C`P . y, y' \in C`Q . z, z' \in C`R :$
$M = z \downarrow (y \downarrow x) . M' = z' \downarrow (y' \downarrow x') :$
$N = (z \downarrow y) \downarrow x . N = (z' \downarrow y') \downarrow x' :$

$[*55\cdot202] \quad \supset : (\exists x, x', y, y', z, z') . M = z \downarrow (y \downarrow x) . M' = z' \downarrow (y' \downarrow x') .$
$x = x' . y = y' . z = z' :$

$[*13\cdot22] \quad \supset : M = M' \quad (1)$

Similarly $\vdash :. \text{Hp} . \supset : MTN . MTN' . \supset . N = N' \quad (2)$

$\vdash . (1) . (2) . \quad \supset \vdash : \text{Hp} . \supset . T \in 1 \rightarrow 1 \quad (3)$

$\vdash . *21\cdot33 . *13\cdot19 . \supset \vdash : \text{Hp} . \supset .$
$\text{Cl}`T = \hat{N}\{(\exists x, y, z) . x \in C`P . y \in C`Q . z \in C`R . N = (z \downarrow y) \downarrow x\}$

$[*113\cdot101] \quad = C`P \times (C`Q \times C`R)$

$[*166\cdot12] \quad = C`\{P \times (Q \times R)\} \quad (4)$

$\vdash . *166\cdot401 . \supset \vdash :: \text{Hp} . \supset :. M\{T^{;}(P \times (Q \times R))\} M' . \equiv : (\exists x, x', y, y', z, z', N, N') :$
$x, x' \in C`P . y, y' \in C`Q . z, z' \in C`R . N = (z \downarrow y) \downarrow x . N' = (z' \downarrow y') \downarrow x' .$
$M = z \downarrow (y \downarrow x) . M' = z' \downarrow (y' \downarrow x') :$
$xPx' . \vee . x = x' . yQy' . \vee . x = x' . y = y' . zRz' :$

$[*13\cdot19 . *166\cdot4] \equiv : M\{(P \times Q) \times R\} M' \quad (5)$

$\vdash . (3) . (4) . (5) . *151\cdot11 . \supset \vdash . \text{Prop}$

***166·42.** $\vdash . (P \times Q) \times R \operatorname{smor} P \times (Q \times R)$ [*166·41]

This is the associative law for the kind of multiplication concerned in this number.

***166·421.** $P \times Q \times R = (P \times Q) \times R$ Df

This definition serves merely for the avoidance of brackets.

The two following propositions give the distributive law. In relation-arithmetic, this is in general only true in one of its two forms, *i.e.* we have

$$(Q \dotplus R) \times P = (Q \times P) \dotplus (R \times P),$$

but not $$P \times (Q \dotplus R) = (P \times Q) \dotplus (P \times R).$$

The latter is true for finite series, but not for infinite series or (except in exceptional cases) for relations which are not serial.

***166·44.** $\vdash . \Sigma\text{‘} \times P ; Q = (\Sigma\text{‘}Q) \times P$

Dem.

$$\begin{aligned}
&\vdash . *166{\cdot}1 . *38{\cdot}11 . *150{\cdot}1 . \supset \vdash . \Sigma\text{‘} \times P ; Q = \Sigma\text{‘}\Sigma ; (P \underset{;}{\downarrow}) \dagger ; Q \\
&[*162{\cdot}34{\cdot}35] \quad = \Sigma\text{‘}P \underset{;}{\downarrow} ; \Sigma\text{‘}Q \\
&[*166{\cdot}1] \quad = (\Sigma\text{‘}Q) \times P . \supset \vdash . \text{Prop}
\end{aligned}$$

***166·45.** $\vdash . (Q \dotplus R) \times P = (Q \times P) \dotplus (R \times P)$

Dem.

$$\begin{aligned}
&\vdash . *166{\cdot}1 . \supset \vdash . (Q \times P) \dotplus (R \times P) = \Sigma\text{‘}P \underset{;}{\downarrow} ; Q \dotplus \Sigma\text{‘}P \underset{;}{\downarrow} ; R \\
&[*162{\cdot}31] \quad = \Sigma\text{‘}(P \underset{;}{\downarrow} ; Q \dotplus P \underset{;}{\downarrow} ; R) \\
&[*162{\cdot}36] \quad = \Sigma\text{‘}P \underset{;}{\downarrow} ; (Q \dotplus R) \\
&[*166{\cdot}1] \quad = (Q \dotplus R) \times P . \supset \vdash . \text{Prop}
\end{aligned}$$

The following propositions (*166·46—·472) exhibit the failure of the distributive law in the form $P \times (Q \dotplus R) = (P \times Q) \dotplus (P \times R)$, and give certain results for special cases. They are not referred to except in this number.

***166·46.** $\vdash . (P \cup Q) \underset{;}{\downarrow} z = P \underset{;}{\downarrow} z \cup Q \underset{;}{\downarrow} z$ [*165·01 . *150·3]

***166·461.** $\vdash . \dot{s}\text{‘}C\text{‘‘}(P \cup Q) \underset{;}{\downarrow} ; R = \dot{s}\text{‘}C\text{‘}P \underset{;}{\downarrow} ; R \cup \dot{s}\text{‘}C\text{‘}Q \underset{;}{\downarrow} ; R$
[*41·6 . *165·12 . *166·46]

***166·462.** $\vdash . F ; (P \cup Q) \underset{;}{\downarrow} ; R = F ; P \underset{;}{\downarrow} ; R \cup F ; Q \underset{;}{\downarrow} ; R \cup \hat{M}\hat{N} \{(\exists x, y, z, w) :$
$zRw : x \in C\text{‘}P . y \in C\text{‘}Q . \mathbf{v} . x \in C\text{‘}Q . y \in C\text{‘}P : M = x \downarrow z . N = y \downarrow w\}$

Dem.

$\vdash . *165{\cdot}161 . \supset \vdash . F ; (P \cup Q) \underset{;}{\downarrow} ; R$

$$\begin{aligned}
&= \hat{M}\hat{N} \{(\exists x, y, z, w) . x, y \in C\text{‘}P \cup C\text{‘}Q . zRw . M = x \downarrow z . N = y \downarrow w\} \\
[*22{\cdot}34] \quad &= \hat{M}\hat{N} \{(\exists x, y, z, w) : x, y \in C\text{‘}P . \mathbf{v} . x, y \in C\text{‘}Q . \mathbf{v} . \\
&\quad x \in C\text{‘}P . y \in C\text{‘}Q . \mathbf{v} . x \in C\text{‘}Q . y \in C\text{‘}P : zRw . M = x \downarrow z . N = y \downarrow w\}
\end{aligned}$$

[∗11·41.∗165·161] $= F;P\underset{;}{\downarrow};R \cup F;Q\underset{;}{\downarrow};R \cup \hat{M}\hat{N}\{(\exists x, y, z, w):$
$x \epsilon C'P \,.\, y \epsilon C'Q \,.\lor.\, x \epsilon C'Q \,.\, y \epsilon C'P :$
$zRw \,.\, M = x \downarrow z \,.\, N = y \downarrow w\} \,.\supset \vdash .$ Prop

∗166·463. $\vdash : C'P \subset C'Q \,.\supset.\, F;P\underset{;}{\downarrow};R \mathrel{G} F;Q\underset{;}{\downarrow};R$ [∗165·161]

∗166·464. $\vdash : C'P \subset C'Q \,.\supset.\, F;(P \cup Q)\underset{;}{\downarrow};R = F;Q\underset{;}{\downarrow};R = F;P\underset{;}{\downarrow};R \cup F;Q\underset{;}{\downarrow};R$

Dem.

$\vdash .$ ∗166·463 $.\supset \vdash : \mathrm{Hp} \,.\supset.\, F;P\underset{;}{\downarrow};R \mathrel{G} F;Q\underset{;}{\downarrow};R$ (1)

$\vdash .$ ∗33·262 $.\supset \vdash : \mathrm{Hp} \,.\supset.\, C'(P \cup Q) = C'Q \,.$

[∗166·463] $\supset.\, F;(P \cup Q)\underset{;}{\downarrow};R = F;Q\underset{;}{\downarrow};R$ (2)

$\vdash . (1) . (2) . \supset \vdash .$ Prop

∗166·47. $\vdash . R \times (P \cup Q) = (R \times P) \cup (R \times Q) \cup \hat{M}\hat{N}\{(\exists x, y, z, w):$
$x \epsilon C'P \,.\, y \epsilon C'Q \,.\lor.\, x \epsilon C'Q \,.\, y \epsilon C'P : zRw \,.\, M = x \downarrow z \,.\, N = y \downarrow w\}$
[∗166·461·462·1 . ∗162·1]

∗166·471. $\vdash : C'P \subset C'Q \,.\supset.\, R \times (P \cup Q) = (R \times P) \cup (R \times Q)$
[∗166·461·464]

∗166·472. $\vdash . R \times (P \mathrel{\updownarrow} Q) = (R \times P) \cup (R \times Q) \cup R \times (C'P \downarrow C'Q)$

Dem.

$\vdash .$ ∗166·471 . ∗35·85 $. \supset$

$\vdash :. \dot{\exists} ! Q \,.\supset: R \times (P \mathrel{\updownarrow} Q) = (R \times P) \cup R \times \{Q \cup (C'P \uparrow C'Q)\} :$

[∗166·471.∗35·86]

$\supset : \dot{\exists} ! P \,.\supset.\, R \times (P \mathrel{\updownarrow} Q) = (R \times P) \cup (R \times Q) \cup R \times (C'P \uparrow C'Q)$ (1)

$\vdash .$ ∗160·21 . ∗166·13 $. \supset$

$\vdash : Q = \dot{\Lambda} \,.\supset.\, P \mathrel{\updownarrow} Q = P \,.\, R \times Q = \dot{\Lambda} \,.\, R \times (C'P \uparrow C'Q) = \dot{\Lambda} \,.$

[∗25·24] $\supset.\, R \times (P \mathrel{\updownarrow} Q) = (R \times P) \cup (R \times Q) \cup R \times (C'P \downarrow C'Q)$ (2)

Similarly

$\vdash : P = \dot{\Lambda} \,.\supset.\, R \times (P \mathrel{\updownarrow} Q) = (R \times P) \cup (R \times Q) \cup R \times (C'P \uparrow C'Q)$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash .$ Prop

The following propositions are concerned with the distributive law for the addition of a single term to a relation. This law, in the form in which it holds, is given in ∗166·53·531 (remembering $\mathrm{Nr}'P \underset{;}{\downarrow} y = \mathrm{Nr}'P$). ∗166·54·541 exhibit the failure of the other form.

∗166·5. $\vdash . (Q \cup R) \times P = (Q \times P) \cup (R \times P)$

Dem.

$\vdash .$ ∗166·1 $. \supset \vdash . (Q \cup R) \times P = \Sigma'P\underset{;}{\downarrow};(Q \cup R)$

[∗162·27] $= \Sigma'P\underset{;}{\downarrow};Q \cup \Sigma'P\underset{;}{\downarrow};R$

[∗166·1] $= (Q \times P) \cup (R \times P) \,.\supset \vdash .$ Prop

***166·51.** $\vdash . (Q \nrightarrow y) \times P = (Q \times P) \mathbin{\dot{\cup}} (C`Q \uparrow \iota`y) \times P$ [*166·5 . *161·1]

***166·511.** $\vdash . (y \nleftarrow Q) \times P = (\iota`y \uparrow C`P) \times P \mathbin{\dot{\cup}} (Q \times P)$

***166·52.** $\vdash . P \underset{;}{\downarrow} {}^{;}(Q \nrightarrow y) = P \underset{;}{\downarrow} {}^{;} Q \nrightarrow P \underset{;}{\downarrow} y$ [*161·4 . *165·2]

***166·521.** $\vdash . P \underset{;}{\downarrow} {}^{;}(y \nleftarrow Q) = P \underset{;}{\downarrow} y \nleftarrow P \underset{;}{\downarrow} {}^{;} Q$

***166·53.** $\vdash : \dot{\exists} ! Q . \supset . (Q \nrightarrow y) \times P = (Q \times P) \ddagger (P \underset{;}{\downarrow} y)$

Dem.

$\vdash . *162·43 . *165·243 . \supset \vdash : \mathrm{Hp} . \supset . \Sigma`(P \underset{;}{\downarrow} {}^{;} Q \nrightarrow P \underset{;}{\downarrow} y) = \Sigma`P \underset{;}{\downarrow} {}^{;} Q \ddagger P \underset{;}{\downarrow} y .$

[*166·52] $\supset . \Sigma`P \underset{;}{\downarrow} {}^{;}(Q \nrightarrow y) = \Sigma`P \underset{;}{\downarrow} {}^{;} Q \ddagger P \underset{;}{\downarrow} y$ (1)

$\vdash . (1) . *166·1 . \supset \vdash . \mathrm{Prop}$

***166·531.** $\vdash : \dot{\exists} ! Q . \supset . (y \nleftarrow Q) \times P = (P \underset{;}{\downarrow} y) \ddagger (Q \times P)$

***166·54.** $\vdash . Q \times (P \nrightarrow x) = (Q \times P) \mathbin{\dot{\cup}} Q \times (C`P \uparrow \iota`x)$

Dem.

$\vdash . *161·1 . \supset \vdash . Q \times (P \nrightarrow x) = Q \times \{P \mathbin{\dot{\cup}} (C`P \uparrow \iota`x)\}$

[*35·85.*166·471] $= (Q \times P) \mathbin{\dot{\cup}} Q \times (C`P \uparrow \iota`x) . \supset \vdash . \mathrm{Prop}$

***166·541.** $\vdash . Q \times (x \nleftarrow P) = Q \times (\iota`x \uparrow C`P) \mathbin{\dot{\cup}} (Q \times P)$

SECTION C

THE PRINCIPLE OF FIRST DIFFERENCES, AND THE MULTIPLICATION AND EXPONENTIATION OF RELATIONS

Summary of Section C.

In the present section, we have to consider various forms of a principle which is of the utmost utility in relation-arithmetic. This principle may be called "the principle of first differences." It has been explained and used by Hausdorff in brilliant articles*. The results there obtained by its use give some measure of its importance in relation-arithmetic. It has, however, other uses besides those that are concerned with the multiplication and exponentiation of relation-numbers, as, for example, in the ordering of segments and stretches in a series, or of any other set of classes which are contained in the field of a given relation. In the present section, after the first two numbers, we shall be concerned with its arithmetical uses, but other uses will occur later.

The principle of first differences has various forms which, though analogous, cannot, in the general case, be reduced to one common genus. The simplest of these is the relation P_{cl}, by which the sub-classes of $C‘P$ are ordered. This is defined as follows. If α and β are both contained in $C‘P$, we say that $\alpha P_{\text{cl}} \beta$ if there are terms belonging to α but not to β such that no terms belonging to β and not to α precede them; *i.e.* if, after taking away the terms (if any) which are common to α and β, there are terms left in α which do not come after any of the terms left in β, *i.e.* if $\exists ! \alpha - \beta - \breve{P}‘‘(\beta - \alpha)$. Thus the definition is

$$P_{\text{cl}} = \hat{\alpha}\hat{\beta}\{\alpha, \beta \in \text{Cl}‘C‘P \,.\, \exists ! \alpha - \beta - \breve{P}‘‘(\beta - \alpha)\} \quad \text{Df.}$$

It will be seen that this relation holds if $\beta \subset \alpha \,.\, \beta \neq \alpha$. Thus it holds between any existent member of $\text{Cl}‘C‘P$ and Λ, and between $C‘P$ and any member of $\text{Cl}‘C‘P$ other than $C‘P$ itself. When P is a serial relation (which is the important case for all the relations in this section), P_{cl} is transitive ($P^2_{\text{cl}} \in P_{\text{cl}}$) and asymmetrical ($P_{\text{cl}} \dot{\cap} \breve{P}_{\text{cl}} = \dot{\Lambda}$), but not necessarily *connected*, *i.e.* there may be two members of its field of which neither has the relation P_{cl} to the other. This happens whenever P is not well-ordered; but when P is well-ordered, P_{cl} is connected, and therefore generates a series.

* "Untersuchungen über Ordnungstypen," *Berichte der mathematisch-physischen Klasse der Königlich Sächsischen Gesellschaft der Wissenschaften zu Leipzig*, Feb. 1906 and Feb. 1907. Cf. also his "Grundzüge einer Theorie der geordneten Mengen," *Math. Annalen*, 65 (1908).

To illustrate the order generated by P_{cl} in a simple case, consider a series of three terms, x, y, z. Let us for the moment write $(x \downarrow y \downarrow z)$ for the relation

$$x \downarrow y \cup x \downarrow z \cup y \downarrow z, \quad i.e. \ (x \downarrow y) \rightarrow\!\!\!\!\!+ z,$$

and similarly we will write $(x \downarrow y \downarrow z \downarrow w)$ for $x \downarrow y \downarrow z \rightarrow\!\!\!\!\!+ w$, and so on. Then assuming $x \neq y \,.\, x \neq z \,.\, y \neq z$,

$$(x \downarrow y \downarrow z)_{\text{cl}} = (\iota\text{`}x \cup \iota\text{`}y \cup \iota\text{`}z) \downarrow (\iota\text{`}x \cup \iota\text{`}y) \\ \downarrow (\iota\text{`}x \cup \iota\text{`}z) \downarrow \iota\text{`}x \downarrow (\iota\text{`}y \cup \iota\text{`}z) \downarrow \iota\text{`}y \downarrow \iota\text{`}z \downarrow \Lambda.$$

In this series, a class containing x is always earlier than one not containing x; and of two classes of which both or neither contain x, one containing y is earlier than one not containing y; and of two classes of which both or neither contain x, and both or neither contain y, one containing z is earlier than one not containing z. Thus our relation may be generated as follows: Begin with $(\iota\text{`}z) \downarrow \Lambda$, which is $(z \downarrow z)_{\text{cl}}$. Add before these terms what results from adding $\iota\text{`}y$ to each; then we have $(y \downarrow z)_{\text{cl}}$, which is

$$(\iota\text{`}y \cup \iota\text{`}z) \downarrow \iota\text{`}y \downarrow \iota\text{`}z \downarrow \Lambda.$$

Now add at the beginning what results from adding $\iota\text{`}x$ to each of the above four classes, and we have $(x \downarrow y \downarrow z)_{\text{cl}}$. Thus generally, if $x \sim \epsilon\, C\text{`}P$,

$$(x \leftarrow\!\!\!\!\!+ P)_{\text{cl}} = (\iota\text{`}x \cup)^{;} P_{\text{cl}} \ \ddagger\ P_{\text{cl}}.$$

Thus by adding one term to P, we double the number of terms in P_{cl}.

Again, if P and Q are two relations which have no common terms in their fields, we shall have

$$\alpha P_{\text{cl}} \beta \,.\, \gamma, \delta \,\epsilon\, \text{Cl}\text{`}C\text{`}Q \,.\supset.\, (\alpha \cup \gamma)(P \ddagger Q)_{\text{cl}} (\beta \cup \delta)$$

and $$\alpha \,\epsilon\, \text{Cl}\text{`}C\text{`}P \,.\, \gamma Q_{\text{cl}} \delta \,.\supset.\, (\alpha \cup \gamma)(P \ddagger Q)_{\text{cl}} (\alpha \cup \delta),$$

while conversely

$$\alpha, \beta \,\epsilon\, \text{Cl}\text{`}C\text{`}P \,.\, \gamma, \delta \,\epsilon\, \text{Cl}\text{`}C\text{`}Q \,.\, (\alpha \cup \gamma)(P \ddagger Q)_{\text{cl}} (\beta \cup \delta) \,.\supset:\\ \alpha P_{\text{cl}} \beta \,.\vee.\, \alpha = \beta \,.\, \gamma Q_{\text{cl}} \delta.$$

Hence $$(\alpha \cup \gamma)(P \ddagger Q)_{\text{cl}} (\beta \cup \delta) \,.\equiv.\, (\gamma \downarrow \alpha)(P_{\text{cl}} \times Q_{\text{cl}})(\delta \downarrow \beta) \\ \equiv .\, (\alpha \cup \gamma) \{s^{;} C^{;} (P_{\text{cl}} \times Q_{\text{cl}})\} (\beta \cup \delta),$$

so that $$\text{Nr}\text{`}(P \ddagger Q)_{\text{cl}} = \text{Nr}\text{`}P_{\text{cl}} \times \text{Nr}\text{`}Q_{\text{cl}}.$$

These propositions illustrate the connection of P_{cl} with multiplication.

Besides P_{cl}, we often require (though not in this Part) the relation which is the converse of $(\breve{P})_{\text{cl}}$. This relation we call P_{lc}, so that

$$P_{\text{lc}} = \text{Cnv}\text{`}(\breve{P})_{\text{cl}} \quad \text{Df.}$$

This begins with Λ, and ends with $C\text{`}P$.

Thus we shall have, for example,

$$(x \downarrow y \downarrow z)_{\mathrm{lc}} = \Lambda \downarrow \iota\text{‘}x \downarrow \iota\text{‘}y \downarrow (\iota\text{‘}x \cup \iota\text{‘}y)$$
$$\downarrow \iota\text{‘}z \downarrow (\iota\text{‘}x \cup \iota\text{‘}z) \downarrow (\iota\text{‘}y \cup \iota\text{‘}z) \downarrow (\iota\text{‘}x \cup \iota\text{‘}y \cup \iota\text{‘}z).$$

Here, if we start from $\Lambda \downarrow \iota\text{‘}x$, which is $(x \downarrow x)_{\mathrm{lc}}$, the series grows by adding terms at the end: we add $\iota\text{‘}y$ to each member of $\Lambda \downarrow \iota\text{‘}x$, and put the resulting terms $\iota\text{‘}y$, $\iota\text{‘}x \cup \iota\text{‘}y$ after Λ and $\iota\text{‘}x$; we then add $\iota\text{‘}z$ to each of the four terms we already have, and add the resulting terms at the end; and so we can proceed indefinitely.

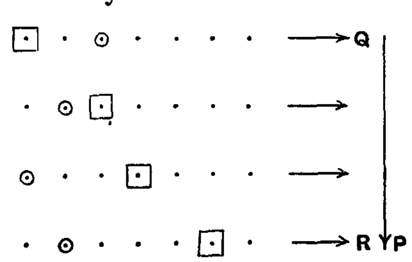
The relation P_{lc} with its field limited arranges the *segments* of P in ascending order of magnitude; if the class of segments is σ, $P_{\mathrm{lc}} \upharpoonright \sigma$ generates what may be called the natural order among the segments (cf. $*212$).

A variant of P_{cl} is afforded by the relation P_{df} ($*171$), which is to hold between two members α, β of $\mathrm{Cl}\text{‘}C\text{‘}P$ when the first term of either which does not belong to both belongs to α, *i.e.* the "first difference" belongs to α. This relation implies P_{cl}, and coincides with it if P is well-ordered; but when P is not well-ordered, P_{cl} may hold between two classes which have no *first* point of difference, *e.g.* (if P is "less than" among rationals) if α consists of rationals between 0 and 1 (both excluded) and β of rationals between 1 and 2 (both excluded). The definition of P_{df} is

$$P_{\mathrm{df}} = \hat{\alpha}\hat{\beta}\,\{\alpha, \beta \in \mathrm{Cl}\text{‘}C\text{‘}P : (\exists z) \,.\, z \in \alpha - \beta \,.\, \overrightarrow{P}\text{‘}z \cap \alpha - \iota\text{‘}z = \overrightarrow{P}\text{‘}z \cap \beta\} \quad \mathrm{Df.}$$

The relation P_{df} has the interesting property that its relation-number is found by raising 2_r to the power $\mathrm{Nr}\text{‘}P$ (cf. $*177$). As the field of P_{df} is $\mathrm{Cl}\text{‘}C\text{‘}P$, this theorem is the ordinal analogue of $\mathrm{Nc}\text{‘}\mathrm{Cl}\text{‘}\alpha = 2^{\mathrm{Nc}\text{‘}\alpha}$ ($*116{\cdot}72$).

A somewhat more complicated form of the relation of first differences arises when we have a series of series. Let us suppose, to begin with, that P is a serial relation whose field consists of mutually exclusive serial relations. Thus in the accompanying figure, each row represents a series, the generating relations of these series being $Q, \ldots R, \ldots$. But the series themselves form a series, which may be regarded as generated by a relation P whose field consists of the relations $Q, \ldots R, \ldots$. (It might be thought more natural to take $C\text{‘}Q$, $C\text{‘}R, \ldots$ as the field of P; but this would lead to confusion in the case when two or more of the series have the same field.) Suppose we now wish to find a relation which will order the multiplicative class of the fields of $Q, R, \ldots$, *i.e.* the class $\mathrm{Prod}\text{‘}C\text{“}C\text{‘}P$. In the case illustrated in the figure, in which P generates a well-ordered series, and all the members of $C\text{‘}P$ are serial, and $P \in \mathrm{Rel}^2\,\mathrm{excl}$, we might use $(\Sigma\text{‘}P)_{\mathrm{cl}}$; this relation, with its field limited to $\mathrm{Prod}\text{‘}C\text{“}C\text{‘}P$, will then give us what we want. This relation will, in the case supposed, put



a selected class μ before another selected class ν if, where they first differ, μ chooses an earlier term than ν. But if the series P is not well-ordered—if it is (say) of the type $\mathrm{Cnv}``\omega$ (cf. $*263$)—there may be no *first* member of the field of P where μ and ν differ. This will happen, for example, if μ consists of all the first terms, and ν of all the second terms. Our ordering relation can be so defined as to put μ before ν in this case also, but if it is so defined, the associative law of multiplication only holds if P is well-ordered. For this reason, we define our ordering relation so that, in such a case, μ comes neither before nor after ν. Again, if P is not a $\mathrm{Rel}^2\,\mathrm{excl}$, a member of a selected class may occur twice, once as the representative of $C`Q$, and once as that of $C`R$, if $C`Q$ and $C`R$ have terms in common. We wish to distinguish these two occurrences. Hence we proceed as follows: If μ and ν are two selected classes of $C``C`P$, let there be one or more members of $C`P$ in which the μ-representative precedes the ν-representative, and which are such that, among all earlier* members of $C`P$, the μ-representative is identical with the ν-representative.

But a further modification is desirable in order to meet the case in which two or more of the members of $C`P$ have the same field. Suppose, for example, we had to deal with a series consisting of all the series that can be formed out of a given set of terms: in this case, we should have to distinguish occurrences of any given term not by the field, but by the generating relation. This requires that we should make an F-selection from $C`P$, not an ϵ-selection from $C``C`P$. Hence we take two members of $F_\Delta`C`P$, say $M$ and $N$, and we arrange them or their domains on the following principle: We put $M$ before $N$ (or $\mathrm{D}`M$ before $\mathrm{D}`N$) if there is a relation $Q$ in the field of $P$ such that the $M$-representative of $Q$, *i.e.* $M`Q$, has the relation Q to the N-representative of Q, and such that, if R is any earlier member of $C`P$, then $M`R$ is identical with $N`R$. That is, M precedes N if

$$(\exists Q):(M`Q)\,Q\,(N`Q):RPQ\,.\,R\neq Q\,.\supset_R.\,M`R=N`R.$$

The relation between M and N so defined has the properties required of an arithmetical product; hence we put

$$\Pi`P=\hat{M}\hat{N}\{M,N\,\epsilon\,F_\Delta`C`P:.\;(\exists Q):(M`Q)\,Q\,(N`Q):RPQ\,.\,R\neq Q\,.\supset_R.\,M`R=N`R\}\quad\mathrm{Df.}$$

This relation is the ordinal analogue of $\epsilon_\Delta`\kappa$. The ordinal analogue of $\mathrm{Prod}`\kappa$ is the corresponding relation of the domains of M and N, *i.e.* $\mathrm{D}{;}\Pi`P$; hence we put

$$\mathrm{Prod}`P=\mathrm{D}{;}\Pi`P\quad\mathrm{Df.}$$

In case P is a $\mathrm{Rel}^2\,\mathrm{excl}$, we have $\mathrm{Nr}`\mathrm{Prod}`P=\mathrm{Nr}`\Pi`P$. But when P is not a $\mathrm{Rel}^2\,\mathrm{excl}$, $\mathrm{Prod}`P$ and $\Pi`P$ are in general not ordinally similar. We can, however, always make a $\mathrm{Rel}^2\,\mathrm{excl}$ by replacing the members x, y, etc. of

* Here Q is said to be *earlier* than R if Q has the relation P to R and is not identical with R.

$C'Q$ (where $Q \epsilon C'P$) by $x \downarrow Q$, $y \downarrow Q$, etc. In this way, if x occurs twice in $C'\Sigma'P$, once as a member of $C'Q$, and once as a member of $C'R$, the two occurrences are made to correspond to $x \downarrow Q$ and $x \downarrow R$ respectively, and thus we get a new relation which *is* a Rel^2 excl.

If every member of $C'P$ has a first term, $B \upharpoonright C'P$ will be the first term of $\Pi'P$, and $B``C'P$ will be the first term of Prod$'P$. If further there is a last member of $C'P$, *i.e.* if $E ! B'\breve{P}$, and if this last member has a second term, the second member of $\Pi'P$ is obtained by taking this second term as the representative of $B'\breve{P}$, and leaving all the other representatives unchanged. In any case, if $B'\breve{P}$ exists, the earliest successors of any member of $\Pi'P$ are those obtained by only varying the representative in $B'\breve{P}$. Thus, if $B'\breve{P}$ exists, those members of $\Pi'P$ which have a given set of representatives in all members of $D'P$ form a consecutive stretch of the series, and this stretch is like $B'\breve{P}$. If $B'\breve{P}$ has an immediate predecessor, the stretches obtained by varying only the representative in this predecessor are again consecutive, and form a series like the said predecessor; and so on. This makes it plain why $\Pi'P$ has the properties of a product.

As in the case of cardinals, the definition of exponentiation is derived from that of multiplication. We put

$$P \exp Q = \text{Prod}'P \underset{\bullet}{\downarrow} {}^{;}Q \quad \text{Df.}$$

We put also $\qquad P^Q = \dot{s}{}^{;}(P \exp Q) \quad \text{Df.}$

This is an important relation, which deserves consideration apart from the fact that it is useful in connection with exponentiation. It will be found that

$$P^Q = \hat{M}\hat{N}\{M, N \epsilon (C'P \uparrow C'Q)_\Delta{}'C'Q :. \\ (\exists y) : y \epsilon C'Q \,.\, (M'y) \, P \, (N'y) : xQy \,.\, x \neq y \,.\, \supset_x \,.\, M'x = N'x\}.$$

This is a form of the principle of first differences which is appropriate when *two* relations are concerned, instead of only one as in P_{cl}. The principle, in this case, is as follows: Let M, N be any two one-many relations which relate part (or the whole) of $C'P$ to the whole of $C'Q$. That is, each of the two relations assigns a representative in $C'P$ to every term of $C'Q$, but different terms of $C'Q$ may have the same representative. Then in travelling along the series Q, there is to be, sooner or later, a term y whose M-representative is earlier than its N-representative, and terms which come earlier than y in Q are all to have their M-representatives identical with their N-representatives.

The relation P^Q may be subjected to various restrictions which give important results. This subject has been treated by Hausdorff. For

example, if $P = x \downarrow y$ (where $x \neq y$), and Q is of the ordinal type which Cantor calls ω, *i.e.* the type of progressions (generated by transitive relations), then if z is any member of $C`Q$, $M`z$ is always either x or y. If we impose the condition that $M`z$ is to be $x$ except for a finite number of values of $z$, the resulting series is of the type of the rationals in order of magnitude, *i.e.* the type called $\eta$. If we impose the condition that there are to be an infinite number of values of $z$ for which $M`z = y$, the resulting series is a continuum, *i.e.* it is of the ordinal type called θ; in this case, the contained "rational" series consists of those M's for which there are only a finite number of z's having $M`z = x$. If we impose no limitation, P^Q is of the type presented by the real numbers when decimals ending in 9 recurring are counted separately from the terminating decimals having the same value.

We may generalize P^Q, instead of restricting it. To begin with, we may allow our M and N to have only part of $C`Q$ for their converse domain, and remove the assumption that there is a *first* member of $C`Q$ for which $M`y$ and $N`y$ differ; this leads to the relation

$$\hat{M}\hat{N}\{M, N \in (1 \rightarrow \mathrm{Cls}) \cap \mathrm{Rl}`(C`P \uparrow C`Q) :.$$
$$(\exists y) : (M`y)\, P\, (N`y) : xQy \,.\, x \in \text{Ɑ}`N \,.\, \supset_x .\, (M`x)(P \cup I)(N`x)\}.$$

Further, we may drop the restriction to one-many relations. It will be observed that if $(M`y)\, P\, (N`y)$, we have $y\, (\breve{M} \,|\, P \,|\, N)\, y$. Thus we may consider the relation

$$MN [\hat{M}, \hat{N} \in \mathrm{Rl}`(C`P \uparrow C`Q) :.$$
$$(\exists y) : y\, (\breve{M} \,|\, P \,|\, N)\, y : xQy \,.\, \supset_x .\, x \{\breve{M} \,|\, (P \cup I) \,|\, N\}\, x].$$

This relation has for its field all relations contained in $C`P \uparrow C`Q$. We may, if we like, drop even this restriction, and consider

$$\hat{M}\hat{N}\, [(\exists y) : y \in C`Q \,.\, y\, (\breve{M} \,|\, P \,|\, N)\, y : xQy \,.\, \supset_x .\, x \{\breve{M} \,|\, (P \cup I \upharpoonright C`P) \,|\, N\}\, x].$$

This represents the most general form of the principle of first differences as applied to a couple of relations P and Q. In ordinal arithmetic, however, P^Q is sufficiently general for the uses we wish to make of it.

The formal laws, as far as they are true, can be proved without excessive difficulty. We have

$$\vdash : P \neq Q \,.\, \supset .\, \mathrm{Nr}`\Pi`(P \downarrow Q) = \mathrm{Nr}`(P \times Q),$$

which connects the two kinds of multiplication;

$$\vdash : P \text{ smor smor } Q \,.\, \supset .\, \mathrm{Nr}`\Pi`P = \mathrm{Nr}`\Pi`Q,$$
$$\vdash : P \in \mathrm{Rel}^2 \text{ excl} \,.\, P \subset J \,.\, \supset .\, \mathrm{Nr}`\Pi`\Pi;P = \mathrm{Nr}`\Pi`\Sigma`P,$$

which is one form of the associative law, of which another form is

$$\vdash : P \neq Q \,.\, \supset .\, \mathrm{Nr}`(\Pi`P \times \Pi`Q) = \mathrm{Nr}`\Pi`(P \mathbin{\updownarrow} Q).$$

Also

$$\vdash : P, \Sigma\text{‘}P \in \mathrm{Rel}^2\,\mathrm{excl} \,.\, P \subset J \,.\, \supset .\, \mathrm{Nr}\text{‘}\mathrm{Prod}\text{‘}\mathrm{Prod};P = \mathrm{Nr}\text{‘}\mathrm{Prod}\text{‘}\Sigma\text{‘}P = \mathrm{Nr}\text{‘}\Pi\text{‘}\Sigma\text{‘}P,$$

which is the associative law for "Prod." We have

$$\vdash : C\text{‘}Q \cap C\text{‘}R = \Lambda \,.\, \supset .\, \mathrm{Nr}\text{‘}(P^Q \times P^R) = \mathrm{Nr}\text{‘}P^{Q \ddagger R},$$

$$\vdash .\, \mathrm{Nr}\text{‘}(P^Q)^R = \mathrm{Nr}\text{‘}(P^{R \times Q}).$$

But we do not have in general

$$\mathrm{Nr}\text{‘}(P^R \times Q^R) = \mathrm{Nr}\text{‘}(P \times Q)^R,$$

which obviously would require the commutative law for multiplication, and therefore does not hold in general in spite of the fact that its cardinal analogue does always hold.

As regards the connection with cardinals, we have

$$\vdash : P \in \mathrm{Rel}^2\,\mathrm{excl} \,.\, \supset .\, C\text{‘}\mathrm{Prod}\text{‘}P = \mathrm{Prod}\text{‘}C\text{“}C\text{‘}P,$$

$$\vdash : \dot{\exists}\,!\,Q \,.\, \supset .\, C\text{‘}(P \exp Q) = (C\text{‘}P) \exp (C\text{‘}Q),$$

and we have already had

$$\vdash .\, C\text{‘}(P \times Q) = C\text{‘}P \times C\text{‘}Q.$$

Moreover the correlators by which similarity is established in cardinals generally suffice to establish likeness in the analogous cases in relation-arithmetic. Thus we have

$$\vdash : S \in P \,\overline{\mathrm{smor}}\, Q \,.\, \supset .\, S_\epsilon \restriction \mathrm{Cl}\text{‘}C\text{‘}Q \in P_{\mathrm{cl}} \,\overline{\mathrm{smor}}\, Q_{\mathrm{cl}},$$

$$\vdash : P, Q \in \mathrm{Rel}^2\,\mathrm{excl} \,.\, S \restriction C\text{‘}\Sigma\text{‘}Q \in P \,\overline{\mathrm{smor}}\,\overline{\mathrm{smor}}\, Q \,.\, \supset . \\ S_\epsilon \restriction C\text{‘}\mathrm{Prod}\text{‘}Q \in (\mathrm{Prod}\text{‘}P) \,\overline{\mathrm{smor}}\, (\mathrm{Prod}\text{‘}Q),$$

$$\vdash : S \restriction C\text{‘}P' \in P \,\overline{\mathrm{smor}}\, P' \,.\, T \restriction C\text{‘}Q' \in Q \,\overline{\mathrm{smor}}\, Q' \,.\, \supset . \\ (S \parallel \breve{T}) \restriction C\text{‘}(P' \exp Q') \in (P \exp Q) \,\overline{\mathrm{smor}}\, (P' \exp Q'),$$

which are all closely analogous to propositions which were proved in cardinals.

The applications of the propositions of this section are almost wholly to series, and it is convenient to imagine our relations to be serial. But the hypothesis that they are serial is not necessary to the truth of any of the propositions of the present section, and it is a remarkable fact that so many of the formal laws of ordinal arithmetic hold for relations in general.

It should be observed that $\Pi\text{‘}P$ is not always a series when P is a series and all the relations in the field of P are series. A series (cf. *204) is a relation P which is (1) contained in diversity, (2) transitive, (3) connected, *i.e.* such that every term of the field of P has the relation P or the relation $\breve{P}$ to every other term of the field. It is the third condition which may fail for $\Pi\text{‘}P$, and which in fact does fail whenever P is not well-ordered. Thus suppose, for the sake of simplicity, that P is of the type $\mathrm{Cnv}\text{“}\omega$, which we will call a *regression*, *i.e.* the converse of a progression (cf. *263); and suppose that the field of P consists entirely of couples. Take a selection M

which chooses the first term of every odd couple, and the second term of every even couple; and take another selection N which chooses the second term of every odd couple, and the first term of every even couple. Neither of these two selections has the relation $\Pi‘P$ to the other, for whatever term Q of $C‘P$ we choose, if M is the selection which chooses the first term of Q, there is an earlier term of $C‘P$ (namely the immediate predecessor of Q) in which N chooses the first term while M chooses the second. Hence there is no such Q as is required for $M(\Pi‘P)N$; and a similar argument holds against $N(\Pi‘P)M$. In such a case, $\Pi‘P$ generates a number of different series, and by suitable restrictions of the field, one of these series can be extracted. Exactly similar remarks apply to P^Q.

$*170$. ON THE RELATION OF FIRST DIFFERENCES AMONG THE SUB-CLASSES OF A GIVEN CLASS

Summary of $*170$.

The definition to be given in this number of the relation of first differences among the sub-classes of a given class is by no means the only one possible, in fact a different definition will be considered in $*171$. In the present number, the definition we choose is this: α is said to precede β according to this definition when α has at least one member which neither belongs to β nor follows any term belonging to β and not to α (α and β being both sub-classes of $C\text{‘}P$). In other words, if we consider the two classes $\alpha - \beta$ and $\beta - \alpha$, there are members of $\alpha - \beta$ which are not preceded by any members of $\beta - \alpha$. Pictorially, we may conceive the relation as follows (P being supposed serial): α and β each pick out terms from $C\text{‘}P$, and these terms have an order conferred by P; we suppose that the earlier terms selected by α and β are perhaps the same, but sooner or later, if $\alpha \neq \beta$, we must come to terms which belong to one but not to the other. We assume that the earliest terms of this sort belong to α, not to β; in this case, α has to β the relation P_{cl}. That is, where α and β begin to differ, it is terms of α that we come to, not terms of β. We do not assume that there is a *first* term which belongs to α and not to β, since this would introduce undesirable restrictions in case P is not well-ordered.

A few of the propositions of the present number will be used in the next number, which deals with a slightly different form of the relation of first differences, but with this exception the propositions of this number will not be referred to again until we come to series. Their chief use occurs in the section on compact series, rational series, and continuous series (Part V, Section F), especially in $*274$ and $*276$, which respectively establish the existence of rational series (assuming the axiom of infinity) and the fact that the cardinal number of terms in a continuous series is the same as the number of classes contained in the field of a progression, *i.e.* $2^{\aleph_0}$. The definitions and a few of the simpler propositions are also used in connection with the series of segments of a series, since, as explained above, the segments of a series P are arranged in the series generated by P_{1c}.

The propositions of this number which will be used in dealing with series are the following:

$*170{\cdot}1$. $\vdash : \alpha P_{cl} \beta . \equiv . \alpha, \beta \in \text{Cl}\text{‘}C\text{‘}P . \exists ! \alpha - \beta - \breve{P}\text{‘‘}(\beta - \alpha)$

$*170{\cdot}101$. $\vdash . P_{1c} = \text{Cnv}\text{‘}(\breve{P})_{cl}$

$*170{\cdot}102$. $\vdash : \alpha P_{1c} \beta . \equiv . \alpha, \beta \in \text{Cl}\text{‘}C\text{‘}P . \exists ! \beta - \alpha - P\text{‘‘}(\alpha - \beta)$

(These propositions merely embody the definitions.)

$*170\cdot11.\ \vdash :. \alpha P_{cl} \beta . \equiv : \alpha, \beta \in \mathrm{Cl}'C'P : (\exists y) . y \in \alpha - \beta . \overrightarrow{P}'y \cap \beta \subset \alpha$

This form is often more convenient than $*170\cdot1$.

$*170\cdot16.\ \vdash : \alpha \subset C'P . \beta \subset \alpha . \beta \neq \alpha . \supset . \alpha P_{cl} \beta$

I.e. every sub-class of $C'P$ has the relation P_{cl} to every proper part of itself.

$*170\cdot17.\ \vdash . P_{cl} \Subset J . P_{lc} \Subset J$

$*170\cdot2.\ \vdash :. \alpha, \beta \in \mathrm{Cl}'C'P : (\exists y) . y \in \alpha - \beta . \overrightarrow{P}'y \cap \alpha = \overrightarrow{P}'y \cap \beta : \supset . \alpha P_{cl} \beta$

This proposition deals with the case where there is a definite first term y which belongs to α and not to β, and whose predecessors all belong to both or neither.

$$*170\cdot23.\ \vdash :. \alpha \subset C'P . y \in \alpha - \beta - \breve{P}``(\beta - \alpha) . \supset : y \min_P (\alpha - \beta) . \equiv . \overrightarrow{P}'y \cap \alpha = \overrightarrow{P}'y \cap \beta$$

This proposition is useful in case P is well-ordered, since then $\alpha - \beta$ must have a minimum if it exists (α and β being supposed sub-classes of $C'P$).

$*170\cdot31.\ \vdash : \beta \subset C'P . \beta \neq C'P . \equiv . (C'P) P_{cl} \beta$

This follows from $*170\cdot16$, as does the following proposition:

$*170\cdot32.\ \vdash : \alpha \subset C'P . \exists ! \alpha . \equiv . \alpha P_{cl} \Lambda$

$*170\cdot35.\ \vdash . \dot{\Lambda}_{cl} = \dot{\Lambda}$

$*170\cdot38.\ \vdash : \dot{\exists} ! P . \supset . B'P_{cl} = C'P . B'\mathrm{Cnv}'P_{cl} = \Lambda$

$*170\cdot6.\ \vdash : \Lambda P_{lc} \beta . \equiv . \beta \subset C'P . \exists ! \beta$

Besides the above, the following propositions should be noted:

$*170\cdot36.\ \vdash . \mathrm{D}'P_{cl} = \mathrm{Cl\ ex}'C'P . \mathrm{Q}'P_{cl} = \mathrm{Cl}'C'P - \iota'C'P$

$*170\cdot37.\ \vdash : \dot{\exists} ! P . \supset . C'P_{cl} = \mathrm{Cl}'C'P$

$*170\cdot44.\ \vdash : P \operatorname{smor} Q . \supset . P_{cl} \operatorname{smor} Q_{cl}$

$*170\cdot64.\ \vdash : x \sim \in C'P . \supset . (x \leftarrow\!\!\!+ P)_{cl} = (\iota'x \cup)^{;} P_{cl} \ \hat{+}\ P_{cl}$

This proposition shows that every term added to P doubles the number of terms in P_{cl}; hence it is not surprising that P_{cl} (when P is well-ordered) has a power of 2_r for its relation-number (cf. $*177$).

$*170\cdot67.\ \vdash : \dot{\exists} ! P . \dot{\exists} ! Q . C'P \cap C'Q = \Lambda . \supset . (P \ \hat{+}\ Q)_{cl} = s^{;}C^{;}(P_{cl} \times Q_{cl})$
whence

$*170\cdot69.\ \vdash : \dot{\exists} ! P . \dot{\exists} ! Q . C'P \cap C'Q = \Lambda . \supset . (P \ \hat{+}\ Q)_{cl} \operatorname{smor} (P_{cl} \times Q_{cl})$

$*170\cdot01.\ P_{cl} = \hat{\alpha}\hat{\beta} \{\alpha, \beta \in \mathrm{Cl}'C'P . \exists ! \alpha - \beta - \breve{P}``(\beta - \alpha)\} \quad \mathrm{Df}$

$*170\cdot02.\ P_{lc} = \mathrm{Cnv}'(\breve{P})_{cl} \quad \mathrm{Df}$

$*170\cdot 1$. $\vdash : \alpha P_{cl} \beta . \equiv . \alpha, \beta \in \mathrm{Cl}`C`P . \exists ! \alpha - \beta - \breve{P}``(\beta - \alpha)$ [$(*170\cdot 01)$]

$*170\cdot 101$. $\vdash . P_{1c} = \mathrm{Cnv}`(\breve{P})_{cl}$ [$(*170\cdot 02)$]

$*170\cdot 102$. $\vdash : \alpha P_{1c} \beta . \equiv . \alpha, \beta \in \mathrm{Cl}`C`P . \exists ! \beta - \alpha - P``(\alpha - \beta)$ [$*170\cdot 1\cdot 101$]

Thus $\alpha P_{1c} \beta$ means, roughly speaking, that $\beta - \alpha$ goes on longer than $\alpha - \beta$, just as $\alpha P_{cl} \beta$ means that $\alpha - \beta$ begins sooner. Thus if P is the relation of earlier and later in time, and α and β are the times when A and B respectively are out of bed, "$\alpha P_{cl} \beta$" will mean that A gets up earlier than B, and "$\alpha P_{1c} \beta$" will mean that B goes to bed later than A.

$*170\cdot 103$. $\vdash : y \sim \in \breve{P}``(\beta - \alpha) . \equiv . \overrightarrow{P}`y \cap \beta \subset \alpha$

Dem.

$\vdash . *37\cdot 105 . \supset \vdash :. y \sim \in \breve{P}``(\beta - \alpha) . \equiv : \sim (\exists x) . x \in \beta - \alpha . xPy :$

$[*10\cdot 51] \quad \equiv : x \in \beta . xPy . \supset_x . x \in \alpha :$

$[*32\cdot 18] \quad \equiv : \overrightarrow{P}`y \cap \beta \subset \alpha :. \supset \vdash . \mathrm{Prop}$

$*170\cdot 11$. $\vdash :. \alpha P_{cl} \beta . \equiv : \alpha, \beta \in \mathrm{Cl}`C`P : (\exists y) . y \in \alpha - \beta . \overrightarrow{P}`y \cap \beta \subset \alpha$
[$*170\cdot 1\cdot 103$]

$*170\cdot 12$. $\vdash : \alpha P_{cl} \beta . \equiv . \alpha, \beta \in \mathrm{Cl}`C`P . \exists ! \alpha - (\alpha \cap \beta) - \breve{P}``\{\beta - (\alpha \cap \beta)\}$
[$*170\cdot 1 . *22\cdot 93$]

$*170\cdot 121$. $\vdash :. \alpha P_{cl} \beta . \equiv . \alpha, \beta \in \mathrm{Cl}`C`P . \exists ! (\alpha \cup \beta) - \beta - \breve{P}``\{(\alpha \cup \beta) - \alpha\}$
[$*170\cdot 1 . *22\cdot 9$]

$*170\cdot 13$. $\vdash :. \alpha P_{cl} \beta . \equiv : (\exists \rho, \sigma, \gamma) . \rho, \sigma, \gamma \in \mathrm{Cl}`C`P .$
$\rho \cap \gamma = \Lambda . \sigma \cap \gamma = \Lambda . \rho \cap \sigma = \Lambda . \alpha = \gamma \cup \rho . \beta = \gamma \cup \sigma . \exists ! \rho - \breve{P}``\sigma$

Dem.

$\vdash . *24\cdot 24 . *22\cdot 69 . \supset \vdash : \rho \cap \sigma = \Lambda . \alpha = \gamma \cup \rho . \beta = \gamma \cup \sigma . \supset . \alpha \cap \beta = \gamma$ (1)

$\vdash . *24\cdot 4 . \quad \supset \vdash :. \alpha = \gamma \cup \rho . \supset : \rho \cap \gamma = \Lambda . \equiv . \alpha - \gamma = \rho$ (2)

$\vdash . *24\cdot 4 . \quad \supset \vdash :. \beta = \gamma \cup \sigma . \supset : \sigma \cap \gamma = \Lambda . \equiv . \beta - \gamma = \sigma$ (3)

$\vdash . (1) . (2) . (3) . \quad \supset \vdash :. \rho \cap \sigma = \Lambda . \alpha = \gamma \cup \rho . \beta = \gamma \cup \sigma . \supset :$
$\rho \cap \gamma = \Lambda . \sigma \cap \gamma = \Lambda . \equiv . \alpha - (\alpha \cap \beta) = \rho . \beta - (\alpha \cap \beta) = \sigma .$

$[*22\cdot 93] \quad \equiv . \alpha - \beta = \rho . \beta - \alpha = \sigma$ (4)

$\vdash . (1) . (4) . \supset \vdash : (\exists \rho, \sigma, \gamma) . \rho, \sigma, \gamma \in \mathrm{Cl}`C`P . \rho \cap \gamma = \Lambda . \sigma \cap \gamma = \Lambda . \rho \cap \sigma = \Lambda .$
$\alpha = \gamma \cup \rho . \beta = \gamma \cup \sigma . \exists ! \rho - \breve{P}``\sigma . \equiv .$
$(\exists \rho, \sigma, \gamma) . \rho, \sigma, \gamma \in \mathrm{Cl}`C`P . \rho \cap \sigma = \Lambda . \alpha = \gamma \cup \rho . \beta = \gamma \cup \sigma .$
$\alpha \cap \beta = \gamma . \alpha - \beta = \rho . \beta - \alpha = \sigma . \exists ! \rho - \breve{P}``\sigma .$

$[*13\cdot 22] \quad \equiv . \alpha - \beta, \beta - \alpha, \alpha \cap \beta \in \mathrm{Cl}`C`P . \exists ! \alpha - \beta - \breve{P}``(\beta - \alpha) .$

$[*60\cdot 43 . *24\cdot 41] \equiv . \alpha, \beta \in \mathrm{Cl}`C`P . \exists ! \alpha - \beta - \breve{P}``(\beta - \alpha) .$

$[*170\cdot 1] \quad \equiv . \alpha P_{cl} \beta : \supset \vdash . \mathrm{Prop}$

***170·14.** $\vdash :. \alpha, \beta \epsilon \mathrm{Cl}`C`P . \supset : \alpha \dot{-} P_{\mathrm{cl}} \beta . \equiv . \alpha - \beta \subset \breve{P}``(\beta - \alpha)$
[*170·1 . *24·55]

***170·141.** $\vdash :. \alpha, \beta \epsilon \mathrm{Cl}`C`P . \supset : \alpha \dot{-} P_{\mathrm{lc}} \beta . \equiv . \beta - \alpha \subset P``(\alpha - \beta)$
[*170·14·101]

***170·15.** $\vdash : \alpha P_{\mathrm{cl}} \beta . \supset . \beta \cap p`\overrightarrow{P}``(\alpha - \beta) \subset \alpha$

Dem.

$\vdash . *40·12 . \supset \vdash : y \epsilon \alpha - \beta . \supset . p`\overrightarrow{P}``(\alpha - \beta) \subset \overrightarrow{P}`y .$
[*22·48] $\supset . \beta \cap p`\overrightarrow{P}``(\alpha - \beta) \subset \beta \cap \overrightarrow{P}`y :$
[*22·44] $\supset \vdash : y \epsilon \alpha - \beta . \beta \cap \overrightarrow{P}`y \subset \alpha . \supset . \beta \cap p`\overrightarrow{P}``(\alpha - \beta) \subset \alpha :$
[*10·11·23] $\supset \vdash : (\exists y) . y \epsilon \alpha - \beta . \beta \cap \overrightarrow{P}`y \subset \alpha . \supset . \beta \cap p`\overrightarrow{P}``(\alpha - \beta) \subset \alpha$ (1)
$\vdash . (1) . *170·11 . \supset \vdash .$ Prop

***170·16.** $\vdash : \alpha \subset C`P . \beta \subset \alpha . \beta \neq \alpha . \supset . \alpha P_{\mathrm{cl}} \beta$

Dem.

$\vdash . *24·6 . \supset \vdash : \mathrm{Hp} . \supset . \exists ! \alpha - \beta$ (1)
$\vdash . *24·3 . \supset \vdash : \mathrm{Hp} . \supset . \beta - \alpha = \Lambda .$
[*37·29] $\supset . \breve{P}``(\beta - \alpha) = \Lambda .$
[*24·101·26] $\supset . \alpha - \beta - \breve{P}``(\beta - \alpha) = \alpha - \beta$ (2)
$\vdash . (1) . (2) . *170·1 . \supset \vdash .$ Prop

***170·161.** $\vdash : \alpha \subset C`P . \beta \subset \alpha . \beta \neq \alpha . \supset . \beta P_{\mathrm{lc}} \alpha$

Dem.

$\vdash . *170·16 . \supset \vdash : \mathrm{Hp} . \supset . \alpha (\breve{P})_{\mathrm{cl}} \beta .$
[*170·101] $\supset . \beta P_{\mathrm{lc}} \alpha : \supset \vdash .$ Prop

***170·17.** $\vdash . P_{\mathrm{cl}} \in J . P_{\mathrm{lc}} \in J$

Dem.

$\vdash . *170·1 . \supset \vdash : \alpha P_{\mathrm{cl}} \beta . \supset . \exists ! \alpha - \beta .$
[*24·55 . *22·42] $\supset . \alpha \neq \beta .$
[*50·11] $\supset . \alpha J \beta : \supset \vdash .$ Prop

In order that P_{cl} should be serial, we need further that it should be transitive and connected. P_{cl} is transitive if P is transitive and connected. But P_{cl} may still not be connected: there may be many distinct families in its field, though all of them must begin with $C`P$ and end with Λ. For example, if P is a regression, the class which takes every odd member does not have either of the relations $P_{\mathrm{cl}}, \breve{P}_{\mathrm{cl}}$ to the class which takes every even member. In order that P_{cl} should be serial, we require that P should be not only serial, but well-ordered, *i.e.* that every existent sub-class of $C`P$ should have a first term. When P is serial but not well-ordered, P_{cl} will, however, generate various series contained in it by imposing suitable limitations on the field.

∗170·2. $\vdash :. \alpha, \beta \in \mathrm{Cl}\text{‘}C\text{‘}P : (\exists y) . y \in \alpha - \beta . \overrightarrow{P}\text{‘}y \cap \alpha = \overrightarrow{P}\text{‘}y \cap \beta : \supset . \alpha P_{\mathrm{cl}} \beta$
[∗170·11 . ∗22·43]

∗170·21. $\vdash :. \alpha \subset C\text{‘}P . \supset : y \min_P (\alpha - \beta) . \equiv . y \in \alpha - \beta . \overrightarrow{P}\text{‘}y \cap \alpha \subset \beta$

Dem.

$\vdash . ∗93·11 . \supset \vdash :. \mathrm{Hp} . \supset : y \min_P (\alpha - \beta) . \equiv . y \in \alpha - \beta - \breve{P}\text{“}(\alpha - \beta) .$

[∗170·103] $\equiv . y \in \alpha - \beta . \overrightarrow{P}\text{‘}y \cap \alpha \subset \beta :. \supset \vdash . \mathrm{Prop}$

∗170·22. $\vdash :. \alpha \subset C\text{‘}P . y \min_P (\alpha - \beta) . \supset : \overrightarrow{P}\text{‘}y \cap \beta \subset \alpha . \equiv . \overrightarrow{P}\text{‘}y \cap \alpha = \overrightarrow{P}\text{‘}y \cap \beta$

Dem.

$\vdash . ∗170·21 . ∗4·73 . \supset \vdash :. \mathrm{Hp} . \supset : \overrightarrow{P}\text{‘}y \cap \beta \subset \alpha . \equiv . \overrightarrow{P}\text{‘}y \cap \alpha \subset \beta . \overrightarrow{P}\text{‘}y \cap \beta \subset \alpha .$

[∗22·74] $\equiv . \overrightarrow{P}\text{‘}y \cap \alpha = \overrightarrow{P}\text{‘}y \cap \beta :. \supset \vdash . \mathrm{Prop}$

∗170·23. $\vdash :. \alpha \subset C\text{‘}P . y \in \alpha - \beta - \breve{P}\text{“}(\beta - \alpha) . \supset :$

$y \min_P (\alpha - \beta) . \equiv . \overrightarrow{P}\text{‘}y \cap \alpha = \overrightarrow{P}\text{‘}y \cap \beta$

Dem.

$\vdash . ∗170·103·21 . \supset \vdash :. \mathrm{Hp} \supset :$

$y \min_P (\alpha - \beta) . \equiv . y \in \alpha - \beta . \overrightarrow{P}\text{‘}y \cap \beta \subset \alpha . \overrightarrow{P}\text{‘}y \cap \alpha \subset \beta .$

[∗22·74.∗4·73] $\equiv . y \in \alpha - \beta . \overrightarrow{P}\text{‘}y \cap \beta \subset \alpha . \overrightarrow{P}\text{‘}y \cap \alpha = \overrightarrow{P}\text{‘}y \cap \beta .$

[∗170·103] $\equiv . y \in \alpha - \beta - \breve{P}\text{“}(\beta - \alpha) . \overrightarrow{P}\text{‘}y \cap \alpha = \overrightarrow{P}\text{‘}y \cap \beta$ (1)

$\vdash . (1) . ∗5·32 . \supset \vdash . \mathrm{Prop}$

∗170·3. $\vdash : \alpha \in \mathrm{Cl}\text{‘}C\text{‘}P . \beta \subset \alpha . \exists ! \alpha - \beta . \supset . \alpha P_{\mathrm{cl}} \beta$ [∗170·16]

∗170·31. $\vdash : \beta \subset C\text{‘}P . \beta \neq C\text{‘}P . \equiv . (C\text{‘}P) P_{\mathrm{cl}} \beta$ [∗170·16]

∗170·32. $\vdash : \alpha \subset C\text{‘}P . \exists ! \alpha . \equiv . \alpha P_{\mathrm{cl}} \Lambda$ [∗170·3]

∗170·33. $\vdash : \dot{\exists} ! P . \equiv . (C\text{‘}P) P_{\mathrm{cl}} \Lambda$

Dem.

$\vdash . ∗33·24 . ∗170·32 . \supset \vdash : \dot{\exists} ! P . \supset . (C\text{‘}P) P_{\mathrm{cl}} \Lambda$ (1)

$\vdash . ∗170·1 . \supset \vdash : (C\text{‘}P) P_{\mathrm{cl}} \Lambda . \supset . \exists ! (C\text{‘}P) - \Lambda .$

[∗33·24] $\supset . \dot{\exists} ! P$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

∗170·34. $\vdash : \dot{\exists} ! P . \equiv . \dot{\exists} ! P_{\mathrm{cl}}$

Dem.

$\vdash . ∗170·33 . \supset \vdash : \dot{\exists} ! P . \supset . \dot{\exists} ! P_{\mathrm{cl}}$ (1)

$\vdash . ∗170·1 . \supset \vdash : \dot{\exists} ! P_{\mathrm{cl}} . \supset . (\exists \alpha, \beta) . \alpha, \beta \in \mathrm{Cl}\text{‘}C\text{‘}P . \exists ! \alpha - \beta .$

[∗24·561] $\supset . (\exists \alpha) . \alpha \in \mathrm{Cl}\text{‘}C\text{‘}P . \exists ! \alpha .$

[∗60·361] $\supset . \exists ! C\text{‘}P .$

[∗33·24] $\supset . \dot{\exists} ! P$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***170·35.** $\vdash . \dot{\Lambda}_{cl} = \dot{\Lambda}$ [*170·34 . Transp]

***170·36.** $\vdash . \mathrm{D}‘P_{cl} = \mathrm{Cl\,ex}‘C‘P . \text{Ɑ}‘P_{cl} = \mathrm{Cl}‘C‘P - \iota‘C‘P$

Dem.

$\vdash . *170·32 . \supset \vdash . \mathrm{Cl\,ex}‘C‘P \subset \mathrm{D}‘P_{cl}$ (1)

$\vdash . *170·31 . \supset \vdash . \mathrm{Cl}‘C‘P - \iota‘C‘P \subset \text{Ɑ}‘P_{cl}$ (2)

$\vdash . *170·1 . \supset \vdash : \alpha \in \mathrm{D}‘P_{cl} . \supset . (\exists \beta) . \alpha, \beta \in \mathrm{Cl}‘C‘P . \exists ! \alpha - \beta .$

[*24·561] $\supset . \alpha \in \mathrm{Cl}‘C‘P . \exists ! \alpha$ (3)

$\vdash . *170·1 . \supset \vdash : \alpha \in \text{Ɑ}‘P_{cl} . \supset . (\exists \beta) . \alpha, \beta \in \mathrm{Cl}‘C‘P . \exists ! \beta - \alpha .$

[*60·2] $\supset . \alpha \in \mathrm{Cl}‘C‘P . \exists ! C‘P - \alpha .$

[*24·6] $\supset . \alpha \in \mathrm{Cl}‘C‘P - \iota‘C‘P$ (4)

$\vdash . (1) . (2) . (3) . (4) . \supset \vdash . \mathrm{Prop}$

***170·37.** $\vdash : \dot{\exists} ! P . \supset . C‘P_{cl} = \mathrm{Cl}‘C‘P$ [*170·36]

***170·371.** $\vdash . C‘P_{cl} \subset \mathrm{Cl}‘C‘P$ [*170·37·35 . *33·241]

***170·38.** $\vdash : \dot{\exists} ! P . \supset . B‘P_{cl} = C‘P . B‘\mathrm{Cnv}‘P_{cl} = \Lambda$ [*170·36]

The following propositions lead up to *170·44.

***170·4.** $\vdash : S \in 1 \rightarrow 1 . C‘Q = \text{Ɑ}‘S . \supset . (S ; Q)_{cl} = S_\epsilon ; Q_{cl}$

Dem.

$\vdash . *170·1 . *150·4 . *37·11 . \supset \vdash : \alpha (S_\epsilon ; Q_{cl}) \beta . \equiv .$

$(\exists \gamma, \delta) . \gamma, \delta \in \mathrm{Cl}‘C‘Q . \alpha = S‘‘\gamma . \beta = S‘‘\delta . \exists ! \gamma - \delta - \breve{Q}‘‘(\delta - \gamma)$ (1)

$\vdash . (1) . \supset \vdash :. \mathrm{Hp} . \supset :$

$\alpha (S_\epsilon ; Q_{cl}) \beta . \equiv . (\exists \gamma, \delta) . \gamma, \delta \in \mathrm{Cl}‘\text{Ɑ}‘S . \alpha = S‘‘\gamma . \beta = S‘‘\delta .$

$\exists ! \gamma - \delta - \breve{Q}‘‘(\delta - \gamma) .$

[*37·43] $\equiv . (\exists \gamma, \delta) . \gamma, \delta \in \mathrm{Cl}‘\text{Ɑ}‘S . \alpha = S‘‘\gamma . \beta = S‘‘\delta .$

$\exists ! S‘‘\{\gamma - \delta - \breve{Q}‘‘(\delta - \gamma)\} .$

[*71·381] $\equiv . (\exists \gamma, \delta) . \gamma, \delta \in \mathrm{Cl}‘\text{Ɑ}‘S . \alpha = S‘‘\gamma . \beta = S‘‘\delta .$

$\exists ! S‘‘\gamma - S‘‘\delta - S‘‘\breve{Q}‘‘(\delta - \gamma) .$

[*72·511.*71·38] $\equiv . (\exists \gamma, \delta) . \gamma, \delta \in \mathrm{Cl}‘\text{Ɑ}‘S . \alpha = S‘‘\gamma . \beta = S‘‘\delta .$

$\exists ! S‘‘\gamma - S‘‘\delta - S‘‘\breve{Q}‘‘\breve{S}‘‘(\beta - \alpha) .$

[*13·193.*37·33] $\equiv . (\exists \gamma, \delta) . \gamma, \delta \in \mathrm{Cl}‘\text{Ɑ}‘S . \alpha = S‘‘\gamma . \beta = S‘‘\delta .$

$\exists ! \alpha - \beta - (S ; \breve{Q})‘‘(\beta - \alpha) .$

[*71·48.*37·23] $\equiv . \alpha, \beta \in \mathrm{Cl}‘\mathrm{D}‘S . \exists ! \alpha - \beta - (S ; \breve{Q})‘‘(\beta - \alpha) .$

[*150·23] $\equiv . \alpha, \beta \in \mathrm{Cl}‘C‘(S ; Q) . \exists ! \alpha - \beta - (S ; \breve{Q})‘‘(\beta - \alpha) .$

[*150·12.*170·1] $\equiv . \alpha (S ; Q)_{cl} \beta :. \supset \vdash . \mathrm{Prop}$

***170·41.** $\vdash . (S \upharpoonright C`Q)_\epsilon {}^{\circ\!\!,} Q_{cl} = S_\epsilon {}^{\circ\!\!,} Q_{cl}$ [*150·95 . *170·371]

***170·42.** $\vdash : S \upharpoonright C`Q \,\epsilon\, 1 \to 1 \,.\, C`Q \subset \text{Ǝ}`S \,.\, \supset .\, (S {}^{\circ\!\!,} Q)_{cl} = S_\epsilon {}^{\circ\!\!,} Q_{cl}$

Dem.

$\vdash . *150·32 . \quad \supset \vdash . (S {}^{\circ\!\!,} Q)_{cl} = \{(S \upharpoonright C`Q) {}^{\circ\!\!,} Q\}_{cl}$ (1)

$\vdash . (1) . *170·4 . \supset \vdash : \text{Hp} . \supset . (S {}^{\circ\!\!,} Q)_{cl} = (S \upharpoonright C`Q)_\epsilon {}^{\circ\!\!,} Q_{cl}$

[*170·41] $= S_\epsilon {}^{\circ\!\!,} Q_{cl} : \supset \vdash . \text{Prop}$

***170·43.** $\vdash : S \upharpoonright C`Q \,\epsilon\, P \,\overline{\text{smor}}\, Q \,.\, \supset .\, S_\epsilon \upharpoonright C`Q_{cl} \,\epsilon\, P_{cl} \,\overline{\text{smor}}\, Q_{cl}$

Dem.

$\vdash . *151·22 . *170·42 . \supset \vdash : \text{Hp} . \supset . P_{cl} = S_\epsilon {}^{\circ\!\!,} Q_{cl}$ (1)

$\vdash . *74·131 . *170·371 . \supset \vdash : \text{Hp} . \supset . S_\epsilon \upharpoonright C`Q_{cl} \,\epsilon\, 1 \to 1$ (2)

$\vdash . *37·231 . \quad \supset \vdash . C`Q_{cl} \subset \text{Ǝ}`S_\epsilon$ (3)

$\vdash . (1) . (2) . (3) . *151·22 . \supset \vdash . \text{Prop}$

***170·44.** $\vdash : P \,\text{smor}\, Q \,.\, \supset .\, P_{cl} \,\text{smor}\, Q_{cl}$ [*170·43 . *151·23·12]

***170·5.** $\vdash . (x \downarrow x)_{cl} = (\iota`x) \downarrow \Lambda$

Dem.

$\vdash . *170·36 . *55·15 . \supset \vdash . \text{D}`(x \downarrow x)_{cl} = \text{Cl ex}`\iota`x$

[*60·37] $= \iota`\iota`x$ (1)

$\vdash . *170·36 . *55·15 . \supset \vdash . \text{Ǝ}`(x \downarrow x)_{cl} = \text{Cl}`\iota`x - \iota`\iota`x$

[*60·362] $= \iota`\Lambda$ (2)

$\vdash . (1) . (2) . *55·16 . \supset \vdash . \text{Prop}$

***170·51.** $\vdash : x \neq y . \supset . (x \downarrow y)_{cl} = (\iota`x \cup \iota`y) \downarrow \iota`x \,\dot\cup\, (\iota`x \cup \iota`y) \downarrow \iota`y \,\dot\cup\, (\iota`x \cup \iota`y) \downarrow \Lambda$
$\dot\cup\, \iota`x \downarrow \iota`y \,\dot\cup\, \iota`x \downarrow \Lambda \,\dot\cup\, \iota`y \downarrow \Lambda$

Dem.

$\vdash . *55·13 . \supset \vdash : \text{Hp} . \supset . \overrightarrow{x \downarrow y}`x = \Lambda . \overrightarrow{x \downarrow y}`y = \iota`x$ (1)

$\vdash . *170·11 . *55·15 . \supset \vdash :: \text{Hp} . \supset :. \alpha (x \downarrow y)_{cl} \beta . \equiv :$

$\alpha, \beta \,\epsilon\, \text{Cl}`(\iota`x \cup \iota`y) : (\exists z) . z \,\epsilon\, \alpha - \beta . \overrightarrow{x \downarrow y}`z \cap \beta \subset \alpha$

[*60·39] $\equiv : \alpha = \iota`x \cup \iota`y . \mathsf{v} . \alpha = \iota`x . \mathsf{v} . \alpha = \iota`y : \beta \subset \iota`x \cup \iota`y :$

$(\exists z) . z \,\epsilon\, \alpha - \beta . \overrightarrow{x \downarrow y}`z \cap \beta \subset \alpha$ (2)

$\vdash . *51·235 . \supset \vdash :: \alpha = \iota`x \cup \iota`y . \supset :. (\exists z) . z \,\epsilon\, \alpha - \beta . \overrightarrow{x \downarrow y}`z \cap \beta \subset \alpha . \equiv :$

$x \,\epsilon\, \alpha - \beta . \overrightarrow{x \downarrow y}`x \cap \beta \subset \alpha . \mathsf{v} . y \,\epsilon\, \alpha - \beta . \overrightarrow{x \downarrow y}`y \cap \beta \subset \alpha :$

[(1)] $\equiv : x \,\epsilon\, \alpha - \beta . \mathsf{v} . y \,\epsilon\, \alpha - \beta . \iota`x \cap \beta \subset \alpha :$

[Hp.*22·43·58] $\equiv : x \,\epsilon\, \alpha - \beta . \mathsf{v} . y \,\epsilon\, \alpha - \beta :$

[*51·232.*4·73] $\equiv : x \sim\epsilon\, \beta . \mathsf{v} . y \sim\epsilon\, \beta$ (3)

$\vdash . *54·4 . \supset \vdash :: \text{Hp} . \supset :. \beta \subset \iota`x \cup \iota`y . x \sim\epsilon\, \beta . \equiv : \beta = \iota`y . \mathsf{v} . \beta = \Lambda$ (4)

$\vdash . *54·4 . \supset \vdash :: \text{Hp} . \supset :. \beta \subset \iota`x \cup \iota`y . y \sim\epsilon\, \beta . \equiv : \beta = \iota`x . \mathsf{v} . \beta = \Lambda$ (5)

$\vdash . (2) . (3) . (4) . (5) . \supset$

$\vdash :: \alpha = \iota`x \cup \iota`y . \supset :. \alpha (x \downarrow y)_{\text{cl}} \beta . \equiv : \beta = \iota`x . \mathrm{v} . \beta = \iota`y . \mathrm{v} . \beta = \Lambda$ (6)

$\vdash . (1) . (2) . \supset \vdash ::. \text{Hp} . \supset :: \alpha = \iota`x . \supset :. \alpha (x \downarrow y)_{\text{cl}} \beta . \equiv : \beta \subset \iota`x \cup \iota`y . x \sim\in \beta .$

[(4)] $\equiv : \beta = \iota`y . \mathrm{v} . \beta = \Lambda$ (7)

$\vdash . (1) . (2) . \supset \vdash :: \text{Hp} . \supset :. \alpha = \iota`y . \supset : \alpha (x \downarrow y)_{\text{cl}} \beta . \equiv .$

$\beta \subset \iota`x \cup \iota`y . y \sim\in \beta . \iota`x \cap \beta \subset \alpha .$

[$*$51·211] $\equiv . \beta \subset \iota`x \cup \iota`y . y \sim\in \beta . x \sim\in \beta .$

[$*$54·4] $\equiv . \beta = \Lambda$ (8)

$\vdash . (2) . (6) . (7) . (8) . \supset \vdash . \text{Prop}$

$*$170·52. $\vdash : x \neq y . \supset . (x \downarrow y)_{\text{cl}} = (\iota`x \cup \iota`y) \downarrow \iota`x \mathbin{\uparrow\!\!\!\!+} \iota`y \downarrow \Lambda$

Dem.

$\vdash . *55{\cdot}15 . \supset \vdash . C`\{(\iota`x \cup \iota`y) \downarrow \iota`x\} \uparrow C`\{\iota`y \downarrow \Lambda\} =$

$\{\iota`(\iota`x \cup \iota`y) \cup \iota`\iota`x\} \uparrow \{\iota`\iota`y \cup \iota`\Lambda\}$

[$*$55·52] $= (\iota`x \cup \iota`y) \downarrow \iota`y \mathbin{\dot\cup} (\iota`x \cup \iota`y) \downarrow \iota`\Lambda \mathbin{\dot\cup} \iota`x \downarrow \iota`y \mathbin{\dot\cup} \iota`x \downarrow \Lambda$ (1)

$\vdash . (1) . *170{\cdot}51 . *160{\cdot}1 . \supset \vdash . \text{Prop}$

$*$170·6. $\vdash : \Lambda P_{\text{lc}} \beta . \equiv . \beta \subset C`P . \exists ! \beta$ [$*$170·32·101]

$*$170·601. $\vdash : \alpha P_{\text{lc}} (C`P) . \equiv . \alpha \subset C`P . \alpha \neq C`P$ [$*$170·31·101]

$*$170·61. $\vdash :. x \sim\in C`P . \dot{\exists} ! P . x \in \alpha \cap \beta . \supset :$

$\alpha (x \leftarrow\!\!\!\!+ P)_{\text{cl}} \beta . \equiv . \alpha \{(\iota`x \cup)_{;} P_{\text{cl}}\} \beta . \equiv . (\alpha - \iota`x) P_{\text{cl}} (\beta - \iota`x)$

This and the following propositions are lemmas for

$x \sim\in C`P . \supset . (x \leftarrow\!\!\!\!+ P)_{\text{cl}} = (\iota`x \cup)_{;} P_{\text{cl}} \mathbin{\uparrow\!\!\!\!+} P_{\text{cl}}$ ($*$170·64).

Dem.

$\vdash . *161{\cdot}111 . \supset \vdash :: \text{Hp} . \supset :. y \in \beta . y (x \leftarrow\!\!\!\!+ P) z . \supset_y . y \in \alpha : \equiv :$

$y \in \beta . yPz . \supset_y . y \in \alpha : y \in \beta . y = x . z \in C`P . \supset_y . y \in \alpha :$

[$*$13·191.$*$33·17] $\equiv : y \in \beta - \iota`x . yPz . \supset_y . y \in \alpha - \iota`x : x \in \beta . z \in C`P . \supset . x \in \alpha :$

[Hp] $\equiv : y \in \beta - \iota`x . yPz . \supset_y . y \in \alpha - \iota`x$ (1)

$\vdash . *51{\cdot}34 . \supset \vdash : \text{Hp} . \supset . -\beta = -\iota`x \cap -\beta .$

[$*$22·481] $\supset . \alpha - \beta = \alpha - \iota`x - \beta$

[$*$24·21] $= \alpha - \iota`x \cap (\iota`x \cup -\beta)$

[$*$22·86] $= \alpha - \iota`x - (\beta - \iota`x)$ (2)

$\vdash . *170{\cdot}11 . *161{\cdot}101{\cdot}14 . \supset \vdash :: \text{Hp} . \supset :. \alpha (x \leftarrow\!\!\!\!+ P)_{\text{cl}} \beta . \equiv :$

$\alpha, \beta \in \text{Cl}`(C`P \cup \iota`x) : (\exists z) : z \in \alpha - \beta : y \in \beta . y (x \leftarrow\!\!\!\!+ P) z . \supset_z . y \in \alpha :$

[(1).(2)] $\equiv : \alpha, \beta \in \text{Cl}`(C`P \cup \iota`x) : (\exists z) : z \in \alpha - \iota`x - (\beta - \iota`x) : y \in \beta - \iota`x . yPz . \supset_y .$

$y \in \alpha - \iota`x :$

[$*$24·43] $\equiv : \alpha - \iota`x, \beta - \iota`x \in \text{Cl}`C`P : (\exists z) . z \in \alpha - \iota`x - (\beta - \iota`x) .$

$\overrightarrow{P}`z \cap (\beta - \iota`x) \subset \alpha - \iota`x :$

[$*$170·11] $\equiv : (\alpha - \iota`x) P_{\text{cl}} (\beta - \iota`x) :$

[$*$51·221] $\equiv : \alpha \{(\iota`x \cup)_{;} P_{\text{cl}}\} \beta :: \supset \vdash . \text{Prop}$

***170·62.** $\vdash :. x \sim\epsilon C`P . \dot{\exists} ! P . x \epsilon \alpha - \beta . \supset :$

$\alpha (x \leftarrow\!\!\!\!+ P)_{cl} \beta . \equiv . \alpha \subset \iota`x \cup C`P . \beta \subset C`P$

Dem.

$\vdash . *161·13 . \supset \vdash :. \text{Hp} . \supset : x \sim\epsilon \text{Cl}`(x \leftarrow\!\!\!\!+ P) :$

$[*10·53] \quad \supset : y \epsilon \beta . y (x \leftarrow\!\!\!\!+ P) x . \supset_y . y \epsilon \alpha :$

$[\text{Hp}] \quad \supset : x \epsilon \alpha - \beta : y \epsilon \beta . y (x \leftarrow\!\!\!\!+ P) x . \supset_y . y \epsilon \alpha :$

$[*170·11] \quad \supset : \alpha, \beta \epsilon \text{Cl}`C`(x \leftarrow\!\!\!\!+ P) . \supset . \alpha (x \leftarrow\!\!\!\!+ P)_{cl} \beta :$

$[*161·14.\text{Hp}.*24·49] \quad \supset : \alpha \subset \iota`x \cup C`P . \beta \subset C`P . \supset . \alpha (x \leftarrow\!\!\!\!+ P)_{cl} \beta \quad (1)$

$\vdash . *170·11 . *161·14 . \supset$

$\vdash :. \text{Hp} . \supset : \alpha (x \leftarrow\!\!\!\!+ P)_{cl} \beta . \supset . \alpha, \beta \epsilon \text{Cl}`(\iota`x \cup C`P) .$

$[*24·49.\text{Hp}] \quad \supset . \alpha \subset \iota`x \cup C`P . \beta \subset C`P \quad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***170·63.** $\vdash :. x \sim\epsilon (\alpha \cup \beta) . \supset : \alpha (x \leftarrow\!\!\!\!+ P)_{cl} \beta . \equiv . \alpha P_{cl} \beta$

Dem.

$\vdash . *24·49 . *161·14 . \supset \vdash :. \text{Hp} . \supset : \alpha, \beta \epsilon \text{Cl}`C`(x \leftarrow\!\!\!\!+ P) . \equiv . \alpha, \beta \epsilon \text{Cl}`C`P \quad (1)$

$\vdash . *13·14 . \supset \vdash :: \text{Hp} . \supset :. y \epsilon \beta . \supset : y \neq x :$

$[*161·111] \quad \supset : y (x \leftarrow\!\!\!\!+ P) z . \equiv . y P z \quad (2)$

$\vdash . *170·11 . \supset \vdash :: \text{Hp} . \supset :. \alpha (x \leftarrow\!\!\!\!+ P)_{cl} \beta . \equiv :$

$\alpha, \beta \epsilon \text{Cl}`(x \leftarrow\!\!\!\!+ P) : (\exists z) : z \epsilon \alpha - \beta : y \epsilon \beta . y (x \leftarrow\!\!\!\!+ P) z . \supset_y . y \epsilon \alpha :$

$[(1).(2)] \quad \equiv : \alpha, \beta \epsilon \text{Cl}`C`P : (\exists z) : z \epsilon \alpha - \beta : y \epsilon \beta . y P z . \supset_y . y \epsilon \alpha :$

$[*170·11] \equiv : \alpha P_{cl} \beta :: \supset \vdash . \text{Prop}$

***170·64.** $\vdash : x \sim\epsilon C`P . \supset . (x \leftarrow\!\!\!\!+ P)_{cl} = (\iota`x \cup)`` P_{cl} \,\hat{+}\, P_{cl}$

Dem.

$\vdash . *170·61·62·63·37 . \supset$

$\vdash :: \text{Hp} . \dot{\exists} ! P . \supset :. \alpha (x \leftarrow\!\!\!\!+ P)_{cl} \beta . \equiv :$

$x \epsilon \alpha \cap \beta . \alpha \{(\iota`x \cup)`` P_{cl}\} \beta . \vee . x \epsilon \alpha - \beta . \alpha \epsilon C`(\iota`x \cup)`` P_{cl} . \beta \epsilon C`P_{cl} . \vee .$

$x \sim\epsilon (\alpha \cup \beta) . \alpha P_{cl} \beta \quad (1)$

$\vdash . *150·4 . \quad \supset \vdash : \alpha \{(\iota`x \cup)`` P\} \beta . \supset . x \epsilon \alpha \cap \beta \quad (2)$

$\vdash . *150·22 . *170·37 . \supset \vdash :. \text{Hp} . \supset : \alpha \epsilon C`(\iota`x \cup)`` P_{cl} . \beta \epsilon C`P_{cl} . \supset . x \epsilon \alpha - \beta \quad (3)$

$\vdash . *170·1 . \quad \supset \vdash :. \text{Hp} . \supset : \alpha P_{cl} \beta . \supset . x \sim\epsilon (\alpha \cap \beta) \quad (4)$

$\vdash . (1) . (2) . (3) . (4) . \supset \vdash :: \text{Hp} . \dot{\exists} ! P . \supset :.$

$\alpha (x \leftarrow\!\!\!\!+ P)_{cl} \beta . \equiv : \alpha \{(\iota`x \cup)`` P_{cl}\} \beta . \vee . \alpha \epsilon C`(\iota`x \cup)`` P_{cl} . \beta \epsilon C`P_{cl} . \vee . \alpha P_{cl} \beta :$

$[*160·11] \quad \equiv : \alpha \{(\iota`x \cup)`` P_{cl} \,\hat{+}\, P_{cl}\} \beta \quad (5)$

$\vdash . *161·201 . \supset \vdash : P = \dot{\Lambda} . \supset . x \leftarrow\!\!\!\!+ P = \dot{\Lambda} .$

$[*170·35] \quad \supset . (x \leftarrow\!\!\!\!+ P)_{cl} = \Lambda \quad (6)$

$\vdash . *150·42 . *160·22 . *170·35 . \supset \vdash : P = \dot{\Lambda} . \supset . (\iota`x \cup)`` P_{cl} \,\hat{+}\, P_{cl} = \dot{\Lambda} \quad (7)$

$\vdash . (6) . (7) . \supset \vdash : P = \dot{\Lambda} . \supset . (x \leftarrow\!\!\!\!+ P)_{cl} = (\iota`x \cup)`` P_{cl} \,\hat{+}\, P_{cl} \quad (8)$

$\vdash . (5) . (8) . \supset \vdash . \text{Prop}$

The following propositions are lemmas for $*170{\cdot}67$, *i.e.*

$$\dot{\exists}\,!\,P\,.\,\dot{\exists}\,!\,Q\,.\,C\text{‘}P \cap C\text{‘}Q = \Lambda\,.\supset.\,(P \ddagger Q)_{\text{cl}} = s\text{;}C\text{;}(P_{\text{cl}} \times Q_{\text{cl}}),$$

which itself leads to $*170{\cdot}69$, *i.e.*

$$\dot{\exists}\,!\,P\,.\,\dot{\exists}\,!\,Q\,.\,C\text{‘}P \cap C\text{‘}Q = \Lambda\,.\supset.\,(P \ddagger Q)_{\text{cl}}\,\text{smor}\,(P_{\text{cl}} \times Q_{\text{cl}}).$$

$*170{\cdot}65$. $\vdash :.\, \rho\,(P \ddagger Q)_{\text{cl}}\,\sigma\,.\equiv:(\exists\alpha,\beta,\gamma,\delta):\alpha,\beta\,\epsilon\,\text{Cl}\text{‘}C\text{‘}P\,.\,\gamma,\delta\,\epsilon\,\text{Cl}\text{‘}C\text{‘}Q\,.$

$\rho = \alpha \cup \gamma\,.\,\sigma = \beta \cup \delta : (\exists y)\,.\,y\,\epsilon\,(\alpha \cup \gamma) - (\beta \cup \delta)\,.\,\overrightarrow{P \ddagger Q}\text{‘}y \cap (\beta \cup \delta) \subset \alpha \cup \gamma$

Dem.

$\vdash .\,*13{\cdot}193\,.\supset\vdash:.\,(\exists\alpha,\beta,\gamma,\delta):\alpha,\beta\,\epsilon\,\text{Cl}\text{‘}C\text{‘}P\,.\,\gamma,\delta\,\epsilon\,\text{Cl}\text{‘}C\text{‘}Q\,.\,\rho = \alpha \cup \gamma\,.\,\sigma = \beta \cup \delta :$

$(\exists y)\,.\,y\,\epsilon\,(\alpha \cup \gamma) - (\beta \cup \delta)\,.\,\overrightarrow{P \ddagger Q}\text{‘}y \cap (\beta \cup \delta) \subset \alpha \cup \gamma :$

$\equiv:(\exists\alpha,\beta,\gamma,\delta):\alpha,\beta\,\epsilon\,\text{Cl}\text{‘}C\text{‘}P\,.\,\gamma,\delta\,\epsilon\,\text{Cl}\text{‘}C\text{‘}Q\,.\,\rho = \alpha \cup \gamma\,.\,\sigma = \beta \cup \delta :$

$(\exists y)\,.\,y\,\epsilon\,\rho - \sigma\,.\,\overrightarrow{P \ddagger Q}\text{‘}y \cap \sigma \subset \rho :$

$[*60{\cdot}45]\ \equiv:\rho,\sigma\,\epsilon\,\text{Cl}\text{‘}(C\text{‘}P \cup C\text{‘}Q):(\exists y)\,.\,y\,\epsilon\,\rho - \sigma\,.\,\overrightarrow{P \ddagger Q}\text{‘}y \cap \sigma \subset \rho :$

$[*160{\cdot}14]\equiv:\rho,\sigma\,\epsilon\,\text{Cl}\text{‘}C\text{‘}(P \ddagger Q):(\exists y)\,.\,y\,\epsilon\,\rho - \sigma\,.\,\overrightarrow{P \ddagger Q}\text{‘}y \cap \sigma \subset \rho :$

$[*170{\cdot}11]\equiv:\rho\,(P \ddagger Q)_{\text{cl}}\,\sigma :.\supset\vdash.\,\text{Prop}$

$*170{\cdot}651$. $\vdash :.\, C\text{‘}P \cap C\text{‘}Q = \Lambda\,.\,\alpha,\beta\,\epsilon\,\text{Cl}\text{‘}C\text{‘}P\,.\,\gamma,\delta\,\epsilon\,\text{Cl}\text{‘}C\text{‘}Q\,.\,y\,\epsilon\,\alpha\,.\supset:$

$y\,\epsilon\,(\alpha \cup \gamma) - (\beta \cup \delta)\,.\,\overrightarrow{P \ddagger Q}\text{‘}y \cap (\beta \cup \delta) \subset \alpha \cup \gamma\,.\equiv.\,y\,\epsilon\,\alpha - \beta\,.\,\overrightarrow{P}\text{‘}y \cap \beta \subset \alpha$

Dem.

$\vdash .\,*24{\cdot}402{\cdot}313\,.\supset\vdash:\text{Hp}\,.\supset.\,(\alpha \cup \gamma) - (\beta \cup \delta) = (\alpha - \beta) \cup (\gamma - \delta)$ (1)

$\vdash .\,*160{\cdot}11\,.\quad\supset\vdash:\text{Hp}\,.\supset.\,\overrightarrow{P \ddagger Q}\text{‘}y = \overrightarrow{P}\text{‘}y$ (2)

$\vdash .\,*24{\cdot}402\,.\quad\supset\vdash:.\,\text{Hp}\,.\supset:y \sim\epsilon\,\gamma :$

$[(1)]\qquad\supset:y\,\epsilon\,(\alpha \cup \gamma) - (\beta \cup \delta)\,.\equiv.\,y\,\epsilon\,\alpha - \beta$ (3)

$\vdash .\,*33{\cdot}15{\cdot}161\,.\quad\supset\vdash.\,\overrightarrow{P}\text{‘}y \subset C\text{‘}P\,.$

$[*24{\cdot}402]\qquad\supset\vdash:\text{Hp}\,.\supset.\,\overrightarrow{P}\text{‘}y \cap \delta = \Lambda\,.$

$[(2)]\qquad\supset.\,\overrightarrow{P \ddagger Q}\text{‘}y \cap (\beta \cup \delta) = \overrightarrow{P}\text{‘}y \cap \beta\,.$ (4)

$[*24{\cdot}402]\qquad\supset.\,\overrightarrow{P \ddagger Q}\text{‘}y \cap (\beta \cup \delta) \cap \gamma = \Lambda$ (5)

$\vdash .\,(4)\,.\,(5)\,.\,*24{\cdot}49\,.\supset\vdash:.\,\text{Hp}\,.\supset:\overrightarrow{P \ddagger Q}\text{‘}y \cap (\beta \cup \delta) \subset \alpha \cup \gamma\,.\equiv.$

$\overrightarrow{P}\text{‘}y \cap \beta \subset \alpha$ (6)

$\vdash .\,(3)\,.\,(6)\,.\supset\vdash.\,\text{Prop}$

$*170{\cdot}652$. $\vdash :.\, C\text{‘}P \cap C\text{‘}Q = \Lambda\,.\,\alpha,\beta\,\epsilon\,\text{Cl}\text{‘}C\text{‘}P\,.\,\gamma,\delta\,\epsilon\,\text{Cl}\text{‘}C\text{‘}Q\,.\,y\,\epsilon\,\gamma\,.\supset:$

$y\,\epsilon\,(\alpha \cup \gamma) - (\beta \cup \delta)\,.\,\overrightarrow{P \ddagger Q}\text{‘}y \cap (\beta \cup \delta) \subset \alpha \cup \gamma\,.\equiv.$

$\beta \subset \alpha\,.\,y\,\epsilon\,\gamma - \delta\,.\,\overrightarrow{Q}\text{‘}y \cap \delta \subset \gamma$

Dem.

$\vdash .\,*24{\cdot}402{\cdot}313\,.\supset\vdash:\text{Hp}\,.\supset.\,(\alpha \cup \gamma) - (\beta \cup \delta) = (\alpha - \beta) \cup (\gamma - \delta)$ (1)

$\vdash .\,*24{\cdot}402\,.\quad\supset\vdash:\text{Hp}\,.\supset.\,y \sim\epsilon\,\alpha$ (2)

$\vdash.(1).(2). \supset\vdash:. \mathrm{Hp}.\supset: y\in(\alpha\cup\gamma)-(\beta\cup\delta).\equiv.y\in\gamma-\delta$ (3)

$\vdash.*160\cdot11.\supset\vdash:\mathrm{Hp}.\supset.\overrightarrow{P\ddagger Q}`y=C`P\cup\overrightarrow{Q}`y.$

$[*22\cdot621.*24\cdot402]\quad\supset.\overrightarrow{P\ddagger Q}`y\cap(\beta\cup\delta)=\beta\cup(\overrightarrow{Q}`y\cap\delta)$ (4)

$\vdash.*24\cdot49.\supset\vdash:.\mathrm{Hp}.\supset:\beta\subset\alpha\cup\gamma.\equiv.\beta\subset\alpha:$

$\overrightarrow{Q}`y\cap\delta\subset\alpha\cup\gamma.\equiv.\overrightarrow{Q}`y\cap\delta\subset\gamma$ (5)

$\vdash.(4).(5).\supset\vdash:.\mathrm{Hp}.\supset:$

$\overrightarrow{P\ddagger Q}`y\cap(\beta\cup\delta)\subset\alpha\cup\gamma.\equiv.\beta\subset\alpha.\overrightarrow{Q}`y\cap\delta\subset\gamma$ (6)

$\vdash.(3).(6).\supset\vdash.\mathrm{Prop}$

***170·653.** $\vdash:: C`P\cap C`Q=\Lambda.\alpha,\beta\in\mathrm{Cl}`C`P.\gamma,\delta\in\mathrm{Cl}`C`Q.\supset:.$

$(\alpha\cup\gamma)(P\ddagger Q)_{\mathrm{cl}}(\beta\cup\delta).\equiv:\alpha P_{\mathrm{cl}}\beta.\vee.\alpha=\beta.\gamma Q_{\mathrm{cl}}\delta$

Dem.

$\vdash.*170\cdot11.\supset\vdash::\mathrm{Hp}.\supset:.(\alpha\cup\gamma)(P\ddagger Q)_{\mathrm{cl}}(\beta\cup\delta).\equiv:$

$(\exists y).y\in(\alpha\cup\gamma)-(\beta\cup\delta).\overrightarrow{P\ddagger Q}`y\cap(\beta\cup\delta)\subset\alpha\cup\gamma:$

$[*170\cdot651\cdot652]\equiv:(\exists y).y\in\alpha-\beta.\overrightarrow{P}`y\cap\beta\subset\alpha:\vee:$

$\beta\subset\alpha:(\exists y).y\in\gamma-\delta.\overrightarrow{Q}`y\cap\delta\subset\beta:$

$[*170\cdot11]\quad\equiv:\alpha P_{\mathrm{cl}}\beta.\vee.\beta\subset\alpha.\gamma Q_{\mathrm{cl}}\delta:$

$[*170\cdot16]\quad\equiv:\alpha P_{\mathrm{cl}}\beta.\vee.\alpha P_{\mathrm{cl}}\beta.\gamma Q_{\mathrm{cl}}\delta.\vee.\alpha=\beta.\gamma Q_{\mathrm{cl}}\delta:$

$[*4\cdot44]\quad\equiv:\alpha P_{\mathrm{cl}}\beta.\vee.\alpha=\beta.\gamma Q_{\mathrm{cl}}\delta::\supset\vdash.\mathrm{Prop}$

***170·66.** $\vdash:.\dot{\exists}!P.\dot{\exists}!Q.C`P\cap C`Q=\Lambda.\supset:$

$\rho(P\ddagger Q)\sigma.\equiv.(\exists\alpha,\beta,\gamma,\delta).(\gamma\downarrow\alpha)(P_{\mathrm{cl}}\times Q_{\mathrm{cl}})(\delta\downarrow\beta).\rho=\alpha\cup\gamma.\sigma=\beta\cup\delta$

Dem.

$\vdash.*170\cdot65\cdot11.\supset$

$\vdash:\rho(P\ddagger Q)_{\mathrm{cl}}\sigma.\equiv.(\exists\alpha,\beta,\gamma,\delta).\alpha,\beta\in\mathrm{Cl}`C`P.\gamma,\delta\in\mathrm{Cl}`C`Q.$

$\rho=\alpha\cup\gamma.\sigma=\beta\cup\delta.(\alpha\cup\gamma)(P\ddagger Q)_{\mathrm{cl}}(\beta\cup\delta)$ (1)

$\vdash.(1).*170\cdot653.\supset\vdash::\mathrm{Hp}.\supset:.$

$\rho(P\ddagger Q)_{\mathrm{cl}}\sigma.\equiv:(\exists\alpha,\beta,\gamma,\delta):\alpha,\beta\in\mathrm{Cl}`C`P.\gamma,\delta\in\mathrm{Cl}`C`Q.\rho=\alpha\cup\gamma.\sigma=\beta\cup\delta:$

$\alpha P_{\mathrm{cl}}\beta.\vee.\alpha=\beta.\gamma Q_{\mathrm{cl}}\delta:$

$[*170\cdot37]\quad\equiv:(\exists\alpha,\beta,\gamma,\delta):\alpha,\beta\in C`P_{\mathrm{cl}}.\gamma,\delta\in C`Q_{\mathrm{cl}}.\rho=\alpha\cup\gamma.\sigma=\beta\cup\delta:$

$\alpha P_{\mathrm{cl}}\beta.\vee.\alpha=\beta.\gamma Q_{\mathrm{cl}}\delta:$

$[*166\cdot112]\quad\equiv:(\exists\alpha,\beta,\gamma,\delta).(\gamma\downarrow\alpha)(P_{\mathrm{cl}}\times Q_{\mathrm{cl}})(\delta\downarrow\beta).\rho=\alpha\cup\gamma.\sigma=\beta\cup\delta::$

$\supset\vdash.\mathrm{Prop}$

***170·67.** $\vdash:\dot{\exists}!P.\dot{\exists}!Q.C`P\cap C`Q=\Lambda.\supset.(P\ddagger Q)_{\mathrm{cl}}=s`C;(P_{\mathrm{cl}}\times Q_{\mathrm{cl}})$

Dem.

$\vdash.*170\cdot66.*13\cdot22.\supset\vdash::\mathrm{Hp}.\supset:.\rho(P\ddagger Q)\sigma.\equiv:$

$(\exists\alpha,\beta,\gamma,\delta,R,S).R=\gamma\downarrow\alpha.S=\delta\downarrow\beta.\rho=\alpha\cup\gamma.\sigma=\beta\cup\delta.$

$R(P_{\mathrm{cl}}\times Q_{\mathrm{cl}})S:$

[$*55\cdot15.*53\cdot11$] $\equiv : (\exists \alpha, \beta, \gamma, \delta, R, S) . R = \gamma \downarrow \alpha . S = \delta \downarrow \beta .$
$\rho = s`C`R . \sigma = s`C`S . R (P_{cl} \times Q_{cl}) S :$

[$*166\cdot111$] $\equiv : (\exists R, S) . \rho = s`C`R . \sigma = s`C`S . R (P_{cl} \times Q_{cl}) S :$

[$*150\cdot4$] $\equiv : \rho \{s;C;(P_{cl} \times Q_{cl})\} \sigma :: \supset \vdash .$ Prop

$*170\cdot68$. $\vdash : \dot{\exists} ! P . \dot{\exists} ! Q . C`P \cap C`Q = \Lambda . \supset .$
$(s \mid C) \upharpoonright C`(P_{cl} \times Q_{cl}) \in (P \ddagger Q)_{cl} \overline{\text{smor}} (P_{cl} \times Q_{cl})$

Dem.

$\vdash . *55\cdot15 . *53\cdot11 . \supset$

$\vdash :. R = \gamma \downarrow \alpha . S = \delta \downarrow \beta . s`C`R = s`C`S . \supset . \alpha \cup \gamma = \beta \cup \delta$ (1)

$\vdash . (1) . *24\cdot48 . \supset$

$\vdash ::$ Hp $. \supset :. \alpha, \beta \in \text{Cl}`C`P . \gamma, \delta \in \text{Cl}`C`Q . R = \gamma \downarrow \alpha . S = \delta \downarrow \beta . s`C`R = s`C`S . \supset .$
$\alpha = \beta . \gamma = \delta .$

[$*55\cdot202$] $\supset . R = S$ (2)

$\vdash . (2) . *166\cdot12 . *170\cdot37 . \supset$

$\vdash :.$ Hp $. \supset : R, S \in C`(P_{cl} \times Q_{cl}) . s`C`R = s`C`S . \supset . R = S$ (3)

$\vdash . (3) . *151\cdot24 . *170\cdot67 . \supset \vdash .$ Prop

$*170\cdot69$. $\vdash : \dot{\exists} ! P . \dot{\exists} ! Q . C`P \cap C`Q = \Lambda . \supset . (P \ddagger Q)_{cl} \text{ smor } (P_{cl} \times Q_{cl})$
[$*170\cdot68$]

*171. THE PRINCIPLE OF FIRST DIFFERENCES (*continued*)

Summary of *171.

In this number, we shall consider a more restricted form of the principle of first differences, which is applicable when there is a definite first member of one class not belonging to the other class. In this case, if z is the first differing member, the part of α which precedes z is to be the same as the part of β which precedes z. If z belongs to α and not to β, we put α before β; in the converse case, we put β before α. In case zPz, z itself is not to be counted among its own predecessors; thus the predecessors of z are to be $\overrightarrow{P}\text{‘}z - \iota\text{‘}z$. Hence the relation in question will hold between two sub-classes (α and β) of $C\text{‘}P$ when there is a z such that

$$z \epsilon \alpha - \beta \,.\, \overrightarrow{P}\text{‘}z - \iota\text{‘}z \cap \alpha = \overrightarrow{P}\text{‘}z - \iota\text{‘}z \cap \beta,$$

or, what comes to the same thing (owing to $z \sim\epsilon \beta$),

$$z \epsilon \alpha - \beta \,.\, \overrightarrow{P}\text{‘}z \cap \alpha - \iota\text{‘}z = \overrightarrow{P}\text{‘}z \cap \beta.$$

This relation between α and β we denote by "P_{df}," where "df" stands for "difference."

Thus our definition is

$$P_{\text{df}} = \hat{\alpha}\hat{\beta}\,\{\alpha, \beta \epsilon \text{Cl}\text{‘}C\text{‘}P : (\exists z) \,.\, z \epsilon \alpha - \beta \,.\, \overrightarrow{P}\text{‘}z \cap \alpha - \iota\text{‘}z = \overrightarrow{P}\text{‘}z \cap \beta\} \quad \text{Df.}$$

On the analogy of P_{lc}, we put also

$$P_{\text{fd}} = \text{Cnv}\text{‘}(\breve{P})_{\text{df}}.$$

When P is well-ordered, P_{df} and P_{fd} coincide respectively with P_{cl} and P_{lc}. Their properties are closely analogous to those of P_{cl} and P_{lc}. Thus *e.g.* the following propositions remain true when P_{df} is substituted for P_{cl}:

*170·17·35·36·37·38·44·5·51·52·64·67·68·69.

The only new propositions to be noted in this number are

***171·2.** $\vdash : P \in J \,.\, \supset .$

$$P_{\text{df}} = \hat{\alpha}\hat{\beta}\,\{\alpha, \beta \epsilon \text{Cl}\text{‘}C\text{‘}P : (\exists z) \,.\, z \epsilon \alpha - \beta \,.\, \overrightarrow{P}\text{‘}z \cap \alpha = \overrightarrow{P}\text{‘}z \cap \beta\}$$

***171·21.** $\vdash . P_{\text{df}} \in P_{\text{cl}}$

and the following formulae suggesting an inductive identification of P_{cl} and P_{df} in cases to which such induction is applicable:

***171·7.** $\vdash : P_{\text{df}} = P_{\text{cl}} \,.\, x \sim\epsilon C\text{‘}P \,.\, \supset .\, (x \leftarrow\!\!\!+ P)_{\text{df}} = (x \leftarrow\!\!\!+ P)_{\text{cl}}$

***171·71.** $\vdash : C\text{‘}P \cap C\text{‘}Q = \Lambda \,.\, P_{\text{df}} = P_{\text{cl}} \,.\, Q_{\text{df}} = Q_{\text{cl}} \,.\, \supset .\, (P \updownarrow Q)_{\text{df}} = (P \updownarrow Q)_{\text{cl}}$

These propositions are however superseded (at a later stage) by the proof that P_{cl} and P_{df} coincide if P is well-ordered (*251·37).

The chief property of P_{df} is that its relation-number is 2_r to the power $\text{Nr}\text{‘}P$. This will be proved in *177 and *186·4.

***171·01.** $P_{df} = \hat{\alpha}\hat{\beta}\{\alpha, \beta \epsilon \text{Cl}`C`P : (\exists z) . z \epsilon \alpha - \beta . \overrightarrow{P}`z \cap \alpha - \iota`z = \overrightarrow{P}`z \cap \beta\}$ Df

***171·02.** $P_{fd} = \text{Cnv}`(\breve{P})_{df}$ Df

***171·1.** $\vdash :. \alpha P_{df} \beta . \equiv : \alpha, \beta \epsilon \text{Cl}`C`P : (\exists z) . z \epsilon \alpha - \beta . \overrightarrow{P}`z \cap \alpha - \iota`z = \overrightarrow{P}`z \cap \beta$ [(*171·01)]

***171·101.** $\vdash . P_{fd} = \text{Cnv}`(\breve{P})_{df}$ [(*171·02)]

***171·102.** $\vdash :. \alpha P_{fd} \beta . \equiv : \alpha, \beta \epsilon \text{Cl}`C`P : (\exists z) . z \epsilon \beta - \alpha . \overleftarrow{P}`z \cap \beta - \iota`z = \overleftarrow{P}`z \cap \alpha$ [*171·1·101]

***171·11.** $\vdash ::. \alpha P_{df} \beta . \equiv :: \alpha, \beta \epsilon \text{Cl}`C`P ::$
$(\exists z) :. z \epsilon \alpha - \beta :. yPz . y \neq z . \supset_y : y \epsilon \alpha . \equiv . y \epsilon \beta$ [*171·1]

***171·12.** $\vdash :. \alpha P_{df} \beta . \equiv : \alpha, \beta \epsilon \text{Cl}`C`P :$
$(\exists z) . z \epsilon \alpha - \beta . \overrightarrow{P}`z \cap \alpha - \iota`z = \overrightarrow{P}`z \cap \beta - \iota`z$ [*171·1 . *51·222]

***171·13.** $\vdash . C`P_{df} \subset \text{Cl}`C`P$ [*171·1]

***171·14.** $\vdash : \alpha \subset C`P . z \epsilon \alpha . \supset . \alpha P_{df} (\alpha - \iota`z)$

Dem.

$\vdash .$ *51·21 $. \supset \vdash : \text{Hp} . \supset . z \epsilon \alpha - (\alpha - \iota`z) .$

[*13·15] $\supset . z \epsilon \alpha - (\alpha - \iota`z) . \overrightarrow{P}`z \cap \alpha - \iota`z = \overrightarrow{P}`z \cap (\alpha - \iota`z) .$

[*171·12] $\supset . \alpha P_{df} (\alpha - \iota`z) : \supset \vdash . \text{Prop}$

***171·15.** $\vdash : \beta \subset C`P . z \epsilon C`P - \beta . \supset . (\beta \cup \iota`z) P_{df} \beta$

Dem.

$\vdash .$ *51·16 $. \supset \vdash : \text{Hp} . \supset . z \epsilon (\beta \cup \iota`z) - \beta$ (1)

$\vdash .$ *51·211·22 $. \supset \vdash : \text{Hp} . \supset . (\beta \cup \iota`z) - \iota`z = \beta .$

[*22·481] $\supset . \overrightarrow{P}`z \cap (\beta \cup \iota`z) - \iota`z = \overrightarrow{P}`z \cap \beta$ (2)

$\vdash . (1) . (2) .$ *171·1 $. \supset \vdash . \text{Prop}$

***171·16.** $\vdash . \text{D}`P_{df} = \text{Cl ex}`C`P . \text{Ⴇ}`P_{df} = \text{Cl}`C`P - \iota`C`P$

Dem.

$\vdash .$ *171·14 $. \supset \vdash : \alpha \epsilon \text{Cl ex}`C`P . \supset . \alpha \epsilon \text{D}`P_{df}$ (1)

$\vdash .$ *171·1 $. \supset \vdash : \alpha \epsilon \text{D}`P_{df} . \supset . \alpha \epsilon \text{Cl ex}`C`P$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{D}`P_{df} = \text{Cl ex}`C`P$ (3)

$\vdash .$ *171·15 $. \supset \vdash : \beta \epsilon \text{Cl}`C`P . \exists ! C`P - \beta . \supset . \beta \epsilon \text{Ⴇ}`P_{df} :$

[*24·6] $\supset \vdash : \beta \epsilon \text{Cl}`C`P - \iota`C`P . \supset . \beta \epsilon \text{Ⴇ}`P_{df}$ (4)

$\vdash .$ *171·1 $. \supset \vdash : \beta \epsilon \text{Ⴇ}`P_{df} . \supset . \beta \epsilon \text{Cl}`C`P . \exists ! C`P - \beta .$

[*24·6] $\supset . \beta \epsilon \text{Cl}`C`P - \iota`C`P$ (5)

$\vdash . (4) . (5) . \supset \vdash . \text{Ⴇ}`P_{df} = \text{Cl}`C`P - \iota`C`P$ (6)

$\vdash . (3) . (6) . \supset \vdash . \text{Prop}$

***171·17.** $\vdash : \dot{\exists} ! P . \supset . C`P_{df} = \mathrm{Cl}`C`P$

Dem.

$\vdash . *171·16 . \supset \vdash : \alpha \epsilon \mathrm{Cl}`C`P . \alpha \neq \Lambda . \supset . \alpha \epsilon \mathrm{D}`P_{df}$ (1)

$\vdash . *171·16 . \supset \vdash : \alpha \epsilon \mathrm{Cl}`C`P . \alpha \neq C`P . \supset . \alpha \epsilon \mathrm{G}`P_{df}$ (2)

$\vdash . (1) . (2) . \supset \vdash : \alpha \epsilon \mathrm{Cl}`C`P . \sim(\alpha = \Lambda . \alpha = C`P) . \supset . \alpha \epsilon C`P_{df} :$

[*13·171] $\supset \vdash : \alpha \epsilon \mathrm{Cl}`C`P . C`P \neq \Lambda . \supset . \alpha \epsilon C`P_{df}$ (3)

$\vdash . (3) . *33·24 . \supset \vdash .$ Prop

***171·18.** $\vdash : \dot{\exists} ! P . \supset . B`P_{df} = C`P . B`\mathrm{Cnv}`P_{df} = \Lambda$

Dem.

$\vdash . *171·16 . \quad \supset \vdash . \overrightarrow{B}`P_{df} = \mathrm{Cl\,ex}`C`P - (\mathrm{Cl}`C`P - \iota`C`P)$

[*24·3] $= \mathrm{Cl\,ex}`C`P \cap \iota`C`P$ (1)

$\vdash . (1) . *60·35 . \supset \vdash : \dot{\exists} ! P . \supset . \overrightarrow{B}`P_{df} = \iota`C`P$ (2)

$\vdash . *171·16 . \quad \supset \vdash . \overrightarrow{B}`\mathrm{Cnv}`P_{df} = \mathrm{Cl}`C`P - \iota`C`P - \mathrm{Cl\,ex}`C`P$

[*60·24] $= \iota`\Lambda - \iota`C`P$ (3)

$\vdash . (3) . *33·24 . \supset \vdash : \dot{\exists} ! P . \supset . \overrightarrow{B}`\mathrm{Cnv}`P_{df} = \iota`\Lambda$ (4)

$\vdash . (2) . (4) . \supset \vdash .$ Prop

***171·19.** $\vdash : P = \dot{\Lambda} . \supset . P_{df} = \dot{\Lambda}$

Dem.

$\vdash . *60·33 . *171·16 . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{D}`P_{df} = \Lambda .$

[*33·241] $\supset . P_{df} = \dot{\Lambda} : \supset \vdash .$ Prop

***171·2.** $\vdash : P \in J . \supset . P_{df} = \hat{\alpha}\hat{\beta}\{\alpha, \beta \epsilon \mathrm{Cl}`C`P : (\exists z) . z \epsilon \alpha - \beta . \overrightarrow{P}`z \cap \alpha = \overrightarrow{P}`z \cap \beta\}$

Dem.

$\vdash . *50·11 . *32·19 . \supset \vdash : \mathrm{Hp} . \supset . \overrightarrow{P}`z \subset -\iota`z .$

[*22·621] $\supset . \overrightarrow{P}`z \cap \alpha - \iota`z = \overrightarrow{P}`z \cap \alpha$ (1)

$\vdash . (1) . *171·1 . \supset \vdash .$ Prop

***171·21.** $\vdash . P_{df} \in P_{cl}$

Dem.

$\vdash . *171·1 . *22·43 . \supset$

$\vdash :. \alpha P_{df} \beta . \supset : \alpha, \beta \epsilon \mathrm{Cl}`C`P : (\exists z) . z \epsilon \alpha - \beta . \overrightarrow{P}`z \cap \beta \subset \alpha :$

[*170·11] $\supset : \alpha P_{cl} \beta :. \supset \vdash .$ Prop

***171·22.** $\vdash . P_{df} \in J$ [*170·17 . *171·21]

***171·4.** $\vdash : S \epsilon 1 \to 1 . C`Q = \mathrm{G}`S . \supset . (S;Q)_{df} = S_\epsilon ; Q_{df}$ [Proof as in *170·4]

***171·41.** $\vdash : (S \upharpoonright C`Q)_\epsilon ; Q_{df} = S_\epsilon ; Q_{df}$ [Proof as in *170·41]

***171·42.** $\vdash : S \upharpoonright C`Q \epsilon 1 \to 1 . C`Q \subset \mathrm{G}`S . \supset . (S;Q)_{df} = S_\epsilon ; Q_{df}$ [*171·4·41]

***171·43.** $\vdash : S \upharpoonright C`Q \epsilon P \,\overline{\mathrm{smor}}\, Q . \supset . S_\epsilon \upharpoonright C`Q_{df} \epsilon P_{df} \,\overline{\mathrm{smor}}\, Q_{df}$

[Proof as in *170·43]

***171·44.** $\vdash : P \operatorname{smor} Q . \supset . P_{\mathrm{df}} \operatorname{smor} Q_{\mathrm{df}}$ [*171·43]

***171·5.** $\vdash . (x \downarrow x)_{\mathrm{df}} = (\iota\text{‘}x) \downarrow \Lambda = (x \downarrow x)_{\mathrm{cl}}$

Dem.

$\vdash . *171\cdot1 . *55\cdot15 . \supset$

$\vdash :. \alpha (x \downarrow x)_{\mathrm{df}} \beta . \equiv : \alpha, \beta \in \mathrm{Cl}\text{‘}\iota\text{‘}x : (\exists z) . z \in \alpha - \beta . \overrightarrow{x \downarrow x}\text{‘}z \cap \alpha - \iota\text{‘}z = \overrightarrow{x \downarrow x}\text{‘}z \cap \beta :$

[*171·16] $\equiv : \alpha \in \mathrm{Cl\,ex}\text{‘}\iota\text{‘}x . \beta \in \mathrm{Cl}\text{‘}\iota\text{‘}x - \iota\text{‘}\iota\text{‘}x :$

$(\exists z) . z \in \alpha - \beta . \overrightarrow{x \downarrow x}\text{‘}z \cap \alpha - \iota\text{‘}z = \overrightarrow{x \downarrow x}\text{‘}z \cap \beta :$

[*60·362·37] $\equiv : \alpha = \iota\text{‘}x . \beta = \Lambda : (\exists z) . z \in \iota\text{‘}x . \overrightarrow{x \downarrow x}\text{‘}z \cap \alpha - \iota\text{‘}z = \overrightarrow{x \downarrow x}\text{‘}z \cap \beta :$

[*13·195] $\equiv : \alpha = \iota\text{‘}x . \beta = \Lambda . \iota\text{‘}x \cap \alpha - \iota\text{‘}x = \iota\text{‘}x \cap \beta :$

[*24·21·23] $\equiv : \alpha = \iota\text{‘}x . \beta = \Lambda . \Lambda = \Lambda :$

[*13·15.*55·13] $\equiv : \alpha \{(\iota\text{‘}x) \downarrow \Lambda\} \beta$ (1)

$\vdash . (1) . *170\cdot5 . \supset \vdash .$ Prop

***171·51.** $\vdash . (x \downarrow y)_{\mathrm{df}} = (x \downarrow y)_{\mathrm{cl}}$

Dem.

$\vdash . *171\cdot1 . \supset \vdash :. \alpha (x \downarrow y)_{\mathrm{df}} \beta . \equiv : \alpha, \beta \in \mathrm{Cl}\text{‘}(\iota\text{‘}x \cup \iota\text{‘}y) :$

$(\exists z) . z \in \alpha - \beta . \overrightarrow{x \downarrow y}\text{‘}z \cap \alpha - \iota\text{‘}z = \overrightarrow{x \downarrow y}\text{‘}z \cap \beta :$

[*171·16] $\equiv : \alpha \in \mathrm{Cl\,ex}\text{‘}(\iota\text{‘}x \cup \iota\text{‘}y) . \beta \in \mathrm{Cl}\text{‘}(\iota\text{‘}x \cup \iota\text{‘}y) - \iota\text{‘}(\iota\text{‘}x \cup \iota\text{‘}y) :$

$(\exists z) . z \in \alpha - \beta . \overrightarrow{x \downarrow y}\text{‘}z \cap \alpha - \iota\text{‘}z = \overrightarrow{x \downarrow y}\text{‘}z \cap \beta :$

[*60·39] $\equiv : \alpha = \iota\text{‘}x \cup \iota\text{‘}y . \vee . \alpha = \iota\text{‘}x . \vee . \alpha = \iota\text{‘}y : \beta = \iota\text{‘}x . \vee . \beta = \iota\text{‘}y . \vee . \beta = \Lambda :$

$(\exists z) . z \in \alpha - \beta . \overrightarrow{x \downarrow y}\text{‘}z \cap \alpha - \iota\text{‘}z = \overrightarrow{x \downarrow y}\text{‘}z \cap \beta$ (1)

$\vdash . *55\cdot13 . \supset \vdash :. x \neq y . \supset : \overrightarrow{x \downarrow y}\text{‘}y = \iota\text{‘}x . \overrightarrow{x \downarrow y}\text{‘}x = \Lambda :$ (2)

[*51·222] $\supset : \alpha = \iota\text{‘}x \cup \iota\text{‘}y . \beta = \iota\text{‘}x . \supset .$

$y \in \alpha - \beta . \overrightarrow{x \downarrow y}\text{‘}y \cap \alpha - \iota\text{‘}y = \iota\text{‘}x = \overrightarrow{x \downarrow y}\text{‘}y \cap \beta .$

[(1)] $\supset . \alpha P_{\mathrm{df}} \beta$ (3)

$\vdash . (2) . \supset \vdash : x \neq y . \alpha = \iota\text{‘}x \cup \iota\text{‘}y . \beta = \iota\text{‘}y . \supset .$

$x \in \alpha - \beta . \overrightarrow{x \downarrow y}\text{‘}x \cap \alpha - \iota\text{‘}x = \Lambda = \overrightarrow{x \downarrow y}\text{‘}x \cap \beta .$

[(1)] $\supset . \alpha P_{\mathrm{df}} \beta$ (4)

$\vdash . (2) . \supset \vdash : x \neq y . \alpha = \iota\text{‘}x \cup \iota\text{‘}y . \beta = \Lambda . \supset .$

$x \in \alpha - \beta . \overrightarrow{x \downarrow y}\text{‘}x \cap \alpha - \iota\text{‘}x = \Lambda = \overrightarrow{x \downarrow y}\text{‘}x \cap \beta .$

[(1)] $\supset . \alpha P_{\mathrm{df}} \beta$ (5)

$\vdash . (2) . \supset \vdash :. x \neq y . \alpha = \iota\text{‘}x : \beta = \Lambda . \vee . \beta = \iota\text{‘}y : \supset .$

$x \in \alpha - \beta . \overrightarrow{x \downarrow y}\text{‘}x \cap \alpha - \iota\text{‘}x = \Lambda = \overrightarrow{x \downarrow y}\text{‘}x \cap \beta .$

$\supset . \alpha P_{\mathrm{df}} \beta$ (6)

$\vdash . (2) . *24\cdot23 . \supset \vdash : x \neq y . \alpha = \iota\text{‘}y . \beta = \Lambda . \supset .$

$y \in \alpha - \beta . \overrightarrow{x \downarrow y}\text{‘}x \cap \alpha - \iota\text{‘}y = \Lambda = \overrightarrow{x \downarrow y}\text{‘}y \cap \beta$ (7)

$\vdash . (3) . (4) . (5) . (6) . (7) . *170\cdot51 . \supset \vdash : x \neq y . \supset . (x \downarrow y)_{\mathrm{cl}} \subset (x \downarrow y)_{\mathrm{df}} .$

[*171·21] $\supset . (x \downarrow y)_{\mathrm{df}} = (x \downarrow y)_{\mathrm{cl}}$ (8)

$\vdash . (8) . *171\cdot5 . \supset \vdash .$ Prop

***171·52.** $\vdash : x \neq y . \supset . (x \downarrow y)_{df} = (\iota`x \cup \iota`y) \downarrow (\iota`x) \ddagger (\iota`y) \downarrow \Lambda$
[*171·51 . *170·52]

***171·64.** $\vdash : x \sim \epsilon C`P . \supset . (x \leftarrow\!\!\!+ P)_{df} = (\iota`x \cup)`;P_{df} \ddagger P_{df}$

The proof proceeds by the same stages as the proof of *170·64.

***171·67.** $\vdash : \dot{\exists} ! P . \dot{\exists} ! Q . C`P \cap C`Q = \Lambda . \supset . (P \ddagger Q)_{df} = s`;C`;(P_{df} \times Q_{df})$
[Proof as in *170·67]

***171·68.** $\vdash : \dot{\exists} ! P . \dot{\exists} ! Q . C`P \cap C`Q = \Lambda . \supset .$
$s \,|\, C \upharpoonright (P_{df} \times Q_{df}) \,\epsilon\, (P \ddagger Q)_{df} \,\overline{\text{smor}}\, (P_{df} \times Q_{df})$
[Proof as in *170·68]

***171·69.** $\vdash : \dot{\exists} ! P . \dot{\exists} ! Q . C`P \cap C`Q = \Lambda . \supset . (P \ddagger Q)_{df} \text{ smor } (P_{df} \times Q_{df})$
[*171·68]

***171·7.** $\vdash : P_{df} = P_{cl} . x \sim \epsilon C`P . \supset . (x \leftarrow\!\!\!+ P)_{df} = (x \leftarrow\!\!\!+ P)_{cl}$
[*171·64 . *170·64]

***171·71.** $\vdash : C`P \cap C`Q = \Lambda . P_{df} = P_{cl} . Q_{df} = Q_{cl} . \supset . (P \ddagger Q)_{df} = (P \ddagger Q)_{cl}$
[*170·67 . *171·67 . *160·21·22]

$*172$. THE PRODUCT OF THE RELATIONS OF A FIELD

Summary of $*172$.

In this number we have to consider the form of product which is applicable to any relation of relations, whether mutually exclusive or not. If our relation were a $\mathrm{Rel}^2\,\mathrm{excl}$, we could take $C``C`P$, and order selected classes from $C``C`P$ by first differences. This would give us a relation whose field would be $\mathrm{Prod}`C``C`P$. But if any two fields overlap, this method fails. We might substitute $\epsilon_\Delta`C``C`P$ for $\mathrm{Prod}`C``C`P$, and order the members of $\epsilon_\Delta`C``C`P$ by first differences; but this method will not give what we want if two or more members of $C`P$ have the same field. In order to avoid any confusion due to repetition, we must, if $Q \,\epsilon\, C`P$ and $x \,\epsilon\, C`Q$, consider x in connection with Q, not merely with $C`Q$. That is, the relations in the field of the product of $P$ must be such as concern themselves with the ordered couple $x \downarrow Q$, not merely with $x$. The simplest way of effecting this is to consider $F_\Delta`C`P$. A member of $F_\Delta`C`P$, say $M$, is a relation which picks out a representative of $Q$ from the field of every $Q$ which is a member of $C`P$; that is, whenever $Q \,\epsilon\, C`P$, $M`Q \,\epsilon\, C`Q$. Since we have $M`Q$, not $M`C`Q$, two relations may have the same field and yet we can distinguish the occurrence of a given term as the representative of the one from its occurrence as the representative of the other. Thus no degree of overlapping will cause confusion.

The relations which compose $F_\Delta`C`P$ are to be ordered by first differences, but in order to distinguish different occurrences of a given term, we must give a slightly different form to the principle of first differences from that employed in $*170$ or $*171$. The new form of the principle is as follows: Consider two relations M and N which are members of $F_\Delta`C`P$. Let Q be a member of $C`P$ in which $M$ chooses a representative which precedes that of $N$, *i.e.* in which $(M`Q)\,Q\,(N`Q)$; and let all earlier relations than $Q$, *i.e.* all relations $R$ such that $RPQ$ and $R \neq Q$, have $M`R = N`R$. Then we say that $M$ precedes $N$. This principle may also be stated as follows: We may divide the members of $C`P$ into four classes, not in general mutually exclusive, namely:

(1) those in which $(M`Q)\,Q\,(N`Q)$, *i.e.* in which the M-representative precedes the N-representative;

(2) those in which $(N`Q)\,Q\,(M`Q)$,

(3) those in which $M`Q = N`Q$,

(4) those in which no one of the above three relations of $M`Q$ and $N`Q$ occurs.

Then we shall say that M precedes N if there is a member of class (1) whose predecessors all belong to class (3).

In case all the members of $C'P$ are serial, the fourth of the above classes is null, and the other three are mutually exclusive. If, further, P is well-ordered, any two different members of $F_\Delta{}'C'P$ must be such that one precedes the other in the above-defined order. Thus in this case the product of a series of series is a series (cf. $*251$).

The definition of the product $\Pi'P$ is

$$\Pi'P = \hat{M}\hat{N}\{M, N \in F_\Delta{}'C'P :. (\exists Q) : (M'Q)\, Q\, (N'Q) : RPQ \,.\, R \neq Q \,.\, \supset_R \,.\, M'R = N'R\} \quad \text{Df.}$$

Owing to the complication of this definition, the proofs of propositions of the present number are apt to be long.

Various other definitions might be adopted for $\Pi'P$, but we have found the above definition on the whole the best.

We might, for example, drop the condition $R \neq Q$ in the definition; we could then write our definition in the simpler form:

$$\Pi'P = \hat{M}\hat{N}\{M, N \in F_\Delta{}'C'P : (\exists Q) \,.\, (M'Q)\, Q\, (N'Q) \,.\, M \upharpoonright \overrightarrow{P}'Q = N \upharpoonright \overrightarrow{P}'Q\},$$

which, with our definition, is only available when $P \subset J$. But if we adopt this simplification, we no longer have

$$\Pi'(P \downarrow P) = P \underset{\centerdot\centerdot}{\downarrow} P \quad (*172{\cdot}2),$$

which is a very useful proposition, required in the proofs of $*183{\cdot}13$, $*185{\cdot}21$ and other important propositions.

On the other hand, we might frame our definition on the analogy of P_{cl} rather than, as above, on the analogy of P_{df}. The definition would then be:

$$\Pi'P = \hat{M}\hat{N}\{M, N \in F_\Delta{}'C'P :. (\exists Q) : (M'Q)\, Q\, (N'Q) : RPQ \,.\, \supset_R \,.\, (M'R)\,(R \cup I)\,(N'R)\}.$$

This definition does not assume that there is a *first* relation Q for which the M-representative precedes the N-representative. Thus it might be thought that it would give better results in cases where P is not well-ordered. But in fact this is not the case. If P is not well-ordered, it may happen that every Q for which $(M'Q)\, Q\, (N'Q)$ is preceded by one for which $(N'Q)\, Q\, (M'Q)$, *and vice versa*; in this case, we shall have neither $M\,(\Pi'P)\,N$ nor $N\,(\Pi'P)\,M$. Thus our suggested new definition does not secure that $\Pi'P$ shall be a series whenever P and all the members of $C'P$ are series, and therefore has no substantial advantage over the simpler definition which we have adopted, and has the disadvantage of greater complication.

In the present number, we first prove that $\Pi'\dot{\Lambda} = \dot{\Lambda}$ ($*172{\cdot}13$) and that $\dot{\Lambda} \in C'P \,.\, \supset \,.\, \Pi'P = \dot{\Lambda}$ ($*172{\cdot}14$), so that a product is null if any one of its factors is null. We then proceed to propositions about $C'\Pi'P$, $\overrightarrow{B}'\Pi'P$, etc. We have

$$*172{\cdot}162. \vdash : \dot{\exists} ! P \,.\, \supset \,.\, \overrightarrow{B}'\Pi'P = B_\Delta{}'C'P \,.\, \overrightarrow{B}'\text{Cnv}'\Pi'P = B_\Delta{}'\text{Cnv}''C'P$$

*172·17. $\vdash : \dot{\exists} ! P . \supset . C`\Pi`P = F_\Delta`C`P$

Hence we derive propositions as to the existence of $\Pi`P$. We have

*172·181. $\vdash :. \text{Mult ax} . \supset : \Lambda \sim\epsilon C`P . \dot{\exists} ! P . \equiv . \dot{\exists} ! \Pi`P$

Thus assuming the multiplicative axiom, a product which has factors none of which are null is not null.

We then consider $\Pi`(P \downarrow P)$, and $\Pi`(P \downarrow Q)$ where $P \neq Q$. We have

*172·2. $\vdash . \Pi`(P \downarrow P) = P \underset{\cdot\cdot}{\downarrow} P$

which is a useful proposition, and

*172·23. $\vdash : P \neq Q . \supset . \Pi`(P \downarrow Q) \operatorname{smor} P \times Q$

which connects the two definitions of multiplication, showing that they lead to equivalent results for any finite number of factors, *i.e.* whenever the definition of *166 is applicable.

We next consider $\Pi`(P \nrightarrow Z)$ and $\Pi`(P \ddagger Q)$, proving

*172·32. $\vdash : Z \sim\epsilon C`P . \supset . \Pi`(P \nrightarrow Z) \operatorname{smor} \Pi`P \times Z$

with a similar proposition for $Z \nleftarrow P$ (*172·321), and

*172·35. $\vdash : \dot{\exists} ! P . \dot{\exists} ! Q . C`P \cap C`Q = \Lambda . \supset . \Pi`(P \ddagger Q) \operatorname{smor} \Pi`P \times \Pi`Q$

which is a form of the associative law using both kinds of multiplication. The kind which uses only Π will be proved in *174.

We have next the proof (with its immediate consequences) that if P and Q have double likeness, $\Pi`P \operatorname{smor} \Pi`Q$. We prove

*172·43. $\vdash : T \upharpoonright C`\Sigma`Q \,\epsilon\, P \,\overline{\operatorname{smor}}\, \overline{\operatorname{smor}}\, Q . \supset .$

$$(T \| \operatorname{Cnv}`T\dagger) \upharpoonright C`\Pi`Q \,\epsilon\, (\Pi`P) \,\overline{\operatorname{smor}}\, (\Pi`Q)$$

This proposition should be compared with *114·51, which is its cardinal analogue. It will be seen that the correlator only differs by the substitution of $T\dagger$ for T_ϵ. From *172·43 we obtain

*172·44. $\vdash : P \operatorname{smor} \operatorname{smor} Q . \supset . \Pi`P \operatorname{smor} \Pi`Q$

whence

*172·45. $\vdash :. \text{Mult ax} . \supset : P, Q \,\epsilon\, \text{Rel}^2 \text{excl} . \exists ! P \,\overline{\operatorname{smor}}\, Q \cap \text{Rl}`\text{smor} . \supset .$

$$\Pi`P \operatorname{smor} \Pi`Q$$

Other propositions about $\Pi`P$ will be given in *174.

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*172·01. $\Pi`P = \hat{M}\hat{N} \{M, N \,\epsilon\, F_\Delta`C`P :.$

$$(\exists Q) : (M`Q)\, Q\, (N`Q) : RPQ . R \neq Q . \supset_R . M`R = N`R\} \quad \text{Df}$$

*172·1. $\vdash :: M (\Pi`P) N . \equiv :. M, N \,\epsilon\, F_\Delta`C`P :.$

$$(\exists Q) : (M`Q)\, Q\, (N`Q) : RPQ . R \neq Q . \supset_R . M`R = N`R$$

[(*172·01)]

***172·11.** $\vdash :: M(\Pi\text{‘}P)N \,.\equiv:.\, M,N \,\epsilon\, F_\Delta\text{‘}C\text{‘}P :.$

$(\exists Q) : Q \,\epsilon\, C\text{‘}P \,.\, (M\text{‘}Q)\,Q\,(N\text{‘}Q) : RPQ \,.\, R \neq Q \,.\supset_R .\, M\text{‘}R = N\text{‘}R$

Dem.

$\vdash .\, *14{\cdot}21 \,.\quad \supset \vdash : (M\text{‘}Q)\,Q\,(N\text{‘}Q) \,.\supset.\, \mathrm{E}\,!\, M\text{‘}Q \,.$

$[*33{\cdot}43] \qquad \supset .\, Q \,\epsilon\, \text{Ɑ}\text{‘}M \qquad (1)$

$\vdash .\, (1) \,.\, *80{\cdot}14 \,.\supset \vdash :.\, M \,\epsilon\, F_\Delta\text{‘}C\text{‘}P \,.\supset: (M\text{‘}Q)\,Q\,(N\text{‘}Q) \,.\supset.\, Q \,\epsilon\, C\text{‘}P :$

$[*4{\cdot}73] \qquad \supset : (M\text{‘}Q)\,Q\,(N\text{‘}Q) \,.\equiv.\, Q \,\epsilon\, C\text{‘}P \,.\, (M\text{‘}Q)\,Q\,(N\text{‘}Q) \qquad (2)$

$\vdash .\, (2) \,.\, *172{\cdot}1 \,.\supset \vdash .\,$ Prop

***172·12.** $\vdash .\, C\text{‘}\Pi\text{‘}P \subset F_\Delta\text{‘}C\text{‘}P$

Dem.

$\vdash .\, *172{\cdot}1 \,.\supset \vdash : M(\Pi\text{‘}P)N \,.\supset.\, M,N \,\epsilon\, F_\Delta\text{‘}C\text{‘}P \qquad (1)$

$\vdash .\, (1) \,.\, *33{\cdot}352 \,.\supset \vdash .\,$ Prop

***172·13.** $\vdash .\, \Pi\text{‘}\dot{\Lambda} = \dot{\Lambda}$

Dem.

$\vdash .\, *172{\cdot}11 \,.\quad \supset \vdash : M(\Pi\text{‘}P)N \,.\supset.\, \exists\,!\, C\text{‘}P \,.$

$[*33{\cdot}24] \qquad \supset .\, \dot{\exists}\,!\, P \qquad (1)$

$\vdash .\, (1) \,.\,$ Transp $.\supset \vdash : P = \dot{\Lambda} \,.\supset.\, (M,N) \,.\sim \{M(\Pi\text{‘}P)N\} : \supset \vdash .\,$ Prop

***172·14.** $\vdash : \dot{\Lambda} \,\epsilon\, C\text{‘}P \,.\supset.\, \Pi\text{‘}P = \dot{\Lambda}$

Dem.

$\vdash .\, *33{\cdot}24{\cdot}5 \,.\supset \vdash .\, \overrightarrow{F}\text{‘}\dot{\Lambda} = \Lambda \,.$

$[*33{\cdot}41] \qquad \supset \vdash .\, \dot{\Lambda} \sim\epsilon\, \text{Ɑ}\text{‘}F \,.$

$[*80{\cdot}21] \qquad \supset \vdash : \dot{\Lambda} \,\epsilon\, C\text{‘}P \,.\supset.\, F_\Delta\text{‘}C\text{‘}P = \Lambda \,.$

$[*172{\cdot}12 . *33{\cdot}24] \qquad \supset .\, \Pi\text{‘}P = \dot{\Lambda} : \supset \vdash .\,$ Prop

***172·141.** $\vdash :.\, \dot{\exists}\,!\, \Pi\text{‘}P \,.\supset: Q \,\epsilon\, C\text{‘}P \,.\supset_Q .\, \dot{\exists}\,!\, Q \qquad [*172{\cdot}14 \,.\,$ Transp$]$

The following propositions are concerned with $C\text{‘}\Pi\text{‘}P$, $\overrightarrow{B}\text{‘}\Pi\text{‘}P$, etc. *172·15·151·16·161 are lemmas for *172·162·17.

***172·15.** $\vdash : M \,\epsilon\, F_\Delta\text{‘}C\text{‘}P \,.\, Q \,\epsilon\, C\text{‘}P \,.\, (M\text{‘}Q)\,Qy \,.\supset.\, M(\Pi\text{‘}P)\{M \upharpoonright -\iota\text{‘}Q \cup y \downarrow Q\}$

Dem.

$\vdash .\, *80{\cdot}41 \,.\supset \vdash : \mathrm{Hp} \,.\supset.\, M \upharpoonright -\iota\text{‘}Q \cup y \downarrow Q \,\epsilon\, F_\Delta\text{‘}C\text{‘}P \qquad (1)$

$\vdash .\, *35{\cdot}101 \,.\, *55{\cdot}13 \,.\supset$

$\vdash :.\, z\{M \upharpoonright -\iota\text{‘}Q \cup y \downarrow Q\}R \,.\equiv: R \neq Q \,.\, zMR \,.\vee.\, R = Q \,.\, z = y \qquad (2)$

$\vdash .\, (2) \,.\, *80{\cdot}3 \,.\supset$

$\vdash :.\, \mathrm{Hp} \,.\supset: R = Q \,.\supset.\, \{M \upharpoonright -\iota\text{‘}Q \cup y \downarrow Q\}\text{‘}R = y :$

$R \,\epsilon\, C\text{‘}P \,.\, R \neq Q \,.\supset.\, \{M \upharpoonright -\iota\text{‘}Q \cup y \downarrow Q\}\text{‘}R = M\text{‘}R :$

$[\mathrm{Hp}] \qquad \supset : (M\text{‘}Q)\,Q\,\{M \upharpoonright -\iota\text{‘}Q \cup y \downarrow Q\}\text{‘}Q :$

$R \,\epsilon\, C\text{‘}P \,.\, R \neq Q \,.\supset.\, \{M \upharpoonright -\iota\text{‘}Q \cup y \downarrow Q\}\text{‘}R = M\text{‘}R :$

$[*33{\cdot}17] \supset : (M\text{‘}Q)\,Q\,\{M \upharpoonright -\iota\text{‘}Q \cup y \downarrow Q\}\text{‘}Q :$

$RPQ \,.\, R \neq Q \,.\supset_R .\, M\text{‘}R = \{M \upharpoonright -\iota\text{‘}Q \cup y \downarrow Q\}\text{‘}R :$

$[*172{\cdot}1 . (1)] \supset : M(\Pi\text{‘}P)\{M \upharpoonright -\iota\text{‘}Q \cup y \downarrow Q\} :. \supset \vdash .\,$ Prop

$*172\cdot151.\ \vdash : N \in F_\Delta{}^\text{‘}C\text{‘}P \,.\, Q \in C\text{‘}P \,.\, yQ(N\text{‘}Q) \,.\, \supset .$
$\{N \upharpoonright - \iota\text{‘}Q \cup y \downarrow Q\}(\Pi\text{‘}P)N$ [Proof as in $*172\cdot15$]

$*172\cdot16.\ \vdash : M \in F_\Delta{}^\text{‘}C\text{‘}P \,.\, \dot{\exists}\,!\, M \dot{-} B \,.\, \supset .\, M \in \text{ᗡ}\text{‘}\Pi\text{‘}P$

Dem.

$\vdash .\, *72\cdot93 .\, *80\cdot14 .\, \supset \vdash :: \mathrm{Hp} .\, \supset :.\, M \Subset B .\, \equiv : Q \in C\text{‘}P .\, \supset_Q .\, (M\text{‘}Q)\, BQ :.$

$[\mathrm{Transp}]\quad \supset :.\, \dot{\exists}\,!\, M \dot{-} B .\, \equiv : (\exists Q) .\, Q \in C\text{‘}P .\, \sim\{(M\text{‘}Q)\, BQ\} :$

$[*93\cdot1 . *80\cdot3]\quad \supset : (\exists Q) .\, Q \in C\text{‘}P .\, M\text{‘}Q \in \text{ᗡ}\text{‘}Q :$

$[*33\cdot131]\quad \supset : (\exists Q, y) .\, Q \in C\text{‘}P .\, yQ(M\text{‘}Q) :$

$[*172\cdot151]\quad \supset : M \in \text{ᗡ}\text{‘}\Pi\text{‘}P :: \supset \vdash .\, \mathrm{Prop}$

$*172\cdot161.\ \vdash : M \in F_\Delta{}^\text{‘}C\text{‘}P \,.\, \dot{\exists}\,!\, M \dot{-} B \,|\, \mathrm{Cnv} \,.\, \supset .\, M \in \mathrm{D}\text{‘}\Pi\text{‘}P$

Dem.

$\vdash .\, *72\cdot93 .\, *80\cdot14 .\, \supset \vdash :: \mathrm{Hp} .\, \supset :.$
$M \Subset B \,|\, \mathrm{Cnv} .\, \equiv : Q \in C\text{‘}P .\, \supset_Q .\, (M\text{‘}Q)(B \,|\, \mathrm{Cnv})\, Q :$

$[*71\cdot7]\quad \equiv : Q \in C\text{‘}P .\, \supset_Q .\, (M\text{‘}Q)\, B\breve{Q} :.$

$[\mathrm{Transp}]\quad \supset :.\, \dot{\exists}\,!\, M \dot{-} B \,|\, \mathrm{Cnv} .\, \equiv : (\exists Q) .\, Q \in C\text{‘}P .\, \sim\{(M\text{‘}Q)\, B\breve{Q}\} :$

$[*93\cdot1 . *80\cdot3]\quad \supset : (\exists Q) .\, Q \in C\text{‘}P .\, M\text{‘}Q \in \mathrm{D}\text{‘}Q :$

$[*33\cdot13]\quad \supset : (\exists Q, y) .\, Q \in C\text{‘}P .\, (M\text{‘}Q)\, Qy :$

$[*172\cdot15]\quad \supset : M \in \mathrm{D}\text{‘}\Pi\text{‘}P :: \supset \vdash .\, \mathrm{Prop}$

The following proposition is important. It shows that, if $C\text{‘}P$ consists of series, if any member of $C\text{‘}P$ has no first term, $\Pi\text{‘}P$ has no first term, but if every member of $C\text{‘}P$ has a first term, the selection of all these first terms is the first term of $\Pi\text{‘}P$.

$*172\cdot162.\ \vdash : \dot{\exists}\,!\, P \,.\, \supset .\, \overrightarrow{B}\text{‘}\Pi\text{‘}P = B_\Delta{}^\text{‘}C\text{‘}P \,.\, \overrightarrow{B}\text{‘}\mathrm{Cnv}\text{‘}\Pi\text{‘}P = B_\Delta{}^\text{‘}\mathrm{Cnv}\text{‘‘}C\text{‘}P$

Dem.

$\vdash .\, *93\cdot103 .\, *172\cdot12\cdot16 .\, \mathrm{Transp} .\, \supset \vdash .\, \overrightarrow{B}\text{‘}\Pi\text{‘}P \subset F_\Delta{}^\text{‘}C\text{‘}P \cap \mathrm{Rl}\text{‘}B \qquad (1)$

$\vdash .\, *72\cdot93 .\, \supset$

$\vdash :.\, M \in F_\Delta{}^\text{‘}C\text{‘}P .\, M \Subset B .\, Q \in C\text{‘}P .\, \supset : (M\text{‘}Q)\, BQ :$

$[*93\cdot1]\quad \supset : (M\text{‘}Q) \in \mathrm{D}\text{‘}Q :$

$[*33\cdot13]\quad \supset : (\exists y) .\, (M\text{‘}Q)\, Qy :$

$[*172\cdot15]\quad \supset : M \in \mathrm{D}\text{‘}\Pi\text{‘}P \qquad (2)$

$\vdash .\, (2) .\, *10\cdot11\cdot23\cdot35 .\, \supset \vdash :.\, \dot{\exists}\,!\, P .\, \supset : M \in F_\Delta{}^\text{‘}C\text{‘}P \cap \mathrm{Rl}\text{‘}B .\, \supset .\, M \in \mathrm{D}\text{‘}\Pi\text{‘}P \qquad (3)$

$\vdash .\, *172\cdot11 .\quad \supset \vdash : N \in \text{ᗡ}\text{‘}\Pi\text{‘}P .\, \supset .\, (\exists Q, M) .\, Q \in C\text{‘}P .\, (M\text{‘}Q)\, Q\, (N\text{‘}Q) .$

$[*93\cdot1]\quad \supset .\, (\exists Q) .\, Q \in C\text{‘}P .\, \sim\{(N\text{‘}Q)\, BQ\} .$

$[*72\cdot93]\quad \supset .\, \sim(N \Subset B) :$

$\left[\mathrm{Transp} .\, \dfrac{M}{N}\right]\quad \supset \vdash : M \Subset B .\, \supset .\, M \sim\in \text{ᗡ}\text{‘}\Pi\text{‘}P \qquad (4)$

$\vdash .\, (1) .\, (3) .\, (4) .\quad \supset \vdash : \mathrm{Hp} .\, \supset .\, \overrightarrow{B}\text{‘}\Pi\text{‘}P = F_\Delta{}^\text{‘}C\text{‘}P \cap \mathrm{Rl}\text{‘}B$

$[*80\cdot17]\quad = B_\Delta{}^\text{‘}C\text{‘}P \qquad (5)$

Similarly $\quad \vdash : \mathrm{Hp} .\, \supset .\, \overrightarrow{B}\text{‘}\mathrm{Cnv}\text{‘}\Pi\text{‘}P = B_\Delta{}^\text{‘}\mathrm{Cnv}\text{‘‘}C\text{‘}P \qquad (6)$

$\vdash .\, (5) .\, (6) .\, \supset \vdash .\, \mathrm{Prop}$

The following proposition is much used.

***172·17.** $\vdash : \dot{\exists} ! P . \supset . C`\Pi`P = F_\Delta`C`P$

Dem.

$\vdash . *172·16·162 . \supset$

$\vdash :. \text{Hp} . M \in F_\Delta`C`P . \supset : \dot{\exists} ! M \dot{-} B . \supset . M \in \text{Cl}`\Pi`P : M \subseteq B . \supset . M \in \overrightarrow{B}`\Pi`P :$

[*93·11.*25·55] $\supset : M \in C`\Pi`P$ (1)

$\vdash . (1) . *172·12 . \supset \vdash . \text{Prop}$

***172·171.** $\vdash : \dot{\exists} ! P . \supset . \text{D}`\Pi`P = F_\Delta`C`P - B_\Delta`\text{Cnv}``C`P .$
$\text{Cl}`\Pi`P = F_\Delta`C`P - B_\Delta`C`P$ [*172·162·17]

***172·18.** $\vdash :. \dot{\exists} ! P . \supset : \dot{\exists} ! \Pi`P . \equiv . \exists ! F_\Delta`C`P$ [*172·17]

***172·181.** $\vdash :. \text{Mult ax} . \supset : \dot{\Lambda} \sim\in C`P . \dot{\exists} ! P . \equiv . \dot{\exists} ! \Pi`P$

Dem.

$\vdash . *88·361 . *172·18 . \supset \vdash :: \text{Hp} . \supset :. \dot{\exists} ! P . \supset : \dot{\exists} ! \Pi`P . \equiv . C`P \subset \text{Cl}`F .$

[*33·41·5] $\equiv . C`P \subset \hat{Q}(\exists ! C`Q) .$

[*33·241] $\equiv . \dot{\Lambda} \sim\in C`P$ (1)

$\vdash . *172·13 . \quad \supset \vdash : \dot{\exists} ! \Pi`P . \supset . \dot{\exists} ! P$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***172·182.** $\vdash :: \text{Mult ax} . \supset :. \dot{\Lambda} \in C`P . \text{v} . P = \dot{\Lambda} : \equiv . \Pi`P = \dot{\Lambda}$
[*172·181 . Transp]

***172·19.** $\vdash : \dot{\exists} ! \Pi`P . \supset . \dot{s}`C`\Pi`P = F \upharpoonright C`P$ [*172·17 . *80·42]

Note that we cannot proceed to $\Sigma`\Pi`P$, because $F;\Pi`P$ is meaningless, owing to the fact that the field of $\Pi`P$ consists of non-homogeneous relations.

***172·191.** $\vdash . \dot{s}`C`\Pi`P \subseteq F \upharpoonright C`P$

Dem.

$\vdash . *172·19 . *23·42 . \supset \vdash : \dot{\exists} ! \Pi`P . \supset . \dot{s}`C`\Pi`P \subseteq F \upharpoonright C`P$ (1)

$\vdash . *41·21 . \quad \supset \vdash : \Pi`P = \dot{\Lambda} . \supset . \dot{s}`C`\Pi`P = \dot{\Lambda} .$

[*25·12] $\supset . \dot{s}`C`\Pi`P \subseteq F \upharpoonright C`P$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***172·192.** $\vdash . \text{Cl}`(F \upharpoonright \beta) = \beta - \iota`\dot{\Lambda}$

Dem.

$\vdash . *35·101 . \supset \vdash : Q \in \text{Cl}`(F \upharpoonright \beta) . \equiv . (\exists x) . xFQ . Q \in \beta .$

[*33·5] $\equiv . \exists ! C`Q . Q \in \beta .$

[*33·24] $\equiv . \dot{\exists} ! Q . Q \in \beta : \supset \vdash . \text{Prop}$

The following proposition is sometimes useful. (It is used in *173·22 . *182·2 . *185·21.)

***172·2.** $\vdash . \Pi`(P \downarrow P) = P \underset{\ddot{}}{\downarrow} P$

Dem.

$\vdash . *172·11 . *55·15 . \supset$

$\vdash :: M \{\Pi`(P \downarrow P)\} N . \equiv :. M, N \in F_\Delta`\iota`P :.$

$(\exists Q) : Q \in \iota`P . (M`Q) Q (N`Q) : R = P . R \neq Q . \supset_R . M`R = N`R :.$

$[*13·195·191] \equiv :. M, N \in F_\Delta`\iota`P :. (M`P) P (N`P) :.$

$[*85·51.*33·5] \equiv :. M, N \in \downarrow P``C`P . (M`P) P (N`P) :.$

$[*38·131] \quad \equiv :. (\exists x, y) . x, y \in C`P . M = x \downarrow P . N = y \downarrow P . (M`P) P (N`P) :.$

$[*55·13] \quad \equiv :. (\exists x, y) . x, y \in C`P . M = x \downarrow P . N = y \downarrow P . xPy :.$

$[*150·11] \quad \equiv :. M (\downarrow P ; P) N :.$

$[*150·6] \quad \equiv :. M (P \underset{\ddot{}}{\downarrow} P) N :: \supset \vdash . \text{Prop}$

The following propositions are concerned with the nature of the connection between $\Pi`(P \downarrow Q)$ and $P \times Q$. The connection is such as might be desired, except when $P = Q$, in which case, as shown above, $\Pi`(P \downarrow P)$ is like P, and is therefore not like $P \times P$.

***172·21.** $\vdash : P \neq Q . \supset . P \times Q = \dagger(Q \downarrow P) ; \Pi`(P \downarrow Q)$

Dem.

$\vdash . *172·11 . *55·15 . \supset$

$\vdash ::. M \{\Pi`(P \downarrow Q)\} N . \equiv :: M, N \in F_\Delta`(\iota`P \cup \iota`Q) :.$

$(\exists R) : R \in \iota`P \cup \iota`Q . (M`R) R (N`R) : S (P \downarrow Q) R . S \neq R . \supset_S . M`S = N`S ::$

$[*51·235] \equiv :: M, N \in F_\Delta`(\iota`P \cup \iota`Q) ::$

$(M`P) P (N`P) : S (P \downarrow Q) P . S \neq P . \supset_S . M`S = N`S :. \vee :.$

$(M`Q) Q (N`Q) : S (P \downarrow Q) Q . S \neq Q . \supset_S . M`S = N`S \quad (1)$

$\vdash . (1) . *55·13 . \supset \vdash :: \text{Hp} . \supset :.$

$M \{\Pi`(P \downarrow Q)\} N . \equiv : M, N \in F_\Delta`(\iota`P \cup \iota`Q) :$

$(M`P) P (N`P) . \vee . (M`Q) Q (N`Q) . M`P = N`P :$

$[*80·9·91] \equiv : (\exists x, x', y, y') : x, x' \in C`P . y, y' \in C`Q .$

$M = x \downarrow P \cup y \downarrow Q . N = x' \downarrow P \cup y' \downarrow Q :$

$xPx' . \vee . x = x' . yQy' \quad (2)$

$\vdash . *150·72 . \supset \vdash : M = x \downarrow P \cup y \downarrow Q . \supset . M ; (Q \downarrow P) = y \downarrow x .$

$[*150·1] \quad \supset . \dagger(Q \downarrow P)`M = y \downarrow x \quad (3)$

$\vdash . *150·4 . \supset \vdash : R \{\dagger(Q \downarrow P) ; \Pi`(P \downarrow Q)\} S . \equiv .$

$(\exists M, N) . M \{\Pi`(P \downarrow Q)\} N . R = \dagger(Q \downarrow P)`M . S = \dagger(Q \downarrow P)`N \quad (4)$

$\vdash . (2) . (3) . (4) . \supset \vdash :: \text{Hp} . \supset :.$

$R \{\dagger(Q \downarrow P) ; \Pi`(P \downarrow Q)\} S . \equiv : (\exists M, N, x, x', y, y') :$

$x, x' \in C`P . y, y' \in C`Q . M = x \downarrow P \cup y \downarrow Q . N = x' \downarrow P \cup y' \downarrow Q :$

$R = y \downarrow x . S = y' \downarrow x' : xPx' . \vee . x = x' . yQy' :$

$[*13·19] \quad \equiv : (\exists x, x', y, y') : x, x' \in C`P . y, y' \in C`Q . R = y \downarrow x . S = y' \downarrow x' :$

$xPx' . \vee . x = x' . yQy' :$

$[*166·111] \equiv : R (P \times Q) S :: \supset \vdash . \text{Prop}$

***172·22.** $\vdash : P \neq Q . \supset . \{\dagger(Q \downarrow P)\} \upharpoonright F_\Delta`(\iota`P \cup \iota`Q) \in (P \times Q) \overline{\text{smor}}\, \Pi`(P \downarrow Q)$

Dem.

$\vdash . *80·9 . *150·71 . \supset \vdash :. \text{Hp} . \supset :$

$$M \in F_\Delta`(\iota`P \cup \iota`Q) . \supset . M;(Q \downarrow P) = (M`Q) \downarrow (M`P) \quad (1)$$

$\vdash . (1) . *150·1 . \supset \vdash :. \text{Hp} . \supset :$

$$M, N \in F_\Delta`(\iota`P \cup \iota`Q) . \dagger(Q \downarrow P)`M = \dagger(Q \downarrow P)`N . \supset .$$

$$(M`Q) \downarrow (M`P) = (N`Q) \downarrow (N`P) .$$

[*55·202] $\supset . M`P = N`P . M`Q = N`Q .$

[*80·91] $\supset . M = N \quad (2)$

$\vdash . (2) . *151·241 . *172·21·17 . \supset \vdash . \text{Prop}$

***172·23.** $\vdash : P \neq Q . \supset . \Pi`(P \downarrow Q) \,\text{smor}\, P \times Q$ [*172·22]

The following propositions are lemmas for *172·32.

***172·3.** $\vdash :. \dot{\exists} ! P . Z \sim\in C`P . \supset : M \{\Pi`(P \dashrightarrow Z)\} N . \equiv .$

$$(\exists S, T, u, v) . (u \downarrow S)(\Pi`P \times Z)(v \downarrow T) . M = S \cup u \downarrow Z . N = T \cup v \downarrow Z$$

Dem.

$\vdash . *80·66·44 . *161·14 . \supset \vdash :. \text{Hp} . \supset : M \in F_\Delta`C`(P \dashrightarrow Z) . \equiv .$

$$(\exists S, u) . S \in F_\Delta`C`P . u \in C`Z . M = S \cup u \downarrow Z \quad (1)$$

$\vdash . *55·13 . *33·14 . *4·73 . \supset \vdash :: M = S \cup u \downarrow Z . \supset :.$

$$xMQ . \equiv : xSQ . Q \in \text{ᗡ}`S . \text{v} . x = u . Q = Z \quad (2)$$

$\vdash . (2) . *80·14 . \supset \vdash :: \text{Hp} . \text{Hp}(2) . S \in F_\Delta`C`P . u \in C`Z . \supset :.$

$$xMQ . \equiv : xSQ . Q \in C`P . \text{v} . x = u . Q = Z :.$$

[*24·37] $\supset :. Q \in C`P . \supset : xMQ . \equiv . xSQ :. Q = Z . \supset : xMQ . \equiv . x = u :.$

[*30·341.*80·3.*30·3] $\supset :. Q \in C`P . \supset . M`Q = S`Q : Q = Z . \supset . M`Q = u \quad (3)$

$\vdash . (1) . *172·11 . *161·14 . *172·17 . \supset$

$\vdash ::. \text{Hp} . \supset :: M \{\Pi`(P \dashrightarrow Z)\} N . \equiv :.$

$(\exists S, T, u, v) :. S, T \in F_\Delta`C`P . u, v \in C`Z . M = S \cup u \downarrow Z . N = T \cup v \downarrow Z :.$

$(\exists Q) : Q \in C`P \cup \iota`Z . (M`Q)\, Q\, (N`Q) : R (P \dashrightarrow Z) Q . R \neq Q . \supset_R . M`R = N`R :.$

[*51·239.(3).*161·11]

$\equiv :. (\exists S, T, u, v) :. S, T \in F_\Delta`C`P . u, v \in C`Z . M = S \cup u \downarrow Z . N = T \cup v \downarrow Z :.$

$(\exists Q) : Q \in C`P . (S`Q)\, Q\, (T`Q) : RPQ . R \neq Q . \supset_R . S`R = T`R : \text{v} :$

$uZv : R \in C`P . \supset_R . S`R = T`R :.$

[*172·11·17.*71·35.*80·14]

$\equiv :. (\exists S, T, u, v) :. S, T \in C`\Pi`P . u, v \in C`Z . M = S \cup u \downarrow Z . N = T \cup v \downarrow Z :$

$S (\Pi`P) T . \text{v} . S = T . uZv :.$

[*166·112] $\equiv :. (\exists S, T, u, v) : (u \downarrow S)(\Pi`P \times Z)(v \downarrow T) .$

$M = S \cup u \downarrow Z . N = T \cup v \downarrow Z ::. \supset \vdash . \text{Prop}$

∗172·31. $\vdash : \dot{\exists} ! P . Z \sim\in C`P .$

$W = \hat{M}\hat{R}\{(\exists S, u) . S \in F_\Delta`C`P . u \in C`Z . R = u \downarrow S . M = S \cup u \downarrow Z\} . \supset .$
$W \in \Pi`(P \dashrightarrow Z) \overline{\text{smor}} (\Pi`P \times Z)$

Dem.

$\vdash . ∗172·3 . \supset \vdash : \text{Hp} . \supset . \Pi`(P \dashrightarrow Z) = W ; (\Pi`P \times Z)$ (1)

$\vdash . ∗21·33 . \supset \vdash :. \text{Hp} . \supset : MWR . M'WR . \equiv .$
$(\exists S, S', u, u') . S, S' \in F_\Delta`C`P . u, u' \in C`Z . R = u \downarrow S = u' \downarrow S' .$
$M = S \cup u \downarrow Z . M' = S' \cup u' \downarrow Z .$

[∗55·202] $\supset . (\exists S, S', u, u') . S, S' \in F_\Delta`C`P . u, u' \in C`Z . S = S' . u = u' .$
$M = S \cup u \downarrow Z . M' = S' \cup u' \downarrow Z .$

[∗13·22·172] $\supset . M = M'$ (2)

$\vdash . ∗21·33 . \supset \vdash :. \text{Hp} . \supset : MWR . MWR' . \equiv .$
$(\exists S, S', u, u') . S, S' \in F_\Delta`C`P . u, u' \in C`Z . R = u \downarrow S . R' = u' \downarrow S' .$
$M = S \cup u \downarrow Z . M = S' \cup u' \downarrow Z .$

[∗80·45·661] $\supset . (\exists S, S', u, u') . R = u \downarrow S . R' = u' \downarrow S' .$
$S = M \upharpoonright C`P . u \downarrow Z = M \upharpoonright \iota`Z . S' = M \upharpoonright C`P . u' \downarrow Z = M \upharpoonright C`P .$

[∗13·172] $\supset . (\exists S, S', u, u') . R = u \downarrow S . R' = u' \downarrow S' . S = S' . u \downarrow Z = u' \downarrow Z .$

[∗55·202] $\supset . R = R'$ (3)

$\vdash . (2) . (3) . \supset \vdash : \text{Hp} . \supset . W \in 1 \rightarrow 1$ (4)

$\vdash . ∗166·12 . ∗113·101 . ∗172·17 . \supset \vdash : \text{Hp} . \supset . \text{C\!I}`W = C`(\Pi`P \times Z)$ (5)

$\vdash . (1) . (4) . (5) . ∗151·11 . \supset \vdash . \text{Prop}$

∗172·32. $\vdash : Z \sim\in C`P . \supset . \Pi`(P \dashrightarrow Z) \text{ smor } \Pi`P \times Z$

Dem.

$\vdash . ∗172·31 . \quad \supset \vdash : \text{Hp} . \dot{\exists} ! P . \supset . \Pi`(P \dashrightarrow Z) \text{ smor } \Pi`P \times Z$ (1)

$\vdash . ∗172·13 . ∗161·2 . \quad \supset \vdash : \sim\dot{\exists} ! P . \supset . \Pi`(P \dashrightarrow Z) = \dot{\Lambda}$ (2)

$\vdash . ∗172·13 . ∗166·13 . \supset \vdash : \sim\dot{\exists} ! P . \supset . \Pi`P \times Z = \dot{\Lambda}$ (3)

$\vdash . (2) . (3) . ∗153·101 . \supset \vdash : \sim\dot{\exists} ! P . \supset . \Pi`(P \dashrightarrow Z) \text{ smor } \Pi`P \times Z$ (4)

$\vdash . (1) . (4) . \supset \vdash . \text{Prop}$

∗172·321. $\vdash : Z \sim\in C`P . \supset . \Pi`(Z \leftarrow\!\!\vdash C`P) \text{ smor } Z \times \Pi`P$

[Proof by similar stages to those in proof of ∗172·32]

The following proposition is a lemma for ∗172·34, which is required in proving ∗172·35 (as well as ∗176·4).

∗172·33. $\vdash :: \dot{\exists} ! P . \dot{\exists} ! Q . C`P \cap C`Q = \Lambda . \supset :.$
$M\{\Pi`(P \ddagger Q)\} N . \equiv : (\exists S, T, S', T') : S, S' \in F_\Delta`C`P . T, T' \in F_\Delta`C`Q :$
$S(\Pi`P) S' . \vee . S = S' . T(\Pi`Q) T' : M = S \cup T . M' = S' \cup T'$

Dem.

$\vdash . ∗80·66 . \supset \vdash :. \text{Hp} . \supset : M \in F_\Delta`(C`P \cup C`Q) . \equiv .$
$(\exists S, T) . S \in F_\Delta`C`P . T \in F_\Delta`C`Q . M = S \cup T$ (1)

$\vdash . *80{\cdot}661 . *35{\cdot}7 . \supset \vdash :. \mathrm{Hp} . S \in F_\Delta\lq C\lq P . T \in F_\Delta\lq C\lq Q . M = S \dot\cup T . \supset :$

$R \in C\lq P . \supset . M\lq R = S\lq R : R \in C\lq Q . \supset . M\lq R = T\lq R \quad (2)$

$\vdash . (1) . *172{\cdot}11{\cdot}17 . *160{\cdot}14 . \supset \vdash ::. \mathrm{Hp} . \supset :: M \{\Pi\lq(P \Uparrow Q)\} N . \equiv :.$

$(\exists S, T, S', T') :. S, S' \in F_\Delta\lq C\lq P . T, T' \in F_\Delta\lq C\lq Q . M = S \dot\cup T . N = S' \dot\cup T' :.$

$(\exists R) : R \in C\lq P \cup C\lq Q . (M\lq R) R (N\lq R) : R'(P \Uparrow Q) R . R \neq R' . \supset_{R'} . M\lq R' = N\lq R' :.$

$[(2).*160{\cdot}11] \equiv :. (\exists S, T, S', T') :. S, S' \in F_\Delta\lq C\lq P . T, T' \in F_\Delta\lq C\lq Q .$

$M = S \dot\cup T . N = S' \dot\cup T' :.$

$(\exists R) : R \in C\lq P . (S\lq R) R (S'\lq R) : R'PR . R' \neq R . \supset_{R'} . S\lq R' = S'\lq R' : \mathsf{v} :$

$(\exists R) : R \in C\lq Q . (T\lq R) R (T'\lq R) : R'QR . R' \neq R . \supset_{R'} . T\lq R' = T'\lq R' :$

$R' \in C\lq P . \supset_{R'} . S\lq R' = S'\lq R' :.$

$[*10{\cdot}35] \equiv :. (\exists S, T, S', T') :. S, S' \in F_\Delta\lq C\lq P . T, T' \in F_\Delta\lq C\lq Q . M = S \dot\cup T . N = S' \dot\cup T' :.$

$(\exists R) : R \in C\lq P . (S\lq R) R (S'\lq R) : R'PR . R' \neq R . \supset_{R'} . S\lq R' = S'\lq R' : \mathsf{v} :$

$R' \in C\lq P . \supset_{R'} . S\lq R' = S'\lq R' :$

$(\exists R) : R \in C\lq Q . (T\lq R) R (T'\lq R) : R'QR . R' \neq R . \supset_{R'} . T\lq R' = T'\lq R' :.$

$[*172{\cdot}11.*71{\cdot}35.*80{\cdot}14]$

$\equiv :. (\exists S, T, S', T') :. S, S' \in F_\Delta\lq C\lq P . T, T' \in F_\Delta\lq C\lq Q . M = S \dot\cup T, M' = S' \dot\cup T' :.$

$S (\Pi\lq P) S' . \mathsf{v} . S = S' . T (\Pi\lq Q) T' ::. \supset \vdash . \mathrm{Prop}$

$*172{\cdot}34$. $\vdash : \dot\exists ! P . \dot\exists ! Q . C\lq P \cap C\lq Q = \Lambda . \supset .$

$(\dot{s} | C) \in \{\Pi\lq(P \Uparrow Q)\} \,\overline{\mathrm{smor}}\, \{\Pi\lq P \times \Pi\lq Q\}$

Dem.

$\vdash . *172{\cdot}33 . *55{\cdot}15 . *53{\cdot}13 . \supset$

$\vdash :: \mathrm{Hp} . \supset :. M \{\Pi\lq(P \Uparrow Q)\} N . \equiv : (\exists S, T, S', T', R, R') :$

$S, S' \in F_\Delta\lq C\lq P . T, T' \in F_\Delta\lq C\lq Q . R = T \downarrow S . R' = T' \downarrow S' . M = \dot{s}\lq C\lq R . N = \dot{s}\lq C\lq R' :$

$S (\Pi\lq P) S' . \mathsf{v} . S = S' . T (\Pi\lq Q) T' :$

$[*166{\cdot}11.*172{\cdot}17] \equiv : (\exists R, R') . R \{\Pi\lq P \times \Pi\lq Q\} R' . M = \dot{s}\lq C\lq R . N = \dot{s}\lq C\lq R' :$

$[*150{\cdot}4] \qquad \equiv : M \{\dot{s};C;(\Pi\lq P \times \Pi\lq Q)\} N \quad (1)$

$\vdash . *113{\cdot}153 . *172{\cdot}19 . *166{\cdot}12 . \supset \vdash : \mathrm{Hp} . \supset . (\dot{s}|C) \restriction C\lq(\Pi\lq P \times \Pi\lq Q) \in 1 \to 1 \quad (2)$

$\vdash . (1) . (2) . *151{\cdot}231 . \supset \vdash . \mathrm{Prop}$

$*172{\cdot}35$. $\vdash : \dot\exists ! P . \dot\exists ! Q . C\lq P \cap C\lq Q = \Lambda . \supset . \Pi\lq(P \Uparrow Q) \,\mathrm{smor}\, \Pi\lq P \times \Pi\lq Q$
$[*172{\cdot}34]$

The above proposition is important, being a form of the associative law.

The following propositions are extensions of $*172{\cdot}23$. It is obvious that they may be extended to any finite number of factors.

$*172{\cdot}36$. $\vdash : X \neq Y . X \neq Z . Y \neq Z . \supset . \Pi\lq\{(X \downarrow Y) \mathbin{\dagger\!\!\to} Z\} \,\mathrm{smor}\, X \times Y \times Z$

Dem.

$\vdash . *172{\cdot}32 . \supset \vdash : \mathrm{Hp} . \supset . \Pi\lq\{(X \downarrow Y) \mathbin{\dagger\!\!\to} Z\} \,\mathrm{smor}\, \Pi\lq(X \downarrow Y) \times Z \quad (1)$

$\vdash . *172{\cdot}23 . *166{\cdot}23 . \supset \vdash : \mathrm{Hp} . \supset . \Pi\lq(X \downarrow Y) \times Z \,\mathrm{smor}\, X \times Y \times Z \quad (2)$

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

$*172{\cdot}361$. $\vdash : X \neq Y . X \neq Z . Y \neq Z . \supset . \Pi`\{X \dotplus\!\!\!\dotplus (Y \downarrow Z)\} \,\mathrm{smor}\, X \times Y \times Z$

[Proof as in $*172{\cdot}36$]

$*172{\cdot}37$. $\vdash : X \neq Y . X \neq Z . X \neq W . Y \neq Z . Y \neq W . Z \neq W . \supset .$

$$\Pi`\{(X \downarrow Y) \Uparrow (Z \downarrow W)\} \,\mathrm{smor}\, X \times Y \times Z \times W$$

Dem.

$\vdash . *172{\cdot}35 . \supset$

$\vdash : \mathrm{Hp} . \supset . \Pi`\{(X \downarrow Y) \Uparrow (Z \downarrow W)\} \,\mathrm{smor}\, \Pi`(X \downarrow Y) \times \Pi`(Z \downarrow W)$ (1)

$\vdash . *172{\cdot}23 . *166{\cdot}23 . \supset$

$\vdash : \mathrm{Hp} . \supset . \Pi`(X \downarrow Y) \times \Pi`(Z \downarrow W) \,\mathrm{smor}\, (X \times Y) \times (Z \times W)$ (2)

$\vdash . (1) . (2) . *166{\cdot}42 . \supset \vdash . \mathrm{Prop}$

The following propositions are concerned with the construction of a correlator of $\Pi`P$ with $\Pi`Q$ when we are given a double correlator of P with Q. If the double correlator is T or $T \upharpoonright C`\Sigma`Q$, the correlator of $\Pi`P$ with $\Pi`Q$ is

$$\{T \| \mathrm{Cnv}`T\dagger\} \upharpoonright C`\Pi`Q.$$

$*172{\cdot}4$. $\vdash : T \,\epsilon\, P \,\overline{\mathrm{smor}}\,\overline{\mathrm{smor}}\, Q . \supset . \{T \| \mathrm{Cnv}`T\dagger\} \upharpoonright C`\Pi`Q \,\epsilon\, 1 \to 1$

Dem.

$\vdash . *164{\cdot}15 . \supset$

$\vdash : \mathrm{Hp} . \supset . T \upharpoonright C`\Sigma`Q, T\dagger \upharpoonright C`Q \,\epsilon\, 1 \to 1 . C`\Sigma`Q = \text{Ꭰ}`T . C`Q \subset \text{Ꭰ}`T\dagger$ (1)

$\vdash . *41{\cdot}43 . \supset \vdash . s`\mathrm{D}``C`\Pi`Q = \mathrm{D}`s`C`\Pi`Q .$

$[*172{\cdot}191] \quad \supset \vdash . s`\mathrm{D}``C`\Pi`Q \subset \mathrm{D}`(F \upharpoonright C`Q)$

$[*37{\cdot}401 . *162{\cdot}23] \quad \subset C`\Sigma`Q$ (2)

$\vdash . *41{\cdot}44 . \supset \vdash . s`\text{Ꭰ}``C`\Pi`Q = \text{Ꭰ}`s`C`\Pi`Q .$

$[*172{\cdot}191] \quad \supset \vdash . s`\text{Ꭰ}``C`\Pi`Q \subset \text{Ꭰ}`(F \upharpoonright C`Q)$

$[*35{\cdot}64] \quad \subset C`Q$ (3)

$\vdash . (1) . (2) . (3) . *74{\cdot}773 \dfrac{T, T\dagger, C`\Sigma`Q, C`Q, C`\Pi`Q}{Q, R, \alpha, \beta, \lambda} . \supset \vdash . \mathrm{Prop}$

$*172{\cdot}401$. $\vdash : T \,\epsilon\, P \,\overline{\mathrm{smor}}\,\overline{\mathrm{smor}}\, Q . N \,\epsilon\, F_\Delta`C`Q . S \,\epsilon\, C`Q . \supset .$

$$\{(T \| \mathrm{Cnv}`T\dagger)`N\}`T;S = T`N`S$$

Dem.

$\vdash . *43{\cdot}112 . *150{\cdot}1 . \supset \vdash . \{(T \| \mathrm{Cnv}`T\dagger)`N\}`T;S = (T | N | \mathrm{Cnv}`T\dagger)`T\dagger`S$ (1)

$\vdash . (1) . *35{\cdot}7{\cdot}48 . *80{\cdot}14 . \supset$

$\vdash : \mathrm{Hp} . \supset . \{(T \| \mathrm{Cnv}`T\dagger)`N\}`T;S = \{T | N | \mathrm{Cnv}`(T\dagger \upharpoonright C`Q)\}`(T\dagger \upharpoonright C`Q)`S$

$[*34{\cdot}41 . *72{\cdot}601 . *164{\cdot}13] \quad = (T | N)`S$

$[*34{\cdot}41] \quad = T`N`S : \supset \vdash . \mathrm{Prop}$

$*172{\cdot}402$. $\vdash : T \,\epsilon\, P \,\overline{\mathrm{smor}}\,\overline{\mathrm{smor}}\, Q . N, N' \,\epsilon\, F_\Delta`C`Q . S \,\epsilon\, C`Q . M = (T \| \mathrm{Cnv}`T\dagger)`N .$

$$M' = (T \| \mathrm{Cnv}`T\dagger)`N' . R = T;S . \supset :$$

$$N`S = N'`S . \equiv . M`R = M'`R : (N`S) \,S\, (N'`S) . \equiv . (M`R) \,R\, (M'`R)$$

Dem.

$\vdash . *172{\cdot}401 . \supset \vdash : \mathrm{Hp} . \supset . M`R = T`N`S . M'`R = T`N'`S$ (1)

$\vdash . *162{\cdot}22 . *40{\cdot}13 . \supset \vdash : \text{Hp} . \supset . C`S \subset C`\Sigma`Q .$

$[*164{\cdot}1] \qquad \supset . C`S \subset \text{Ɑ}`T \qquad (2)$

$\vdash . *80{\cdot}31 . *33{\cdot}5 . \quad \supset \vdash :. \text{Hp} . \supset : N`S, N'`S \in C`S :$

$[(2)] \qquad \supset : N`S, N'`S \in \text{Ɑ}`T :$

$[*71{\cdot}56] \qquad \supset : N`S = N'`S . \equiv . T`N`S = T`N'`S .$

$[(1)] \qquad \equiv . M`R = M'`R \qquad (3)$

$\vdash . (1) . \supset \vdash :. \text{Hp} . \supset : (M`R) R (M'`R) . \equiv . (T`N`S)(T;S)(T`N'`S) .$

$[*150{\cdot}41] \qquad \equiv . (N`S)(\breve{T};T;S)(N'`S) .$

$[*151{\cdot}252.(2)] \qquad \equiv . (N`S) S (N'`S) \qquad (4)$

$\vdash . (3) . (4) . \supset \vdash . \text{Prop}$

$*172{\cdot}403.$ $\vdash : T \in P \,\overline{\text{smor}}\, \overline{\text{smor}}\, Q . \supset . (T \,\|\, \text{Cnv}`T\dagger)``F_\Delta`C`Q \subset F_\Delta`C`P$

Dem.

$\vdash . *80{\cdot}14 . *35{\cdot}48 . \supset$

$\vdash :. \text{Hp} . \supset : N \in F_\Delta`C`Q . \supset . T \,|\, N \,|\, \text{Cnv}`T\dagger = T \,|\, N \,|\, \text{Cnv}`(T\dagger \upharpoonright C`Q) .$

$[*80{\cdot}14.*164{\cdot}13] \qquad \supset . (T \,|\, N \,|\, \text{Cnv}`T\dagger) \in 1 \rightarrow \text{Cls} \qquad (1)$

$\vdash . *37{\cdot}32 . \supset \vdash :. \text{Hp} . \supset : N \in F_\Delta`C`Q . \supset . \text{Ɑ}`(T \,|\, N \,|\, \text{Cnv}`T\dagger) = T\dagger``\breve{N}``\text{Ɑ}`T$

$[*37{\cdot}271.*164{\cdot}1.*80{\cdot}33.*162{\cdot}23] \qquad = T\dagger``\text{Ɑ}`N$

$[*80{\cdot}14] \qquad = T\dagger``C`Q$

$[*164{\cdot}1.*150{\cdot}22] \qquad = C`P \qquad (2)$

$\vdash . *80{\cdot}14 . \supset \vdash :. \text{Hp} . \supset : N \in F_\Delta`C`Q . x (T \,|\, N \,|\, \text{Cnv}`T\dagger) R . \supset .$

$(\exists y, S) . xTy . yFS . R = T;S . S \in C`Q .$

$[*33{\cdot}51.*37{\cdot}1] \qquad \supset . (\exists S) . x \in T``C`S . R = T;S . S \in C`Q .$

$[*150{\cdot}22.*164{\cdot}1] \qquad \supset . x \in C`R :$

$[*33{\cdot}51] \qquad \supset : N \in F_\Delta`C`Q . \supset . T \,|\, N \,|\, \text{Cnv}`T\dagger \in F \qquad (3)$

$\vdash . (1) . (2) . (3) . *80{\cdot}14 . \supset$

$\vdash :. \text{Hp} . \supset : N \in F_\Delta`C`Q . \supset . (T \,|\, N \,|\, \text{Cnv}`T\dagger) \in F_\Delta`C`P \qquad (4)$

$\vdash . (4) . *43{\cdot}112 . \supset \vdash . \text{Prop}$

$*172{\cdot}404.$ $\vdash :. T \in P \,\overline{\text{smor}}\, \overline{\text{smor}}\, Q . \supset : N \in F_\Delta`C`Q . M = T \,|\, N \,|\, \text{Cnv}`T\dagger . \equiv .$

$M \in F_\Delta`C`P . N = \breve{T} \,|\, M \,|\, \text{Cnv}`\breve{T}\dagger$

Dem.

$\vdash . *164{\cdot}1 . *162{\cdot}23 . *80{\cdot}33 . \supset \vdash :. \text{Hp} . \supset : N \in F_\Delta`C`Q . \supset . \text{D}`N \subset \text{Ɑ}`T .$

$[*71{\cdot}191.*50{\cdot}63] \qquad \supset . \breve{T} \,|\, T \,|\, N = N :$

$[*34{\cdot}28] \supset : N \in F_\Delta`C`Q . M = T \,|\, N \,|\, \text{Cnv}`T\dagger . \supset . \breve{T} \,|\, M = N \,|\, \text{Cnv}`T\dagger .$

$[*34{\cdot}27] \qquad \supset . \breve{T} \,|\, M \,|\, \text{Cnv}`\breve{T}\dagger = N \,|\, \text{Cnv}`T\dagger \,|\, \text{Cnv}`\breve{T}\dagger \qquad (1)$

$\vdash . *80{\cdot}14 . \quad \supset \vdash : N \in F_\Delta`C`Q . S \in \text{Ɑ}`N . \supset . S \in C`Q .$

$[*40{\cdot}13] \qquad \supset . C`S \subset s`C``C`Q \qquad (2)$

$\vdash . (2) . *164{\cdot}1 . \supset \vdash :. \text{Hp} . \supset : N \in F_\Delta`C`Q . S \in \text{Ɑ}`N . \supset . C`S \subset \text{Ɑ}`T \qquad (3)$

$\vdash . (1) . *150{\cdot}1 . \supset \vdash :. \text{Hp} . N \in F_\Delta\text{‘}C\text{‘}Q . M = T \,|\, N \,|\, \text{Cnv}\text{‘}T\dagger . \supset :$

$y\,(\breve{T} \,|\, M \,|\, \text{Cnv}\text{‘}\breve{T}\dagger)\, Y . \equiv . (\exists S, R) . yNS . R = T;S . Y = \breve{T};R .$

$[(3).*151{\cdot}25] \qquad \equiv . (\exists S, R) . yNS . R = T;S . Y = S .$

$[*13{\cdot}19{\cdot}195] \qquad \equiv . yNS \qquad (4)$

$\vdash . (4) . *172{\cdot}403 . \supset \vdash :. \text{Hp} . \supset : N \in F_\Delta\text{‘}C\text{‘}Q . M = T \,|\, N \,|\, \text{Cnv}\text{‘}T\dagger . \supset .$

$M \in F_\Delta\text{‘}C\text{‘}P . N = \breve{T} \,|\, M \,|\, \text{Cnv}\text{‘}\breve{T}\dagger \qquad (5)$

$\vdash . (5) \dfrac{\breve{T}, Q, P}{T, P, Q} . *164{\cdot}21 . \supset \vdash :. \text{Hp} . \supset : M \in F_\Delta\text{‘}C\text{‘}P . N = \breve{T} \,|\, M \,|\, \text{Cnv}\text{‘}\breve{T}\dagger . \supset .$

$N \in F_\Delta\text{‘}C\text{‘}Q . M = T \,|\, N \,|\, \text{Cnv}\text{‘}T\dagger \qquad (6)$

$\vdash . (5) . (6) . \supset \vdash . \text{Prop}$

$*172{\cdot}41$. $\vdash : T \in P \,\overline{\text{smor}}\, \overline{\text{smor}}\, Q . \supset . F_\Delta\text{‘}C\text{‘}P = (T \,\|\, \text{Cnv}\text{‘}T\dagger)\text{“}F_\Delta\text{‘}C\text{‘}Q$

Dem.

$\vdash . *172{\cdot}404 . *43{\cdot}112 . \supset \vdash :. \text{Hp} . \supset : M \in F_\Delta\text{‘}C\text{‘}P . \supset .$

$\breve{T} \,|\, M \,|\, \text{Cnv}\text{‘}\breve{T}\dagger \in F_\Delta\text{‘}C\text{‘}Q . M = (T \,\|\, \text{Cnv}\text{‘}T\dagger)\text{‘}(\breve{T} \,|\, M \,|\, \text{Cnv}\text{‘}\breve{T}\dagger) .$

$[*37{\cdot}6] \qquad \supset . M \in (T \,\|\, \text{Cnv}\text{‘}T\dagger)\text{“}F_\Delta\text{‘}C\text{‘}Q \qquad (1)$

$\vdash . (1) . *172{\cdot}403 . \supset \vdash . \text{Prop}$

The following proposition is important, since it gives the required correlator of $\Pi\text{‘}P$ with $\Pi\text{‘}Q$.

$*172{\cdot}42$. $\vdash : T \in P \,\overline{\text{smor}}\, \overline{\text{smor}}\, Q . \supset . (T \,\|\, \text{Cnv}\text{‘}T\dagger) \upharpoonright C\text{‘}\Pi\text{‘}Q \in (\Pi\text{‘}P) \,\overline{\text{smor}}\, (\Pi\text{‘}Q)$

Dem.

$\vdash . *164{\cdot}1 . *150{\cdot}22 . \supset \vdash :. \text{Hp} . \supset : C\text{‘}P = T\dagger\text{“}C\text{‘}Q :$

$[*37{\cdot}6 . *150{\cdot}1] \qquad \supset : R \in C\text{‘}P . \equiv . (\exists S) . S \in C\text{‘}Q . R = T;S \qquad (1)$

$\vdash . *164{\cdot}1 . \supset \vdash :. \text{Hp} . \supset : R'PR . \equiv . (\exists S', Y) . R' = T;S' . R = T;Y . S'QY \qquad (2)$

$\vdash . *151{\cdot}31 . *164{\cdot}1 . \supset \vdash :: \text{Hp} . \supset :. S, Y \in C\text{‘}Q . R = T;S . R = T;Y . \supset . S = Y :. \qquad (3)$

$[*13{\cdot}13] \supset :. S \in C\text{‘}Q . R = T;S . \supset : Y \in C\text{‘}Q . R = T;Y . \equiv . S = Y \qquad (4)$

$\vdash . (2) . (4) . *33{\cdot}17 . \supset$

$\vdash :. \text{Hp} . S \in C\text{‘}Q . R = T;S . \supset : R'PR . \equiv . (\exists S', Y) . R' = T;S' . S = Y . S'QY .$

$[*13{\cdot}195] \qquad \equiv . (\exists S') . R' = T;S' . S'QS \qquad (5)$

$\vdash . *150{\cdot}4 . *172{\cdot}11 . \supset \vdash ::. \text{Hp} . \supset :: M \{(T \,\|\, \text{Cnv}\text{‘}T\dagger);\Pi\text{‘}Q\}\, M' . \equiv :.$

$(\exists N, N') :. M = (T \,\|\, \text{Cnv}\text{‘}T\dagger)\text{‘}N . M' = (T \,\|\, \text{Cnv}\text{‘}T\dagger)\text{‘}N' . N, N' \in F_\Delta\text{‘}C\text{‘}Q :.$

$(\exists S) : S \in C\text{‘}Q . (N\text{‘}S)\, S\, (N'\text{‘}S) : S'QS . S' \neq S . \supset_{S'} . N\text{‘}S' = N'\text{‘}S' :.$

$[*172{\cdot}41{\cdot}402] \equiv :. M, M' \in F_\Delta\text{‘}C\text{‘}P :. (\exists S, R) : S \in C\text{‘}Q . R = T;S . (M\text{‘}R)\, R\, (M'\text{‘}R) :$

$S'QS . R' = T;S . S' \neq S . \supset_{S', R'} . M\text{‘}R' = M'\text{‘}R' :.$

$[*10{\cdot}23 . (3) . (5)] \equiv :. M, M' \in F_\Delta\text{‘}C\text{‘}P :. (\exists S, R) : S \in C\text{‘}Q . R = T;S . (M\text{‘}R)\, R\, (M'\text{‘}R) :$

$R' \neq R . R'PR . \supset_{R'} . M\text{‘}R' = M'\text{‘}R' :.$

$[(1) . *172{\cdot}11] \quad \equiv :. M\, (\Pi\text{‘}P)\, M' \qquad (6)$

$\vdash . (6) . *43{\cdot}302 . *172{\cdot}4 . *151{\cdot}22 . \supset \vdash . \text{Prop}$

The following proposition is a lemma for $*172\cdot43$.

$*172\cdot421$. $\vdash : S = T \upharpoonright C‘\Sigma‘Q \,.\, S \in P \,\overline{\text{smor}}\, \overline{\text{smor}}\, Q \,.\supset.$

$$(S \parallel \text{Cnv}‘S\dagger) \upharpoonright F_\Delta‘C‘Q = (T \parallel \text{Cnv}‘T\dagger) \upharpoonright F_\Delta‘C‘Q$$

Dem.

$\vdash \,.\, *80\cdot33 \,.\, *164\cdot18 \,.\, *162\cdot23 \,.\supset \vdash :.\, \text{Hp} \,.\, N \in F_\Delta‘C‘Q \,.\supset.\, \text{D}‘N \subset C‘\Sigma‘Q \,.$

$[*35\cdot481] \quad \supset .\, T \mid N = S \mid N \quad (1)$

$\vdash \,.\, *80\cdot14 \,.\supset \vdash :.\, \text{Hp} \,.\, N \in F_\Delta‘C‘Q \,.\, Y \in Ɑ‘N \,.\supset :\, Y \in C‘Q :$

$[*150\cdot33 . *162\cdot22] \supset : X = T;Y \,.\equiv.\, X = S;Y \,.$

$[*150\cdot1] \quad \supset : Y (\text{Cnv}‘T\dagger) X \,.\equiv.\, Y (\text{Cnv}‘S\dagger) X \quad (2)$

$\vdash \,.\, (2) \,.\, *33\cdot14 \,.\supset \vdash :.\, \text{Hp} \,.\, N \in F_\Delta‘C‘Q \,.\supset :$

$$(\exists Y) \,.\, yNY \,.\, Y (\text{Cnv}‘T\dagger) X \,.\equiv.\, (\exists Y) \,.\, yNY \,.\, Y (\text{Cnv}‘S\dagger) X :$$

$[*34\cdot1] \quad \supset : N \mid \text{Cnv}‘T\dagger = N \mid \text{Cnv}‘S\dagger \quad (3)$

$\vdash \,.\, (1) \,.\, (3) \,.\supset \vdash :.\, \text{Hp} \,.\supset : N \in F_\Delta‘C‘Q \,.\supset.\, T \mid N \mid \text{Cnv}‘T\dagger = S \mid N \mid \text{Cnv}‘S\dagger :$

$[*43\cdot112 . *35\cdot71] \supset : (T \parallel \text{Cnv}‘T\dagger) \upharpoonright F_\Delta‘C‘Q = (S \parallel \text{Cnv}‘S\dagger) \upharpoonright F_\Delta‘C‘Q :. \supset \vdash .\, \text{Prop}$

$*172\cdot43$. $\vdash : T \upharpoonright C‘\Sigma‘Q \in P \,\overline{\text{smor}}\, \overline{\text{smor}}\, Q \,.\supset.$

$$(T \parallel \text{Cnv}‘T\dagger) \upharpoonright C‘\Pi‘Q \in (\Pi‘P) \,\overline{\text{smor}}\, (\Pi‘Q)$$

$[*172\cdot42\cdot421]$

$*172\cdot44$. $\vdash : P \,\text{smor}\,\text{smor}\, Q \,.\supset.\, \Pi‘P \,\text{smor}\, \Pi‘Q \quad [*172\cdot42]$

$*172\cdot45$. $\vdash :.\, \text{Mult ax} \,.\supset : P, Q \in \text{Rel}^2 \text{excl} \,.\, \exists\,!\, P \,\overline{\text{smor}}\, Q \cap \text{Rl}‘\text{smor} \,.\supset.$

$$\Pi‘P \,\text{smor}\, \Pi‘Q$$

$[*164\cdot44 \,.\, *172\cdot44]$

The following proposition shows that if two relations have the same field, and if the parts of them that are contained in diversity are the same, they have the same product. Thus *e.g.* $\Pi‘P_{\text{po}} = \Pi‘P_*$, in virtue of $*91\cdot541$.

$*172\cdot5$. $\vdash : C‘P = C‘Q \,.\, P \dot{\cap} J = Q \dot{\cap} J \,.\supset.\, \Pi‘P = \Pi‘Q$

Dem.

$\vdash \,.\, *50\cdot11 \,.\supset \vdash :.\, \text{Hp} \,.\supset : RPS \,.\, R \neq S \,.\equiv.\, RQS \,.\, R \neq S \quad (1)$

$\vdash \,.\, (1) \,.\, *172\cdot11 \,.\supset \vdash .\, \text{Prop}$

The following proposition is used in $*182\cdot42$.

$*172\cdot51$. $\vdash .\, \Pi‘P = \Pi‘(P \cup I \upharpoonright C‘P) \quad [*172\cdot5]$

$*172\cdot52$. $\vdash :.\, Q \in Ɑ‘P \,.\supset_Q .\, (\exists R) \,.\, RPQ \,.\, R \neq Q : \supset .\, \Pi‘P = \Pi‘(P \dot{\cap} J)$

Dem.

$\vdash \,.\, *50\cdot11 \,. \quad \supset \vdash : \text{Hp} \,.\supset.\, Ɑ‘P \subset Ɑ‘(P \dot{\cap} J) \quad (1)$

$\vdash \,.\, *33\cdot14 \,.\, *93\cdot101 \,.\, \text{Transp} \,.\supset \vdash : QPQ \,.\supset.\, Q \sim\in \overrightarrow{B}‘P :$

$[\text{Transp}.*33\cdot13] \quad \supset \vdash : Q \in \overrightarrow{B}‘P \,.\supset.\, (\exists R) \,.\, QPR \,.\, R \neq Q \,.$

$[*50\cdot11] \quad \supset .\, Q \in C‘(P \dot{\cap} J) \quad (2)$

$\vdash \,.\, (1) \,.\, (2) \,.\, *93\cdot103 \,. \quad \supset \vdash : \text{Hp} \,.\supset.\, C‘P \subset C‘(P \dot{\cap} J) \,.$

$[*33\cdot265] \quad \supset .\, C‘P = C‘(P \dot{\cap} J) \quad (3)$

$\vdash \,.\, (3) \,.\, *172\cdot5 \,.\supset \vdash .\, \text{Prop}$

Thus we shall always have $\Pi‘P = \Pi‘(P \dot{\cap} J)$ unless there are members of $Ɑ‘P$ which have no referent except themselves.

$*173$. THE PRODUCT OF THE RELATIONS OF A FIELD (*continued*)

Summary of $*173$.

In this number, we shall consider the relation between the domains of relations related by $\Pi‘P$, *i.e.* we shall consider $\mathrm{D};\Pi‘P$. This relation bears to $\Pi‘P$ a relation analogous to that which $\mathrm{Prod}‘\kappa$ bears to $\epsilon_\Delta‘\kappa$. We shall denote it by "$\mathrm{Prod}‘P$." When $P \in \mathrm{Rel}^2\,\mathrm{excl}$, $\mathrm{Prod}‘P$ is like $\Pi‘P$, and is often more convenient than $\Pi‘P$. When $P \in \mathrm{Rel}^2\,\mathrm{excl}$, $\mathrm{Prod}‘P$ arranges the multiplicative class of $C‘‘C‘P$ by first differences, taking first differences to mean that the earliest member Q of $C‘P$ for which $\mu \cap C‘Q \neq \nu \cap C‘Q$ has the μ-member earlier than the ν-member in the Q-series.

The properties of $\mathrm{Prod}‘P$ all result immediately from those of $\Pi‘P$, and offer no difficulty of any kind. The most important of them are:

$*173{\cdot}14$. $\vdash : \dot{\exists}\,!\,P\,.\,C \upharpoonright C‘P \in 1 \to 1\,.\,\supset .\,C‘\mathrm{Prod}‘P = \mathrm{Prod}‘C‘‘C‘P$

I.e. if P is not null, and no two members of $C‘P$ have the same field, then the field of $\mathrm{Prod}‘P$ is the product of the fields of $C‘P$. Observe that $C \upharpoonright C‘P \in 1 \to 1$ if $P \in \mathrm{Rel}^2\,\mathrm{excl}$.

$*173{\cdot}16$. $\vdash : P \in \mathrm{Rel}^2\,\mathrm{excl}\,.\,\supset .$

$$\mathrm{Prod}‘P \;\mathrm{smor}\; \Pi‘P\,.\,\mathrm{D} \upharpoonright C‘\Pi‘P \in (\mathrm{Prod}‘P)\,\overline{\mathrm{smor}}\,(\Pi‘P)$$

$*173{\cdot}2$. $\vdash .\,\mathrm{Prod}‘\dot{\Lambda} = \dot{\Lambda}$

$*173{\cdot}22$. $\vdash .\,\mathrm{Prod}‘(P \downarrow P) = \iota;P$

$*173{\cdot}23$. $\vdash : P \neq Q\,.\,\supset .\,\mathrm{Prod}‘(P \downarrow Q) = C;(P \times Q)$

$*173{\cdot}3$. $\vdash : T \upharpoonright C‘\Sigma‘Q \in P\,\overline{\mathrm{smor}}\,\overline{\mathrm{smor}}\,Q\,.\,\supset .$

$$T_\epsilon \upharpoonright C‘\mathrm{Prod}‘Q \in (\mathrm{Prod}‘P)\,\overline{\mathrm{smor}}\,(\mathrm{Prod}‘Q)$$

$*173{\cdot}31$. $\vdash : P \;\mathrm{smor}\;\mathrm{smor}\; Q\,.\,\supset .\,\mathrm{Prod}‘P \;\mathrm{smor}\; \mathrm{Prod}‘Q$

$*173{\cdot}01$. $\mathrm{Prod}‘P = \mathrm{D};\Pi‘P \quad \mathrm{Df}$

$*173{\cdot}1$. $\vdash .\,\mathrm{Prod}‘P = \mathrm{D};\Pi‘P \quad [(*175{\cdot}01)]$

$*173{\cdot}11$. $\vdash : \mu\,(\mathrm{Prod}‘P)\,\nu\,.\equiv .\,(\exists M, N)\,.\,M\,(\Pi‘P)\,N\,.\,\mu = \mathrm{D}‘M\,.\,\nu = \mathrm{D}‘N$
$[*173{\cdot}1\,.\,*150{\cdot}51]$

$*173{\cdot}12$. $\vdash .\,C‘\mathrm{Prod}‘P \subset \mathrm{D}‘‘F_\Delta‘C‘P \quad [*172{\cdot}12\,.\,*150{\cdot}202]$

$*173{\cdot}121$. $\vdash .\,C‘\mathrm{Prod}‘P = \mathrm{D}‘‘C‘\Pi‘P \quad [*173{\cdot}1\,.\,*150{\cdot}22]$

$*173{\cdot}13$. $\vdash : \dot{\exists}\,!\,P\,.\,\supset .\,C‘\mathrm{Prod}‘P = \mathrm{D}‘‘F_\Delta‘C‘P \quad [*172{\cdot}17\,.\,*173{\cdot}121]$

***173·14.** $\vdash : \dot{\exists} ! P . C \upharpoonright C'P \in 1 \to 1 . \supset . C'\text{Prod}'P = \text{Prod}'C``C'P$

Dem.

$\vdash . *85{\cdot}12 . *33{\cdot}5 . \supset \vdash : \text{Hp} . \supset . \text{D}``F_\Delta{}'C'P = \text{D}``\epsilon_\Delta{}'C``C'P .$

$[*173{\cdot}13 . *115{\cdot}1] \quad \supset . C'\text{Prod}'P = \text{Prod}'C``C'P : \supset \vdash . \text{Prop}$

***173·15.** $\vdash : \text{D} \upharpoonright F_\Delta{}'C'P \in 1 \to 1 . \supset . \text{D} \upharpoonright C'\Pi'P \in (\text{Prod}'P) \overline{\text{smor}} (\Pi'P)$

$[*173{\cdot}1 . *172{\cdot}12 . *151{\cdot}231]$

***173·151.** $\vdash : \text{D} \upharpoonright F_\Delta{}'C'P \in 1 \to 1 . \supset . \text{Prod}'P \text{ smor } \Pi'P \quad [*173{\cdot}15]$

***173·16.** $\vdash : P \in \text{Rel}^2 \text{ excl} . \supset .$

$\text{Prod}'P \text{ smor } \Pi'P . \text{D} \upharpoonright C'\Pi'P \in (\text{Prod}'P) \overline{\text{smor}} (\Pi'P)$

Dem.

$\vdash . *163{\cdot}12 . \supset \vdash : \text{Hp} . \supset . F \upharpoonright C'P \in \text{Cls} \to 1 .$

$[*81{\cdot}21] \quad \supset . \text{D} \upharpoonright F_\Delta{}'C'P \in 1 \to 1 \quad (1)$

$\vdash . (1) . *173{\cdot}151{\cdot}15 . \supset \vdash . \text{Prop}$

***173·161.** $\vdash : P \in \text{Rel}^2 \text{ excl} . \dot{\exists} ! P . \supset . C'\text{Prod}'P = \text{Prod}'C``C'P$

$[*173{\cdot}14 . *163{\cdot}14]$

***173·17.** $\vdash : \dot{\exists} ! \text{Prod}'P . \supset . s'C'\text{Prod}'P = C'\Sigma'P$

Dem.

$\vdash . *173{\cdot}13 . \supset \vdash : \text{Hp} . \supset . s'C'\text{Prod}'P = s'\text{D}``F_\Delta{}'C'P$

$[*41{\cdot}43 . *80{\cdot}42] \quad = \text{D}'F \upharpoonright C'P$

$[*37{\cdot}401 . *162{\cdot}23] \quad = C'\Sigma'P : \supset \vdash . \text{Prop}$

***173·2.** $\vdash . \text{Prod}'\dot{\Lambda} = \dot{\Lambda} \quad [*172{\cdot}13 . *150{\cdot}42]$

***173·21.** $\vdash : \dot{\exists} ! \text{Prod}'P . \equiv . \dot{\exists} ! \Pi'P \quad [*173{\cdot}1 . *150{\cdot}24 . *33{\cdot}12]$

***173·22.** $\vdash . \text{Prod}'(P \downarrow P) = \iota ; P$

Dem.

$\vdash . *172{\cdot}2 . \supset \vdash . \text{Prod}'(P \downarrow P) = \text{D} ; \downarrow P ; P$

$[*150{\cdot}4] \quad = \hat{\mu}\hat{\nu} \{(\exists x, y) . xPy . \mu = \text{D}'(x \downarrow P) . \nu = \text{D}'(y \downarrow P)\}$

$[*55{\cdot}16] \quad = \hat{\mu}\hat{\nu} \{(\exists x, y) . xPy . \mu = \iota'x . \nu = \iota'y\}$

$[*150{\cdot}4] \quad = \iota ; P . \supset \vdash . \text{Prop}$

***173·23.** $\vdash : P \neq Q . \supset . \text{Prod}'(P \downarrow Q) = C ; (P \times Q)$

Dem.

$\vdash . *172{\cdot}21 . \quad \supset \vdash : \text{Hp} . \supset . C ; (P \times Q) = C ; \dagger(Q \downarrow P) ; \Pi'(P \downarrow Q) \quad (1)$

$\vdash . *80{\cdot}14 . *150{\cdot}23 . \supset \vdash : M \in F_\Delta{}'C'(Q \downarrow P) . \supset . C'M ; (Q \downarrow P) = \text{D}'M :$

$[*55{\cdot}15 . *150{\cdot}1] \quad \supset \vdash : M \in F_\Delta{}'C'(P \downarrow Q) . \supset . C`\dagger(Q \downarrow P)'M = \text{D}'M :$

$[*172{\cdot}12] \quad \supset \vdash : M \in C'\Pi'(P \downarrow Q) . \supset . C`\dagger(Q \downarrow P)'M = \text{D}'M :$

$[*150{\cdot}35] \quad \supset \vdash . C ; \dagger(Q \downarrow P) ; \Pi'(P \downarrow Q) = \text{D} ; \Pi'(P \downarrow Q) \quad (2)$

$\vdash . (1) . (2) . *173{\cdot}1 . \supset \vdash . \text{Prop}$

***173·24.** $\vdash : C\text{‘}P \cap C\text{‘}Q = \Lambda . \supset . C \upharpoonright C\text{‘}(P \times Q) \in \{\text{Prod‘}(P \downarrow Q)\} \overline{\text{smor}} (P \times Q) .$
$\text{Prod‘}(P \downarrow Q) \text{ smor } P \times Q$

Dem.

$\vdash . *166\cdot12 . \quad \supset \vdash . C \upharpoonright C\text{‘}(P \times Q) = C \upharpoonright (C\text{‘}P \times C\text{‘}Q) \quad (1)$
$\vdash . (1) . *113\cdot148 . \supset \vdash : \text{Hp} . \supset . C \upharpoonright C\text{‘}(P \times Q) \in 1 \rightarrow 1 \quad (2)$
$\vdash . (2) . *173\cdot23 . \supset \vdash . \text{Prop}$

***173·25.** $\vdash : P \in \text{Rel}^2 \text{excl} . Z \sim\in C\text{‘}P . C\text{‘}Z \cap C\text{‘}\Sigma\text{‘}P = \Lambda . \supset .$
$\text{Prod‘}(P \nrightarrow Z) \text{ smor } (\text{Prod‘}P \times Z) . \text{Prod‘}(Z \nleftarrow P) \text{ smor } (Z \times \text{Prod‘}P)$

Dem.

$\vdash . *163\cdot451 . \supset \vdash : \text{Hp} . \supset . P \nrightarrow Z \in \text{Rel}^2 \text{excl} .$
[*173·16] $\supset . \text{Prod‘}(P \nrightarrow Z) \text{ smor } \Pi\text{‘}(P \nrightarrow Z) .$
[*172·32] $\supset . \text{Prod‘}(P \nrightarrow Z) \text{ smor } \Pi\text{‘}P \times Z .$
[*173·16.*166·23] $\supset . \text{Prod‘}(P \nrightarrow Z) \text{ smor Prod‘}P \times Z \quad (1)$
Similarly $\vdash : \text{Hp} . \supset . \text{Prod‘}(Z \nleftarrow P) \text{ smor } Z \times \text{Prod‘}P \quad (2)$
$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***173·26.** $\vdash : P, Q \in \text{Rel}^2 \text{excl} . \dot{\exists} ! P . \dot{\exists} ! Q . C\text{‘}P \cap C\text{‘}Q = \Lambda . C\text{‘}\Sigma\text{‘}P \cap C\text{‘}\Sigma\text{‘}Q = \Lambda . \supset .$
$\text{Prod‘}(P \ddagger Q) \text{ smor Prod‘}P \times \text{Prod‘}Q$

Dem.

$\vdash . *163\cdot441 . *173\cdot16 . \supset \vdash : \text{Hp} . \supset . \text{Prod‘}(P \ddagger Q) \text{ smor } \Pi\text{‘}(P \ddagger Q) .$
[*172·35] $\supset . \text{Prod‘}(P \ddagger Q) \text{ smor } \Pi\text{‘}P \times \Pi\text{‘}Q .$
[*173·16.*166·23] $\supset . \text{Prod‘}(P \ddagger Q) \text{ smor Prod‘}P \times \text{Prod‘}Q : \supset \vdash . \text{Prop}$

***173·27.** $\vdash : C\text{‘}P \cap C\text{‘}Q = \Lambda . C\text{‘}P \cap C\text{‘}R = \Lambda . C\text{‘}Q \cap C\text{‘}R = \Lambda . \supset .$
$\text{Prod‘}\{(P \downarrow Q) \nrightarrow R\} \text{ smor } P \times Q \times R$

Dem.

$\vdash . *173\cdot25 . \supset \vdash : \text{Hp} . R \neq P . R \neq Q . \supset .$
$\text{Prod‘}\{(P \downarrow Q) \nrightarrow R\} \text{ smor } \{\text{Prod‘}(P \downarrow Q)\} \times R .$
[*173·24] $\supset . \text{Prod‘}\{(P \downarrow Q) \nrightarrow R\} \text{ smor } P \times Q \times R \quad (1)$
$\vdash . *33\cdot241 . \supset \vdash : \text{Hp} . R = P . \supset . R = \dot{\Lambda} . P = \dot{\Lambda} .$
[*172·14.*166·13] $\supset . \Pi\text{‘}\{(P \downarrow Q) \nrightarrow R\} = \dot{\Lambda} . P \times Q \times R = \dot{\Lambda} .$
[*173·1.*150·42] $\supset . \text{Prod‘}\{(P \downarrow Q) \nrightarrow R\} = \dot{\Lambda} . P \times Q \times R = \dot{\Lambda} .$
[*153·101] $\supset . \text{Prod‘}\{(P \downarrow Q) \nrightarrow R\} \text{ smor } (P \times Q \times R) \quad (2)$
Similarly $\vdash : \text{Hp} . R = Q . \supset . \text{Prod‘}\{(P \downarrow Q) \nrightarrow R\} \text{ smor } (P \times Q \times R) \quad (3)$
$\vdash . (1) . (2) . (3) . \supset \vdash . \text{Prop}$

The following proposition gives a correlator of Prod‘P and Prod‘Q when we are given a double correlator of P and Q.

***173·3.** $\vdash : T \upharpoonright C\text{‘}\Sigma\text{‘}Q \in P \,\overline{\text{smor}}\,\overline{\text{smor}}\, Q \,.\supset .$
$$T_\epsilon \upharpoonright C\text{‘}\text{Prod}\text{‘}Q \in (\text{Prod}\text{‘}P)\,\overline{\text{smor}}\,(\text{Prod}\text{‘}Q)$$

Dem.

$\vdash . *173{\cdot}11 . *172{\cdot}43 . \supset$

$\vdash :. \text{Hp} . \supset : \mu\, (\text{Prod}\text{‘}P)\, \mu' . \equiv . (\exists N, N') . N\, (\Pi\text{‘}Q)\, N' . \mu = \text{D}\text{‘}(T \mid N \mid \text{Cnv}\text{‘}T\dagger) .$
$\mu' = \text{D}\text{‘}(T \mid N' \mid \text{Cnv}\text{‘}T\dagger) .$

$[*37{\cdot}32{\cdot}321] \equiv . (\exists N, N') . N\, (\Pi\text{‘}Q)\, N' . \mu = T\text{‘‘}\text{D}\text{‘}N . \mu' = T\text{‘‘}\text{D}\text{‘}N' .$

$[*173{\cdot}11] \quad \equiv . (\exists \nu, \nu') . \nu\, (\text{Prod}\text{‘}Q)\, \nu' . \mu = T\text{‘‘}\nu . \mu' = T\text{‘‘}\nu' .$

$[*37{\cdot}101] \quad \equiv . \mu\, (T_\epsilon \text{;} \text{Prod}\text{‘}Q)\, \mu'$ (1)

$\vdash . *173{\cdot}17 . \supset \vdash . s\text{‘}C\text{‘}\text{Prod}\text{‘}Q \subset C\text{‘}\Sigma\text{‘}Q$ (2)

$[*111{\cdot}12] \quad \supset \vdash . (T \upharpoonright C\text{‘}\Sigma\text{‘}Q)_\epsilon \upharpoonright C\text{‘}\text{Prod}\text{‘}Q = T_\epsilon \upharpoonright C\text{‘}\text{Prod}\text{‘}Q$ (3)

$\vdash . (2) . (3) . *72{\cdot}451 . \supset \vdash : \text{Hp} . \supset . T_\epsilon \upharpoonright C\text{‘}\text{Prod}\text{‘}Q \in 1 \rightarrow 1$ (4)

$\vdash . (1) . (4) . *151{\cdot}231 . \supset \vdash . \text{Prop}$

***173·31.** $\vdash : P \,\text{smor}\,\text{smor}\, Q . \supset . \text{Prod}\text{‘}P \,\text{smor}\, \text{Prod}\text{‘}Q$ [*173·3]

***173·32.** $\vdash : R \upharpoonright C\text{‘}\Sigma\text{‘}Q \in 1 \rightarrow 1 . C\text{‘}\Sigma\text{‘}Q \subset \text{ᗡ}\text{‘}R . \supset . \text{Prod}\text{‘}R\dagger\text{;}Q = R_\epsilon \text{;} \text{Prod}\text{‘}Q$

Dem.

$\vdash . *164{\cdot}18 . \supset \vdash : \text{Hp} . \supset . R \upharpoonright C\text{‘}\Sigma\text{‘}Q \in (R\dagger\text{;}Q)\,\overline{\text{smor}}\,\overline{\text{smor}}\, Q .$

$[*173{\cdot}3] \qquad \supset . R_\epsilon \upharpoonright C\text{‘}\text{Prod}\text{‘}Q \in (\text{Prod}\text{‘}R\dagger\text{;}Q)\,\overline{\text{smor}}\,(\text{Prod}\text{‘}Q) .$

$[*151{\cdot}22] \qquad \supset . \text{Prod}\text{‘}R\dagger\text{;}Q = R_\epsilon \text{;} \text{Prod}\text{‘}Q : \supset \vdash . \text{Prop}$

***173·33.** $\vdash : \text{D} \upharpoonright C\text{‘}\Sigma\text{‘}Q \in 1 \rightarrow 1 . \supset . \text{Prod}\text{‘}\text{D}\dagger\text{;}Q = \text{D}_\epsilon \text{;} \text{Prod}\text{‘}Q \left[*173{\cdot}32 \dfrac{\text{D}}{R} \right].$

The above proposition is used in proving the associative law for "Prod" (*174·401).

*174. THE ASSOCIATIVE LAW OF RELATIONAL MULTIPLICATION

Summary of *174.

In the present number, we have to prove the associative law for Π and for Prod, *i.e.* we have to prove (with a suitable hypothesis)

$$\Pi\text{‘}\Pi\,;P \text{ smor } \Pi\text{‘}\Sigma\text{‘}P$$

and $$\text{Prod‘Prod}\,;P \text{ smor Prod‘}\Sigma\text{‘}P.$$

The first of these requires $P \epsilon \text{Rel}^2\text{excl}$ and either $P \mathrel{\mathsf{G}} J$ or

$$QPQ \,.\supset_Q .\, C\text{‘}Q \,\epsilon\, 0 \cup 1\,;$$

the second requires not only this, but also $\Sigma\text{‘}P \,\epsilon\, \text{Rel}^2\text{excl}$. When both P and $\Sigma\text{‘}P$ are relations of mutually exclusive relations, we call P an *arithmetical* relation, which we denote by "Rel^3arithm." Arithmetical relations serve exactly analogous purposes to those served by arithmetical classes in cardinal arithmetic.

The proof of the associative law for Π consists in showing that, under a suitable hypothesis, $\dot{s}\,|\,\text{D}$ (with its converse domain limited) is a correlator of $\Pi\text{‘}\Sigma\text{‘}P$ and $\Pi\text{‘}\Pi\,;P$ (*174·221·23). To prove this, we first prove

***174·17.** $\vdash : P \epsilon \text{Rel}^2\text{excl} \,.\supset .\, \dot{s}\text{“D“}C\text{‘}\Pi\text{‘}\Pi\,;P = C\text{‘}\Pi\text{‘}\Sigma\text{‘}P$

and

***174·19.** $\vdash : P \epsilon \text{Rel}^2\text{excl} \,.\supset .\, (\dot{s}\,|\,\text{D}) \upharpoonright C\text{‘}\Pi\text{‘}\Pi\,;P \,\epsilon\, 1 \to 1$

This gives what we may call the cardinal part of the proof, *i.e.* it shows that $(\dot{s}\,|\,\text{D}) \upharpoonright C\text{‘}\Pi\text{‘}\Pi\,;P$ is a cardinal correlator of the fields of $\Pi\text{‘}\Sigma\text{‘}P$ and $\Pi\text{‘}\Pi\,;P$. We then prove that if M and N belong to the field of $\Pi\text{‘}\Pi\,;P$, they have the relation $\Pi\text{‘}\Pi\,;P$ when the relational sums of their domains have the relation $\Pi\text{‘}\Sigma\text{‘}P$. Here, in addition to the hypothesis $P \epsilon \text{Rel}^2\text{excl}$, we require that if any relation Q has the relation P to itself, then $C\text{‘}Q$ is not to have more than one term. Thus we have

***174·215.** $\vdash :. P \epsilon \text{Rel}^2\text{excl} : QPQ \,.\supset_Q .\, C\text{‘}Q \,\epsilon\, 0 \cup 1 : \supset :$
$$M\,(\Pi\text{‘}\Pi\,;P)\,N \,.\equiv .\, M, N \,\epsilon\, F_\Delta\text{‘}\Pi\text{“}C\text{‘}P \,.\, (\dot{s}\text{‘D‘}M)\,(\Pi\text{‘}\Sigma\text{‘}P)\,(\dot{s}\text{‘D‘}N)$$

The hypothesis $QPQ \,.\supset_Q .\, C\text{‘}Q \,\epsilon\, 0 \cup 1$ is verified if $P \mathrel{\mathsf{G}} J$ (*174·216); thus for most purposes it is more convenient to substitute the simpler hypothesis $P \mathrel{\mathsf{G}} J$ for $QPQ \,.\supset_Q .\, C\text{‘}Q \,\epsilon\, 0 \cup 1$. We shall, however, have occasion to use the hypothesis $QPQ \,.\supset_Q .\, C\text{‘}Q \,\epsilon\, 0 \cup 1$ in *182·42·43·431, where our P is a relation whose field consists entirely of relations of the form $Q \downarrow Q$, whose fields are always unit classes, so that our P satisfies the above hypothesis even if P is not contained in J.

The proof of $*174{\cdot}215$ (above) is effected by first proving

$*174{\cdot}2.$ $\vdash : P \in \text{Rel}^2\,\text{excl} . Q \in C'P . M \in C'\Pi'\Pi;P . \supset . M'\Pi'Q = (\dot{s}'\text{D}'M) \upharpoonright C'Q$

From $*174{\cdot}17{\cdot}19{\cdot}215$ we deduce

$*174{\cdot}221.$ $\vdash :. P \in \text{Rel}^2\,\text{excl} : QPQ . \supset_Q . C'Q \in 0 \cup 1 : \supset .$
$\Pi'\Sigma'P = \dot{s};\text{D};\Pi'\Pi;P . (\dot{s}|\text{D}) \upharpoonright C'\Pi'\Pi;P \in (\Pi'\Sigma'P)\,\overline{\text{smor}}\,(\Pi'\Pi;P)$

whence we obtain the more convenient proposition

$*174{\cdot}23.$ $\vdash : P \in \text{Rel}^2\,\text{excl} . P \subset J . \supset .$
$\Pi'\Sigma'P = \dot{s};\text{D};\Pi'\Pi;P . (\dot{s}|\text{D}) \upharpoonright C'\Pi'\Pi;P \in (\Pi'\Sigma'P)\,\overline{\text{smor}}\,(\Pi'\Pi;P)$

Thus if the hypothesis of $*174{\cdot}221$ or of $*174{\cdot}23$ holds, the associative law holds for Π ($*174{\cdot}241{\cdot}25$).

To prove the associative law for Prod, *i.e.*

$$P \in \text{Rel}^3\,\text{arithm} . P \subset J . \supset . \text{Prod}'\Sigma'P \text{ smor Prod}'\text{Prod};P,$$

we observe that, since $\Pi'\Sigma'P = \dot{s};\text{D};\Pi'\Pi;P$ ($*174{\cdot}23$)
$= \dot{s};\text{Prod}'\Pi;P$, by the definition of Prod,
we have ($*174{\cdot}41$) $\text{Prod}'\Sigma'P = \text{D};\dot{s};\text{Prod}'\Pi;P$
$= s;\text{D}_\epsilon;\text{Prod}'\Pi;P$, by $*41{\cdot}33$,
$= s;\text{Prod}'\text{D}\dagger;\Pi;P$, by $*173{\cdot}33$,
$= s;\text{Prod}'\text{Prod};P$, by the definition of Prod.

Also $s \upharpoonright C'\text{Prod}'\text{Prod};P \in 1 \to 1$, by $*115{\cdot}46$. Hence the associative law follows ($*174{\cdot}43$). It will be observed that in this case the correlator is simply s with its converse domain limited ($*174{\cdot}42$).

As in the case of Π, "$P \subset J$" is a stronger hypothesis than we really need: what we need is $QPQ . \supset_Q . C'Q \in 0 \cup 1$.

$*174{\cdot}01.$ $\text{Rel}^3\,\text{arithm} = \hat{P}(P, \Sigma'P \in \text{Rel}^2\,\text{excl})$ Df

$*174{\cdot}12.$ $\vdash : C \upharpoonright C'P \in 1 \to 1 . \supset . \Pi;P \in \text{Rel}^2\,\text{excl}$

Dem.

$\vdash . *150{\cdot}202 . \supset$
$\vdash : M, N \in C'\Pi;P . \exists ! C'M \cap C'N . \supset . M, N \in \Pi``C'P . \exists ! C'M \cap C'N .$
$[*37{\cdot}6]$ $\supset . (\exists Q, R) . Q, R \in C'P . M = \Pi'Q . N = \Pi'R . \exists ! C'M \cap C'N .$
$[*172{\cdot}12] \supset . (\exists Q, R) . Q, R \in C'P . M = \Pi'Q . N = \Pi'R . \exists ! F_\Delta{}'C'Q \cap F_\Delta{}'C'R .$
$[*80{\cdot}82.\text{Transp}] \supset . (\exists Q, R) . Q, R \in C'P . M = \Pi'Q . N = \Pi'R . C'Q = C'R$ (1)
$\vdash . (1) . *71{\cdot}59 . \supset \vdash :. \text{Hp} . \supset :$
$M, N \in C'\Pi;P . \exists ! C'M \cap C'N . \supset . (\exists Q, R) . Q = R . M = \Pi'Q . N = \Pi'R .$
$[*13{\cdot}195{\cdot}172]$ $\supset . M = N$ (2)
$\vdash . (2) . *163{\cdot}11 . \supset \vdash . \text{Prop}$

$*174{\cdot}13.$ $\vdash : P \in \text{Rel}^2\,\text{excl} . \supset . \Pi;P \in \text{Rel}^2\,\text{excl}$ $[*174{\cdot}12 . *163{\cdot}14]$

***174·16.** $\vdash : \dot{\exists} ! P . \supset . C`\Pi`\Pi \mathbin{;} P = F_\Delta`\Pi``C`P$

Dem.

$\vdash . *150·25 . \supset \vdash : \text{Hp} . \supset . \dot{\exists} ! \Pi \mathbin{;} P .$

$[*172·17] \qquad \supset . C`\Pi`\Pi \mathbin{;} P = F_\Delta`C`\Pi \mathbin{;} P$

$[*150·22] \qquad = F_\Delta`\Pi``C`P : \supset \vdash . \text{Prop}$

***174·161.** $\vdash : \dot{\exists} ! P . P \in \text{Rel}^2 \text{excl} . \supset .$

$C`\text{Prod}`\Pi \mathbin{;} P = \text{D}``C`\Pi`\Pi \mathbin{;} P = \text{Prod}`F_\Delta``C``C`P$

Dem.

$\vdash . *173·121 . \quad \supset \vdash . C`\text{Prod}`\Pi \mathbin{;} P = \text{D}``C`\Pi`\Pi \mathbin{;} P \quad (1)$

$\vdash . *173·161 . \quad \supset \vdash : \text{Hp} . \supset . C`\text{Prod}`\Pi \mathbin{;} P = \text{Prod}`C``C`\Pi \mathbin{;} P$

$[*150·22] \qquad = \text{Prod}`C``\Pi``C`P \quad (2)$

$\vdash . *172·17 . \quad \supset \vdash : \dot{\Lambda} \sim \in C`P . \supset . C``\Pi``C`P = F_\Delta``C``C`P \quad (3)$

$\vdash . *172·14 . *173·21 . \supset \vdash : \dot{\Lambda} \in C`P . \supset . C`\text{Prod}`\Pi \mathbin{;} P = \Lambda \quad (4)$

$\vdash . *80·26 . *83·11 . \quad \supset \vdash : \dot{\Lambda} \in C`P . \supset . \text{Prod}`F_\Delta``C``C`P = \Lambda \quad (5)$

$\vdash . (2) . (3) . \supset \vdash : \text{Hp} . \dot{\Lambda} \sim \in C`P . \supset . C`\text{Prod}`\Pi \mathbin{;} P = \text{Prod}`F_\Delta``C``C`P \quad (6)$

$\vdash . (4) . (5) . \supset \vdash : \text{Hp} . \dot{\Lambda} \in C`P . \supset . C`\text{Prod}`\Pi \mathbin{;} P = \text{Prod}`F_\Delta``C``C`P \quad (7)$

$\vdash . (1) . (6) . (7) . \supset \vdash . \text{Prop}$

***174·162.** $\vdash : \dot{\exists} ! P . P \in \text{Rel}^2 \text{excl} . \supset . \dot{s}``\text{D}``C`\Pi`\Pi \mathbin{;} P = C`\Pi`\Sigma`P = F_\Delta`C`\Sigma`P$

Dem.

$\vdash . *174·161 . *115·1 . \supset \vdash : \text{Hp} . \supset . \dot{s}``\text{D}``C`\Pi`\Pi \mathbin{;} P = \dot{s}``\text{D}``\epsilon_\Delta`F_\Delta``C``C`P$

$[*85·27 . *163·16] \qquad = F_\Delta`s`C``C`P$

$[*162·22] \qquad = F_\Delta`C`\Sigma`P \quad (1)$

$\vdash . (1) . *172·17 . \supset \vdash : \text{Hp} . \dot{\exists} ! \Sigma`P . \supset . \dot{s}``\text{D}``C`\Pi`\Pi \mathbin{;} P = C`\Pi`\Sigma`P \quad (2)$

$\vdash . *162·45 . \quad \supset \vdash : \text{Hp} . \Sigma`P = \dot{\Lambda} . \supset . P = \dot{\Lambda} \downarrow \dot{\Lambda} .$

$[*172·13 . *150·71] \qquad \supset . \Pi \mathbin{;} P = \dot{\Lambda} \downarrow \Lambda .$

$[*172·14] \qquad \supset . \Pi`\Pi \mathbin{;} P = \dot{\Lambda} .$

$[*33·241] \qquad \supset . \dot{s}``\text{D}``C`\Pi`\Pi \mathbin{;} P = \Lambda \quad (3)$

$\vdash . *172·13 . *33·241 . \supset \vdash : \Sigma`P = \dot{\Lambda} . \supset . C`\Pi`\Sigma`P = \Lambda \quad (4)$

$\vdash . (3) . (4) . \supset \vdash : \text{Hp} . \Sigma`P = \dot{\Lambda} . \supset . \dot{s}``\text{D}``C`\Pi`\Pi \mathbin{;} P = C`\Pi`\Sigma`P \quad (5)$

$\vdash . (1) . (2) . (5) . \supset \vdash . \text{Prop}$

***174·17.** $\vdash : P \in \text{Rel}^2 \text{excl} . \supset . \dot{s}``\text{D}``C`\Pi`\Pi \mathbin{;} P = C`\Pi`\Sigma`P$

Dem.

$\vdash . *150·42 . *172·13 . \supset \vdash : P = \dot{\Lambda} . \supset . \dot{s}``\text{D}``C`\Pi`\Pi \mathbin{;} P = \Lambda \quad (1)$

$\vdash . *162·4 . *172·13 . \supset \vdash : P = \dot{\Lambda} . \supset . C`\Pi`\Sigma`P = \Lambda \quad (2)$

$\vdash . (1) . (2) . *174·162 . \supset \vdash . \text{Prop}$

***174·18.** $\vdash : P \in \text{Rel}^2 \text{excl} . \supset . \text{D} \restriction C`\Pi`\Pi \mathbin{;} P \in 1 \to 1$

Dem.

$\vdash . *174·12 . *163·14·12 . \supset \vdash : \text{Hp} . \supset . F \restriction C`\Pi \mathbin{;} P \in \text{Cls} \to 1 .$

$[*81·21] \qquad \supset . \text{D} \restriction F_\Delta`C`\Pi \mathbin{;} P \in 1 \to 1 .$

$[*172·12] \qquad \supset . \text{D} \restriction C`\Pi`\Pi \mathbin{;} P \in 1 \to 1 : \supset \vdash . \text{Prop}$

$*174\cdot19$. $\vdash : P \epsilon \text{Rel}^2 \text{excl} . \supset . (\dot{s}|\text{D}) \upharpoonright C`\Pi`\Pi;P \epsilon 1 \rightarrow 1$

Dem.

$\vdash . *163\cdot1 . *35\cdot14 . \supset \vdash :. \text{Hp} . \supset :$

$Q, R \epsilon C`P . Q \neq R . \supset_{Q,R} . F \upharpoonright C`Q \dot{\frown} F \upharpoonright C`R = \dot{\Lambda} .$

$[*172\cdot191] \quad \supset_{Q,R} . \dot{s}`C`\Pi`Q \dot{\frown} \dot{s}`C`\Pi`R = \dot{\Lambda} \quad (1)$

$\vdash . (1) . *33\cdot5 . *85\cdot31 \dfrac{F, \Pi``C`P}{P, \quad \alpha} . \supset$

$\vdash :. \text{Hp} . \supset : M, N \epsilon F_\Delta`\Pi``C`P . \dot{s}`\text{D}`M = \dot{s}`\text{D}`N . \supset . M = N :$

$[*172\cdot12.*150\cdot22] \supset : M, N \epsilon C`\Pi`\Pi;P . \dot{s}`\text{D}`M = \dot{s}`\text{D}`N . \supset . M = N :. \supset \vdash . \text{Prop}$

$*174\cdot191$. $\vdash : P \epsilon \text{Rel}^2 \text{excl} . \supset . \dot{s} \upharpoonright C`\text{Prod}`\Pi;P \epsilon 1 \rightarrow 1$

Dem.

$\vdash . *174\cdot19 . \supset \vdash :. \text{Hp} . \supset : M, N \epsilon C`\Pi`\Pi;P . \dot{s}`\text{D}`M = \dot{s}`\text{D}`N . \supset . M = N .$

$[*30\cdot37] \quad \supset . \text{D}`M = \text{D}`N :$

$[*37\cdot63] \quad \supset : \mu, \nu \epsilon \text{D}``C`\Pi`\Pi;P . \dot{s}`\mu = \dot{s}`\nu . \supset . \mu = \nu :$

$[*173\cdot121] \quad \supset : \mu, \nu \epsilon C`\text{Prod}`\Pi;P . \dot{s}`\mu = \dot{s}`\nu . \supset . \mu = \nu :. \supset \vdash . \text{Prop}$

$*174\cdot2$. $\vdash : P \epsilon \text{Rel}^2 \text{excl} . Q \epsilon C`P . M \epsilon C`\Pi`\Pi;P . \supset . M`\Pi`Q = (\dot{s}`\text{D}`M) \upharpoonright C`Q$

Dem.

$\vdash . *172\cdot12 . *150\cdot22 . \supset \vdash : \text{Hp} . \supset . M \epsilon F_\Delta`\Pi``C`P . \quad (1)$

$[*80\cdot31.*33\cdot5] \quad \supset . M`\Pi`Q \epsilon C`\Pi`Q .$

$[*172\cdot12] \quad \supset . M`\Pi`Q \epsilon F_\Delta`C`Q .$

$[*80\cdot14] \quad \supset . \text{G}`M`\Pi`Q = C`Q \quad (2)$

$\vdash . (1) . *80\cdot3 . *41\cdot13 . \supset \vdash : \text{Hp} . \supset . M`\Pi`Q \subset \dot{s}`\text{D}`M \quad (3)$

$\vdash . *174\cdot17 . \quad \supset \vdash : \text{Hp} . \supset . \dot{s}`\text{D}`M \epsilon C`\Pi`\Sigma`P .$

$[*172\cdot12.*80\cdot14] \quad \supset . \dot{s}`\text{D}`M \epsilon 1 \rightarrow \text{Cls} \quad (4)$

$\vdash . (3) . (4) . *72\cdot92 . \quad \supset \vdash : \text{Hp} . \supset . M`\Pi`Q = (\dot{s}`\text{D}`M) \upharpoonright \text{G}`M`\Pi`Q$

$[(2)] \quad = (\dot{s}`\text{D}`M) \upharpoonright C`Q : \supset \vdash . \text{Prop}$

$*174\cdot21$. $\vdash :: P \epsilon \text{Rel}^2 \text{excl} . Q \epsilon C`P . M, N \epsilon C`\Pi`\Pi;P . \supset :.$

$M`\Pi`Q = N`\Pi`Q . \equiv : R \epsilon C`Q . \supset_R . (\dot{s}`\text{D}`M)`R = (\dot{s}`\text{D}`N)`R$

Dem.

$\vdash . *71\cdot35 . *80\cdot14 . *172\cdot12 . \supset \vdash :: \text{Hp} . \supset :.$

$M`\Pi`Q = N`\Pi`Q . \equiv : R \epsilon C`Q . \supset_R . (M`\Pi`Q)`R = (N`\Pi`Q)`R :$

$[*174\cdot2] \quad \equiv : R \epsilon C`Q . \supset_R . \{(\dot{s}`\text{D}`M) \upharpoonright C`Q\}`R = \{(\dot{s}`\text{D}`N) \upharpoonright C`Q\}`R :$

$[*35\cdot7] \quad \equiv : R \epsilon C`Q . \supset_R . (\dot{s}`\text{D}`M)`R = (\dot{s}`\text{D}`N)`R :: \supset \vdash . \text{Prop}$

$*174\cdot211$. $\vdash :.: P \epsilon \text{Rel}^2 \text{excl} . \supset :: M (\Pi`\Pi;P) N . \equiv :.$

$M, N \epsilon F_\Delta`\Pi``C`P :. (\exists Q, S) : Q \epsilon C`P . S \epsilon C`Q . \{(M`\Pi`Q)`S\} S \{(N`\Pi`Q)`S\} :$

$TQS . T \neq S . \supset_T . (M`\Pi`Q)`T = (N`\Pi`Q)`T :$

$RPQ . R \neq Q . T \epsilon C`R . \supset_{R,T} . (M`\Pi`R)`T = (N`\Pi`R)`T$

Dem.

$\vdash . *172\cdot11 . *150\cdot22 . \supset \vdash :: M (\Pi`\Pi;P) N . \equiv :.$

$M, N \epsilon F_\Delta`\Pi``C`P :. (\exists Q) : Q \epsilon C`P . (M`\Pi`Q) (\Pi`Q) (N`\Pi`Q) :$

$RPQ . \Pi`R \neq \Pi`Q . \supset_R . M`\Pi`R = N`\Pi`R \quad (1)$

$$\vdash . *80\cdot31 . \supset \vdash :. M \in F_\Delta{}'\Pi''C'P . \supset : Q \in C'P . \supset_Q . M'\Pi'Q \in C'\Pi'Q . \quad (2)$$

$$[*33\cdot24] \qquad \supset_Q . \dot{\exists} ! \Pi'Q . \quad (3)$$

$$[*172\cdot19] \qquad \supset_Q . \dot{s}'C'\Pi'Q = F \upharpoonright C'Q \quad (4)$$

$$\vdash . (3) . (4) . \supset \vdash : M \in F_\Delta{}'\Pi''C'P . Q, R \in C'P . \Pi'Q = \Pi'R . \supset .$$
$$F \upharpoonright C'Q = F \upharpoonright C'R . \dot{\exists} ! \Pi'Q . \dot{\exists} ! \Pi'R .$$

$$[*172\cdot141\cdot192] \qquad \supset . C'Q = C'R \quad (5)$$

$$\vdash . (5) . *163\cdot14 . \supset \vdash :: \text{Hp} . \supset :. M \in F_\Delta{}'\Pi''C'P . Q, R \in C'P . \supset :$$
$$\Pi'Q = \Pi'R . \supset . Q = R :$$

$$[*30\cdot37.\text{Transp}] \qquad \supset : \Pi'Q \neq \Pi'R . \equiv . Q \neq R \quad (6)$$

$$\vdash . *71\cdot35 . (2) . *172\cdot12 . *80\cdot14 . \supset$$

$$\vdash :: M, N \in F_\Delta{}'\Pi''C'P . R \in C'P . \supset :.$$
$$M'\Pi'R = N'\Pi'R . \equiv : T \in C'R . \supset_T . (M'\Pi'R)'T = (N'\Pi'R)'T \quad (7)$$

$$\vdash . *172\cdot11 . \supset \vdash :: (M'\Pi'Q)(\Pi'Q)(N'\Pi'Q) . \equiv :. M'\Pi'Q, N'\Pi'Q \in F_\Delta{}'C'Q :.$$
$$(\exists S) : S \in C'Q . \{(M'\Pi'Q)'S\} S \{(N'\Pi'Q)'S\} :$$
$$TQS . T \neq S . \supset_T . (M'\Pi'Q)'T = (N'\Pi'Q)'T \quad (8)$$

$$\vdash . (1) . (2) . (6) . (7) . (8) . \supset \vdash . \text{Prop}$$

***174·212.** $\vdash ::. P \in \text{Rel}^2 \text{excl} . \supset :: M (\Pi'\Pi{}_{;}P) N . \equiv :.$

$$M, N \in F_\Delta{}'\Pi''C'P :. (\exists Q, S) : Q \in C'P . S \in C'Q . \{(\dot{s}'\text{D}'M)'S\} S \{(\dot{s}'\text{D}'N)'S\} :$$
$$TQS . T \neq S . \supset_T . (\dot{s}'\text{D}'M)'T = (\dot{s}'\text{D}'N)'T :$$
$$RPQ . R \neq Q . T \in C'R . \supset_{R,T} . (\dot{s}'\text{D}'M)'T = (\dot{s}'\text{D}'N)'T$$

[*174·2·211 . *35·7]

***174·213.** $\vdash :. RPQ . S \in C'Q . T \in C'R . S \neq T . \supset_{Q,R,S,T} . R \neq Q : P \in \text{Rel}^2\text{excl} : \supset :$

$$RPQ . S \in C'Q . T \in C'R . R \neq Q . \equiv . RPQ . S \in C'Q . T \in C'R . S \neq T$$

Dem.

$$\vdash . *163\cdot1 . \supset \vdash :. \text{Hp} . \supset : RPQ . R \neq Q . S \in C'Q . T \in C'R . \supset . S \neq T \quad (1)$$

$$\vdash . *11\cdot1 . \supset \vdash :. \text{Hp} . \supset : RPQ . S \in C'Q . T \in C'R . S \neq T . \supset . R \neq Q \quad (2)$$

$$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$$

***174·214.** $\vdash :: P \in \text{Rel}^2 \text{excl} : QPQ . \supset_Q . C'Q \in 0 \cup 1 : \supset :.$

$$(\exists Q) : Q \in C'P : TQS . T \neq S . \mathbf{v} . (\exists R) . RPQ . R \neq Q . S \in C'Q . T \in C'R : \equiv .$$
$$T (\Sigma'P) S . T \neq S$$

Dem.

$$\vdash . *52\cdot41 . \supset \vdash :. \text{Hp} . \supset : S, T \in C'Q . S \neq T . \supset . \sim(QPQ) :$$

$$[*13\cdot12.\text{Transp}] \qquad \supset : RPQ . S \in C'Q . T \in C'R . S \neq T . \supset . Q \neq R \quad (1)$$

$$\vdash . (1) . *174\cdot213 . \supset$$

$$\vdash :: \text{Hp} . \supset :. RPQ . S \in C'Q . T \in C'R . R \neq Q . \equiv . RPQ . S \in C'Q . T \in C'R . S \neq T :.$$

$$[*4\cdot37 . *11\cdot341] \supset :.$$
$$(\exists Q) . Q \in C'P . TQS . T \neq S . \mathbf{v} . (\exists Q, R) . RPQ . S \in C'Q . T \in C'R . R \neq Q : \equiv :$$
$$(\exists Q) . Q \in C'P . TQS . T \neq S . \mathbf{v} . (\exists Q, R) . RPQ . S \in C'Q . T \in C'R . T \neq S :$$

$$[*162\cdot13] \qquad \equiv : T (\Sigma'P) S . T \neq S \quad (2)$$

$$\vdash . (2) . *33\cdot17 . \supset \vdash . \text{Prop}$$

∗174·215. $\vdash :. P \epsilon \text{Rel}^2 \text{excl} : QPQ . \supset_Q . C'Q \epsilon 0 \cup 1 : \supset :$
$M(\Pi'\Pi;P) N . \equiv . M, N \epsilon F_\Delta'\Pi''C'P . (\dot{s}'\text{D}'M)(\Pi'\Sigma'P)(\dot{s}'\text{D}'N)$

Dem.

$\vdash . ∗174·212·214 . \supset \vdash ::. \text{Hp} . \supset :: M(\Pi'\Pi;P) N . \equiv :.$
$M, N \epsilon F_\Delta'\Pi''C'P :. (\exists Q, S) : Q \epsilon C'P . S \epsilon C'Q . \{(\dot{s}'\text{D}'M)'S\} S \{(\dot{s}'\text{D}'N)'S\} :$
$T(\Sigma'P) S . T \neq S . \supset_T . (\dot{s}'\text{D}'M)'T = (\dot{s}'\text{D}'N)'T$ (1)

$\vdash . ∗172·13 . ∗152·42 . \supset \vdash : M(\Pi'\Pi;P) N . \supset . \dot{\exists} ! P .$
[∗172·162] $\supset . \dot{s}'\text{D}'M, \dot{s}'\text{D}'N \epsilon F_\Delta'C'\Sigma'P$ (2)

$\vdash . ∗162·22 . \supset \vdash : (\exists Q) . Q \epsilon C'P . S \epsilon C'Q . \equiv . S \epsilon C'\Sigma'P$ (3)

$\vdash . (1) . (2) . (3) . ∗172·11 . \supset \vdash . \text{Prop}$

∗174·216. $\vdash :. P \subseteq J . \supset : QPQ . \supset_Q . C'Q \epsilon 0 \cup 1$

Dem.

$\vdash . ∗50·24 . \supset \vdash :. \text{Hp} . \supset : (Q) . \sim(QPQ) :$
[∗10·53] $\supset : QPQ . \supset_Q . C'Q \epsilon 0 \cup 1 :. \supset \vdash . \text{Prop}$

∗174·22. $\vdash :. P \epsilon \text{Rel}^2 \text{excl} . P \subseteq J . \supset :$
$M(\Pi'\Pi;P) N . \equiv . M, N \epsilon F_\Delta'\Pi''C'P . (\dot{s}'\text{D}'M)(\Pi'\Sigma'P)(\dot{s}'\text{D}'N)$
[∗174·215·216]

∗174·221. $\vdash :. P \epsilon \text{Rel}^2 \text{excl} : QPQ . \supset_Q . C'Q \epsilon 0 \cup 1 : \supset .$
$\Pi'\Sigma'P = \dot{s};\text{D};\Pi'\Pi;P . (\dot{s} \mid \text{D}) \upharpoonright C'\Pi'\Pi;P \epsilon (\Pi'\Sigma'P) \overline{\text{smor}} (\Pi'\Pi;P)$

Dem.

$\vdash . ∗174·215 . ∗150·41 . \supset$

$\vdash : \text{Hp} . T = (\dot{s} \mid \text{D}) \upharpoonright C'\Pi'\Pi;P . \supset . \Pi'\Pi;P = \breve{T};\Pi'\Sigma'P$ (1)

$\vdash . ∗174·19 . \quad \supset \vdash : \text{Hp}(1) . \supset . T \epsilon 1 \rightarrow 1$ (2)

$\vdash . ∗174·17 . \quad \supset \vdash : \text{Hp}(1) . \supset . \text{D}'T = C'\Pi'\Sigma'P$ (3)

$\vdash . (1) . (2) . (3) . ∗151·11 . \supset \vdash : \text{Hp}(1) . \supset . \breve{T} \epsilon (\Pi'\Pi;P) \overline{\text{smor}} (\Pi'\Sigma'P) .$
[∗151·131] $\supset . T \epsilon (\Pi'\Sigma'P) \overline{\text{smor}} (\Pi'\Pi;P)$ (4)

$\vdash . (4) . ∗151·22 . \supset \vdash . \text{Prop}$

∗174·23. $\vdash : P \epsilon \text{Rel}^2 \text{excl} . P \subseteq J . \supset . \Pi'\Sigma'P = \dot{s};\text{D};\Pi'\Pi;P .$
$(\dot{s} \mid \text{D}) \upharpoonright C'\Pi'\Pi;P \epsilon (\Pi'\Sigma'P) \overline{\text{smor}} (\Pi'\Pi;P)$ [∗174·221·216]

∗174·231. $\vdash :. P \epsilon \text{Rel}^2 \text{excl} : QPQ . \supset_Q . C'Q \epsilon 0 \cup 1 : \supset .$
$\dot{s} \upharpoonright C'\text{Prod}'\Pi;P \epsilon (\Pi'\Sigma'P) \overline{\text{smor}} (\text{Prod}'\Pi;P)$

Dem.

$\vdash . ∗174·221 . ∗173·1 . \supset \vdash : \text{Hp} . \supset . \Pi'\Sigma'P = \dot{s};\text{Prod}'\Pi;P$ (1)

$\vdash . (1) . ∗174·191 . ∗151·231 . \supset \vdash . \text{Prop}$

∗174·24. $\vdash : P \epsilon \text{Rel}^2 \text{excl} . P \subseteq J . \supset .$
$\dot{s} \upharpoonright C'\text{Prod}'\Pi;P \epsilon (\Pi'\Sigma'P) \overline{\text{smor}} (\text{Prod}'\Pi;P)$ [∗174·231·216]

∗174·241. $\vdash :. P \epsilon \text{Rel}^2 \text{excl} : QPQ . \supset_Q . C'Q \epsilon 0 \cup 1 : \supset .$
$\Pi'\Sigma'P \,\text{smor}\, \Pi'\Pi;P . \Pi'\Sigma'P \,\text{smor}\, \text{Prod}'\Pi;P$ [∗174·221·231]

∗174·25. $\vdash : P \epsilon \text{Rel}^2 \text{excl} . P \subseteq J . \supset .$
$\Pi'\Sigma'P \,\text{smor}\, \Pi'\Pi;P . \Pi'\Sigma'P \,\text{smor}\, \text{Prod}'\Pi;P$ [∗174·23·24]

This proposition gives the associative law for Π. It remains to prove the associative law for Prod.

The following propositions are concerned with various properties of "arithmetical" relations, down to $*174\cdot4$, where the proof of the associative law for Prod begins.

$*174\cdot3.\quad \vdash : P \epsilon \text{Rel}^3 \text{arithm} . \equiv . P, \Sigma\text{‘}P \epsilon \text{Rel}^2 \text{excl} \qquad [(*174\cdot01)]$

$*174\cdot31.\quad \vdash :. P \epsilon \text{Rel}^3 \text{arithm} . \equiv : Q, Q' \epsilon C\text{‘}P . Q \neq Q' . \supset_{Q,Q'} . C\text{‘}Q \cap C\text{‘}Q' = \Lambda :$
$R, R' \epsilon C\text{‘}\Sigma\text{‘}P . R \neq R' . \supset_{R,R'} . C\text{‘}R \cap C\text{‘}R' = \Lambda \quad [*174\cdot3 . *163\cdot1]$

$*174\cdot311.\quad \vdash :. P \epsilon \text{Rel}^3 \text{arithm} . \equiv : Q, Q' \epsilon C\text{‘}P . \exists ! C\text{‘}Q \cap C\text{‘}Q' . \supset_{Q,Q'} . Q = Q' :$
$R, R' \epsilon C\text{‘}\Sigma\text{‘}P . \exists ! C\text{‘}R \cap C\text{‘}R' . \supset_{R,R'} . R = R' \quad [*174\cdot3 . *163\cdot11]$

$*174\cdot32.\quad \vdash : P \epsilon \text{Rel}^3 \text{arithm} . \equiv . F \upharpoonright C\text{‘}P, F \upharpoonright C\text{‘}\Sigma\text{‘}P \epsilon \text{Cls} \to 1$
$[*174\cdot3 . *163\cdot12]$

$*174\cdot321.\quad \vdash : P \epsilon \text{Rel}^3 \text{arithm} . \supset . C \upharpoonright C\text{‘}P, C \upharpoonright C\text{‘}\Sigma\text{‘}P \epsilon 1 \to 1 \quad [*174\cdot3 . *163\cdot14]$

$*174\cdot322.\quad \vdash : P \epsilon \text{Rel}^3 \text{arithm} . Q, Q' \epsilon C\text{‘}P . \exists ! C\text{“}C\text{‘}Q \cap C\text{“}C\text{‘}Q' . \supset . Q = Q'$

Dem.

$\vdash . *37\cdot6 . \supset \vdash : \text{Hp} . \supset . (\exists R, R') . R \epsilon C\text{‘}Q . R' \epsilon C\text{‘}Q' . C\text{‘}R = C\text{‘}R' .$
$[*174\cdot321] \quad \supset . (\exists R, R') . R \epsilon C\text{‘}Q . R' \epsilon C\text{‘}Q' . R = R' .$
$[*13\cdot195] \quad \supset . \exists ! C\text{‘}Q \cap C\text{‘}Q' .$
$[*174\cdot311] \quad \supset . Q = Q' : \supset \vdash . \text{Prop}$

$*174\cdot33.\quad \vdash : P \epsilon \text{Rel}^3 \text{arithm} . \supset . C\text{“‘}C\text{“}C\text{‘}P \epsilon \text{Cls}^3 \text{arithm}$

Dem.

$\vdash . *174\cdot322 . \supset$
$\vdash :. \text{Hp} . \supset : Q, Q' \epsilon C\text{‘}P . \exists ! C\text{“}C\text{‘}Q \cap C\text{“}C\text{‘}Q' . \supset_{Q,Q'} . C\text{“}C\text{‘}Q = C\text{“}C\text{‘}Q' :$
$[*37\cdot63] \quad \supset : \gamma, \delta \epsilon C\text{“‘}C\text{“}C\text{‘}P . \exists ! \gamma \cap \delta . \supset_{\gamma,\delta} . \gamma = \delta :$
$[*84\cdot11] \quad \supset : C\text{“‘}C\text{“}C\text{‘}P \epsilon \text{Cls}^2 \text{excl} \quad (1)$
$\vdash . *174\cdot3 . *163\cdot16 . *162\cdot22 . \supset \vdash : \text{Hp} . \supset . C\text{“}s\text{‘}C\text{“}C\text{‘}P \epsilon \text{Cls}^2 \text{excl} .$
$[*40\cdot38] \quad \supset . s\text{‘}C\text{“‘}C\text{“}C\text{‘}P \epsilon \text{Cls}^2 \text{excl} \quad (2)$
$\vdash . (1) . (2) . *115\cdot2 . \supset \vdash . \text{Prop}$

$*174\cdot34.\quad \vdash : P \epsilon \text{Rel}^3 \text{arithm} . \equiv .$
$C\text{“‘}C\text{“}C\text{‘}P \epsilon \text{Cls}^3 \text{arithm} . C \upharpoonright C\text{‘}P, C \upharpoonright C\text{‘}\Sigma\text{‘}P \epsilon 1 \to 1$

Dem.

$\vdash . *174\cdot321\cdot33 . \supset$
$\vdash : P \epsilon \text{Rel}^3 \text{arithm} . \supset . C\text{“‘}C\text{“}C\text{‘}P \epsilon \text{Cls}^3 \text{arithm} . C \upharpoonright C\text{‘}P, C \upharpoonright C\text{‘}\Sigma\text{‘}P \epsilon 1 \to 1 \quad (1)$
$\vdash . *115\cdot2 . \supset \vdash : C\text{“‘}C\text{“}C\text{‘}P \epsilon \text{Cls}^3 \text{arithm} . C \upharpoonright C\text{‘}\Sigma\text{‘}P \epsilon 1 \to 1 . \supset .$
$s\text{‘}C\text{“‘}C\text{“}C\text{‘}P \epsilon \text{Cls}^2 \text{excl} . C \upharpoonright C\text{‘}\Sigma\text{‘}P \epsilon 1 \to 1 .$
$[*40\cdot38 . *162\cdot22] \quad \supset . C\text{“}C\text{‘}\Sigma\text{‘}P \epsilon \text{Cls}^2 \text{excl} . C \upharpoonright C\text{‘}\Sigma\text{‘}P \epsilon 1 \to 1 .$
$[*163\cdot17] \quad \supset . \Sigma\text{‘}P \epsilon \text{Rel}^2 \text{excl} \quad (2)$
$\vdash . *37\cdot62 . \supset \vdash : Q, Q' \epsilon C\text{‘}P . R \epsilon C\text{‘}Q \cap C\text{‘}Q' . \supset . C\text{‘}R \epsilon C\text{“}C\text{‘}Q \cap C\text{“}C\text{‘}Q' \quad (3)$
$\vdash . (3) . *115\cdot2 . *84\cdot11 . \supset$

$\vdash : C``C``C`P \in \text{Cls}^3\,\text{arithm} . Q, Q' \in C`P . \exists ! C`Q \cap C`Q' . \supset .$

$C``C`Q = C``C`Q'$ (4)

$\vdash . (4) . *72\cdot481 . *37\cdot421 . \supset$

$\vdash : C``C``C`P \in \text{Cls}^3\,\text{arithm} . C \upharpoonright C`\Sigma`P \in 1 \rightarrow 1 .$

$Q, Q' \in C`P . \exists ! C`Q \cap C`Q' . \supset . C`Q = C`Q'$ (5)

$\vdash . (5) . *71\cdot59 . \supset$

$\vdash :. C``C``C`P \in \text{Cls}^3\,\text{arithm} . C \upharpoonright C`\Sigma`P \in 1 \rightarrow 1 . C \upharpoonright C`P \in 1 \rightarrow 1 . \supset :$

$Q, Q' \in C`P . \exists ! C`Q \cap C`Q' . \supset_{Q,Q'} . Q = Q' :$

[$*163\cdot11$] $\supset : P \in \text{Rel}^2\,\text{excl}$ (6)

$\vdash . (2) . (6) . *174\cdot3 . \supset$

$\vdash : C``C``C`P \in \text{Cls}^3\,\text{arithm} . C \upharpoonright C`\Sigma`P \in 1 \rightarrow 1 . C \upharpoonright C`P \in 1 \rightarrow 1 . \supset .$

$P \in \text{Rel}^3\,\text{arithm}$ (7)

$\vdash . (1) . (7) . \supset \vdash . \text{Prop}$

$*174\cdot35$. $\vdash : P \in \text{Rel}^3\,\text{arithm} . Q, Q' \in C`P . Q \neq Q' . \supset . C`\Sigma`Q \cap C`\Sigma`Q' = \Lambda$

Dem.

$\vdash . *174\cdot3 . *163\cdot1 . \supset \vdash :. \text{Hp} . \supset : R \in C`Q . R' \in C`Q' . \supset_{R,R'} . R \neq R'$ (1)

$\vdash . *162\cdot22 . \supset \vdash :. \text{Hp} . \supset : R \in C`Q . R' \in C`Q' . \supset_{R,R'} . R, R' \in C`\Sigma`P$ (2)

$\vdash . (1) . (2) . *174\cdot31 . \supset \vdash :. \text{Hp} . \supset : R \in C`Q . R' \in C`Q' . \supset_{R,R'} . C`R \cap C`R' = \Lambda :$

[$*40\cdot27$] $\supset : s`C``C`Q \cap s`C``C`Q' = \Lambda :$

[$*162\cdot22$] $\supset : C`\Sigma`Q \cap C`\Sigma`Q' = \Lambda :. \supset \vdash . \text{Prop}$

$*174\cdot36$. $\vdash : P \in \text{Rel}^3\,\text{arithm} . \supset . \Sigma;P \in \text{Rel}^2\,\text{excl}$

Dem.

$\vdash . *174\cdot35 . *37\cdot63 . *150\cdot22 . \supset$

$\vdash :. \text{Hp} . \supset : R, R' \in C`\Sigma;P . R \neq R' . \supset . C`R \cap C`R' = \Lambda$ (1)

$\vdash . (1) . *163\cdot1 . \supset \vdash . \text{Prop}$

$*174\cdot361$. $\vdash : P \in \text{Rel}^3\,\text{arithm} . \supset . C`P \subset \text{Rel}^2\,\text{excl}$

Dem.

$\vdash . *162\cdot1 . \supset \vdash : Q \in C`P . \supset . Q \,\mathrm{G}\, \Sigma`P$ (1)

$\vdash . *174\cdot3 . \supset \vdash : \text{Hp} . \supset . \Sigma`P \in \text{Rel}^2\,\text{excl}$ (2)

$\vdash . (1) . (2) . *163\cdot43 . \supset \vdash . \text{Prop}$

$*174\cdot362$. $\vdash : P \in \text{Rel}^3\,\text{arithm} . Q, Q' \in C`P . C``C`Q = C``C`Q' . \supset . Q = Q'$

Dem.

$\vdash . *174\cdot322 . \supset \vdash : \text{Hp} . \exists ! C``C`Q . \supset . Q = Q'$ (1)

$\vdash . *37\cdot45 . \supset \vdash : C``C`Q = \Lambda . \supset . C`Q = \Lambda .$

[$*33\cdot241$] $\supset . Q = \dot{\Lambda}$ (2)

$\vdash . (2) . *13\cdot172 . \supset \vdash : \text{Hp} . C``C`Q = \Lambda . \supset . Q = Q'$ (3)

$\vdash . (1) . (3) . \supset \vdash . \text{Prop}$

***174·363.** $\vdash : P \in \text{Rel}^3\,\text{arithm} \,.\supset .\, \text{Prod};P \in \text{Rel}^2\,\text{excl}$

Dem.

$\vdash .\, {*}173{\cdot}161 \,.\, {*}174{\cdot}361 \,.\, {*}173{\cdot}2 \,.\, \text{Transp} \,.\supset$

$\vdash :.\, \text{Hp} \,.\supset : Q, Q' \in C\text{‘}P \,.\, \exists !\, C\text{‘}\text{Prod}\text{‘}Q \cap C\text{‘}\text{Prod}\text{‘}Q' \,.\supset .$

$\qquad \exists !\, \text{Prod}\text{‘}C\text{“}C\text{‘}Q \cap \text{Prod}\text{‘}C\text{“}C\text{‘}Q' \,.$

$[{*}115{\cdot}23 . {*}174{\cdot}33] \qquad \supset .\, C\text{“}C\text{‘}Q = C\text{“}C\text{‘}Q' \,.$

$[{*}174{\cdot}362] \qquad \supset .\, Q = Q' \,.$

$[{*}30{\cdot}37] \qquad \supset .\, \text{Prod}\text{‘}Q = \text{Prod}\text{‘}Q' \qquad (1)$

$\vdash .\, (1) \,.\, {*}163{\cdot}11 \,.\, {*}150{\cdot}22 \,.\supset \vdash .\, \text{Prop}$

***174·4.** $\vdash : P \in \text{Rel}^2\,\text{excl} \,.\, P \mathrel{\text{⋐}} J \,.\supset .$

$\qquad \text{Prod}\text{‘}\Sigma\text{‘}P = \text{D};\dot{s};\text{Prod}\text{‘}\Pi;P = \dot{s};\text{D}_\epsilon;\text{Prod}\text{‘}\Pi;P$

Dem.

$\vdash .\, {*}173{\cdot}1 \,. \qquad \supset \vdash .\, \text{Prod}\text{‘}\Sigma\text{‘}P = \text{D};\Pi\text{‘}\Sigma\text{‘}P \qquad (1)$

$\vdash .\, (1) \,.\, {*}174{\cdot}24 \,.\supset \vdash : P \in \text{Rel}^2\,\text{excl} \,.\, P \mathrel{\text{⋐}} J \,.\supset .\, \text{Prod}\text{‘}\Sigma\text{‘}P = \text{D};\dot{s};\text{Prod}\text{‘}\Pi;P$

$[{*}41{\cdot}43] \qquad = \dot{s};\text{D}_\epsilon;\text{Prod}\text{‘}\Pi;P : \supset \vdash .\, \text{Prop}$

***174·401.** $\vdash : P \in \text{Rel}^3\,\text{arithm} \,.\supset .\, \text{Prod}\text{‘}\text{Prod};P = \text{D}_\epsilon;\text{Prod}\text{‘}\Pi;P$

Dem.

$\vdash .\, {*}80{\cdot}33 \,.\, {*}162{\cdot}23 \,.\supset \vdash : R \in F_\Delta\text{‘}C\text{‘}Q \,.\supset .\, \text{D}\text{‘}R \subset C\text{‘}\Sigma\text{‘}Q \qquad (1)$

$\vdash .\, (1) \,.\supset \vdash : R \in F_\Delta\text{‘}C\text{‘}Q \,.\, R' \in F_\Delta\text{‘}C\text{‘}Q' \,.\, \exists !\, \text{D}\text{‘}R \cap \text{D}\text{‘}R' \,.\supset .$

$\qquad \exists !\, C\text{‘}\Sigma\text{‘}Q \cap C\text{‘}\Sigma\text{‘}Q' \qquad (2)$

$\vdash .\, (2) \,.\, {*}174{\cdot}35 \,.\supset \vdash :.\, \text{Hp} \,.\supset :$

$\qquad Q, Q' \in C\text{‘}P \,.\, R \in F_\Delta\text{‘}C\text{‘}Q \,.\, R' \in F_\Delta\text{‘}C\text{‘}Q' \,.\, \text{D}\text{‘}R = \text{D}\text{‘}R' \,.\, \exists !\, \text{D}\text{‘}R \,.\supset .$

$\qquad Q = Q' \,.$

$[{*}81{\cdot}21 . {*}174{\cdot}361 . {*}163{\cdot}12] \qquad \supset .\, R = R' \qquad (3)$

$\vdash .\, (3) \,.\, {*}33{\cdot}241 \,.\supset$

$\vdash :.\, \text{Hp} \,.\supset : Q, Q' \in C\text{‘}P \,.\, R \in F_\Delta\text{‘}C\text{‘}Q \,.\, R' \in F_\Delta\text{‘}C\text{‘}Q' \,.\, \text{D}\text{‘}R = \text{D}\text{‘}R' \,.\supset .\, R = R' :$

$[{*}172{\cdot}12 . {*}150{\cdot}22] \supset : \text{D} \upharpoonright s\text{‘}C\text{“}\Pi\text{“}C\text{‘}P \in 1 \to 1 :$

$[{*}162{\cdot}22] \qquad \supset : \text{D} \upharpoonright C\text{‘}\Sigma\text{‘}\Pi;P \in 1 \to 1 :$

$[{*}173{\cdot}33] \qquad \supset : \text{D}_\epsilon;\text{Prod}\text{‘}\Pi;P = \text{Prod}\text{‘}\text{D}\dagger;\Pi;P$

$[{*}173{\cdot}1] \qquad = \text{Prod}\text{‘}\text{Prod};P :. \supset \vdash .\, \text{Prop}$

***174·41.** $\vdash : P \in \text{Rel}^3\,\text{arithm} \,.\, P \mathrel{\text{⋐}} J \,.\supset .\, \text{Prod}\text{‘}\Sigma\text{‘}P = \dot{s};\text{Prod}\text{‘}\text{Prod};P$

$[{*}174{\cdot}4{\cdot}401]$

***174·42.** $\vdash : P \in \text{Rel}^3\,\text{arithm} \,.\, P \mathrel{\text{⋐}} J \,.\supset .$

$\qquad s \upharpoonright (C\text{‘}\text{Prod}\text{‘}\text{Prod};P) \in (\text{Prod}\text{‘}\Sigma\text{‘}P)\,\overline{\text{smor}}\,(\text{Prod}\text{‘}\text{Prod};P)$

Dem. $\vdash .\, {*}173{\cdot}161{\cdot}2 \,.\, {*}174{\cdot}363 \,.\supset$

$\vdash : \text{Hp} \,.\supset .\, C\text{‘}\text{Prod}\text{‘}\text{Prod};P \subset \text{Prod}\text{‘}C\text{“}C\text{‘}\text{Prod};P$

$[{*}150{\cdot}22] \qquad \subset \text{Prod}\text{‘}C\text{“}\text{Prod}\text{“}C\text{‘}P$

$[{*}173{\cdot}161 . {*}174{\cdot}361] \qquad \subset \text{Prod}\text{‘}\text{Prod}\text{“}C\text{“‘}C\text{“}C\text{‘}P \qquad (1)$

$\vdash .\, (1) \,.\, {*}174{\cdot}33 \,.\, {*}115{\cdot}46 \,.\supset \vdash : \text{Hp} \,.\supset .\, s \upharpoonright C\text{‘}\text{Prod}\text{‘}\text{Prod};P \in 1 \to 1 \qquad (2)$

$\vdash .\, (2) \,.\, {*}174{\cdot}41 \,.\, {*}151{\cdot}231 \,.\supset \vdash .\, \text{Prop}$

***174·43.** $\vdash : P \in \text{Rel}^3\,\text{arithm} \,.\, P \mathrel{\text{⋐}} J \,.\supset .\, \text{Prod}\text{‘}\Sigma\text{‘}P \;\text{smor}\; \text{Prod}\text{‘}\text{Prod};P$

$[{*}174{\cdot}42]$

This is the associative law for Prod.

***174·44.** $\vdash : P \in \text{Rel}^3\,\text{arithm} . \supset . \text{Prod}`\text{Prod};P = \text{D}_\epsilon;\text{D};\Pi`\Pi;P$
[*174·401 . *173·1]

***174·45.** $\vdash : P \in \text{Rel}^3\,\text{arithm} . \supset .$
$$(\text{D}_\epsilon \mid \text{D}) \upharpoonright C`\Pi`\Pi;P \in (\text{Prod}`\text{Prod};P)\,\overline{\text{smor}}\,(\Pi`\Pi;P)$$

Dem.

$\vdash . *174·18 . \supset \vdash : \text{Hp} . \supset . \text{D} \upharpoonright C`\Pi`\Pi;P \in 1 \to 1$ (1)

$\vdash . *80·33 . *162·23 . \supset$

$\vdash : R \in F_\Delta`C`Q . R' \in F_\Delta`C`Q' . \exists ! \text{D}`R \cap \text{D}`R' . \supset . \exists ! C`\Sigma`Q \cap C`\Sigma`Q'$ (2)

$\vdash . (2) . *174·35 . \supset$

$\vdash :. \text{Hp} . \supset : Q, Q' \in C`P . R \in F_\Delta`C`Q . R' \in F_\Delta`C`Q' . \exists ! \text{D}`R . \text{D}`R = \text{D}`R' . \supset .$
$Q = Q' .$

[*81·21.*174·361.*163·12] $\supset . R = R'$ (3)

$\vdash . (3) . *33·241 . \supset$

$\vdash :. \text{Hp} . \supset : Q, Q' \in C`P . R \in F_\Delta`C`Q . R' \in F_\Delta`C`Q' . \text{D}`R = \text{D}`R' . \supset . R = R' :$

[*172·12] $\supset : Q, Q' \in C`P . R \in C`\Pi`Q . R' \in C`\Pi`Q' . \text{D}`R = \text{D}`R' . \supset . R = R'$ (4)

$\vdash . *173·161 . *37·6 . *173·2 . \text{Transp} . \supset$

$\vdash :: \text{Hp} . \mu, \nu \in C`\text{Prod}`\Pi;P . \text{D}``\mu = \text{D}``\nu . \supset :.$

$R \in \mu . \supset : (\exists Q) : Q \in C`P . R \in C`\Pi`Q :$

$(\exists Q', R') . Q' \in C`P . R' \in C`\Pi`Q' . R' \in \nu . \text{D}`R = \text{D}`R' :$

[(4)] $\supset : (\exists R') . R' \in \nu . R = R' :$

[*13·195] $\supset : R \in \nu$ (5)

Similarly $\vdash :. \text{Hp}(5) . \supset : R \in \nu . \supset . R \in \mu$ (6)

$\vdash . (5) . (6) . \supset \vdash :. \text{Hp} . \supset : \mu, \nu \in C`\text{Prod}`\Pi;P . \text{D}``\mu = \text{D}``\nu . \supset . \mu = \nu :$

[*71·55] $\supset : \text{D}_\epsilon \upharpoonright C`\text{Prod}`\Pi;P \in 1 \to 1 :$

[*150·22.*173·1] $\supset : \text{D}_\epsilon \upharpoonright \text{D}``C`\Pi`\Pi;P \in 1 \to 1$ (7)

$\vdash . (1) . (7) . *35·481 . \supset \vdash : \text{Hp} . \supset . (\text{D}_\epsilon \mid \text{D}) \upharpoonright C`\Pi`\Pi;P \in 1 \to 1$ (8)

$\vdash . (8) . *174·44 . \supset \vdash . \text{Prop}$

***174·46.** $\vdash : P \in \text{Rel}^3\,\text{arithm} . \supset . \text{Prod}`\text{Prod};P \,\text{smor}\, \Pi`\Pi;P$ [*174·45]

***174·461.** $\vdash : P \in \text{Rel}^3\,\text{arithm} . P \subset J . \supset . \text{Prod}`\text{Prod};P \,\text{smor}\, \Pi`\Sigma`P$
[*174·46·25]

***174·462.** $\vdash : P \in \text{Rel}^3\,\text{arithm} . \supset . \Pi`\text{Prod};P \,\text{smor}\, \text{Prod}`\text{Prod};P$
[*174·363 . *173·16]

The two following propositions merely sum up previous results.

***174·47.** $\vdash : P \in \text{Rel}^3\,\text{arithm} . P \subset J . \supset .$
$\text{Prod}`\Sigma`P = s;\text{Prod}`\text{Prod};P = s;\text{D}_\epsilon;\text{D};\Pi`\Pi;P = \text{D};\dot{s};\text{Prod}`\Pi;P .$
$s \upharpoonright C`\text{Prod}`\text{Prod};P, s \mid \text{D}_\epsilon \mid \text{D} \upharpoonright C`\Pi`\Pi;P, \text{D} \mid \dot{s} \upharpoonright C`\text{Prod}`\Pi;P \in 1 \to 1$
[*174·42·45·24 . *41·43]

***174·48.** $\vdash : P \in \text{Rel}^3\,\text{arithm} . P \subset J . \supset .$
$\text{Nr}`\text{Prod}`\text{Prod};P = \text{Nr}`\text{Prod}`\Sigma`P = \text{Nr}`\Pi`\Sigma`P = \text{Nr}`\Pi`\Pi;P$
$= \text{Nr}`\text{Prod}`\Pi;P = \text{Nr}`\Pi`\text{Prod};P$
[*174·43·46·25·462 . *152·321]

$*176$. EXPONENTIATION

Summary of $*176$.

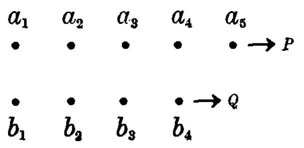
The definition of exponentiation is framed on the analogy of the definition in cardinals, *i.e.* we put

$$P \exp Q = \text{Prod}`P \mathbin{\dot{\downarrow}} {}^{\prime\prime}Q \quad \text{Df.}$$

We put also, what is often a more convenient form,

$$P^Q = \dot{s}{}^{\prime\prime}(P \exp Q) \quad \text{Df.}$$

The relation P^Q has for its field (unless $Q = \dot{\Lambda}$) the class of Cantor's "Belegungen," *i.e.* the class $(C`P \uparrow C`Q)_\Delta`C`Q$. It arranges these by a form of the principle of first differences, namely as follows: Suppose M and N are two members of $(C`P \uparrow C`Q)_\Delta`C`Q$, and suppose there is in $C`Q$ a term $y$ for which the $M$-representative $(M`y)$ precedes the N-representative $(N`y)$, *i.e.* for which $(M`y) \, P \, (N`y)$, and suppose further that all terms in $C`Q$ which are earlier than y, *i.e.* for which $zQy \,.\, z \neq y$, have their M-representative and their N-representative identical; in this case we say that M has to N the relation P^Q. This may be stated as follows, provided we assume that P and Q are series: Let M and N be two one-valued functions whose possible arguments are all the members of $C`Q$, while their values are some or all of the members of $C`P$. Then we say that M has to N the relation P^Q if the first argument for which the two functions do not have the same value gives an earlier value to M than to N. Thus for example let P be the series a_1, a_2, a_3, a_4, a_5, and let Q be the series b_1, b_2, b_3, b_4. Then M and N are to be such that $M`b$ or $N`b$ is defined when, and only when, b is b_1 or b_2 or b_3 or b_4, and the value of $M`b$ or $N`b$ is a_1 or a_2 or a_3 or a_4 or a_5. Then if $M`b_1 = a_1$ and $N`b_1 \neq a_1$, M precedes N; if $M`b_1 = N`b_1 = a_1$, and $M`b_2 = a_1 \,.\, N`b_2 \neq a_1$, M precedes N; and so on. Thus in this case the first term of the series generated by P^Q is the one for which $M`b = a_1$ when $b$ has any of the values $b_1, b_2, b_3, b_4$. Thus the first term of the series is $\iota`a_1 \uparrow C`Q$, *i.e.* $\iota`B`P \uparrow C`Q$. The next term will be



$$\iota`a_1 \uparrow (\iota`b_1 \cup \iota`b_2 \cup \iota`b_3) \cup \iota`a_2 \uparrow \iota`b_4,$$

i.e. $$\iota`B`P \uparrow \text{D}`Q \cup 2_P \downarrow B`\breve{Q}.$$

The next is $$\iota`B`P \uparrow \text{D}`Q \cup 3_P \downarrow B`\breve{Q},$$

and so on. This makes it evident that our series has the structure required of a series which is to represent the Qth power of P.

The two relations $P \exp Q$ and P^Q are ordinally similar, since $\dot{s}$ is one-one when its field is limited to $C`(P \exp Q)$. This follows from $*116 \cdot 131$, together with

$$\exists ! \, Q \,.\, \supset \,.\, C`(P \exp Q) = (C`P) \exp (C`Q).$$

If S is a correlator of P and P', and T is a correlator of Q and Q', then $(S \| \breve{T})_\epsilon$ and $(S \| \breve{T})$, with their converse domains limited, are respectively correlators of $(P \exp Q)$ with $(P' \exp Q')$ and of P^Q with $P'^{Q'}$. This shows that the relation-number of $(P \exp Q)$ depends only upon those of P and Q, which is of course essential if $(P \exp Q)$ is to afford a definition of exponentiation.

If the multiplicative axiom is assumed, then if R is a relation which is like Q, and whose field consists of relations which are like P, and $R \in \mathrm{Rel}^2 \,\mathrm{excl}$, the product of R is like $(P \exp Q)$. That is, if we put $\mu = \mathrm{Nr}\text{‘}P \,.\, \nu = \mathrm{Nr}\text{‘}Q$, so that R consists of ν terms each of which has μ terms, the product of R has μ^ν terms. This gives the connection of multiplication with exponentiation.

There are two formal laws of exponentiation which hold for relation-numbers, namely

$$P^Q \times P^R \,\mathrm{smor}\, P^{Q \dotplus R}$$

and

$$(P^Q)^R \,\mathrm{smor}\, P^{R \times Q}.$$

They both need a hypothesis: the first needs

$$\dot{\exists} \,!\, Q \,.\, \dot{\exists} \,!\, R \,.\, C\text{‘}Q \cap C\text{‘}R = \Lambda,$$

while the second needs $\qquad R \in J$

because it is proved by means of the associative law (*174·43).

The first of the above formal laws can be generalized, by putting $\Sigma\text{‘}S$ in place of $Q \dotplus R$, and taking the product of the various powers

$$P \exp Q,\; P \exp Q', \ldots,$$

where $Q, Q', \ldots \in C\text{‘}S$, and the products are taken in the order determined by S. The resulting generalization is

$$S \in \mathrm{Rel}^2 \,\mathrm{excl} \,.\, S \in J \,.\, \supset \,.\, \{\mathrm{Prod}\text{‘}(P \exp)^{;}S\} \,\mathrm{smor}\, \{P \exp (\Sigma\text{‘}S)\}.$$

The proof of this proposition results immediately from *174·43 and *162·35.

The proof of the second of the formal laws is more difficult. We observe, to begin with, that

$$P \exp (R \times Q) = \mathrm{Prod}\text{‘}P \underset{\cdot\cdot}{\downarrow}{}^{;}\Sigma\text{‘}Q \underset{\cdot\cdot}{\downarrow}{}^{;}R.$$

Assuming suitable hypotheses, this, by *162·35,

$$= \mathrm{Prod}\text{‘}\Sigma\text{‘}(P \underset{\cdot\cdot}{\downarrow})\dagger^{;}Q \underset{\cdot\cdot}{\downarrow}{}^{;}R,$$

which is like $\qquad \mathrm{Prod}\text{‘}\mathrm{Prod}^{;}(P \underset{\cdot\cdot}{\downarrow})\dagger^{;}Q \underset{\cdot\cdot}{\downarrow}{}^{;}R$, by *174·43.

But $\qquad (P \exp Q) \exp R = \mathrm{Prod}\text{‘}\{\mathrm{Prod}\text{‘}P \underset{\cdot\cdot}{\downarrow}{}^{;}Q\} \underset{\cdot\cdot}{\downarrow}{}^{;}R.$

Thus our result will follow if we can prove

$$\{\mathrm{Prod}^{;}(P \underset{\cdot\cdot}{\downarrow})\dagger^{;}Q \underset{\cdot\cdot}{\downarrow}{}^{;}R\} \,\mathrm{smor\,smor}\, \{(\mathrm{Prod}\text{‘}P \underset{\cdot\cdot}{\downarrow}{}^{;}Q) \underset{\cdot\cdot}{\downarrow}{}^{;}R\}.$$

Now one member of the field of $\mathrm{Prod}^{;}(P \underset{\cdot\cdot}{\downarrow})\dagger^{;}Q \underset{\cdot\cdot}{\downarrow}{}^{;}R$ will be

$$\mathrm{Prod}\text{‘}P \underset{\cdot\cdot}{\downarrow}{}^{;}Q \underset{\cdot\cdot}{\downarrow} z, \text{ where } z \in C\text{‘}R.$$

This is like Prod$`P \underset{\cdot,}{\downarrow} {}^{\flat}Q$, because $Q \underset{\cdot,}{\downarrow} z$ smor $Q$. Hence Prod$^{\flat}(P \underset{\cdot,}{\downarrow})\dagger^{\flat} Q \underset{\cdot,}{\downarrow} {}^{\flat}R$ is a series of terms each of which is like Prod$`P \underset{\cdot,}{\downarrow} {}^{\flat}Q$, and the whole series of such terms is like R. If we assumed the multiplicative axiom, this would suffice to prove the result. But it is possible to obtain our result without assuming the multiplicative axiom.

For this purpose, we proceed as follows. The correlator of

$$\text{Prod}`P \underset{\cdot,}{\downarrow} {}^{\flat}Q \underset{\cdot,}{\downarrow} z \text{ and } \text{Prod}`P \underset{\cdot,}{\downarrow} {}^{\flat}Q$$

is $\{|(\text{Cnv}`\downarrow z)\}_\epsilon$, by $*165\cdot361$ and $*172\cdot3$. Call this $M`z$. Then

$$M \epsilon 1 \to 1 : z \epsilon C`R \,.\supset_z .\, (M`z) \epsilon (\text{Prod}`P \underset{\cdot,}{\downarrow} {}^{\flat}Q \underset{\cdot,}{\downarrow} z) \,\overline{\text{smor}}\, (\text{Prod}`P \underset{\cdot,}{\downarrow} {}^{\flat}Q) :$$
$$z, w \epsilon C`R \,.\, \exists ! \,\text{D}`M`z \cap \text{D}`M`w \,.\supset_{z,w} .\, z = w.$$

This, by the help of two or three lemmas, suffices to prove that

$$\{\text{Prod}^{\flat}(P \underset{\cdot,}{\downarrow})\dagger^{\flat} Q \underset{\cdot,}{\downarrow} {}^{\flat}R\} \text{ smor smor } \{(\text{Prod}`P \underset{\cdot,}{\downarrow} {}^{\flat}Q) \underset{\cdot,}{\downarrow} {}^{\flat}R\},$$

whence the result follows.

The principal propositions of the present number are the following:

$*176\cdot1.$ $\vdash . P \exp Q = \text{Prod}`P \underset{\cdot,}{\downarrow} {}^{\flat}Q = \text{D}^{\flat}\Pi`P \underset{\cdot,}{\downarrow} {}^{\flat}Q$

$*176\cdot11.$ $\vdash . P^Q = \dot{s}^{\flat}(P \exp Q) = \dot{s}^{\flat}\text{Prod}`P \underset{\cdot,}{\downarrow} {}^{\flat} Q = \dot{s}^{\flat}\text{D}^{\flat}\Pi`P \underset{\cdot,}{\downarrow} {}^{\flat} Q$

These propositions merely embody the definitions.

$*176\cdot14.$ $\vdash : \dot{\exists} ! Q \,.\supset.\, C`(P \exp Q) = (C`P) \exp (C`Q) \,.\, C`P^Q = (C`P \uparrow C`Q)_\Delta`C`Q$

$*176\cdot151.$ $\vdash :. P = \dot{\Lambda} \,.\vee.\, Q = \dot{\Lambda} : \equiv . P \exp Q = \dot{\Lambda} . \equiv . P^Q = \dot{\Lambda}$

It will be observed that in relation-arithmetic, $\mu^0 = 0$, whereas in cardinal arithmetic $\mu^0 = 1$. The difference is due to the fact that there is no ordinal number 1 (cf. $*153$).

$*176\cdot181.$ $\vdash . P^Q \text{ smor } (P \exp Q)$

$*176\cdot182.$ $\vdash . (P \exp Q) \text{ smor } (\Pi`P \underset{\cdot,}{\downarrow} {}^{\flat} Q)$

$*176\cdot19.$ $\vdash :: S (P^Q) T . \equiv :. S, T \epsilon (C`P \uparrow C`Q)_\Delta`C`Q :.$
$$(\exists y) : y \epsilon C`Q \,.\, (S`y) P (T`y) : y'Qy \,.\, y' \neq y \,.\supset_{y'} .\, S`y' = T`y'$$

$*176\cdot2.$ $\vdash : U \upharpoonright C`R \epsilon P \,\overline{\text{smor}}\, R \,.\, W \upharpoonright C`S \epsilon Q \,\overline{\text{smor}}\, S \,.\supset .$
$$(U \| \breve{W})_\epsilon \upharpoonright C`(R \exp S) \epsilon (P \exp Q) \,\overline{\text{smor}}\, (R \exp S)$$

$*176\cdot21.$ With the same hypothesis, $(U \| \breve{W}) \upharpoonright C`(R^S)$ correlates P^Q and R^S

$*176\cdot22.$ $\vdash : P \text{ smor } R \,.\, Q \text{ smor } S \,.\supset .\, (P \exp Q) \text{ smor } (R \exp S) \,.\, P^Q \text{ smor } R^S$

$*176\cdot24.$ $\vdash :. \text{Mult ax} \,.\supset :$
$$R \epsilon \text{Rel}^2 \text{ excl} \cap \text{Nr}`Q \,.\, C`R \subset \text{Nr}`P \,.\supset .\, \Pi`R \text{ smor } (P \exp Q)$$

This proposition connects multiplication and exponentiation.

$*176\cdot31.$ $\vdash : \dot{\exists} ! Q \,.\supset .\, \overrightarrow{B}`(P \exp Q) = (\overrightarrow{B}`P) \exp (C`Q)$

***176·311·32·321.** Similar propositions for $\overrightarrow{B}$'Cnv'$(P \exp Q)$, $\overrightarrow{B}$'(P^Q), $\overrightarrow{B}$'(Cnv'P^Q)

***176·34.** $\vdash : \dot{\exists} ! Q . E ! B'P . \supset .$

$$B'(P \exp Q) = (B'P) \downarrow "C'Q . B'(P^Q) = (\iota'B'P) \uparrow C'Q$$

We come next to the formal laws. We have

***176·42.** $\vdash : \dot{\exists} ! Q . \dot{\exists} ! R . C'Q \cap C'R = \Lambda . \supset . P^Q \times P^R \text{ smor } P^{Q \dotplus R} .$

$$(P \exp Q) \times (P \exp R) \text{ smor } P \exp (Q \dotplus R)$$

***176·44.** $\vdash : S \epsilon \text{Rel}^2 \text{excl} . S \subset J . \supset . \{\text{Prod}'(P \exp){}^{;}S\} \text{ smor } \{P \exp (\Sigma'S)\}$

This is an extension of *176·42.

***176·57.** $\vdash : R \subset J . \supset . \{(P \exp Q) \exp R\} \text{ smor } \{P \exp (R \times Q)\} .$

$$(P^Q)^R \text{ smor } P^{R \times Q}$$

***176·01.** $P \exp Q = \text{Prod}'P \underset{\cdot\cdot}{\downarrow} {}^{;}Q$ Df

***176·02.** $P^Q = \dot{s}{}^{;}(P \exp Q)$ Df

***176·1.** $\vdash . P \exp Q = \text{Prod}'P \underset{\cdot\cdot}{\downarrow} {}^{;}Q = \text{D}{}^{;}\Pi'P \underset{\cdot\cdot}{\downarrow} {}^{;}Q$ [(*176·01)]

***176·11.** $\vdash . P^Q = \dot{s}{}^{;}(P \exp Q) = \dot{s}{}^{;}\text{Prod}'P \underset{\cdot\cdot}{\downarrow} {}^{;}Q = \dot{s}{}^{;}\text{D}{}^{;}\Pi'P \underset{\cdot\cdot}{\downarrow} {}^{;}Q$ [(*176·02)]

***176·12.** $\vdash :: \mu (P \exp Q) \nu . \equiv :. \mu, \nu \epsilon (C'P) \exp (C'Q) :.$

$(\exists y, x, x') : x \downarrow y \epsilon \mu . x' \downarrow y \epsilon \nu . xPx' : zQy . z \neq y . w \downarrow z \epsilon \mu . \supset_{w,z} . w \downarrow z \epsilon \nu$

Dem.

$\vdash . *165{\cdot}21{\cdot}12 . *163{\cdot}12 . \supset \vdash . F \upharpoonright P \underset{\cdot\cdot}{\downarrow} "C'Q \epsilon \text{Cls} \to 1 .$ (1)

[*85·1.*115·1.*33·5] $\supset \vdash . \text{D}"F_\Delta'P \underset{\cdot\cdot}{\downarrow} "C'Q = \text{Prod}'C"P \underset{\cdot\cdot}{\downarrow} "C'Q$

[*165·12·14.(*116·01)] $= (C'P) \exp (C'Q)$ (2)

$\vdash . *176{\cdot}1 . *173{\cdot}11 . *172{\cdot}11 . *165{\cdot}12 . \supset$

$\vdash ::. \mu (P \exp Q) \nu . \equiv :: (\exists M, N, y) :. M, N \epsilon F_\Delta'P \underset{\cdot\cdot}{\downarrow} "C'Q :$

$y \epsilon C'Q . (M'P \underset{\cdot\cdot}{\downarrow} y) (P \underset{\cdot\cdot}{\downarrow} y) (N'P \underset{\cdot\cdot}{\downarrow} y) :$

$zQy . z \neq y . \supset_z . M'P \underset{\cdot\cdot}{\downarrow} z = N'P \underset{\cdot\cdot}{\downarrow} z : \mu = \text{D}'M . \nu = \text{D}'N ::$

[*81·15.(1).*150·6] $\equiv :: (\exists M, N, y) :. M, N \epsilon F_\Delta'P \underset{\cdot\cdot}{\downarrow} "C'Q . \mu = \text{D}'M . \nu = \text{D}'N :$

$y \epsilon C'Q . \breve{\iota}'(\mu \cap \downarrow y"C'P) (\downarrow y{}^{;}P) \breve{\iota}'(\nu \cap \downarrow y"C'P) :$

$zQy . z \neq y . \supset_z . \breve{\iota}'(\mu \cap \downarrow y"C'P) = \breve{\iota}'(\nu \cap \downarrow y"C'P) ::$

[(2).*150·55] $\equiv :: (\exists y) :: \mu, \nu \epsilon (C'P) \exp (C'Q) . y \epsilon C'Q :. (\exists x, x') :.$

$x \downarrow y = \breve{\iota}'(\mu \cap \downarrow y"C'P) . x' \downarrow y = \breve{\iota}'(\nu \cap \downarrow y"C'P) . xPx' :$

$zQy . z \neq y . w \downarrow z = \breve{\iota}'(\mu \cap \downarrow z"C'P) .$

$w' \downarrow z = \breve{\iota}'(\nu \cap \downarrow z"C'P) . \supset_{z,w,w'} . w = w' ::$

[*116·11] $\equiv :: (\exists y) :: \mu, \nu \epsilon (C'P) \exp (C'Q) :.$

$(\exists x, x') : x \downarrow y \epsilon \mu . x' \downarrow y \epsilon \nu . xPx' :$

$zQy . z \neq y . w \downarrow z \epsilon \mu . \supset_{w,z} . w \downarrow z \epsilon \nu ::. \supset \vdash . \text{Prop}$

The above proposition is used in $*176{\cdot}19$. It has the merit of giving a direct formula for $P \exp Q$, instead of one which proceeds by way of $\Pi`P \downarrow ; Q$.

$*176{\cdot}13$. $\vdash : \dot{\exists} ! (P \exp Q) . \equiv . \dot{\exists} ! P^Q . \equiv . \dot{\exists} ! \Pi`P \downarrow ; Q$ $[*150{\cdot}25 . *176{\cdot}1{\cdot}11]$

$*176{\cdot}131$. $\vdash : Q = \dot{\Lambda} . \supset . P \exp Q = \dot{\Lambda} . P^Q = \dot{\Lambda}$ $[*165{\cdot}241 . *173{\cdot}2 . *150{\cdot}42]$

Owing to this proposition, propositions stating analogies between ordinal and cardinal powers mostly require the hypothesis $\dot{\exists} ! Q$ or its equivalent, because an ordinal power whose index is zero is itself zero, whereas a cardinal power whose index is zero is 1.

$*176{\cdot}132$. $\vdash : P = \dot{\Lambda} . \dot{\exists} ! Q . \supset . P \exp Q = \dot{\Lambda} . P^Q = \dot{\Lambda}$
$[*165{\cdot}244 . *172{\cdot}14 . *176{\cdot}13 . *150{\cdot}42]$

$*176{\cdot}133$. $\vdash . C`P^Q = \dot{s}``C`(P \exp Q)$ $[*176{\cdot}11 . *150{\cdot}22]$

$*176{\cdot}14$. $\vdash : \dot{\exists} ! Q . \supset . C`(P \exp Q) = (C`P) \exp (C`Q) . C`P^Q = (C`P \uparrow C`Q)_\Delta `C`Q$

Dem.

$\vdash . *165{\cdot}243 . \supset \vdash : \text{Hp} . \supset . \dot{\exists} ! P \downarrow ; Q .$

$[*173{\cdot}161 . *165{\cdot}21]$ $\supset . C`\text{Prod}`P \downarrow ; Q = \text{Prod}`C``C`P \downarrow ; Q .$

$[*176{\cdot}1 . *165{\cdot}14]$ $\supset . C`(P \exp Q) = \text{Prod}`(C`P) \downarrow ``(C`Q)$

$[(*116{\cdot}01)]$ $= (C`P) \exp (C`Q)$ (1)

$\vdash . (1) . *176{\cdot}133 . \supset \vdash : \text{Hp} . \supset . C`P^Q = \dot{s}``\{(C`P) \exp (C`Q)\}$

$[*116{\cdot}13]$ $= (C`P \uparrow C`Q)_\Delta `C`Q$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

$*176{\cdot}15$. $\vdash : \dot{\exists} ! P . \dot{\exists} ! Q . \equiv . \dot{\exists} ! (P \exp Q) . \equiv . \dot{\exists} ! P^Q$

Dem.

$\vdash . *176{\cdot}131{\cdot}132 .$ $\supset \vdash : \dot{\exists} ! (P \exp Q) . \supset . \dot{\exists} ! P . \dot{\exists} ! Q$ (1)

$\vdash . *116{\cdot}18 . *176{\cdot}14 . \supset \vdash : \dot{\exists} ! P . \dot{\exists} ! Q . \supset . \dot{\exists} ! C`(P \exp Q) .$

$[*33{\cdot}24]$ $\supset . \dot{\exists} ! (P \exp Q)$ (2)

$\vdash . (1) . (2) . *176{\cdot}13 . \supset \vdash . \text{Prop}$

$*176{\cdot}151$. $\vdash :. P = \dot{\Lambda} . \vee . Q = \dot{\Lambda} : \equiv . P \exp Q = \dot{\Lambda} . \equiv . P^Q = \dot{\Lambda}$ $[*176{\cdot}15]$

$*176{\cdot}16$. $\vdash . C`(P \exp Q) \subset (C`P) \exp (C`Q) . C`P^Q \subset (C`P \uparrow C`Q)_\Delta `C`Q$
$[*176{\cdot}14{\cdot}151]$

$*176{\cdot}18$. $\vdash . \dot{s} \upharpoonright C`(P \exp Q) \in (P^Q) \overline{\text{smor}} (P \exp Q)$

Dem.

$\vdash . *116{\cdot}131 . *176{\cdot}14 . \supset$

$\vdash : \dot{\exists} ! Q . \supset . \dot{s} \upharpoonright C`(P \exp Q) \in (C`P^Q) \overline{\text{sm}} C`(P \exp Q)$ (1)

$\vdash . (1) . *176{\cdot}11 . *151{\cdot}191 . \supset$

$\vdash : \dot{\exists} ! Q . \supset . \dot{s} \upharpoonright C`(P \exp Q) \in (P^Q) \overline{\text{smor}} (P \exp Q)$ (2)

$\vdash . *176{\cdot}151 . *150{\cdot}42 . *72{\cdot}1 . \supset$

$\vdash : Q = \Lambda . \supset . \dot{s} \upharpoonright C`(P \exp Q) \in (P^Q) \overline{\text{smor}} (P \exp Q)$ (3)

$\vdash . (2) . (3) . \supset \vdash . \text{Prop}$

$*176{\cdot}181$. $\vdash . P^Q \operatorname{smor} (P \exp Q)$ $[*176{\cdot}18]$

$*176{\cdot}182$. $\vdash . (P \exp Q) \operatorname{smor} (\Pi ‘ P \downarrow ; Q)$ $[*176{\cdot}1 . *173{\cdot}16 . *165{\cdot}21]$

$*176{\cdot}19$. $\vdash :: S (P^Q) T . \equiv :. S, T \in (C‘P \uparrow C‘Q)_\Delta ‘ C‘Q :.$
$$(\exists y) : y \in C‘Q . (S‘y) P (T‘y) : y' Q y . y' \neq y . \supset_{y'} . S‘y' = T‘y'$$

Dem.

$\vdash . *176{\cdot}11{\cdot}12 . \supset$

$\vdash :: S (P^Q) T . \equiv :. (\exists \mu, \nu) :. \mu, \nu \in (C‘P) \exp (C‘Q) . S = \dot{s}‘\mu . T = \dot{s}‘\nu :.$
$$(\exists y, x, x') : y \in C‘Q . x \downarrow y \in \mu . x' \downarrow y \in \nu . x P x' :$$
$$y' Q y . y' \neq y . w \downarrow y' \in \mu . \supset_{y', w} . w \downarrow y' \in \nu :.$$

$[*56{\cdot}4]$ $\equiv :. (\exists \mu, \nu) :. \mu, \nu \in (C‘P) \exp (C‘Q) . S = \dot{s}‘\mu . T = \dot{s}‘\nu :.$
$$(\exists y, x, x') : y \in C‘Q . x S y . x' T y . x P x' : y' Q y . y' \neq y . w S y' . \supset_{y', w} . w T y' :.$$

$[*116{\cdot}13 . *80{\cdot}3] \equiv :. S, T \in (C‘P \uparrow C‘Q)_\Delta ‘ C‘Q :. (\exists y) : y \in C‘Q . (S‘y) P (T‘y) :$
$$y' Q y . y' \neq y . \supset_{y'} . S‘y' = T‘y' :: \supset \vdash . \text{Prop}$$

The above proposition is often usefnl, since it gives a direct formula for P^Q, not one which passes by way of $P \exp Q$ or $\Pi ‘ P \downarrow ; Q$.

$*176{\cdot}2$. $\vdash : U \upharpoonright C‘R \in P \,\overline{\operatorname{smor}}\, R . W \upharpoonright C‘S \in Q \,\overline{\operatorname{smor}}\, S . \supset .$
$$(U \parallel \breve{W})_\epsilon \upharpoonright C‘(R \exp S) \in (P \exp Q) \,\overline{\operatorname{smor}}\, (R \exp S)$$

Dem.

$\vdash . *165{\cdot}362 . \supset \vdash : \text{Hp} . \supset . (U \parallel \breve{W}) \upharpoonright C‘\Sigma ‘ R \downarrow ; S \in (P \downarrow ; Q) \,\overline{\operatorname{smor}}\, \overline{\operatorname{smor}}\, (R \downarrow ; S)$.

$[*173{\cdot}3] \supset . (U \parallel \breve{W})_\epsilon \upharpoonright C‘\text{Prod}‘R \downarrow ; S \in (\text{Prod}‘P \downarrow ; Q) \,\overline{\operatorname{smor}}\, (\text{Prod}‘R \downarrow ; S)$ (1)

$\vdash . (1) . *176{\cdot}1 . \supset \vdash . \text{Prop}$

$*176{\cdot}21$. $\vdash : U \upharpoonright C‘R \in P \,\overline{\operatorname{smor}}\, R . W \upharpoonright C‘S \in Q \,\overline{\operatorname{smor}}\, S . \supset .$
$$(U \parallel \breve{W}) \upharpoonright C‘(R^S) \in (P^Q) \,\overline{\operatorname{smor}}\, (R^S)$$

Dem.

$\vdash . *176{\cdot}2{\cdot}18 . *151{\cdot}401 . \supset \vdash : \text{Hp} . \supset . \dot{s} ; (U \parallel \breve{W})_\epsilon \upharpoonright C‘(R \exp S) \in (P^Q) \,\overline{\operatorname{smor}}\, (R^S)$

$[*150{\cdot}961]$ $\supset . (\breve{U} \parallel W) \upharpoonright \dot{s}‘‘C‘(R \exp S) \in (P^Q) \,\overline{\operatorname{smor}}\, (R^S)$ (1)

$\vdash . (1) . *176{\cdot}11 . *150{\cdot}22 . \supset \vdash . \text{Prop}$

$*176{\cdot}22$. $\vdash : P \operatorname{smor} R . Q \operatorname{smor} S . \supset . (P \exp Q) \operatorname{smor} (R \exp S) . P^Q \operatorname{smor} R^S$
$[*176{\cdot}2{\cdot}21]$

$*176{\cdot}23$. $\vdash : R \operatorname{smor} \operatorname{smor} P \downarrow ; Q . \supset . \Pi ‘ R \operatorname{smor} (P \exp Q)$

Dem.

$\vdash . *172{\cdot}44 . \supset \vdash : \text{Hp} . \supset . \Pi ‘ R \operatorname{smor} \Pi ‘ P \downarrow ; Q$ (1)

$\vdash . (1) . *176{\cdot}182 . \supset \vdash . \text{Prop}$

$*176{\cdot}24$. $\vdash :. \text{Mult ax} . \supset :$
$$R \in \text{Rel}^2 \text{excl} \cap \text{Nr}‘Q . C‘R \subset \text{Nr}‘P . \supset . \Pi ‘ R \operatorname{smor} (P \exp Q)$$
$[*165{\cdot}38 . *176{\cdot}23]$

***176·3.** $\vdash . \mathrm{Cnv}‘(P^Q) = (\breve{P})^Q$

Dem.

$\vdash . *176·19 . \supset$

$\vdash :: T(\breve{P})^Q S . \equiv :. S, T \in (C‘P \uparrow C‘Q)_\Delta‘C‘Q :.$

$(\exists y) : y \in C‘Q . (T‘y)\, \breve{P}\, (S‘y) : y'Qy . y' \neq y . \supset_{y'} . S‘y' = T‘y' :.$

$[*176·19] \quad \equiv :. S(P^Q)T :: \supset \vdash . \mathrm{Prop}$

***176·31.** $\vdash : \dot{\exists} ! Q . \supset . \overrightarrow{B}‘(P \exp Q) = (\overrightarrow{B}‘P) \exp (C‘Q)$

Dem.

$\vdash . *165·21 . *163·12 . *71·221 . *93·1 . \supset \vdash . B \upharpoonright C‘P \underset{;}{\downarrow} ;Q \in \mathrm{Cls} \rightarrow 1 \qquad (1)$

$\vdash . *165·12·01 . *37·67 . \supset \vdash . \overrightarrow{B}“C‘P \underset{;}{\downarrow} ;Q = \hat{\alpha}\{(\exists z) . z \in C‘Q . \alpha = \overrightarrow{B}‘ \downarrow z;P\}$

$[*165·251.*151·5.*38·3] \quad = (\overrightarrow{B}‘P) \underset{;}{\downarrow} “C‘Q \qquad (2)$

$\vdash . *172·162 . *165·243 . \supset \vdash : \dot{\exists} ! Q . \supset . \overrightarrow{B}‘\Pi‘P \underset{;}{\downarrow} ;Q = B_\Delta‘C‘P \underset{;}{\downarrow} ;Q .$

$[*173·16.*165·21.*151·5] \quad \supset . \overrightarrow{B}‘(P \exp Q) = \mathrm{D}“B_\Delta‘C‘P \underset{;}{\downarrow} ;Q$

$[*85·1.(1).*115·1] \quad = \mathrm{Prod}‘\overrightarrow{B}“C‘P \underset{;}{\downarrow} ;Q$

$[(2)] \quad = \mathrm{Prod}‘(\overrightarrow{B}‘P) \underset{;}{\downarrow} “C‘Q$

$[(*116·01)] \quad = (\overrightarrow{B}‘P) \exp (C‘Q) : \supset \vdash . \mathrm{Prop}$

***176·311.** $\vdash : \dot{\exists} ! Q . \supset . \overrightarrow{B}‘\mathrm{Cnv}‘(P \exp Q) = (\overrightarrow{B}‘\breve{P}) \exp (C‘Q)$

[Proof as in *176·31]

***176·32.** $\vdash : \dot{\exists} ! Q . \supset . \overrightarrow{B}‘(P^Q) = (\overrightarrow{B}‘P \uparrow C‘Q)_\Delta‘C‘Q$

Dem. $\vdash . *176·31·18 . *151·5 . \supset$

$\vdash : \mathrm{Hp} . \supset . \overrightarrow{B}‘(P^Q) = \dot{s}“(\overrightarrow{B}‘P) \exp (C‘Q)$

$[*116·13] \quad = (\overrightarrow{B}‘P \uparrow C‘Q)_\Delta‘C‘Q : \supset \vdash . \mathrm{Prop}$

***176·321.** $\vdash : \dot{\exists} ! Q . \supset . \overrightarrow{B}‘\mathrm{Cnv}‘(P^Q) = (\overrightarrow{B}‘\breve{P} \uparrow C‘Q)_\Delta‘C‘Q \quad [*176·32·3]$

***176·33.** $\vdash :. \dot{\exists} ! Q . \supset : \exists ! \overrightarrow{B}‘(P \exp Q) . \equiv . \exists ! \overrightarrow{B}‘(P^Q) . \equiv . \exists ! \overrightarrow{B}‘P :$

$\exists ! \overrightarrow{B}‘\mathrm{Cnv}‘(P \exp Q) . \equiv . \exists ! \overrightarrow{B}‘\mathrm{Cnv}‘(P^Q) . \equiv . \exists ! \overrightarrow{B}‘\breve{P}$

[*176·31·311·32·321 . *116·18·15]

***176·34.** $\vdash : \dot{\exists} ! Q . \mathrm{E} ! B‘P . \supset .$

$B‘(P \exp Q) = (B‘P) \downarrow “C‘Q . B‘(P^Q) = (\iota‘B‘P) \uparrow C‘Q$

Dem. $\vdash . *176·31 . \supset \vdash : \mathrm{Hp} . \supset . \overrightarrow{B}‘(P \exp Q) = (\iota‘B‘P) \exp (C‘Q)$

$[(*116·01)] \quad = \mathrm{Prod}‘(\iota‘B‘P) \underset{;}{\downarrow} “C‘Q$

$[*38·3.*53·31] \quad = \mathrm{Prod}‘\iota“(B‘P) \downarrow “C‘Q$

$[*115·143] \quad = \iota‘\{(B‘P) \downarrow “C‘Q\} \qquad (1)$

$\vdash . *176{\cdot}32 . \supset \vdash : \mathrm{Hp} . \supset . \overrightarrow{B}{`}(P^Q) = (\iota{`}B{`}P \uparrow C{`}Q)_\Delta{`}C{`}Q$

$[*116{\cdot}12 . *51{\cdot}4] \qquad = \iota{`}\{(\iota{`}B{`}P) \uparrow C{`}Q\}$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***176·341.** $\vdash : \dot{\exists} ! Q . E ! B{`}\breve{P} . \supset .$

$B{`}\mathrm{Cnv}{`}(P \exp Q) = (B{`}\breve{P}) \downarrow {``}C{`}Q . B{`}\mathrm{Cnv}{`}(P^Q) = (\iota{`}B{`}\breve{P}) \uparrow C{`}Q$

[Proof as in *176·34]

***176·35.** $\vdash : P \mathrel{\mathsf{G}} Q . \supset . P^R \mathrel{\mathsf{G}} Q^R$

Dem.

$\vdash . *116{\cdot}12 . \supset \vdash : \mathrm{Hp} . \supset . (C{`}P \uparrow C{`}R)_\Delta{`}C{`}R \subset (C{`}Q \uparrow C{`}R)_\Delta{`}C{`}R$ (1)

$\vdash . (1) . *176{\cdot}19 . \supset \vdash . \mathrm{Prop}$

The above proposition is used in the theory of finite ordinals (*261·64).

The following propositions are concerned in proving (with a suitable hypothesis)

$$P^Q \times P^R \,\mathrm{smor}\, P^{Q \dotplus R}$$

and its extension

$$\{\mathrm{Prod}{`}(P \exp) ; S\} \,\mathrm{smor}\, \{P \exp (\Sigma{`}S)\}.$$

***176·4.** $\vdash : \dot{\exists} ! Q . \dot{\exists} ! R . P{``}C{`}Q \cap P{``}C{`}R = \Lambda . C{`}Q \cup C{`}R \subset \mathrm{G}{`}P . \supset .$

$\dot{s} | C \upharpoonright C{`}\{(\Pi{`}P ; Q) \times (\Pi{`}P ; R)\} \in \{\Pi{`}P ; (Q \dotplus R)\} \,\overline{\mathrm{smor}}\, \{(\Pi{`}P ; Q) \times (\Pi{`}P ; R)\}$

Dem.

$\vdash . *172{\cdot}34 . *150{\cdot}22{\cdot}24 . \supset \vdash : \mathrm{Hp} . \supset .$

$\dot{s} | C \upharpoonright C{`}\{(\Pi{`}P ; Q) \times (\Pi{`}P ; R)\} \in \{\Pi{`}(P ; Q \dotplus P ; R)\} \,\overline{\mathrm{smor}}\, \{(\Pi{`}P ; Q) \times (\Pi{`}P ; R)\}$ (1)

$\vdash . *162{\cdot}36 . \supset \vdash : \mathrm{Hp} . \supset . P ; Q \dotplus P ; R = P ; (Q \dotplus R)$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***176·41.** $\vdash : \dot{\exists} ! Q . \dot{\exists} ! R . P{``}C{`}Q \cap P{``}C{`}R = \Lambda . C{`}Q \cup C{`}R \subset \mathrm{G}{`}P . \supset .$

$\Pi{`}P ; (Q \dotplus R) \,\mathrm{smor}\, (\Pi{`}P ; Q) \times (\Pi{`}P ; R)$ [*176·4]

***176·42.** $\vdash : \dot{\exists} ! Q . \dot{\exists} ! R . C{`}Q \cap C{`}R = \Lambda . \supset . P^Q \times P^R \,\mathrm{smor}\, P^{Q \dotplus R} .$

$(P \exp Q) \times (P \exp R) \,\mathrm{smor}\, P \exp (Q \dotplus R)$

Dem.

$\vdash . *72{\cdot}411 . *165{\cdot}22 . \supset$

$\vdash : \dot{\exists} ! P . C{`}Q \cap C{`}R = \Lambda . \supset . P \downarrow_{;} {``}C{`}Q \cap P \downarrow_{;} {``}C{`}R = \Lambda$ (1)

$\vdash . (1) . *176{\cdot}41 \dfrac{P \downarrow_{;}}{P} . *38{\cdot}12 . *33{\cdot}431 . \supset$

$\vdash : \mathrm{Hp} . \dot{\exists} ! P . \supset . \Pi{`}P \downarrow_{;} ; (Q \dotplus R) \,\mathrm{smor}\, (\Pi{`}P \downarrow_{;} ; Q) \times (\Pi{`}P \downarrow_{;} ; R) .$

$[*176{\cdot}182 . *166{\cdot}23] \supset . P \exp (Q \dotplus R) \,\mathrm{smor}\, (P \exp Q) \times (P \exp R) .$ (2)

$[*176{\cdot}181 . *166{\cdot}23] \supset . P^{Q \dotplus R} \,\mathrm{smor}\, P^Q \times P^R$ (3)

$\vdash . *176{\cdot}151 . *166{\cdot}13 . *153{\cdot}101 . \supset$

$\vdash : P = \dot{\Lambda} . \supset . P \exp (Q \dotplus R) \,\mathrm{smor}\, (P \exp Q) \times (P \exp R) .$

$P^{Q \dotplus R} \,\mathrm{smor}\, P^Q \times P^R$ (4)

$\vdash . (2) . (3) . (4) . \supset \vdash . \mathrm{Prop}$

***176·43.** $\vdash : S \in \text{Rel}^2\,\text{excl} \,.\, S \subset J \,.\, \supset .$
$$s \upharpoonright C\text{‘Prod‘}(P\,\text{exp})\,;S \in \{P\,\text{exp}\,(\Sigma\text{‘}S)\}\;\overline{\text{smor}}\;\text{Prod‘}(P\,\text{exp})\,;S$$

Dem.

$\vdash . *165{\cdot}22 . *163{\cdot}3 . \quad \supset \vdash : \text{Hp} . \dot{\exists} ! P . \supset . (P \underset{;}{\downarrow})\dagger;S \in \text{Rel}^2\,\text{excl} \quad (1)$

$\vdash . *162{\cdot}35 . *38{\cdot}12 . *33{\cdot}431 . \supset \vdash . \Sigma\text{‘}(P \underset{;}{\downarrow})\dagger;S = P \underset{;}{\downarrow}\,;\Sigma\text{‘}S \quad (2)$

$\vdash . *165{\cdot}21 . (2) . \quad \supset \vdash . \Sigma\text{‘}(P \underset{;}{\downarrow})\dagger;S \in \text{Rel}^2\,\text{excl} \quad (3)$

$\vdash . (1) . (3) . *174{\cdot}3 . \quad \supset \vdash : \text{Hp} . \dot{\exists} ! P . \supset . (P \underset{;}{\downarrow})\dagger;S \in \text{Rel}^3\,\text{arithm} \quad (4)$

$\vdash . *165{\cdot}223 . \text{Transp} . \quad \supset \vdash :. \text{Hp} . \dot{\exists} ! P . \supset : Q \neq R . \supset . P \underset{;}{\downarrow}\,;Q \neq P \underset{;}{\downarrow}\,;R :$

$[*150{\cdot}4 . *72{\cdot}14] \quad \supset : (P \underset{;}{\downarrow})\dagger;S \subset J \quad (5)$

$\vdash . *176{\cdot}1 . \quad \supset \vdash . \text{Prod‘}(P\,\text{exp});S = \text{Prod‘Prod};(P \underset{;}{\downarrow})\dagger;S \quad (6)$

$\vdash . (4) . (5) . (6) . *174{\cdot}42 . \supset$

$\vdash : \text{Hp} . \dot{\exists} ! P . \supset .$
$$s \upharpoonright C\text{‘Prod‘}(P\,\text{exp});S \in \{\text{Prod‘}\Sigma\text{‘}(P \underset{;}{\downarrow})\dagger;S\}\;\overline{\text{smor}}\;\{\text{Prod‘}(P\,\text{exp});S\} .$$

$[(2)] \quad \supset . s \upharpoonright C\text{‘Prod‘}(P\,\text{exp});S \in \{\text{Prod‘}P \underset{;}{\downarrow}\,;\Sigma\text{‘}S\}\;\overline{\text{smor}}\;\{\text{Prod‘}(P\,\text{exp});S\} .$

$[*176{\cdot}1] \supset . s \upharpoonright C\text{‘}(P\,\text{exp});S \in \{P\,\text{exp}\,(\Sigma\text{‘}S)\}\;\overline{\text{smor}}\;\{\text{Prod‘}(P\,\text{exp});S\} \quad (7)$

$\vdash . *176{\cdot}151 . *173{\cdot}21 . *172{\cdot}13{\cdot}14 . \supset$

$\vdash : P = \dot{\Lambda} . \supset . \text{Prod‘}(P\,\text{exp});S = \dot{\Lambda} . P\,\text{exp}\,(\Sigma\text{‘}S) = \dot{\Lambda} \quad (8)$

$\vdash . (7) . (8) . *173{\cdot}2 . *164{\cdot}32 . \supset \vdash . \text{Prop}$

***176·44.** $\vdash : S \in \text{Rel}^2\,\text{excl} . S \subset J . \supset . \{\text{Prod‘}(P\,\text{exp});S\}\;\text{smor}\;\{P\,\text{exp}\,(\Sigma\text{‘}S)\}$
$[*176{\cdot}43]$

The following propositions are lemmas for
$$R \subset J . \supset . (P^Q)^R\;\text{smor}\;P^{R \times Q}.$$

***176·5.** $\vdash :. M \upharpoonright C\text{‘}R \in 1 \rightarrow 1 . C\text{‘}R \subset \text{G‘}M . C\text{‘}Q \subset p\text{‘G‘‘}M\text{‘‘}C\text{‘}R .$
$$M\text{‘‘}C\text{‘}R \subset 1 \rightarrow 1 : z, z' \in C\text{‘}R . \exists ! \text{D‘}M\text{‘}z \cap \text{D‘}M\text{‘}z' . \supset_{z,z'} . z = z' :$$
$$T = \hat{x}\hat{X}\,\{(\exists u, z) . u \in C\text{‘}Q . z \in C\text{‘}R . x = (M\text{‘}z)\text{‘}u . X = u \downarrow (M\text{‘}z)\} :$$
$$\supset . T \in 1 \rightarrow 1$$

Dem.

$\vdash . *21{\cdot}33 . \supset \vdash :. \text{Hp} . \supset : xTX . x'TX . \supset .$
$$(\exists u, u', z, z') . u, u' \in C\text{‘}Q . z, z' \in C\text{‘}R . x = (M\text{‘}z)\text{‘}u . x' = (M\text{‘}z')\text{‘}u' .$$
$$X = u \downarrow (M\text{‘}z) . X = u' \downarrow (M\text{‘}z') .$$

$[*55{\cdot}202] \quad \supset . (\exists u, u', z, z') . x = (M\text{‘}z)\text{‘}u . x' = (M\text{‘}z')\text{‘}u' . u = u' . M\text{‘}z = M\text{‘}z' .$

$[*13{\cdot}22] \quad \supset . x = x' \quad (1)$

$\vdash . *21{\cdot}33 . \supset \vdash :. \text{Hp} . \supset : xTX . xTX' . \supset .$
$$(\exists u, u', z, z') . u, u' \in C\text{‘}Q . z, z' \in C\text{‘}R . x = (M\text{‘}z)\text{‘}u . x = (M\text{‘}z')\text{‘}u' .$$
$$X = u \downarrow (M\text{‘}z) . X' = u' \downarrow (M\text{‘}z') .$$

$[*33{\cdot}43 . \text{Hp}] \supset . (\exists u, u', z, z') . u, u' \in C\text{‘}Q . z \in C\text{‘}R . z = z' . x = (M\text{‘}z)\text{‘}u = (M\text{‘}z')\text{‘}u' .$
$$X = u \downarrow (M\text{‘}z) . X' = u' \downarrow (M\text{‘}z') .$$

$[*13\cdot195] \quad \supset . (\exists u, u', z) . u, u' \in C\text{‘}Q . z \in C\text{‘}R . x = (M\text{‘}z)\text{‘}u = (M\text{‘}z)\text{‘}u' .$
$\qquad X = u \downarrow (M\text{‘}z) . X' = u' \downarrow (M\text{‘}z) .$

$[*71\cdot59 . \mathrm{Hp}] \supset . (\exists u, u', z) . u = u' . X = u \downarrow (M\text{‘}z) . X' = u' \downarrow (M\text{‘}z) .$

$[*13\cdot195] \quad \supset . X = X' \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

$*176\cdot501$. $\vdash : \mathrm{Hp}\, *176\cdot5 . \supset . \text{Ɑ}\text{‘}T = C\text{‘}\Sigma\text{‘}Q \underset{..}{\downarrow} ; M ; R$

Dem.

$\vdash . *71\cdot16 . \supset \vdash :. \mathrm{Hp} . \supset : z \in C\text{‘}R . u \in C\text{‘}Q . \supset . E ! (M\text{‘}z)\text{‘}u .$

$[*21\cdot33] \qquad \supset . u \downarrow (M\text{‘}z) \in \text{Ɑ}\text{‘}T \qquad (1)$

$\vdash . *21\cdot33 . \supset$

$\vdash :. \mathrm{Hp} . \supset : X \in \text{Ɑ}\text{‘}T . \supset . (\exists z, u) . z \in C\text{‘}R . u \in C\text{‘}Q . X = u \downarrow (M\text{‘}z) \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash :. \mathrm{Hp} . \supset : X \in \text{Ɑ}\text{‘}T . \equiv . (\exists z, u) . z \in C\text{‘}R . u \in C\text{‘}Q . X = u \downarrow (M\text{‘}z) .$

$[*71\cdot4] \qquad \equiv . (\exists Z, u) . Z \in M\text{“}C\text{‘}R . u \in C\text{‘}Q . X = u \downarrow Z .$

$[*150\cdot22] \qquad \equiv . (\exists Z, u) . Z \in C\text{‘}M ; R . u \in C\text{‘}Q . X = u \downarrow Z .$

$[*165\cdot16 . *113\cdot101] \qquad \equiv . X \in C\text{‘}\Sigma\text{‘}Q \underset{..}{\downarrow} ; M ; R :. \supset \vdash . \mathrm{Prop}$

$*176\cdot502$. $\vdash : \mathrm{Hp}\, *176\cdot5 . z \in C\text{‘}R . \supset . T ; Q \underset{..}{\downarrow}\text{‘}M\text{‘}z = \dagger Q\text{‘}M\text{‘}z$

Dem.

$\vdash . *150\cdot4 . *165\cdot01 . *176\cdot5 . \supset$

$\vdash : \mathrm{Hp} . \supset . T ; Q \underset{..}{\downarrow}\text{‘}M\text{‘}z = \hat{x}\hat{y}\{(\exists u, v) . uQv . x = T\text{‘} \downarrow (M\text{‘}z)\text{‘}u . y = T\text{‘} \downarrow (M\text{‘}z)\text{‘}v\}$

$[\mathrm{Hp} . *176\cdot5\cdot501] \qquad = \hat{x}\hat{y}\{(\exists u, v) . uQv . x = (M\text{‘}z)\text{‘}u . y = (M\text{‘}z)\text{‘}v\}$

$[*150\cdot4] \qquad = (M\text{‘}z) ; Q$

$[*150\cdot1] \qquad = \dagger Q\text{‘}M\text{‘}z : \supset \vdash . \mathrm{Prop}$

$*176\cdot503$. $\vdash : \mathrm{Hp}\, *176\cdot5 . \supset . T \in (\dagger Q ; M ; R)\, \overline{\mathrm{smor}}\, \overline{\mathrm{smor}}\, (Q \underset{..}{\downarrow} ; M ; R)$

Dem.

$\vdash . *176\cdot502 . *150\cdot1\cdot35 . \supset \vdash : \mathrm{Hp} . \supset . T \dagger ; Q \underset{..}{\downarrow} ; M ; R = \dagger Q ; M ; R \qquad (1)$

$\vdash . (1) . *176\cdot5\cdot501 . *164\cdot1 . \supset \vdash . \mathrm{Prop}$

$*176\cdot51$. $\vdash :. M \upharpoonright C\text{‘}R \in 1 \rightarrow 1 . M\text{“}C\text{‘}R \subset 1 \rightarrow 1 .$
$\qquad C\text{‘}R \subset \text{Ɑ}\text{‘}M . C\text{‘}Q \subset p\text{‘}\text{Ɑ}\text{“}M\text{“}C\text{‘}R :$
$z, z' \in C\text{‘}R . \exists ! D\text{‘}M\text{‘}z \cap D\text{‘}M\text{‘}z' . \supset_{z, z'} . z = z' : \supset . \dagger Q ; M ; R \;\mathrm{smor}\;\mathrm{smor}\; Q \underset{..}{\downarrow} ; R$

Dem.

$\vdash . *165\cdot361 . \supset \vdash : \mathrm{Hp} . \supset . Q \underset{..}{\downarrow} ; M ; R \;\mathrm{smor}\;\mathrm{smor}\; Q \underset{..}{\downarrow} ; R \qquad (1)$

$\vdash . (1) . *176\cdot503 . *164\cdot221 . \supset \vdash . \mathrm{Prop}$

$*176\cdot52$. $\vdash :. z \in C\text{‘}R . \supset_z . M\text{‘}z \in (P\text{‘}z)\, \overline{\mathrm{smor}}\, Q : \supset . P ; R = \dagger Q ; M ; R$

Dem.

$\vdash . *151\cdot11 . \supset \vdash :. \mathrm{Hp} . \supset : z \in C\text{‘}R . \supset_z . P\text{‘}z = (M\text{‘}z) ; Q$

$[*150\cdot1] \qquad = \dagger Q\text{‘}M\text{‘}z \qquad (1)$

$\vdash . (1) . *150\cdot35 . \supset \vdash . \mathrm{Prop}$

***176·53.** $\vdash :. M \upharpoonright C'R \in 1 \to 1 : z \in C'R . \supset_z . M'z \in (P'z) \, \overline{\text{smor}} \, Q :$
$z, z' \in C'R . \exists ! C'P'z \cap C'P'z' . \supset_{z,z'} . z = z' : \supset . P ; R \text{ smor smor } Q \underset{;}{\downarrow} ; R$

Dem.

$\vdash . *14·21 . \supset \vdash :. \text{Hp} . \supset : z \in C'R . \supset_z . E ! M'z :$ (1)

$[*33·43] \quad \supset : C'R \subset \text{Cl}'M$ (2)

$\vdash . *151·11 . \supset \vdash :. \text{Hp} . \supset : z \in C'R . \supset_z . M'z \in 1 \to 1 :$

$[*37·61.(1)] \quad \supset : M``C'R \subset 1 \to 1$ (3)

$\vdash . *151·11·131 . \supset \vdash :. \text{Hp} . \supset : z \in C'R . \supset_z . D'M'z = C'P'z :$

$[\text{Hp}] \quad \supset : z, z' \in C'R . \exists ! D'M'z \cap D'M'z' . \supset_{z,z'} . z = z'$ (4)

$\vdash . *151·11 . \quad \supset \vdash :. \text{Hp} . \supset : z \in C'R . \supset . \text{Cl}'M'z = C'Q :$

$[*37·63] \quad \supset : Z \in M``C'R . \supset . \text{Cl}'Z = C'Q :$

$[*40·15] \quad \supset : C'Q \subset p'\text{Cl}``M``C'R$ (5)

$\vdash . (2) . (3) . (4) . (5) . *176·51 . \supset \vdash : \text{Hp} . \supset . \dagger Q ; M ; R \text{ smor smor } Q \underset{;}{\downarrow} ; R$ (6)

$\vdash . (6) . *176·52 . \supset \vdash . \text{Prop}$

***176·54.** $\vdash :. \dot{\exists} ! P . \dot{\exists} ! Q . M = \hat{Z}\hat{z} [z \in C'R . Z = \{|(\text{Cnv}' \downarrow z)\}_\epsilon \upharpoonright C'(P \exp Q)] . \supset :$
$M \in 1 \to 1 : z \in C'R . \supset_z . M'z \in (\text{Prod}'P \underset{;}{\downarrow} ; Q \underset{;}{\downarrow} z) \, \overline{\text{smor}} \, (\text{Prod}'P \underset{;}{\downarrow} ; Q)$

Dem.

$\vdash . *116·606 . *176·14 . \supset \vdash : \text{Hp} . \supset . M \in 1 \to 1$ (1)

$\vdash . *21·33 . *30·3 . \supset \vdash : \text{Hp} . z \in C'R . \supset . M'z = \{|(\text{Cnv}' \downarrow z)\}_\epsilon \upharpoonright C'(P \exp Q)$ (2)

$\vdash . *151·65 . *165·361 . *166·1 . *165·01 . \supset$

$\vdash . \{|(\text{Cnv}' \downarrow z)\} \upharpoonright C'(Q \times P) \in (P \underset{;}{\downarrow} ; Q \underset{;}{\downarrow} z) \, \overline{\text{smor}} \, \overline{\text{smor}} \, (P \underset{;}{\downarrow} ; Q) .$

$[(2).*173·3] \supset \vdash : \text{Hp} . z \in C'R . \supset .$

$M'z \in (\text{Prod}'P \underset{;}{\downarrow} ; Q \underset{;}{\downarrow} z) \, \overline{\text{smor}} \, (\text{Prod}'P \underset{;}{\downarrow} ; Q)$ (3)

$\vdash . (1) . (3) . \supset \vdash . \text{Prop}$

***176·541.** $\vdash . (P \underset{;}{\downarrow}) \dagger ; Q \underset{;}{\downarrow} ; R \in \text{Rel}^3 \text{ arithm} . \Sigma'(P \underset{;}{\downarrow}) \dagger ; Q \underset{;}{\downarrow} ; R = P \underset{;}{\downarrow} ; \Sigma'Q \underset{;}{\downarrow} ; R$

Dem.

$\vdash . *163·3 . *165·21·22 . \supset \vdash : \dot{\exists} ! P . \supset . (P \underset{;}{\downarrow}) \dagger ; Q \underset{;}{\downarrow} ; R \in \text{Rel}^2 \text{ excl}$ (1)

$\vdash . *165·242 . \supset \vdash : P = \dot{\Lambda} . \dot{\exists} ! S . \dot{\exists} ! S' . \supset . P \underset{;}{\downarrow} ; S = \dot{\Lambda} \downarrow \dot{\Lambda} . P \underset{;}{\downarrow} ; S' = \dot{\Lambda} \downarrow \dot{\Lambda} :$

$[\text{Transp}] \quad \supset \vdash :. P = \dot{\Lambda} . P \underset{;}{\downarrow} ; S \neq P \underset{;}{\downarrow} ; S' . \supset : S = \Lambda . \lor . S' = \Lambda :$

$[*165·241] \quad \supset : P \underset{;}{\downarrow} ; S = \dot{\Lambda} . \lor . P \underset{;}{\downarrow} ; S' = \dot{\Lambda} :$

$[*33·241] \quad \supset : C'P \underset{;}{\downarrow} ; S \cap C'P \underset{;}{\downarrow} ; S' = \Lambda$ (2)

$\vdash . *150·22·1 . \supset \vdash :. P = \dot{\Lambda} . \supset :$

$T, T' \in C'(P \underset{;}{\downarrow}) \dagger ; Q \underset{;}{\downarrow} ; R . T \neq T' . \supset .$

$(\exists z, z') . z \neq z' . z, z' \in C'R . T = (P \underset{;}{\downarrow}) ; Q \underset{;}{\downarrow} z . T' = (P \underset{;}{\downarrow}) ; Q \underset{;}{\downarrow} z' .$

$[(2)] \quad \supset . C'T \cap C'T' = \Lambda$ (3)

$\vdash . (1) . (3) . *163·1 . \supset \vdash . (P \underset{;}{\downarrow}) \dagger ; Q \underset{;}{\downarrow} ; R \in \text{Rel}^2 \text{ excl}$ (4

$\vdash . *162\cdot35 . \supset \vdash . \Sigma`(P \underset{;}{\downarrow})\dagger;Q \underset{;}{\downarrow};R = P \underset{;}{\downarrow};\Sigma`Q \underset{;}{\downarrow};R .$ (5)

$[*165\cdot21] \quad \supset \vdash . \Sigma`(P \underset{;}{\downarrow})\dagger;Q \underset{;}{\downarrow};R \in \text{Rel}^2 \text{excl}$ (6)

$\vdash . (4) . (5) . (6) . *174\cdot3 . \supset \vdash . \text{Prop}$

$*176\cdot55$. $\vdash : \dot{\exists} ! P . \dot{\exists} ! Q . \supset . \text{Prod};(P \underset{;}{\downarrow})\dagger;Q \underset{;}{\downarrow};R \,\text{smor smor}\, (\text{Prod}`P \underset{;}{\downarrow};Q) \underset{;}{\downarrow};R$

Dem.

$\vdash . *176\cdot133\cdot15 . *37\cdot44\cdot21 . \supset$

$\vdash : \exists ! C`\text{Prod}`P \underset{;}{\downarrow};Q \underset{;}{\downarrow}z \cap C`\text{Prod}`P \underset{;}{\downarrow};Q \underset{;}{\downarrow}w . \supset .$

$\exists ! C`P^{Q \underset{;}{\downarrow} z} \cap C`P^{Q \underset{;}{\downarrow} w} . \dot{\exists} ! Q \underset{;}{\downarrow} z .$

$[*176\cdot16 . *80\cdot14 . *165\cdot212] \supset . (\exists R) . \text{Ɑ}`R = C`Q \underset{;}{\downarrow} z . \text{Ɑ}`R = C`Q \underset{;}{\downarrow} w . \dot{\exists} ! Q .$

$[*13\cdot171 . *150\cdot22] \quad \supset . \downarrow z``C`Q = \downarrow w``C`Q . \dot{\exists} ! Q .$

$[*55\cdot232] \quad \supset . z = w$ (1)

$\vdash . (1) . *176\cdot54 . *176\cdot53 \dfrac{\text{Prod}`P \underset{;}{\downarrow};Q \underset{;}{\downarrow} z,\ \text{Prod}`P \underset{;}{\downarrow};Q}{P`z,\quad Q} . \supset \vdash . \text{Prop}$

$*176\cdot56$. $\vdash : \dot{\exists} ! P . \dot{\exists} ! Q . R \in J . \supset .$

$\text{Prod}`\Sigma`(P \underset{;}{\downarrow})\dagger;Q \underset{;}{\downarrow};R \,\text{smor}\, \text{Prod}`(\text{Prod}`P \underset{;}{\downarrow};Q) \underset{;}{\downarrow};R$

Dem.

$\vdash . *165\cdot223 . \supset \vdash :. \dot{\exists} ! P . P \underset{;}{\downarrow};Q \underset{;}{\downarrow} z = P \underset{;}{\downarrow};Q \underset{;}{\downarrow} z' . \supset : Q \underset{;}{\downarrow} z = Q \underset{;}{\downarrow} z' :$

$[*165\cdot22] \quad \supset : \dot{\exists} ! Q . \supset . z = z'$ (1)

$\vdash . (1) . \text{Transp} . \supset \vdash :. \text{Hp} . zRz' . \supset . P \underset{;}{\downarrow};Q \underset{;}{\downarrow} z \neq P \underset{;}{\downarrow};Q \underset{;}{\downarrow} z'$ (2)

$\vdash . (2) . *150\cdot4 . \supset \vdash : \text{Hp} . \supset . (P \underset{;}{\downarrow})\dagger;Q \underset{;}{\downarrow};R \in J$ (3)

$\vdash . (3) . *176\cdot541 . *174\cdot43 . \supset$

$\vdash : \text{Hp} . \supset . \text{Prod}`\Sigma`(P \underset{;}{\downarrow})\dagger;Q \underset{;}{\downarrow};R \,\text{smor}\, \text{Prod}`\text{Prod};(P \underset{;}{\downarrow})\dagger;Q \underset{;}{\downarrow};R$ (4)

$\vdash . *176\cdot55 . *173\cdot31 . \supset$

$\vdash : \text{Hp} . \supset . \text{Prod}`\text{Prod};(P \underset{;}{\downarrow})\dagger;Q \underset{;}{\downarrow};R \,\text{smor}\, \text{Prod}`(\text{Prod}`P \underset{;}{\downarrow};Q) \underset{;}{\downarrow};R$ (5)

$\vdash . (4) . (5) . \supset \vdash . \text{Prop}$

$*176\cdot57$. $\vdash : R \in J . \supset . \{(P \exp Q) \exp R\} \,\text{smor}\, \{P \exp (R \times Q)\} . (P^Q)^R \,\text{smor}\, P^{R \times Q}$

Dem.

$\vdash . *176\cdot151 . \quad \supset \vdash :. P = \dot{\Lambda} . \lor . Q = \dot{\Lambda} : \supset . (P \exp Q) \exp R = \dot{\Lambda}$ (1)

$\vdash . *176\cdot151 . *166\cdot13 . \supset \vdash :. P = \dot{\Lambda} . \lor . Q = \dot{\Lambda} : \supset . P \exp (R \times Q) = \dot{\Lambda}$ (2)

$\vdash . (1) . (2) . *153\cdot101 . \supset \vdash :. P = \dot{\Lambda} . \lor . Q = \dot{\Lambda} : \supset .$

$\{(P \exp Q) \exp R\} \,\text{smor}\, \{P \exp (R \times Q)\}$ (3)

$\vdash . *176\cdot56\cdot541\cdot1 . *166\cdot1 . \supset \vdash : \dot{\exists} ! P . \dot{\exists} ! Q . R \in J . \supset .$

$\{(P \exp Q) \exp R\} \,\text{smor}\, \{P \exp (R \times Q)\}$ (4)

$\vdash . (3) . (4) . \supset \vdash : \text{Hp} . \supset . \{(P \exp Q) \exp R\} \,\text{smor}\, \{P \exp (R \times Q)\}$ (5)

$[*176\cdot181\cdot22] \quad \supset . (P^Q)^R \,\text{smor}\, P^{R \times Q}$ (6)

$\vdash . (5) . (6) . \supset \vdash . \text{Prop}$

This completes the proof of the second formal law of exponentiation.

$*177$. PROPOSITIONS CONNECTING P_{df} WITH PRODUCTS AND POWERS

Summary of $*177$.

The principal proposition on this subject is

$*177\cdot13$. $\vdash : x \neq y . \supset . P_{df} \operatorname{smor} \{(x \downarrow y)^P\}$

which is the analogue of $*116\cdot72$, or rather leads to the analogue of $*116\cdot72$ as soon as powers of relation-numbers have been defined; for then it becomes

$$P_{df} \in 2_r^{\mathrm{Nr}`P}.$$

Another proposition is an extension of $*171\cdot69$, namely

$*177\cdot22$. $\vdash : P \in \mathrm{Rel}^2 \operatorname{excl} . P \subset J . \supset . \mathrm{Prod}`\mathrm{df};P \operatorname{smor} (\Sigma`P)_{df}$

where we put $\mathrm{df}`Q = Q_{df}$.

The remaining propositions of this number are lemmas for the above two.

$*177\cdot13$ shows, for example, that all classes of finite integers can be arranged in a series of which the relation-number is 2_r^{ω}, where ω is the relation-number of the series of finite integers. 2_r^{ω} is not the relation-number of the continuum, but is closely allied to it.

$*177\cdot1$. $\vdash : x \neq y . T = \hat{\mu}\hat{R}[R \in \{(\iota`x \cup \iota`y) \uparrow \alpha\}_\Delta`\alpha . \mu = \overleftarrow{R}`x] . \supset .$

$$T \in (\mathrm{Cl}`\alpha) \overline{\mathrm{sm}} \{(\iota`x \cup \iota`y) \uparrow \alpha\}_\Delta`\alpha \quad [*116\cdot712\cdot713\cdot715]$$

In the propositions of $*116$ referred to, Λ and V appear in place of x and y, but no property of Λ and V is used in the proof except $\Lambda \neq \mathrm{V}$.

$*177\cdot11$. $\vdash : \mathrm{Hp}*177\cdot1 . \alpha = C`P . \supset . T;(x \downarrow y)^P = P_{df}$

Dem.

$\vdash . *176\cdot19 . \supset$

$\vdash ::. \mathrm{Hp} . \supset :: \mu \{T;(x \downarrow y)^P\} \nu . \equiv :. (\exists R, S) : R, S \in \{(\iota`x \cup \iota`y) \uparrow C`P\}_\Delta`C`P :$
$(\exists z) : z \in C`P . R`z (x \downarrow y) S`z :$
$wPz . w \neq z . \supset_w . R`w = S`w : \mu = \overleftarrow{R}`x . \nu = \overleftarrow{S}`x :.$

$[*55\cdot13] \equiv :. (\exists R, S) : R, S \in \{(\iota`x \cup \iota`y) \uparrow C`P\}_\Delta`C`P :.$
$(\exists z) :. z \in C`P . R`z = x . S`z = y :. \mu = \overleftarrow{R}`x . \nu = \overleftarrow{S}`x :.$
$wPz . w \neq z . \supset_w : xRw . \equiv . xSw : yRw . \equiv . ySw :.$

$[*71\cdot36] \equiv :. (\exists R, S) : R, S \in \{(\iota`x \cup \iota`y) \uparrow C`P\}_\Delta`C`P :.$
$(\exists z) :. z \in C`P . z \in \mu - \nu . \mu = \overleftarrow{R}`x . \nu = \overleftarrow{S}`x :.$
$wPz . w \neq z . \supset_w : w \in \mu . \equiv . w \in \nu :.$

$[*177\cdot1] \equiv :. \mu, \nu \in \mathrm{Cl}`C`P :. (\exists z) :. z \in C`P . z \in \mu - \nu :.$
$wPz . w \neq z . \supset_w : w \in \mu . \equiv . w \in \nu :.$

$[*171\cdot11] \equiv :. \mu (P_{df}) \nu ::. \supset \vdash . \mathrm{Prop}$

***177·12.** $\vdash : \mathrm{Hp}*177\cdot11 . \supset . T \epsilon P_{\mathrm{df}} \overline{\mathrm{smor}} \{(x \downarrow y)^P\}$ [*177·1·11 . *151·191]

***177·13.** $\vdash : x \neq y . \supset . P_{\mathrm{df}} \mathrm{smor} \{(x \downarrow y)^P\}$ [*177·12]

***177·2.** $\mathrm{df}‘Q = Q_{\mathrm{df}}$ Dft [*177]

***177·21.** $\vdash : P \epsilon \mathrm{Rel}^2 \mathrm{excl} . P \subset J . \supset . s \restriction C‘\mathrm{Prod}‘\mathrm{df};P \epsilon (\Sigma‘P)_{\mathrm{df}} \overline{\mathrm{smor}} (\mathrm{Prod}‘\mathrm{df};P)$

The proof proceeds as the proof of *174·24 proceeds. If $Q \epsilon C‘P$, we shall have, if $M \epsilon F_\Delta‘\mathrm{df}“C‘P$,

$$M‘Q_{\mathrm{df}} = (s‘\mathrm{D}‘M) \cap C‘Q.$$

Hence we easily obtain

$$M(\Pi‘\mathrm{df};P)N . \equiv . M, N \epsilon F_\Delta‘\mathrm{df}“C‘P . (s‘\mathrm{D}‘M)(\Sigma‘P)_{\mathrm{df}}(s‘\mathrm{D}‘N),$$

whence

$$\mu(\mathrm{Prod}‘\mathrm{df};P)\nu . \equiv . \mu, \nu \epsilon \mathrm{Prod}‘\mathrm{Cl}“C“C‘P . (s‘\mu)(\Sigma‘P)_{\mathrm{df}}(s‘\nu),$$

whence the result follows easily.

***177·22.** $\vdash : P \epsilon \mathrm{Rel}^2 \mathrm{excl} . P \subset J . \supset . \mathrm{Prod}‘\mathrm{df};P \,\mathrm{smor}\, (\Sigma‘P)_{\mathrm{df}}$ [*177·21]

SECTION D

ARITHMETIC OF RELATION-NUMBERS

Summary of Section D.

In the present section, we shall be concerned with the arithmetical operations on relation-numbers. Their purely logical properties have been dealt with in Section A; in the present section, it is their arithmetical properties that are to be established. These properties result immediately from the arithmetical properties of relations which have been established in Sections B and C. The subjects treated of in the present section are analogous to those treated of in Section B of Part III, with the exception of such as have already had their analogues discussed in Sections B and C of Part IV. The analogy is sufficiently close to render it often unnecessary to give proofs, since these are often step by step analogous to the proofs of corresponding propositions in Part III, Section B.

The two chief requisites in defining the arithmetical operations with relation-numbers are (1) to take due account of types, (2) to construct what may be called *separated* relations, *i.e.* relations of mutually exclusive relations derived from and ordinally similar to given relations. Each of these points calls for some preliminary explanations.

The sum of two relation-numbers μ, ν will be denoted by "$\mu \dotplus \nu$," in order to distinguish this kind of addition from $\mu + \nu$ (the arithmetical addition of classes) and $\mu +_c \nu$ (the addition of cardinals). In defining $\mu \dotplus \nu$, we have to take account of the following considerations.

Suppose P and Q are two relations which are of the same type, and have mutually exclusive fields. Then obviously we shall want to frame our definition of the sum of two relation-numbers in such a way that the sum of $\mathrm{Nr}\text{‘}P$ and $\mathrm{Nr}\text{‘}Q$ shall be $\mathrm{Nr}\text{‘}(P \uparrow\!\!\!\!+ Q)$. But if P and Q are not of the same type, $P \uparrow\!\!\!\!+ Q$ is meaningless; and if $C\text{‘}P$ and $C\text{‘}Q$ overlap, $P \uparrow\!\!\!\!+ Q$ may be too small to have as its relation-number the sum of the relation-numbers of P and Q. Both these difficulties can be met by observing that, if $\mathrm{Nr}\text{‘}P = \mathrm{Nr}\text{‘}R$ and $\mathrm{Nr}\text{‘}Q = \mathrm{Nr}\text{‘}S$, we must make such definitions as to have

$$\mathrm{Nr}\text{‘}P \dotplus \mathrm{Nr}\text{‘}Q = \mathrm{Nr}\text{‘}R \dotplus \mathrm{Nr}\text{‘}S.$$

Hence, in defining the sum of the relation-numbers of P and Q, we may replace P and Q by any two relations R and S which are respectively like P and Q. Therefore what we require for our definition is to find two relations R and S which (1) are respectively like P and Q, (2) are of the

same type, (3) have mutually exclusive fields. All these three requisites are satisfied if we put

$$R = \downarrow (\Lambda \cap C\text{‘}Q) \,;\, \iota \,;\, P \,.\, S = (\Lambda \cap C\text{‘}P) \downarrow \,;\, \iota \,;\, Q.$$

We then define $P + Q$ as meaning $R \,\ddagger\, S$, and we define the sum of the relation-numbers of P and Q as the relation-number of $P + Q$. This procedure is exactly analogous to that of *110; in fact, we have

$$C\text{‘}(P + Q) = C\text{‘}P + C\text{‘}Q.$$

In defining the sum of the relation-numbers of a field, we do not have to consider types, because the members of a field are necessarily all of the same type. But we do have to consider the question of overlapping. If a term x occurs both in $C\text{‘}Q$ and in $C\text{‘}R$, where $Q, R \in C\text{‘}P$, we want a method of counting x twice over in forming the arithmetical sum. Thus $\mathrm{Nr}\text{‘}\Sigma\text{‘}P$ cannot be taken as the sum of the relation-numbers of members of $C\text{‘}P$, unless $P \in \mathrm{Rel}^2\,\mathrm{excl}$. Suppose, for instance, we have three series

$$(a, b, c), \quad (b, c, a), \quad (c, a, b).$$

These each have three terms; and we want the sum of their relation-numbers to be the relation-number of a series of nine terms. But if we put

$Q = a \downarrow b \downarrow c$ (where $a \downarrow b \downarrow c$ is written for $a \downarrow b \cup a \downarrow c \cup b \downarrow c$),
$R = b \downarrow c \downarrow a$,
$S = c \downarrow a \downarrow b$,

and if we further put

$P = Q \downarrow R \downarrow S$,

so that P places the above three series in the above order, we have

$$\Sigma\text{‘}P = (\iota\text{‘}a \cup \iota\text{‘}b \cup \iota\text{‘}c) \uparrow (\iota\text{‘}a \cup \iota\text{‘}b \cup \iota\text{‘}c),$$

which is not a series, and does not have the relation-number which we require as the sum of the relation-numbers of Q, R, S.

What is wanted is a method of distinguishing the various occurrences of a and b and c. For this reason, when a occurs as a member of the field of Q, we replace it by $a \downarrow Q$; when as a member of the field of R, by $a \downarrow R$; and when as a member of the field of S, by $a \downarrow S$. Thus the series (a, b, c) is replaced by $(a \downarrow Q, b \downarrow Q, c \downarrow Q)$; (b, c, a) is replaced by $(b \downarrow R, c \downarrow R, a \downarrow R)$; and (c, a, b) is replaced by $(c \downarrow S, a \downarrow S, b \downarrow S)$. The sum of these three series then has the relation-number which is required as the sum of the relation-numbers of Q, R, S.

The above process is symbolized as follows. The generating relation of the series $(a \downarrow Q, b \downarrow Q, c \downarrow Q)$ is $\downarrow Q \,;\, Q$; thus the three relations whose sum is to be taken are $\downarrow Q \,;\, Q$, $\downarrow R \,;\, R$, $\downarrow S \,;\, S$, *i.e.* using the notation of *182, according to which we put $\overset{\frown}{\text{♀}}\text{‘}x = x \,\text{♀}\, x$, our three relations are $\overset{\frown}{\underset{;}{\downarrow}}\text{‘}Q$, $\overset{\frown}{\underset{;}{\downarrow}}\text{‘}R$, $\overset{\frown}{\underset{;}{\downarrow}}\text{‘}S$.

But the generating relation of the series $(\overset{\frown}{\underset{\text{;}}{\downarrow}}\text{‘}Q,\ \overset{\frown}{\underset{\text{;}}{\downarrow}}\text{‘}R,\ \overset{\frown}{\underset{\text{;}}{\downarrow}}\text{‘}S)$ is $\overset{\frown}{\underset{\text{;}}{\downarrow}}\,;P$, since $P = (Q \downarrow R \downarrow S)$. Thus $\overset{\frown}{\underset{\text{;}}{\downarrow}}\,;P$ is the relation required for defining the sum of the relation-numbers of members of the field of P; *i.e.* we put

$$\Sigma \mathrm{Nr}\text{‘}P = \mathrm{Nr}\text{‘}\Sigma\text{‘}\overset{\frown}{\underset{\text{;}}{\downarrow}}\,;P \quad \text{Df.}$$

We will call $\overset{\frown}{\underset{\text{;}}{\downarrow}}\,;P$ the *separated* relation corresponding to P. $\overset{\frown}{\underset{\text{;}}{\downarrow}}\,;P$ is constructed, as above, by replacing every member x of $C\text{‘}Q$, where $Q \in C\text{‘}R$, by $x \downarrow Q$; so that if x belongs both to $C\text{‘}Q$ and to $C\text{‘}R$, it is duplicated by being transformed once into $x \downarrow Q$, and once again into $x \downarrow R$.

For the treatment of products, we do not require $\overset{\frown}{\underset{\text{;}}{\downarrow}}\,;P$, because $\Pi\text{‘}P$ has been so defined as to effect the requisite separation. We might, however, by the use of $\overset{\frown}{\underset{\text{;}}{\downarrow}}\,;P$, have dispensed with $\Pi\text{‘}P$ as a fundamental notion, and contented ourselves with $\mathrm{Prod}\text{‘}P$; for we have

$$\Pi\text{‘}P = \dot{s}\,;\mathrm{Prod}\text{‘}\overset{\frown}{\underset{\text{;}}{\downarrow}}\,;P.$$

Thus we might have taken Prod as the fundamental notion, and defined Π by means of it.

The addition of unity to a relation-number has to be treated separately from the addition of two relation-numbers, for the same reasons which necessitate the treatment of $P \mathbin{+\!\!\rightarrow} x$ and $x \mathbin{\leftarrow\!\!+} P$ separately from $P \mathbin{\overset{\uparrow}{+}} Q$. There is no ordinal number 1, but we can define the *addition* of one to a relation-number. If $\mathrm{Nr}\text{‘}P = \mu$ and $x \sim\in C\text{‘}P$, we must have

$$\mathrm{Nr}\text{‘}(P \mathbin{+\!\!\rightarrow} x) = \mu \dot{+} \dot{1},$$

where we write "$\dot{1}$" for unity as an addendum. We do not write "1_r," because we shall, at a later stage, give a general definition of μ_r, in virtue of which, if μ is an inductive cardinal, μ_r will be the corresponding ordinal. This definition entails $1_r = \Lambda$, and therefore we use a different symbol "$\dot{1}$" for 1 as addendum. The symbol $\dot{1}$ is only defined in its uses, and has no significance except in a use which has been specially defined.

We define the product $\mu \dot{\times} \nu$ as the relation-number of $P \times Q$, when $\mu = \mathrm{N_0r}\text{‘}P$ and $\nu = \mathrm{N_0r}\text{‘}Q$. The product so defined obeys the associative law, and obeys the distributive law in the form

$$(\nu \dot{+} \varpi) \dot{\times} \mu = (\nu \dot{\times} \mu) \dot{+} (\varpi \dot{\times} \mu)$$

but not, in general, in the form

$$\mu \dot{\times} (\nu \dot{+} \varpi) = (\mu \dot{\times} \nu) \dot{+} (\mu \dot{\times} \varpi).$$

The latter form holds when μ, ν, ϖ are finite ordinals, as we shall prove at a later stage (*262). The commutative law also does not hold in general for ordinal addition and multiplication, but holds where finite ordinals are concerned.

The product of the numbers of the members of $C\text{‘}P$, in the order generated by P, is defined as being $\mathrm{Nr}\text{‘}\Pi\text{‘}P$, and is denoted by $\Pi\mathrm{Nr}\text{‘}P$. It will be seen that $\Pi\mathrm{Nr}\text{‘}P$ is not a function of $C\text{‘}P$, since the value of a product depends upon the order of the factors; it is also not a function of $\mathrm{Nr}``P$, unless no two members of $C\text{‘}P$ have the same relation-number. The properties of $\Pi\mathrm{Nr}\text{‘}P$ result from $*172$ and $*174$.

"μ to the νth power" is denoted by "$\mu \exp_r \nu$" and is defined as the relation-number of $P \exp Q$, where $\mu = \mathrm{N_0r}\text{‘}P$ and $\nu = \mathrm{N_0r}\text{‘}Q$. Its properties result from the propositions of $*176$ and $*177$.

$*180$. THE SUM OF TWO RELATION-NUMBERS

Summary of $*180$.

In order to define the sum of two relation-numbers, we proceed (as in $*110$) to construct a relation whose relation-number shall be the required sum. For this purpose, we put

$$P + Q = \{\downarrow (\Lambda \cap C\text{`}Q) \,\mathring{;}\, \iota \,\mathring{;}\, P\} \,\dot{\mp}\, \{(\Lambda \cap C\text{`}P) \downarrow \mathring{;}\, \iota \,\mathring{;}\, Q\} \quad \text{Df.}$$

This definition has the following merits: (1) whatever may be the types of P and Q, $\downarrow (\Lambda \cap C\text{`}Q) \,\mathring{;}\, \iota \,\mathring{;}\, P$ is of the same type as $(\Lambda \cap C\text{`}P) \downarrow \mathring{;}\, \iota \,\mathring{;}\, Q$; (2) however the fields of P and Q may overlap, and even if $P = Q$, the fields of $\downarrow (\Lambda \cap C\text{`}Q) \,\mathring{;}\, \iota \,\mathring{;}\, P$ and $(\Lambda \cap C\text{`}P) \downarrow \mathring{;}\, \iota \,\mathring{;}\, Q$ are mutually exclusive; (3) these two relations are respectively similar to P and Q. Hence it is evident that, without placing any restriction upon P and Q, we may take the relation-number of $P + Q$ as defining the sum of the relation-numbers of P and Q. Hence we put

$$\mu \dot{+} \nu = \hat{R} \{(\exists P, Q) \,.\, \mu = \mathrm{N_0r}\text{`}P \,.\, \nu = \mathrm{N_0r}\text{`}Q \,.\, R \,\mathrm{smor}\, (P + Q)\} \quad \text{Df.}$$

From this definition it follows that $\mu \dot{+} \nu$ is null unless μ and ν are homogeneous relation-numbers, but that if they are the homogeneous relation-numbers of P and Q, then $\mu \dot{+} \nu$ is the relation-number of $P + Q$.

In order to be able to deal with typically ambiguous relation-numbers, we put, as in $*110$,

$$\mathrm{Nr}\text{`}P \dot{+} \nu = \mathrm{N_0r}\text{`}P \dot{+} \nu \quad \text{Df,}$$
$$\mu \dot{+} \mathrm{Nr}\text{`}Q = \mu \dot{+} \mathrm{N_0r}\text{`}Q \quad \text{Df.}$$

The principal propositions of the present number are

$*180 \cdot 111$. $\vdash \,.\, C\text{`}(P + Q) = C\text{`}P + C\text{`}Q$

$*180 \cdot 3$. $\vdash \,.\, \mathrm{Nr}\text{`}P \dot{+} \mathrm{Nr}\text{`}Q = \mathrm{N_0r}\text{`}P \dot{+} \mathrm{Nr}\text{`}Q = \mathrm{Nr}\text{`}P \dot{+} \mathrm{N_0r}\text{`}Q$
$\qquad = \mathrm{N_0r}\text{`}P \dot{+} \mathrm{N_0r}\text{`}Q = \mathrm{Nr}\text{`}(P + Q)$

$*180 \cdot 31$. $\vdash : P \,\mathrm{smor}\, R \,.\, Q \,\mathrm{smor}\, S \,.\, \supset .\, \mathrm{Nr}\text{`}P \dot{+} \mathrm{Nr}\text{`}Q = \mathrm{Nr}\text{`}R \dot{+} \mathrm{Nr}\text{`}S$

This proposition is essential, since otherwise $\mathrm{Nr}\text{`}P \dot{+} \mathrm{Nr}\text{`}Q$ would not be a function of $\mathrm{Nr}\text{`}P$ and $\mathrm{Nr}\text{`}Q$, but would depend upon the particular P and Q.

$*180 \cdot 32$. $\vdash : C\text{`}P \cap C\text{`}Q = \Lambda \,.\, \supset .\, \mathrm{Nr}\text{`}P \dot{+} \mathrm{Nr}\text{`}Q = \mathrm{Nr}\text{`}(P \,\dot{\mp}\, Q)$

$*180 \cdot 4$. $\vdash : \exists \,!\, \mu \dot{+} \nu \,.\, \supset .\, \mu, \nu \in \mathrm{NR} - \iota\text{`}\Lambda \,.\, \mu, \nu \in \mathrm{N_0R}$

$*180 \cdot 42$. $\vdash \,.\, \mu \dot{+} \nu \in \mathrm{NR}$

$*180 \cdot 56$. $\vdash \,.\, (\mu \dot{+} \nu) \dot{+} \varpi = \mu \dot{+} (\nu \dot{+} \varpi)$

which is the associative law.

$*180 \cdot 61$. $\vdash \,.\, \mathrm{Nr}\text{`}P \dot{+} 0_r = \mathrm{Nr}\text{`}P = 0_r \dot{+} \mathrm{Nr}\text{`}P$

$*180{\cdot}71.\ \vdash : \mu, \nu \in \mathrm{NR} \,.\supset .\, C\text{“}(\mu \dotplus \nu) = C\text{“}\mu +_c C\text{“}\nu$

This proposition gives the connection of ordinal and cardinal addition. It should be observed that, in virtue of $*154{\cdot}9$, $C\text{“}\mu$ and $C\text{“}\nu$ are cardinals when μ and ν are relation-numbers.

$*180{\cdot}01.\ P + Q = \{\downarrow (\Lambda \cap C\text{‘}Q) ;\iota ; P\} \mathbin{+\!\!\!\!\uparrow} \{(\Lambda \cap C\text{‘}P) \downarrow ;\iota ; Q\}$ Df

$*180{\cdot}02.\ \mu \dotplus \nu = \hat{R}\{(\exists P, Q) \,.\, \mu = \mathrm{N_0r}\text{‘}P \,.\, \nu = \mathrm{N_0r}\text{‘}Q \,.\, R \,\mathrm{smor}\, (P + Q)\}$ Df

$*180{\cdot}03.\ \mathrm{Nr}\text{‘}P \dotplus \nu = \mathrm{N_0r}\text{‘}P \dotplus \nu$ Df

$*180{\cdot}031.\ \mu \dotplus \mathrm{Nr}\text{‘}Q = \mu \dotplus \mathrm{N_0r}\text{‘}Q$ Df

On the purpose of the definitions $*180{\cdot}03{\cdot}031$, see the remarks on the corresponding definitions in $*110$ and II T of the Prefatory Statement.

$*180{\cdot}1.\ \vdash . P + Q = \{\downarrow (\Lambda \cap C\text{‘}Q) ;\iota ; P\} \mathbin{+\!\!\!\!\uparrow} \{(\Lambda \cap C\text{‘}P) \downarrow ;\iota ; Q\}$ $[(*180{\cdot}01)]$

$*180{\cdot}101.\ \vdash . C\text{‘} \downarrow (\Lambda \cap C\text{‘}Q) ;\iota ; P = \downarrow (\Lambda \cap C\text{‘}Q)\text{“}\iota\text{“}C\text{‘}P \,.$
$\quad C\text{‘}(\Lambda \cap C\text{‘}P) \downarrow ;\iota ; Q = (\Lambda \cap C\text{‘}P) \downarrow \text{“}\iota\text{“}C\text{‘}Q$ $[*150{\cdot}22]$

$*180{\cdot}11.\ \vdash . C\text{‘} \downarrow (\Lambda \cap C\text{‘}Q) ;\iota ; P \cap C\text{‘}(\Lambda \cap C\text{‘}P) \downarrow ;\iota ; Q = \Lambda$ $[*180{\cdot}101 \,.\, *110{\cdot}11]$

$*180{\cdot}111.\ \vdash . C\text{‘}(P + Q) = C\text{‘}P + C\text{‘}Q$

Dem.

$\vdash . *180{\cdot}101 \,.\, *160{\cdot}14 \,.\supset$
$\vdash . C\text{‘}(P + Q) = \downarrow (\Lambda \cap C\text{‘}Q)\text{“}\iota\text{“}C\text{‘}P \cup (\Lambda \cap C\text{‘}P) \downarrow \text{“}\iota\text{“}C\text{‘}Q$
$[(*110{\cdot}01)] \quad = C\text{‘}P + C\text{‘}Q \,.\supset \vdash . \mathrm{Prop}$

$*180{\cdot}12.\ \vdash . \downarrow (\Lambda \cap C\text{‘}Q) ;\iota ; P \,\mathrm{smor}\, P \,.\, (\Lambda \cap C\text{‘}P) \downarrow ;\iota ; Q \,\mathrm{smor}\, Q$ $[*151{\cdot}61{\cdot}64{\cdot}65]$

$*180{\cdot}13.\ \vdash : R \,\mathrm{smor}\, P \,.\, S \,\mathrm{smor}\, Q \,.\, C\text{‘}R \cap C\text{‘}S = \Lambda \,.\supset .\, R \mathbin{+\!\!\!\!\uparrow} S \,\mathrm{smor}\, P + Q$

Dem.

$\vdash . *180{\cdot}12 \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, R \,\mathrm{smor} \downarrow (\Lambda \cap C\text{‘}Q) ;\iota ; P \,.\, S \,\mathrm{smor} (\Lambda \cap C\text{‘}P) \downarrow ;\iota ; Q$ (1)
$\vdash . (1) \,.\, *180{\cdot}11 \,.\, *160{\cdot}48 \,.\supset$
$\quad \vdash : \mathrm{Hp} \,.\supset .\, R \mathbin{+\!\!\!\!\uparrow} S \,\mathrm{smor}\, \{\downarrow (\Lambda \cap C\text{‘}Q) ;\iota ; P \mathbin{+\!\!\!\!\uparrow} (\Lambda \cap C\text{‘}P) \downarrow ;\iota ; Q\} \,.$
$[(*180{\cdot}01)] \quad \supset .\, R \mathbin{+\!\!\!\!\uparrow} S \,\mathrm{smor}\, (P + Q) : \supset \vdash . \mathrm{Prop}$

$*180{\cdot}14.\ \vdash : C\text{‘}P \cap C\text{‘}Q = \Lambda \,.\supset .\, P \mathbin{+\!\!\!\!\uparrow} Q \,\mathrm{smor}\, P + Q$ $[*180{\cdot}13 \,.\, *151{\cdot}13]$

$*180{\cdot}15.\ \vdash : R \,\mathrm{smor}\, P \,.\, S \,\mathrm{smor}\, Q \,.\supset .\, R + S \,\mathrm{smor}\, P + Q$

Dem.

$\vdash . *180{\cdot}12 \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, \downarrow (\Lambda \cap C\text{‘}S) ;\iota ; R \,\mathrm{smor}\, P \,.\, (\Lambda \cap C\text{‘}R) \downarrow ;\iota ; S \,\mathrm{smor}\, Q \,.$
$[*180{\cdot}13] \quad \supset .\, \{\downarrow (\Lambda \cap C\text{‘}S) ;\iota ; R \mathbin{+\!\!\!\!\uparrow} (\Lambda \cap C\text{‘}R) \downarrow ;\iota ; S\} \,\mathrm{smor}\, P + Q \,.$
$[(*180{\cdot}01)] \quad \supset .\, R + S \,\mathrm{smor}\, P + Q : \supset \vdash . \mathrm{Prop}$

$*180{\cdot}151.\ \vdash :. C\text{‘}P \cap C\text{‘}Q = \Lambda \,.\supset : Z \,\mathrm{smor}\, (P \mathbin{+\!\!\!\!\uparrow} Q) \,.\equiv .$
$\quad (\exists R, S) \,.\, R \,\mathrm{smor}\, P \,.\, S \,\mathrm{smor}\, Q \,.\, C\text{‘}R \cap C\text{‘}S = \Lambda \,.\, Z = R \mathbin{+\!\!\!\!\uparrow} S$

Dem.

$\vdash . *160{\cdot}48 \,.\supset \vdash :. \mathrm{Hp} \,.\supset : (\exists R, S) \,.\, R \,\mathrm{smor}\, P \,.\, S \,\mathrm{smor}\, Q \,.\, C\text{‘}R \cap C\text{‘}S = \Lambda \,.$
$\quad Z = R \mathbin{+\!\!\!\!\uparrow} S \,.\supset .\, Z \,\mathrm{smor}\, (P \mathbin{+\!\!\!\!\uparrow} Q)$ (1)

$\vdash . *160\cdot44 . \quad \supset \vdash : T \epsilon Z \overline{\text{smor}} (P \ddagger Q) . \supset . Z = T;P \ddagger T;Q$ (2)

$\vdash . *160\cdot14 . *151\cdot11 . \supset \vdash : T \epsilon Z \overline{\text{smor}} (P \ddagger Q) . \supset . C'P \subset \text{Cl}'T . C'Q \subset \text{Cl}'T$ (3)

$\vdash . (3) . *151\cdot21 . \quad \supset \vdash : T \epsilon Z \overline{\text{smor}} (P \ddagger Q) . \supset . T;P \text{ smor } P . T;Q \text{ smor } Q$ (4)

$\vdash . *72\cdot411 . *150\cdot22 . \supset \vdash :. \text{Hp} . \supset :$

$T \epsilon Z \overline{\text{smor}} (P \ddagger Q) . \supset . C'T;P \cap C'T;Q = \Lambda$ (5)

$\vdash . (2) . (4) . (5) . \supset \vdash :. \text{Hp} . \supset : T \epsilon Z \overline{\text{smor}} (P \ddagger Q) . \supset .$

$T;P \text{ smor } P . T;Q \text{ smor } Q . C'T;P \cap C'T;Q = \Lambda . Z = T;P \ddagger T;Q :$

[$*151\cdot12$] $\supset : Z \text{ smor } (P \ddagger Q) . \supset .$

$(\exists R, S) . R \text{ smor } P . S \text{ smor } Q . C'R \cap C'S = \Lambda . Z = R \ddagger S$ (6)

$\vdash . (1) . (6) . \supset \vdash . \text{Prop}$

$*180\cdot152$. $\vdash : Z \text{ smor } (P + Q) . \equiv .$

$(\exists R, S) . R \text{ smor } P . S \text{ smor } Q . C'R \cap C'S = \Lambda . Z = R \ddagger S$

[$*180\cdot151\cdot11\cdot12$]

$*180\cdot16$. $\vdash . \text{Nr}'(P + Q) =$

$\hat{Z} \{(\exists R, S) . R \epsilon \text{Nr}'P . S \epsilon \text{Nr}'Q . C'R \cap C'S = \Lambda . Z = R \ddagger S\}$

[$*180\cdot152 . *152\cdot11$]

$*180\cdot2$. $\vdash : Z \epsilon \mu \dot{+} \nu . \equiv . (\exists P, Q) . \mu = \text{N}_0\text{r}'P . \nu = \text{N}_0\text{r}'Q . Z \text{ smor } (P + Q)$

[($*180\cdot02$)]

$*180\cdot201$. $\vdash :. Z \epsilon \mu \dot{+} \nu . \equiv : \mu, \nu \epsilon \text{N}_0\text{R} : (\exists P, Q) . P \epsilon \mu . Q \epsilon \nu . Z \text{ smor } (P + Q)$

[$*155\cdot27 . *180\cdot2$]

$*180\cdot202$. $\vdash :. Z \epsilon \mu \dot{+} \nu . \equiv :$

$\exists ! \mu . \exists ! \nu : (\exists P, Q) . \mu = \text{Nr}'P . \nu = \text{Nr}'Q . Z \text{ smor } (P + Q)$

Dem.

$\vdash . *155\cdot34\cdot22 . *180\cdot201 . \supset$

$\vdash :. Z \epsilon \mu \dot{+} \nu . \equiv : \exists ! \mu . \exists ! \nu . \mu, \nu \epsilon \text{NR} : (\exists P, Q) . P \epsilon \mu . Q \epsilon \nu . Z \text{ smor } (P + Q) :$

[$*152\cdot44$] $\equiv : \exists ! \mu . \exists ! \nu . \mu, \nu \epsilon \text{NR} : (\exists P, Q) . \mu = \text{Nr}'P . \nu = \text{Nr}'Q . Z \text{ smor } (P + Q) :$

[$*152\cdot41$] $\equiv : \exists ! \mu . \exists ! \nu : (\exists P, Q) . \mu = \text{Nr}'P . \nu = \text{Nr}'Q . Z \text{ smor } (P + Q) :.$

$\supset \vdash . \text{Prop}$

In the following propositions proofs are omitted, since they are exactly analogous to proofs of propositions in $*110$ whose numbers have the same decimal part.

$*180\cdot21$. $\vdash :. \mu, \nu \epsilon \text{NR} . \supset : Z \epsilon \mu \dot{+} \nu . \equiv . (\exists P, Q) . P \epsilon \mu . Q \epsilon \nu . Z \text{ smor } (P + Q)$

$*180\cdot211$. $\vdash :. \mu, \nu \epsilon \text{NR} . \supset : Z \epsilon \mu \dot{+} \nu . \equiv .$

$(\exists R, S) . R \epsilon \text{smor}``\mu . S \epsilon \text{smor}``\nu . C'R \cap C'S = \Lambda . Z = R \ddagger S$

$*180\cdot212$. $\vdash :. \mu, \nu \epsilon \text{NR} . \supset : Z \epsilon \mu \dot{+} \nu . \equiv .$

$(\exists R) . R \epsilon \text{smor}``\mu . R \mathrel{G} Z . Z \upharpoonright (-C'R) \epsilon \text{smor}``\nu$

$*180\cdot22$. $\vdash . \text{N}_0\text{r}'P \dot{+} \text{N}_0\text{r}'Q = \text{Nr}'(P + Q)$

***180·24.** $\vdash : R \operatorname{smor} P \,.\, S \operatorname{smor} Q \,.\, \supset \,.\, \mathrm{N_0r}`R \dot{+} \mathrm{N_0r}`S = \mathrm{N_0r}`P \dot{+} \mathrm{N_0r}`Q$
[*180·15·22]

***180·3.** $\vdash \,.\, \mathrm{Nr}`P \dot{+} \mathrm{Nr}`Q = \mathrm{N_0r}`P \dot{+} \mathrm{Nr}`Q = \mathrm{Nr}`P \dot{+} \mathrm{N_0r}`Q$
$= \mathrm{N_0r}`P \dot{+} \mathrm{N_0r}`Q = \mathrm{Nr}`(P + Q)$ [*180·22 . (*180·03·031)]

***180·31.** $\vdash : P \operatorname{smor} R \,.\, Q \operatorname{smor} S \,.\, \supset \,.\, \mathrm{Nr}`P \dot{+} \mathrm{Nr}`Q = \mathrm{Nr}`R \dot{+} \mathrm{Nr}`S$

***180·32.** $\vdash : C`P \cap C`Q = \Lambda \,.\, \supset \,.\, \mathrm{Nr}`P \dot{+} \mathrm{Nr}`Q = \mathrm{Nr}`(P \ddagger Q)$ [*180·14·3]

***180·4.** $\vdash : \exists ! \, \mu \dot{+} \nu \,.\, \supset \,.\, \mu, \nu \in \mathrm{NR} - \iota`\Lambda \,.\, \mu, \nu \in \mathrm{N_0R}$

***180·42.** $\vdash \,.\, \mu \dot{+} \nu \in \mathrm{NR}$

***180·43.** $\vdash : \mu \dot{+} \nu = \mathrm{N_0r}`Z \,.\, \equiv \,.\, Z \in \mu \dot{+} \nu$

***180·53.** $\vdash \,.\, (P + Q) + R \operatorname{smor} P + (Q + R)$

Dem.

$\vdash \,.\, *160·44 \,.\, (*180·01) \,.\, \supset$
$\vdash : P' = \downarrow (\Lambda \cap C`R) ;\iota; \downarrow (\Lambda \cap C`Q) ;\iota; P \,.\, Q' = \downarrow (\Lambda \cap C`R) ;\iota; (\Lambda \cap C`P) \downarrow ;\iota; Q \,.$
$R' = \{\Lambda \cap C`(P + Q)\} \downarrow ;\iota; R \,.\, \supset \,.\, P' \ddagger Q' = \downarrow (\Lambda \cap C`R) ;\iota; (P + Q) \,.$ (1)
[(*180·01)] $\supset \,.\, (P' \ddagger Q') \ddagger R' = (P + Q) + R \,.$
[*160·31] $\supset \,.\, P' \ddagger (Q' \ddagger R') = (P + Q) + R$ (2)
$\vdash \,.\, (1) \,.\, *180·11 \dfrac{P + Q, R}{P, \; Q} \,.\, *160·14 \,.\, \supset \vdash : \mathrm{Hp}(1) \,.\, \supset \,.$
$C`P' \cap C`R' = \Lambda \,.\, C`Q' \cap C`R' = \Lambda$ (3)
$\vdash \,.\, *180·11 \,.\, *72·411 \,.\, *150·22 \,.\, \supset \vdash : \mathrm{Hp}(1) \,.\, \supset \,.\, C`P' \cap C`Q' = \Lambda$ (4)
$\vdash \,.\, (3) \,.\, (4) \,.\, *160·14 \,.\, \supset \vdash : \mathrm{Hp}(1) \,.\, \supset \,.\, C`P' \cap C`(Q' \ddagger R') = \Lambda$ (5)
$\vdash \,.\, *180·12 \,.\, \supset \vdash : \mathrm{Hp}(1) \,.\, \supset \,.\, P' \operatorname{smor} P \,.\, Q' \operatorname{smor} Q \,.\, R' \operatorname{smor} R$ (6)
$\vdash \,.\, (3) \,.\, (6) \,.\, *180·13 \,.\, \supset \vdash : \mathrm{Hp}(1) \,.\, \supset \,.\, Q' \ddagger R' \operatorname{smor} Q + R \,.$
[(5).(6).*180·13] $\supset \,.\, P' \ddagger (Q' \ddagger R') \operatorname{smor} P + (Q + R) \,.$
[(2)] $\supset \,.\, (P + Q) + R \operatorname{smor} P + (Q + R)$ (7)
$\vdash \,.\, (7) \,.\, *13·19 \,.\, \supset \vdash \,.\, \mathrm{Prop}$

***180·531.** $P + Q + R = (P + Q) + R$ Df

***180·54.** $\vdash \,.\, (\mathrm{Nr}`P \dot{+} \mathrm{Nr}`Q) \dot{+} \mathrm{Nr}`R = \mathrm{Nr}`(P + Q + R)$

***180·541.** $\vdash \,.\, \mathrm{Nr}`P \dot{+} (\mathrm{Nr}`Q \dot{+} \mathrm{Nr}`R) = \mathrm{Nr}`(P + Q + R)$

***180·55.** $\vdash \,.\, (\mathrm{Nr}`P \dot{+} \mathrm{Nr}`Q) \dot{+} \mathrm{Nr}`R = \mathrm{Nr}`P \dot{+} (\mathrm{Nr}`Q \dot{+} \mathrm{Nr}`R)$

***180·551.** $\vdash \,.\, (\mathrm{N_0r}`P \dot{+} \mathrm{N_0r}`Q) \dot{+} \mathrm{N_0r}`R = \mathrm{N_0r}`P \dot{+} (\mathrm{N_0r}`Q \dot{+} \mathrm{N_0r}`R)$

***180·56.** $\vdash \,.\, (\mu \dot{+} \nu) \dot{+} \varpi = \mu \dot{+} (\nu \dot{+} \varpi)$

***180·561.** $\mu \dot{+} \nu \dot{+} \varpi = (\mu \dot{+} \nu) \dot{+} \varpi$ Df

***180·57.** $\vdash \,.\, (\mu \dot{+} \nu) \dot{+} (\varpi \dot{+} \rho) = \mu \dot{+} \nu \dot{+} \varpi \dot{+} \rho$

***180·6.** $\vdash : \mu \in \mathrm{NR} \,.\, \supset \,.\, \mu \dot{+} 0_r = \operatorname{smor}``\mu = 0_r \dot{+} \mu$

Observe that $\mu \dot{+} 0_r = 0_r \dot{+} \mu$ is an equation depending upon the peculiar properties of 0_r. We do not in general have $\mu \dot{+} \nu = \nu \dot{+} \mu$ unless μ and ν are *finite* ordinals.

∗180·61. $\vdash . \mathrm{Nr}`P \dot{+} 0_r = \mathrm{Nr}`P = 0_r \dot{+} \mathrm{Nr}`P$

∗180·62. $\vdash : \mu \dot{+} \nu = 0_r . \equiv . \mu = 0_r . \nu = 0_r$

∗180·64. $\vdash . 0_r \dot{+} 0_r = 0_r$

∗180·642. $\vdash . 2_r \dot{+} 0_r = 0_r \dot{+} 2_r = 2_r$

Note that $\dot{1} \dot{+} 0_r$, which will be defined in ∗181, is 0_r, not $\dot{1}$.

The following propositions, being concerned with the relations of relation-numbers and cardinal numbers, have no analogues in ∗110.

∗180·7. $\vdash . C``\mathrm{Nr}`(P+Q) = C``\mathrm{Nr}`P +_c C``\mathrm{Nr}`Q = \mathrm{Nc}`C`P +_c \mathrm{Nc}`C`Q$

Dem.

$\vdash . ∗152·7 . \supset \vdash . C``\mathrm{Nr}`(P+Q) = \mathrm{Nc}`C`(P+Q)$

[∗180·111] $= \mathrm{Nc}`(C`P + C`Q)$

[∗110·3] $= \mathrm{Nc}`C`P +_c \mathrm{Nc}`C`Q$ (1)

[∗152·7] $= C``\mathrm{Nr}`P +_c C``\mathrm{Nr}`Q$ (2)

$\vdash . (1) . (2) . \supset \vdash .$ Prop

∗180·71. $\vdash : \mu, \nu \in \mathrm{NR} . \supset . C``(\mu \dot{+} \nu) = C``\mu +_c C``\nu$

Dem.

$\vdash . ∗152·4 . \supset \vdash : \mathrm{Hp} . \supset . (\exists P, Q) . \mu = \mathrm{Nr}`P . \nu = \mathrm{Nr}`Q .$

[∗180·3] $\supset . (\exists P, Q) . \mu = \mathrm{Nr}`P . \nu = \mathrm{Nr}`Q . \mu \dot{+} \nu = \mathrm{Nr}`(P+Q) .$

[∗180·7] $\supset . (\exists P, Q) . \mu = \mathrm{Nr}`P . \nu = \mathrm{Nr}`Q .$

$C``(\mu \dot{+} \nu) = C``\mathrm{Nr}`P +_c C``\mathrm{Nr}`Q .$

[∗13·193] $\supset . C``(\mu \dot{+} \nu) = C``\mu +_c C``\nu : \supset \vdash .$ Prop

$*181$. ON THE ADDITION OF UNITY TO A RELATION-NUMBER

Summary of $*181$.

The relation-number $\dot{1}$ has, according to our definitions, no meaning in isolation, because our definitions are framed with a view to series, and a series cannot consist of one term. But we can *add* one term to a series; hence $\dot{1}$ is required as an addendum. In order to get our definitions in the most manageable form, we first construct a relation, which we call $P \,\dot{+\!\!\rightarrow}\, x$, which is such that, whenever P exists, $P \,\dot{+\!\!\rightarrow}\, x$ has one more term in its field than P; the relation-number of this relation is then defined as $\mathrm{Nr}`P \dotplus \dot{1}$. We add also a definition

$$\dot{1} \dotplus \dot{1} = 2_r \quad \mathrm{Df}$$

which is purely formal, and serves to minimize exceptions to the associative law of addition.

The definitions are closely analogous to those of $*180$. We put

$$P \,\dot{+\!\!\rightarrow}\, x = \downarrow \Lambda_x ;\iota ; P \mathbin{+\!\!\rightarrow} (\Lambda \cap C`P) \downarrow \iota`x \quad \mathrm{Df}$$

with a similar definition for $x \leftarrow\!\!\dot{+}\, P$. x and P may be of any relative types, and we have always

$$\downarrow \Lambda_x ;\iota ; P \,\mathrm{smor}\, P \,.\, (\Lambda \cap C`P) \downarrow \iota`x \sim\epsilon\, C` \downarrow \Lambda_x ;\iota ; P \quad (*181\cdot11\cdot12).$$

We put

$$\mu \dotplus \dot{1} = \hat{R}\,\{(\exists P, x) \,.\, \mathrm{N_0r}`P = \mu \,.\, R \,\mathrm{smor}\, (P \,\dot{+\!\!\rightarrow}\, x)\} \quad \mathrm{Df}$$

with a similar definition for $\dot{1} \dotplus \mu$. We also introduce definitions analogous to $*180\cdot03\cdot031$.

The principal propositions of this number are

$*181\cdot3$. $\vdash . \mathrm{Nr}`P \dotplus \dot{1} = \mathrm{N_0r}`P \dotplus \dot{1} = \mathrm{Nr}`(P \,\dot{+\!\!\rightarrow}\, x)$

$*181\cdot31$. $\vdash : P \,\mathrm{smor}\, Q \,.\supset .\, \mathrm{Nr}`P \dotplus \dot{1} = \mathrm{Nr}`Q \dotplus \dot{1}$

$*181\cdot32$. $\vdash : x \sim\epsilon\, C`P \,.\supset .\, \mathrm{Nr}`P \dotplus \dot{1} = \mathrm{Nr}`(P \,\dot{+\!\!\rightarrow}\, x)$

$*181\cdot33$. $\vdash :. \mu, \nu \,\epsilon\, \mathrm{NR} \,.\, \exists ! \mu \dotplus \dot{1} \,.\supset : \mu = \nu \,.\equiv .\, \mu \dotplus \dot{1} = \nu \dotplus \dot{1} \,.\equiv .\, \dot{1} \dotplus \mu = \dot{1} \dotplus \nu$

$*181\cdot4$. $\vdash : \exists ! \mu \dotplus \dot{1} \,.\supset .\, \mu \,\epsilon\, \mathrm{NR} - \iota`\Lambda \,.\, \mu \,\epsilon\, \mathrm{N_0R}$

$*181\cdot42$. $\vdash . \mu \dotplus \dot{1} \,\epsilon\, \mathrm{NR}$

The following propositions are formally forms of the associative law, but they need separate proof on account of the peculiarity of $\dot{1}$.

$*181\cdot54$. $\vdash : \nu \neq 0_r \,.\supset .\, (\mu \dotplus \nu) \dotplus \dot{1} = \mu \dotplus (\nu \dotplus \dot{1})$

$*181\cdot56$. $\vdash : \mu \neq 0_r \,.\supset .\, (\mu \dotplus \dot{1}) \dotplus \dot{1} = \mu \dotplus (\dot{1} \dotplus \dot{1}) = \mu \dotplus 2_r$

$*181\cdot58$. $\vdash : \mu \neq 0_r \,.\, \nu \neq 0_r \,.\supset .\, (\mu \dotplus \dot{1}) \dotplus \nu = \mu \dotplus (\dot{1} \dotplus \nu)$

***181·59.** $\vdash : \mu \neq 0_r \,.\, \nu \neq 0_r \,.\, \supset .\, (\mu \dot{+} \dot{1}) \dot{+} (\dot{1} \dot{+} \nu) = \mu \dot{+} 2_r \dot{+} \nu$

The hypotheses in the above propositions are essential.

***181·6.** $\vdash : \dot{\exists} ! P \,.\, \supset .\, C``\text{Nr}`(P \dot{\leftrightarrow} x) = \text{Nc}`C`P +_c 1$

***181·62.** $\vdash : \mu \in \text{NR} - \iota`0_r \,.\, \supset .\, C``(\mu \dot{+} \dot{1}) = C``(\dot{1} \dot{+} \mu) = C``\mu +_c 1$

These propositions give the connection with cardinals.

***181·01.** $P \dot{\leftrightarrow} x = \downarrow \Lambda_x ; \iota ; P \leftrightarrow (\Lambda \cap C`P) \downarrow \iota`x$ Df

***181·011.** $x \dot{\leftarrow} P = (\iota`x) \downarrow (\Lambda \cap C`P) \leftarrow \Lambda_x \downarrow ; \iota ; P$ Df

***181·02.** $\mu \dot{+} \dot{1} = \hat{R} \{(\exists P, x) \,.\, \text{N}_0\text{r}`P = \mu \,.\, R \,\text{smor}\, (P \dot{\leftrightarrow} x)\}$ Df

***181·021.** $\dot{1} \dot{+} \mu = \hat{R} \{(\exists P, x) \,.\, \text{N}_0\text{r}`P = \mu \,.\, R \,\text{smor}\, (x \dot{\leftarrow} P)\}$ Df

***181·03.** $\text{Nr}`P \dot{+} \dot{1} = \text{N}_0\text{r}`P \dot{+} \dot{1}$ Df

***181·031.** $\dot{1} \dot{+} \text{Nr}`P = \dot{1} \dot{+} \text{N}_0\text{r}`P$ Df

***181·04.** $\dot{1} \dot{+} \dot{1} = 2_r$ Df

Propositions concerning $x \dot{\leftarrow} P$ are omitted in what follows, since they are proved exactly as the analogous propositions concerning $P \dot{\leftrightarrow} x$ are proved.

***181·1.** $\vdash :. R (P \dot{\leftrightarrow} x) S \,.\, \equiv : (\exists y, z) \,.\, yPz \,.\, R = (\iota`y) \downarrow \Lambda_x \,.\, S = (\iota`z) \downarrow \Lambda_x \,.\, \vee .$
$(\exists y) \,.\, y \in C`P \,.\, R = (\iota`y) \downarrow \Lambda_x \,.\, S = (\Lambda \cap C`P) \downarrow \iota`x$ [(*181·01)]

***181·11.** $\vdash .\, (\Lambda \cap C`P) \downarrow \iota`x \sim\in C` \downarrow \Lambda_x ; \iota ; P$

Dem.

$\vdash .\, *150·22 \,.\, \supset \vdash .\, C` \downarrow \Lambda_x ; \iota ; P = \downarrow \Lambda_x `` \iota `` C`P \,.$

[*55·15] $\supset \vdash : Q \in C` \downarrow \Lambda_x ; \iota ; P \,.\, \supset_Q .\, \text{Cl}`Q = \iota`\Lambda_x$ (1)

$\vdash .\, *55·15 \,.\, \supset \vdash .\, \text{Cl}`(\Lambda \cap C`P) \downarrow \iota`x = \iota`\iota`x$ (2)

$\vdash .\, (1) . (2) . *51·161 \,.\, \supset \vdash : Q \in C` \downarrow \Lambda_x ; \iota ; P \,.\, \supset_Q .\, \text{Cl}`Q \neq \text{Cl}`(\Lambda \cap C`P) \downarrow \iota`x \,.$

[*30·37.Transp] $\supset_Q .\, Q \neq (\Lambda \cap C`P) \downarrow \iota`x :$

[*13·196] $\supset \vdash .\, (\Lambda \cap C`P) \downarrow \iota`x \sim\in C` \downarrow \Lambda_x ; \iota ; P \,.\, \supset \vdash .\, \text{Prop}$

***181·12.** $\vdash .\, \downarrow \Lambda_x ; \iota ; P \,\text{smor}\, P$ [*151·61·65]

***181·13.** $\vdash : Q \,\text{smor}\, P \,.\, y \sim\in C`Q \,.\, \supset .\, Q \leftrightarrow y \,\text{smor}\, P \dot{\leftrightarrow} x$

Dem.

$\vdash .\, *181·12 \,.\, \supset \vdash : \text{Hp} \,.\, \supset .\, Q \,\text{smor} \downarrow \Lambda_x ; \iota ; P$ (1)

$\vdash .\, (1) \,.\, *161·31 \,.\, *181·11 \,.\, \supset \vdash .\, \text{Prop}$

***181·2.** $\vdash : Z \in \mu \dot{+} \dot{1} \,.\, \equiv .\, (\exists P, x) \,.\, \mu = \text{N}_0\text{r}`P \,.\, Z \,\text{smor}\, (P \dot{\leftrightarrow} x)$ [(*181·02)]

***181·21.** $\vdash :. \mu \in \text{NR} \,.\, \supset : Z \in \mu \dot{+} \dot{1} \,.\, \equiv .\, (\exists P, x) \,.\, P \in \mu \,.\, Z \,\text{smor}\, (P \dot{\leftrightarrow} x)$
[*181·2 . *155·26]

∗181·22. $\vdash . \mathrm{N_0r}`P \dot{+} \dot{1} = \mathrm{Nr}`(P \mathbin{\dot{+\!\!\!\rightarrow}} x)$

Dem.

$\vdash . *181{\cdot}21 . \supset \vdash : Z \in \mathrm{N_0r}`P \dot{+} \dot{1} . \equiv . (\exists Q, y) . Q \in \mathrm{N_0r}`P . Z \,\mathrm{smor}\, (Q \mathbin{\dot{+\!\!\!\rightarrow}} y)$ (1)

$\vdash . (1) . *155{\cdot}12 . *152{\cdot}11 . \supset \vdash . \mathrm{Nr}`(P \mathbin{\dot{+\!\!\!\rightarrow}} x) \subset \mathrm{N_0r}`P \dot{+} \dot{1}$ (2)

$\vdash . *181{\cdot}12{\cdot}11 . *161{\cdot}31 . \supset \vdash : Q \in \mathrm{N_0r}`P . Z \,\mathrm{smor}\, (Q \mathbin{\dot{+\!\!\!\rightarrow}} y) . \supset . Z \,\mathrm{smor}\, (P \mathbin{\dot{+\!\!\!\rightarrow}} x)$ (3)

$\vdash . (1) . (3) . *152{\cdot}11 . \quad \supset \vdash : Z \in \mathrm{N_0r}`P \dot{+} \dot{1} . \supset . Z \in \mathrm{Nr}`(P \mathbin{\dot{+\!\!\!\rightarrow}} x)$ (4)

$\vdash . (2) . (4) . \supset \vdash . \mathrm{Prop}$

∗181·24. $\vdash : P \,\mathrm{smor}\, Q . \supset . \mathrm{N_0r}`P \dot{+} \dot{1} = \mathrm{N_0r}`Q \dot{+} \dot{1}$ $[*181{\cdot}22{\cdot}12{\cdot}11 . *161{\cdot}31]$

∗181·3. $\vdash . \mathrm{Nr}`P \dot{+} \dot{1} = \mathrm{N_0r}`P \dot{+} \dot{1} = \mathrm{Nr}`(P \mathbin{\dot{+\!\!\!\rightarrow}} x)$ $[*181{\cdot}22 . (*181{\cdot}03)]$

∗181·31. $\vdash : P \,\mathrm{smor}\, Q . \supset . \mathrm{Nr}`P \dot{+} \dot{1} = \mathrm{Nr}`Q \dot{+} \dot{1}$ $[*181{\cdot}3{\cdot}24]$

∗181·32. $\vdash : x \sim\in C`P . \supset . \mathrm{Nr}`P \dot{+} \dot{1} = \mathrm{Nr}`(P \mathbin{+\!\!\!\rightarrow} x)$ $[*181{\cdot}3{\cdot}13]$

∗181·33. $\vdash :. \mu, \nu \in \mathrm{NR} . \exists ! \mu \dot{+} \dot{1} . \supset : \mu = \nu . \equiv . \mu \dot{+} \dot{1} = \nu \dot{+} \dot{1} . \equiv . \dot{1} \dot{+} \mu = \dot{1} \dot{+} \nu$
$[*161{\cdot}33 . *181{\cdot}3{\cdot}11{\cdot}12]$

The above proposition is used in ∗253·23·571.

∗181·4. $\vdash : \exists ! \mu \dot{+} \dot{1} . \supset . \mu \in \mathrm{NR} - \iota`\Lambda . \mu \in \mathrm{N_0R}$ $[*181{\cdot}2 . *155{\cdot}22]$

∗181·42. $\vdash . \mu \dot{+} \dot{1} \in \mathrm{NR}$

Dem.

$\vdash . *181{\cdot}3 . \supset \vdash : \mu \in \mathrm{N_0R} . \supset . (\exists P, x) . \mu \dot{+} \dot{1} = \mathrm{Nr}`(P \mathbin{\dot{+\!\!\!\rightarrow}} x) .$

$[*152{\cdot}4] \qquad \supset . \mu \dot{+} 1 \in \mathrm{NR}$ (1)

$\vdash . *181{\cdot}4 . \supset \vdash : \mu \sim\in \mathrm{N_0R} . \supset . \mu \dot{+} 1 = \Lambda .$

$[*154{\cdot}242] \qquad \supset . \mu \dot{+} 1 \in \mathrm{NR}$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

∗181·43. $\vdash : \mu \dot{+} 1 = \mathrm{N_0r}`Z . \equiv . Z \in \mu \dot{+} 1$ $[*155{\cdot}26 . *181{\cdot}42]$

The following propositions are concerned with the associative law when $\dot{1}$ is one of the addenda.

∗181·53. $\vdash : \dot{\exists} ! P . x \neq y . \supset . (P \mathbin{\dot{+\!\!\!\rightarrow}} x) \mathbin{\dot{+\!\!\!\rightarrow}} y \,\mathrm{smor}\, P + (x \downarrow y)$

Dem.

$\vdash . *13{\cdot}15 . (*181{\cdot}01) . \supset$

$\vdash . (P \mathbin{\dot{+\!\!\!\rightarrow}} x) \mathbin{\dot{+\!\!\!\rightarrow}} y = \downarrow \Lambda_y{}^{;}\iota{}^{;}\{\downarrow \Lambda_x{}^{;}\iota{}^{;}P \mathbin{+\!\!\!\rightarrow} (\Lambda \cap C`P) \downarrow \iota`x\} \mathbin{+\!\!\!\rightarrow} \{\Lambda \cap C`(P \mathbin{\dot{+\!\!\!\rightarrow}} x)\} \downarrow \iota`y$

$[*161{\cdot}4] = \downarrow \Lambda_y{}^{;}\iota{}^{;} \downarrow \Lambda_x{}^{;}\iota{}^{;}P \mathbin{+\!\!\!\rightarrow} \downarrow \Lambda_y`\iota`(\Lambda \cap C`P) \downarrow \iota`x \mathbin{+\!\!\!\rightarrow} \{\Lambda \cap C`(P \mathbin{\dot{+\!\!\!\rightarrow}} x)\} \downarrow \iota`y$ (1)

$\vdash . (1) . *161{\cdot}22 . \supset \vdash : \mathrm{Hp} . \supset . (P \mathbin{\dot{+\!\!\!\rightarrow}} x) \mathbin{\dot{+\!\!\!\rightarrow}} y$

$= \downarrow \Lambda_y{}^{;}\iota{}^{;} \downarrow \Lambda_x{}^{;}\iota{}^{;}P \mathbin{\uparrow\!\!\!+} \{\downarrow \Lambda_y`\iota`(\Lambda \cap C`P) \downarrow \iota`x\} \downarrow [\{\Lambda \cap C`(P \mathbin{\dot{+\!\!\!\rightarrow}} x)\} \downarrow \iota`y]$ (2)

$\vdash . *180{\cdot}13 . *181{\cdot}11 . \supset$

$\vdash : \mathrm{Hp} . \supset . \downarrow \Lambda_y{}^{;}\iota{}^{;} \downarrow \Lambda_x{}^{;}\iota{}^{;}P \mathbin{\uparrow\!\!\!+} \{\downarrow \Lambda_y`\iota`(\Lambda \cap C`P) \downarrow \iota`x\} \downarrow [\{\Lambda \cap C`(P \mathbin{\dot{+\!\!\!\rightarrow}} x)\} \downarrow \iota`y]$
$\mathrm{smor}\, P + (x \downarrow y)$ (3)

$\vdash . (2) . (3) . \supset \vdash . \mathrm{Prop}$

$*181\cdot54$. $\vdash : \nu \neq 0_r . \supset . (\mu \dot{+} \nu) \dot{+} \dot{1} = \mu \dot{+} (\nu \dot{+} \dot{1})$

Dem.

$\vdash . *181\cdot2 . *180\cdot2 . \supset \vdash : Y \in (\mu \dot{+} \nu) \mathbin{\dot{+}\!\!\!\rightarrow} \dot{1} . \equiv .$

$\qquad (\exists P, Q, R, x) . \mathrm{N_0r}`P = \mu . \mathrm{N_0r}`Q = \nu . R \operatorname{smor} P + Q . Y \operatorname{smor} R \mathbin{\dot{+}\!\!\!\rightarrow} x .$

$[*181\cdot22\cdot31] \equiv . (\exists P, Q, x) . \mathrm{N_0r}`P = \mu . \mathrm{N_0r}`Q = \nu . Y \operatorname{smor} (P + Q) \mathbin{\dot{+}\!\!\!\rightarrow} x \quad (1)$

$\vdash . *153\cdot14 . \supset \vdash :. \mathrm{Hp} . \supset : \mathrm{N_0r}`Q = \nu . \supset . \dot{\exists} ! Q \quad (2)$

$\vdash . *160\cdot44 . (*180\cdot01 . *181\cdot01) . \supset$

$\vdash : P' = \downarrow \Lambda_x ; \iota ; \downarrow \Lambda_{C`Q} ; \iota ; P . Q' = \downarrow \Lambda_x ; \iota ; \Lambda_{C`P} \downarrow ; \iota ; Q . X = \{\Lambda \cap C`(P + Q)\} \downarrow \iota`x .$

$\qquad \supset . P' \Uparrow Q' = \downarrow \Lambda_x ; \iota ; (P + Q) . (P' \Uparrow Q') \mathbin{+\!\!\!\rightarrow} X = (P + Q) \mathbin{\dot{+}\!\!\!\rightarrow} x \quad (3)$

$\vdash . *180\cdot12 . *181\cdot12 . \supset \vdash : \mathrm{Hp}(3) . \supset . P' \operatorname{smor} P . Q' \operatorname{smor} Q \quad (4)$

$\vdash . *180\cdot11 . *72\cdot411 . *181\cdot11 . (3) . \supset$

$\vdash : \mathrm{Hp}(3) . \supset . C`P' \cap C`Q' = \Lambda . X \sim\in C`P' . X \sim\in C`Q' \quad (5)$

$\vdash . *161\cdot23 . (4) . \supset \vdash : \mathrm{Hp}(3) . \dot{\exists} ! Q . \supset . (P' \Uparrow Q') \mathbin{+\!\!\!\rightarrow} X = P' \Uparrow (Q' \mathbin{+\!\!\!\rightarrow} X) \quad (6)$

$\vdash . *181\cdot13 . (4) . (5) . \supset \vdash : \mathrm{Hp}(3) . \supset . Q' \mathbin{+\!\!\!\rightarrow} X \operatorname{smor} (Q \mathbin{\dot{+}\!\!\!\rightarrow} x) .$

$[*180\cdot13.(4).(5)] \qquad \supset . P' \Uparrow (Q' \mathbin{+\!\!\!\rightarrow} X) \operatorname{smor} P + (Q \mathbin{\dot{+}\!\!\!\rightarrow} x) \quad (7)$

$\vdash . (1) . (2) . (6) . (7) . \supset \vdash :. \mathrm{Hp} . \mathrm{Hp}(3) . \supset : Y \in (\mu \dot{+} \nu) \dot{+} \dot{1} . \equiv .$

$\qquad (\exists P, Q, x) . \mathrm{N_0r}`P = \mu . \mathrm{N_0r}`Q = \nu . Y \operatorname{smor} P + (Q \mathbin{\dot{+}\!\!\!\rightarrow} x) .$

$[*180\cdot3.*181\cdot3] \equiv . (\exists P, Q, x) . \mathrm{N_0r}`P = \mu . \mathrm{N_0r}`Q = \nu . Y \in \mathrm{N_0r}`P \dot{+} (\mathrm{N_0r}`Q \dot{+} \dot{1}) .$

$[*13\cdot193.*155\cdot2] \equiv . \mu, \nu \in \mathrm{N_0R} . Y \in \mu \dot{+} (\nu \dot{+} \dot{1}) .$

$[*181\cdot4.*180\cdot4] \equiv . Y \in \mu \dot{+} (\nu \dot{+} \dot{1}) \quad (8)$

$\vdash . (8) . *13\cdot19 . \supset \vdash . \mathrm{Prop}$

$*181\cdot55$. $\vdash : \mu \neq 0_r . \supset . \dot{1} \dot{+} (\mu \dot{+} \nu) = (\dot{1} \dot{+} \mu) \dot{+} \nu$ [Proof as in $*181\cdot54$]

$*181\cdot56$. $\vdash : \mu \neq 0_r . \supset . (\mu \dot{+} \dot{1}) \dot{+} \dot{1} = \mu \dot{+} (\dot{1} \dot{+} \dot{1}) = \mu \dot{+} 2_r$

Dem.

$\vdash . *153\cdot2 . *180\cdot2 . \supset$

$\vdash : Z \in \mu \dot{+} 2_r . \equiv . (\exists P, x, y) . \mu = \mathrm{N_0r}`P . x \neq y . Z \operatorname{smor} P \dot{+} (x \downarrow y) \quad (1)$

$\vdash . (1) . *181\cdot53 . \supset \vdash :. \mathrm{Hp} . \supset :$

$\qquad Z \in \mu \dot{+} 2_r . \equiv . (\exists P, x, y) . \mu = \mathrm{N_0r}`P . x \neq y . Z \operatorname{smor} (P \mathbin{\dot{+}\!\!\!\rightarrow} x) \mathbin{\dot{+}\!\!\!\rightarrow} y .$

$[*181\cdot22] \qquad \equiv . (\exists P, x, y) . \mu = \mathrm{N_0r}`P . x \neq y . Z \in (\mathrm{N_0r}`P \dot{+} \dot{1}) \dot{+} \dot{1} .$

$[*181\cdot4] \qquad \equiv . (\exists x, y) . x \neq y . Z \in (\mu \dot{+} \dot{1}) \dot{+} \dot{1} .$

$[*24\cdot1] \qquad \equiv . Z \in (\mu \dot{+} \dot{1}) \dot{+} \dot{1} \quad (2)$

$\vdash . (2) . (*181\cdot04) . \supset \vdash . \mathrm{Prop}$

The last line in the above proof, in which $*24\cdot1$ is used, is legitimate because x and y may be of any type whatever, and therefore the fact that $\Lambda \neq \mathrm{V}$ is sufficient to establish $(\exists x, y) . x \neq y$ in the sense wanted.

$*181\cdot561$. $\mu \dot{+} \dot{1} \dot{+} \dot{1} = \mu \dot{+} (\dot{1} \dot{+} \dot{1})$ Df

This definition adopts the opposite convention to that usually adopted. But it is convenient to have $0_r \dot{+} \dot{1} \dot{+} \dot{1} = 2_r$, and also to have as much similarity as possible between the results of adding $\dot{1}$ at the beginning and end of a relation. Both reasons lead to the adoption of the above convention. (Cf. $*181\cdot57\cdot571$, below.)

$*181\cdot57$. $\vdash : \mu \neq 0_r . \supset . \dot{1} \dot{+} (\dot{1} \dot{+} \mu) = (\dot{1} \dot{+} \dot{1}) \dot{+} \mu = 2_r \dot{+} \mu$

[Proof as in $*181\cdot56$]

$*181\cdot571$. $\dot{1} \dot{+} \dot{1} \dot{+} \mu = (\dot{1} \dot{+} \dot{1}) \dot{+} \mu$ Df

$*181\cdot58$. $\vdash : \mu \neq 0_r . \nu \neq 0_r . \supset . (\mu \dot{+} \dot{1}) \dot{+} \nu = \mu \dot{+} (\dot{1} \dot{+} \nu)$ [$*161\cdot232$]

The proof proceeds in the same way as that of $*181\cdot54$.

$*181\cdot59$. $\vdash : \mu \neq 0_r . \nu \neq 0_r . \supset . (\mu \dot{+} \dot{1}) \dot{+} (\dot{1} \dot{+} \nu) = \mu \dot{+} 2_r \dot{+} \nu$ [$*161\cdot25$]

The above propositions show that, except when one of the summands is zero, the associative law holds for $\dot{1}$ just as if it were a relation-number.

The following propositions are concerned with relations to cardinal addition.

$*181\cdot6$. $\vdash : \dot{\exists} ! P . \supset . C``\text{Nr}`(P \dot{\rightarrow} x) = \text{Nc}`C`P +_c 1$

Dem.

$\vdash . *152\cdot7 . \quad \supset \vdash . C``\text{Nr}`(P \dot{\rightarrow} x) = \text{Nc}`C`(P \dot{\rightarrow} x) .$

$\vdash . (1) . *161\cdot14 . \supset \vdash : \text{Hp} . \supset . C``\text{Nr}`(P \dot{\rightarrow} x)$

$= \text{Nc}`[\downarrow \Lambda_x``\iota``C`P \cup \iota`\{(\Lambda \cap C`P) \downarrow \iota`x\}]$

$[*110\cdot13\cdot3 . *181\cdot11 . *110\cdot12] = \text{Nc}`C`P +_c 1 : \supset \vdash . \text{Prop}$

$*181\cdot61$. $\vdash : \dot{\exists} ! P . \supset . C``\text{Nr}`(x \dot{\leftarrow} P) = 1 +_c \text{Nc}`C`P = \text{Nc}`C`P +_c 1$

[Proof as in $*181\cdot6$]

$*181\cdot62$. $\vdash : \mu \in \text{NR} - \iota`0_r . \supset . C``(\mu \dot{+} \dot{1}) = C``(\dot{1} \dot{+} \mu) = C``\mu +_c 1$

Dem.

$\vdash . *153\cdot16 . *152\cdot4 . \supset \vdash : \text{Hp} . \supset . (\exists P) . \mu = \text{Nr}`P . \dot{\exists} ! P .$

$[*181\cdot3\cdot6] \quad \supset . (\exists P) . \mu = \text{Nr}`P . C``(\text{Nr}`P \dot{+} \dot{1}) = \text{Nc}`C`P +_c 1$

$[*152\cdot7] \quad = C``\text{Nr}`P +_c 1 .$

$[*13\cdot193] \quad \supset . C``(\mu \dot{+} \dot{1}) = C``\mu +_c 1 \quad (1)$

Similarly $\quad \vdash : \text{Hp} . \supset . C``(1 \dot{+} \mu) = 1 +_c C``\mu \quad (2)$

$\vdash . (1) . (2) . *110\cdot51 . \supset \vdash . \text{Prop}$

$*$182. ON SEPARATED RELATIONS

Summary of $*182$.

In this number, we have to consider, as a preliminary to the addition of the relation-numbers of a field, the properties of the relation $\overset{\frown}{\underset{;}{\downarrow}};P$, which is defined as follows. If $x ♀ y$ is any function of two arguments in the sense of $*38$, we put $\overset{\frown}{♀}‘x = x ♀ x$ Df. Thus $\overset{\frown}{\underset{;}{\downarrow}}‘Q = Q \underset{;}{\downarrow} Q$, *i.e.* $\overset{\frown}{\underset{;}{\downarrow}}‘Q = \downarrow Q;Q$. Hence $\overset{\frown}{\underset{;}{\downarrow}};P$ is the relation of $\downarrow Q;Q$ to $\downarrow R;R$ when QPR. Thus the symbol $\overset{\frown}{\underset{;}{\downarrow}};P$ is only significant when P is a relation of relations; when this is the case, $\overset{\frown}{\underset{;}{\downarrow}};P$ is the relation which results when, for every Q which is a member of $C‘P$, every member x of $C‘Q$ is replaced by $x \downarrow Q$. The result is a $\text{Rel}^2\,\text{excl}$, whose arithmetical properties serve to define the arithmetical properties of the sum of the relation-numbers of members of $C‘P$. In the next number, we shall put

$$\Sigma\,\text{Nr}‘P = \text{Nr}‘\Sigma‘\overset{\frown}{\underset{;}{\downarrow}};P \quad \text{Df.}$$

We shall put later

$$\Pi\text{Nr}‘P = \text{Nr}‘\Pi‘P$$

and we shall find

$$\Pi‘P = \dot{s};\text{Prod}‘\overset{\frown}{\underset{;}{\downarrow}};P \,.\, \text{Nr}‘\Pi‘P = \text{Nr}‘\text{Prod}‘\overset{\frown}{\underset{;}{\downarrow}};P.$$

Thus we might have dispensed with $\Pi‘P$ as a fundamental notion, using Prod instead, and putting

$$\Pi‘P = \dot{s};\text{Prod}‘\overset{\frown}{\underset{;}{\downarrow}};P \quad \text{Df.}$$

But this course is on the whole less convenient than that adopted in $*172$ and $*173$.

The notation $\overset{\frown}{♀}$ is thus required in connection with ordinal addition, where it is almost indispensable. It has besides certain minor uses. The object of the notation is to enable us to exhibit as a function of x an expression of the form $x ♀ x$, where ♀ is any descriptive double function which exists for all possible pairs of arguments. Thus for example $x \downarrow x$ is a function of x, but the notations hitherto introduced do not enable us to exhibit it in the form $R‘x$. Hence if we wish (say) to deal with the class

$$\hat{P}\{(\exists x) \,.\, x \epsilon \alpha \,.\, P = x \downarrow x\}$$

we cannot write it in the form $R‘‘\alpha$ unless we introduce a new notation. We put

$$\overset{\frown}{\downarrow}‘x = x \downarrow x$$

whence $$\hat{P}\{(\exists x) \,.\, x \epsilon \alpha \,.\, P = x \downarrow x\} = \overset{\frown}{\downarrow}‘‘\alpha.$$

We introduce the notation generally for all descriptive double functions which exist for all possible pairs of arguments. Thus "φ" in this number corresponds to "φ" in $*38$.

In the present number, we shall begin by a few propositions illustrating possible uses of the notation $\overset{\frown}{\varphi}$. Thus for example if λ is a class of relations, we have hitherto had no simple notation for expressing the class of their squares. But since $R^2 = \overset{\frown}{|}\text{‘}R$, the class of the squares of λ's is $\overset{\frown}{|}\text{“}\lambda$. The notation is, however, introduced chiefly in order to be applied to $\downarrow$ and $\underset{\cdot,}{\downarrow}$. We therefore proceed almost at once to propositions on $\overset{\frown}{\underset{\cdot,}{\downarrow}}$, and especially on $\overset{\frown}{\underset{\cdot,}{\downarrow}}{;}P$. We have

$*182\cdot16\cdot162$. $\vdash . \overset{\frown}{\underset{\cdot,}{\downarrow}}{;}P \in \text{Rel}^2\,\text{excl} . \overset{\frown}{\underset{\cdot,}{\downarrow}} \in 1 \to 1 . \overset{\frown}{\underset{\cdot,}{\downarrow}}{;}P \,\text{smor}\, P$

$*182\cdot2$. $\vdash . \overset{\frown}{\underset{\cdot,}{\downarrow}}\text{‘}Q = \Pi\text{‘}(Q \downarrow Q) = \Pi\text{‘}\overset{\frown}{\downarrow}\text{‘}Q$

$*182\cdot21$. $\vdash . \overset{\frown}{\underset{\cdot,}{\downarrow}}{;}P = \Pi{;}\overset{\frown}{\downarrow}{;}P$

We next prove ($*182\cdot27$) that if $P \in \text{Rel}^2\,\text{excl}$, then P has double likeness to $\overset{\frown}{\underset{\cdot,}{\downarrow}}{;}P$, the double correlator being $\breve{\iota}\,|\,\text{D}$ with its converse domain limited to $C\text{‘}\Sigma\text{‘}\overset{\frown}{\underset{\cdot,}{\downarrow}}{;}P$ ($*182\cdot26$). We then prove ($*182\cdot33$) that if $T \restriction C\text{‘}\Sigma\text{‘}P$ is a double correlator of P with Q, then $T \,\|\, \text{Cnv}\text{‘}T\dagger$ (with its converse domain limited) is a double correlator of $\overset{\frown}{\underset{\cdot,}{\downarrow}}{;}P$ and $\overset{\frown}{\underset{\cdot,}{\downarrow}}{;}Q$, whence we deduce

$*182\cdot34$. $\vdash : P \,\text{smor smor}\, Q . \supset . \overset{\frown}{\underset{\cdot,}{\downarrow}}{;}P \,\text{smor smor}\, \overset{\frown}{\underset{\cdot,}{\downarrow}}{;}Q$

We next proceed to prove

$*182\cdot42$. $\vdash . \Pi\text{‘}P = \dot{s}{;}\text{Prod}\text{‘}\overset{\frown}{\underset{\cdot,}{\downarrow}}{;}P = \dot{s}{;}\text{D}{;}\Pi\text{‘}\overset{\frown}{\underset{\cdot,}{\downarrow}}{;}P = \Pi\text{‘}\Sigma\text{‘}\overset{\frown}{\downarrow}{;}P$

The proof of this is as follows: In virtue of $*182\cdot21$ and the associative law for Π, we have

$$\dot{s}{;}\text{D}{;}\Pi\text{‘}\overset{\frown}{\underset{\cdot,}{\downarrow}}{;}P = \Pi\text{‘}\Sigma\text{‘}\overset{\frown}{\downarrow}{;}P.$$

Now $\quad \Sigma\text{‘}\overset{\frown}{\downarrow}{;}P = P \mathbin{\dot{\cup}} I \restriction C\text{‘}P \quad (*182\cdot413)$,

and $\quad \Pi\text{‘}(P \mathbin{\dot{\cup}} I \restriction C\text{‘}P) = \Pi\text{‘}P \quad (*172\cdot51)$.

Hence our proposition results. Hence we arrive at

$*182\cdot44$. $\vdash . \text{Nr}\text{‘}\Pi\text{‘}P = \text{Nr}\text{‘}\text{Prod}\text{‘}\overset{\frown}{\underset{\cdot,}{\downarrow}}{;}P = \text{Nr}\text{‘}\Pi\text{‘}\overset{\frown}{\underset{\cdot,}{\downarrow}}{;}P$

Finally we have some propositions showing how the notation $\overset{\frown}{\varphi}$ can be applied in cardinals. It is then applied to $\underset{,,}{\downarrow}$, instead of, as above, to $\underset{\cdot,}{\downarrow}$.

We have (*182·5·51·52) $\epsilon\, \overline{\downarrow}\, \alpha = \overset{\frown}{\underset{,,}{\downarrow}}‘\alpha \,.\, \epsilon\, \overline{\downarrow}\, “\kappa = \overset{\frown}{\underset{,,}{\downarrow}}“\kappa \,.\, \Sigma‘\kappa = s‘\overset{\frown}{\underset{,,}{\downarrow}}“\kappa$. Thus the notation of the present number might have been employed in dealing with cardinal addition (*112) instead of the notation $\epsilon\, \overline{\downarrow}\, \alpha$. The general notation $P\, \overline{\downarrow}\, x$ was, however, required for other purposes (cf. *85) and could not have been dispensed with.

In *183 we shall put

$$\Sigma \mathrm{Nr}‘P = \mathrm{Nr}‘\Sigma‘\overset{\frown}{\underset{\cdot,}{\downarrow}};P,$$

and by *182·52 we have

$$\Sigma \mathrm{Nc}‘\kappa = \mathrm{Nc}‘s‘\overset{\frown}{\underset{,,}{\downarrow}}“\kappa.$$

It will be seen that these formulae have the usual kind of analogy.

***182·01.** $\overset{\frown}{♀} = \hat{y}\hat{x}(y = x\, ♀\, x)$ Df

***182·02.** $\vdash : y\, \overset{\frown}{♀}\, x \,.\equiv.\, y = x\, ♀\, x$ [(*182·01)]

***182·021.** $\vdash .\, \overset{\frown}{♀}‘x = x\, ♀\, x$ [*182·02 . *30·3]

***182·022.** $\vdash .\, \mathrm{E}\,!\, \overset{\frown}{♀}‘x$ [*182·021 . *14·21]

***182·023.** $\vdash : \overset{\frown}{♀} \,\epsilon\, 1 \rightarrow \mathrm{Cls} : (\alpha) \,.\, \alpha \subset \mathrm{C\!\!\!I}‘\overset{\frown}{♀}$ [*182·022 . *71·166 . *33·431]

***182·03.** $\vdash .\, \overset{\frown}{|}‘R = R^2$ [*182·021 . (*34·02)]

Thus if λ is a class of relations, the class of their squares is $\overset{\frown}{|}“\lambda$.

***182·031.** $\vdash .\, \overset{\frown}{\uparrow}‘\alpha = \alpha \uparrow \alpha$ [*182·021]

***182·032.** $\vdash .\, \overset{\frown}{\downarrow}‘x = x \downarrow x$ [*182·021]

***182·033.** $\vdash .\, \dot{2} - 2_r = \mathrm{D}‘\overset{\frown}{\downarrow} = 1_s$ [*56·13 . *182·032 . *153·3]

***182·04.** $\vdash .\, \overset{\frown}{\underset{,,}{\downarrow}}‘\alpha = \downarrow \alpha“\alpha$ [*182·021 . *38·2]

Observe that in $\overset{\frown}{\underset{,,}{\downarrow}}$, we first take $\underset{,,}{\downarrow}$, and then put a circumflex over it. If we first took $\overset{\frown}{\downarrow}$, we could not then place two commas under it, because $\overset{\frown}{\downarrow}$ is a relation, not a double descriptive function, and two commas can only significantly be placed under a double descriptive function.

***182·05.** $\vdash .\, \overset{\frown}{\underset{\cdot,}{\downarrow}}‘Q = \downarrow Q;Q = Q \underset{\cdot,}{\downarrow} Q$ [*182·021 . *150·6]

The relation for the sake of which the above notation is chiefly introduced is $\overset{\frown}{\underset{\cdot,}{\downarrow}};P$, where P is a relation of relations. If P relates Q and R, then $\overset{\frown}{\underset{\cdot,}{\downarrow}};P$ relates $\downarrow Q;Q$ and $\downarrow R;R$. This is stated in the following proposition:

***182·1.** $\vdash .\, \overset{\frown}{\underset{\cdot,}{\downarrow}};P = \hat{X}\hat{Y}\{(\exists Q, R) \,.\, QPR \,.\, X = Q \underset{\cdot,}{\downarrow} Q \,.\, Y = R \underset{\cdot,}{\downarrow} R\}$
[*182·023·05 . *150·4]

***182·11.** $\vdash . C`\overset{\frown}{\underset{;}{\downarrow}};P = \overset{\frown}{\underset{;}{\downarrow}}``C`P$ [*150·22]

***182·12.** $\vdash . C`\overset{\frown}{\underset{;}{\downarrow}}`Q = \downarrow Q``C`Q = F\bar{\downarrow} Q$ [*182·05 . *150·22 . *33·5 . (*85·5)]

***182·13.** $\vdash . C``C`\overset{\frown}{\underset{;}{\downarrow}};P = F\bar{\downarrow}``C`P$ [*182·11·12]

***182·14.** $\vdash . F\bar{\downarrow} \epsilon 1 \to 1$

Dem.

$\vdash . *182·12 . \quad \supset \vdash : F\bar{\downarrow} Q = F\bar{\downarrow} R . \supset . \downarrow Q``C`Q = \downarrow R``C`R$ (1)

$\vdash . (1) . *55·232 . *37·45 . \supset \vdash : F\bar{\downarrow} Q = F\bar{\downarrow} R . \dot{\exists} ! Q . \supset . Q = R$ (2)

$\vdash . *37·45 . \quad \supset \vdash : \downarrow Q``C`Q = \downarrow R``C`R . Q = \dot{\Lambda} . \supset . \downarrow R``C`R = \dot{\Lambda} .$

[*37·45.*33·241] $\supset . R = \dot{\Lambda}$ (3)

$\vdash . (1) . (2) . (3) . \quad \supset \vdash : F\bar{\downarrow} Q = F\bar{\downarrow} R . \supset . Q = R : \supset \vdash . \text{Prop}$

***182·15.** $\vdash : \exists ! F\bar{\downarrow} Q \cap F\bar{\downarrow} R . \supset . Q = R$

Dem.

$\vdash . *182·12 . \supset \vdash : \text{Hp} . \supset . \exists ! \downarrow Q``C`Q \cap \downarrow R``C`R .$

[*55·232] $\supset . Q = R : \supset \vdash . \text{Prop}$

***182·16.** $\vdash . \overset{\frown}{\underset{;}{\downarrow}};P \epsilon \text{Rel}^2 \text{excl}$ [*182·12·15]

***182·161.** $\vdash : \overset{\frown}{\underset{;}{\downarrow}}`Q = \overset{\frown}{\underset{;}{\downarrow}}`R . \equiv . Q = R$

Dem.

$\vdash . *182·05 . \supset \vdash : \overset{\frown}{\underset{;}{\downarrow}}`Q = \overset{\frown}{\underset{;}{\downarrow}}`R . \equiv . Q \underset{;}{\downarrow} Q = R \underset{;}{\downarrow} R .$

[*165·23] $\supset . Q = R$ (1)

$\vdash . (1) . *30·37 . \supset \vdash . \text{Prop}$

***182·162.** $\vdash . \overset{\frown}{\underset{;}{\downarrow}} \epsilon 1 \to 1 . \overset{\frown}{\underset{;}{\downarrow}};P \,\text{smor}\, P$ [*182·161 . *71·57 . *151·243]

***182·17.** $\vdash . C`\Sigma`\overset{\frown}{\underset{;}{\downarrow}};P = \hat{S}\{(\exists Q, x) . Q \epsilon C`P . x \epsilon C`Q . S = x \downarrow Q\}$

Dem.

$\vdash . *182·12 . *162·22 . *40·4 . \supset$

$\vdash . C`\Sigma`\overset{\frown}{\underset{;}{\downarrow}};P = \hat{S}\{(\exists Q) . Q \epsilon C`P . S \epsilon \downarrow Q``C`Q\}$

[*55·231] $= \hat{S}\{(\exists Q, x) . Q \epsilon C`P . x \epsilon C`Q . S = x \downarrow Q\} . \supset \vdash . \text{Prop}$

***182·18.** $\vdash . \dot{s}`C`\Sigma`\overset{\frown}{\underset{;}{\downarrow}};P = F\underset{\bullet}{\upharpoonright} C`P$

Dem.

$\vdash . *182·17 . \supset$

$\vdash . \dot{s}`C`\Sigma`\overset{\frown}{\underset{;}{\downarrow}};P = \hat{y}\hat{R}\{(\exists Q, x) . Q \epsilon C`P . x \epsilon C`Q . y(x \downarrow Q) R\}$

$[*55\cdot13] \quad = \hat{y}\hat{R}\{(\exists Q, x) . Q \in C`P . x \in C`Q . y = x . R = Q\}$

$[*13\cdot22] \quad = \hat{y}\hat{R}\{R \in C`P . y \in C`R\}$

$[*33\cdot51] \quad = \hat{y}\hat{R}\{R \in C`P . yFR\}$

$[*35\cdot101] \quad = F \upharpoonright C`P . \supset \vdash . \text{Prop}$

$*182\cdot19$. $\vdash . s`\text{D}``C`\Sigma`\underset{;}{\hat{\downarrow}};P = C`\Sigma`P . s`\text{ɑ}``C`\Sigma`\underset{;}{\hat{\downarrow}};P = C`P - \iota`\Lambda$

Dem.

$\vdash . *41\cdot43 . *182\cdot18 . \supset \vdash . s`\text{D}``C`\Sigma`\underset{;}{\hat{\downarrow}};P = \text{D}`(F \upharpoonright C`P)$

$[*162\cdot23] \quad = C`\Sigma`P \qquad (1)$

$\vdash . *41\cdot44 . *182\cdot18 . \supset \vdash . s`\text{ɑ}``C`\Sigma`\underset{;}{\hat{\downarrow}};P = \text{ɑ}`(F \upharpoonright C`P)$

$[*172\cdot192] \quad = C`P - \iota`\dot{\Lambda} \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

$*182\cdot2$. $\vdash . \underset{;}{\hat{\downarrow}}`Q = \Pi`(Q \downarrow Q) = \Pi`\hat{\downarrow}`Q \quad [*182\cdot05\cdot032 . *172\cdot2]$

$*182\cdot21$. $\vdash . \underset{;}{\hat{\downarrow}};P = \Pi;\hat{\downarrow};P \quad [*182\cdot2]$

The following propositions lead up to $*182\cdot26\cdot27$.

$*182\cdot22$. $\vdash . \text{D};\underset{;}{\hat{\downarrow}}`Q = \text{Prod}`\hat{\downarrow}`Q = \iota;Q \quad [*182\cdot2 . *173\cdot1\cdot22]$

$*182\cdot23$. $\vdash . \breve{\iota};\text{D};\underset{;}{\hat{\downarrow}}`Q = Q \quad [*182\cdot22 . *151\cdot252]$

$*182\cdot24$. $\vdash . \breve{\iota}\dagger;\text{D}\dagger;\underset{;}{\hat{\downarrow}};P = P \quad [*182\cdot23]$

$*182\cdot25$. $\vdash . \breve{\iota};\text{D};\Sigma`\underset{;}{\hat{\downarrow}};P = \Sigma`P . C`\Sigma`\underset{;}{\hat{\downarrow}};P \subset \text{ɑ}`(\breve{\iota} \mid \text{D})$

Dem.

$\vdash . *55\cdot15 . \supset \vdash . \breve{\iota}`\text{D}`\downarrow Q`y = y .$

$[*33\cdot43] \quad \supset \vdash . \downarrow Q`y \in \text{ɑ}`(\breve{\iota} \mid \text{D}) .$

$[*182\cdot12] \quad \supset \vdash . C`\underset{;}{\hat{\downarrow}}`Q \subset \text{ɑ}`(\breve{\iota} \mid \text{D}) .$

$[*162\cdot22] \quad \supset \vdash . C`\Sigma`\underset{;}{\hat{\downarrow}};P \subset \text{ɑ}`(\breve{\iota} \mid \text{D}) .$

$[*162\cdot35] \quad \supset \vdash . \breve{\iota};\text{D};\Sigma`\underset{;}{\hat{\downarrow}};P = \Sigma`\breve{\iota}\dagger;\text{D}\dagger;\underset{;}{\hat{\downarrow}};P$

$[*182\cdot24] \quad = \Sigma`P . \supset \vdash . \text{Prop}$

$*182\cdot26$. $\vdash : P \in \text{Rel}^2 \text{ excl} . \supset . \breve{\iota} \mid \text{D} \upharpoonright C`\Sigma`\underset{;}{\hat{\downarrow}};P \in P \,\overline{\text{smor}}\, \overline{\text{smor}}\, \underset{;}{\hat{\downarrow}};P$

Dem.

$\vdash . *182\cdot24\cdot25 . \supset \vdash . P = (\breve{\iota} \mid \text{D})\dagger;\underset{;}{\hat{\downarrow}};P . C`\Sigma`\underset{;}{\hat{\downarrow}};P \subset \text{ɑ}`(\breve{\iota} \mid \text{D}) \qquad (1)$

$\vdash . *182\cdot17 . *55\cdot15 . \supset$

$\vdash : S, T \in C\text{‘}\Sigma\text{‘}\overset{\frown}{\underset{\cdot\cdot}{\downarrow}} ;P . \breve{\iota}\text{‘}\mathrm{D}\text{‘}S = \breve{\iota}\text{‘}\mathrm{D}\text{‘}T . \supset .$

$\quad (\exists Q, R, x, y) . Q, R \in C\text{‘}P . x \in C\text{‘}Q . y \in C\text{‘}R . x = y . S = x \downarrow Q . T = y \downarrow R .$

$[*13\cdot195] \supset . (\exists Q, R, x) . Q, R \in C\text{‘}P . x \in C\text{‘}Q \cap C\text{‘}R . S = x \downarrow Q . T = x \downarrow R \quad (2)$

$\vdash . (2) . *163\cdot11 . \supset$

$\vdash : \mathrm{Hp} . S, T \in C\text{‘}\Sigma\text{‘}\overset{\frown}{\underset{\cdot\cdot}{\downarrow}} ;P . \breve{\iota}\text{‘}\mathrm{D}\text{‘}S = \breve{\iota}\text{‘}\mathrm{D}\text{‘}T . \supset .$

$\quad (\exists Q, R, x) . Q = R . S = x \downarrow Q . T = x \downarrow R .$

$[*13\cdot195\cdot172] \supset . S = T \quad (3)$

$\vdash . (3) . *71\cdot55 . \supset \vdash : \mathrm{Hp} . \supset . \breve{\iota} \,|\, \mathrm{D} \upharpoonright C\text{‘}\Sigma\text{‘}\overset{\frown}{\underset{\cdot\cdot}{\downarrow}} ;P \in 1 \to 1 \quad (4)$

$\vdash . (1) . (4) . *164\cdot18 . \supset \vdash . \mathrm{Prop}$

$*182\cdot27$. $\vdash : P \in \mathrm{Rel}^2\,\mathrm{excl} . \supset . P \,\mathrm{smor}\,\mathrm{smor}\, \overset{\frown}{\underset{\cdot\cdot}{\downarrow}} ;P \quad [*182\cdot26]$

The following propositions lead up to $*182\cdot33\cdot34$.

$*182\cdot3$. $\vdash : T \upharpoonright C\text{‘}\Sigma\text{‘}Q \in P \,\overline{\mathrm{smor}}\,\overline{\mathrm{smor}}\, Q . \supset . (T \,\|\, \mathrm{Cnv}\text{‘}T\dagger) \upharpoonright C\text{‘}\Sigma\text{‘}\overset{\frown}{\underset{\cdot\cdot}{\downarrow}} ;Q \in 1 \to 1$

Dem.

$\vdash . *74\cdot775 \dfrac{T, T\dagger, C\text{‘}\Sigma\text{‘}\overset{\frown}{\underset{\cdot\cdot}{\downarrow}} ;Q}{Q, \ R, \quad \lambda} . \supset$

$\vdash : T \upharpoonright s\text{‘}\mathrm{D}\text{“}C\text{‘}\Sigma\text{‘}\overset{\frown}{\underset{\cdot\cdot}{\downarrow}} ;Q, T\dagger \upharpoonright s\text{‘}\mathrm{G}\text{“}C\text{‘}\Sigma\text{‘}\overset{\frown}{\underset{\cdot\cdot}{\downarrow}} ;Q \in \mathrm{Cls} \to 1 .$

$\quad s\text{‘}\mathrm{D}\text{“}C\text{‘}\Sigma\text{‘}\overset{\frown}{\underset{\cdot\cdot}{\downarrow}} ;Q \subset \mathrm{G}\text{‘}T . s\text{‘}\mathrm{G}\text{“}C\text{‘}\Sigma\text{‘}\overset{\frown}{\underset{\cdot\cdot}{\downarrow}} ;Q \subset \mathrm{G}\text{‘}T\dagger . \supset .$

$\quad\quad (T \,\|\, \mathrm{Cnv}\text{‘}\breve{T}) \upharpoonright C\text{‘}\Sigma\text{‘}\overset{\frown}{\underset{\cdot\cdot}{\downarrow}} ;Q \in 1 \to 1 \quad (1)$

$\vdash . *182\cdot19 . *164\cdot18 . \supset \vdash : \mathrm{Hp} . \supset . T \upharpoonright s\text{‘}\mathrm{D}\text{“}C\text{‘}\Sigma\text{‘}\overset{\frown}{\underset{\cdot\cdot}{\downarrow}} ;Q \in 1 \to 1 \quad (2)$

$\vdash . *164\cdot18\cdot13 . \quad \supset \vdash : \mathrm{Hp} . \supset . T\dagger \upharpoonright C\text{‘}Q \in 1 \to 1 .$

$[*182\cdot19] \quad \supset . T\dagger \upharpoonright s\text{‘}\mathrm{G}\text{“}C\text{‘}\Sigma\text{‘}\overset{\frown}{\underset{\cdot\cdot}{\downarrow}} ;Q \in 1 \to 1 \quad (3)$

$\vdash . *164\cdot18 . *182\cdot19 . \supset \vdash : \mathrm{Hp} . \supset . s\text{‘}\mathrm{D}\text{“}C\text{‘}\Sigma\text{‘}\overset{\frown}{\underset{\cdot\cdot}{\downarrow}} ;Q \subset \mathrm{G}\text{‘}T \quad (4)$

$\vdash . *150\cdot1 . *33\cdot431 . \supset \vdash . s\text{‘}\mathrm{G}\text{“}C\text{‘}\Sigma\text{‘}\overset{\frown}{\underset{\cdot\cdot}{\downarrow}} ;Q \subset \mathrm{G}\text{‘}T\dagger \quad (5)$

$\vdash . (1) . (2) . (3) . (4) . (5) . \supset \vdash . \mathrm{Prop}$

$*182\cdot31$. $\vdash : \mathrm{E}\,!\,!\, T\text{“}C\text{‘}S . \supset . \overset{\frown}{\underset{\cdot\cdot}{\downarrow}}\text{‘}T ;S = (T \,\|\, \mathrm{Cnv}\text{‘}T\dagger) ; \overset{\frown}{\underset{\cdot\cdot}{\downarrow}}\text{‘}S$

Dem. $\vdash . *182\cdot05 . \supset \vdash . \overset{\frown}{\underset{\cdot\cdot}{\downarrow}}\text{‘}T ;S = (T ;S) \underset{\cdot\cdot}{\downarrow} (T ;S) \quad (1)$

$\vdash . (1) . *165\cdot31 . \supset$

$\vdash : \mathrm{Hp} . \supset . \overset{\frown}{\underset{\cdot\cdot}{\downarrow}}\text{‘}T ;S = T \,|\, ;S \underset{\cdot\cdot}{\downarrow} (T ;S)$

$[*150\cdot1 . *165\cdot321] = T \,|\, ;(\,|\, \mathrm{Cnv}\text{‘}T\dagger) ;S \underset{\cdot\cdot}{\downarrow} S$

$[*150\cdot13 . *182\cdot05] = (T \,\|\, \mathrm{Cnv}\text{‘}T\dagger) ; \overset{\frown}{\underset{\cdot\cdot}{\downarrow}}\text{‘}S : \supset \vdash . \mathrm{Prop}$

$*182{\cdot}32$. $\vdash : E!!\, T``C`\Sigma`Q \,.\supset .\, \overset{\frown}{\underset{;}{\downarrow}} \,;T\dagger ;Q = (T \parallel \mathrm{Cnv}`T\dagger)\dagger ; \overset{\frown}{\underset{;}{\downarrow}} \,;Q$

Dem.

$\vdash .\, *162{\cdot}22 \,.\supset \vdash :.\, \mathrm{Hp} \,.\supset : S \in C`Q \,.\supset_S .\, E!!\, T``C`S \,.$

$[*182{\cdot}31] \qquad \supset_S .\, \overset{\frown}{\underset{;}{\downarrow}} `T;S = (T \parallel \mathrm{Cnv}`T\dagger); \overset{\frown}{\underset{;}{\downarrow}} `S :$

$[*150{\cdot}35{\cdot}1] \qquad \supset : \overset{\frown}{\underset{;}{\downarrow}} \,; T\dagger ;S = (T \parallel \mathrm{Cnv}`T\dagger)\dagger ; \overset{\frown}{\underset{;}{\downarrow}} \,;Q :. \supset \vdash .\, \mathrm{Prop}$

$*182{\cdot}33$. $\vdash : T \upharpoonright C`\Sigma`P \in P \,\overline{\mathrm{smor}}\, \overline{\mathrm{smor}}\, Q \,.\supset .$

$$(T \parallel \mathrm{Cnv}`T\dagger) \upharpoonright C`\Sigma` \overset{\frown}{\underset{;}{\downarrow}} \,;Q \in (\overset{\frown}{\underset{;}{\downarrow}} \,;P)\, \overline{\mathrm{smor}}\, \overline{\mathrm{smor}}\, (\overset{\frown}{\underset{;}{\downarrow}} \,;Q)$$

Dem.

$\vdash .\, *164{\cdot}18 \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, \mathrm{Cl}`T \subset C`\Sigma`Q \,.\, T \upharpoonright C`\Sigma`P \in 1 \rightarrow 1 \,.\, P = T\dagger ;Q \,.$

$[*74{\cdot}11] \qquad \supset .\, E!!\, T``C`\Sigma`Q \,.\, P = T\dagger ;Q \,.$

$[*182{\cdot}32] \qquad \supset .\, \overset{\frown}{\underset{;}{\downarrow}} \,;P = (T \parallel \mathrm{Cnv}`T\dagger)\dagger ; \overset{\frown}{\underset{;}{\downarrow}} \,;Q \qquad (1)$

$\vdash .\, (1) \,.\, *182{\cdot}3 \,.\, *164{\cdot}18 \,.\supset \vdash .\, \mathrm{Prop}$

$*182{\cdot}34$. $\vdash : P \,\mathrm{smor}\,\mathrm{smor}\, Q \,.\supset .\, \overset{\frown}{\underset{;}{\downarrow}} \,;P \,\mathrm{smor}\,\mathrm{smor}\, \overset{\frown}{\underset{;}{\downarrow}} \,;Q \quad [*182{\cdot}33]$

The converse of the above proposition is false. For example, if $Q = \overset{\frown}{\underset{;}{\downarrow}} \,;P$, we shall have $\overset{\frown}{\underset{;}{\downarrow}} \,;P \,\mathrm{smor}\,\mathrm{smor}\, \overset{\frown}{\underset{;}{\downarrow}} \,;Q$, by $*182{\cdot}16{\cdot}27$, but we shall not have $P \,\mathrm{smor}\,\mathrm{smor}\, Q$ unless $P \in \mathrm{Rel}^2\,\mathrm{excl}$, as appears from $*182{\cdot}16$ and $*164{\cdot}23$.

$*182{\cdot}411{\cdot}412$ are lemmas for $*182{\cdot}413$. All the following propositions lead up to $*182{\cdot}42$, which leads to $*182{\cdot}44$.

$*182{\cdot}411$. $\vdash .\, \dot{s}`C` \overset{\frown}{\underset{;}{\downarrow}} \,;P = I \upharpoonright C`P$

Dem. $\vdash .\, *150{\cdot}22 \,.\supset \vdash .\, \dot{s}`C` \overset{\frown}{\underset{;}{\downarrow}} \,;P = \dot{s}` \overset{\frown}{\underset{;}{\downarrow}} ``C`P$

$[*182{\cdot}032] \qquad = \dot{s}`\hat{R}\,\{(\exists y) \,.\, y \in C`P \,.\, R = y \downarrow y\}$

$[*41{\cdot}11] \qquad = \hat{x}\hat{z}\,\{(\exists y) \,.\, y \in C`P \,.\, x\,(y \downarrow y)\,z\}$

$[*55{\cdot}13 . *13{\cdot}195] \qquad = \hat{x}\hat{z}\,\{z \in C`P \,.\, x = z\}$

$[*50{\cdot}1 . *35{\cdot}101] \qquad = I \upharpoonright C`P \,.\supset \vdash .\, \mathrm{Prop}$

$*182{\cdot}412$. $\vdash .\, F ; \overset{\frown}{\underset{;}{\downarrow}} \,;P = P$

Dem.

$\vdash .\, *150{\cdot}11 \,.\, *182{\cdot}032 \,.\supset \vdash .\, F ; \overset{\frown}{\underset{;}{\downarrow}} \,;P = \hat{x}\hat{y}\,\{(\exists z, w) \,.\, zPw \,.\, xF(z \downarrow z) \,.\, yF(w \downarrow w)\}$

$[*33{\cdot}51 . *55{\cdot}15] \qquad = \hat{x}\hat{y}\,\{(\exists z, w) \,.\, zPw \,.\, x = z \,.\, y = w\}$

$[*13{\cdot}22] \qquad = P \,.\supset \vdash .\, \mathrm{Prop}$

$*182{\cdot}413$. $\vdash .\, \Sigma` \overset{\frown}{\underset{;}{\downarrow}} \,;P = P \cup I \upharpoonright C`P \quad [*182{\cdot}411{\cdot}412 \,.\, *162{\cdot}1]$

$*182{\cdot}414$. $\vdash .\, \overset{\frown}{\underset{;}{\downarrow}} \,;P \in \mathrm{Rel}^2\,\mathrm{excl}$

Dem.

$\vdash .\, *150{\cdot}22 \,.\supset \vdash .\, C` \overset{\frown}{\underset{;}{\downarrow}} \,;P = \overset{\frown}{\underset{;}{\downarrow}} ``C`P$

$[*182{\cdot}032] \qquad = \hat{Q}\,\{(\exists x) \,.\, x \in C`P \,.\, Q = x \downarrow x\} \qquad (1)$

$\vdash . (1) . *55 \cdot 15 . \supset \vdash : Q, R \in C`\overset{\frown}{\downarrow};P . \exists ! C`Q \cap C`R . \supset .$

$(\exists x, y) . x, y \in C`P . Q = x \downarrow x . R = y \downarrow y . \exists ! \iota`x \cap \iota`y .$

$[*51 \cdot 231 . \text{Transp}] \supset . (\exists x, y) . Q = x \downarrow x . R = y \downarrow y . x = y .$

$[*13 \cdot 195 \cdot 172] \quad \supset . Q = R : \supset \vdash . \text{Prop}$

$*182 \cdot 415$. $\vdash : Q \in C`\overset{\frown}{\downarrow};P . \supset . C`Q \in 1$

Dem.

$\vdash . *150 \cdot 22 . \supset \vdash : \text{Hp} . \supset . (\exists x) . x \in C`P . Q = x \downarrow x .$

$[*55 \cdot 15] \quad \supset . (\exists x) . x \in C`P . C`Q = \iota`x .$

$[*52 \cdot 1] \quad \supset . C`Q \in 1 : \supset \vdash . \text{Prop}$

The purpose of the above proposition is to enable us to apply $*174 \cdot 221 \cdot 231$ to $\Pi`\Pi;\overset{\frown}{\downarrow};P$, as is done in $*182 \cdot 42 \cdot 43 \cdot 431$ below.

$*182 \cdot 42$. $\vdash . \Pi`P = \dot{s};\text{Prod}`\overset{\frown}{\underset{;}{\downarrow}};P = \dot{s};\text{D};\Pi`\overset{\frown}{\underset{;}{\downarrow}};P = \Pi`\Sigma`\overset{\frown}{\downarrow};P$

Dem.

$\vdash . *182 \cdot 21 . \supset \vdash . \dot{s};\text{D};\Pi`\overset{\frown}{\underset{;}{\downarrow}};P = \dot{s};\text{D};\Pi`\Pi;\overset{\frown}{\downarrow};P$

$[*174 \cdot 221 . *182 \cdot 414 \cdot 415] \quad = \Pi`\Sigma`\overset{\frown}{\downarrow};P \qquad (1)$

$[*182 \cdot 413] \quad = \Pi`(P \cup I \upharpoonright C`P)$

$[*172 \cdot 51] \quad = \Pi`P \qquad (2)$

$\vdash . (1) . (2) . *173 \cdot 1 . \supset \vdash . \text{Prop}$

$*182 \cdot 43$. $\vdash . \dot{s} \upharpoonright (C`\text{Prod}`\overset{\frown}{\underset{;}{\downarrow}};P) \in (\Pi`P) \,\overline{\text{smor}}\, (\text{Prod}`\overset{\frown}{\underset{;}{\downarrow}};P)$

Dem.

$\vdash . *174 \cdot 231 . *182 \cdot 414 \cdot 415 . \supset$

$\vdash . \dot{s} \upharpoonright (C`\text{Prod}`\Pi;\overset{\frown}{\downarrow};P) \in (\Pi`\Sigma`\overset{\frown}{\downarrow};P) \,\overline{\text{smor}}\, (\text{Prod}`\Pi;\overset{\frown}{\downarrow};P) \qquad (1)$

$\vdash . (1) . *182 \cdot 21 \cdot 42 . \supset \vdash . \text{Prop}$

$*182 \cdot 431$. $\vdash . \dot{s} | \text{D} \upharpoonright (C`\Pi`\overset{\frown}{\underset{;}{\downarrow}};P) \in (\Pi`P) \,\overline{\text{smor}}\, (\Pi`\overset{\frown}{\underset{;}{\downarrow}};P)$

$[*174 \cdot 221 . *182 \cdot 414 \cdot 415 \cdot 21 \cdot 42]$

$*182 \cdot 44$. $\vdash . \text{Nr}`\Pi`P = \text{Nr}`\text{Prod}`\overset{\frown}{\underset{;}{\downarrow}};P = \text{Nr}`\Pi`\overset{\frown}{\underset{;}{\downarrow}};P \quad [*182 \cdot 43 \cdot 431 . *152 \cdot 321]$

$*182 \cdot 45$. $\vdash : P \in \text{Rel}^2 \,\text{excl} . \supset . \text{Nr}`\text{Prod}`P = \text{Nr}`\text{Prod}`\overset{\frown}{\underset{;}{\downarrow}};P \quad [*182 \cdot 44 . *173 \cdot 16]$

The following propositions are concerned with cardinals. They show how to express the propositions and definitions of $*112$ in the notation of this number, and they thereby illustrate the analogy of cardinal and ordinal addition.

$*182 \cdot 5$. $\vdash . \epsilon \downarrow \alpha = \overset{\frown}{\underset{,,}{\downarrow}}`\alpha \quad [*182 \cdot 04 . *85 \cdot 601]$

$*182 \cdot 51$. $\vdash . \epsilon \downarrow ``\kappa = \overset{\frown}{\underset{,,}{\downarrow}}``\kappa \quad [*182 \cdot 5]$

***182·52.** $\vdash . \Sigma`\kappa = s`\underset{,,}{\overset{\frown}{\downarrow}}``\kappa . \Sigma\mathrm{Nc}`\kappa = \mathrm{Nc}`s`\underset{,,}{\overset{\frown}{\downarrow}}``\kappa$ [*182·51 . *112·1·101]

***182·53.** $\vdash : C \upharpoonright C`P \in 1 \rightarrow 1 . \supset .$

$$(|\breve{C}) \upharpoonright (C`\Sigma`\underset{\cdot,}{\overset{\frown}{\downarrow}} ; P) \in (\underset{,,}{\overset{\frown}{\downarrow}}``C``C`P) \,\overline{\mathrm{sm}}\,\overline{\mathrm{sm}}\, (C``C`\underset{\cdot,}{\overset{\frown}{\downarrow}} ; P)$$

Dem.

$\vdash . *182·18 . *41·44 . \supset \vdash . s`\text{Ꭰ}``C`\Sigma`\underset{\cdot,}{\overset{\frown}{\downarrow}} ; P = \text{Ꭰ}`(F \upharpoonright C`P)$

[*35·64] $\subset C`P$ (1)

$\vdash . (1) . *74·75 . \supset \vdash : \mathrm{Hp} . \supset . (|\breve{C}) \upharpoonright C`\Sigma`\underset{\cdot,}{\overset{\frown}{\downarrow}} ; P \in 1 \rightarrow 1$ (2)

$\vdash . *55·581 \dfrac{\breve{C}}{S} . \supset \vdash . (x \downarrow Q) | \breve{C} = x \downarrow C`Q .$

[*38·11] $\supset \vdash . |\breve{C}` \downarrow Q`x = \downarrow (C`Q)`x .$

[*182·12·04] $\supset \vdash . |\breve{C}``C`\underset{\cdot,}{\overset{\frown}{\downarrow}}`Q = \underset{,,}{\overset{\frown}{\downarrow}}`C`Q .$

[*150·22] $\supset \vdash . |\breve{C}```C``C`\underset{\cdot,}{\overset{\frown}{\downarrow}} ; P = \underset{,,}{\overset{\frown}{\downarrow}}``C``C`P$ (3)

$\vdash . (2) . (3) . *111·14 . *162·22 . \supset \vdash .$ Prop

***182·54.** $\vdash : C \upharpoonright C`P \in 1 \rightarrow 1 . \supset . \mathrm{Nc}`C`\Sigma`\underset{\cdot,}{\overset{\frown}{\downarrow}} ; P = \Sigma\mathrm{Nc}`C``C`P$

[*182·52·53 . *111·44]

$*183$. THE SUM OF THE RELATION-NUMBERS OF A FIELD

Summary of $*183$.

In this number we have to define and consider the sum of the relation-numbers of the members of $C\text{‘}P$, where P is a relation of relations. Since relational sums are not commutative, we cannot define the sum of the relation-numbers of members of a class of relations λ: it is necessary that λ should be given as the field of a relation P, where P determines the order in which the summation is to be effected.

In order to avoid repetition, we replace P by $\overset{\frown}{\underset{;}{\downarrow}}\,;P$, so that if Q is a member of $C\text{‘}P$, Q is replaced by $\overset{\frown}{\underset{;}{\downarrow}}\,\text{‘}Q$, *i.e.* by $\downarrow Q;Q$. This relation is like Q, and its field has no members in common with the field of $\downarrow R;R$, unless $Q = R$. Hence we are led to the following definition:

$*183\cdot01$. $\Sigma\text{Nr}\text{‘}P = \text{Nr}\text{‘}\Sigma\text{‘}\overset{\frown}{\underset{;}{\downarrow}}\,;P$ Df

This definition is analogous to $*112\cdot01$, as appears from $*182\cdot52$, and the propositions of the present number are analogous to some of the propositions of $*112$.

We have not merely

$*183\cdot11$. $\vdash : P \text{ smor smor } Q \,.\supset.\, \Sigma\text{Nr}\text{‘}P = \Sigma\text{Nr}\text{‘}Q$

but also

$*183\cdot15$. $\vdash : \overset{\frown}{\underset{;}{\downarrow}}\,;P \text{ smor smor } \overset{\frown}{\underset{;}{\downarrow}}\,;Q \,.\supset.\, \Sigma\text{Nr}\text{‘}P = \Sigma\text{Nr}\text{‘}Q$

which is a proposition with a weaker hypothesis than that of $*183\cdot11$ (cf. note to $*182\cdot34$).

Important propositions in this number are

$*183\cdot13$. $\vdash : P \in \text{Rel}^2\text{excl} \,.\supset.\, \text{Nr}\text{‘}\Sigma\text{‘}P = \Sigma\text{Nr}\text{‘}P$

$*183\cdot2$. $\vdash : \Sigma\text{Nr}\text{‘}P = 0_r \,.\equiv.\, \dot{\Sigma}\text{‘}P = \dot{\Lambda}$

I.e. a sum is only zero when there is no summand except (at most) zero. (Cf. $*162\cdot4\cdot45$.)

$*183\cdot25$. $\vdash .\, \Sigma\text{Nr}\text{‘}P \underset{;}{\downarrow}\,;Q = \text{Nr}\text{‘}(Q \times P)$

$*183\cdot26$. $\vdash :.\, \text{Mult ax} \,.\supset : P \in \text{Nr}\text{‘}R \,.\, C\text{‘}P \subset \text{Nr}\text{‘}S \,.\supset.\, \Sigma\text{Nr}\text{‘}P = \text{Nr}\text{‘}(R \times S)$

This proposition connects addition and multiplication.

$*183\cdot31$. $\vdash : P \mathbin{\ddagger} Q \,.\supset.\, \Sigma\text{Nr}\text{‘}(P \downarrow Q) = \text{Nr}\text{‘}P \dotplus \text{Nr}\text{‘}Q$

This proposition connects the two kinds of addition. We have also

$*183\cdot33$. $\vdash : \dot{\exists} ! P \,.\, Z \sim\in C\text{‘}P \,.\dot{\supset}.\, \Sigma\text{Nr}\text{‘}(P \rightarrow\!\!\!\!+ Z) = \Sigma\text{Nr}\text{‘}P \dotplus \text{Nr}\text{‘}Z$

The associative law of addition in a very general form is

$*183{\cdot}43.\quad \vdash : P \in \mathrm{Rel}^2\,\mathrm{excl}\,.\supset .\,\Sigma\mathrm{Nr}\text{‘}\Sigma\,;\,\overset{\frown}{\underset{\cdot\,;}{\downarrow}}\,\dagger\,;P = \Sigma\mathrm{Nr}\text{‘}\Sigma\text{‘}P$

Finally the connection of ordinal and cardinal addition is given by

$*183{\cdot}5.\quad \vdash : C \restriction C\text{‘}P \in 1 \to 1\,.\supset .\,C\text{“}\Sigma\mathrm{Nr}\text{‘}P = \Sigma\mathrm{Nc}\text{‘}C\text{“}C\text{‘}P$

$*183{\cdot}01.\quad \Sigma\mathrm{Nr}\text{‘}P = \mathrm{Nr}\text{‘}\Sigma\text{‘}\overset{\frown}{\underset{\cdot\,;}{\downarrow}}\,;P \quad \mathrm{Df}$

$*183{\cdot}1.\quad \vdash .\,\Sigma\mathrm{Nr}\text{‘}P = \mathrm{Nr}\text{‘}\Sigma\text{‘}\overset{\frown}{\underset{\cdot\,;}{\downarrow}}\,;P \quad [(*183{\cdot}01)]$

$*183{\cdot}11.\quad \vdash : P\,\mathrm{smor\,smor}\,Q\,.\supset .\,\Sigma\mathrm{Nr}\text{‘}P = \Sigma\mathrm{Nr}\text{‘}Q$

Dem.

$\vdash .\,*182{\cdot}34\,.\supset\vdash : \mathrm{Hp}\,.\supset .\,(\overset{\frown}{\underset{\cdot\,;}{\downarrow}}\,;P)\,\mathrm{smor\,smor}\,(\overset{\frown}{\underset{\cdot\,;}{\downarrow}}\,;Q)\,.$

$[*164{\cdot}151] \qquad \supset .\,(\Sigma\text{‘}\overset{\frown}{\underset{\cdot\,;}{\downarrow}}\,;P)\,\mathrm{smor}\,(\Sigma\text{‘}\overset{\frown}{\underset{\cdot\,;}{\downarrow}}\,;Q)\,.$

$[*183{\cdot}1.*152{\cdot}321] \qquad \supset .\,\Sigma\mathrm{Nr}\text{‘}P = \Sigma\mathrm{Nr}\text{‘}Q : \supset\vdash .\,\mathrm{Prop}$

$*183{\cdot}12.\quad \vdash : P\,\mathrm{smor\,smor}\,\overset{\frown}{\underset{\cdot\,;}{\downarrow}}\,;Q\,.\supset .\,\mathrm{Nr}\text{‘}\Sigma\text{‘}P = \Sigma\mathrm{Nr}\text{‘}Q \quad [*164{\cdot}151\,.\,*183{\cdot}1]$

$*183{\cdot}13.\quad \vdash : P \in \mathrm{Rel}^2\,\mathrm{excl}\,.\supset .\,\mathrm{Nr}\text{‘}\Sigma\text{‘}P = \Sigma\mathrm{Nr}\text{‘}P \quad [*182{\cdot}27\,.\,*183{\cdot}12]$

$*183{\cdot}14.\quad \vdash .\,\Sigma\mathrm{Nr}\text{‘}P = \Sigma\mathrm{Nr}\text{‘}\overset{\frown}{\underset{\cdot\,;}{\downarrow}}\,;P$

Dem.

$\vdash .\,*182{\cdot}16\,.\,*183{\cdot}13\,.\supset\vdash .\,\mathrm{Nr}\text{‘}\Sigma\text{‘}\overset{\frown}{\underset{\cdot\,;}{\downarrow}}\,;P = \Sigma\mathrm{Nr}\text{‘}\overset{\frown}{\underset{\cdot\,;}{\downarrow}}\,;P \qquad (1)$

$\vdash .\,(1)\,.\,*183{\cdot}1\,.\supset\vdash .\,\mathrm{Prop}$

$*183{\cdot}15.\quad \vdash : \overset{\frown}{\underset{\cdot\,;}{\downarrow}}\,;P\,\mathrm{smor\,smor}\,\overset{\frown}{\underset{\cdot\,;}{\downarrow}}\,;Q\,.\supset .\,\Sigma\mathrm{Nr}\text{‘}P = \Sigma\mathrm{Nr}\text{‘}Q \quad [*183{\cdot}11{\cdot}14]$

$*183{\cdot}2.\quad \vdash : \Sigma\mathrm{Nr}\text{‘}P = 0_r\,.\equiv .\,\Sigma\text{‘}P = \dot\Lambda$

Dem.

$\vdash .\,*183{\cdot}1\,.\,*153{\cdot}17\,.\supset$

$\vdash :.\,\Sigma\mathrm{Nr}\text{‘}P = 0_r\,.\equiv : \Sigma\text{‘}\overset{\frown}{\underset{\cdot\,;}{\downarrow}}\,;P = \dot\Lambda :$

$[*162{\cdot}42] \qquad \equiv : C\text{‘}\overset{\frown}{\underset{\cdot\,;}{\downarrow}}\,;P \subset \iota\text{‘}\dot\Lambda :$

$[*182{\cdot}05] \qquad \equiv : Q \in C\text{‘}P\,.\supset_Q .\,\downarrow Q\,;Q = \dot\Lambda :$

$[*151{\cdot}65.*153{\cdot}101] \qquad \equiv : Q \in C\text{‘}P\,.\supset_Q .\,Q = \dot\Lambda :$

$[*162{\cdot}42] \qquad \equiv : \Sigma\text{‘}P = \dot\Lambda :.\supset\vdash .\,\mathrm{Prop}$

$*183{\cdot}22.\quad \vdash :.\,\mathrm{Mult\,ax}\,.\supset : \exists\,!\,(\overset{\frown}{\underset{\cdot\,;}{\downarrow}}\,;P)\,\overline{\mathrm{smor}}\,(\overset{\frown}{\underset{\cdot\,;}{\downarrow}}\,;Q) \cap \mathrm{Rl}\text{‘}\mathrm{smor}\,.\supset .\,\Sigma\mathrm{Nr}\text{‘}P = \Sigma\mathrm{Nr}\text{‘}Q$

Dem.

$\vdash .\,*164{\cdot}46\,.\,*182{\cdot}16\,.\supset$

$\vdash :.\,\mathrm{Mult\,ax}\,.\supset : \exists\,!\,(\overset{\frown}{\underset{\cdot\,;}{\downarrow}}\,;P)\,\overline{\mathrm{smor}}\,(\overset{\frown}{\underset{\cdot\,;}{\downarrow}}\,;Q) \cap \mathrm{Rl}\text{‘}\mathrm{smor}\,.\supset .\,\Sigma\text{‘}\overset{\frown}{\underset{\cdot\,;}{\downarrow}}\,;P\,\mathrm{smor}\,\Sigma\text{‘}\overset{\frown}{\underset{\cdot\,;}{\downarrow}}\,;Q\,.$

$[*183{\cdot}1.*152{\cdot}321] \qquad \supset .\,\Sigma\mathrm{Nr}\text{‘}P = \Sigma\mathrm{Nr}\text{‘}Q : \supset\vdash .\,\mathrm{Prop}$

$*183{\cdot}23.\quad \vdash :. \text{Mult ax} . \supset : P, Q \in \text{Rel}^2 \text{excl} . \exists ! P \,\overline{\text{smor}}\, Q \cap \text{Rl}`\text{smor} . \supset .$
$\Sigma\text{Nr}`P = \Sigma\text{Nr}`Q \quad [*164{\cdot}46 . *183{\cdot}13]$

$*183{\cdot}231.\quad \vdash : P \in \text{Nr}`R . C`P \subset \text{Nr}`S . \equiv . \overset{\frown}{\downarrow} \dot{;} P \in \text{Nr}`R . C`\overset{\frown}{\downarrow} \dot{;} P \subset \text{Nr}`S$

Dem.

$\vdash . *182{\cdot}162 . *152{\cdot}31{\cdot}321 . \supset \vdash : P \in \text{Nr}`R . \equiv . \overset{\frown}{\downarrow} \dot{;} P \in \text{Nr}`R \quad (1)$

$\vdash . *182{\cdot}05{\cdot}11 . \supset \vdash : C`\overset{\frown}{\downarrow} \dot{;} P \subset \text{Nr}`S . \equiv : Q \in C`P . \supset_Q . \downarrow Q ; Q \in \text{Nr}`S :$

$[*151{\cdot}65] \quad \equiv : Q \in C`P . \supset_Q . Q \in \text{Nr}`S :$

$[*22{\cdot}1] \quad \equiv : C`P \subset \text{Nr}`S \quad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

$*183{\cdot}24.\quad \vdash :. \text{Mult ax} . \supset : P, Q \in \text{Nr}`R . C`P, C`Q \in \text{Cl}`\text{Nr}`S . \supset . \Sigma\text{Nr}`P = \Sigma\text{Nr}`Q$

Dem.

$\vdash . *183{\cdot}231 . \supset \vdash :. P, Q \in \text{Nr}`R . C`P, C`Q \in \text{Cl}`\text{Nr}`S . \supset :$
$\overset{\frown}{\downarrow} \dot{;} P, \overset{\frown}{\downarrow} \dot{;} Q \in \text{Nr}`R . C`\overset{\frown}{\downarrow} \dot{;} P, C`\overset{\frown}{\downarrow} \dot{;} Q \in \text{Cl}`\text{Nr}`S :$

$[*164{\cdot}48 . *182{\cdot}16] \supset : \text{Mult ax} . \supset . \overset{\frown}{\downarrow} \dot{;} P \text{ smor smor } \overset{\frown}{\downarrow} \dot{;} Q .$

$[*183{\cdot}15] \quad \supset . \Sigma\text{Nr}`P = \Sigma\text{Nr}`Q \quad (1)$

$\vdash . (1) . \text{Comm} . \supset \vdash . \text{Prop}$

$*183{\cdot}25.\quad \vdash . \Sigma\text{Nr}`P \downarrow \dot{;} Q = \text{Nr}`(Q \times P)$

Dem.

$\vdash . *165{\cdot}21 . *183{\cdot}13 . \supset \vdash . \Sigma\text{Nr}`P \downarrow \dot{;} Q = \text{Nr}`\Sigma`P \downarrow \dot{;} Q$

$[*166{\cdot}1] \quad = \text{Nr}`(Q \times P) . \supset \vdash . \text{Prop}$

$*183{\cdot}26.\quad \vdash :. \text{Mult ax} . \supset : P \in \text{Nr}`R . C`P \subset \text{Nr}`S . \supset . \Sigma\text{Nr}`P = \text{Nr}`(R \times S)$

Dem.

$\vdash . *165{\cdot}27 . *183{\cdot}24 . \supset$

$\vdash :. \text{Mult ax} . \supset : \dot{\exists} ! S . P \in \text{Nr}`R . C`P \subset \text{Nr}`S . \supset . \Sigma\text{Nr}`P = \Sigma\text{Nr}`S \downarrow \dot{;} R$

$[*183{\cdot}13 . *166{\cdot}1] \quad = \text{Nr}`(R \times S) \quad (1)$

$\vdash . *153{\cdot}11{\cdot}101 . \supset$

$\vdash : S = \dot{\Lambda} . P \in \text{Nr}`R . C`P \subset \text{Nr}`S . \supset . C`P \subset \iota`\dot{\Lambda} .$

$[*162{\cdot}42] \quad \supset . \Sigma`P = \dot{\Lambda} .$

$[*183{\cdot}2] \quad \supset . \Sigma\text{Nr}`P = 0_r \quad (2)$

$\vdash . *166{\cdot}13 . \supset \vdash : S = \dot{\Lambda} . \supset . R \times S = \dot{\Lambda} \quad (3)$

$\vdash . (2) . (3) . *153{\cdot}17 . \supset$

$\vdash : S = \dot{\Lambda} . P \in \text{Nr}`R . C`P \subset \text{Nr}`S . \supset . \Sigma\text{Nr}`P = \text{Nr}`(R \times S) \quad (4)$

$\vdash . (1) . (4) . \supset \vdash . \text{Prop}$

$*183{\cdot}3.\quad \vdash . \Sigma\text{Nr}`\dot{\Lambda} = 0_r \quad [*183{\cdot}2 . *162{\cdot}4]$

$*183{\cdot}301.\quad \vdash . \Sigma\text{Nr}`(\dot{\Lambda} \downarrow \dot{\Lambda}) = 0_r \quad [*183{\cdot}2 . *162{\cdot}41]$

∗183·302. $\vdash . \Sigma\mathrm{Nr}‘(P \downarrow P) = \mathrm{Nr}‘(C‘P \uparrow C‘P)$

Dem.

$\vdash . {*}183{\cdot}13 . {*}163{\cdot}41 . \supset \vdash . \Sigma\mathrm{Nr}‘(P \downarrow P) = \mathrm{Nr}‘\Sigma‘(P \downarrow P)$

[∗162·3.∗160·1] $\quad = \mathrm{Nr}‘(C‘P \uparrow C‘P) . \supset \vdash . \mathrm{Prop}$

∗183·31. $\vdash : P \ddagger Q . \supset . \Sigma\mathrm{Nr}‘(P \downarrow Q) = \mathrm{Nr}‘P \dotplus \mathrm{Nr}‘Q$

Dem.

$\vdash . {*}183{\cdot}1 . \supset \vdash . \Sigma\mathrm{Nr}‘(P \downarrow Q) = \mathrm{Nr}‘\Sigma‘\overset{\frown}{\underset{;}{\downarrow}};(P \downarrow Q)$

[∗150·71] $\quad = \mathrm{Nr}‘\Sigma‘\{(\overset{\frown}{\underset{;}{\downarrow}}‘P) \downarrow (\overset{\frown}{\underset{;}{\downarrow}}‘Q)\}$

[∗162·3] $\quad = \mathrm{Nr}‘(\overset{\frown}{\underset{;}{\downarrow}}‘P \mathbin{\overset{\uparrow}{+}} \overset{\frown}{\underset{;}{\downarrow}}‘Q) \qquad (1)$

$\vdash . (1) . {*}180{\cdot}32 . {*}182{\cdot}12{\cdot}15 . \supset \vdash : \mathrm{Hp} . \supset .$

$\Sigma\,\mathrm{Nr}‘(P \downarrow Q) = \mathrm{Nr}‘\overset{\frown}{\underset{;}{\downarrow}}‘P \dotplus \mathrm{Nr}‘\overset{\frown}{\underset{;}{\downarrow}}‘Q$

[∗182·05.∗151·65.∗180·31] $\quad = \mathrm{Nr}‘P \dotplus \mathrm{Nr}‘Q : \supset \vdash . \mathrm{Prop}$

∗183·32. $\vdash : C‘P \cap C‘Q = \Lambda . \supset . \Sigma\mathrm{Nr}‘(P \mathbin{\overset{\uparrow}{+}} Q) = \Sigma\mathrm{Nr}‘P \dotplus \Sigma\mathrm{Nr}‘Q$

Dem.

$\vdash . {*}183{\cdot}1 . \supset \vdash . \Sigma\mathrm{Nr}‘(P \mathbin{\overset{\uparrow}{+}} Q) = \mathrm{Nr}‘\Sigma‘\overset{\frown}{\underset{;}{\downarrow}};(P \mathbin{\overset{\uparrow}{+}} Q)$

[∗162·31.∗160·44] $\quad = \mathrm{Nr}‘(\Sigma‘\overset{\frown}{\underset{;}{\downarrow}};P \mathbin{\overset{\uparrow}{+}} \Sigma‘\overset{\frown}{\underset{;}{\downarrow}};Q) \qquad (1)$

$\vdash . {*}182{\cdot}17 . {*}55{\cdot}202 . \supset$

$\vdash : \exists ! C‘\Sigma‘\overset{\frown}{\underset{;}{\downarrow}};P \cap C‘\Sigma‘\overset{\frown}{\underset{;}{\downarrow}};Q . \equiv . (\exists S, R, x) . R \in C‘P \cap C‘Q . x \in C‘R . S = x \downarrow R .$

[∗10·5] $\quad \supset . \exists ! C‘P \cap C‘Q \qquad (2)$

$\vdash . (2) . \mathrm{Transp} . \supset \vdash : \mathrm{Hp} . \supset . C‘\Sigma‘\overset{\frown}{\underset{;}{\downarrow}};P \cap C‘\Sigma‘\overset{\frown}{\underset{;}{\downarrow}};Q = \Lambda .$

[∗180·32] $\quad \supset . \mathrm{Nr}‘(\Sigma‘\overset{\frown}{\underset{;}{\downarrow}};P \mathbin{\overset{\uparrow}{+}} \Sigma‘\overset{\frown}{\underset{;}{\downarrow}};Q) = \mathrm{Nr}‘\Sigma‘\overset{\frown}{\underset{;}{\downarrow}};P \dotplus \mathrm{Nr}‘\Sigma‘\overset{\frown}{\underset{;}{\downarrow}};Q$

[∗183·1] $\quad = \Sigma\mathrm{Nr}‘P \dotplus \Sigma\mathrm{Nr}‘Q \qquad (3)$

$\vdash . (1) . (3) . \supset \vdash . \mathrm{Prop}$

∗183·33. $\vdash : \dot{\exists} ! P . Z \sim\in C‘P . \supset . \Sigma\mathrm{Nr}‘(P \mathbin{+\!\!\rightarrow} Z) = \Sigma\mathrm{Nr}‘P \dotplus \mathrm{Nr}‘Z$

Dem.

$\vdash . {*}183{\cdot}1 . \supset \vdash . \Sigma\mathrm{Nr}‘(P \mathbin{+\!\!\rightarrow} Z) = \mathrm{Nr}‘\Sigma‘\overset{\frown}{\underset{;}{\downarrow}};(P \mathbin{+\!\!\rightarrow} Z)$

[∗161·4] $\quad = \mathrm{Nr}‘\Sigma‘(\overset{\frown}{\underset{;}{\downarrow}};P \mathbin{+\!\!\rightarrow} \overset{\frown}{\underset{;}{\downarrow}}‘Z) \qquad (1)$

$\vdash . (1) . {*}162{\cdot}43 . \supset \vdash : \mathrm{Hp} . \supset . \Sigma\mathrm{Nr}‘(P \mathbin{+\!\!\rightarrow} Z) = \mathrm{Nr}‘(\Sigma‘\overset{\frown}{\underset{;}{\downarrow}};P \mathbin{\overset{\uparrow}{+}} \overset{\frown}{\underset{;}{\downarrow}}‘Z)$

[∗182·12·15.∗180·32] $\quad = \mathrm{Nr}‘\Sigma‘\overset{\frown}{\underset{;}{\downarrow}};P \dotplus \mathrm{Nr}‘\overset{\frown}{\underset{;}{\downarrow}}‘Z$

[∗183·1.∗182·05.∗151·65] $\quad = \Sigma\mathrm{Nr}‘P \dotplus \mathrm{Nr}‘Z : \supset \vdash . \mathrm{Prop}$

∗183·331. $\vdash : \dot{\exists} ! P . Z \sim\in C‘P . \supset . \Sigma\mathrm{Nr}‘(Z \mathbin{\leftarrow\!\!+} P) = \mathrm{Nr}‘Z \dotplus \Sigma\mathrm{Nr}‘P$

[Proof as in ∗183·33]

$*183{\cdot}42$. $\vdash : P \in \mathrm{Rel}^2\,\mathrm{excl} \,.\supset .\, \overset{\frown}{\underset{\cdot,}{\downarrow}}\dagger{}^{;}P \in \mathrm{Rel}^3\,\mathrm{arithm}$

Dem.

$$\vdash .\, *163{\cdot}3 \,.\, *182{\cdot}162 \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, \overset{\frown}{\underset{\cdot,}{\downarrow}}\dagger{}^{;}P \in \mathrm{Rel}^2\,\mathrm{excl} \qquad (1)$$

$$\vdash .\, *162{\cdot}35 \,. \qquad \supset \vdash .\, \Sigma{}^{\backprime}\overset{\frown}{\underset{\cdot,}{\downarrow}}\dagger{}^{;}P = \overset{\frown}{\underset{\cdot,}{\downarrow}}{}^{;}\Sigma{}^{\backprime}P \,.$$

$$[*182{\cdot}16] \qquad \supset \vdash .\, \Sigma{}^{\backprime}\overset{\frown}{\underset{\cdot,}{\downarrow}}\dagger{}^{;}P \in \mathrm{Rel}^2\,\mathrm{excl} \qquad (2)$$

$$\vdash .\, (1) \,.\, (2) \,.\, *174{\cdot}3 \,.\supset \vdash .\, \mathrm{Prop}$$

$*183{\cdot}43$. $\vdash : P \in \mathrm{Rel}^2\,\mathrm{excl} \,.\supset .\, \Sigma\mathrm{Nr}{}^{\backprime}\Sigma{}^{;}\overset{\frown}{\underset{\cdot,}{\downarrow}}\dagger{}^{;}P = \Sigma\mathrm{Nr}{}^{\backprime}\Sigma{}^{\backprime}P$

This is a form of the associative law of addition.

Dem.

$$\vdash .\, *183{\cdot}42 \,.\, *174{\cdot}36 \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, \Sigma{}^{;}\overset{\frown}{\underset{\cdot,}{\downarrow}}\dagger{}^{;}P \in \mathrm{Rel}^2\,\mathrm{excl} \,.$$

$$[*183{\cdot}13] \qquad \supset .\, \Sigma\mathrm{Nr}{}^{\backprime}\Sigma{}^{;}\overset{\frown}{\underset{\cdot,}{\downarrow}}\dagger{}^{;}P = \mathrm{Nr}{}^{\backprime}\Sigma{}^{\backprime}\Sigma{}^{;}\overset{\frown}{\underset{\cdot,}{\downarrow}}\dagger{}^{;}P$$

$$[*162{\cdot}34] \qquad = \mathrm{Nr}{}^{\backprime}\Sigma{}^{\backprime}\Sigma{}^{\backprime}\overset{\frown}{\underset{\cdot,}{\downarrow}}\dagger{}^{;}P$$

$$[*162{\cdot}35] \qquad = \mathrm{Nr}{}^{\backprime}\Sigma{}^{\backprime}\overset{\frown}{\underset{\cdot,}{\downarrow}}{}^{;}\Sigma{}^{\backprime}P$$

$$[*183{\cdot}1] \qquad = \Sigma\mathrm{Nr}{}^{\backprime}\Sigma{}^{\backprime}P : \supset \vdash .\, \mathrm{Prop}$$

$*183{\cdot}5$. $\vdash : C \upharpoonright C{}^{\backprime}P \in 1 \to 1 \,.\supset\dot{}\, C{}^{\backprime\backprime}\Sigma\mathrm{Nr}{}^{\backprime}P = \Sigma\mathrm{Nc}{}^{\backprime}C{}^{\backprime\backprime}C{}^{\backprime}P$

Dem.

$$\vdash .\, *152{\cdot}7 \,.\, *183{\cdot}1 \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, C{}^{\backprime\backprime}\Sigma\mathrm{Nr}{}^{\backprime}P = \mathrm{Nc}{}^{\backprime}C{}^{\backprime}\Sigma{}^{\backprime}\overset{\frown}{\underset{\cdot,}{\downarrow}}{}^{;}P$$

$$[*182{\cdot}54] \qquad = \Sigma\mathrm{Nc}{}^{\backprime}C{}^{\backprime\backprime}C{}^{\backprime}P \,.\supset \vdash .\, \mathrm{Prop}$$

$*184$. THE PRODUCT OF TWO RELATION-NUMBERS

Summary of $*184$.

The propositions of this number are for the most part analogous to those of the propositions of $*113$ which are concerned with $\mu \times_0 \nu$. Those of $*113$ which are concerned with $\alpha \times \beta$ have their analogues in $*166$. We put

$*184{\cdot}01$. $\mu \dot{\times} \nu = \hat{R}\{(\exists P, Q) . \mu = \mathrm{N_0r}`P . \nu = \mathrm{N_0r}`Q . R \operatorname{smor} (P \times Q)\}$ Df

$*184{\cdot}02$. $\mathrm{Nr}`P \dot{\times} \nu = \mathrm{N_0r}`P \dot{\times} \nu$ Df

$*184{\cdot}03$. $\mu \dot{\times} \mathrm{Nr}`Q = \mu \dot{\times} \mathrm{N_0r}`Q$ Df

We prove that $\mu \dot{\times} \nu$ is only zero when one of its factors is zero ($*184{\cdot}16$); we prove the associative law ($*184{\cdot}31$), and the distributive law in the forms

$*184{\cdot}33$. $\vdash : P \in \mathrm{Rel}^2 \mathrm{excl} . \supset . \Sigma \mathrm{Nr}`P \dot{\times} \mathrm{Nr}`R = \Sigma \mathrm{Nr}`(\times R) ; P$

$*184{\cdot}35$. $\vdash . (\nu \dot{+} \varpi) \dot{\times} \mu = (\nu \dot{\times} \mu) \dot{+} (\varpi \dot{\times} \mu)$

and we prove $2_r \dot{\times} \mu = \mu \dot{+} \mu$ ($*184{\cdot}4$). Also we extend the distributive law to the case where one of the summands is $\dot{1}$, *i.e.* we prove

$*184{\cdot}41$. $\vdash : \nu \neq 0_r . \supset . (\nu \dot{+} \dot{1}) \dot{\times} \mu = (\nu \dot{\times} \mu) \dot{+} \mu$

$*184{\cdot}42$. $\vdash : \nu \neq 0_r . \supset . (\dot{1} \dot{+} \nu) \dot{\times} \mu = \mu \dot{+} (\nu \dot{\times} \mu)$

and the connection of cardinal and ordinal multiplication is given by

$*184{\cdot}5$. $\vdash : \mu, \nu \in \mathrm{NR} . \supset . C``(\mu \dot{\times} \nu) = C``\mu \times_0 C``\nu$

$*184{\cdot}01$. $\mu \dot{\times} \nu = \hat{R}\{(\exists P, Q) . \mu = \mathrm{N_0r}`P . \nu = \mathrm{N_0r}`Q . R \operatorname{smor} (P \times Q)\}$ Df

$*184{\cdot}02$. $\mathrm{Nr}`P \dot{\times} \nu = \mathrm{N_0r}`P \dot{\times} \nu$ Df

$*184{\cdot}03$. $\mu \dot{\times} \mathrm{Nr}`Q = \mu \dot{\times} \mathrm{N_0r}`Q$ Df

$*184{\cdot}1$. $\vdash : R \in \mu \dot{\times} \nu . \equiv . (\exists P, Q) . \mu = \mathrm{N_0r}`P . \nu = \mathrm{N_0r}`Q . R \operatorname{smor} (P \times Q)$
[($*184{\cdot}01$)]

The proofs of the following propositions are omitted, since they are analogous to those of the corresponding propositions of $*113$.

$*184{\cdot}11$. $\vdash : \exists ! \mu \dot{\times} \nu . \supset . \mu, \nu \in \mathrm{N_0R} . \exists ! \mu . \exists ! \nu$

$*184{\cdot}111$. $\vdash : \sim(\mu, \nu \in \mathrm{N_0R}) . \supset . \mu \dot{\times} \nu = \Lambda$

$*184{\cdot}12$. $\vdash :. \mu, \nu \in \mathrm{NR} . \supset : R \in \mu \dot{\times} \nu . \equiv . (\exists P, Q) . P \in \mu . Q \in \nu . R \operatorname{smor} (P \times Q)$

$*184{\cdot}13$. $\vdash . \mathrm{Nr}`P \dot{\times} \mathrm{Nr}`Q = \mathrm{N_0r}`P \dot{\times} \mathrm{Nr}`Q = \mathrm{Nr}`P \dot{\times} \mathrm{N_0r}`Q$
$= \mathrm{N_0r}`P \dot{\times} \mathrm{N_0r}`Q = \mathrm{Nr}`(P \times Q)$

$*184{\cdot}14$. $\vdash : P \operatorname{smor} R . Q \operatorname{smor} S . \supset . \mathrm{Nr}`P \dot{\times} \mathrm{Nr}`Q = \mathrm{Nr}`R \dot{\times} \mathrm{Nr}`S$

***184·15.** $\vdash . \mu \dot{\times} \nu \in \mathrm{NR}$

***184·16.** $\vdash :. \mu \dot{\times} \nu = 0_r . \equiv : \mu, \nu \in \mathrm{NR} - \iota`\Lambda : \mu = 0_r . \vee . \nu = 0_r$

***184·2.** $\vdash :. \text{Mult ax} . \supset : P \in \mathrm{Nr}`R . C`P \subset \mathrm{Nr}`S . \supset . \Sigma\mathrm{Nr}`P = \mathrm{Nr}`R \dot{\times} \mathrm{Nr}`S$
[*183·26 . *184·13]

***184·21.** $\vdash :. \text{Mult ax} . \supset : \mu, \nu \in \mathrm{NR} . \nu \neq \Lambda . P \in \mu . C`P \subset \nu . \supset . \Sigma\mathrm{Nr}`P = \mu \dot{\times} \nu$

Dem.

$\vdash . *152·45 . \supset \vdash : \mu \in \mathrm{NR} . P \in \mu . \supset . \mu = \mathrm{Nr}`P$ (1)

$\vdash . *152·45 . \supset \vdash : \nu \in \mathrm{NR} . C`P \subset \nu . S \in C`P . \supset . \nu = \mathrm{Nr}`S . C`P \subset \mathrm{Nr}`S$ (2)

$\vdash . (1) . (2) . *184·2 . \supset$

$\vdash :. \text{Mult ax} . \supset : \mu, \nu \in \mathrm{NR} . P \in \mu . C`P \subset \nu . S \in C`P . \supset .$

$\Sigma\mathrm{Nr}`P = \mathrm{Nr}`P \dot{\times} \mathrm{Nr}`S . \mu = \mathrm{Nr}`P . \nu = \mathrm{Nr}`S .$

[*13·13] $\supset . \Sigma\mathrm{Nr}`P = \mu \dot{\times} \nu$ (3)

$\vdash . (3) . *10·11·21·23 . \supset$

$\vdash :. \text{Mult ax} . \supset : \mu, \nu \in \mathrm{NR} . P \in \mu . C`P \subset \nu . \exists ! C`P . \supset . \Sigma\mathrm{Nr}`P = \mu \dot{\times} \nu$ (4)

$\vdash . *183·2 . *162·4 . \quad \supset \vdash : P = \dot{\Lambda} . \supset . \Sigma\mathrm{Nr}`P = 0_r$ (5)

$\vdash . *153·16 . \text{Transp} . \supset \vdash :. \mu \in \mathrm{NR} . P \in \mu . P = \dot{\Lambda} . \supset : \mu = 0_r :$

[*184·16] $\supset : \nu \in \mathrm{NR} - \iota`\Lambda . \supset . \mu \dot{\times} \nu = 0_r$ (6)

$\vdash . (5) . (6) . \supset \vdash : \mu, \nu \in \mathrm{NR} . \nu \neq \Lambda . P \in \mu . C`P \subset \nu . P = \dot{\Lambda} . \supset .$

$\Sigma\mathrm{Nr}`P = \mu \dot{\times} \nu$ (7)

$\vdash . (4) . (7) . \supset \vdash . \text{Prop}$

***184·3.** $\vdash . (\mathrm{Nr}`P \dot{\times} \mathrm{Nr}`Q) \dot{\times} \mathrm{Nr}`R = \mathrm{Nr}`P \dot{\times} (\mathrm{Nr}`Q \dot{\times} \mathrm{Nr}`R) = \mathrm{Nr}`(P \times Q \times R)$

Dem.

$\vdash . *184·13 . \supset \vdash . (\mathrm{Nr}`P \dot{\times} \mathrm{Nr}`Q) \dot{\times} \mathrm{Nr}`R = \mathrm{Nr}`(P \times Q) \dot{\times} \mathrm{Nr}`R$

[*184·13] $= \mathrm{Nr}`(P \times Q \times R)$

[*166·42] $= \mathrm{Nr}`\{P \times (Q \times R)\}$

[*184·13] $= \mathrm{Nr}`P \dot{\times} (\mathrm{Nr}`Q \dot{\times} \mathrm{Nr}`R) . \supset \vdash . \text{Prop}$

***184·31.** $\vdash . (\mu \dot{\times} \nu) \dot{\times} \varpi = \mu \dot{\times} (\nu \dot{\times} \varpi)$

Dem.

$\vdash . *184·111 . \supset \vdash : \sim(\mu, \nu, \varpi \in \mathrm{N_0R}) . \supset . (\mu \dot{\times} \nu) \dot{\times} \varpi = \Lambda . \mu \dot{\times} (\nu \dot{\times} \varpi) = \Lambda$ (1)

$\vdash . *155·2 . \quad \supset \vdash : \mu, \nu, \varpi \in \mathrm{N_0R} . \supset .$

$(\exists P, Q, R) . \mu = \mathrm{N_0r}`P . \nu = \mathrm{N_0r}`Q . \varpi = \mathrm{N_0r}`R$ (2)

$\vdash . *184·13 . \supset$

$\vdash : \mu = \mathrm{N_0r}`P . \nu = \mathrm{N_0r}`Q . \varpi = \mathrm{N_0r}`R . \supset . (\mu \dot{\times} \nu) \dot{\times} \varpi = \mathrm{Nr}`\{(P \times Q) \times R\}$

[*184·3] $= \mathrm{Nr}`\{P \times (Q \times R)\}$

[*184·13] $= \mu \dot{\times} (\nu \dot{\times} \varpi)$ (3)

$\vdash . (2) . (3) . \supset \vdash : \mu, \nu, \varpi \in \mathrm{N_0R} . \supset . (\mu \dot{\times} \nu) \dot{\times} \varpi = \mu \dot{\times} (\nu \dot{\times} \varpi)$ (4)

$\vdash . (1) . (4) . \supset \vdash . \text{Prop}$

***184·32.** $\mu \dot{\times} \nu \dot{\times} \varpi = (\mu \dot{\times} \nu) \dot{\times} \varpi \quad \text{Df}$

***184·33.** $\vdash : P \in \text{Rel}^2\,\text{excl} . \supset . \Sigma\text{Nr}`P \dot{\times} \text{Nr}`R = \Sigma\text{Nr}`(\times R) ; P$

Dem.

$\vdash . *183{\cdot}13 . \supset \vdash : \text{Hp} . \supset . \Sigma\text{Nr}`P \dot{\times} \text{Nr}`R = \text{Nr}`\Sigma`P \dot{\times} \text{Nr}`R$

$[*184{\cdot}13] \quad = \text{Nr}`(\Sigma`P \times R)$

$[*166{\cdot}44] \quad = \text{Nr}`\Sigma`(\times R) ; P \qquad (1)$

$\vdash . *166{\cdot}3 . \supset \vdash : \text{Hp} . \supset . (\times R) ; P \in \text{Rel}^2\,\text{excl} .$

$[*183{\cdot}13] \quad \supset . \text{Nr}`\Sigma`(\times R) ; P = \Sigma\text{Nr}`(\times R) ; P \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***184·34.** $\vdash . (\text{Nr}`P \dot{+} \text{Nr}`Q) \dot{\times} \text{Nr}`R = (\text{Nr}`P \dot{\times} \text{Nr}`R) \dot{+} (\text{Nr}`Q \dot{\times} \text{Nr}`R)$

Dem.

$\vdash . *180{\cdot}3 . *184{\cdot}13 . \supset$

$\vdash . (\text{Nr}`P \dot{+} \text{Nr}`Q) \dot{\times} \text{Nr}`R = \text{Nr}`\{(P + Q) \times R\}$

$[*166{\cdot}45 . *180{\cdot}1] \quad = \text{Nr}`[\{\downarrow (\Lambda \cap C`Q) ; \iota ; P\} \times R \,\ddagger\, \{(\Lambda \cap C`P) \downarrow ; \iota ; Q\} \times R]$

$[*166{\cdot}3 . *180{\cdot}11{\cdot}32] = \text{Nr}`[\{\downarrow (\Lambda \cap C`Q) ; \iota ; P\} \times R] \dot{+} \text{Nr}`[\{(\Lambda \cap C`P) \downarrow ; \iota ; Q\} \times R]$

$[*184{\cdot}14 . *180{\cdot}12] \quad = \text{Nr}`(P \times R) \dot{+} \text{Nr}`(Q \times R)$

$[*184{\cdot}13] \quad = (\text{Nr}`P \dot{\times} \text{Nr}`Q) \dot{+} (\text{Nr}`Q \dot{\times} \text{Nr}`R) . \supset \vdash . \text{Prop}$

***184·35.** $\vdash . (\nu \dot{+} \varpi) \dot{\times} \mu = (\nu \dot{\times} \mu) \dot{+} (\varpi \dot{\times} \mu) \quad [*184{\cdot}34]$

The proof proceeds as in *184·31.

***184·4.** $\vdash . 2_r \dot{\times} \mu = \mu \dot{+} \mu$

Dem.

$\vdash . *184{\cdot}111 . *180{\cdot}4 . \supset \vdash : \mu \sim\in \text{N}_0\text{R} . \supset . 2_r \dot{\times} \mu = \Lambda . \mu \dot{+} \mu = \Lambda \qquad (1)$

$\vdash . *153{\cdot}24 . *184{\cdot}13 . \supset$

$\vdash : \mu = \text{N}_0\text{r}`P . \supset . 2_r \dot{\times} \mu = \text{Nr}`\{\Lambda \downarrow (\iota`x) \times P\} \qquad (2)$

$\vdash . *166{\cdot}1 . \supset \vdash . \Lambda \downarrow (\iota`x) \times P = \Sigma`P \underset{\cdot\cdot}{\downarrow} ; (\Lambda \downarrow \iota`x)$

$[*150{\cdot}71] \quad = \Sigma`\{(P \underset{\cdot\cdot}{\downarrow} \Lambda) \downarrow (P \underset{\cdot\cdot}{\downarrow} \iota`x)\}$

$[*162{\cdot}3] \quad = (P \underset{\cdot\cdot}{\downarrow} \Lambda) \ddagger (P \underset{\cdot\cdot}{\downarrow} \iota`x) \qquad (3)$

$\vdash . *180{\cdot}31{\cdot}32 . *165{\cdot}251{\cdot}211 . \text{Transp} . \supset$

$\vdash . \text{Nr}`\{(P \underset{\cdot\cdot}{\downarrow} \Lambda) \ddagger (P \underset{\cdot\cdot}{\downarrow} \iota`x)\} = \text{Nr}`P \dot{+} \text{Nr}`P \qquad (4)$

$\vdash . (2) . (3) . (4) . \supset \vdash : \mu = \text{N}_0\text{r}`P . \supset . 2_r \dot{\times} \mu = \text{Nr}`P \dot{+} \text{Nr}`P$

$[*180{\cdot}3] \quad = \mu \dot{+} \mu \qquad (5)$

$\vdash . (1) . (5) . \supset \vdash . \text{Prop}$

***184·41.** $\vdash : \nu \neq 0_r . \supset . (\nu \dot{+} \dot{1}) \dot{\times} \mu = (\nu \dot{\times} \mu) \dot{+} \mu$

Dem.

$\vdash . *166{\cdot}53 . *180{\cdot}32 . *165{\cdot}251 . \supset$

$\vdash : \dot{\exists} ! Q . y \sim\in C`Q . \supset . \text{Nr}`\{(Q \rightarrow y) \times P\} = \text{Nr}`(Q \times P) \dot{+} \text{Nr}`P \qquad (1)$

$\vdash . (1) . *181{\cdot}32 . *184{\cdot}13 . \supset$

$\vdash : \mu = \text{Nr}`P . \nu = \text{Nr}`Q . \nu \neq 0_r . y \sim\in C`Q . \supset . (\nu \dot{+} \dot{1}) \dot{\times} \mu = (\nu \dot{\times} \mu) \dot{+} \mu \qquad (2)$

$\vdash . *181{\cdot}11{\cdot}12 . \supset \vdash : \nu \in \text{NR} - \iota`\Lambda . \supset . (\exists Q, y) . \nu = \text{Nr}`Q . y \sim\in C`Q \qquad (3)$

$\vdash . (2) . (3) . \supset$

$\vdash : \mu \in \mathrm{NR} \,.\, \nu \in \mathrm{NR} - \iota`\Lambda \,.\, \nu \neq 0_r \,.\, \supset \,.\, (\nu \dot{+} \dot{1}) \dot{\times} \mu = (\nu \dot{\times} \mu) \dot{+} \mu$ (4)

$\vdash \,.\, *184{\cdot}111 \,.\, *181{\cdot}4 \,.\, \supset$

$\vdash : \sim(\mu \in \mathrm{NR} \,.\, \nu \in \mathrm{NR} - \iota`\Lambda) \,.\, \supset \,.\, (\nu \dot{+} 1) \dot{\times} \mu = \Lambda \,.\, (\nu \dot{\times} \mu) \dot{+} \mu = \Lambda$ (5)

$\vdash \,.\, (4) \,.\, (5) \,.\, \supset \vdash \,.\,$ Prop

***184·42.** $\vdash : \nu \neq 0_r \,.\, \supset \,.\, (\dot{1} \dot{+} \nu) \dot{\times} \mu = \mu \dot{+} (\nu \dot{\times} \mu)$ [Proof as in *184·41]

***184·5.** $\vdash : \mu, \nu \in \mathrm{NR} \,.\, \supset \,.\, C``(\mu \dot{\times} \nu) = C``\mu \times_0 C``\nu$

Dem.

$\vdash \,.\, *184{\cdot}13 \,.\, \supset \vdash : \mathrm{Hp} \,.\, P \in \mu \,.\, Q \in \nu \,.\, \supset \,.\, C``(\mu \dot{\times} \nu) = C``\mathrm{Nr}`(P \times Q)$

[*152·7.*166·12] $= \mathrm{Nc}`(C`P \times C`Q)$

[*152·7.*113·25] $= C``\mathrm{Nr}`P \times_0 C``\mathrm{Nr}`Q$

[*152·45] $= C``\mu \times_0 C``\nu$ (1)

$\vdash \,.\, *184{\cdot}11 \,.\, *113{\cdot}204 \,.\, \supset$

$\vdash : \sim(\exists ! \mu \,.\, \exists ! \nu) \,.\, \supset \,.\, C``(\mu \dot{\times} \nu) = \Lambda \,.\, C``\mu \times_0 C``\nu = \Lambda$ (2)

$\vdash \,.\, (1) \,.\, (2) \,.\, \supset \vdash \,.\,$ Prop

*185. THE PRODUCT OF THE RELATION-NUMBERS OF A FIELD

Summary of *185.

The subject of this number is analogous to part of the subject of *114. The propositions concerned are immediate consequences of previously proved properties of $\Pi{}^{\backprime}P$, and offer no difficulty of any kind.

***185·01.** $\Pi\text{Nr}{}^{\backprime}P = \text{Nr}{}^{\backprime}\Pi{}^{\backprime}P$ Df

***185·1.** $\vdash . \Pi\text{Nr}{}^{\backprime}P = \text{Nr}{}^{\backprime}\Pi{}^{\backprime}P$ [(*185·01)]

***185·11.** $\vdash : P \,\text{smor smor}\, Q . \supset . \Pi\text{Nr}{}^{\backprime}P = \Pi\text{Nr}{}^{\backprime}Q$ [*172·44]

***185·12.** $\vdash . \Pi\text{Nr}{}^{\backprime}P = \text{Nr}{}^{\backprime}\text{Prod}{}^{\backprime}\overset{\frown}{\downarrow};P = \text{Nr}{}^{\backprime}\Pi{}^{\backprime}\overset{\frown}{\downarrow};P$ [*182·44]

***185·2.** $\vdash . \Pi\text{Nr}{}^{\backprime}\dot{\Lambda} = 0_r$ [*172·13]

***185·21.** $\vdash . \Pi\text{Nr}{}^{\backprime}(P \downarrow P) = \text{Nr}{}^{\backprime}P$ [*172·2 . *165·251]

***185·22.** $\vdash . \Pi\text{Nr}{}^{\backprime}(\dot{\Lambda} \downarrow \dot{\Lambda}) = 0_r$ [*185·21]

***185·23.** $\vdash : \dot{\Lambda} \in C{}^{\backprime}P . \supset . \Pi\text{Nr}{}^{\backprime}P = 0_r$ [*172·14]

***185·25.** $\vdash :: \text{Mult ax} . \supset :. \Pi\text{Nr}{}^{\backprime}P = 0_r . \equiv : \dot{\Lambda} \in C{}^{\backprime}P . \vee . P = \dot{\Lambda}$ [*172·182]

***185·27.** $\vdash :. \text{Mult ax} . \supset : P, Q \in \text{Rel}^2\,\text{excl} . \exists ! P \,\overline{\text{smor}}\, Q \cap \text{Rl}{}^{\backprime}\text{smor} . \supset .$
$\Pi\text{Nr}{}^{\backprime}P = \Pi\text{Nr}{}^{\backprime}Q$ [*172·45]

***185·28.** $\vdash :. \text{Mult ax} . \supset : P, Q \in \text{Rel}^2\,\text{excl} . P, Q \in \text{Nr}{}^{\backprime}R . C{}^{\backprime}P, C{}^{\backprime}Q \in \text{Cl}{}^{\backprime}\text{Nr}{}^{\backprime}S . \supset .$
$\Pi\text{Nr}{}^{\backprime}P = \Pi\text{Nr}{}^{\backprime}Q$ [*164·48 . *185·11]

***185·29.** $\vdash :. \text{Mult ax} . \supset : P \in \text{Rel}^2\,\text{excl} . P \in \text{Nr}{}^{\backprime}R . C{}^{\backprime}P \subset \text{Nr}{}^{\backprime}S . \supset .$
$\Pi\text{Nr}{}^{\backprime}P = \text{Nr}{}^{\backprime}(S \exp R)$ [*176·24]

***185·31.** $\vdash : \dot{\exists} ! P . \dot{\exists} ! Q . C{}^{\backprime}P \cap C{}^{\backprime}Q = \Lambda . \supset . \Pi\text{Nr}{}^{\backprime}(P \dotplus Q) = \Pi\text{Nr}{}^{\backprime}P \dot{\times} \Pi\text{Nr}{}^{\backprime}Q$
[*172·35]

***185·32.** $\vdash : Z \sim\in C{}^{\backprime}P . \supset . \Pi\text{Nr}{}^{\backprime}(P \nrightarrow Z) = \Pi\text{Nr}{}^{\backprime}P \dot{\times} \text{Nr}{}^{\backprime}Z$ [*172·32]

***185·321.** $\vdash : Z \sim\in C{}^{\backprime}P . \supset . \Pi\text{Nr}{}^{\backprime}(Z \nleftarrow P) = \text{Nr}{}^{\backprime}Z \dot{\times} \Pi\text{Nr}{}^{\backprime}P$ [*172·321]

***185·35.** $\vdash : P \not\!\!+ Q . \supset . \Pi\text{Nr}{}^{\backprime}(P \downarrow Q) = \text{Nr}{}^{\backprime}P \dot{\times} \text{Nr}{}^{\backprime}Q$ [*172·23]

***185·4.** $\vdash :. P \in \text{Rel}^2\,\text{excl} : QPQ . \supset_Q . C{}^{\backprime}Q \in 0 \cup 1 : \supset . \Pi\text{Nr}{}^{\backprime}\Pi;P = \Pi\text{Nr}{}^{\backprime}\Sigma{}^{\backprime}P$
[*174·241]

***185·41.** $\vdash : P \in \text{Rel}^2\,\text{excl} . P \Subset J . \supset . \Pi\text{Nr}{}^{\backprime}\Pi;P = \Pi\text{Nr}{}^{\backprime}\Sigma{}^{\backprime}P$ [*174·25]

The following proposition gives the connection between ordinal and cardinal multiplication.

***185·5.** $\vdash : P \epsilon \text{Rel}^2 \text{ excl} . \dot{\exists} ! P . \supset . C``\Pi\text{Nr}`P = \Pi\text{Nc}`C``C`P$

Dem.

$\vdash . *173\cdot16 . \supset \vdash : \text{Hp} . \supset . C``\Pi\text{Nr}`P = C``\text{Nr}`\text{Prod}`P$

$[*152\cdot7] \qquad = \text{Nc}`C`\text{Prod}`P$

$[*173\cdot161] \qquad = \text{Nc}`\text{Prod}`C``C`P$

$[*163\cdot16 . *115\cdot12] \qquad = \Pi\text{Nc}`C``C`P : \supset \vdash . \text{Prop}$

$*186$. POWERS OF RELATION-NUMBERS

Summary of $*186$.

For "μ to the νth power," where ordinal powers are concerned, we use the notation "$\mu \exp_r \nu$." We cannot use "μ^ν" or "$\mu \exp \nu$" because these have been already used for cardinals and classes ($*116$). We therefore put a suffix r to "exp" to show that it is *relational* powers that we are dealing with. We put

$$\mu \exp_r \nu = \hat{R}\{(\exists P, Q) \,.\, \mu = \mathrm{N_0r}`P \,.\, \nu = \mathrm{N_0r}`Q \,.\, R \operatorname{smor} (P \exp Q)\} \quad \mathrm{Df}$$

The following are the principal propositions of this number:

$*186{\cdot}2$. $\vdash : \mu \in \mathrm{N_0R} \,.\, \supset \,.\, 0_r \exp_r \mu = 0_r \,.\, \mu \exp_r 0_r = 0_r$

We do not have $\mu \exp_r 0_r = 1$, because there is no ordinal 1.

$*186{\cdot}21$. $\vdash . \mu \exp_r 2_r = \mu \dot{\times} \mu$

$*186{\cdot}22$. $\vdash . \alpha \exp_r (\beta \dot{+} \dot{1}) = (\alpha \exp_r \beta) \dot{\times} \alpha$

$*186{\cdot}23$. $\vdash . \alpha \exp_r (\dot{1} \dot{+} \beta) = \alpha \dot{\times} (\alpha \exp_r \beta)$

$*186{\cdot}14$. $\vdash : \nu \neq 0_r \,.\, \varpi \neq 0_r \,.\, \supset \,.\, \mu \exp_r (\nu \dot{+} \varpi) = (\mu \exp_r \nu) \dot{\times} (\mu \exp_r \varpi)$

$*186{\cdot}15$. $\vdash : \varpi \subset \mathrm{Rl}`J \,.\, \supset \,.\, \mu \exp_r (\varpi \dot{\times} \nu) = (\mu \exp_r \nu) \exp_r \varpi$

$*186{\cdot}31$. $\vdash :. \mathrm{Mult\,ax} \,.\, \supset : \mu, \nu \in \mathrm{NR} - \iota`\Lambda \,.\, P \in \mathrm{Rel^2\,excl} \cap \mu \,.\, C`P \subset \nu \,.\, \supset \,.$
$\Pi \mathrm{Nr}`P = \mu \exp_r \nu$

which connects exponentiation with multiplication.

$*186{\cdot}4$. $\vdash . \mathrm{Nr}`P_{\mathrm{df}} = 2_r \exp_r (\mathrm{Nr}`P)$ (cf. $*177$)

$*186{\cdot}5$. $\vdash : \mu, \nu \in \mathrm{N_0R} \,.\, \nu \neq 0_r \,.\, \supset \,.\, C``(\mu \exp_r \nu) = (C``\mu)^{C``\nu}$

which connects ordinal and cardinal exponentiation.

$*186{\cdot}01$. $\mu \exp_r \nu = \hat{R}\{(\exists P, Q) \,.\, \mu = \mathrm{N_0r}`P \,.\, \nu = \mathrm{N_0r}`Q \,.\, R \operatorname{smor} (P \exp Q)\}$ Df

$*186{\cdot}02$. $(\mathrm{Nr}`P) \exp_r \nu = (\mathrm{N_0r}`P) \exp_r \nu$ Df

$*186{\cdot}03$. $\mu \exp_r (\mathrm{Nr}`Q) = \mu \exp_r (\mathrm{N_0r}`Q)$ Df

$*186{\cdot}1$. $\vdash : R \in \mu \exp_r \nu \,.\, \equiv \,.\, (\exists P, Q) \,.\, \mu = \mathrm{N_0r}`P \,.\, \nu = \mathrm{N_0r}`Q \,.\, R \operatorname{smor} (P \exp Q)$
[($*186{\cdot}01$)]

$*186{\cdot}11$. $\vdash . \exists ! \mu \exp_r \nu \,.\, \supset \,.\, \mu, \nu \in \mathrm{N_0R} \,.\, \mu, \nu \in \mathrm{NR} - \iota`\Lambda$

$*186{\cdot}111$. $\vdash : \sim(\mu, \nu \in \mathrm{N_0R}) \,.\, \supset \,.\, \mu \exp_r \nu = \Lambda$

$*186{\cdot}12$. $\vdash : R \in \mu \exp_r \nu \,.\, \equiv \,.\, (\exists P, Q) \,.\, \mu = \mathrm{N_0r}`P \,.\, \nu = \mathrm{N_0r}`Q \,.\, R \operatorname{smor} P^Q$
[$*176{\cdot}181 \,.\, *186{\cdot}1$]

***186·13.** $\vdash . (\mathrm{Nr}\text{‘}P) \exp_r (\mathrm{Nr}\text{‘}Q) = (\mathrm{N_0r}\text{‘}P) \exp_r (\mathrm{Nr}\text{‘}Q) = (\mathrm{Nr}\text{‘}P) \exp_r (\mathrm{N_0r}\text{‘}Q)$
$= (\mathrm{N_0r}\text{‘}P) \exp_r (\mathrm{N_0r}\text{‘}Q) = \mathrm{Nr}\text{‘}(P \exp Q) = \mathrm{Nr}\text{‘}(P^Q)$
[Proof as in *180·3]

***186·14.** $\vdash : \nu \neq 0_r . \varpi \neq 0_r . \supset . \mu \exp_r (\nu \dot{+} \varpi) = (\mu \exp_r \nu) \dot{\times} (\mu \exp_r \varpi)$

Dem.

$\vdash . *180\cdot4 . *186\cdot111 . \supset$
$\vdash : \sim(\mu, \nu, \varpi \in \mathrm{N_0R}) . \supset . \mu \exp_r (\nu \dot{+} \varpi) = \Lambda . (\mu \exp_r \nu) \dot{\times} (\mu \exp_r \varpi) = \Lambda \quad (1)$
$\vdash . *186\cdot13 . *180\cdot3 . \supset$
$\vdash : \mu = \mathrm{N_0r}\text{‘}P . \nu = \mathrm{N_0r}\text{‘}Q . \varpi = \mathrm{N_0r}\text{‘}R . \supset . \mu \exp_r (\nu \dot{+} \varpi) = \mathrm{Nr}\text{‘}P^{Q+R} \quad (2)$
$\vdash . *176\cdot42 . *180\cdot11 . \supset$
$\vdash : \mathrm{Hp} . \mathrm{Hp}(2) . \supset . \mathrm{Nr}\text{‘}P^{Q+R} = \mathrm{Nr}\text{‘}(P^{\downarrow(\Lambda \cap C\text{‘}P);;Q} \times P^{(\Lambda \cap C\text{‘}P)\downarrow;;R})$
$[*180\cdot12 . *176\cdot22 . *166\cdot23] \quad = \mathrm{Nr}\text{‘}(P^Q \times P^R)$
$[*186\cdot13 . *184\cdot13] \quad = (\mu \exp_r \nu) \dot{\times} (\mu \exp_r \varpi) \quad (3)$
$\vdash . (2) . (3) . *155\cdot2 . \supset \vdash : \mu, \nu, \varpi \in \mathrm{N_0R} . \nu \neq 0_r . \varpi \neq 0_r . \supset .$
$\mu \exp_r (\nu \dot{+} \varpi) = (\mu \exp_r \nu) \dot{\times} (\mu \exp_r \varpi) \quad (4)$
$\vdash . (1) . (4) . \supset \vdash . \mathrm{Prop}$

***186·15.** $\vdash : \varpi \subset \mathrm{Rl}\text{‘}J . \supset . \mu \exp_r (\varpi \dot{\times} \nu) = (\mu \exp_r \nu) \exp_r \varpi$

Dem.

$\vdash . *186\cdot111 . *184\cdot111 . \supset$
$\vdash : \sim(\mu, \nu, \varpi \in \mathrm{N_0R}) . \supset . \mu \exp_r (\varpi \dot{\times} \nu) = \Lambda . (\mu \exp_r \nu) \exp_r \varpi = \Lambda \quad (1)$
$\vdash . *186\cdot13 . *184\cdot13 . \supset$
$\vdash : \mu = \mathrm{N_0r}\text{‘}P . \nu = \mathrm{N_0r}\text{‘}Q . \varpi = \mathrm{N_0r}\text{‘}R . \supset . \mu \exp_r (\varpi \dot{\times} \nu) = \mathrm{Nr}\text{‘}(P^{R \times Q}) \quad (2)$
$\vdash . *176\cdot57 . \supset \vdash : \mathrm{Hp} . \mathrm{Hp}(2) . \supset . \mathrm{Nr}\text{‘}(P^{R \times Q}) = \mathrm{Nr}\text{‘}(P^Q)^R$
$[*186\cdot13] \quad = \{(\mathrm{N_0r}\text{‘}P) \exp_r (\mathrm{N_0r}\text{‘}Q)\} \exp_r (\mathrm{N_0r}\text{‘}R)$
$[\mathrm{Hp}] \quad = (\mu \exp_r \nu) \exp_r \varpi \quad (3)$
$\vdash . (2) . (3) . *155\cdot2 . \supset$
$\vdash : \mu, \nu, \varpi \in \mathrm{N_0R} . \varpi \subset \mathrm{Rl}\text{‘}J . \supset . \mu \exp_r (\varpi \dot{\times} \nu) = (\mu \exp_r \nu) \exp_r \varpi \quad (4)$
$\vdash . (1) . (4) . \supset \vdash . \mathrm{Prop}$

***186·2.** $\vdash : \mu \in \mathrm{N_0R} . \supset . 0_r \exp_r \mu = 0_r . \mu \exp_r 0_r = 0_r \quad [*176\cdot151]$

***186·21.** $\vdash . \mu \exp_r 2_r = \mu \dot{\times} \mu$

Dem.

$\vdash . *186\cdot111 . *184\cdot111 . \supset \vdash : \mu \sim\in \mathrm{N_0R} . \supset . \mu \exp_r 2_r = \Lambda . \mu \dot{\times} \mu = \Lambda \quad (1)$
$\vdash . *186\cdot13 . *176\cdot1 . \supset \vdash : \mu = \mathrm{N_0r}\text{‘}P . x \neq y . \supset .$
$\mu \exp_r 2_r = \mathrm{Nr}\text{‘}\mathrm{Prod}\text{‘}P \underset{;}{\downarrow} {}^{;}(x \downarrow y)$
$[*150\cdot71] \quad = \mathrm{Nr}\text{‘}\mathrm{Prod}\text{‘}\{(P \underset{;}{\downarrow} x) \downarrow (P \underset{;}{\downarrow} y)\}$
$[*173\cdot24 . *165\cdot211 . \mathrm{Transp}] = \mathrm{Nr}\text{‘}\{(P \underset{;}{\downarrow} x) \times (P \underset{;}{\downarrow} y)\}$
$[*165\cdot251 . *166\cdot23] \quad = \mathrm{Nr}\text{‘}(P \times P)$
$[*184\cdot13] \quad = \mu \dot{\times} \mu \quad (2)$
$\vdash . (2) . *155\cdot2 . *24\cdot1 . \supset \vdash : \mu \in \mathrm{N_0R} . \supset . \mu \exp_r 2_r = \mu \dot{\times} \mu \quad (3)$
$\vdash . (1) . (3) . \supset \vdash . \mathrm{Prop}$

***186·22.** $\vdash . \alpha \exp_r (\beta \dot{+} \dot{1}) = (\alpha \exp_r \beta) \dot{\times} \alpha$

Dem.

$\vdash . *186·111 . *181·4 . \supset$

$\vdash : \sim(\alpha, \beta \in N_0R) . \supset . \alpha \exp_r (\beta \dot{+} \dot{1}) = \Lambda . (\alpha \exp_r \beta) \dot{\times} \alpha = \Lambda$ (1)

$\vdash . *186·13 . *181·22 . \supset$

$\vdash : \alpha = N_0r\text{‘}P . \beta = N_0r\text{‘}Q . \supset . \alpha \exp_r (\beta \dot{+} \dot{1}) = Nr\text{‘}\{P \exp (Q \dot{+\!\!\!\rightarrow} z)\} .$

$(\alpha \exp_r \beta) \dot{\times} \alpha = Nr\text{‘}(P \exp Q) \dot{\times} Nr\text{‘}P$ (2)

$\vdash . (2) . *176·151 . *166·13 . \supset$

$\vdash : Hp (2) . P = \dot{\Lambda} . \supset . \alpha \exp_r (\beta \dot{+} \dot{1}) = 0_r . (\alpha \exp_r \beta) \dot{\times} \alpha = 0_r$ (3)

$\vdash . *165·2 . *161·4 . *176·1 . (*181·01) . \supset$

$\vdash . Nr\text{‘}\{P \exp (Q \dot{+\!\!\!\rightarrow} z)\} = Nr\text{‘}Prod\text{‘}[P \downarrow_{;} ; \downarrow \Lambda_x ; \iota ; Q +\!\!\!\rightarrow P \downarrow_{;} \{(\Lambda \cap C\text{‘}P) \downarrow \iota\text{‘}x\}]$ (4)

$\vdash . *165·221·222 . *181·11 . *162·22 . \supset$

$\vdash : \dot{\exists} ! P . \supset . P \downarrow_{;} \{(\Lambda \cap C\text{‘}P) \downarrow \iota\text{‘}x\} \sim\in C\text{‘}P \downarrow_{;} ; \downarrow \Lambda_x ; \iota ; Q .$

$C\text{‘}P \downarrow_{;} \{(\Lambda \cap C\text{‘}P) \downarrow \iota\text{‘}x\} \cap C\text{‘}\Sigma\text{‘}P \downarrow_{;} ; \downarrow \Lambda_x ; \iota ; Q = \Lambda$ (5)

$\vdash . (4) . (5) . *165·21 . *173·25 . \supset \vdash :. \dot{\exists} ! P . \supset .$

$Nr\text{‘}\{P \exp (Q \dot{+\!\!\!\rightarrow} z)\} = Nr\text{‘}[(Prod\text{‘}P \downarrow_{;} ; \downarrow \Lambda_x ; \iota ; Q) \times P \downarrow_{;} \{(\Lambda \cap C\text{‘}P) \downarrow \iota\text{‘}x\}]$

$[*181·12.*165·251.*176·1·22.*184·13] = Nr\text{‘}(P \exp Q) \dot{\times} Nr\text{‘}P$ (6)

$\vdash . (2) . (6) . \supset \vdash : Hp (2) . \dot{\exists} ! P . \supset . \alpha \exp_r (\beta \dot{+} \dot{1}) = (\alpha \exp_r \beta) \dot{\times} \alpha$ (7)

$\vdash . (1) . (3) . (7) . \supset \vdash . Prop$

***186·23.** $\vdash . \alpha \exp_r (\dot{1} \dot{+} \beta) = \alpha \dot{\times} (\alpha \exp_r \beta)$ [Proof as in *186·22]

***186·3.** $\vdash :. Mult\ ax . \supset : P \in Rel^2 excl \cap Nr\text{‘}R . C\text{‘}P \subset Nr\text{‘}S . \supset .$

$\Pi Nr\text{‘}P = (Nr\text{‘}P) \exp_r (Nr\text{‘}S)$ [*185·29]

***186·31.** $\vdash :. Mult\ ax . \supset : \mu, \nu \in NR - \iota\text{‘}\Lambda . P \in Rel^2 excl \cap \mu . C\text{‘}P \subset \nu . \supset .$

$\Pi Nr\text{‘}P = \mu \exp_r \nu$ [*186·3]

***186·4.** $\vdash . Nr\text{‘}P_{df} = 2_r \exp_r (Nr\text{‘}P)$ [*177·13]

***186·5.** $\vdash : \mu, \nu \in N_0R . \nu \neq 0_r . \supset . C\text{‘‘}(\mu \exp_r \nu) = (C\text{‘‘}\mu)^{C\text{‘‘}\nu}$

Dem.

$\vdash . *152·7 . *186·13 . \supset \vdash : \mu = N_0r\text{‘}P . \nu = N_0r\text{‘}Q . \supset .$

$C\text{‘‘}(\mu \exp_r \nu) = Nc\text{‘}C\text{‘}(P \exp Q)$ (1)

$\vdash . (1) . *176·14 . \supset \vdash : Hp (1) . \nu \neq 0_r . \supset . C\text{‘‘}(\mu \exp_r \nu) = Nc\text{‘}\{(C\text{‘}P) \exp (C\text{‘}Q)\}$

$[*116·222] \quad = (N_0c\text{‘}C\text{‘}P)^{N_0c\text{‘}C\text{‘}Q}$

$[*155·6] \quad = (C\text{‘‘}N_0r\text{‘}P)^{C\text{‘‘}N_0r\text{‘}Q}$

$[Hp] \quad = (C\text{‘‘}\mu)^{C\text{‘‘}\nu} : \supset \vdash . Prop$

PART V

SERIES

SUMMARY OF PART V

A RELATION is said to be *serial*, or to generate a series, when it possesses three different properties, namely (1) being contained in diversity, (2) transitiveness, (3) connexity, *i.e.* the property that the relation or its converse holds between any two different members of its field. Thus P is a serial relation if (1) $P \subseteq J$, (2) $P^2 \subseteq P$, (3) $x, y \in C'P \,.\, x \neq y \,.\, \supset_{x,y} : xPy \,.\, \vee \,.\, yPx$. The third characteristic, that of connexity, may be written more shortly

$$x \in C'P \,.\, \supset_x \,.\, \overrightarrow{P}'x \cup \iota'x \cup \overleftarrow{P}'x = C'P,$$

i.e.

$$x \in C'P \,.\, \supset_x \,.\, \overleftrightarrow{P}'x = C'P,$$

using the notation of $*97$; and this, in virtue of $*97{\cdot}23$, is equivalent to

$$\overleftrightarrow{P}''C'P \in 0 \cup 1.$$

In virtue of $*50{\cdot}47$, the first two characteristics are equivalent to

$$P \dot{\cap} \breve{P} = \dot{\Lambda} \,.\, P^2 \subseteq P.$$

When $P \dot{\cap} \breve{P} = \dot{\Lambda}$, we say that P is "asymmetrical." Thus serial relations are such as are asymmetrical, transitive, and connected.

It might be thought that a serial relation need not be contained in diversity, since we commonly speak of series in which there are repetitions, *i.e.* in which an earlier term is identical with a later term. Thus, *e.g.*

$$a,\ b,\ c,\ a,\ e,\ f,\ b,\ g,\ h$$

would be called a series of letters, although the letters a and b recur. But in all such cases, there is some means (in the above case, position in space) by which one *occurrence* of a given term is distinguished from another occurrence, and this will be found to mean that there is some other series (in the above case, the series of positions in a line) free from repetitions, with which our pseudo-series has a one-many correlation. Thus, in the above instance, we have a series of nine positions, which we may call

$$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,$$

which form a true series without repetitions; we have a one-many relation, that of *occupying* these positions, by means of which we distinguish occurrences of a, the first occurrence being a as the correlate of 1, the second being a as the correlate of 4. All series in which there are repetitions (which we may call pseudo-series) are thus obtained by correlation with true series, *i.e.* with series in which there is no repetition. That is to say, a pseudo-series has as its generating relation a relation of the form $S | P$, where P is a serial relation, and S is a one-many relation whose converse domain contains the field of P. Thus what we may call self-subsistent series must be series without repetitions, *i.e.* series whose generating relations are contained in diversity.

For our purposes, there is no use in distinguishing a series from its generating relation. A series is not a class, since it has a definite order, while a class has no order, but is capable of many orders (unless it contains only one term or none). The generating relation determines the order, and also the class of terms ordered, since this class is the field of the generating relation. Hence the generating relation completely determines the series, and may, for all mathematical purposes, be taken to *be* the series.

When P is transitive, we have

$$P_{po} = P \,.\, P_* = P \cup I \upharpoonright C'P.$$

Hence all the propositions of Part II, Section E become greatly simplified when applied to series.

Also, since the field of a connected relation consists of a single family, a series has one first term or none, and one last term or none.

In the case of a serial relation P, the relation P_1 (defined in $*121{\cdot}02$) becomes $P \dot{-} P^2$, *i.e.* the relation "immediately preceding." In a *discrete* series, the terms in general immediately precede other terms. A *compact* series, on the contrary, is defined as one in which there are terms between any two: in such a series, $P_1 = \dot{\Lambda}$.

It very frequently occurs that we wish to consider the relations of various series which are all contained in some one series; for example, we may wish to consider various series of real numbers, all arranged in order of magnitude. In such a case, if P is the series in which all the others are contained, and $\alpha, \beta, \gamma, \ldots$ are the fields of the contained series, the contained series themselves are $P \upharpoonright \alpha$, $P \upharpoonright \beta$, $P \upharpoonright \gamma, \ldots$. Thus when series are given as contained in a given series, they are completely determined by their fields.

In what follows, Section A deals with the elementary properties of series, including maximum and minimum points, sequent points and limits.

Section B will deal with the theory of segments and kindred topics; in this section we shall define "Dedekindian" series, and shall prove the important proposition that the series of segments of a series is always Dedekindian, *i.e.* that every class of segments has either a maximum or a limit.

Section C, which stands outside the main developments of the book, is concerned with convergence and the limits of functions and the definition of a continuous function. Its purpose is to show how these notions can be expressed, and many of their properties established, in a much more general way than is usually done, and without assuming that the arguments or values of the functions concerned are either numerical or numerically measurable.

Section D will deal with "well-ordered" series, *i.e.* series in which every class containing members of the field has a first term. The properties of

well-ordered series are many and important; most of them depend upon the fact that an extended variety of mathematical induction is possible in dealing with well-ordered series. The term "ordinal number" is confined by usage to the relation-number of a well-ordered series; ordinal numbers will also be considered in our fourth section.

Section E will deal with finite and infinite. We shall show that the distinction between "inductive" and "non-reflexive" does not arise in well-ordered series.

Section F will deal with "compact" series, *i.e.* series in which there is a term between any two, *i.e.* in which $P^2 = P$. In particular we shall consider "rational" series (*i.e.* series like the series of rationals in order of magnitude) and continuous series (*i.e.* series like the series of real numbers in order of magnitude). Our treatment of this subject will follow Cantor closely.

SECTION A

GENERAL THEORY OF SERIES

Summary of Section A.

In the present section, we shall be concerned with the properties common to all series. Such properties, for the most part, are very simple, and present no difficulties of any kind. Many of the properties of series do not require all the three characteristics by which serial relations are defined, but only one or two of these properties: we therefore begin with numbers in which, though the properties proved derive their chief importance from their applicability to series, the hypotheses are only that the relations in question have one or two of the properties of serial relations. Thence we proceed to the most elementary properties peculiar to series, and thence to the theory of minimum and maximum members of classes contained in a series, and of the successors and limits of classes. We then proceed to the correlation of a series with part of itself. The ground covered is familiar, and the difficulties encountered are less than in most previous sections.

It will be observed that where series are concerned, if α is an existent class contained in $C'P$, $p'\overleftarrow{P}``\alpha$ is correlative to $P``\alpha$ (which is $s'\overrightarrow{P}``\alpha$): $P``\alpha$ is "predecessors of some α," and $p'\overleftarrow{P}``\alpha$ is "successors of all $\alpha$'s." If $\alpha$ is an existent class contained in $C'P$, the whole of $C'P$, with the exception of the last term of $\alpha$ (if there is such a term), belongs to one or other of the classes $P``\alpha$, $p'\overleftarrow{P}``\alpha$, of which the first wholly precedes the second. The division of $C'P$ into these two classes is the Dedekind "cut" defined by $\alpha$. But when only part of $\alpha$ is contained in $C'P$, we must replace $p'\overleftarrow{P}``\alpha$ by $p'\overleftarrow{P}``(\alpha \cap C'P)$, since $p'\overleftarrow{P}``\alpha = \Lambda$ if α has any member not belonging to $C'P$. Again, if $\alpha \cap C'P = \Lambda$, we have $p'\overleftarrow{P}``(\alpha \cap C'P) = \mathrm{V}$. But what we want is the complement to $P``\alpha$, which in this case is null. Hence we must replace $p'\overleftarrow{P}``(\alpha \cap C'P)$ by $C'P \cap p'\overleftarrow{P}``(\alpha \cap C'P)$: this is $C'P$ when $P``\alpha = \Lambda$, *i.e.* when $\alpha \cap C'P = \Lambda$. In any other event it is equal to $p'\overleftarrow{P}``(\alpha \cap C'P)$. If α is contained in $C'P$ and is not null, $C'P \cap p'\overleftarrow{P}``(\alpha \cap C'P) = p'\overleftarrow{P}``\alpha$. Thus the Dedekind "cut" defined by a class α, whether or not this class is contained in whole or part in $C'P$, is always the two classes

$$P``\alpha,\ C'P \cap p'\overleftarrow{P}``(\alpha \cap C'P).$$

Throughout the elementary propositions of this section, we have been careful to avoid stronger hypotheses than are required: we have not assumed

P to be serial, if our conclusion would follow (*e.g.*) from the hypothesis that P is transitive and connected. It will be found that many properties of series depend upon the fact that, if x, y are two different terms of a series P, then $xPy . \equiv . \sim(yPx)$ (*204·3). Here the implication $xPy . \supset . \sim(yPx)$ requires that P should be asymmetrical, *i.e.* that we should have $P \dot{\wedge} \breve{P} = \dot{\Lambda}$ or $P^2 \subseteq J$. The implication $\sim(yPx) . \supset . xPy$ requires that P should be connected. Thus the hypothesis required is not that P should be serial, but that P should be connected and asymmetrical (*202·5).

Again, consider the proposition that if P is a series, $P_1 = P \dot{-} P^2$. This relation P_1 is the very useful relation "immediately preceding"; thus the above proposition is important, as is the further proposition that if P is a series, P_1 is a one-one relation. It will be remembered that (by *121) "xP_1y" means that $P(x \mapsto y)$ consists of two terms. It was shown in *121·304·305 that if P_{po} is contained in diversity, "xP_1y" implies "xPy," and is equivalent to the statement that x and y constitute the whole interval $P(x \mapsto y)$ and are not identical. Also by *121·254, $P_1 = (P_{po})_1$. It is evident that, if P_{po} is contained in diversity, and xP_1y, we cannot have xP^2y, because there is no term other than x and y in the interval $P(x \mapsto y)$, and we cannot have xPx or yPy. Hence if $P_{po} \subseteq J$, we have $P_1 \subseteq \dot{-} P^2$. Hence by what was said above (*121·305), if $P_{po} \subseteq J$, we shall have $P_1 \subseteq P \dot{-} P^2$. On the other hand, if P is transitive, we have $P \dot{-} P^2 \subseteq P_1$ (*201·61). Combining these two facts, and remembering that if P is transitive, $P = P_{po}$ (*201·18), we find that $P_1 = P \dot{-} P^2$ if P is transitive and contained in diversity. We find further (*202·7) that if P is connected, $P \dot{-} P^2$ is one-one. Hence we need the full hypothesis that P is a series in order to prove that P_1 is a one-one (*204·7). This is a good example of the way in which the various separate characteristics that make up the definition of series are relevant in proving the properties of series.

$*200$. RELATIONS CONTAINED IN DIVERSITY

Summary of $*200$.

Some of the propositions of this number are repetitions or immediate consequences of previous propositions, especially those of the propositions of $*50$ which deal with diversity. But we are chiefly concerned here with propositions which will be useful in the theory of series; this leads us to introduce propositions on $p\text{'}\overleftarrow{P}\text{''}\alpha$ and on matters connected with relation-arithmetic and other topics. It will be seen that "$P^2 \subset J$" (*i.e.* "P is asymmetrical") is an important hypothesis, as is also $P_{\text{po}} \subset J$, of the use of which we have already had examples in $*96$ and $*121$.

The following are among the most useful propositions in this number:

$*200\cdot12$. $\vdash : P \in \text{Rl}\text{'}J \,.\supset .\, C\text{'}P \sim\in 1$

This is the proposition which makes it impossible to define an ordinal number 1 which shall take its place among relation-numbers applicable to series.

$*200\cdot35$. $\vdash : P \subset J \,.\, \alpha \in 1 \,.\supset .\, P \upharpoonright \alpha = \dot{\Lambda}$

This is a consequence of $*200\cdot12$.

$*200\cdot36$. $\vdash : P^2 \subset J \,.\supset .\, P \subset J$

$*200\cdot361$. $\vdash : P^2 \subset J \,.\supset .\, \overrightarrow{P}\text{'}x \cap (\iota\text{'}x \cup \overleftarrow{P}\text{'}x) = \Lambda \,.\, \overleftarrow{P}\text{'}x \cap (\overrightarrow{P}\text{'}x \cup \iota\text{'}x) = \Lambda$

I.e. if $P^2 \subset J$, no term precedes itself or any of its predecessors, and no term succeeds itself or any of its successors.

$*200\cdot38$. $\vdash : P_{\text{po}} \subset J \,.\supset .\, P_{\text{po}} = P_* \dot{\cap} J$

$*200\cdot39$. $\vdash : P_{\text{po}} \subset J \,.\, x \in C\text{'}P \,.\supset .\, \overrightarrow{P_*}\text{'}x \cap \overleftarrow{P_*}\text{'}x = \iota\text{'}x$

We then have a collection of propositions concerned with relation-arithmetic.

$*200\cdot211$. $\vdash : P \subset J \,.\, P \,\text{smor}\, Q \,.\supset .\, Q \subset J$

I.e. the property of being contained in diversity is invariant for likeness-transformations;

$*200\cdot4$. $\vdash : P \dagger Q \in \text{Rl}\text{'}J \,.\equiv .\, P, Q \in \text{Rl}\text{'}J \,.\, C\text{'}P \cap C\text{'}Q = \Lambda$

$*200\cdot41$. $\vdash : P \rightarrow\!\!\!+ x \subset J \,.\equiv .\, x \leftarrow\!\!\!+ P \subset J \,.\equiv .\, P \subset J \,.\, x \sim\in C\text{'}P$

and other such propositions.

We then have a set of propositions concerned with $p\text{'}\overrightarrow{P}\text{''}\alpha$ and $p\text{'}\overleftarrow{P}\text{''}\alpha$. The most important are

$*200\cdot5.\quad \vdash : P \mathrel{\mathsf{G}} J . \supset . \alpha \cap p`\overrightarrow{P}``\alpha = \Lambda . \alpha \cap p`\overleftarrow{P}``\alpha = \Lambda$

$*200\cdot52.\quad \vdash : P \mathrel{\mathsf{G}} J . \supset . C`P \sim \epsilon \overrightarrow{P}``C`P$

$*200\cdot53.\quad \vdash : P^2 \mathrel{\mathsf{G}} J . \supset . P``\alpha \cap p`\overleftarrow{P}``\alpha = \Lambda . \breve{P}``\alpha \cap p`\overrightarrow{P}``\alpha = \Lambda$

I.e. if P is asymmetrical, the terms which precede part of α do not succeed the whole of α, and vice versa.

$*200\cdot11.\quad \vdash : P \epsilon \mathrm{Rl}`J . \equiv . \breve{P} \epsilon \mathrm{Rl}`J \quad [*50\cdot23]$

$*200\cdot12.\quad \vdash : P \epsilon \mathrm{Rl}`J . \supset . C`P \sim \epsilon 1$

Dem.

$\vdash . *50\cdot11 . *33\cdot17 . \supset \vdash :. \mathrm{Hp} : xPy . \mathsf{v} . yPx : \supset . y \neq x . y \epsilon C`P \qquad (1)$

$\vdash . (1) . *33\cdot132 . \quad \supset \vdash :. \mathrm{Hp} . \supset : x \epsilon C`P . \supset . (\exists y) . y \neq x . y \epsilon C`P :$

$[*52\cdot181] \qquad \supset : C`P \sim \epsilon 1 :. \supset \vdash . \mathrm{Prop}$

$*200\cdot2.\quad \vdash : T \epsilon 1 \to 1 . \supset . T;(P \dot{\cap} J) = T;P \dot{\cap} J$

Dem.

$\vdash . *150\cdot4 . \supset \vdash :. \mathrm{Hp} . \supset :$

$x \{T;(P \dot{\cap} J)\} y . \equiv . (\exists z, w) . x = T`z . y = T`w . zPw . z \neq w .$

$[*71\cdot56] \qquad \equiv . (\exists z, w) . x \neq y . x = T`z . y = T`w . zPw .$

$[*150\cdot4] \qquad \equiv . x \{T;P \dot{\cap} J\} y :. \supset \vdash . \mathrm{Prop}$

$*200\cdot21.\quad \vdash : T \epsilon \mathrm{Cls} \to 1 . P \mathrel{\mathsf{G}} J . \supset . T;P \mathrel{\mathsf{G}} J$

Dem.

$\vdash . *150\cdot1 . *50\cdot24 . \supset \vdash :. \mathrm{Hp} . \supset : x (T;P) y . \supset . (\exists z, w) . xTz . yTw . z \neq w .$

$[*71\cdot171.\mathrm{Transp}] \qquad \supset . x \neq y :. \supset \vdash . \mathrm{Prop}$

$*200\cdot211.\quad \vdash : P \mathrel{\mathsf{G}} J . P \mathrm{smor} Q . \supset . Q \mathrel{\mathsf{G}} J \quad [*200\cdot21 . *151\cdot1]$

The properties of relations are very frequently common to all relations which are like a given relation, and this applies specially to the kinds of properties with which we are most concerned. The above proposition is an illustration of this fact: it shows that the property of being contained in diversity is invariant for likeness-transformations.

$*200\cdot22.\quad \vdash : P \mathrel{\mathsf{G}} J . \equiv . \mathrm{N_0r}`P \subset \mathrm{Rl}`J . \equiv . \exists ! \mathrm{N_0r}`P \cap \mathrm{Rl}`J$

Dem. $\vdash . *155\cdot11 . *200\cdot211 . \supset \vdash : P \mathrel{\mathsf{G}} J . \supset . \mathrm{N_0r}`P \subset \mathrm{Rl}`J \qquad (1)$

$\vdash . *155\cdot12 . \qquad \supset \vdash : \mathrm{N_0r}`P \subset \mathrm{Rl}`J . \supset . P \mathrel{\mathsf{G}} J \qquad (2)$

$\vdash . *155\cdot12 . \qquad \supset \vdash : P \mathrel{\mathsf{G}} J . \supset . \exists ! \mathrm{N_0r}`P \cap \mathrm{Rl}`J \qquad (3)$

$\vdash . *155\cdot11 . *200\cdot211 . \supset \vdash : \exists ! \mathrm{N_0r}`P \cap \mathrm{Rl}`J . \supset . P \mathrel{\mathsf{G}} J \qquad (4)$

$\vdash . (1) . (2) . (3) . (4) . \supset \vdash . \mathrm{Prop}$

We have, without the need of typical definiteness,

$$\vdash : P \mathrel{\mathsf{G}} J . \supset . \mathrm{Nr}`P \subset \mathrm{Rl}`J$$

and

$$\vdash : \exists ! \mathrm{Nr}`P \cap \mathrm{Rl}`J . \supset . P \mathrel{\mathsf{G}} J,$$

both of which are immediate consequences of $*200{\cdot}211$. The converse implications, however, fail if $\text{Nr}{}^{\backprime}P$ is taken in a type in which $\text{Nr}{}^{\backprime}P = \Lambda$.

$*200{\cdot}3$. $\vdash . \dot{\Lambda} \in \text{Rl}{}^{\backprime}J$ [$*25{\cdot}12$]

$*200{\cdot}31$. $\vdash : x \neq y . \equiv . x \downarrow y \in \text{Rl}{}^{\backprime}J$ [$*55{\cdot}3$]

$*200{\cdot}32$. $\vdash : \alpha \uparrow \beta \mathrel{\mathsf{G}} J . \equiv . \alpha \cap \beta = \Lambda$ [$*50{\cdot}55$]

$*200{\cdot}33$. $\vdash : P \mathrel{\mathsf{G}} J . \supset . P \upharpoonright \alpha \mathrel{\mathsf{G}} J$ [$*35{\cdot}442$]

$*200{\cdot}34$. $\vdash : P \upharpoonright \alpha \mathrel{\mathsf{G}} J . \equiv . P \restriction \alpha \mathrel{\mathsf{G}} J . \equiv . \alpha \upharpoonleft P \mathrel{\mathsf{G}} J$ [$*50{\cdot}58$]

$*200{\cdot}35$. $\vdash : P \mathrel{\mathsf{G}} J . \alpha \in 1 . \supset . P \upharpoonright \alpha = \dot{\Lambda}$

Dem.

$\vdash . *52{\cdot}16 . \supset \vdash :. \text{Hp} . \supset : x, y \in \alpha . \supset_{x,y} . \sim (xJy) .$

[$*23{\cdot}81$] $\supset_{x,y} . \sim (xPy) :$

[$*11{\cdot}521$] $\supset : (x, y) . \sim \{x, y \in \alpha . xPy\} :. \supset \vdash . \text{Prop}$

$*200{\cdot}36$. $\vdash : P^2 \mathrel{\mathsf{G}} J . \supset . P \mathrel{\mathsf{G}} J$ [$*50{\cdot}45$]

$*200{\cdot}361$. $\vdash : P^2 \mathrel{\mathsf{G}} J . \supset . \overrightarrow{P}{}^{\backprime}x \cap (\iota{}^{\backprime}x \cup \overleftarrow{P}{}^{\backprime}x) = \Lambda . \overleftarrow{P}{}^{\backprime}x \cap (\overrightarrow{P}{}^{\backprime}x \cup \iota{}^{\backprime}x) = \Lambda$

Dem.

$\vdash . *51{\cdot}15 . \supset \vdash : y \in \overrightarrow{P}{}^{\backprime}x \cap \iota{}^{\backprime}x . \supset . xPx$ (1)

$\vdash . *200{\cdot}36 . \supset \vdash : \text{Hp} . \supset . \sim (xPx) .$

[(1).Transp] $\supset . \overrightarrow{P}{}^{\backprime}x \cap \iota{}^{\backprime}x = \Lambda$ (2)

$\vdash . *34{\cdot}11 . \supset \vdash : \exists ! \overrightarrow{P}{}^{\backprime}x \cap \overleftarrow{P}{}^{\backprime}x . \equiv . xP^2x$ (3)

$\vdash . (3) . \text{Transp} . \supset \vdash : \text{Hp} . \supset . \overrightarrow{P}{}^{\backprime}x \cap \overleftarrow{P}{}^{\backprime}x = \Lambda$ (4)

$\vdash . (2) . (4) . \supset \vdash : \text{Hp} . \supset . \overrightarrow{P}{}^{\backprime}x \cap (\iota{}^{\backprime}x \cap \overleftarrow{P}{}^{\backprime}x) = \Lambda$ (5)

Similarly $\vdash : \text{Hp} . \supset . \overleftarrow{P}{}^{\backprime}x \cap (\overrightarrow{P}{}^{\backprime}x \cup \iota{}^{\backprime}x) = \Lambda$ (6)

$\vdash . (5) . (6) . \supset \vdash . \text{Prop}$

$*200{\cdot}37$. $\vdash : \exists ! \text{Pot}{}^{\backprime}P \cap \text{Rl}{}^{\backprime}J . \supset . P \mathrel{\mathsf{G}} J$

Dem.

$\vdash . *91{\cdot}373 \dfrac{xSx}{\phi S} . \supset$

$\vdash :: xPx : S \in \text{Pot}{}^{\backprime}P . xSx . \supset_S . x(S \mid P)x : \supset : Q \in \text{Pot}{}^{\backprime}P . \supset_Q . xQx$ (1)

$\vdash . *3{\cdot}2 . \supset \vdash :. xPx . \supset : xSx . \supset . xSx . xPx .$

[$*34{\cdot}1$] $\supset . x(S \mid P)x$ (2)

$\vdash . (1) . (2) . \supset \vdash :. xPx . \supset : Q \in \text{Pot}{}^{\backprime}P . \supset_Q . xQx .$

[$*50{\cdot}24$] $\supset_Q . \sim (Q \mathrel{\mathsf{G}} J)$ (3)

$\vdash . (3) . \text{Transp} . \supset \vdash : (\exists Q) . Q \in \text{Pot}{}^{\backprime}P . Q \mathrel{\mathsf{G}} J . \supset . \sim (xPx) .$

[$*50{\cdot}24$] $\supset . P \mathrel{\mathsf{G}} J : \supset \vdash . \text{Prop}$

$*200{\cdot}38$. $\vdash : P_{\text{po}} \mathrel{\mathsf{G}} J . \supset . P_{\text{po}} = P_* \mathbin{\dot{\cap}} J$ [$*91{\cdot}541$]

***200·381.** $\vdash : P_{po} \mathrel{\dot{\subset}} J . \supset . \overrightarrow{P_{po}}\text{‘}x \cap \overleftarrow{P_*}\text{‘}x = \Lambda . \overleftarrow{P_{po}}\text{‘}x \cap \overrightarrow{P_*}\text{‘}x = \Lambda$

Dem.

$\vdash . *91{\cdot}56 . \supset \vdash : \text{Hp} . \supset . P_{po}{}^2 \mathrel{\dot{\subset}} J .$

$[*200{\cdot}361] \quad \supset . \overrightarrow{P_{po}}\text{‘}x \cap (\iota\text{‘}x \cup \overleftarrow{P_{po}}\text{‘}x) = \Lambda . \overleftarrow{P_{po}}\text{‘}x \cap (\overrightarrow{P_{po}}\text{‘}x \cup \iota\text{‘}x) = \Lambda .$

$[*91{\cdot}54] \quad \supset . \overrightarrow{P_{po}}\text{‘}x \cap \overleftarrow{P_*}\text{‘}x = \Lambda . \overleftarrow{P_{po}}\text{‘}x \cap \overrightarrow{P_*}\text{‘}x = \Lambda : \supset \vdash . \text{Prop}$

***200·39.** $\vdash : P_{po} \mathrel{\dot{\subset}} J . x \in C\text{‘}P . \supset . \overrightarrow{P_*}\text{‘}x \cap \overleftarrow{P_*}\text{‘}x = \iota\text{‘}x$

Dem.

$\vdash . *91{\cdot}54 . \quad \supset \vdash : \text{Hp} . \supset . \overrightarrow{P_*}\text{‘}x \cap \overleftarrow{P_*}\text{‘}x = (\overrightarrow{P_{po}}\text{‘}x \cup \iota\text{‘}x) \cap (\overleftarrow{P_{po}}\text{‘}x \cup \iota\text{‘}x)$

$[*22{\cdot}69] \quad = (\overrightarrow{P_{po}}\text{‘}x \cap \overleftarrow{P_{po}}\text{‘}x) \cup \iota\text{‘}x \qquad (1)$

$\vdash . *91{\cdot}56 . \quad \supset \vdash : y \in \overrightarrow{P_{po}}\text{‘}x \cap \overleftarrow{P_{po}}\text{‘}x . \supset . y P_{po} y \qquad (2)$

$\vdash . (2) . \text{Transp} . \supset \vdash : \text{Hp} . \supset . \overrightarrow{P_{po}}\text{‘}x \cap \overleftarrow{P_{po}}\text{‘}x = \Lambda \qquad (3)$

$\vdash . (1) . (3) . \supset \vdash . \text{Prop}$

***200·391.** $\vdash : P_{po} \mathrel{\dot{\subset}} J . \supset . \overrightarrow{P_*} ; P \,\text{smor}\, P . \overrightarrow{P_*} \upharpoonright C\text{‘}P \in (\overrightarrow{P_*} ; P) \,\overline{\text{smor}}\, P$

Dem.

$\vdash . *90{\cdot}12 . \supset \vdash : \text{Hp} . x, y \in C\text{‘}P . \overrightarrow{P_*}\text{‘}x = \overrightarrow{P_*}\text{‘}y . \supset . x P_* y . y P_* x .$

$[*200{\cdot}39] \quad \supset . x = y \qquad (1)$

$\vdash . (1) . *151{\cdot}24 . \supset \vdash . \text{Prop}$

The above proposition is useful in the theory of segments.

The following propositions are concerned with the ideas of relation-arithmetic. Analogous propositions will be proved for transitiveness and connection in *201 and *202, whence analogous propositions concerning series will be deduced in *204.

***200·4.** $\vdash : P \mathbin{\dot{\updownarrow}} Q \in \text{Rl}\text{‘}J . \equiv . P, Q \in \text{Rl}\text{‘}J . C\text{‘}P \cap C\text{‘}Q = \Lambda$

Dem.

$\vdash . *23{\cdot}59 . *160{\cdot}1 . \supset$

$\vdash : P \mathbin{\dot{\updownarrow}} Q \in \text{Rl}\text{‘}J . \equiv . P, Q \in \text{Rl}\text{‘}J . C\text{‘}P \uparrow C\text{‘}Q \mathrel{\dot{\subset}} J .$

$[*200{\cdot}32] \quad \equiv . P, Q \in \text{Rl}\text{‘}J . C\text{‘}P \cap C\text{‘}Q = \Lambda : \supset \vdash . \text{Prop}$

This proposition is part of the proof that the sum of two mutually exclusive series is a series.

***200·41.** $\vdash : P \mathbin{\dot{+}\!\!\to} x \mathrel{\dot{\subset}} J . \equiv . x \mathbin{\leftarrow\!\!\dot{+}} P \mathrel{\dot{\subset}} J . \equiv . P \mathrel{\dot{\subset}} J . x \sim\in C\text{‘}P \quad [*23{\cdot}59 . *200{\cdot}32]$

***200·42.** $\vdash : \Sigma\text{‘}P \mathrel{\dot{\subset}} J . \equiv . C\text{‘}P \subset \text{Rl}\text{‘}J . F ; P \mathrel{\dot{\subset}} J$

Dem.

$\vdash . *23{\cdot}59 . *162{\cdot}1 . \supset \vdash : \Sigma\text{‘}P \mathrel{\dot{\subset}} J . \equiv . \dot{s}\text{‘}C\text{‘}P \mathrel{\dot{\subset}} J . F ; P \mathrel{\dot{\subset}} J .$

$[*61{\cdot}52] \quad \equiv . C\text{‘}P \subset \text{Rl}\text{‘}J . F ; P \mathrel{\dot{\subset}} J : \supset \vdash . \text{Prop}$

The following propositions ($*200{\cdot}421{\cdot}422{\cdot}423$) are lemmas for $*204{\cdot}53$.

$*200{\cdot}421$. $\vdash : P \in \mathrm{Rel}^2\,\mathrm{excl}\,.\,P \mathrel{G} J\,.\,Q \in C\text{‘}P\,.\,\supset\,.\,Q = (\Sigma\text{‘}P) \upharpoonright C\text{‘}Q$

Dem.

$\vdash\,.\,*163{\cdot}11\,.\,*162{\cdot}13\,.\,\supset \vdash :: \mathrm{Hp}\,.\,\supset :.\, x\{(\Sigma\text{‘}P)\upharpoonright C\text{‘}Q\}\,y\,.\equiv :$

$(\exists R)\,.\,R \in C\text{‘}P\,.\,x, y \in C\text{‘}Q\,.\,xRy\,.\,R = Q\,.\,\mathrm{v}\,.$

$(\exists R, S)\,.\,RPS\,.\,x, y \in C\text{‘}Q\,.\,x \in C\text{‘}R\,.\,y \in C\text{‘}S\,.\,R = Q\,.\,S = Q :$

$[*13{\cdot}195{\cdot}22]\quad \equiv : xQy\,.\,\mathrm{v}\,.\,QPQ\,.\,x, y \in C\text{‘}Q :$

$[*50{\cdot}24.\mathrm{Hp}]\quad \equiv : xQy :: \supset \vdash\,.\,\mathrm{Prop}$

$*200{\cdot}422$. $\vdash : \Sigma\text{‘}P \mathrel{G} J\,.\,\supset\,.\,P \upharpoonright (-\iota\text{‘}\dot\Lambda) \mathrel{G} J$

Dem.

$\vdash\,.\,*162{\cdot}13\,.\,*50{\cdot}24\,.\,\supset \vdash :: \mathrm{Hp}\,.\,\supset :.\, QPR\,.\,\supset : x \in C\text{‘}Q\,.\,y \in C\text{‘}R\,.\,\supset\,.\,x \neq y :$

$[*24{\cdot}37]\quad \supset : C\text{‘}Q \cap C\text{‘}R = \Lambda :$

$[*24{\cdot}57]\quad \supset : \dot{\exists}\,!\,Q\,.\,\supset\,.\,C\text{‘}Q \neq C\text{‘}R\,.$

$[*30{\cdot}37]\quad \supset\,.\,Q \neq R :: \supset \vdash\,.\,\mathrm{Prop}$

$*200{\cdot}423$. $\vdash :.\, P \in \mathrm{Rel}^2\,\mathrm{excl}\,.\,\Lambda \sim\in C\text{‘}P\,.\,\supset : \Sigma\text{‘}P \mathrel{G} J\,.\equiv .\,P \mathrel{G} J\,.\,C\text{‘}P \subset \mathrm{Rl}\text{‘}J$

Dem.

$\vdash\,.\,*200{\cdot}422{\cdot}42\,.\quad \supset \vdash : \mathrm{Hp}\,.\,\Sigma\text{‘}P \mathrel{G} J\,.\,\supset\,.\,P \mathrel{G} J\,.\,C\text{‘}P \subset \mathrm{Rl}\text{‘}J \quad (1)$

$\vdash\,.\,*61{\cdot}52\,.\quad \supset \vdash : C\text{‘}P \subset \mathrm{Rl}\text{‘}J\,.\,\supset\,.\,\dot{s}\text{‘}C\text{‘}P \mathrel{G} J \quad (2)$

$\vdash\,.\,*163{\cdot}12\,.\,*200{\cdot}21\,.\,\supset \vdash : \mathrm{Hp}\,.\,P \mathrel{G} J\,.\,\supset\,.\,F \,\dot{;}\, P \mathrel{G} J \quad (3)$

$\vdash\,.\,(2)\,.\,(3)\,.\,*162{\cdot}1\,.\quad \supset \vdash : \mathrm{Hp}\,.\,P \mathrel{G} J\,.\,C\text{‘}P \subset \mathrm{Rl}\text{‘}J\,.\,\supset\,.\,\Sigma\text{‘}P \mathrel{G} J \quad (4)$

$\vdash\,.\,(1)\,.\,(4)\,.\,\supset \vdash\,.\,\mathrm{Prop}$

$*200{\cdot}43$. $\vdash : P \mathrel{G} J\,.\,\supset\,.\,\Pi\text{‘}P =$

$$\hat{M}\hat{N}\{M, N \in F_\Delta\text{‘}C\text{‘}P : (\exists Q)\,.\,(M\text{‘}Q)\,Q\,(N\text{‘}Q)\,.\,M \upharpoonright \overrightarrow{P}\text{‘}Q = N \upharpoonright \overrightarrow{P}\text{‘}Q\}$$

Dem.

$\vdash\,.\,*4{\cdot}71\,.\,*172{\cdot}1\,.\,\supset \vdash : \mathrm{Hp}\,.\,\supset\,.$

$\Pi\text{‘}P = \hat{M}\hat{N}\{M, N \in F_\Delta\text{‘}C\text{‘}P :.\, (\exists Q) : (M\text{‘}Q)\,Q\,(N\text{‘}Q) : RPQ\,.\,\supset_R\,.\,M\text{‘}R = N\text{‘}R\}$

$[*35{\cdot}71.*71{\cdot}35]$

$= \hat{M}\hat{N}\{M, N \in F_\Delta\text{‘}C\text{‘}P : (\exists Q)\,.\,(M\text{‘}Q)\,Q\,(N\text{‘}Q)\,.\,M \upharpoonright \overrightarrow{P}\text{‘}Q = N \upharpoonright \overrightarrow{P}\text{‘}Q\}\,.\,\supset \vdash\,.\,\mathrm{Prop}$

The following propositions, with the exception of $*200{\cdot}52$, are concerned with $p\text{‘}\overrightarrow{P}\text{‘‘}\alpha$ and $p\text{‘}\overleftarrow{P}\text{‘‘}\alpha$, *i.e.* the class of terms preceding (or succeeding) the whole of α.

$*200{\cdot}5$. $\vdash : P \mathrel{G} J\,.\,\supset\,.\,\alpha \cap p\text{‘}\overrightarrow{P}\text{‘‘}\alpha = \Lambda\,.\,\alpha \cap p\text{‘}\overleftarrow{P}\text{‘‘}\alpha = \Lambda$

Dem. $\vdash\,.\,*40{\cdot}51\,.\,\supset \vdash :.\, x \in \alpha \cap p\text{‘}\overrightarrow{P}\text{‘‘}\alpha\,.\,\supset : x \in \alpha : y \in \alpha\,.\,\supset_y\,.\,xPy :$

$[*10{\cdot}26]\quad \supset : xPx :$

$[*50{\cdot}24]\quad \supset : \sim(P \mathrel{G} J) \quad (1)$

$\vdash\,.\,(1)\,.\,\mathrm{Transp}\,.\,\supset \vdash : \mathrm{Hp}\,.\,\supset\,.\,\alpha \cap p\text{‘}\overrightarrow{P}\text{‘‘}\alpha = \Lambda \quad (2)$

Similarly $\quad \vdash : \mathrm{Hp}\,.\,\supset\,.\,\alpha \cap p\text{‘}\overleftarrow{P}\text{‘‘}\alpha = \Lambda \quad (3)$

$\vdash\,.\,(2)\,.\,(3)\,.\,\supset \vdash\,.\,\mathrm{Prop}$

***200·51.** $\vdash : P \Subset J \,.\, \dot{\exists} ! P \,.\, \supset \,.\, p^{\backprime}\overrightarrow{P}^{\backprime\backprime}C^{\backprime}P = \Lambda \,.\, p^{\backprime}\overleftarrow{P}^{\backprime\backprime}C^{\backprime}P = \Lambda$

Dem. $\vdash \,.\, *40\cdot62 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, p^{\backprime}\overrightarrow{P}^{\backprime\backprime}C^{\backprime}P \subset C^{\backprime}P \,.$

$[*22\cdot621] \qquad \supset \,.\, p^{\backprime}\overrightarrow{P}^{\backprime\backprime}C^{\backprime}P = C^{\backprime}P \cap p^{\backprime}\overrightarrow{P}^{\backprime\backprime}C^{\backprime}P$

$[*200\cdot5] \qquad = \Lambda \qquad (1)$

Similarly $\vdash : \mathrm{Hp} \,.\, \supset \,.\, p^{\backprime}\overleftarrow{P}^{\backprime\backprime}C^{\backprime}P = \Lambda \qquad (2)$

$\vdash \,.\, (1) \,.\, (2) \,.\, \supset \vdash \,.\, \mathrm{Prop}$

***200·52.** $\vdash : P \Subset J \,.\, \supset \,.\, C^{\backprime}P \sim\in \overrightarrow{P}^{\backprime\backprime}C^{\backprime}P$

Dem. $\vdash \,.\, *50\cdot24 \,.\, \supset \vdash :. \mathrm{Hp} \,.\, \supset : x \in C^{\backprime}P \,.\, \supset_x \,.\, x \sim\in \overrightarrow{P}^{\backprime}x \,.$

$[*13\cdot14] \qquad \supset_x \,.\, C^{\backprime}P \neq \overrightarrow{P}^{\backprime}x :$

$[*37\cdot7.\mathrm{Transp}] \qquad \supset : C^{\backprime}P \sim\in \overrightarrow{P}^{\backprime\backprime}C^{\backprime}P :. \supset \vdash \,.\, \mathrm{Prop}$

This proposition is often used in the theory of well-ordered series.

***200·53.** $\vdash : P^2 \Subset J \,.\, \supset \,.\, P^{\backprime\backprime}\alpha \cap p^{\backprime}\overleftarrow{P}^{\backprime\backprime}\alpha = \Lambda \,.\, \breve{P}^{\backprime\backprime}\alpha \cap p^{\backprime}\overrightarrow{P}^{\backprime\backprime}\alpha = \Lambda$

Dem.

$\vdash \,.\, *37\cdot1 \,.\, *40\cdot53 \,.\, \supset \vdash :. x \in P^{\backprime\backprime}\alpha \cap p^{\backprime}\overleftarrow{P}^{\backprime\backprime}\alpha \,.\, \supset : (\exists y) \,.\, y \in \alpha \,.\, xPy : y \in \alpha \,.\, \supset_y \,.\, yPx :$

$[*10\cdot56] \qquad \supset : (\exists y) \,.\, xPy \,.\, yPx :$

$[*34\cdot5] \qquad \supset : xP^2x :$

$[*50\cdot24] \qquad \supset : \sim(P^2 \Subset J) \qquad (1)$

$\vdash \,.\, (1) \,.\, \mathrm{Transp} \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, (x) \,.\, x \sim\in P^{\backprime\backprime}\alpha \cap p^{\backprime}\overleftarrow{P}^{\backprime\backprime}\alpha \qquad (2)$

Similarly $\vdash : \mathrm{Hp} \,.\, \supset \,.\, (x) \,.\, x \sim\in \breve{P}^{\backprime\backprime}\alpha \cap p^{\backprime}\overrightarrow{P}^{\backprime\backprime}\alpha \qquad (3)$

$\vdash \,.\, (2) \,.\, (3) \,.\, \supset \vdash \,.\, \mathrm{Prop}$

The above proposition is frequently used. If α is an existent class contained in $C^{\backprime}P$, $P^{\backprime\backprime}\alpha$ and $p^{\backprime}\overleftarrow{P}^{\backprime\backprime}\alpha$ are the two parts of the Dedekind "cut" determined by α (excluding the maximum of α, if any). The above proposition shows that these two parts are mutually exclusive.

***200·54.** $\vdash : P \Subset J \,.\, \dot{\exists} ! P \,.\, \supset \,.\, p^{\backprime}\overrightarrow{P}^{\backprime\backprime}\{C^{\backprime}P \cap p^{\backprime}\overleftarrow{P}^{\backprime\backprime}\alpha\} = p^{\backprime}\overrightarrow{P}^{\backprime\backprime}p^{\backprime}\overleftarrow{P}^{\backprime\backprime}\alpha$

Dem. $\vdash \,.\, *40\cdot62 \,.\, \supset \vdash : \exists ! \alpha \,.\, \supset \,.\, C^{\backprime}P \cap p^{\backprime}\overleftarrow{P}^{\backprime\backprime}\alpha = p^{\backprime}\overleftarrow{P}^{\backprime\backprime}\alpha \qquad (1)$

$\vdash \,.\, *40\cdot2 \,.\, \supset \vdash : \alpha = \Lambda \,.\, \supset \,.\, p^{\backprime}\overleftarrow{P}^{\backprime\backprime}\alpha = \mathrm{V} \,. \qquad (2)$

$[*40\cdot16] \qquad \supset \,.\, p^{\backprime}\overrightarrow{P}^{\backprime\backprime}p^{\backprime}\overleftarrow{P}^{\backprime\backprime}\alpha \subset p^{\backprime}\overrightarrow{P}^{\backprime\backprime}C^{\backprime}P \qquad (3)$

$\vdash \,.\, (3) \,.\, *200\cdot51 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \alpha = \Lambda \,.\, \supset \,.\, p^{\backprime}\overrightarrow{P}^{\backprime\backprime}p^{\backprime}\overleftarrow{P}^{\backprime\backprime}\alpha = \Lambda \qquad (4)$

$\vdash \,.\, (2) \,.\, *24\cdot26 \,.\, \supset \vdash : \alpha = \Lambda \,.\, \supset \,.\, C^{\backprime}P \cap p^{\backprime}\overleftarrow{P}^{\backprime\backprime}\alpha = C^{\backprime}P \qquad (5)$

$\vdash \,.\, (5) \,.\, *200\cdot51 \,.\, \supset \vdash : \mathrm{Hp} . \alpha = \Lambda . \supset . p^{\backprime}\overrightarrow{P}^{\backprime\backprime}(C^{\backprime}P \cap p^{\backprime}\overleftarrow{P}^{\backprime\backprime}\alpha) = \Lambda \qquad (6)$

$\vdash \,.\, (1) \,.\, (4) \,.\, (6) \,.\, \supset \vdash \,.\, \mathrm{Prop}$

This proposition is a lemma whose purpose is to avoid the necessity of introducing the hypothesis $\exists ! \alpha$ in proofs in which it is not really necessary. The first use of this proposition occurs in *206·551.

$*201$. TRANSITIVE RELATIONS

Summary of $*201$.

There are two main varieties of transitive relations, namely those that are symmetrical ($P = \breve{P}$), and those that are asymmetrical ($P \mathrel{\dot{\cap}} \breve{P} = \dot{\Lambda}$). Transitive *symmetrical* relations have the formal properties of equality; examples of such relations have occurred above, *e.g.* identity, similarity, and likeness. The propositions of the present number, however, are rather such as will be useful in connection with transitive *asymmetrical* relations, since they are intended to be applied to series.

We denote the class of transitive relations by "trans"; thus

$$\text{trans} = \hat{P}(P^2 \mathrel{\dot{\subset}} P) \quad \text{Df.}$$

Many propositions of this number are analogous to propositions whose numbers have the same decimal part in $*200$. Such are: If P is transitive, so is its converse ($*201{\cdot}11$), and so is any relation which is like P ($*201{\cdot}211$); $\dot{\Lambda}$ and $x \downarrow y$ are transitive ($*201{\cdot}3{\cdot}31$); if P is transitive, so is $P \upharpoonright\!\!\!\!\upharpoonleft \alpha$ ($*201{\cdot}33$). The propositions $*201{\cdot}4$—$\cdot 42$, which deal with the ideas of relation-arithmetic, are also analogous to $*200{\cdot}4$—$\cdot 42$.

Most of the other propositions of this number, however, have no analogues in $*200$. Among the most important of these are the following:

$*201{\cdot}14$. $\vdash : P \in \text{trans} \,.\, xPy \,.\, \supset \,.\, \overrightarrow{P}\text{‘}x \subset \overrightarrow{P}\text{‘}y$

$*201{\cdot}15$. $\vdash \,.\, R_* \in \text{trans}$

$*201{\cdot}18$. $\vdash : P^2 \mathrel{\dot{\subset}} P \,.\, \supset \,.\, P_{\text{po}} = P \,.\, P_* = P \mathrel{\dot{\cup}} I \upharpoonright C\text{‘}P$

This proposition is very important, since it effects an immense simplification in the use of all propositions involving P_{po} or P_*, when these propositions are to be applied to transitive relations. Owing to the above proposition, P_{po} drops out where transitive relations are concerned. P_*, on the other hand, remains useful: if $y \in C\text{‘}P$, "xP_*y" will mean "x precedes or is y," which, if P generates a series of which x and y are members, is equivalent to "x does not follow y."

We have a series of propositions ($*201{\cdot}5$—$\cdot 56$) on $P\text{“}\alpha$ and $p\text{‘}\overrightarrow{P}\text{“}\alpha$. The chief of these are

$*201{\cdot}5$. $\vdash : P \in \text{trans} \,.\, \supset \,.\, P\text{“}P\text{“}\alpha \subset P\text{“}\alpha$

$*201{\cdot}501$. $\vdash : P \in \text{trans} \,.\, \supset \,.\, P\text{“}\overrightarrow{P}\text{‘}x \subset \overrightarrow{P}\text{‘}x$

These two propositions express the fact that a predecessor of a predecessor is a predecessor.

$*201\cdot52$. $\vdash : P \in \text{trans} . \supset . P_*``\alpha = P``\alpha \cup (\alpha \cap C`P)$

Thus if $\alpha \subset C`P$, $P_*``\alpha$ consists of α together with the predecessors of its members.

$*201\cdot521$. $\vdash : P \in \text{trans} . x \in C`P . \supset . \overrightarrow{P_*}`x = \overrightarrow{P}`x \cup \iota`x$

$*201\cdot55$. $\vdash : P \in \text{trans} . \supset . P``(\alpha \cup P``\alpha) = P``\alpha$

We have next a set of important propositions on $P \dot{-} P^2$ and P_1. The chief are

$*201\cdot63$. $\vdash : P \in \text{trans} \cap \text{Rl}`J . \supset . P_1 = P \dot{-} P^2$

$*201\cdot65$. $\vdash :. P \in \text{trans} \cap \text{Rl}`J . \supset : P_1 = \dot{\Lambda} . \equiv . P^2 = P$

On these two propositions, see the notes appended to them below.

$*201\cdot01$. $\text{trans} = \hat{P}(P^2 \mathrel{\dot{\subset}} P)$ Df

$*201\cdot1$. $\vdash : P \in \text{trans} . \equiv . P^2 \mathrel{\dot{\subset}} P$ $[(*201\cdot01)]$

$*201\cdot11$. $\vdash : P \in \text{trans} . \equiv . \breve{P} \in \text{trans}$

Dem.

$\vdash . *201\cdot1 . *31\cdot4 . \supset \vdash : P \in \text{trans} . \equiv . \text{Cnv}`P^2 \mathrel{\dot{\subset}} \breve{P} .$

$[*34\cdot63 . *201\cdot1] \qquad \equiv . \breve{P} \in \text{trans} : \supset \vdash . \text{Prop}$

$*201\cdot12$. $\vdash :. P \in \text{trans} . \supset : P \mathrel{\dot{\subset}} J . \equiv . P^2 \mathrel{\dot{\subset}} J . \equiv . P \mathbin{\dot{\cap}} \breve{P} = \dot{\Lambda}$ $[*50\cdot47]$

In virtue of this proposition, being contained in diversity is equivalent (where transitive relations are concerned) to asymmetry. This is not in general the case with relations which are not transitive; thus *e.g.* diversity itself is contained in diversity, but is symmetrical.

$*201\cdot13$. $\vdash . \text{Rl}`I \subset \text{trans}$

Dem.

$\vdash . *34\cdot34 . \supset \vdash : R \mathrel{\dot{\subset}} I . \supset . R^2 \mathrel{\dot{\subset}} R \mid I .$

$[*50\cdot4] \qquad \supset . R^2 \mathrel{\dot{\subset}} R : \supset \vdash . \text{Prop}$

$*201\cdot14$. $\vdash : P \in \text{trans} . xPy . \supset . \overrightarrow{P}`x \subset \overrightarrow{P}`y$

Dem.

$\vdash . *201\cdot1 . \supset \vdash : \text{Hp} . zPx . \supset . zPy \qquad (1)$

$\vdash . (1) . *32\cdot18 . \supset \vdash . \text{Prop}$

The following propositions ($*201\cdot15$—$\cdot19$) are concerned with R_* and R_{po}.

$*201\cdot15$. $\vdash . R_* \in \text{trans}$ $[*90\cdot17]$

$*201\cdot16$. $\vdash . R_{po} \in \text{trans}$ $[*91\cdot56]$

This proposition is important, since it often happens that a series is given as defined by a one-one relation R, as in $*122$ for example, and in such cases R_{po} is a serial relation in our present sense. By the above proposition, R_{po}

is always transitive; by $*96{\cdot}421$, R_{po} is connected when confined to the posterity of a given term, provided $R \in \text{Cls} \to 1$; by $*96{\cdot}23$, if $R \in 1 \to \text{Cls}$ and xBR, R_{po} is contained in diversity throughout the posterity of x. Thus if R is a one-one, R_{po} confined to any family which has a beginning will be a serial relation.

***201·17.** $\vdash : P^2 \subseteq P \,.\, Q \in \text{Pot}`P \,.\, \supset .\, Q \subseteq P$

Dem. $\vdash .\, *34{\cdot}34 \,.\, \supset \vdash :.\, \text{Hp} \,.\, \supset : S \subseteq P \,.\, \supset_S .\, S \,|\, P \subseteq P$ (1)

$\vdash .\, *91{\cdot}171 \dfrac{S \subseteq P}{\phi S} .\, \supset$

$\vdash :.\, Q \in \text{Pot}`P : S \subseteq P \,.\, \supset_S .\, S \,|\, P \subseteq P : P \subseteq P : \supset .\, Q \subseteq P$ (2)

$\vdash .\, (1) \,.\, (2) \,.\, *23{\cdot}42 \,.\, \supset \vdash .\, \text{Prop}$

***201·18.** $\vdash : P^2 \subseteq P \,.\, \supset .\, P_{\text{po}} = P \,.\, P_* = P \cup I \upharpoonright C`P$

Dem. $\vdash .\, *201{\cdot}17 \,.\, *41{\cdot}151 \,.\, (*91{\cdot}05) \,.\, \supset \vdash : \text{Hp} \,.\, \supset .\, P_{\text{po}} \subseteq P$ (1)

$\vdash .\, (1) \,.\, *91{\cdot}502 \,.\quad \supset \vdash : \text{Hp} \,.\, \supset .\, P_{\text{po}} = P$ (2)

$\vdash .\, (2) \,.\, *91{\cdot}54 \,.\, \supset \vdash .\, \text{Prop}$

This proposition is important, since it simplifies all propositions concerning P_{po} and P_* in case P is transitive. The following proposition is an instance of this simplification.

***201·19.** $\vdash : P \in \text{trans} \,.\, \supset .\, P(x - y) = \overleftarrow{P}`x \cap \overrightarrow{P}`y$ [$*201{\cdot}18 \,.\, (*121{\cdot}01)$]

The following propositions ($*201{\cdot}2$—$\cdot 22$) are concerned in proving that transitiveness is unaffected by likeness-transformations, and therefore belongs to every member of a relation-number or to none.

***201·2.** $\vdash : S \in \text{Cls} \to 1 \,.\, \text{C}`Q \subset \text{C}`S \,.\, \supset .\, (S;Q)^2 = S;Q^2$

Dem. $\vdash .\, *150{\cdot}1 \,.\, \supset \vdash .\, (S;Q)^2 = S \,|\, Q \,|\, \breve{S} \,|\, S \,|\, Q \,|\, \breve{S}$ (1)

$\vdash .\, *72{\cdot}601 \,.\, \supset \vdash : \text{Hp} \,.\, \supset .\, Q \,|\, \breve{S} \,|\, S = Q$ (2)

$\vdash .\, (1) \,.\, (2) \,.\, \supset \vdash : \text{Hp} \,.\, \supset .\, (S;Q)^2 = S \,|\, Q^2 \,|\, \breve{S} : \supset \vdash .\, \text{Prop}$

***201·201.** $\vdash : S \in \text{Cls} \to 1 \,.\, \text{D}`Q \subset \text{C}`S \,.\, \supset .\, (S;Q)^2 = S;Q^2$
[Proof as in $*201{\cdot}2$]

***201·21.** $\vdash : S \in \text{Cls} \to 1 \,.\, Q \in \text{trans} \,.\, \supset .\, S;Q \in \text{trans}$

Dem. $\vdash .\, *150{\cdot}36 \,.\, *35{\cdot}452 \,.\, \supset \vdash .\, S;Q = S;Q \upharpoonright \text{C}`S$ (1)

$\vdash .\, (1) \,.\, *201{\cdot}2 \,.\quad \supset \vdash : \text{Hp} \,.\, \supset .\, (S;Q)^2 = S;(Q \upharpoonright \text{C}`S)^2 \,.$

[$*150{\cdot}31 . *201{\cdot}1$] $\supset .\, (S;Q)^2 \subseteq S;Q : \supset \vdash .\, \text{Prop}$

***201·211.** $\vdash : P \in \text{trans} \,.\, Q \,\text{smor}\, P \,.\, \supset .\, Q \in \text{trans}$ [$*201{\cdot}21 \,.\, *151{\cdot}1$]

This shows that transitiveness is a property which is unchanged by likeness-transformations. Hence

***201·212.** $\vdash : P \in \text{trans} \,.\, \supset .\, \text{Nr}`P \subset \text{trans}$ [$*201{\cdot}211$]

***201·22.** $\vdash : P \epsilon \text{trans} . \equiv . \text{N}_0\text{r}'P \subset \text{trans} . \equiv . \exists ! \text{N}_0\text{r}'P \cap \text{trans}$
[Proof as in *200·22]

***201·3.** $\vdash . \dot{\Lambda} \epsilon \text{trans}$

Dem. $\vdash . *34{\cdot}32 . \quad \supset \vdash . \dot{\Lambda}^2 = \dot{\Lambda}$ (1)

$\vdash . (1) . *23{\cdot}42 . \supset \vdash . \dot{\Lambda}^2 \mathrel{\dot\subset} \dot{\Lambda} . \supset \vdash . \text{Prop}$

***201·31.** $\vdash . x \downarrow y \epsilon \text{trans}$

Dem. $\vdash . *55{\cdot}13 . \supset \vdash : z (x \downarrow y)^2 w . \equiv . (\exists u) . z = x . u = y . u = x . w = y .$

$[*10{\cdot}35] \qquad \supset . z = x . w = y .$

$[*55{\cdot}13] \qquad \supset . z (x \downarrow y) w : \supset \vdash . \text{Prop}$

Unless $x = y$, $(x \downarrow y)^2 = \dot{\Lambda}$. A relation whose square is $\dot{\Lambda}$ is transitive, because $\dot{\Lambda}$ is contained in every relation.

***201·32.** $\vdash . \alpha \uparrow \beta \epsilon \text{trans}$

Dem. $\vdash . *35{\cdot}103 . \supset \vdash : x (\alpha \uparrow \beta)^2 z . \equiv . (\exists y) . x \epsilon \alpha . y \epsilon \beta . y \epsilon \alpha . z \epsilon \beta .$

$[*10{\cdot}35] \qquad \supset . x \epsilon \alpha . z \epsilon \beta .$

$[*35{\cdot}103] \qquad \supset . x (\alpha \uparrow \beta) z : \supset \vdash . \text{Prop}$

***201·33.** $\vdash : P \epsilon \text{trans} . \supset . P \upharpoonright \alpha \epsilon \text{trans}$

Dem. $\vdash . *36{\cdot}13 . \supset \vdash : x (P \upharpoonright \alpha)^2 z . \equiv . (\exists y) . x, y, z \epsilon \alpha . xPy . yPz$ (1)

$\vdash . (1) . \quad \supset \vdash :. \text{Hp} . \supset : x (P \upharpoonright \alpha)^2 z . \supset . (\exists y) . x, y, z \epsilon \alpha . xPz .$

$[*10{\cdot}35 . *36{\cdot}13] \qquad \supset . x (P \upharpoonright \alpha) z :. \supset \vdash . \text{Prop}$

The following propositions (*201·4—·42) are concerned with the ideas of relation-arithmetic.

***201·4.** $\vdash : P, Q \epsilon \text{trans} . C'P \cap C'Q = \Lambda . \supset . P \mathbin{\dot{+}} Q \epsilon \text{trans}$

Dem.

$\vdash . *160{\cdot}51 . \supset \vdash : \text{Hp} . \supset . (P \mathbin{\dot{+}} Q)^2 = P^2 \mathbin{\dot\cup} Q^2 \mathbin{\dot\cup} \text{D}'P \uparrow C'Q \mathbin{\dot\cup} C'P \uparrow \text{Ꮖ}'Q$ (1)

$\vdash . *201{\cdot}1 . \quad \supset \vdash : \text{Hp} . \supset . P^2 \mathrel{\dot\subset} P . Q^2 \mathrel{\dot\subset} Q$ (2)

$\vdash . *35{\cdot}432{\cdot}82 . \quad \supset \vdash . \text{D}'P \uparrow C'Q \mathrel{\dot\subset} C'P \uparrow C'Q . C'P \uparrow \text{Ꮖ}'Q \mathrel{\dot\subset} C'P \uparrow C'Q$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash : \text{Hp} . \supset . (P \mathbin{\dot{+}} Q)^2 \mathrel{\dot\subset} P \mathbin{\dot\cup} Q \mathbin{\dot\cup} C'P \uparrow C'Q : \supset \vdash . \text{Prop}$

***201·401.** $\vdash :. C'P \cap C'Q = \Lambda . \supset : P \mathbin{\dot{+}} Q \epsilon \text{trans} . \equiv . P, Q \epsilon \text{trans}$

Dem.

$\vdash . *160{\cdot}51 . \supset$

$\vdash :. \text{Hp} . \supset : P \mathbin{\dot{+}} Q \epsilon \text{trans} . \equiv . P^2 \mathbin{\dot\cup} Q^2 \mathbin{\dot\cup} \text{D}'P \uparrow C'Q \mathbin{\dot\cup} C'P \uparrow \text{Ꮖ}'Q \mathrel{\dot\subset} P \mathbin{\dot{+}} Q .$

$[*160{\cdot}1] \qquad \equiv . P^2 \mathbin{\dot\cup} Q^2 \mathrel{\dot\subset} P \mathbin{\dot{+}} Q .$

$[*160{\cdot}5] \qquad \supset . (P^2 \mathbin{\dot\cup} Q^2) \upharpoonright C'P \mathrel{\dot\subset} P . (P^2 \mathbin{\dot\cup} Q^2) \upharpoonright C'Q \mathrel{\dot\subset} Q .$

$[*36{\cdot}4 . *34{\cdot}56] \qquad \supset . P^2 \mathrel{\dot\subset} P . Q^2 \mathrel{\dot\subset} Q$ (1)

$\vdash . (1) . *201{\cdot}4 . \supset \vdash . \text{Prop}$

$*201{\cdot}41$. $\vdash :. x \sim\epsilon C{`}P . \supset : P \epsilon \text{trans} . \equiv . P \mapsto x \epsilon \text{trans} . \equiv . x \leftarrow P \epsilon \text{trans}$

Dem.

$\vdash . *34{\cdot}301 . \supset \vdash : \text{Hp} . \supset . (C{`}P \uparrow \iota{`}x) | P = \dot{\Lambda} .$

$[*161{\cdot}1] \quad \supset . (P \mapsto x)^2 = P^2 \cup (C{`}P \uparrow \iota{`}x)^2 \cup P | (C{`}P \uparrow \iota{`}x)$

$[*35{\cdot}881] \quad = P^2 \cup (C{`}P \uparrow \iota{`}x)^2 \cup (D{`}P \uparrow \iota{`}x)$

$[*35{\cdot}895] \quad = P^2 \cup (D{`}P \uparrow \iota{`}x) \quad (1)$

$\vdash . (1) . *201{\cdot}1 . \supset$

$\vdash :. \text{Hp} . \supset : (P \mapsto x) \epsilon \text{trans} . \equiv . P^2 \cup (D{`}P \uparrow \iota{`}x) \mathrel{\dot{\subset}} P \cup (C{`}P \uparrow \iota{`}x) .$

$[*35{\cdot}432{\cdot}82] \quad \equiv . P^2 \mathrel{\dot{\subset}} P \cup (C{`}P \uparrow \iota{`}x) \quad (2)$

$\vdash . *33{\cdot}33 . *34{\cdot}56 . *35{\cdot}86 . \supset \vdash : \text{Hp} . \supset . P^2 \mathbin{\dot{\cap}} (C{`}P \uparrow \iota{`}x) = \dot{\Lambda} \quad (3)$

$\vdash . (2) . (3) . *25{\cdot}49 . \supset \vdash :. \text{Hp} . \supset : P \mapsto x \epsilon \text{trans} . \equiv . P^2 \mathrel{\dot{\subset}} P .$

$[*201{\cdot}1] \quad \equiv . P \epsilon \text{trans} \quad (4)$

Similarly $\quad \vdash :. \text{Hp} . \supset : x \leftarrow P \epsilon \text{trans} . \equiv . P \epsilon \text{trans} \quad (5)$

$\vdash . (4) . (5) . \supset \vdash . \text{Prop}$

$*201{\cdot}411$. $\vdash : z \neq x . z \neq y . \supset . x \downarrow y \mapsto z \epsilon \text{trans} \quad [*201{\cdot}41{\cdot}31]$

$*201{\cdot}42$. $\vdash : P \epsilon \text{trans} \cap \text{Rel}^2 \text{excl} . C{`}P \subset \text{trans} . \supset . \Sigma{`}P \epsilon \text{trans}$

Dem.

$\vdash . *162{\cdot}1 . \supset$

$\vdash . (\Sigma{`}P)^2 = (\dot{s}{`}C{`}P)^2 \cup (F{;}P)^2 \cup (\dot{s}{`}C{`}P) | (F{;}P) \cup (F{;}P) | (\dot{s}{`}C{`}P) \quad (1)$

$\vdash . *41{\cdot}11 . \supset \vdash : x (\dot{s}{`}C{`}P)^2 z . \equiv . (\exists Q, R, y) . Q, R \epsilon C{`}P . xQy . yRz .$

$[*33{\cdot}17] \quad \equiv . (\exists Q, R, y) . Q, R \epsilon C{`}P . xQy . yRz . \exists ! C{`}Q \cap C{`}R \quad (2)$

$\vdash . (2) . *163{\cdot}11 . \supset$

$\vdash :. \text{Hp} . \supset : x (\dot{s}{`}C{`}P)^2 z . \supset . (\exists Q, R, y) . Q, R \epsilon C{`}P . xQy . yRz . Q = R .$

$[*13{\cdot}195] \quad \supset . (\exists Q) . Q \epsilon C{`}P . xQ^2z .$

$[*201{\cdot}1.\text{Hp}] \quad \supset . (\exists Q) . Q \epsilon C{`}P . xQz .$

$[*41{\cdot}11] \quad \supset . x (\dot{s}{`}C{`}P) z \quad (3)$

$\vdash . *201{\cdot}21 . *163{\cdot}12 . \supset \vdash : \text{Hp} . \supset . (F{;}P)^2 \mathrel{\dot{\subset}} F{;}P \quad (4)$

$\vdash . *34{\cdot}1 . *41{\cdot}11 . *150{\cdot}52 . \supset$

$\vdash : x (\dot{s}{`}C{`}P) | (F{;}P) z . \supset . (\exists Q, R, S, y) . Q \epsilon C{`}P . xQy . RPS . y \epsilon C{`}R . z \epsilon C{`}S \quad (5)$

$\vdash . (5) . *163{\cdot}11 . *13{\cdot}195 . \supset$

$\vdash :. \text{Hp} . \supset : x (\dot{s}{`}C{`}P) | (F{;}P) z . \supset . (\exists Q, S, y) . Q \epsilon C{`}P . xQy . QPS . z \epsilon C{`}S .$

$[*33{\cdot}17 . *150{\cdot}52] \quad \supset . x (F{;}P) z \quad (6)$

Similarly $\quad \vdash :. \text{Hp} . \supset : x (F{;}P) | (\dot{s}{`}C{`}P) z . \supset . x (F{;}P) z \quad (7)$

$\vdash . (1) . (3) . (4) . (6) . (7) . \supset$

$\vdash : \text{Hp} . \supset . (\Sigma{`}P)^2 \mathrel{\dot{\subset}} \dot{s}{`}C{`}P \cup F{;}P : \supset \vdash . \text{Prop}$

The following propositions ($*201{\cdot}5$—$\cdot 56$) are concerned with $P{``}\alpha$ and $p{`}\overrightarrow{P}{``}\alpha$, *i.e.* with the predecessors of some part of a class and the predecessors of the whole of a class.

$*201{\cdot}5$. $\vdash : P \epsilon \text{trans} . \supset . P{``}P{``}\alpha \subset P{``}\alpha \quad [*37{\cdot}33{\cdot}201]$

***201·501.** $\vdash : P \,\epsilon\, \text{trans} \,.\supset.\, P\text{‘‘}\overrightarrow{P}\text{‘}x \subset \overrightarrow{P}\text{‘}x$ [*53·301 . *201·5]

***201·51.** $\vdash : P \,\epsilon\, \text{trans} \,.\supset.\, P\text{‘‘}p\text{‘}\overrightarrow{P}\text{‘‘}\alpha \subset p\text{‘}\overrightarrow{P}\text{‘‘}\alpha$

Dem.

$\vdash . *37\cdot1 . *40\cdot51 . \supset \vdash :. x \,\epsilon\, P\text{‘‘}p\text{‘}\overrightarrow{P}\text{‘‘}\alpha \,.\equiv: (\exists y) : z \,\epsilon\, \alpha \,.\supset_z.\, yPz : xPy :$

[*5·31] $\supset : z \,\epsilon\, \alpha \,.\supset_z.\, xP^2z$ (1)

$\vdash . (1) . *201\cdot1 . \supset \vdash :. \text{Hp} \,.\supset: x \,\epsilon\, P\text{‘‘}p\text{‘}\overrightarrow{P}\text{‘‘}\alpha \,.\supset: z \,\epsilon\, \alpha \,.\supset_z.\, xPz :$

[*40·51] $\supset : x \,\epsilon\, p\text{‘}\overrightarrow{P}\text{‘‘}\alpha :. \supset \vdash . \text{Prop}$

***201·52.** $\vdash : P \,\epsilon\, \text{trans} \,.\supset.\, P_*\text{‘‘}\alpha = P\text{‘‘}\alpha \cup (\alpha \cap C\text{‘}P)$ [*91·543 . *201·18]

***201·521.** $\vdash : P \,\epsilon\, \text{trans} \,.\, x \,\epsilon\, C\text{‘}P \,.\supset.\, \overrightarrow{P_*}\text{‘}x = \overrightarrow{P}\text{‘}x \cup \iota\text{‘}x$ [*201·52 . *53·301]

***201·53.** $\vdash : P \,\epsilon\, \text{trans} \,.\supset.\, P_*\text{‘‘}P\text{‘‘}\alpha = P\text{‘‘}\alpha$ [*201·5·52 . *37·265]

***201·54.** $\vdash : P \,\epsilon\, \text{trans} \,.\supset.\, P_*\text{‘‘}p\text{‘}\overrightarrow{P}\text{‘‘}\alpha \subset p\text{‘}\overrightarrow{P}\text{‘‘}\alpha$ [*201·51·52]

***201·55.** $\vdash : P \,\epsilon\, \text{trans} \,.\supset.\, P\text{‘‘}(\alpha \cup P\text{‘‘}\alpha) = P\text{‘‘}\alpha$

Dem.

$\vdash . *201\cdot5 . \supset \vdash : \text{Hp} \,.\supset.\, P\text{‘‘}\alpha = P\text{‘‘}\alpha \cup P\text{‘‘}P\text{‘‘}\alpha$

[*37·22] $= P\text{‘‘}(\alpha \cup P\text{‘‘}\alpha) : \supset \vdash . \text{Prop}$

The following proposition is a lemma which is used in *205·192 and *206·24.

***201·56.** $\vdash : P \,\epsilon\, \text{trans} \,.\, \beta \subset P\text{‘‘}\alpha \,.\supset.$

$$P\text{‘‘}(\alpha \cup \beta) = P\text{‘‘}\alpha \,.\, p\text{‘}\overleftarrow{P}\text{‘‘}\{(\alpha \cup \beta) \cap C\text{‘}P\} = p\text{‘}\overleftarrow{P}\text{‘‘}(\alpha \cap C\text{‘}P)$$

Dem.

$\vdash . *37\cdot22 . \supset \vdash . P\text{‘‘}(\alpha \cup \beta) = P\text{‘‘}\alpha \cup P\text{‘‘}\beta$ (1)

$\vdash . *37\cdot2 . \supset \vdash : \text{Hp} \,.\supset.\, P\text{‘‘}\beta \subset P\text{‘‘}P\text{‘‘}\alpha \,.$

[*201·5] $\supset . P\text{‘‘}\beta \subset P\text{‘‘}\alpha$ (2)

$\vdash . (1) . (2) . \supset \vdash : \text{Hp} \,.\supset.\, P\text{‘‘}(\alpha \cup \beta) = P\text{‘‘}\alpha$ (3)

$\vdash . *40\cdot51 . *37\cdot265 . \supset$

$\vdash :: \text{Hp} \,.\supset:. z \,\epsilon\, p\text{‘}\overleftarrow{P}\text{‘‘}(\alpha \cap C\text{‘}P) \,.\, x \,\epsilon\, \beta \cap C\text{‘}P \,.\supset:$

$y \,\epsilon\, \alpha \cap C\text{‘}P \,.\supset_y.\, yPz : (\exists y) . y \,\epsilon\, \alpha \cap C\text{‘}P . xPy :$

[*10·56] $\supset : (\exists y) . xPy . yPz :$

[*34·5.Hp] $\supset : xPz$ (4)

$\vdash . (4) . *40\cdot51 . \supset \vdash : \text{Hp} \,.\supset.\, p\text{‘}\overleftarrow{P}\text{‘‘}(\alpha \cap C\text{‘}P) \subset p\text{‘}\overleftarrow{P}\text{‘‘}(\beta \cap C\text{‘}P) \,.$

[*22·621] $\supset . p\text{‘}\overleftarrow{P}\text{‘‘}(\alpha \cap C\text{‘}P) = p\text{‘}\overleftarrow{P}\text{‘‘}(\alpha \cap C\text{‘}P) \cap p\text{‘}\overleftarrow{P}\text{‘‘}(\beta \cap C\text{‘}P)$

[*40·18.*37·22] $= p\text{‘}\overleftarrow{P}\text{‘‘}\{(\alpha \cup \beta) \cap C\text{‘}P\}$ (5)

$\vdash . (3) . (5) . \supset \vdash . \text{Prop}$

The following propositions, to the end of the number, are concerned with the relation P_1 defined in *121. We may regard P_1 as meaning "immediately precedes." *201·6·61·62 are lemmas for *201·63.

***201·6.** $\vdash : P \epsilon \text{trans} . \sim(xPx) . \sim(yPy) . xP_1 y . \supset . x(P \dot{-} P^2) y$

Dem.

$$\begin{aligned}
&\vdash . *121\cdot32\cdot242 . \supset \vdash : \text{Hp} . \supset . P(x \mapsto y) = \iota`x \cup \iota`y \cup P(x - y) \\
&[*201\cdot19] \qquad = \iota`x \cup \iota`y \cup \overleftarrow{P}`x \cap \overrightarrow{P}`y \quad (1) \\
&\vdash . *121\cdot321 . *201\cdot18 . \quad \supset \vdash : \text{Hp} . \supset . xPy \quad (2) \\
&\vdash . (2) . *13\cdot14 . \quad \supset \vdash : \text{Hp} . \supset . x \neq y \quad (3) \\
&\vdash . (1) . (3) . *54\cdot53 . *121\cdot11 . \supset \vdash : \text{Hp} . \supset . \overleftarrow{P}`x \cap \overrightarrow{P}`y \subset \iota`x \cup \iota`y \quad (4) \\
&\vdash . *32\cdot18\cdot181 . \quad \supset \vdash : \text{Hp} . \supset . x \sim\epsilon \overleftarrow{P}`x . y \sim\epsilon \overrightarrow{P}`y \quad (5) \\
&\vdash . (4) . (5) . \quad \supset \vdash : \text{Hp} . \supset . \overleftarrow{P}`x \cap \overrightarrow{P}`y = \Lambda . \\
&[*34\cdot11] \qquad \supset . \sim(xP^2 y) \quad (6) \\
&\vdash . (2) . (6) . \supset \vdash . \text{Prop}
\end{aligned}$$

***201·61.** $\vdash : P \epsilon \text{trans} . \supset . P \dot{-} P^2 \subseteq P_1$

Dem.

$$\begin{aligned}
&\vdash . *121\cdot242 . *90\cdot151 . \supset \vdash : xPy . \supset . P(x \mapsto y) = \iota`x \cup \iota`y \cup P(x - y) \quad (1) \\
&\vdash . (1) . *201\cdot19 . \supset \vdash :. \text{Hp} . \supset : xPy . \supset . P(x \mapsto y) = \iota`x \cup \iota`y \cup (\overleftarrow{P}`x \cap \overrightarrow{P}`y) \quad (2) \\
&\vdash . *34\cdot11 . \quad \supset \vdash : \sim(xP^2 y) . \supset . \overleftarrow{P}`x \cap \overrightarrow{P}`y = \Lambda \quad (3) \\
&\vdash . *34\cdot54 . \quad \supset \vdash : xPy . \sim(xP^2 y) . \supset . x \neq y \quad (4) \\
&\vdash . (2) . (3) . (4) . \supset \vdash :. \text{Hp} . \supset : xPy . \sim(xP^2 y) . \supset . P(x \mapsto y) = \iota`x \cup \iota`y . x \neq y . \\
&[*54\cdot101] \qquad \supset . P(x \mapsto y) \epsilon 2 . \\
&[*121\cdot11] \qquad \supset . xP_1 y :. \supset \vdash . \text{Prop}
\end{aligned}$$

***201·62.** $\vdash :. P \epsilon \text{trans} . \sim(xPx) . \sim(yPy) . \supset : xP_1 y . \equiv . x(P \dot{-} P^2) y$ [*201·6·61]

***201·63.** $\vdash : P \epsilon \text{trans} \cap \text{Rl}`J . \supset . P_1 = P \dot{-} P^2$ [*201·62]

The above proposition is of fundamental importance. The relation P_1 (defined in *121) plays a great part in the theory of series. It is the relation "immediately preceding." Its domain consists of those terms which have immediate successors; its converse domain, of those that have immediate predecessors. In well-ordered series, $\text{D}`P_1 = \text{D}`P$, while $\text{Ɑ}`P_1$ consists of all terms (except the first) which do not belong to the first derivative (cf. *216). In any series, $\text{Ɑ}`P - \text{Ɑ}`P_1$ consists of all the terms which are limits of ascending series, and $\text{D}`P - \text{D}`P_1$ consists of all the terms which are limits of descending series.

***201·64.** $\vdash :. P \epsilon \text{trans} . \supset : P \dot{-} P^2 = \dot{\Lambda} . \equiv . P^2 = P$

Dem.

$$\begin{aligned}
&\vdash . *23\cdot41 . \supset \vdash :. \text{Hp} . \supset : P^2 = P . \equiv . P \subseteq P^2 . \\
&[*25\cdot3] \qquad \equiv . P \dot{-} P^2 = \dot{\Lambda} :. \supset \vdash . \text{Prop}
\end{aligned}$$

***201·65.** $\vdash :. P \epsilon \text{trans} \cap \text{Rl}`J . \supset : P_1 = \dot{\Lambda} . \equiv . P^2 = P$ [*201·64·63]

When P is a series, $P^2 = P$ is the condition for its being a *compact* series, *i.e.* one in which there are terms between any two. In virtue of $*201{\cdot}65$, this condition is equivalent to $P_1 = \dot{\Lambda}$, which states that no term has an immediate predecessor.

The following proposition is first used in $*253{\cdot}521$.

$*201{\cdot}66$. $\vdash : P \epsilon \mathrm{trans} . \mathrm{E} ! P‘x . P‘x \neq x . \supset . (P‘x) P_1 x$

Dem.

$\vdash . *201{\cdot}521 . *121{\cdot}11 . \supset$

$\vdash :. \mathrm{Hp} . \supset : (P‘x) P_1 x . \equiv . (\iota‘P‘x \cup \overleftarrow{P}‘P‘x) \cap (\iota‘x \cup \overrightarrow{P}‘x) \epsilon 2$ (1)

$\vdash . *53{\cdot}31 . \supset \vdash : \mathrm{Hp} . \supset .$

$(\iota‘P‘x \cup \overleftarrow{P}‘P‘x) \cap (\iota‘x \cup \overrightarrow{P}‘x) = (\iota‘P‘x \cup \overleftarrow{P}‘P‘x) \cap (\iota‘x \cup \iota‘P‘x)$

[$*30{\cdot}32 . *22{\cdot}68$] $= \iota‘x \cup \iota‘P‘x$ (2)

$\vdash . *54{\cdot}26 . \supset \vdash : \mathrm{Hp} . \supset . (\iota‘x \cup \iota‘P‘x) \epsilon 2$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash . \mathrm{Prop}$

$*201{\cdot}661$. $\vdash : P \epsilon \mathrm{trans} . \text{Ɑ}‘P \epsilon 1 . \exists ! \mathrm{D}‘P - \text{Ɑ}‘P . \supset . \text{Ɑ}‘P \subset \text{Ɑ}‘P_1$

Dem.

$\vdash . *33{\cdot}151{\cdot}4 . *60{\cdot}38 . \supset$

$\vdash : \mathrm{Hp} . y \epsilon \mathrm{D}‘P - \text{Ɑ}‘P . \supset . \overleftarrow{P}‘y \epsilon 1 . y \sim\epsilon \overleftarrow{P}‘y . \overleftarrow{P}‘y = \text{Ɑ}‘P .$

[$*53{\cdot}3$] $\supset . \mathrm{E} ! \breve{P}‘y . y \neq \breve{P}‘y . \iota‘\breve{P}‘y = \text{Ɑ}‘P .$

[$*201{\cdot}66{\cdot}11 . *121{\cdot}26$] $\supset . y P_1 (\breve{P}‘y) . \iota‘\breve{P}‘y = \text{Ɑ}‘P : \supset \vdash . \mathrm{Prop}$

The above proposition is a lemma for the following.

$*201{\cdot}662$. $\vdash : P \epsilon \mathrm{trans} . \exists ! \overrightarrow{B}‘P . \exists ! \text{Ɑ}‘P - \text{Ɑ}‘P_1 . \supset . \text{Ɑ}‘P \sim\epsilon 1$

[$*201{\cdot}661$. Transp]

This proposition is first used in $*253{\cdot}521$.

*202. CONNECTED RELATIONS

Summary of *202.

A relation is said to be *connected* when either it or its converse holds between any two different members of its field, *i.e.* when, if $x, y \in C'P \,.\, x \neq y$, we have $xPy \,.\vee.\, yPx$. Thus the field of a connected relation consists of a single family, unless the relation is null, in which case it has no families. Conversely, a relation which has one family or none is connected. Connection is necessary, in addition to transitiveness and asymmetry, in order that a relation may generate a single series. If λ is a class of transitive or asymmetrical relations, $\dot{s}'\lambda$ is transitive or asymmetrical; but if λ is a class of connected relations, $\dot{s}'\lambda$ is not in general connected. Hence if λ is a class of series, $\dot{s}'\lambda$ is not one series, but many detached series. This is one reason why the arithmetical sum of a relation of relations is not defined as $\dot{s}'C'P$, but as $\dot{s}'C'P \,\dot{\cup}\, F \,;\, P$ (cf. *162), because the latter, but not in general the former, is connected when P and all the members of $C'P$ are connected (*202·42).

When P is connected, if α is any class contained in $C'P$, we have

$$C'P = P''\alpha \cup \alpha \cup (C'P \cap p'\overleftarrow{P}''\alpha),$$

and there is at most one member of α belonging neither to $P''\alpha$ nor to $C'P \cap p'\overleftarrow{P}''\alpha$. This member of α, if it exists, is the maximum of α. If, further, $P^2 \subset J$ (*i.e.* if P is asymmetrical), $(P''\alpha \cup \alpha) \cap (C'P \cap p'\overleftarrow{P}''\alpha) = \Lambda$. Thus when P is both connected and asymmetrical, $P''\alpha \cup \alpha$ and $C'P \cap p'\overleftarrow{P}''\alpha$ are each other's complements, and the two together constitute the Dedekind cut defined by α, $P''\alpha \cup \alpha$ being all the terms that do not follow the whole of α, and $C'P \cap p'\overleftarrow{P}''\alpha$ being all the terms that do follow the whole of α.

More generally, if α is any class, not necessarily contained in $C'P$, then when P is connected, we have

$$C'P - p'\overleftarrow{P}''(\alpha \cap C'P) \subset P''\alpha \cup (\alpha \cap C'P),$$

and when P is asymmetrical, we have

$$P''\alpha \cup (\alpha \cap C'P) \subset C'P - p'\overleftarrow{P}''\alpha.$$

Thus when both conditions are fulfilled, we have (*202·503)

$$C'P - p'\overleftarrow{P}''(\alpha \cap C'P) = P''\alpha \cup (\alpha \cap C'P).$$

The above inclusions and the consequent equality will be constantly required throughout what follows. The division of $C'P$ into the two mutually exclusive parts

$$P''\alpha \cup (\alpha \cap C'P) \text{ and } C'P \cap p'\overleftarrow{P}''(\alpha \cap C'P)$$

is the Dedekind "cut" defined by the class α. If $\alpha \subset C\text{‘}P$, the two parts become, as above mentioned,

$$P\text{“}\alpha \cup \alpha \text{ and } C\text{‘}P \cap p\text{‘}\overleftarrow{P}\text{“}\alpha.$$

If, further, α is not null, they become

$$P\text{“}\alpha \cup \alpha \text{ and } p\text{‘}\overleftarrow{P}\text{“}\alpha.$$

If α is contained in $C\text{‘}P$ and contains all its own predecessors, they become

$$\alpha \text{ and } C\text{‘}P \cap p\text{‘}\overleftarrow{P}\text{“}\alpha.$$

In this simplified form, Dedekind "cuts" will be considered later (*211).

We take as our definition

$$\text{connex} = \hat{P}\{x \in C\text{‘}P \,.\supset_x .\, \overleftrightarrow{P}\text{‘}x = C\text{‘}P\} \quad \text{Df.}$$

Some of the propositions of the present number are analogues of propositions in *200 and *201. Such are: If P is connected, so is $\breve{P}$ (*202·11); if P is connected, so is any similar relation (*202·211); $\dot{\Lambda}$ and $x \downarrow y$ are connected (*202·3·31); if P is connected, so is $P \upharpoonright \alpha$ (*202·33); and various propositions connected with relation-arithmetic (*202·4—·42). The majority of the propositions of this number, however, deal with properties peculiar to connexity. Among the most important of these are:

*202·101. $\vdash :. P \in \text{connex} \,.\equiv :\, x \in C\text{‘}P \,.\supset_x .\, \overrightarrow{P}\text{‘}x \cup \iota\text{‘}x \cup \overleftarrow{P}\text{‘}x = C\text{‘}P$

*202·103. $\vdash :: P \in \text{connex} \,.\equiv :.\, x, y \in C\text{‘}P \,.\supset_{x,y} :\, xPy \,.\vee.\, x = y \,.\vee.\, yPx$

These are merely alternative forms of the definition.

*202·13. $\vdash : R_* \in \text{connex} \,.\equiv.\, R_{\text{po}} \in \text{connex}$

*202·5. $\vdash :. P \in \text{connex} \,.\, P^2 \mathrel{\text{ɢ}} J \,.\, x, y \in C\text{‘}P \,.\supset :\, x \neq y \,.\, \sim(xPy) \,.\equiv.\, yPx$

*202·501. $\vdash : P \in \text{connex} \,.\supset.\, C\text{‘}P - \alpha - P\text{“}\alpha \subset p\text{‘}\overleftarrow{P}\text{“}(\alpha \cap C\text{‘}P)$

*202·503. $\vdash : P \in \text{connex} \,.\, P^2 \mathrel{\text{ɢ}} J \,.\supset.\, C\text{‘}P - p\text{‘}\overleftarrow{P}\text{“}(\alpha \cap C\text{‘}P) = (\alpha \cap C\text{‘}P) \cup P\text{“}\alpha$

*202·505. $\vdash : P \in \text{connex} \,.\supset.\, C\text{‘}P = P\text{“}\alpha \cup (\alpha \cap C\text{‘}P) \cup \{C\text{‘}P \cap p\text{‘}\overleftarrow{P}\text{“}(\alpha \cap C\text{‘}P)\}$

*202·52. $\vdash : P \in \text{connex} \,.\supset.\, \overrightarrow{B}\text{‘}P, \overrightarrow{B}\text{‘}\breve{P} \in 0 \cup 1$

*202·524. $\vdash : P \in \text{connex} \,.\, \exists ! \overrightarrow{B}\text{‘}P \,.\supset.\, \text{Ɑ}\text{‘}P = \overleftarrow{P}\text{‘}B\text{‘}P$

*202·55. $\vdash : P \upharpoonright \alpha \in \text{connex} \,.\, \alpha \subset C\text{‘}P \,.\, \alpha \sim\in 1 \,.\supset.\, C\text{‘}P \upharpoonright \alpha = \alpha$

In virtue of this proposition (and others) if P is a series and α is a class (not a unit class) contained in $C\text{‘}P$, $P \upharpoonright \alpha$ is the generating relation of the series consisting of the class α in the order which it has in the series P.

*202·7. $\vdash : P \in \text{connex} \,.\supset.\, P \dot{-} P^2 \in 1 \to 1$

This proposition is to be taken in connection with *201·63. The two together show that when P is a series, P_1 is one-one.

$*202\cdot01$. $\text{connex} = \hat{P}\{x \epsilon C`P \,.\supset_x .\, \overleftrightarrow{P}`x = C`P\}$ Df

For the definition of $\overleftrightarrow{P}`x$, see $*97\cdot01$.

$*202\cdot1$. $\vdash :. P \epsilon \text{connex} \,.\equiv :\, x \epsilon C`P \,.\supset_x .\, \overleftrightarrow{P}`x = C`P$ $[(*202\cdot01)]$

$*202\cdot101$. $\vdash :. P \epsilon \text{connex} \,.\equiv :\, x \epsilon C`P \,.\supset_x .\, \overrightarrow{P}`x \cup \iota`x \cup \overleftarrow{P}`x = C`P$
$[*202\cdot1 \,.\, *97\cdot1]$

$*202\cdot102$. $\vdash : P \epsilon \text{connex} \,.\equiv .\, \overleftrightarrow{P}``C`P \epsilon 0 \cup 1$ $[*97\cdot231 \,.\, *202\cdot101]$

$*202\cdot103$. $\vdash :: P \epsilon \text{connex} \,.\equiv :.\, x, y \epsilon C`P \,.\supset_{x,y} : xPy \,.\vee .\, x = y \,.\vee .\, yPx$
$[*97\cdot23 \,.\, *202\cdot102]$

$*202\cdot104$. $\vdash :: P \epsilon \text{connex} \,.\equiv :.\, x, y \epsilon C`P \,.\, x \neq y \,.\supset_{x,y} : xPy \,.\vee .\, yPx$
$[*202\cdot103 \,.\, *5\cdot6]$

$*202\cdot11$. $\vdash : P \epsilon \text{connex} \,.\equiv .\, \breve{P} \epsilon \text{connex}$ $[*202\cdot104 \,.\, *33\cdot22]$

$*202\cdot12$. $\vdash :. \dot{\exists} ! P \,.\supset : P \epsilon \text{connex} \,.\equiv .\, \overleftrightarrow{P}``C`P \epsilon 1 \,.\equiv .\, \overleftrightarrow{P}``C`P = \iota`C`P$

Dem.

$\vdash . *202\cdot1 \,.\supset \vdash : P \epsilon \text{connex} \,.\equiv .\, \overleftrightarrow{P}``C`P \subset \iota`C`P$ (1)

$\vdash . *37\cdot45 \,.\supset \vdash :. \text{Hp} \,.\supset : \exists ! \overleftrightarrow{P}``C`P :$

$[*54\cdot102]$ $\supset : \overleftrightarrow{P}``C`P \sim\epsilon 0 :$

$[*202\cdot102]$ $\supset : P \epsilon \text{connex} \,.\supset .\, \overleftrightarrow{P}``C`P \epsilon 1$ (2)

$\vdash . *202\cdot102 \,.$ $\supset \vdash : \overleftrightarrow{P}``C`P \epsilon 1 \,.\supset .\, P \epsilon \text{connex}$ (3)

$\vdash . (2) . (3) \,.$ $\supset \vdash :. \text{Hp} \,.\supset : P \epsilon \text{connex} \,.\equiv .\, \overleftrightarrow{P}``C`P \epsilon 1$ (4)

$\vdash . (1) . (4) . *52\cdot46 \,.\supset \vdash :. \text{Hp} \,.\supset : P \epsilon \text{connex} \,.\supset .\, \overleftrightarrow{P}``C`P = \iota`C`P$ (5)

$\vdash . (1) . *22\cdot42 \,.$ $\supset \vdash : \overleftrightarrow{P}``C`P = \iota`C`P \,.\supset .\, P \epsilon \text{connex}$ (6)

$\vdash . (4) . (5) . (6) \,.\supset \vdash .$ Prop

The following propositions, down to $*202\cdot181$ inclusive (excepting $*202\cdot16\cdot161$), are concerned with R_* and R_{po}. It often happens that these are connected when R is not so, *e.g.* if R is the relation $+_c 1$ among inductive cardinals.

$*202\cdot13$. $\vdash : R_* \epsilon \text{connex} \,.\equiv .\, R_{po} \epsilon \text{connex}$

Dem.

$\vdash . *202\cdot104 \,.\supset$

$\vdash :: R_* \epsilon \text{connex} \,.\equiv :.\, x, y \epsilon C`R_* \,.\, x \neq y \,.\supset_{x,y} : xR_*y \,.\vee .\, yR_*x :$

$[*91\cdot542]$ $\equiv :.\, x, y \epsilon C`R_* \,.\, x \neq y \,.\supset_{x,y} : xR_{po}y \,.\vee .\, yR_{po}x :.$

$[*90\cdot14 . *91\cdot504]$ $\equiv :.\, x, y \epsilon C`R_{po} \,.\, x \neq y \,.\supset_{x,y} : xR_{po}y \,.\vee .\, yR_{po}x :.$

$[*202\cdot104]$ $\equiv :.\, R_{po} \epsilon \text{connex} :: \supset \vdash .$ Prop

$*202\cdot131$. $\vdash : P \epsilon \text{connex} \,.\, C`P = C`Q \,.\, P \subseteq Q \,.\supset .\, Q \epsilon \text{connex}$ $[*202\cdot103]$

$*202{\cdot}132$. $\vdash : P \,\epsilon\, \text{connex} \,.\supset.\, P_{\text{po}}, P_* \,\epsilon\, \text{connex}$
$[*202{\cdot}131 \,.\, *90{\cdot}14{\cdot}151 \,.\, *91{\cdot}502{\cdot}504]$

$*202{\cdot}133$. $\vdash :: I \upharpoonright C\text{'}P \mathrel{\dot\subset} P \,.\supset:.\, P \,\epsilon\, \text{connex} \,.\equiv:\, x \,\epsilon\, C\text{'}P \,.\supset_x.\, C\text{'}P = \overrightarrow{P}\text{'}x \cup \overleftarrow{P}\text{'}x$

Dem.

$\vdash .\, *35{\cdot}101 \,.\supset \vdash :.\, \text{Hp} \,.\supset:\, x \,\epsilon\, C\text{'}P \,.\supset.\, \iota\text{'}x \subset \overrightarrow{P}\text{'}x$ (1)

$\vdash .\, (1) \,.\, *202{\cdot}101 \,.\supset \vdash .\, \text{Prop}$

$*202{\cdot}134$. $\vdash ::.\, I \upharpoonright C\text{'}P \mathrel{\dot\subset} P \,.\supset ::\, P \,\epsilon\, \text{connex} \,.\equiv:.\, x, y \,\epsilon\, C\text{'}P \,.\supset_{x,y}:\, xPy \,.\vee.\, yPx$
$[*202{\cdot}103]$

$*202{\cdot}135$. $\vdash : P \,\epsilon\, \text{connex} \,.\equiv.\, P \mathbin{\dot\cup} I \upharpoonright C\text{'}P \,\epsilon\, \text{connex}$

Dem.

$\vdash .\, *202{\cdot}134 \,.\supset \vdash ::\, P \mathbin{\dot\cup} I \upharpoonright C\text{'}P \,\epsilon\, \text{connex} \,.\equiv:.$

$x, y \,\epsilon\, C\text{'}P \,.\supset_{x,y}:\, x\,(P \mathbin{\dot\cup} I \upharpoonright C\text{'}P)\,y \,.\vee.\, y\,(P \mathbin{\dot\cup} I \upharpoonright C\text{'}P)\,x :.$

$[*202{\cdot}103]$ $\equiv:.\, P \,\epsilon\, \text{connex} :: \supset \vdash .\, \text{Prop}$

$*202{\cdot}136$. $\vdash :.\, P_* \,\epsilon\, \text{connex} \,.\equiv:\, x \,\epsilon\, C\text{'}P \,.\supset_x.\, C\text{'}P = \overrightarrow{P_*}\text{'}x \cup \overleftarrow{P_*}\text{'}x$
$[*202{\cdot}133 \,.\, *90{\cdot}14{\cdot}15]$

$*202{\cdot}137$. $\vdash ::\, P_* \,\epsilon\, \text{connex} \,.\equiv:.\, x, y \,\epsilon\, C\text{'}P \,.\supset_{x,y}:\, xP_*y \,.\vee.\, yP_*x$
$[*202{\cdot}134 \,.\, *90{\cdot}15]$

$*202{\cdot}138$. $\vdash :.\, P \,\epsilon\, \text{trans} \,.\supset:\, P \,\epsilon\, \text{connex} \,.\equiv.\, P_* \,\epsilon\, \text{connex}$ $[*202{\cdot}13 \,.\, *201{\cdot}18]$

$*202{\cdot}14$. $\vdash : R \,\epsilon\, \text{Cls} \rightarrow 1 \,.\supset.\, R_{\text{po}} \mathbin{\text{£}} \overleftarrow{R_*}\text{'}x \,\epsilon\, \text{connex}$ $[*96{\cdot}303 \,.\, *202{\cdot}104]$

$*202{\cdot}141$. $\vdash : R \,\epsilon\, 1 \rightarrow \text{Cls} \,.\supset.\, R_{\text{po}} \mathbin{\text{£}} \overrightarrow{R_*}\text{'}x \,\epsilon\, \text{connex}$ $\left[*202{\cdot}14 \dfrac{\breve{R}}{R} \,.\, *202{\cdot}11\right]$

$*202{\cdot}15$. $\vdash : R \,\epsilon\, 1 \rightarrow 1 \,.\supset.\, R_{\text{po}} \mathbin{\text{£}} \overleftrightarrow{R_*}\text{'}x \,\epsilon\, \text{connex}$

Dem.

$\vdash .\, *97{\cdot}13 \,.\supset \vdash :.\, y, z \,\epsilon\, \overleftrightarrow{R_*}\text{'}x \,.\supset_{y,z}:\, y, z \,\epsilon\, \overrightarrow{R_*}\text{'}x \,.\vee.\, y, z \,\epsilon\, \overleftarrow{R_*}\text{'}x \,.\vee.$

$y \,\epsilon\, \overrightarrow{R_*}\text{'}x \,.\, z \,\epsilon\, \overleftarrow{R_*}\text{'}x \,.\vee.\, y \,\epsilon\, \overleftarrow{R_*}\text{'}x \,.\, z \,\epsilon\, \overrightarrow{R_*}\text{'}x$ (1)

$\vdash .\, *202{\cdot}141{\cdot}104 \,.\supset \vdash ::\, \text{Hp} \,.\supset:.\, y, z \,\epsilon\, \overrightarrow{R_*}\text{'}x \,.\, y \neq z \,.\supset:\, yR_{\text{po}}x \,.\vee.\, xR_{\text{po}}y$ (2)

$\vdash .\, *202{\cdot}14{\cdot}104 \,.\supset \vdash ::\, \text{Hp} \,.\supset:.\, y, z \,\epsilon\, \overleftarrow{R_*}\text{'}x \,.\, y \neq z \,.\supset:\, yR_{\text{po}}x \,.\vee.\, xR_{\text{po}}y$ (3)

$\vdash .\, *90{\cdot}17 \,.\supset \vdash :\, y \,\epsilon\, \overrightarrow{R_*}\text{'}x \,.\, z \,\epsilon\, \overleftarrow{R_*}\text{'}x \,.\, y \neq z \,.\supset.\, yR_*z \,.\, y \neq z \,.$

$[*91{\cdot}542]$ $\supset.\, yR_{\text{po}}z$ (4)

Similarly $\vdash :\, y \,\epsilon\, \overleftarrow{R_*}\text{'}x \,.\, z \,\epsilon\, \overrightarrow{R_*}\text{'}x \,.\, y \neq z \,.\supset.\, zR_{\text{po}}y$ (5)

$\vdash .\, (1) \,.\, (2) \,.\, (3) \,.\, (4) \,.\, (5) \,.\supset$

$\vdash ::\, \text{Hp} \,.\supset:.\, y, z \,\epsilon\, \overleftrightarrow{R_*}\text{'}x \,.\, y \neq z \,.\supset_{y,z}:\, yR_{\text{po}}z \,.\vee.\, zR_{\text{po}}y$ (6)

$\vdash .\, (6) \,.\, *202{\cdot}104 \,.\supset \vdash .\, \text{Prop}$

The above proposition is used in the ordinal theory of finite and infinite ($*260{\cdot}4$).

$*202{\cdot}16$. $\vdash : P \epsilon \text{connex} \,.\, x, y \epsilon C‘P \,.\, \sim(xPx) \,.\, \sim(yPy) \,.\, \overrightarrow{P}‘x = \overrightarrow{P}‘y \,.\, \supset \,.\, x = y$

Dem.

$\vdash \,.\, *32{\cdot}18{\cdot}181 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, \sim(xPy) \,.\, \sim(yPx) \,.$

$[*202{\cdot}103] \qquad \supset \,.\, x = y : \supset \vdash \,.\, \text{Prop}$

$*202{\cdot}161$. $\vdash : P \epsilon \text{connex} \cap \text{Rl}‘J \,.\, \supset \,.\, \overrightarrow{P} \upharpoonleft C‘P \epsilon 1 \to 1 \,.\, \overrightarrow{P} \upharpoonleft C‘P \epsilon (\overrightarrow{P} \,;\, P) \,\overline{\text{smor}}\, P$

Dem.

$\vdash \,.\, *202{\cdot}16 \,.\, \supset \vdash :. \text{Hp} \,.\, \supset : x, y \epsilon C‘P \,.\, \overrightarrow{P}‘x = \overrightarrow{P}‘y \,.\, \supset \,.\, x = y \qquad (1)$

$\vdash \,.\, (1) \,.\, *71{\cdot}55 \,.\, *151{\cdot}24 \,.\, \supset \vdash \,.\, \text{Prop}$

$*202{\cdot}162$. $\vdash : P \epsilon \text{connex} \,.\, P_{\text{po}} \subseteq J \,.\, \supset \,.\, P \sqsubset \,;\, \overrightarrow{P}_* \,;\, P \,\text{smor}\, P \,.\, P \sqsubset | \overrightarrow{P}_* \upharpoonleft C‘P \epsilon 1 \to 1$

Dem.

$\vdash \,.\, *36{\cdot}13 \,.\, \supset \vdash :. P \sqsubset \overrightarrow{P}_*‘x = P \sqsubset \overrightarrow{P}_*‘y \,.\, \equiv :$

$\qquad uPv \,.\, u, v \epsilon \overrightarrow{P}_*‘x \,.\, \equiv_{u,v} \,.\, uPv \,.\, u, v \epsilon \overrightarrow{P}_*‘y \qquad (1)$

$\vdash \,.\, (1) \,.\, *11{\cdot}1 \,.\, *90{\cdot}12 \,.\, \supset$

$\vdash :. x, y \epsilon C‘P \,.\, P \sqsubset \overrightarrow{P}_*‘x = P \sqsubset \overrightarrow{P}_*‘y \,.\, \supset : xPy \,.\, yP_*x \,.\, \equiv \,.\, xPy \,.\, xP_*y :$

$\qquad yPx \,.\, yP_*x \,.\, \equiv \,.\, yPx \,.\, xP_*y :$

$[*90{\cdot}151 . *91{\cdot}52] \supset : xPy \,.\, \supset \,.\, xP_{\text{po}}x : yPx \,.\, \supset \,.\, yP_{\text{po}}y \qquad (2)$

$\vdash \,.\, (2) \,.\, \supset \vdash : \text{Hp} \,.\, x, y \epsilon C‘P \,.\, P \sqsubset \overrightarrow{P}_*‘x = P \sqsubset \overrightarrow{P}_*‘y \,.\, \supset \,.\, \sim(xPy) \,.\, \sim(yPx) \,.$

$[*202{\cdot}103] \qquad \supset \,.\, x = y : \supset \vdash \,.\, \text{Prop}$

$*202{\cdot}17$. $\vdash : P_{\text{po}} \epsilon \text{connex} \,.\, y \epsilon P(x \mapsto z) \,.\, \supset \,.\, P(x \mapsto y) \cup P(y \mapsto z) = P(x \mapsto z)$

Dem.

$\vdash \,.\, *201{\cdot}14{\cdot}15 \,.\, *121{\cdot}103 \,.\, \supset$

$\vdash : \text{Hp} \,.\, \supset \,.\, P(x \mapsto y) \subset P(x \mapsto z) \,.\, P(y \mapsto z) \subset P(x \mapsto z) \qquad (1)$

$\vdash \,.\, *202{\cdot}13{\cdot}137 \,.\, *121{\cdot}103 \,.\, \supset$

$\vdash :. \text{Hp} \,.\, w \epsilon P(x \mapsto z) \,.\, \supset : wP_*y \,.\, \vee \,.\, yP_*w : xP_*w \,.\, wP_*z :$

$[*121{\cdot}103] \qquad \supset : w \epsilon P(x \mapsto y) \cup P(y \mapsto z) \qquad (2)$

$\vdash \,.\, (1) \,.\, (2) \,.\, \supset \vdash \,.\, \text{Prop}$

$*202{\cdot}171$. $\vdash : P_{\text{po}} \epsilon \text{connex} \,.\, y \epsilon P(x \mapsto z) \,.\, \supset \,.$

$P(x \dashv z) = P(x \dashv y) \cup P(y \dashv z) \,.\, P(x \vdash z) = P(x \vdash y) \cup P(y \vdash z)$

[Proof as in $*202{\cdot}17$]

$*202{\cdot}172$. $\vdash : P_{\text{po}} \epsilon \text{connex} \,.\, y \epsilon P(x - z) \,.\, \supset \,.$

$P(x - y) = P(x \dashv y) \cup P(y - z) = P(x - y) \cup P(y \vdash z)$

[Proof as in $*202{\cdot}17$]

$*202{\cdot}18$. $\vdash : P_{\text{po}} \epsilon \text{connex} \,.\, \text{E} ! B‘P \,.\, \supset \,.\, C‘P = \overleftarrow{P}_*‘B‘P$

Dem.

$\vdash \,.\, *202{\cdot}1 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, C‘P = \overleftrightarrow{P}_{\text{po}}‘B‘P$

$[*97{\cdot}2 . *91{\cdot}504] \qquad = \overleftarrow{P}_*‘B‘P : \supset \vdash \,.\, \text{Prop}$

***202·181.** $\vdash : P_{po} \epsilon \text{connex} . \text{E} ! B`P . \text{E} ! B`\breve{P} . \supset . C`P = P(B`P \mapsto B`\breve{P})$

Dem.

$\vdash . *202\cdot18 . \supset \vdash : \text{Hp} . \supset . C`P = \overleftarrow{P_*}`B`P \cap \overrightarrow{P_*}`B`\breve{P}$

$[*121\cdot103] \quad = P(B`P \mapsto B`\breve{P}) : \supset \vdash . \text{Prop}$

The above proposition is used in the ordinal theory of finite and infinite (*261·2).

The following proposition is a lemma for *202·211, which shows that if a relation is connected, so are all similar relations.

***202·21.** $\vdash : P \epsilon \text{connex} . S \epsilon 1 \rightarrow \text{Cls} . \supset . S ; P \epsilon \text{connex}$

Dem.

$\vdash . *150\cdot202 . \supset \vdash :: \text{Hp} . \supset :. x, y \epsilon C`S;P . x \neq y . \supset : x, y \epsilon S``C`P . x \neq y :$
$[*71\cdot4.*30\cdot37] \supset : (\exists z, w) . z, w \epsilon C`P . x = S`z . y = S`w . z \neq w :$
$[*202\cdot104] \quad \supset : (\exists z, w) : x = S`z . y = S`w : zPw . \vee . wPz :$
$[*150\cdot4] \quad \supset : x (S;P) y . \vee . y (S;P) x \quad (1)$
$\vdash . (1) . *202\cdot104 . \supset \vdash . \text{Prop}$

The proofs of the three following propositions proceed like the proofs of the analogous propositions in *200 and *201.

***202·211.** $\vdash : P \epsilon \text{connex} . Q \,\text{smor}\, P . \supset . Q \epsilon \text{connex}$

***202·212.** $\vdash : P \epsilon \text{connex} . \supset . \text{Nr}`P \subset \text{connex}$

***202·22.** $\vdash : P \epsilon \text{connex} . \equiv . \text{N}_0\text{r}`P \subset \text{connex} . \equiv . \exists ! \text{N}_0\text{r}`P \cap \text{connex}$

***202·3.** $\vdash . \dot{\Lambda} \epsilon \text{connex}$

Dem.

$\vdash . *37\cdot29 . \supset \vdash : P = \dot{\Lambda} . \supset . \overleftrightarrow{P}``C`P = \Lambda .$
$[*202\cdot102] \quad \supset . P \epsilon \text{connex} : \supset \vdash . \text{Prop}$

***202·31.** $\vdash . x \downarrow y \epsilon \text{connex}$

Dem.

$\vdash . *55\cdot15 . \supset \vdash :. z, w \epsilon C`(x \downarrow y) . \supset :$
$z, w \epsilon \iota`x . \vee . z, w \epsilon \iota`y . \vee . z \epsilon \iota`x . w \epsilon \iota`y . \vee . z \epsilon \iota`y . w \epsilon \iota`x :$
$[*51\cdot15.*13\cdot172] \supset : z = w . \vee . z = x . w = y . \vee . z = y . w = x :$
$[*55\cdot15] \quad \supset : z = w . \vee . z (x \downarrow y) w . \vee . w (x \downarrow y) z \quad (1)$
$\vdash . (1) . *202\cdot103 . \supset \vdash . \text{Prop}$

***202·33.** $\vdash : P \epsilon \text{connex} . \supset . P \upharpoonright \alpha \epsilon \text{connex}$

Dem.

$\vdash . *37\cdot41 . \supset \vdash : x, y \epsilon C`P \upharpoonright \alpha . \supset . x, y \epsilon \alpha . x, y \epsilon C`P \quad (1)$
$\vdash . (1) . *202\cdot103 . \supset$
$\vdash :: \text{Hp} . \supset :. x, y \epsilon C`P \upharpoonright \alpha . \supset : x, y \epsilon \alpha : xPy . \vee . x = y . \vee . yPx :$
$[*36\cdot13] \quad \supset : x (P \upharpoonright \alpha) y . \vee . x = y . \vee . y (P \upharpoonright \alpha) x \quad (2)$
$\vdash . (2) . *202\cdot103 . \supset \vdash . \text{Prop}$

The following propositions (∗202·4—·42) are concerned with applications of relation-arithmetic.

∗202·4. $\vdash : P, Q \epsilon \text{connex} . \supset . P \mathbin{\updownarrow} Q \epsilon \text{connex}$

Dem.

$\vdash . *160{\cdot}14 . \supset \vdash :. x, y \epsilon C'(P \mathbin{\updownarrow} Q) . \equiv :$

$x, y \epsilon C'P . \mathsf{v} . x, y \epsilon C'Q . \mathsf{v} . x \epsilon C'P . y \epsilon C'Q . \mathsf{v} . x \epsilon C'Q . y \epsilon C'P$ (1)

$\vdash . *202{\cdot}103 . \supset \vdash :: \text{Hp} . \supset :. x, y \epsilon C'P . \supset : xPy . \mathsf{v} . x = y . \mathsf{v} . yPx :$

[∗160·1] $\supset : x (P \mathbin{\updownarrow} Q) y . \mathsf{v} . x = y . \mathsf{v} . y (P \mathbin{\updownarrow} Q) x$ (2)

Similarly $\vdash :: \text{Hp} . \supset :. x, y \epsilon C'Q . \supset : x (P \mathbin{\updownarrow} Q) y . \mathsf{v} . x = y . \mathsf{v} . y (P \mathbin{\updownarrow} Q) x$ (3)

$\vdash . *160{\cdot}1 . *35{\cdot}103 . \supset \vdash : x \epsilon C'P . y \epsilon C'Q . \supset . x (P \mathbin{\updownarrow} Q) y$ (4)

$\vdash . *160{\cdot}1 . *35{\cdot}103 . \supset \vdash : x \epsilon C'Q . y \epsilon C'P . \supset . y (P \mathbin{\updownarrow} Q) x$ (5)

$\vdash . (1) . (2) . (3) . (4) . (5) . \supset$

$\vdash :: \text{Hp} . \supset :. x, y \epsilon C'(P \mathbin{\updownarrow} Q) . \supset : x (P \mathbin{\updownarrow} Q) y . \mathsf{v} . x = y . \mathsf{v} . y (P \mathbin{\updownarrow} Q) x$ (6)

$\vdash . (6) . *202{\cdot}103 . \supset \vdash . \text{Prop}$

The above proposition illustrates the reasons for defining $P \mathbin{\updownarrow} Q$ as was done in ∗160. When P and Q are connected, $P \cup Q$ is in general not connected: it is the additional term $C'P \uparrow C'Q$ which insures connection.

∗202·401. $\vdash :. C'P \cap C'Q = \Lambda . \supset : P \mathbin{\updownarrow} Q \epsilon \text{connex} . \equiv . P, Q \epsilon \text{connex}$

Dem.

$\vdash . *202{\cdot}33 . \supset \vdash : P \mathbin{\updownarrow} Q \epsilon \text{connex} . \supset . (P \mathbin{\updownarrow} Q) \upharpoonright C'P, (P \mathbin{\updownarrow} Q) \upharpoonright C'Q \epsilon \text{connex}$ (1)

$\vdash . (1) . *160{\cdot}5 . \supset \vdash :. \text{Hp} . \supset : P \mathbin{\updownarrow} Q \epsilon \text{connex} . \supset . P, Q \epsilon \text{connex}$ (2)

$\vdash . (2) . *202{\cdot}4 . \supset \vdash . \text{Prop}$

∗202·41. $\vdash : P \epsilon \text{connex} . \supset . P \mathbin{\leftrightarrow\!\!\!\!\!+} z \epsilon \text{connex} . z \mathbin{+\!\!\!\!\leftarrow} P \epsilon \text{connex}$

Dem.

$\vdash . *161{\cdot}14{\cdot}2 . \supset \vdash :. x, y \epsilon C'(P \mathbin{+\!\!\!\!\rightarrow} z) . x \neq y . \supset : x, y \epsilon (C'P \cup \iota'z) . x \neq y :$

[∗51·236] $\supset : x, y \epsilon C'P . x \neq y . \mathsf{v} . x \epsilon C'P . y = z . \mathsf{v} . y \epsilon C'P . x = z$ (1)

$\vdash . (1) . *202{\cdot}104 . \supset$

$\vdash :: P \epsilon \text{connex} . \supset :. x, y \epsilon C'(P \mathbin{+\!\!\!\!\rightarrow} z) . x \neq y . \supset :$

$xPy . \mathsf{v} . yPx . \mathsf{v} . x \epsilon C'P . y = z . \mathsf{v} . y \epsilon C'P . x = z :$

[∗161·11] $\supset : x (P \mathbin{+\!\!\!\!\rightarrow} z) y . \mathsf{v} . y (P \mathbin{+\!\!\!\!\rightarrow} z) x :.$

[∗202·104] $\supset :. P \mathbin{+\!\!\!\!\rightarrow} z \epsilon \text{connex}$ (2)

Similarly $\vdash : P \epsilon \text{connex} . \supset . z \mathbin{\leftarrow\!\!\!\!+} P \epsilon \text{connex}$ (3)

$\vdash . (2) . (3) . \supset \vdash . \text{Prop}$

∗202·411. $\vdash . x \downarrow y \mathbin{+\!\!\!\!\rightarrow} z \epsilon \text{connex}$ [∗202·41·31]

∗202·412. $\vdash :. z \sim \epsilon C'P . \supset : P \epsilon \text{connex} . \equiv . P \mathbin{+\!\!\!\!\rightarrow} z \epsilon \text{connex} . \equiv . z \mathbin{\leftarrow\!\!\!\!+} P \epsilon \text{connex}$

Dem.

$\vdash . *161{\cdot}16 . \supset \vdash :. \text{Hp} . \supset : P = (P \mathbin{+\!\!\!\!\rightarrow} z) \upharpoonright C'P :$

[∗202·33] $\supset : P \mathbin{+\!\!\!\!\rightarrow} z \epsilon \text{connex} . \supset . P \epsilon \text{connex}$ (1)

Similarly $\vdash :. \text{Hp} . \supset . z \mathbin{\leftarrow\!\!\!\!+} P \epsilon \text{connex} . \supset . P \epsilon \text{connex}$ (2)

$\vdash . (1) . (2) . *202{\cdot}41 . \supset \vdash . \text{Prop}$

***202·42.** $\vdash : P \in \text{connex} \,.\, C'P \subset \text{connex} \,.\, \supset \,.\, \Sigma'P \in \text{connex}$

Dem.

$\vdash \,.\, {*}162\cdot22 \,.\, \supset \vdash : x, y \in C'\Sigma'P \,.\, \equiv \,.\, (\exists Q, R) \,.\, Q, R \in C'P \,.\, x \in C'Q \,.\, y \in C'R$ (1)

$\vdash \,.\, (1) \,.\, {*}202\cdot103 \,.\, \supset$

$\vdash :: P \in \text{connex} \,.\, \supset :. \, x, y \in C'\Sigma'P \,.\, \supset :$

$(\exists Q, R) : QPR \,.\, \mathsf{v} \,.\, Q = R \,.\, Q, R \in C'P \,.\, \mathsf{v} \,.\, RPQ : x \in C'Q \,.\, y \in C'R$ (2)

$\vdash \,.\, {*}162\cdot13 \,.\, \supset \vdash :. \, QPR \,.\, \mathsf{v} \,.\, RPQ : x \in C'Q \,.\, y \in C'R : \supset :$

$x (\Sigma'P) y \,.\, \mathsf{v} \,.\, y (\Sigma'P) x$ (3)

$\vdash \,.\, {*}13\cdot195 \,.\, \supset \vdash : (\exists Q, R) \,.\, Q = R \,.\, Q, R \in C'P \,.\, x \in C'Q \,.\, y \in C'R \,.\, \supset \,.$

$(\exists Q) \,.\, Q \in C'P \,.\, x, y \in C'Q$ (4)

$\vdash \,.\, {*}202\cdot103 \,.\, \supset \vdash :: C'P \subset \text{connex} \,.\, \supset :. \, (\exists Q) \,.\, Q \in C'P \,.\, x, y \in C'Q \,.\, \supset :$

$(\exists Q) : Q \in C'P : xQy \,.\, \mathsf{v} \,.\, x = y \,.\, \mathsf{v} \,.\, yQx :$

[*162·13] $\supset : x (\Sigma'P) y \,.\, \mathsf{v} \,.\, x = y \,.\, \mathsf{v} \,.\, y (\Sigma'P) x$ (5)

$\vdash \,.\, (4) \,.\, (5) \,.\, \supset$

$\vdash :: C'P \subset \text{connex} \,.\, \supset :. \, (\exists Q, R) \,.\, Q = R \,.\, Q, R \in C'P \,.\, x \in C'Q \,.\, y \in C'R \,.\, \supset :$

$x (\Sigma'P) y \,.\, \mathsf{v} \,.\, x = y \,.\, \mathsf{v} \,.\, y (\Sigma'P) x$ (6)

$\vdash \,.\, (2) \,.\, (3) \,.\, (6) \,.\, \supset \vdash :: \text{Hp} \,.\, \supset :.$

$x, y \in C'\Sigma'P \,.\, \supset : x (\Sigma'P) y \,.\, \mathsf{v} \,.\, x = y \,.\, \mathsf{v} \,.\, y (\Sigma'P) x$ (7)

$\vdash \,.\, (7) \,.\, {*}202\cdot103 \,.\, \supset \vdash \,.\, \text{Prop}$

***202·5.** $\vdash :. \, P \in \text{connex} \,.\, P^2 \mathrel{G} J \,.\, x, y \in C'P \,.\, \supset : x \neq y \,.\, \sim (xPy) \,.\, \equiv \,.\, yPx$

Dem.

$\vdash \,.\, {*}50\cdot43 \,.\quad \supset \vdash :. \, P^2 \mathrel{G} J \,.\, \supset : yPx \,.\, \supset \,.\, \sim (xPy)$ (1)

$\vdash \,.\, {*}200\cdot36 \,.\quad \supset \vdash :. \, P^2 \mathrel{G} J \,.\, \supset : yPx \,.\, \supset \,.\, x \neq y$ (2)

$\vdash \,.\, {*}202\cdot104 \,.\, \supset \vdash :. \, P \in \text{connex} \,.\, x, y \in C'P \,.\, \supset : x \neq y \,.\, \sim (xPy) \,.\, \supset \,.\, yPx$ (3)

$\vdash \,.\, (1) \,.\, (2) \,.\, (3) \,.\, \supset \vdash \,.\, \text{Prop}$

The following propositions (*202·501—·51) are concerned with the relations of $P``\alpha$ and $p'\overleftarrow{P}``(\alpha \cap C'P)$. They are important, and *202·501·503·505 will be often used.

***202·501.** $\vdash : P \in \text{connex} \,.\, \supset \,.\, C'P - \alpha - P``\alpha \subset p'\overleftarrow{P}``(\alpha \cap C'P)$

Dem.

$\vdash \,.\, {*}13\cdot14 \,.\, {*}37\cdot1 \,.\, \supset \vdash :. \, y \in C'P - \alpha - P``\alpha \,.\, x \in \alpha \,.\, \supset \,.\, x \neq y \,.\, \sim (yPx)$ (1)

$\vdash \,.\, (1) \,.\, {*}202\cdot103 \,.\, \supset \vdash :. \, \text{Hp} \,.\, \supset : y \in C'P - \alpha - P``\alpha \,.\, x \in \alpha \cap C'P \,.\, \supset \,.\, xPy :$

[*40·53] $\supset : y \in C'P - \alpha - P``\alpha \,.\, \supset \,.\, y \in p'\overleftarrow{P}``(\alpha \cap C'P) :. \supset \vdash \,.\, \text{Prop}$

***202·502.** $\vdash :. \, P \in \text{connex} \,.\, P^2 \mathrel{G} J \,.\, \exists ! \, \alpha \cap C'P \,.\, \supset \,.\, C'P - \alpha - P``\alpha = p'\overleftarrow{P}``(\alpha \cap C'P)$

Dem.

$\vdash \,.\, {*}40\cdot62 \,.\quad \supset \vdash : \text{Hp} \,.\, \supset \,.\, p'\overleftarrow{P}``(\alpha \cap C'P) \subset C'P$ (1)

$\vdash \,.\, {*}200\cdot5 \,.\quad \supset \vdash : \text{Hp} \,.\, \supset \,.\, p'\overleftarrow{P}``(\alpha \cap C'P) \subset -\alpha$ (2)

$\vdash \,.\, {*}200\cdot53 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, p'\overleftarrow{P}``(\alpha \cap C'P) \subset -P``\alpha$ (3)

$\vdash \,.\, (1) \,.\, (2) \,.\, (3) \,.\, {*}202\cdot501 \,.\, \supset \vdash \,.\, \text{Prop}$

∗202·503. $\vdash : P \in \text{connex} \,.\, P^2 \mathrel{\unicode{x22D0}} J \,.\, \supset \,.\, C\text{‘}P - p\text{‘}\overleftarrow{P}\text{“}(\alpha \cap C\text{‘}P) = (\alpha \cap C\text{‘}P) \cup P\text{“}\alpha$

Dem.

$\vdash \,.\, *202{\cdot}501 \,.\, *24{\cdot}43 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, C\text{‘}P - p\text{‘}\overleftarrow{P}\text{“}(\alpha \cap C\text{‘}P) \subset \alpha \cup P\text{“}\alpha \quad (1)$

$\vdash \,.\, (1) \,.\, *22{\cdot}43 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, C\text{‘}P - p\text{‘}\overleftarrow{P}\text{“}(\alpha \cap C\text{‘}P) \subset (\alpha \cup P\text{“}\alpha) \cap C\text{‘}P$

$[*22{\cdot}68 . *37{\cdot}15] \qquad \subset (\alpha \cap C\text{‘}P) \cup P\text{“}\alpha \quad (2)$

$\vdash \,.\, *200{\cdot}5{\cdot}36 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, \alpha \cap C\text{‘}P \subset - p\text{‘}\overleftarrow{P}\text{“}(\alpha \cap C\text{‘}P) \quad (3)$

$\vdash \,.\, *200{\cdot}53 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, P\text{“}\alpha \subset - p\text{‘}\overleftarrow{P}\text{“}(\alpha \cap C\text{‘}P) \quad (4)$

$\vdash \,.\, *22{\cdot}43 \,.\, *37{\cdot}15 \,.\, \supset \vdash \,.\, \alpha \cap C\text{‘}P \subset C\text{‘}P \,.\, P\text{“}\alpha \subset C\text{‘}P \quad (5)$

$\vdash \,.\, (3) \,.\, (4) \,.\, (5) \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, (\alpha \cap C\text{‘}P) \cup P\text{“}\alpha \subset C\text{‘}P - p\text{‘}\overleftarrow{P}\text{“}(\alpha \cap C\text{‘}P) \quad (6)$

$\vdash \,.\, (2) \,.\, (6) \,.\, \supset \vdash \,.\, \text{Prop}$

∗202·504. $\vdash : P \in \text{connex} \,.\, P^2 \mathrel{\unicode{x22D0}} J \,.\, \supset \,.\, C\text{‘}P \cap p\text{‘}\overleftarrow{P}\text{“}(\alpha \cap C\text{‘}P) = C\text{‘}P - \alpha - P\text{“}\alpha$

Dem.

$\vdash \,.\, *200{\cdot}5{\cdot}36 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, p\text{‘}\overleftarrow{P}\text{“}(\alpha \cap C\text{‘}P) \subset - \alpha \quad (1)$

$\vdash \,.\, *200{\cdot}53 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, p\text{‘}\overleftarrow{P}\text{“}(\alpha \cap C\text{‘}P) \subset - P\text{“}\alpha \quad (2)$

$\vdash \,.\, (1) \,.\, (2) \,.\, *22{\cdot}48 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, C\text{‘}P \cap p\text{‘}\overleftarrow{P}\text{“}(\alpha \cap C\text{‘}P) \subset C\text{‘}P - \alpha - P\text{“}\alpha \quad (3)$

$\vdash \,.\, (3) \,.\, *202{\cdot}501 \,.\, \supset \vdash \,.\, \text{Prop}$

∗202·505. $\vdash : P \in \text{connex} \,.\, \supset \,.\, C\text{‘}P = P\text{“}\alpha \cup (\alpha \cap C\text{‘}P) \cup \{C\text{‘}P \cap p\text{‘}\overleftarrow{P}\text{“}(\alpha \cap C\text{‘}P)\}$

Dem.

$\vdash \,.\, *202{\cdot}501 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, C\text{‘}P - \alpha - P\text{“}\alpha \subset p\text{‘}\overleftarrow{P}\text{“}(\alpha \cap C\text{‘}P) \,.$

$[*24{\cdot}43] \qquad \supset \,.\, C\text{‘}P \subset \alpha \cup P\text{“}\alpha \cup \{p\text{‘}\overleftarrow{P}\text{“}(\alpha \cap C\text{‘}P)\} \,.$

$[*22{\cdot}621 . *37{\cdot}15] \supset \,.\, C\text{‘}P = (\alpha \cap C\text{‘}P) \cup P\text{“}\alpha \cup \{C\text{‘}P \cap p\text{‘}\overleftarrow{P}\text{“}(\alpha \cap C\text{‘}P)\} : \supset \vdash \,.\, \text{Prop}$

∗202·51. $\vdash : P \in \text{connex} \,.\, \alpha \subset C\text{‘}P \,.\, \exists ! \alpha \,.\, \supset .$

$$C\text{‘}P = P\text{“}\alpha \cup \alpha \cup p\text{‘}\overleftarrow{P}\text{“}\alpha = \breve{P}\text{“}\alpha \cup \alpha \cup p\text{‘}\overrightarrow{P}\text{“}\alpha$$

Dem.

$\vdash \,.\, *40{\cdot}62 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, p\text{‘}\overleftarrow{P}\text{“}\alpha \subset C\text{‘}P \quad (1)$

$\vdash \,.\, *22{\cdot}621 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, \alpha = \alpha \cap C\text{‘}P \quad (2)$

$\vdash \,.\, (1) \,.\, (2) \,.\, *202{\cdot}505 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, C\text{‘}P = P\text{“}\alpha \cup \alpha \cup p\text{‘}\overleftarrow{P}\text{“}\alpha \quad (3)$

$\vdash \,.\, (3) \frac{\breve{P}}{P} \,.\, *202{\cdot}11 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, C\text{‘}P = \breve{P}\text{“}\alpha \cup \alpha \cup p\text{‘}\overrightarrow{P}\text{“}\alpha \quad (4)$

$\vdash \,.\, (3) \,.\, (4) \,.\, \supset \vdash \,.\, \text{Prop}$

The following propositions (∗202·511—·524) are concerned with $\overrightarrow{B}\text{‘}P$. ∗202·52 shows that if $P \in$ connex, P cannot have more than one first term or more than one last term, and ∗202·523 shows that this still holds if only P_* is connected. ∗202·511 shows that if P is a connected relation which has a first term, then if α is any class, there are predecessors of the whole of $\alpha \cap C\text{‘}P$ when and only when $B\text{‘}P$ is such a predecessor, and when and only

when $B`P \sim\epsilon\, \alpha$. *202·524 shows that if $P$ is connected and has a first term, $\text{Ɑ}`P$ consists of the successors of the first term. These propositions are much used.

***202·511.** $\vdash :. P \epsilon \text{connex} . E ! B`P . \supset :$

$$\exists ! p`\overrightarrow{P}``(\alpha \cap C`P) . \equiv . B`P \sim\epsilon\, \alpha . \equiv . B`P \epsilon\, p`\overrightarrow{P}``(\alpha \cap C`P)$$

Dem.

$\vdash . *202{\cdot}104 . *93{\cdot}1 . \supset \vdash :. \text{Hp} . B`P \sim\epsilon\, \alpha . \supset : x \epsilon (\alpha \cap C`P) . \supset_x . (B`P) P x :$

[*40·51] $\supset : B`P \epsilon\, p`\overrightarrow{P}``(\alpha \cap C`P) :$ (1)

[*10·24] $\supset : \exists ! p`\overrightarrow{P}``(\alpha \cap C`P)$ (2)

$\vdash . *93{\cdot}1 . \supset \vdash : \text{Hp} . B`P \epsilon\, \alpha . \supset . (x) . \sim \{xP(B`P)\} . B`P \epsilon\, \alpha \cap C`P .$

[*40·51] $\supset . p`\overrightarrow{P}``(\alpha \cap C`P) = \Lambda .$ (3)

[*24·105] $\supset . B`P \sim\epsilon\, p`\overrightarrow{P}``(\alpha \cap C`P)$ (4)

$\vdash . (2) . (3) . \supset \vdash :. \text{Hp} . \supset : B`P \sim\epsilon\, \alpha . \equiv . \exists ! p`\overrightarrow{P}``(\alpha \cap C`P)$ (5)

$\vdash . (1) . (4) . \supset \vdash :. \text{Hp} . \supset : B`P \sim\epsilon\, \alpha . \equiv . B`P \epsilon\, p`\overrightarrow{P}``(\alpha \cap C`P)$ (6)

$\vdash . (5) . (6) . \supset \vdash . \text{Prop}$

***202·52.** $\vdash : P \epsilon \text{connex} . \supset . \overrightarrow{B}`P, \overrightarrow{B}`\breve{P} \epsilon\, 0 \cup 1$

Dem.

$\vdash . *93{\cdot}103 . \quad \supset \vdash : x, y \epsilon \overrightarrow{B}`P . \supset . x, y \epsilon C`P . x \sim\epsilon\, \text{Ɑ}`P . y \sim\epsilon\, \text{Ɑ}`P .$

[*33·14] $\supset . x, y \epsilon C`P . \sim(xPy) . \sim(yPx)$ (1)

$\vdash . (1) . *202{\cdot}103 . \supset \vdash :. \text{Hp} . \supset : x, y \epsilon \overrightarrow{B}`P . \supset . x = y :$

[*52·4] $\supset : \overrightarrow{B}`P \epsilon\, 0 \cup 1$ (2)

$\vdash . (2) . *202{\cdot}11 . \quad \supset \vdash : \text{Hp} . \supset . \overrightarrow{B}`\breve{P} \epsilon\, 0 \cup 1$ (3)

$\vdash . (2) . (3) . \supset \vdash . \text{Prop}$

***202·521.** $\vdash : P_* \epsilon \text{connex} . \supset . \overrightarrow{B}`P \subset p`\overrightarrow{P_*}``C`P$

Dem.

$\vdash . *202{\cdot}13{\cdot}103 . \supset$

$\vdash :: \text{Hp} . \supset :. x \epsilon \overrightarrow{B}`P . y \epsilon C`P . \supset : x P_{\text{po}} y . \vee . x = y . \vee . y P_{\text{po}} x$ (1)

$\vdash . *91{\cdot}504 . \supset \vdash : x \epsilon \overrightarrow{B}`P . \supset . \sim(y P_{\text{po}} x)$ (2)

$\vdash . (1) . (2) . \supset \vdash :: \text{Hp} . \supset :. x \epsilon \overrightarrow{B}`P . y \epsilon C`P . \supset : x P_{\text{po}} y . \vee . x = y :$

[*91·54] $\supset : x P_* y :: \supset \vdash . \text{Prop}$

***202·522.** $\vdash . \overrightarrow{B}`P = \overrightarrow{B}`P_{\text{po}}$ [*91·504]

***202·523.** $\vdash : P_* \epsilon \text{connex} . \supset . \overrightarrow{B}`P \epsilon\, 0 \cup 1$ [*202·13·52·522]

***202·524.** $\vdash : P \epsilon \text{connex} . \exists ! \overrightarrow{B}`P . \supset . \text{Ɑ}`P = \overleftarrow{P}`B`P$

Dem.

$\vdash . *202{\cdot}52 . \supset \vdash :. \text{Hp} . \supset : E ! B`P :$

[*202·104.*93·103] $\supset : x \epsilon\, \text{Ɑ}`P . \supset . (B`P) P x$ (1)

$\vdash . (1) . *33{\cdot}151 . \supset \vdash . \text{Prop}$

The following propositions (*202·53—·55) are concerned with relations with limited fields. Such relations are constantly used in the theory of series.

***202·53.** $\vdash : Q \,\epsilon\, \text{connex} \,.\, P^2 \mathrel{\subseteq} J \,.\, Q \mathrel{\subseteq} P \,.\, \supset \,.\, Q = P \upharpoonright C\text{‘}Q$

Dem.

$\vdash \,.\, *33\cdot17 \,.\, *36\cdot13 \,.\quad \supset \vdash :. \text{Hp} \,.\, \supset : xQy \,.\, \supset \,.\, x\,(P \upharpoonright C\text{‘}Q)\,y \quad (1)$

$\vdash \,.\, *50\cdot43 \,.\quad \supset \vdash :. \text{Hp} \,.\, \supset : xPy \,.\, \supset \,.\, \sim(yPx) \,.$

$[*23\cdot81] \quad \supset \,.\, \sim(yQx) \,. \quad (2)$

$\vdash \,.\, *200\cdot36 \,.\quad \supset \vdash :. \text{Hp} \,.\, \supset : xPy \,.\, \supset \,.\, x \neq y \quad (3)$

$\vdash \,.\, (2) \,.\, (3) \,.\, *202\cdot104 \,.\, \supset \vdash :. \text{Hp} \,.\, \supset : x, y \,\epsilon\, C\text{‘}Q \,.\, xPy \,.\, \supset \,.\, xQy :$

$[*36\cdot13] \quad \supset : x\,(P \upharpoonright C\text{‘}Q)\,y \,.\, \supset \,.\, xQy \quad (4)$

$\vdash \,.\, (1) \,.\, (4) \,.\, \supset \vdash \,.\, \text{Prop}$

This proposition is important in series. If P and Q are serial relations, and $Q \mathrel{\subseteq} P$, they verify the above hypothesis; hence if Q is a series contained in a given series P, Q is simply P with its field limited. Thus series contained in a given series are completely determined by their fields.

***202·54.** $\vdash : P \upharpoonright \alpha \,\epsilon\, \text{connex} \,.\, \alpha \cap C\text{‘}P \sim\epsilon\, 1 \,.\, \supset \,.\, C\text{‘}P \upharpoonright \alpha = \alpha \cap C\text{‘}P$

Dem.

$\vdash \,.\, *52\cdot181 \,.\, \supset$

$\vdash :: \text{Hp} \,.\, \supset :. x \,\epsilon\, \alpha \cap C\text{‘}P \,.\, \supset_x : (\exists y) \,.\, y \,\epsilon\, \alpha \cap C\text{‘}P \,.\, y \neq x :$

$[*202\cdot104] \quad \supset_x : (\exists y) : y \,\epsilon\, \alpha \cap C\text{‘}P : xPy \,.\, \vee \,.\, yPx :$

$[*36\cdot13] \quad \supset_x : (\exists y) : x\,(P \upharpoonright \alpha)\,y \,.\, \vee \,.\, y\,(P \upharpoonright \alpha)\,x :$

$[*33\cdot132] \quad \supset_x : x \,\epsilon\, C\text{‘}P \upharpoonright \alpha \quad (1)$

$\vdash \,.\, *37\cdot41\cdot15\cdot16 \,.\, \supset \vdash \,.\, C\text{‘}P \upharpoonright \alpha \subset \alpha \cap C\text{‘}P \quad (2)$

$\vdash \,.\, (1) \,.\, (2) \,.\, \supset \vdash \,.\, \text{Prop}$

The above proposition is frequently used. *202·55, which is an immediate consequence of *202·54, is used incessantly.

The following proposition is used in *232·14.

***202·541.** $\vdash : P \,\epsilon\, \text{trans} \cap \text{connex} \,.\, \alpha \cap C\text{‘}P \sim\epsilon\, 1 \,.\, \supset \,.\, (P \upharpoonright \alpha)_* = P_* \upharpoonright \alpha$

Dem.

$\vdash \,.\, *201\cdot18\cdot33 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, (P \upharpoonright \alpha)_* = P \upharpoonright \alpha \cup I \upharpoonright (C\text{‘}P \upharpoonright \alpha)$

$[*202\cdot54] \quad = P \upharpoonright \alpha \cup I \upharpoonright (C\text{‘}P \cap \alpha)$

$[*201\cdot18 \,.\, *36\cdot23 \,.\, *50\cdot5] \quad = P_* \upharpoonright \alpha$

***202·55.** $\vdash : P \upharpoonright \alpha \,\epsilon\, \text{connex} \,.\, \alpha \subset C\text{‘}P \,.\, \alpha \sim\epsilon\, 1 \,.\, \supset \,.\, C\text{‘}P \upharpoonright \alpha = \alpha \quad [*202\cdot54]$

***202·56.** $\vdash : P \,\epsilon\, \text{connex} \,.\, P \mathrel{\subseteq} J \,.\, x \,\epsilon\, C\text{‘}P \,.\, \beta \subset C\text{‘}P \,.\, P\text{‘‘}\beta \subset \overrightarrow{P}\text{‘}x \,.\, \supset \,.\, \beta \subset \overrightarrow{P}\text{‘}x \cup \iota\text{‘}x$

Dem.

$\vdash \,.\, *37\cdot1 \,.\quad \supset \vdash : P\text{‘‘}\beta \subset \overrightarrow{P}\text{‘}x \,.\, y \,\epsilon\, \beta \,.\, xPy \,.\, \supset \,.\, xPx \quad (1)$

$\vdash \,.\, (1) \,.\, \text{Transp} \,.\, \supset \vdash : \text{Hp} \,.\, y \,\epsilon\, \beta \,.\, \supset \,.\, \sim(xPy) \quad (2)$

$\vdash \,.\, (2) \,.\, *32\cdot18 \,.\, \supset \vdash : \text{Hp} \,.\, y \,\epsilon\, \beta - \overrightarrow{P}\text{‘}x \,.\, \supset \,.\, \sim(xPy) \,.\, \sim(yPx) \,.$

$[*202\cdot103] \quad \supset \,.\, y = x : \supset \vdash \,.\, \text{Prop}$

The above proposition is used in *212·652.

***202·6.** $\vdash :: P \in \text{connex} . P \mathrel{G} J . \supset :. x, y \in C`P . x \neq y . \equiv : xPy . \vee . yPx$

Dem.

$\vdash . *202·104 . \quad \supset \vdash :: \text{Hp} . \supset :. x, y \in C`P . x \neq y . \supset : xPy . \vee . yPx$ (1)

$\vdash . *50·11 . *33·17 . \supset \vdash :: \text{Hp} . \supset :. xPy . \vee . yPx : \supset . x, y \in C`P . x \neq y$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

The following proposition is a lemma for *202·62, which is itself a lemma for *204·52.

***202·61.** $\vdash :: P \in \text{connex} . P \mathrel{G} J : \phi(x, y) . \equiv_{x,y} . \phi(y, x) : \supset :.$
$xPy . \supset_{x,y} . \phi(x, y) : \equiv : x, y \in C`P . x \neq y . \supset_{x,y} . \phi(x, y)$

Dem.

$\vdash . *202·6 . \supset \vdash ::. \text{Hp} . \supset :: x, y \in C`P . x \neq y . \supset_{x,y} . \phi(x, y) : \equiv :.$
$xPy . \vee . yPx : \supset_{x,y} . \phi(x, y) :.$

[*4·77] $\equiv :. xPy . \supset_{x,y} . \phi(x, y) : yPx . \supset_{x,y} . \phi(x, y) :.$

[*4·85.Hp] $\equiv :. xPy . \supset_{x,y} . \phi(x, y) : yPx . \supset_{x,y} . \phi(y, x) :.$

[*4·24] $\equiv :. xPy . \supset_{x,y} . \phi(x, y) ::. \supset \vdash . \text{Prop}$

***202·611.** $\vdash :. P \in \text{connex} . P \mathrel{G} J . R = \breve{R} . \supset : P \mathrel{G} R . \equiv . J \upharpoonright C`P \mathrel{G} R$

$\left[*202·61 \dfrac{xRy}{\phi(x, y)}\right]$

***202·62.** $\vdash :. P \in \text{connex} . P \mathrel{G} J . \supset : P \in \text{Rel}^2 \text{excl} . \equiv . F ; P \mathrel{G} J$

Dem.

$\vdash . *202·61 . *163·1 . \supset \vdash :: \text{Hp} . \supset :.$
$P \in \text{Rel}^2 \text{excl} . \equiv : QPR . \supset_{Q,R} . C`Q \cap C`R = \Lambda :$

[*24·37] $\equiv : QPR . x \in C`Q . y \in C`R . \supset_{Q,R,x,y} . x \neq y :$

[*150·52] $\equiv . x(F ; P) y . \supset_{x,y} . x \neq y :: \supset \vdash . \text{Prop}$

The three following propositions (*202·7—·72) are concerned with $P \dot{-} P^2$. Of these, *202·7 is important: it shows that if P is connected, no term can have more than one immediate predecessor or successor. *202·72 is used in *204·71, which is an important proposition.

***202·7.** $\vdash : P \in \text{connex} . \supset . P \dot{-} P^2 \in 1 \rightarrow 1$

Dem.

$\vdash . *34·5 . \text{Transp} . \supset \vdash : zPx . \sim(yP^2x) . \supset . \sim(yPz)$ (1)

Similarly $\vdash : yPx . \sim(zP^2x) . \supset . \sim(zPy)$ (2)

$\vdash . (1) . (2) . \supset \vdash : y(P \dot{-} P^2) x . z(P \dot{-} P^2) x . \supset . \sim(yPz) . \sim(zPy)$ (3)

$\vdash . (3) . *202·103 . \supset \vdash :. \text{Hp} . \supset : y(P \dot{-} P^2) x . z(P \dot{-} P^2) x . \supset . y = z$ (4)

Similarly $\vdash :. \text{Hp} . \supset : x(P \dot{-} P^2) y . x(P \dot{-} P^2) z . \supset . y = z$ (5)

$\vdash . (4) . (5) . \supset \vdash . \text{Prop}$

***202·71.** $\vdash : P \in \text{connex} . x(P \dot{-} P^2) y . \supset . \overrightarrow{P_{\text{po}}}`y = \overrightarrow{P_*}`x$

Dem.

$\vdash . *91·52 . \supset \vdash : \text{Hp} . \supset . \overrightarrow{P_*}`x \subset \overrightarrow{P_{\text{po}}}`y$ (1)

$\vdash . *91{\cdot}57 . \supset \vdash :. zP_{po}y . \supset : zPy . \vee . zP_{po}|Py :$

[*25·41] $\supset : z(P \dot{-} P^2)y . \vee . z(P \dot{\cap} P^2)y . \vee . z(P_{po}|P)y :$

[*91·502] $\supset : z(P \dot{-} P^2)y . \vee . z(P_{po}|P)y$ (2)

$\vdash . *202{\cdot}7 . \supset \vdash : \text{Hp} . z(P \dot{-} P^2)y . \supset . z = x$ (3)

$\vdash . (2) . (3) . \supset \vdash : \text{Hp} . zP_{po}y . z \neq x . \supset . z(P_{po}|P)y .$

[*34·1] $\supset . (\exists w) . zP_{po}w . wPy$ (4)

$\vdash . *34{\cdot}5 . \supset \vdash : wPy . xPw . \supset . xP^2y$ (5)

$\vdash . (5) . \text{Transp} . \supset \vdash :. \text{Hp} . wPy . \supset : \sim(xPw) :$

[*202·103] $\supset : wPx . \vee . w = x$ (6)

$\vdash . (4) . (6) . \supset \vdash :. \text{Hp} . zP_{po}y . z \neq x . \supset : zP_{po}x . \vee . (\exists w) . zP_{po}w . wPx :$

[*91·511] $\supset : zP_{po}x$ (7)

$\vdash . (7) . *91{\cdot}54 . \supset \vdash : \text{Hp} . \supset . \overrightarrow{P_{po}}`y \subset \overrightarrow{P_*}`x$ (8)

$\vdash . (1) . (8) . \supset \vdash . \text{Prop}$

***202·72.** $\vdash : P \epsilon \text{trans} \cap \text{connex} . x(P \dot{-} P^2)y . \supset . \overrightarrow{P}`y = \overrightarrow{P}`x \cup \iota`x$

[*202·71 . *201·18·521]

***202·8.** $\vdash : Q \epsilon \text{connex} . S \epsilon P \,\overline{\text{smor}}\, Q . C`Q \cap \beta \sim\epsilon 1 . \supset .$

$S \upharpoonright \beta \,\epsilon\, (P \upharpoonright S``\beta) \,\overline{\text{smor}}\, Q \upharpoonright \beta$

Dem.

$\vdash . *71{\cdot}29 . \supset \vdash : \text{Hp} . \supset . S \upharpoonright \beta \,\epsilon\, 1 \to 1$ (1)

$\vdash . *35{\cdot}64 . *151{\cdot}11 . \supset \vdash : \text{Hp} . \supset . \text{ᗡ}`(S \upharpoonright \beta) = C`Q \cap \beta$

[*202·54] $= C`(Q \upharpoonright \beta)$ (2)

$\vdash . *150{\cdot}37 . \supset \vdash : \text{Hp} . \supset . (S \upharpoonright \beta) ; Q = P \upharpoonright S``\beta$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash . \text{Prop}$

***202·81.** $\vdash : Q \epsilon \text{connex} . S \epsilon P \,\overline{\text{smor}}\, Q . \supset . (P \upharpoonright S``\beta) \text{ smor } Q \upharpoonright \beta$

Dem.

$\vdash . *202{\cdot}8 . \supset \vdash : \text{Hp} . C`Q \cap \beta \sim\epsilon 1 . \supset . (P \upharpoonright S``\beta) \text{ smor } Q \upharpoonright \beta$ (1)

$\vdash . *36{\cdot}13 . *33{\cdot}17 . \supset \vdash : C`Q \cap \beta = \iota`y . \supset . Q \upharpoonright \beta \,\mathsf{G}\, y \downarrow y$ (2)

$\vdash . *36{\cdot}13 . \supset \vdash :. \text{Hp}(2) . \supset : y(Q \upharpoonright \beta)y . \equiv . yQy$ (3)

$\vdash . (2) . (3) . *55{\cdot}341 . \supset \vdash : \text{Hp}(2) . yQy . \supset . Q \upharpoonright \beta = y \downarrow y$ (4)

$\vdash . *35{\cdot}64 . *151{\cdot}11 . \supset \vdash : \text{Hp} . \supset . \text{ᗡ}`(S \upharpoonright \beta) = C`Q \cap \beta$ (5)

$\vdash . (4) . (5) . \supset \vdash : \text{Hp}(4) . \supset . \text{ᗡ}`(S \upharpoonright \beta) = C`(Q \upharpoonright \beta)$ (6)

$\vdash . *71{\cdot}29 . *150{\cdot}37 . \supset \vdash : \text{Hp} . \supset . S \upharpoonright \beta \,\epsilon\, 1 \to 1 . P \upharpoonright S``\beta = S ; (Q \upharpoonright \beta)$ (7)

$\vdash . (6) . (7) . *151{\cdot}1 . \supset \vdash : \text{Hp}(4) . \supset . (P \upharpoonright S``\beta) \text{ smor } Q \upharpoonright \beta$ (8)

$\vdash . (2) . (3) . *55{\cdot}341 . \supset \vdash : \text{Hp}(2) . \sim(yQy) . \supset . Q \upharpoonright \beta = \dot{\Lambda} .$ (9)

[(7).*150·42] $\supset . P \upharpoonright S``\beta = \dot{\Lambda}$ (10)

$\vdash . (9) . (10) . *153{\cdot}101 . \supset \vdash : \text{Hp}(9) . \supset . (P \upharpoonright S``\beta) \text{ smor } Q \upharpoonright \beta$ (11)

$\vdash . (8) . (11) . *52{\cdot}1 . \supset \vdash : \text{Hp} . C`Q \cap \beta \,\epsilon\, 1 . \supset . (P \upharpoonright S``\beta) \text{ smor } Q \upharpoonright \beta$ (12)

$\vdash . (1) . (12) . \supset \vdash . \text{Prop}$

The above proposition shows that if Q is connected, and any class β is picked out of $C`Q$, and P is similar to Q, then Q arranges β in an order which is similar to that in which P arranges the correlates of β.

$*204$. ELEMENTARY PROPERTIES OF SERIES

Summary of $*204$.

In this number we give the definition and a few of the simpler properties of series. Most of the propositions of this number result immediately from those of $*200$, $*201$, and $*202$. Our definition is

$$\text{Ser} = \text{Rl}`J \cap \text{trans} \cap \text{connex} \quad \text{Df.}$$

We have

$*204\cdot16$. $\vdash : P \epsilon \text{Ser} . \equiv . P \epsilon \text{connex} . P^2 \mathrel{\subset\!\!\!\cdot} J . P^3 \mathrel{\subset\!\!\!\cdot} J . \equiv . P \epsilon \text{connex} . P_{\text{po}} \mathrel{\subset\!\!\!\cdot} J$

either of which might have been taken as the definition.

After a few propositions giving other possible forms of the definition of series, we proceed to a set of propositions which follow immediately from those of $*200$, $*201$, and $*202$. Such are

$*204\cdot2$. $\vdash : P \epsilon \text{Ser} . \equiv . \breve{P} \epsilon \text{Ser}$

$*204\cdot21$. $\vdash : P \epsilon \text{Ser} . P \,\text{smor}\, Q . \supset . Q \epsilon \text{Ser}$

$*204\cdot24$. $\vdash . \dot{\Lambda} \epsilon \text{Ser}$

$*204\cdot25$. $\vdash : x \neq y . \equiv . x \downarrow y \epsilon \text{Ser}$

Another important proposition on couples is

$*204\cdot272$. $\vdash :. P \epsilon \text{Ser} . \supset : \text{D}`P \epsilon 1 . \equiv . P \epsilon 2_r . \equiv . \text{G}`P \epsilon 1$

so that couples are the only series having unit classes for their domains or converse domains.

We then proceed to a set of propositions on $\overrightarrow{P}`x$. We have

$*204\cdot33$. $\vdash :. P \epsilon \text{Ser} . x, y \epsilon C`P . \supset : x \neq y . \overrightarrow{P}`y \subset \overrightarrow{P}`x . \equiv . yPx$

Also, if $P \epsilon \text{Ser}$, $\overrightarrow{P} \upharpoonright C`P$ is a one-one and $\overrightarrow{P} ; P$ smor P ($*204\cdot34\cdot35$).

We then have some propositions ($*204\cdot4$—$\cdot44$) on relations with limited fields. The most important of these are

$*204\cdot4$. $\vdash : P \epsilon \text{Ser} . \supset . P \upharpoonright \alpha \epsilon \text{Ser}$

$*204\cdot41$. $\vdash : P, Q \epsilon \text{Ser} . Q \mathrel{\subset\!\!\!\cdot} P . \supset . Q = P \upharpoonright C`Q$

This proposition is important, since it shows that any series contained in a given series is wholly determined when its field is given.

We have next a number of propositions ($*204\cdot45$—$\cdot59$) applying relation-arithmetic to series. The first set of these ($*204\cdot45$—$\cdot483$) are concerned with the proof that if a "cut" is made in a series, the series is the sum of the two parts into which the cut divides it, where the sum is taken in the sense of $*160$ or $*161$, according as one part of the cut does not or does

consist of a single term. Most of these propositions do not require the full hypothesis that P is a series, but only some part of it. Thus we have for instance

$*204{\cdot}46.\quad \vdash : P \epsilon \text{connex} \,.\, \text{E}\,!\, B'P \,.\, \text{Ɑ}'P \sim\epsilon 1 \,.\supset .$
$$P = B'P \leftarrow\!\!\!+ P \upharpoonright \text{Ɑ}'P \,.\, \text{Nr}'P = \dot{1} \dotplus \text{Nr}'(P \upharpoonright \text{Ɑ}'P)$$

with a similar proposition for $B'\breve{P}$ and $\text{D}'P$ ($*204{\cdot}461$).

We next prove that if P, Q are mutually exclusive series, their sum ($P \mathbin{\dot{+}} Q$) is a series, and vice versa ($*204{\cdot}5$); that if P is a series to which x does not belong, $P \rightarrow\!\!\!+ x$ and $x \leftarrow\!\!\!+ P$ are series, and vice versa ($*204{\cdot}51$); that if P is a series of mutually exclusive series, its sum $\Sigma'P$ is a series ($*204{\cdot}52$); that if P, Q are series, so is $P \times Q$ ($*204{\cdot}55$); that if P is a series of series, $\Pi'P$ is contained in diversity and is transitive ($*204{\cdot}561$), while if P is also well-ordered, *i.e.* such that every existent sub-class of $C'P$ has a first term, then $\Pi'P$ is a series ($*204{\cdot}57$); and that if P and Q are series, and Q is well-ordered, then P^Q and $P \exp Q$ are series ($*204{\cdot}59$). These propositions are essential to ordinal arithmetic, but they will not be referred to again until we reach that stage (Sections D and E of this Part).

We have next a collection of propositions ($*204{\cdot}6$—$\cdot 65$) on $p'\overrightarrow{P}``\alpha$ for various values of α, and finally three propositions on P_1. Two of these are much used, namely

$*204{\cdot}7.\quad \vdash : P \epsilon \text{Ser} \,.\supset .\, P_1 \epsilon 1 \rightarrow 1$

$*204{\cdot}71.\quad \vdash : P \epsilon \text{Ser} \,.\, xP_1 y \,.\supset .\, \overrightarrow{P}'y = \overrightarrow{P}'x \cup \iota'x$

$*204{\cdot}01.\quad \text{Ser} = \text{Rl}'J \cap \text{trans} \cap \text{connex} \quad \text{Df}$

$*204{\cdot}1.\quad \vdash : P \epsilon \text{Ser} \,.\equiv .\, P \subseteq J \,.\, P^2 \subseteq P \,.\, P \epsilon \text{connex} \,.$
$\equiv .\, P \epsilon \text{Rl}'J \,.\, P \epsilon \text{trans} \,.\, P \epsilon \text{connex} \quad [(*204{\cdot}01)]$

$*204{\cdot}11.\quad \vdash :. P \epsilon \text{Ser} \,.\equiv : P \subseteq J \,.\, P^2 \subseteq P : x \epsilon C'P \,.\supset_x .\, \overrightarrow{P}'x \cup \iota'x \cup \overleftarrow{P}'x = C'P$
$[*204{\cdot}1 \,.\, *202{\cdot}101]$

$*204{\cdot}12.\quad \vdash :: P \epsilon \text{Ser} \,.\equiv :. P \subseteq J \,.\, P^2 \subseteq P :. x, y \epsilon C'P \,.\supset_{x,y} :$
$xPy \,.\vee .\, x = y \,.\vee .\, yPx \quad [*204{\cdot}1 \,.\, *202{\cdot}103]$

$*204{\cdot}121.\quad \vdash :: P \epsilon \text{Ser} \,.\equiv :. P \subseteq J \,.\, P^2 \subseteq P :. x, y \epsilon C'P \,.\, x \neq y \,.\supset_{x,y} :$
$xPy \,.\vee .\, yPx \quad [*204{\cdot}1 \,.\, *202{\cdot}104]$

$*204{\cdot}13.\quad \vdash : P \epsilon \text{Ser} \,.\supset .\, P^2 \subseteq J \,.\, P \mathbin{\dot{\cap}} \breve{P} = \dot{\Lambda}$

Dem.

$\vdash .\, *204{\cdot}1 \,.\, *23{\cdot}44 \,.\supset \vdash : P \epsilon \text{Ser} \,.\supset .\, P^2 \subseteq J \,.\, P \epsilon \text{trans} \qquad (1)$
$\vdash .\, (1) \,.\, *201{\cdot}12 \,.\supset \vdash .\, \text{Prop}$

$*204{\cdot}14.\quad \vdash : P \epsilon \text{Ser} \,.\equiv .\, P \mathbin{\dot{\cap}} \breve{P} = \dot{\Lambda} \,.\, P^2 \subseteq P \,.\, P \epsilon \text{connex}$
$[*204{\cdot}1 \,.\, *50{\cdot}47]$

***204·15.** $\vdash : P \epsilon \text{connex} . P^2 \subset J . P^3 \subset J . \supset . P \epsilon \text{trans}$

Dem.

$\vdash . *34·5 . \supset \vdash :. P^2 \subset J . \supset : xPy . yPz . \supset . x \neq z$ (1)

$\vdash . *50·41 . \supset \vdash :. P^3 \subset J . \supset : xPy . yPz . \supset . \sim(zPx)$ (2)

$\vdash . (1) . (2) . \supset \vdash :. \text{Hp} . \supset : xPy . yPz . \supset . x \neq z . \sim(zPx) .$

$[*202·103] \qquad \supset . xPz :. \supset \vdash . \text{Prop}$

***204·151.** $\vdash : P \epsilon \text{connex} . P_{po} \subset J . \supset . P \epsilon \text{trans}$
$[*204·15 . *91·502·503·511]$

***204·16.** $\vdash : P \epsilon \text{Ser} . \equiv . P \epsilon \text{connex} . P^2 \subset J . P^3 \subset J . \equiv . P \epsilon \text{connex} . P_{po} \subset J$
$[*204·15·151 . *200·36 . *201·18]$

We have also

$$\vdash : P \epsilon \text{Ser} . \equiv . P \epsilon \text{connex} . P^6 \subset J.$$

For, by *200·37, since $P^6 = (P^2)^3 = (P^3)^2$, it follows that

$$P^6 \subset J . \supset . P^2 \subset J . P^3 \subset J.$$

A relation such as $x \downarrow y \cup y \downarrow z \cup z \downarrow x$, where $x \neq y . y \neq z . z \neq x$, satisfies $P \epsilon \text{connex} . P^2 \subset J$, but not $P^3 \subset J$. On the other hand,

$$x \downarrow y \cup y \downarrow z \cup z \downarrow w \cup w \downarrow x$$

satisfies $P^2 \subset J . P^3 \subset J$, but not $P \epsilon \text{connex}$.

***204·2.** $\vdash : P \epsilon \text{Ser} . \equiv . \breve{P} \epsilon \text{Ser} \quad [*200·11 . *201·11 . *202·11]$

***204·21.** $\vdash : P \epsilon \text{Ser} . P \text{ smor } Q . \supset . Q \epsilon \text{Ser}$
$[*200·211 . *201·211 . *202·211]$

***204·22.** $\vdash : P \epsilon \text{Ser} . \supset . \text{Nr}‘P \subset \text{Ser} \quad [*204·21]$

***204·23.** $\vdash : P \epsilon \text{Ser} . \equiv . \text{N}_0\text{r}‘P \subset \text{Ser} . \equiv . \exists ! \text{N}_0\text{r}‘P \cap \text{Ser}$
$[*200·22 . *201·22 . *202·22]$

***204·24.** $\vdash . \dot{\Lambda} \epsilon \text{Ser} \qquad [*200·3 . *201·3 . *202·3]$

***204·25.** $\vdash : x \neq y . \equiv . x \downarrow y \epsilon \text{Ser} \quad [*200·31 . *201·31 . *202·31]$

***204·26.** $\vdash : x \neq y . x \neq z . y \neq z . \supset . x \downarrow y \mathbin{+\!\!\!\to} z \epsilon \text{Ser}$
$[*200·31·41 . *201·411 . *202·411]$

The three following propositions deal with couples. Couples often require special treatment, owing to the fact that, if P is a couple, $P \upharpoonright \text{D}‘P = \dot{\Lambda}$, so that $C‘(P \upharpoonright \text{D}‘P) \neq \text{D}‘P$, whereas in any other case, if P is a series, $C‘(P \upharpoonright \text{D}‘P) = \text{D}‘P$. Hence the following propositions are often required.

***204·27.** $\vdash : P \epsilon \text{Ser} . xPy . \text{D}‘P = \iota‘x . \supset . P = x \downarrow y$

Dem.

$\vdash . *33·14 . \supset \vdash : \text{Hp} . zPw . \supset . z = x$ (1)

$\vdash . (1) . *50·24 . \supset \vdash : \text{Hp} . zPw . \supset . w \neq x .$

$\left[(1).\text{Transp} \frac{w, y}{z, w}\right] \qquad \supset . \sim(wPy)$ (2)

$\vdash . (1) . \text{Transp} . *50 \cdot 24 . \supset \vdash : \text{Hp} . \supset . \sim (yPw)$ (3)

$\vdash . (2) . (3) . *204 \cdot 12 . \quad \supset \vdash : \text{Hp} . zPw . \supset . y = w$ (4)

$\vdash . (1) . (4) . \supset \vdash :. \text{Hp} . \supset : zPw . \supset . z = x . y = w$ (5)

$\vdash . (5) . *55 \cdot 34 . \supset \vdash . \text{Prop}$

***204·271.** $\vdash : P \in \text{Ser} . \text{D}`P \in 1 . \supset . P \in 2_r$

Dem.

$\vdash . *204 \cdot 27 . \supset \vdash : \text{Hp} . \supset . (\exists x, y) . P = x \downarrow y .$

$[*204 \cdot 25] \qquad \supset . (\exists x, y) . x \neq y . P = x \downarrow y .$

$[*56 \cdot 11] \qquad \supset . P \in 2_r : \supset \vdash . \text{Prop}$

***204·272.** $\vdash :. P \in \text{Ser} . \supset : \text{D}`P \in 1 . \equiv . P \in 2_r . \equiv . \text{ᗡ}`P \in 1$
$[*204 \cdot 271 \cdot 2 . *56 \cdot 111]$

***204·3.** $\vdash :. P \in \text{Ser} . x, y \in C`P . \supset : x \neq y . \sim (yPx) . \equiv . xPy$
$[*202 \cdot 5 . *204 \cdot 13]$

***204·32.** $\vdash :. P \in \text{Ser} . x, y \in C`P . \supset : \overrightarrow{P}`y \subset \overrightarrow{P}`x . \equiv . y \in \overrightarrow{P}`x \cup \iota`x$

Dem.

$\vdash . *204 \cdot 1 . \supset \vdash :. \text{Hp} . \supset : yPx . zPy . \supset . zPx :$

$[*32 \cdot 18] \qquad \supset : y \in \overrightarrow{P}`x . \supset . \overrightarrow{P}`y \subset \overrightarrow{P}`x$ (1)

$\vdash . *22 \cdot 42 . \supset \vdash : y = x . \supset . \overrightarrow{P}`y \subset \overrightarrow{P}`x$ (2)

$\vdash . (1) . (2) . \supset \vdash :. \text{Hp} . \supset : y \in \overrightarrow{P}`x \cup \iota`x . \supset . \overrightarrow{P}`y \subset \overrightarrow{P}`x$ (3)

$\vdash . *204 \cdot 11 . \supset \vdash :. \text{Hp} . \supset : y \sim \in \overrightarrow{P}`x \cup \iota`x . \supset . y \in \overleftarrow{P}`x .$

$[*32 \cdot 18 \cdot 181] \qquad \supset . x \in \overrightarrow{P}`y$ (4)

$\vdash . *50 \cdot 24 . \supset \vdash : \text{Hp} . \supset . x \sim \in \overrightarrow{P}`x$ (5)

$\vdash . (4) . (5) . \supset \vdash :. \text{Hp} . \supset : y \sim \in \overrightarrow{P}`x \cup \iota`x . \supset . \sim (\overrightarrow{P}`y \subset \overrightarrow{P}`x)$ (6)

$\vdash . (3) . (6) . \supset \vdash . \text{Prop}$

***204·33.** $\vdash :. P \in \text{Ser} . x, y \in C`P . \supset : x \neq y . \overrightarrow{P}`y \subset \overrightarrow{P}`x . \equiv . yPx$

Dem.

$\vdash . *204 \cdot 32 . \supset \vdash :. \text{Hp} . \supset : x \neq y . \overrightarrow{P}`y \subset \overrightarrow{P}`x . \equiv . x \neq y . y \in \overrightarrow{P}`x \cup \iota`x .$

$[*51 \cdot 15] \qquad \equiv . x \neq y . y \in \overrightarrow{P}`x .$

$[\text{Hp}.*4 \cdot 71] \qquad \equiv . yPx :. \supset \vdash . \text{Prop}$

The three following propositions only require $P \in \text{Rl}`J \cap \text{connex}$, but are required for application to series, and are therefore convenient in the form here given.

***204·331.** $\vdash :. P \in \text{Ser} . x, y \in C`P . \supset : \overrightarrow{P}`x = \overrightarrow{P}`y . \equiv . x = y$
$[*202 \cdot 161 . *71 \cdot 55]$

***204·34.** $\vdash : P \in \text{Ser} . \supset . \overrightarrow{P} \upharpoonright C`P \in 1 \to 1 . \overrightarrow{P} \upharpoonright C`P \in (\overrightarrow{P} ; P) \overline{\text{smor}}\, P \quad [*202 \cdot 161]$

$*204\cdot35.\quad \vdash : P \in \mathrm{Ser} \,.\supset .\, \overrightarrow{P} ; P \ \mathrm{smor}\ P \quad [*204\cdot34]$

This proposition shows that the series of segments which have immediate successors is like the original series, for a segment whose immediate successor is x is $\overrightarrow{P}\text{‘}x$, and the series of such segments is $\overrightarrow{P};P$.

The following propositions ($*204\cdot4$—$\cdot44$) are concerned with relations with limited fields.

$*204\cdot4.\quad \vdash : P \in \mathrm{Ser} \,.\supset .\, P \upharpoonright \alpha \in \mathrm{Ser} \quad [*200\cdot33 \,.\, *201\cdot33 \,.\, *202\cdot33]$

$*204\cdot41.\quad \vdash : P, Q \in \mathrm{Ser} \,.\, Q \mathrel{\dot\subset} P \,.\supset .\, Q = P \upharpoonright C\text{‘}Q \quad [*202\cdot53 \,.\, *204\cdot13]$

In virtue of the above two propositions, the series contained in a given series are the relations resulting from limitations of the field; the process of limiting the field is merely the process of selecting a part of the original series without changing the order.

$*204\cdot42.\quad \vdash :. P \in \mathrm{Ser} \,.\supset : Q \in \mathrm{Ser} \,.\, Q \mathrel{\dot\subset} P \,.\equiv .\, (\exists \alpha) \,.\, Q = P \upharpoonright \alpha \,.\equiv .\, Q \in \mathrm{D}\text{‘}P\upharpoonright$
$[*204\cdot4\cdot41]$

$*204\cdot421.\quad \vdash : P \in \mathrm{Ser} \,.\supset .\, \mathrm{Ser} \cap \mathrm{Rl}\text{‘}P = \mathrm{D}\text{‘}P\upharpoonright \quad [*204\cdot42]$

$*204\cdot43.\quad \vdash : P^2 \mathrel{\dot\subset} P \,.\, P \mathrel{\dot\subset} J \,.\, Q \mathrel{\dot\subset} P \,.\, Q \in \mathrm{connex} \,.\supset .\, Q \in \mathrm{Ser}$

Dem.

$\vdash .\, *23\cdot1 \,.\, *34\cdot55 \,.\supset \vdash :. \mathrm{Hp} \,.\supset : xQy \,.\, yQz \,.\supset .\, xPz \,.$

$[*50\cdot43.\mathrm{Hp}] \qquad \supset .\, \sim(zPx) \,.\, x \neq z \,.$

$[*23\cdot81.\mathrm{Hp}] \qquad \supset .\, \sim(zQx) \,.\, x \neq z \,.$

$[*202\cdot103] \qquad \supset .\, xQz :$

$[*34\cdot55] \qquad \supset : Q^2 \mathrel{\dot\subset} Q \qquad (1)$

$\vdash .\, *23\cdot44 \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, Q \mathrel{\dot\subset} J \qquad (2)$

$\vdash .\, (1) \,.\, (2) \,.\, *204\cdot1 \,.\supset \vdash .\, \mathrm{Prop}$

$*204\cdot44.\quad \vdash : P \in \mathrm{Rl}\text{‘}J \cap \mathrm{trans} \,.\supset .\, \mathrm{Rl}\text{‘}P \cap \mathrm{connex} \subset \mathrm{Ser} \quad [*204\cdot43]$

The following propositions ($*204\cdot45$—$\cdot483$) are concerned with the division of a series into two parts, one of which wholly precedes the other. The case where one of the parts consists of a single term requires special treatment, and so does the case where both parts consist of single terms, *i.e.* where the series is a couple.

$*204\cdot45.\quad \vdash : P \in \mathrm{connex} \,.\, \alpha \in \mathrm{Cl}\text{‘}C\text{‘}P - 1 \,.\, P\text{“}\alpha \subset \alpha \,.\, \beta = C\text{‘}P - \alpha \,.\, \beta \sim\in 1 \,.\supset .$
$P = P \upharpoonright \alpha \mathbin{\overset{\uparrow}{+}} P \upharpoonright \beta \,.\, \mathrm{Nr}\text{‘}P = \mathrm{Nr}\text{‘}P\upharpoonright\alpha \dotplus \mathrm{Nr}\text{‘}P\upharpoonright\beta$

Dem.

$\vdash .\, *24\cdot411 \,.\, *33\cdot17 \,.\supset \vdash :: \mathrm{Hp} \,.\supset :.$

$xPy \,.\equiv : y \in \alpha \,.\, xPy \,.\vee .\, x \in \alpha \,.\, y \in \beta \,.\, xPy \,.\vee .\, x, y \in \beta \,.\, xPy \qquad (1)$

$\vdash .\, *37\cdot17 \,. \qquad \supset \vdash :. \mathrm{Hp} \,.\supset : y \in \alpha \,.\, xPy \,.\supset .\, x \in \alpha \qquad (2)$

$\vdash .\, (2) \,.\, \mathrm{Transp} \,.\, *202\cdot103 \,.\supset \vdash :. \mathrm{Hp} \,.\supset : y \in \alpha \,.\, x \in \beta \,.\supset .\, yPx \qquad (3)$

$\vdash .\, *202\cdot55 \,. \qquad \supset \vdash : \mathrm{Hp} \,.\supset .\, \alpha = C\text{‘}P\upharpoonright\alpha \,.\, \beta = C\text{‘}P\upharpoonright\beta \qquad (4)$

$\vdash . (1) . (2) . (3) . (4) . \supset \vdash :: \text{Hp} . \supset :.$

$xPy . \equiv : x (P \upharpoonright \alpha) y . \vee . x \in C\text{‘}P \upharpoonright \alpha . y \in C\text{‘}P \upharpoonright \beta . \vee . x (P \upharpoonright \beta) y :$

$[*160\cdot1] \quad \equiv : x \{P \upharpoonright \alpha \mathbin{\dot{+}} P \upharpoonright \beta\} y \qquad (5)$

$\vdash . (5) . *180\cdot32 . \supset \vdash : \text{Hp} . \supset . \text{Nr‘}P = \text{Nr‘}P \upharpoonright \alpha \dotplus \text{Nr‘}P \upharpoonright \beta \qquad (6)$

$\vdash . (5) . (6) . \supset \vdash . \text{Prop}$

$*204\cdot46$. $\vdash : P \in \text{connex} . \text{E} ! B\text{‘}P . \text{Ɑ‘}P \sim\in 1 . \supset .$

$P = B\text{‘}P \leftarrow\!\!\!+ P \upharpoonright \text{Ɑ‘}P . \text{Nr‘}P = \dot{1} \dotplus \text{Nr‘}(P \upharpoonright \text{Ɑ‘}P)$

Dem.

$\vdash . *202\cdot524 . \quad \supset \vdash :. \text{Hp} . \supset : x = B\text{‘}P . y \in \text{Ɑ‘}P . \supset . xPy \qquad (1)$

$\vdash . (1) . *161\cdot111 . \supset \vdash :: \text{Hp} . \supset :. x (B\text{‘}P \leftarrow\!\!\!+ P \upharpoonright \text{Ɑ‘}P) y . \equiv :$

$x = B\text{‘}P . y \in \text{Ɑ‘}P . xPy . \vee . x, y \in \text{Ɑ‘}P . xPy :$

$[*93\cdot103] \quad \equiv : x \in C\text{‘}P . y \in \text{Ɑ‘}P . xPy :$

$[*33\cdot14\cdot17] \quad \equiv : xPy \qquad (2)$

$\vdash . (2) . *181\cdot32 . \quad \supset \vdash : \text{Hp} . \supset . \text{Nr‘}P = \dot{1} \dotplus \text{Nr‘}(P \upharpoonright \text{Ɑ‘}P) \qquad (3)$

$\vdash . (2) . (3) . \supset \vdash . \text{Prop}$

$*204\cdot461$. $\vdash : P \in \text{connex} . \text{E} ! B\text{‘}\breve{P} . \text{D‘}P \sim\in 1 . \supset .$

$P = P \upharpoonright \text{D‘}P \mathbin{+\!\!\!\rightarrow} B\text{‘}\breve{P} . \text{Nr‘}P = \text{Nr‘}(P \upharpoonright \text{D‘}P) \dotplus \dot{1}$

[Proof as in $*204\cdot46$]

$*204\cdot462$. $\vdash :. P, Q \in \text{connex} . \text{E} ! B\text{‘}P . \text{Ɑ‘}P \sim\in 1 . \text{E} ! B\text{‘}Q . \text{Ɑ‘}Q \sim\in 1 . \supset :$

$P \text{ smor } Q . \equiv . P \upharpoonright \text{Ɑ‘}P \text{ smor } Q \upharpoonright \text{Ɑ‘}Q \quad [*161\cdot33 . *204\cdot46]$

$*204\cdot463$. $\vdash : P, Q \in \text{Rl‘}J . \text{E} ! B\text{‘}P . \text{Ɑ‘}P \in 1 . \text{E} ! B\text{‘}Q . \text{Ɑ‘}Q \in 1 . \supset .$

$P \text{ smor } Q . P \upharpoonright \text{Ɑ‘}P \text{ smor } Q \upharpoonright \text{Ɑ‘}Q$

Dem.

$\vdash . *56\cdot37 . \quad \supset \vdash : \text{Hp} . \supset . P, Q \in 2_r \qquad (1)$

$\vdash . *200\cdot35 . \supset \vdash : \text{Hp} . \supset . P \upharpoonright \text{Ɑ‘}P = \dot{\Lambda} . Q \upharpoonright \text{Ɑ‘}Q = \dot{\Lambda} \qquad (2)$

$\vdash . (1) . (2) . *153\cdot202\cdot101 . \supset$

$\vdash : \text{Hp} . \supset . P \text{ smor } Q . P \upharpoonright \text{Ɑ‘}P \text{ smor } Q \upharpoonright \text{Ɑ‘}Q : \supset \vdash . \text{Prop}$

$*204\cdot47$. $\vdash :. P, Q \in \text{connex} \cap \text{Rl‘}J . \text{E} ! B\text{‘}P . \text{E} ! B\text{‘}Q . \supset :$

$P \text{ smor } Q . \equiv . P \upharpoonright \text{Ɑ‘}P \text{ smor } Q \upharpoonright \text{Ɑ‘}Q$

Dem.

$\vdash . *151\cdot18 . *200\cdot35 . *202\cdot55 . *153\cdot102 . \supset$

$\vdash : \text{Hp} . \text{Ɑ‘}P \in 1 . \text{Ɑ‘}Q \sim\in 1 . \supset . \sim(P \text{ smor } Q) . \sim(P \upharpoonright \text{Ɑ‘}P \text{ smor } Q \upharpoonright \text{Ɑ‘}Q) \qquad (1)$

$\vdash . (1) . *204\cdot462\cdot463 . \supset \vdash . \text{Prop}$

$*204\cdot48$. $\vdash :: P \in \text{Ser} . \supset :.$

$\text{E} ! B\text{‘}P . \equiv : (\exists Q) . \dot{\exists} ! Q . \text{Nr‘}P = \dot{1} \dotplus \text{Nr‘}Q . \vee . \text{Nr‘}P = 2_r$

Dem.

$\vdash . *204\cdot46 . \quad \supset \vdash : \text{Hp} . \text{E} ! B\text{‘}P . \text{Ɑ‘}P \sim\in 1 . \supset . (\exists Q) . \text{Nr‘}P = \dot{1} \dotplus \text{Nr‘}Q \qquad (1)$

$\vdash . *161\cdot2 . \quad \supset \vdash : \dot{\exists} ! P . \text{Nr‘}P = \dot{1} \dotplus \text{Nr‘}Q . \supset . \dot{\exists} ! Q \qquad (2)$

$\vdash . (1) . (2) . \quad \supset \vdash : \text{Hp} . \text{E} ! B\text{‘}P . \text{Ɑ‘}P \sim\in 1 . \supset .$

$(\exists Q) . \dot{\exists} ! Q . \text{Nr‘}P = \dot{1} \dotplus \text{Nr‘}Q \qquad (3)$

$\vdash . *204{\cdot}272 . \supset \vdash : \text{Hp} . \text{G}`P \in 1 . \supset . P \in 2_r$ (4)

$\vdash . (3) . (4) . \quad \supset \vdash : \text{Hp} . \text{E} ! B`P . \supset :$

$(\exists Q) . \dot{\exists} ! Q . \text{Nr}`P = \dot{1} \dot{+} \text{Nr}`Q . \text{v} . \text{Nr}`P = 2_r$ (5)

$\vdash . *181{\cdot}11{\cdot}12{\cdot}32 . \supset \vdash : \text{Nr}`P = \dot{1} \dot{+} \text{Nr}`Q . \supset .$

$(\exists R, z) . R \,\text{smor}\, Q . z \sim\in C`R . \text{Nr}`P = \text{Nr}`(z \leftrightarrow R)$ (6)

$\vdash . *161{\cdot}15{\cdot}12 . \supset \vdash :. \dot{\exists} ! R . z \sim\in C`R . \supset : \text{E} ! B`(z \leftrightarrow R) :$

[*151·5] $\supset : \text{Nr}`P = \text{Nr}`(z \leftrightarrow R) . \supset . \text{E} ! B`P$ (7)

$\vdash . (6) . (7) . \quad \supset \vdash : \text{Nr}`P = \dot{1} \dot{+} \text{Nr}`Q . \dot{\exists} ! Q . \supset . \text{E} ! B`P$ (8)

$\vdash . *153{\cdot}281 . \supset \vdash : P \in 2_r . \supset . \text{E} ! B`P$ (9)

$\vdash . (5) . (8) . (9) . \supset \vdash . \text{Prop}$

***204·481.** $\vdash :: P \in \text{Ser} . \supset :.$

$\text{E} ! B`\breve{P} . \equiv : (\exists Q) . \dot{\exists} ! Q . \text{Nr}`P = \text{Nr}`Q \dot{+} \dot{1} . \text{v} . \text{Nr}`P = 2_r$

[Proof as in *204·48]

***204·482.** $\vdash :: \alpha \in \text{N}_0\text{r}``\text{Ser} . \supset :. \alpha \subset \text{G}`B : \equiv : \exists ! \alpha \cap \text{G}`B :$

$\equiv : (\exists \beta) . \beta \in \text{NR} - \iota`0_r . \alpha = \dot{1} \dot{+} \beta . \text{v} . \alpha = 2_r$

Dem.

$\vdash . *151{\cdot}5 . *155{\cdot}13 . \supset \vdash :. \text{Hp} . \supset : \alpha \subset \text{G}`B . \equiv . \exists ! \alpha \cap \text{G}`B$ (1)

$\vdash . *204{\cdot}23{\cdot}48 . \quad \supset \vdash :: \text{Hp} . P \in \alpha . \supset :.$

$\text{E} ! B`P . \equiv : (\exists \beta) . \beta \in \text{NR} - \iota`0_r . \alpha = \dot{1} \dot{+} \beta . \text{v} . \alpha = 2_r$ (2)

$\vdash . (1) . (2) . *202{\cdot}52 . \supset \vdash . \text{Prop}$

***204·483.** $\vdash :: \alpha \in \text{N}_0\text{r}``\text{Ser} . \supset :. \alpha \subset \text{G}`(B \,|\, \text{Cnv}) : \equiv : \exists ! \alpha \cap \text{G}`(B \,|\, \text{Cnv}) :$

$\equiv : (\exists \beta) . \beta \in \text{NR} - \iota`0_r . \alpha = \beta \dot{+} \dot{1} . \text{v} . \alpha = 2_r$

[Proof as in *204·482]

The following propositions are concerned with the application of relation-arithmetic to series.

***204·5.** $\vdash : P, Q \in \text{Ser} . C`P \cap C`Q = \Lambda . \equiv . P \dotplus Q \in \text{Ser}$

[*200·4 . *201·401 . *202·401]

***204·51.** $\vdash : P \in \text{Ser} . x \sim\in C`P . \equiv . P \rightarrow x \in \text{Ser} . \equiv . x \leftarrow P \in \text{Ser}$

[*200·41 . *201·41 . *202·412]

***204·52.** $\vdash : P \in \text{Rel}^2\,\text{excl} \cap \text{Ser} . C`P \subset \text{Ser} . \supset . \Sigma`P \in \text{Ser}$

Dem.

$\vdash . *200{\cdot}42 . *202{\cdot}62 . \supset \vdash : \text{Hp} . \supset . \Sigma`P \in J$ (1)

$\vdash . (1) . *201{\cdot}42 . *202{\cdot}42 . \supset \vdash . \text{Prop}$

***204·53.** $\vdash :. P \in \text{Rel}^2\,\text{excl} . \dot{\Lambda} \sim\in C`P . \supset : \Sigma`P \in \text{Ser} . \equiv . P \in \text{Ser} . C`P \subset \text{Ser}$

Dem.

$\vdash . *200{\cdot}423 . \supset \vdash :. \text{Hp} . \Sigma`P \in \text{Ser} . \supset : P \in J :$ (1)

[*200·421] $\supset : Q \in C`P . \supset . Q = (\Sigma`P) \upharpoonright C`Q .$

[*204·4] $\supset . Q \in \text{Ser}$ (2)

$\vdash . *162\cdot13 . \supset$

$\vdash :. \mathrm{Hp} . \Sigma\text{‘}P \in \mathrm{Ser} . QPR . RPS . x \in C\text{‘}Q . y \in C\text{‘}R . z \in C\text{‘}S . \supset : x(\Sigma\text{‘}P)z :$ (3)

[*162·13.*163·11] $\supset : (\exists M, N) . MPN . x \in C\text{‘}M . z \in C\text{‘}N . M = Q . N = S . \lor .$

$(\exists M) . M \in C\text{‘}P . xMz . M = Q . M = S :$

[*13·22·195] $\supset : QPS . \lor . Q = S$ (4)

$\vdash . (3) . *50\cdot24 . *24\cdot37 . \supset \vdash : \mathrm{Hp}(3) . \supset . C\text{‘}Q \cap C\text{‘}S = \Lambda .$

[*24·57.*30·37] $\supset . Q \neq S$ (5)

$\vdash . (4) . (5) . \supset \vdash :. \mathrm{Hp} . \Sigma\text{‘}P \in \mathrm{Ser} . \supset : QPR . RPS . \supset . QPS$ (6)

$\vdash . *162\cdot1 . \supset \vdash :. \mathrm{Hp} . \Sigma\text{‘}P \in \mathrm{Ser} . Q, R \in C\text{‘}P . x \in C\text{‘}P . y \in C\text{‘}Q . Q \neq R . \supset : x \neq y :$

[*202·104] $\supset : x(\Sigma\text{‘}P)y . \lor . y(\Sigma\text{‘}P)x :$

[*162·13.*163·11] $\supset : QPR . \lor . RPQ$ (7)

$\vdash . (6) . (7) . \supset \vdash : \mathrm{Hp} . \Sigma\text{‘}P \in \mathrm{Ser} . \supset . P \in \mathrm{trans} \cap \mathrm{connex}$ (8)

$\vdash . (1) . (2) . (8) . *204\cdot52 . \supset \vdash . \mathrm{Prop}$

***204·54.** $\vdash : P \in \mathrm{Rel}^3\,\mathrm{arithm} \cap \mathrm{Ser} . C\text{‘}P \subset \mathrm{Ser} . C\text{‘}\Sigma\text{‘}P \subset \mathrm{Ser} . \supset . \Sigma\text{‘}\Sigma\text{‘}P \in \mathrm{Ser}$

Dem.

$\vdash . *204\cdot52 . \supset \vdash : \mathrm{Hp} . \supset . \Sigma\text{‘}P \in \mathrm{Ser}$ (1)

$\vdash . *174\cdot3 . \supset \vdash : \mathrm{Hp} . \supset . \Sigma\text{‘}P \in \mathrm{Rel}^2\,\mathrm{excl}$ (2)

$\vdash . (1) . (2) . *204\cdot52 . \supset \vdash . \mathrm{Prop}$

***204·55.** $\vdash : P, Q \in \mathrm{Ser} . \supset . Q \times P \in \mathrm{Ser}$

Dem.

$\vdash . *165\cdot27 . *204\cdot22 . \supset \vdash :. \mathrm{Hp} . \supset : \dot{\exists} ! P . \supset . P \downarrow \text{;}Q \in \mathrm{Ser}$ (1)

$\vdash . *165\cdot26 . *204\cdot22 . \supset \vdash : \mathrm{Hp} . \supset . C\text{‘}P \downarrow \text{;}Q \subset \mathrm{Ser}$ (2)

$\vdash . (1) . (2) . *165\cdot21 . *204\cdot52 . \supset \vdash : \mathrm{Hp} . \dot{\exists} ! P . \supset . \Sigma\text{‘}P \downarrow \text{;}Q \in \mathrm{Ser} .$

[*166·1] $\supset . Q \times P \in \mathrm{Ser}$ (3)

$\vdash . *166\cdot13 . *204\cdot24 . \supset \vdash : P = \dot{\Lambda} . \supset . Q \times P \in \mathrm{Ser}$ (4)

$\vdash . (3) . (4) . \supset \vdash . \mathrm{Prop}$

***204·551.** $\vdash :. \dot{\exists} ! P . \dot{\exists} ! Q . \supset : P \times Q \in \mathrm{Ser} . \equiv . P, Q \in \mathrm{Ser}$

Dem.

$\vdash . *165\cdot21\cdot212 . \supset \vdash :. \mathrm{Hp} . \supset : P \downarrow \text{;}Q \in \mathrm{Rel}^2\,\mathrm{excl} . \dot{\Lambda} \sim\in C\text{‘}P \downarrow \text{;}Q :$

[*204·53.*166·1] $\supset : P \times Q \in \mathrm{Ser} . \equiv . P \downarrow \text{;}Q \in \mathrm{Ser} . C\text{‘}P \downarrow \text{;}Q \subset \mathrm{Ser} .$

[*165·27.*204·22] $\equiv . P, Q \in \mathrm{Ser} :. \supset \vdash . \mathrm{Prop}$

***204·56.** $\vdash : C\text{‘}P \subset \mathrm{Rl}\text{‘}J . \supset . \Pi\text{‘}P \Subset J$

Dem.

$\vdash . *172\cdot11 . \supset \vdash : M(\Pi\text{‘}P)N . \supset . (\exists Q) . Q \in C\text{‘}P . (M\text{‘}Q)\, Q\, (N\text{‘}Q)$ (1)

$\vdash . (1) . \supset \vdash :. \mathrm{Hp} . \supset : M(\Pi\text{‘}P)N . \supset . (\exists Q) . M\text{‘}Q \neq N\text{‘}Q .$

[*30·37.Transp] $\supset . M \neq N :. \supset \vdash . \mathrm{Prop}$

∗204·561. $\vdash : P \epsilon \mathrm{Ser} . C\text{‘}P \subset \mathrm{Ser} . \supset . \Pi\text{‘}P \epsilon \mathrm{Rl}\text{‘}J \cap \mathrm{trans}$

Dem.

$\vdash . *200\cdot43 . \supset \vdash :: \mathrm{Hp} . \supset :. L(\Pi\text{‘}P) M . M(\Pi\text{‘}P) N . \supset :$

$(\exists Q, R) . Q, R \epsilon C\text{‘}P . (L\text{‘}Q) Q (M\text{‘}Q) . (M\text{‘}R) R (N\text{‘}R) . L \upharpoonright \overrightarrow{P}\text{‘}Q = M \upharpoonright \overrightarrow{P}\text{‘}Q .$
$M \upharpoonright \overrightarrow{P}\text{‘}R = N \upharpoonright \overrightarrow{P}\text{‘}R :$

[∗204·12]
$\supset : (\exists Q, R) : Q = R . \mathrm{v} . QPR . \mathrm{v} . RPQ : (L\text{‘}Q) Q (M\text{‘}Q) . (M\text{‘}R) R (N\text{‘}R) .$
$L \upharpoonright \overrightarrow{P}\text{‘}Q = M \upharpoonright \overrightarrow{P}\text{‘}Q . M \upharpoonright \overrightarrow{P}\text{‘}R = N \upharpoonright \overrightarrow{P}\text{‘}R$ (1)

$\vdash . *204\cdot1 . \supset \vdash : \mathrm{Hp} . L(\Pi\text{‘}P) M . M(\Pi\text{‘}P) N .$
$Q = R . (L\text{‘}Q) Q (M\text{‘}Q) . (M\text{‘}R) R (N\text{‘}R) . L \upharpoonright \overrightarrow{P}\text{‘}Q = M \upharpoonright \overrightarrow{P}\text{‘}Q .$
$M \upharpoonright \overrightarrow{P}\text{‘}R = N \upharpoonright \overrightarrow{P}\text{‘}R . \supset . (L\text{‘}Q) Q (N\text{‘}Q) . L \upharpoonright \overrightarrow{P}\text{‘}Q = N \upharpoonright \overrightarrow{P}\text{‘}Q$ (2)

$\vdash . *204\cdot33 . \supset$
$\vdash : \mathrm{Hp} . L(\Pi\text{‘}P) M . M(\Pi\text{‘}P) N . QPR . (L\text{‘}Q) Q (M\text{‘}Q) . (M\text{‘}R) R (N\text{‘}R) .$
$L \upharpoonright \overrightarrow{P}\text{‘}Q = M \upharpoonright \overrightarrow{P}\text{‘}Q . M \upharpoonright \overrightarrow{P}\text{‘}R = N \upharpoonright \overrightarrow{P}\text{‘}R . \supset .$
$L \upharpoonright \overrightarrow{P}\text{‘}Q = N \upharpoonright \overrightarrow{P}\text{‘}Q . M\text{‘}Q = N\text{‘}Q .$

[∗13·12] $\supset . L \upharpoonright \overrightarrow{P}\text{‘}Q = N \upharpoonright \overrightarrow{P}\text{‘}Q . (L\text{‘}Q) Q (N\text{‘}Q)$ (3)

$\vdash . *204\cdot33 . \supset$
$\vdash : \mathrm{Hp} . L(\Pi\text{‘}P) M . M(\Pi\text{‘}P) N . RPQ . (L\text{‘}Q) Q (M\text{‘}Q) . (M\text{‘}R) R (N\text{‘}R) .$
$L \upharpoonright \overrightarrow{P}\text{‘}Q = M \upharpoonright \overrightarrow{P}\text{‘}Q . M \upharpoonright \overrightarrow{P}\text{‘}R = N \upharpoonright \overrightarrow{P}\text{‘}R . \supset .$
$L \upharpoonright \overrightarrow{P}\text{‘}R = N \upharpoonright \overrightarrow{P}\text{‘}R . L\text{‘}R = M\text{‘}R .$

[∗13·12] $\supset . L \upharpoonright \overrightarrow{P}\text{‘}R = N \upharpoonright \overrightarrow{P}\text{‘}R . (L\text{‘}R) R (N\text{‘}R)$ (4)

$\vdash . (1) . (2) . (3) . (4) . *200\cdot43 . \supset$
$\vdash :. \mathrm{Hp} . \supset : L(\Pi\text{‘}P) M . M(\Pi\text{‘}P) N . \supset . L(\Pi\text{‘}P) N$ (5)

$\vdash . (5) . *204\cdot56 . \supset \vdash . \mathrm{Prop}$

In order to prove that $\Pi\text{‘}P$ is connected, we require a further hypothesis, namely that P is *well-ordered, i.e.* that every class contained in $C\text{‘}P$ and not null has a first term.

∗204·562. $\vdash :. C\text{‘}P \subset \mathrm{Ser} : \alpha \subset C\text{‘}P . \exists ! \alpha . \supset_\alpha . \exists ! \alpha - \breve{P}\text{‘‘}\alpha : \supset . \Pi\text{‘}P \epsilon \mathrm{connex}$

Dem.

$\vdash . *172\cdot11 . *33\cdot45 . \mathrm{Transp} . \supset$
$\vdash :: \mathrm{Hp} . \supset :. M, N \epsilon C\text{‘}\Pi\text{‘}P . M \neq N . \supset : (\exists Q) . Q \epsilon C\text{‘}P . M\text{‘}Q \neq N\text{‘}Q :$

[Hp] $\supset : (\exists Q) : Q \epsilon C\text{‘}P . M\text{‘}Q \neq N\text{‘}Q : RPQ . \supset_R . M\text{‘}R = N\text{‘}R :$

[∗204·121.∗172·12] $\supset : (\exists Q) : Q \epsilon C\text{‘}P : (M\text{‘}Q) Q (N\text{‘}Q) . \mathrm{v} . (N\text{‘}Q) Q (M\text{‘}Q) :$
$RPQ . \supset_R . M\text{‘}R = N\text{‘}R :$

[∗172·11] $\supset : M(\Pi\text{‘}P) N . \mathrm{v} . N(\Pi\text{‘}P) M$ (1)

$\vdash . (1) . *202\cdot104 . \supset \vdash . \mathrm{Prop}$

∗204·57. $\vdash :. P \epsilon \mathrm{Ser} . C\text{‘}P \subset \mathrm{Ser} : \alpha \subset C\text{‘}P . \exists ! \alpha . \supset_\alpha . \exists ! \alpha - \breve{P}\text{‘‘}\alpha : \supset . \Pi\text{‘}P \epsilon \mathrm{Ser}$
[∗204·561·562]

***204·58.** $\vdash :. P \in \mathrm{Ser} . C\text{‘}P \subset \mathrm{Ser} . C\text{‘}\Sigma\text{‘}P \subset \mathrm{Ser} . P \in \mathrm{Rel}^2\,\mathrm{excl} :$
$\alpha \subset C\text{‘}\Sigma\text{‘}P . \exists ! \alpha . \supset_\alpha . \exists ! \alpha - (\mathrm{Cnv}\text{‘}\Sigma\text{‘}P)\text{“}\alpha : \supset . \Pi\text{‘}\Sigma\text{‘}P, \Pi\text{‘}\Pi\,;P \in \mathrm{Ser}$

Dem.

$\vdash . *204{\cdot}52 . \quad \supset \vdash : \mathrm{Hp} . \supset . \Sigma\text{‘}P \in \mathrm{Ser}$ (1)

$\vdash . (1) . *204{\cdot}57 . \quad \supset \vdash : \mathrm{Hp} . \supset . \Pi\text{‘}\Sigma\text{‘}P \in \mathrm{Ser}$ (2)

$\vdash . *174{\cdot}25 . \quad \supset \vdash : \mathrm{Hp} . \supset . \Pi\text{‘}\Sigma\text{‘}P \;\mathrm{smor}\; \Pi\text{‘}\Pi\,;P$ (3)

$\vdash . (2) . (3) . *204{\cdot}21 . \supset \vdash : \mathrm{Hp} . \supset . \Pi\text{‘}\Pi\,;P \in \mathrm{Ser}$ (4)

$\vdash . (2) . (4) . \supset \vdash . \mathrm{Prop}$

***204·581.** $\vdash : \mathrm{Hp}\, *204{\cdot}58 . \Sigma\text{‘}P \in \mathrm{Rel}^2\,\mathrm{excl} . \supset . \mathrm{Prod}\text{‘}\mathrm{Prod}\,;P, \mathrm{Prod}\text{‘}\Sigma\text{‘}P \in \mathrm{Ser}$
$[*174{\cdot}461{\cdot}43 . *204{\cdot}58{\cdot}21]$

***204·59.** $\vdash :. P, Q \in \mathrm{Ser} : \alpha \subset C\text{‘}Q . \exists ! \alpha . \supset_\alpha . \exists ! \alpha - \breve{Q}\text{“}\alpha : \supset .$
$P^Q \in \mathrm{Ser} . (P \exp Q) \in \mathrm{Ser}$

Dem.

$\vdash . *165{\cdot}27{\cdot}241 . *204{\cdot}22{\cdot}24 . \supset \vdash : \mathrm{Hp} . \supset . P\underset{;}{\downarrow}\,;Q \in \mathrm{Ser}$ (1)

$\vdash . *165{\cdot}26 . *204{\cdot}22 . \quad \supset \vdash : \mathrm{Hp} . \supset . C\text{‘}P\underset{;}{\downarrow}\,;Q \subset \mathrm{Ser}$ (2)

$\vdash . *150{\cdot}22 . *71{\cdot}47 . \supset \vdash : \beta \subset C\text{‘}P\underset{;}{\downarrow}\,;Q . \exists ! \beta . \supset . (\exists \alpha) . \alpha \subset C\text{‘}Q . \exists ! \alpha . \beta = P\underset{;}{\downarrow}\text{“}\alpha :$

$[\mathrm{Hp}] \quad \supset \vdash : \mathrm{Hp} . \beta \subset C\text{‘}P\underset{;}{\downarrow}\,;Q . \exists ! \beta . \supset . (\exists \alpha) . \exists ! \alpha - \breve{Q}\text{“}\alpha . \beta = P\underset{;}{\downarrow}\text{“}\alpha$ (3)

$\vdash . *37{\cdot}45 . \supset \vdash : \exists ! \alpha - \breve{Q}\text{“}\alpha . \equiv . \exists ! P\underset{;}{\downarrow}\text{“}(\alpha - \breve{Q}\text{“}\alpha)$ (4)

$\vdash . (4) . *71{\cdot}381 . *165{\cdot}22 . \supset \vdash : \dot{\exists} ! P . \exists ! \alpha - \breve{Q}\text{“}\alpha . \supset . \exists ! P\underset{;}{\downarrow}\text{“}\alpha - P\underset{;}{\downarrow}\text{“}\breve{Q}\text{“}\alpha$ (5)

$\vdash . *72{\cdot}503 . *165{\cdot}22 . \quad \supset \vdash : \dot{\exists} ! P . \supset . \alpha = (\mathrm{Cnv}\text{‘}P\underset{;}{\downarrow})\text{“}P\underset{;}{\downarrow}\text{“}\alpha$ (6)

$\vdash . (5) . (6) . \supset \vdash : \dot{\exists} ! P . \exists ! \alpha - \breve{Q}\text{“}\alpha . \supset . \exists ! P\underset{;}{\downarrow}\text{“}\alpha - P\underset{;}{\downarrow}\text{“}\breve{Q}\text{“}(\mathrm{Cnv}\text{‘}P\underset{;}{\downarrow})\text{“}P\underset{;}{\downarrow}\text{“}\alpha .$

$[*165{\cdot}18] \quad \supset . \exists ! P\underset{;}{\downarrow}\text{“}\alpha - (\mathrm{Cnv}\text{‘}P\underset{;}{\downarrow}\,;Q)\text{“}P\underset{;}{\downarrow}\text{“}\alpha$ (7)

$\vdash . (3) . (7) . \supset \vdash :. \mathrm{Hp} . \dot{\exists} ! P . \supset :$
$\beta \subset C\text{‘}P\underset{;}{\downarrow}\,;Q . \exists ! \beta . \supset . \exists ! \beta - (\mathrm{Cnv}\text{‘}P\underset{;}{\downarrow}\,;Q)\text{“}\beta$ (8)

$\vdash . (1) . (2) . (8) . *204{\cdot}57 . \quad \supset \vdash : \mathrm{Hp} . \dot{\exists} ! P . \supset . \Pi\text{‘}P\underset{;}{\downarrow}\,;Q \in \mathrm{Ser}$ (9)

$\vdash . (9) . *176{\cdot}182 . *204{\cdot}21 . \supset \vdash : \mathrm{Hp} . \dot{\exists} ! P . \supset . (P \exp Q) \in \mathrm{Ser}$ (10)

$\vdash . *176{\cdot}151 . *204{\cdot}24 . \quad \supset \vdash : P = \dot{\Lambda} . \supset . (P \exp Q) \in \mathrm{Ser}$ (11)

$\vdash . (10) . (11) . \quad \supset \vdash : \mathrm{Hp} . \supset . (P \exp Q) \in \mathrm{Ser}$ (12)

$\vdash . (12) . *176{\cdot}181 . *204{\cdot}21 . \supset \vdash : \mathrm{Hp} . \supset . P^Q \in \mathrm{Ser}$ (13)

$\vdash . (12) . (13) . \supset \vdash . \mathrm{Prop}$

The two following propositions are lemmas for *204·62.

***204·6.** $\vdash : P \in \mathrm{trans} . \supset . \alpha \cup \breve{P}\text{“}\alpha \subset p\text{‘}\overleftarrow{P}\text{“}p\text{‘}\overrightarrow{P}\text{“}\alpha$

Dem.

$\vdash . *40{\cdot}53 . \supset \vdash :: x \in p\text{‘}\overleftarrow{P}\text{“}p\text{‘}\overrightarrow{P}\text{“}\alpha . \equiv :. y \in p\text{‘}\overrightarrow{P}\text{“}\alpha . \supset_y . yPx :.$

$[*40{\cdot}51] \quad \equiv :. z \in \alpha . \supset_z . yPz : \supset_y . yPx$ (1)

$\vdash . *10{\cdot}26 . \supset \vdash :. x \epsilon \alpha : z \epsilon \alpha . \supset_z . yPz : \supset . yPx :$

$[\text{Exp.}(1)] \quad \supset \vdash : x \epsilon \alpha . \supset . x \epsilon p\text{‘}\overleftarrow{P}\text{“}p\text{‘}\overrightarrow{P}\text{“}\alpha$ (2)

$\vdash . *10{\cdot}1 . \quad \supset \vdash :. u \epsilon \alpha . uPx : z \epsilon \alpha . \supset_z . yPz : \supset . yPu . uPx$ (3)

$\vdash . (3) . *201{\cdot}1 . \supset \vdash :: \text{Hp} . \supset :. u \epsilon \alpha . uPx : z \epsilon \alpha . \supset_z . yPz : \supset . yPx :.$

$[*37{\cdot}105] \quad \supset :. x \epsilon \breve{P}\text{“}\alpha : z \epsilon \alpha . \supset_z . yPz : \supset . yPx :.$

$[\text{Exp.}(1)] \quad \supset :. x \epsilon \breve{P}\text{“}\alpha . \supset . x \epsilon p\text{‘}\overleftarrow{P}\text{“}p\text{‘}\overrightarrow{P}\text{“}\alpha$ (4)

$\vdash . (2) . (4) . \supset \vdash . \text{Prop}$

***204·61.** $\vdash : P \epsilon \text{Rl}\text{‘}J \cap \text{connex} . \supset . C\text{‘}P \cap p\text{‘}\overleftarrow{P}\text{“}p\text{‘}\overrightarrow{P}\text{“}(\alpha \cap C\text{‘}P) \subset \alpha \cup \breve{P}\text{“}\alpha$

Dem.

$\vdash . *200{\cdot}5 . \supset \vdash : \text{Hp} . \supset . p\text{‘}\overrightarrow{P}\text{“}(\alpha \cap C\text{‘}P) \cap p\text{‘}\overleftarrow{P}\text{“}p\text{‘}\overrightarrow{P}\text{“}(\alpha \cap C\text{‘}P) = \Lambda .$

$[*24{\cdot}311] \quad \supset . p\text{‘}\overleftarrow{P}\text{“}p\text{‘}\overrightarrow{P}\text{“}(\alpha \cap C\text{‘}P) \subset - p\text{‘}\overrightarrow{P}\text{“}(\alpha \cap C\text{‘}P) .$

$[*22{\cdot}48] \quad \supset . C\text{‘}P \cap p\text{‘}\overleftarrow{P}\text{“}p\text{‘}\overrightarrow{P}\text{“}(\alpha \cap C\text{‘}P) \subset C\text{‘}P - p\text{‘}\overrightarrow{P}\text{“}(\alpha \cap C\text{‘}P)$

$[*24{\cdot}43 . *202{\cdot}505] \quad \subset \alpha \cup \breve{P}\text{“}\alpha : \supset \vdash . \text{Prop}$

***204·62.** $\vdash : P \epsilon \text{Ser} . \supset . C\text{‘}P \cap p\text{‘}\overleftarrow{P}\text{“}p\text{‘}\overrightarrow{P}\text{“}(\alpha \cap C\text{‘}P) = (\alpha \cap C\text{‘}P) \cup \breve{P}\text{“}\alpha$

Dem.

$\vdash . *204{\cdot}6 . *37{\cdot}265 . \supset \vdash : \text{Hp} . \supset . (\alpha \cap C\text{‘}P) \cup \breve{P}\text{“}\alpha \subset p\text{‘}\overleftarrow{P}\text{“}p\text{‘}\overrightarrow{P}\text{“}(\alpha \cap C\text{‘}P)$ (1)

$\vdash . *37{\cdot}16 . *22{\cdot}43 . \quad \supset \vdash . (\alpha \cap C\text{‘}P) \cup \breve{P}\text{“}\alpha \subset C\text{‘}P$ (2)

$\vdash . *204{\cdot}61 . *22{\cdot}43 . \supset \vdash : \text{Hp} . \supset . C\text{‘}P \cap p\text{‘}\overleftarrow{P}\text{“}p\text{‘}\overrightarrow{P}\text{“}(\alpha \cap C\text{‘}P) \subset (\alpha \cup \breve{P}\text{“}\alpha) \cap C\text{‘}P .$

$[*37{\cdot}16] \quad \supset . C\text{‘}P \cap p\text{‘}\overleftarrow{P}\text{“}p\text{‘}\overrightarrow{P}\text{“}(\alpha \cap C\text{‘}P) \subset (\alpha \cap C\text{‘}P) \cup \breve{P}\text{“}\alpha$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash . \text{Prop}$

***204·63.** $\vdash : P \epsilon \text{Ser} . \exists ! p\text{‘}\overrightarrow{P}\text{“}\alpha . \supset . p\text{‘}\overleftarrow{P}\text{“}p\text{‘}\overrightarrow{P}\text{“}\alpha = \alpha \cup \breve{P}\text{“}\alpha$

Dem.

$\vdash . *40{\cdot}65 . \text{Transp} . \supset \vdash : \text{Hp} . \supset . \alpha \subset C\text{‘}P$ (1)

$\vdash . *40{\cdot}62 . \quad \supset \vdash : \text{Hp} . \supset . p\text{‘}\overleftarrow{P}\text{“}p\text{‘}\overrightarrow{P}\text{“}\alpha \subset C\text{‘}P$ (2)

$\vdash . (1) . (2) . *204{\cdot}62 . \supset \vdash . \text{Prop}$

***204·64.** $\vdash : P \epsilon \text{Ser} . x \epsilon \text{D}\text{‘}P . \supset . p\text{‘}\overrightarrow{P}\text{“}\overleftarrow{P}\text{‘}x = \overrightarrow{P_*}\text{‘}x$

Dem.

$\vdash . *40{\cdot}62 . \quad \supset \vdash : \text{Hp} . \supset . p\text{‘}\overrightarrow{P}\text{“}\overleftarrow{P}\text{‘}x \subset C\text{‘}P$ (1)

$\vdash . *40{\cdot}51 . \quad \supset \vdash :. z \epsilon p\text{‘}\overrightarrow{P}\text{“}\overleftarrow{P}\text{‘}x . \equiv : xPy . \supset_y . zPy$ (2)

$\vdash . (2) . *50{\cdot}11 . \supset \vdash :: \text{Hp} . \supset :. z \epsilon p\text{‘}\overrightarrow{P}\text{“}\overleftarrow{P}\text{‘}x . \supset : xPy . \supset_y . z \neq y :$

$[(1) . *202{\cdot}103] \quad \supset : zPx . \vee . z = x$ (3)

$\vdash . *201{\cdot}521 . \quad \supset \vdash :: \text{Hp} . \supset :. z \epsilon \overrightarrow{P_*}\text{‘}x . \equiv : zPx . \vee . z = x :$ (4)

$[*201{\cdot}1 . *13{\cdot}12] \quad \supset : xPy . \supset . zPy :$

$[(2)] \quad \supset : z \epsilon p\text{‘}\overrightarrow{P}\text{“}\overleftarrow{P}\text{‘}x$ (5)

$\vdash . (3) . (4) . (5) . \supset \vdash . \text{Prop}$

The following proposition is used in $*234\cdot101$.

$*204\cdot65$. $\vdash : P \epsilon \mathrm{Ser} \,.\, x \epsilon C'P \,.\, \supset \,.\, p'\overrightarrow{P}``\overleftarrow{P}'x \cap C'P = \overrightarrow{P_*}'x$

Dem.

$\vdash \,.\, *40\cdot2 \,.\, \supset \vdash : \mathrm{Hp} \,.\, x \sim\epsilon \mathrm{D}'P \,.\, \supset \,.\, p'\overrightarrow{P}``\overleftarrow{P}'x \cap C'P = C'P$

$[*204\cdot11] \qquad = \overrightarrow{P}'x \cup \iota'x$

$[*201\cdot521] \qquad = \overrightarrow{P_*}'x \qquad (1)$

$\vdash \,.\, *40\cdot62 \,.\, *204\cdot64 \,.\, \supset \vdash : \mathrm{Hp} \,.\, x \epsilon \mathrm{D}'P \,.\, \supset \,.\, p'\overrightarrow{P}``\overleftarrow{P}'x \cap C'P = \overrightarrow{P_*}'x \qquad (2)$

$\vdash \,.\, (1) \,.\, (2) \,.\, \supset \vdash \,.\, \mathrm{Prop}$

$*204\cdot7$. $\vdash : P \epsilon \mathrm{Ser} \,.\, \supset \,.\, P_1 \epsilon 1 \rightarrow 1 \qquad [*201\cdot63 \,.\, *202\cdot7]$

On this proposition, compare the remarks preceding $*201\cdot6$.

$*204\cdot71$. $\vdash : P \epsilon \mathrm{Ser} \,.\, xP_1y \,.\, \supset \,.\, \overrightarrow{P}'y = \overrightarrow{P}'x \cup \iota'x \qquad [*202\cdot72 \,.\, *201\cdot63]$

$*204\cdot72$. $\vdash :: P \epsilon \mathrm{Ser} \,.\, \supset :. \, xP_1y \,.\, \equiv : xPy : xPz \,.\, z \neq y \,.\, \supset_z \,.\, yPz$

Dem.

$\vdash \,.\, *201\cdot63 \,.\, \qquad \supset \vdash :. \, \mathrm{Hp} \,.\, \supset : xP_1y \,.\, \supset \,.\, xPy \qquad (1)$

$\vdash \,.\, *204\cdot71 \,.\, *121\cdot26 \,.\, \supset \vdash :. \, \mathrm{Hp} \,.\, xP_1y \,.\, \supset : \overleftarrow{P}'x = \overleftarrow{P}'y \cup \iota'y :$

$[*24\cdot43 \,.\, *32\cdot181] \qquad \supset : xPz \,.\, z \neq y \,.\, \supset_z \,.\, yPz \qquad (2)$

$\vdash \,.\, *24\cdot43 \,.\, *32\cdot181 \,.\, \supset \vdash :. \, xPz \,.\, z \neq y \,.\, \supset_z \,.\, yPz : \supset \,.\, \overleftarrow{P}'x \subset \iota'y \cup \overleftarrow{P}'y \qquad (3)$

$\vdash \,.\, (3) \,.\, *200\cdot361 \,.\, \quad \supset \vdash :. \, \mathrm{Hp} : xPz \,.\, z \neq y \,.\, \supset_z \,.\, yPz : \supset \,.\, \overleftarrow{P}'x \cap \overrightarrow{P}'y = \Lambda \,.$

$[*34\cdot11] \qquad \supset \,.\, \sim(xP^2y) \qquad (4)$

$\vdash \,.\, (4) \,.\, \mathrm{Fact} \,.\, \supset \vdash :. \, \mathrm{Hp} : xPy : xPz \,.\, z \neq y \,.\, \supset_z \,.\, yPz : \supset \,.\, x(P \dot{-} P^2)y \,.$

$[*201\cdot63] \qquad \supset \,.\, xP_1y \qquad (5)$

$\vdash \,.\, (1) \,.\, (2) \,.\, (5) \,.\, \supset \vdash \,.\, \mathrm{Prop}$

The above proposition is used in $*274\cdot23$.

$*205$. MAXIMUM AND MINIMUM POINTS

Summary of $*205$.

The minimum points of a class α with respect to a relation P are those members of α which belong to the field of P but to which no members of α have the relation P; that is, they are those members of α which belong to $C`P$ but have no predecessors in $\alpha$. Similarly the maximum points of $\alpha$ are those members of $\alpha$ which belong to $C`P$ but have no successors in α. Both these notions have been already defined in $*93$, but they were there only used for the special purpose of studying generations. Their chief utility is in connection with *series*, and it is in this connection that we shall now consider them. Many of the properties of maxima and minima in series do not demand the whole hypothesis "$P \epsilon \mathrm{Ser}$," but only "$P \epsilon \mathrm{connex}$." This is the case, in particular, with the fundamental property of maxima and minima in series, namely that each class has at most one maximum and one minimum. The minimum of a class, if it exists, is the first term of the class, and the maximum, if it exists, is the last term. The maxima with respect to P are the minima with respect to $\breve{P}$; hence properties of maxima result immediately from the corresponding properties of minima, and will be set down without proof in what follows.

It will be seen that the maxima and minima of α depend only upon $\alpha \cap C`P$: the part of $\alpha$ (if any) which is not contained in $C`P$ is irrelevant.

In accordance with the definitions of $*93$, the class of minima of α is denoted by $\overrightarrow{\min_P}`\alpha$, where

$$\overrightarrow{\min_P}`\alpha = (\alpha \cap C`P) - \breve{P}``\alpha,$$

the definition being

$$\min_P = \hat{x}\hat{\alpha}\,\{x \,\epsilon\, (\alpha \cap C`P) - \breve{P}``\alpha\}.$$

Thus $\min_P$ is a relation contained in ϵ. When P is connected, we have $\overrightarrow{\min_P}`\alpha \,\epsilon\, 0 \cup 1$, *i.e.* (by $*71\cdot12$)

$$\min_P \epsilon\, 1 \rightarrow \mathrm{Cls}.$$

It follows that, if κ is a set of classes which all have minima, $\min_P \restriction \kappa$ is a selective relation for κ, *i.e.*

$$\min_P \restriction \kappa \,\epsilon\, \epsilon_\Delta`\kappa.$$

Owing to this fact, the existence of selections can sometimes be proved in dealing with series (especially with well-ordered series), in cases where such proof would be impossible if no serial arrangement were given.

The definition of $\min_P$ is so chosen as to exclude from $\overrightarrow{\min_P}`\alpha$ whatever part of $\alpha$ is not contained in $C`P$, and to make $\overrightarrow{\min_P}`\iota`x = \iota`x$, *i.e.* $\min_P`\iota`x = x$,

provided $x \in C'P . \sim(xPx)$. For these two reasons we have to reject two simpler definitions which might otherwise be thought preferable. One of these would give

$$\overrightarrow{\min_P}'\alpha = \alpha - \breve{P}''\alpha,$$

which might be obtained by putting

$$\min_P = \epsilon \dot{-} \epsilon \,|\, \breve{P} \quad \text{Df.}$$

This agrees with our definition whenever $\alpha \subset C'P$, but not otherwise, since it includes in $\overrightarrow{\min_P}'\alpha$ any part of α not contained in $C'P$. Hence it necessitates the hypothesis $\alpha \subset C'P$ in many propositions which, with our definition, do not require this hypothesis, and in particular in the proposition

$$P \in \text{connex} . \supset . \overrightarrow{\min_P}'\alpha \in 0 \cup 1,$$

so that instead of having (as with our definition)

$$P \in \text{connex} . \supset . \min_P \in 1 \to \text{Cls}$$

we should only have

$$P \in \text{connex} . \supset . \min_P \upharpoonright \text{Cl}'C'P \in 1 \to \text{Cls}.$$

For these reasons, this definition is less convenient than the one we have adopted.

The other definition which suggests itself is one which will give

$$\overrightarrow{\min_P}'\alpha = \overrightarrow{B}'P \upharpoonright \alpha.$$

If this definition were adopted, we might dispense with a special notation altogether, using $\overrightarrow{B}'P \upharpoonright \alpha$, $B'P \upharpoonright \alpha$ in place of $\overrightarrow{\min_P}'\alpha$, $\min_P'\alpha$. This definition, however, has the drawback that, if $\alpha \in 1$ and $P \in J$,

$$P \upharpoonright \alpha = \dot{\Lambda},$$

so that we have

$$\overrightarrow{\min_P}'\alpha = \dot{\Lambda} \text{ when } \alpha \in 1 . \alpha \subset C'P.$$

This necessitates the addition of the hypothesis $\alpha \sim \in 1$ (as in $*204{\cdot}45$ above, for example) in cases where, with our definition, no such hypothesis is required. If we take $\overrightarrow{B}'\alpha \upharpoonleft P$, instead of $\overrightarrow{B}'P \upharpoonright \alpha$, as the class of minimum points, we secure $\min_P'\iota'x = x$ when $P \in J$ and $x \in \text{D}'P$, but not when $x \in \overrightarrow{B}'\breve{P}$ Thus we still have exceptions to provide against which do not arise with the definition we have adopted.

The first few propositions of this number have already been proved in $*93$, but are repeated here for convenience of reference.

The propositions of this number are numerous and much used. Among the elementary properties of $\max_P$ and $\min_P$ with which the number begins, the following should be noted:

$*205{\cdot}12$. $\vdash . \overrightarrow{B}'P = \overrightarrow{\min_P}'\text{D}'P = \overrightarrow{\min_P}'C'P$

$*205\cdot123.\ \vdash : \overrightarrow{\max}_P{}^\backprime\alpha = \Lambda . \equiv . \alpha \cap C{}^\backprime P \subset P{}^{\backprime\backprime}\alpha$

$*205\cdot14.\ \ \vdash . \overrightarrow{\min}_P{}^\backprime\alpha = \hat{x}\{x \epsilon \alpha \cap C{}^\backprime P . \alpha \cap \overrightarrow{P}{}^\backprime x = \Lambda\}$

$*205\cdot15.\ \ \vdash . \overrightarrow{\min}_P{}^\backprime(\alpha \cap C{}^\backprime P) = \overrightarrow{\min}_P{}^\backprime\alpha$

$*205\cdot16.\ \ \vdash . \overrightarrow{\min}_P{}^\backprime\Lambda = \Lambda$

$*205\cdot18.\ \ \vdash : \sim(xPx) . x \epsilon C{}^\backprime P . \supset . \min_P{}^\backprime\iota{}^\backprime x = \max_P{}^\backprime\iota{}^\backprime x = x$

$*205\cdot19.\ \ \vdash : P \epsilon \text{trans} . \supset . \overrightarrow{\min}_P{}^\backprime\alpha = \overrightarrow{\min}_P{}^\backprime(\alpha \cup \breve{P}{}^{\backprime\backprime}\alpha) = \overrightarrow{\min}_P{}^\backprime\breve{P}_*{}^{\backprime\backprime}\alpha$

$*205\cdot194.\ \vdash : x \min_P \alpha . \supset . \sim(xPx)$

Owing to this proposition, we can sometimes dispense with the hypothesis $P \in J$ in propositions about minima which would otherwise require this hypothesis.

$*205\cdot197.\ \vdash :. P \epsilon \text{Rl}{}^\backprime J \cap \text{trans} . \supset : x \epsilon C{}^\backprime P . \equiv . x = \max_P{}^\backprime(\overrightarrow{P}{}^\backprime x \cup \iota{}^\backprime x)$

Our next set of propositions ($*205\cdot2$—$\cdot27$) introduces the hypothesis that P is connected, or transitive and connected. The chief of them are

$*205\cdot21.\ \ \vdash : P \epsilon \text{connex} . \text{E}\,! \min_P{}^\backprime\alpha . y \epsilon \alpha \cap C{}^\backprime P - \iota{}^\backprime\min_P{}^\backprime\alpha . \supset . \min_P{}^\backprime\alpha\, Py$

I.e. if the minimum of α exists, it precedes every other member of $\alpha \cap C{}^\backprime P$.

$*205\cdot22.\ \ \vdash : P \epsilon \text{trans} \cap \text{connex} . \text{E}\,! \min_P{}^\backprime\alpha . \supset . \breve{P}{}^{\backprime\backprime}\alpha = \overleftarrow{P}{}^\backprime\min_P{}^\backprime\alpha$

I.e. the terms which come after some part of α are those that come after its minimum (when the minimum exists).

$*205\cdot25.\ \ \vdash . \overrightarrow{\min}_P{}^\backprime\overleftarrow{P}{}^\backprime x = (\overleftarrow{P \dot{-} P^2}){}^\backprime x$

We have next the fundamental proposition:

$*205\cdot3.\ \ \ \vdash : P \epsilon \text{connex} . \supset . \overrightarrow{\min}_P{}^\backprime\alpha \epsilon 0 \cup 1 . \overrightarrow{\max}_P{}^\backprime\alpha \epsilon 0 \cup 1$

whence

$*205\cdot31.\ \ \vdash : P \epsilon \text{connex} . \supset . \min_P, \max_P \epsilon 1 \rightarrow \text{Cls}$

which leads to

$*205\cdot33.\ \ \vdash : P \epsilon \text{connex} . \kappa \subset \text{Cl}{}^\backprime\min_P . \supset . \min_P \upharpoonright \kappa \epsilon \epsilon_\Delta{}^\backprime\kappa$

This proposition is useful in the theory of well-ordered series. Observe that "$\kappa \subset \text{Cl}{}^\backprime\min_P$" means that κ consists of classes which have minima.

We have next a set of propositions ($*205\cdot4$—$\cdot44$) dealing with the relations of $\min_P{}^\backprime\alpha$ to $B{}^\backprime P \upharpoonright \alpha$ and $B{}^\backprime\alpha \rceil P$; next we have propositions on the relations of the minima of two different classes, of which the most useful is

$*205\cdot55.\ \ \vdash : P \epsilon \text{connex} . B{}^\backprime P \epsilon \alpha . \supset . B{}^\backprime P = \min_P{}^\backprime\alpha$

We have next various propositions on $p{}^\backprime\overrightarrow{P}{}^{\backprime\backprime}(\alpha \cap C{}^\backprime P)$, of which the chief is

$*205\cdot65.\ \ \vdash : P \epsilon \text{trans} \cap \text{connex} . \text{E}\,! \min_P{}^\backprime\alpha . \supset . p{}^\backprime\overrightarrow{P}{}^{\backprime\backprime}(\alpha \cap C{}^\backprime P) = \overrightarrow{P}{}^\backprime\min_P{}^\backprime\alpha$

I.e. the predecessors of the whole of a class contained in $C{}^\backprime P$ are the predecessors of its minimum (if it has one).

A useful proposition is

$*205{\cdot}68.\ \ \vdash : \breve{P}\text{“}\alpha \subset \alpha\ .\ \supset .\ \overrightarrow{\min}_P\text{‘}\alpha = \overrightarrow{\min}\,(P_{\text{po}})\text{‘}\alpha$

I.e. if α is a hereditary class, its minima with respect to P are the same as its minima with respect to P_{po}.

We prove next that if $P\text{“}\alpha$ has a maximum, so has α ($*205{\cdot}7$), and that if $P \epsilon \text{connex}$, only a unit class can have its maximum identical with its minimum ($*205{\cdot}73$).

$*205{\cdot}8$—$\cdot85$ are concerned with relation-arithmetic. The chief proposition here is

$*205{\cdot}8.\ \ \vdash : S \epsilon P \,\overline{\text{smor}}\, Q\ .\ \supset .\ \overrightarrow{\min}_P\text{‘}\alpha = S\text{“}\overrightarrow{\min}_Q\text{‘}\breve{S}\text{“}\alpha$

I.e. in any correlation, the minima of the correlates of a class are the correlates of the minima.

We end with two propositions on relations with limited fields. The more useful of these is

$*205{\cdot}9.\ \ \vdash : P \epsilon \text{connex}\ .\ \kappa \subset C\text{‘}P\ .\ \kappa \sim\epsilon 1\ .\ \supset .\ \overrightarrow{\min}\,(P \upharpoonright \kappa)\text{‘}\alpha = \overrightarrow{\min}_P\text{‘}(\alpha \cap \kappa)$

$*205{\cdot}1.\ \ \vdash : x \min_P \alpha\ .\ \equiv .\ x \epsilon \alpha \cap C\text{‘}P - \breve{P}\text{“}\alpha$ $\quad[*93{\cdot}11]$

$*205{\cdot}101.\ \ \vdash : x \max_P \alpha\ .\ \equiv .\ x \epsilon \alpha \cap C\text{‘}P - P\text{“}\alpha\ .\ \equiv .\ x \min\,(\breve{P})\,\alpha$ $\quad[*93{\cdot}115]$

$*205{\cdot}102.\ \ \vdash .\ \max_P = \min\,(\breve{P})$ $\quad[*93{\cdot}114]$

$*205{\cdot}11.\ \ \vdash .\ \overrightarrow{\min}_P\text{‘}\alpha = \alpha \cap C\text{‘}P - \breve{P}\text{“}\alpha$ $\quad[*93{\cdot}111]$

$*205{\cdot}111.\ \ \vdash .\ \overrightarrow{\max}_P\text{‘}\alpha = \alpha \cap C\text{‘}P - P\text{“}\alpha$ $\quad[*93{\cdot}116]$

$*205{\cdot}12.\ \ \vdash .\ \overrightarrow{B}\text{‘}P = \overrightarrow{\min}_P\text{‘}\text{D}\text{‘}P = \overrightarrow{\min}_P\text{‘}C\text{‘}P$ $\quad[*93{\cdot}112]$

$*205{\cdot}121.\ \ \vdash .\ \overrightarrow{B}\text{‘}\breve{P} = \overrightarrow{\max}_P\text{‘}\text{Ɑ}\text{‘}P = \overrightarrow{\max}_P\text{‘}C\text{‘}P$ $\quad[*93{\cdot}117]$

$*205{\cdot}122.\ \ \vdash : \overrightarrow{\min}_P\text{‘}\alpha = \Lambda\ .\ \equiv .\ \alpha \cap C\text{‘}P \subset \breve{P}\text{“}\alpha$ $\quad[*205{\cdot}11\ .\ *24{\cdot}3]$

$*205{\cdot}123.\ \ \vdash : \overrightarrow{\max}_P\text{‘}\alpha = \Lambda\ .\ \equiv .\ \alpha \cap C\text{‘}P \subset P\text{“}\alpha$

$*205{\cdot}13.\ \ \vdash .\ \overrightarrow{\min}_P\text{‘}\alpha \cup \breve{P}\text{“}\alpha = (\alpha \cap C\text{‘}P) \cup \breve{P}\text{“}\alpha$ $\quad[*22{\cdot}91\ .\ *205{\cdot}11]$

$*205{\cdot}131.\ \ \vdash .\ \overrightarrow{\max}_P\text{‘}\alpha \cup P\text{“}\alpha = (\alpha \cap C\text{‘}P) \cup P\text{“}\alpha$

$*205{\cdot}14.\ \ \vdash .\ \overrightarrow{\min}_P\text{‘}\alpha = \hat{x}\{x \epsilon \alpha \cap C\text{‘}P\ .\ \alpha \cap \overrightarrow{P}\text{‘}x = \Lambda\}$ $\quad[*37{\cdot}462\ .\ *205{\cdot}11]$

$*205{\cdot}141.\ \ \vdash .\ \overrightarrow{\max}_P\text{‘}\alpha = \hat{x}\{x \epsilon \alpha \cap C\text{‘}P\ .\ \alpha \cap \overleftarrow{P}\text{‘}x = \Lambda\}$

$*205{\cdot}15.\ \ \vdash .\ \overrightarrow{\min}_P\text{‘}(\alpha \cap C\text{‘}P) = \overrightarrow{\min}_P\text{‘}\alpha$ $\quad[*37{\cdot}265\ .\ *205{\cdot}11]$

$*205{\cdot}151.\ \ \vdash .\ \overrightarrow{\max}_P\text{‘}(\alpha \cap C\text{‘}P) = \overrightarrow{\max}_P\text{‘}\alpha$

$*205{\cdot}16.\ \ \vdash .\ \overrightarrow{\min}_P\text{‘}\Lambda = \Lambda$ $\quad[*205{\cdot}11\ .\ *24{\cdot}23]$

✳205·161. $\vdash . \overrightarrow{\max}_P\text{‘}\Lambda = \Lambda$

✳205·17. $\vdash :. x \in (\alpha \cap C\text{‘}P) . \supset_x . \sim(xPx) : \alpha \cap C\text{‘}P \in 1 : \supset .$

$$\overrightarrow{\min}_P\text{‘}\alpha = \overrightarrow{\max}_P\text{‘}\alpha = \alpha \cap C\text{‘}P$$

Dem.

$\vdash . ✳13\cdot14 . \quad \supset \vdash :. \mathrm{Hp} . \supset : x \in \alpha . xPy . \supset_{x,y} . x \neq y \qquad (1)$

$\vdash . ✳52\cdot16 . \quad \supset \vdash :. \mathrm{Hp} . \supset : x, y \in \alpha \cap C\text{‘}P . \supset_{x,y} . x = y \qquad (2)$

$\vdash . (1) . (2) . ✳33\cdot17 . \supset \vdash :. \mathrm{Hp} . \supset : x \in \alpha . xPy . \supset_{x,y} . y \sim\in \alpha :$

$[✳37\cdot1] \qquad \supset : \breve{P}\text{‘‘}\alpha \subset -\alpha :$

$[✳22\cdot811] \qquad \supset : \alpha \subset -\breve{P}\text{‘‘}\alpha \qquad (3)$

$\vdash . (3) . ✳205\cdot11 . \supset \vdash . \mathrm{Prop}$

✳205·18. $\vdash : \sim(xPx) . x \in C\text{‘}P . \supset . \min_P\text{‘}\iota\text{‘}x = \max_P\text{‘}\iota\text{‘}x = x$

Dem.

$\vdash . ✳205\cdot17 . \supset \vdash : \mathrm{Hp} . \supset . \overrightarrow{\min}_P\text{‘}\iota\text{‘}x = \overrightarrow{\max}_P\text{‘}\iota\text{‘}x = \iota\text{‘}x \qquad (1)$

$\vdash . (1) . ✳53\cdot4 . \supset \vdash . \mathrm{Prop}$

✳205·181. $\vdash : xPy . \sim(xPx) . \sim(yPx) . \supset . \min_P\text{‘}(\iota\text{‘}x \cup \iota\text{‘}y) = x$

Dem.

$\vdash . ✳37\cdot105 . \supset \vdash : \mathrm{Hp} . \supset . x \sim\in \breve{P}\text{‘‘}(\iota\text{‘}x \cup \iota\text{‘}y) . y \in \breve{P}\text{‘‘}(\iota\text{‘}x \cup \iota\text{‘}y) \qquad (1)$

$\vdash . ✳33\cdot17 . \supset \vdash : \mathrm{Hp} . \supset . \iota\text{‘}x \cup \iota\text{‘}y \subset C\text{‘}P \qquad (2)$

$\vdash . (1) . (2) . ✳205\cdot11 . \supset \vdash : \mathrm{Hp} . \supset . \overrightarrow{\min}_P\text{‘}(\iota\text{‘}x \cup \iota\text{‘}y) = \iota\text{‘}x : \supset \vdash . \mathrm{Prop}$

✳205·182. $\vdash : P^2 \mathrel{\subset\!\!\!\!\cdot} J . xPy . \supset . \min_P\text{‘}(\iota\text{‘}x \cup \iota\text{‘}y) = x$

Dem.

$\vdash . ✳200\cdot36 . ✳50\cdot43 . \supset \vdash : \mathrm{Hp} . \supset . \sim(xPx) . \sim(yPx) \qquad (1)$

$\vdash . (1) . ✳205\cdot181 . \supset \vdash . \mathrm{Prop}$

✳205·183. $\vdash :. P^2 \mathrel{\subset\!\!\!\!\cdot} J . P \in \mathrm{connex} . x, y \in C\text{‘}P . \supset :$

$$\min_P\text{‘}(\iota\text{‘}x \cup \iota\text{‘}y) = x . \vee . \min_P\text{‘}(\iota\text{‘}x \cup \iota\text{‘}y) = y$$

Dem.

$\vdash . ✳202\cdot103 . \supset \vdash :. \mathrm{Hp} . \supset : x = y . \vee . xPy . \vee . yPx \qquad (1)$

$\vdash . ✳205\cdot18 . \quad \supset \vdash : \mathrm{Hp} . x = y . \supset . \min_P\text{‘}(\iota\text{‘}x \cup \iota\text{‘}y) = x \qquad (2)$

$\vdash . ✳205\cdot182 . \supset \vdash : \mathrm{Hp} . xPy . \supset . \min_P\text{‘}(\iota\text{‘}x \cup \iota\text{‘}y) = x \qquad (3)$

$\vdash . ✳205\cdot182 . \supset \vdash : \mathrm{Hp} . yPx . \supset . \min_P(\iota\text{‘}x \cup \iota\text{‘}y) = y \qquad (4)$

$\vdash . (1) . (2) . (3) . (4) . \supset \vdash . \mathrm{Prop}$

✳205·19. $\vdash : P \in \mathrm{trans} . \supset . \overrightarrow{\min}_P\text{‘}\alpha = \overrightarrow{\min}_P\text{‘}(\alpha \cup \breve{P}\text{‘‘}\alpha) = \overrightarrow{\min}_P\text{‘}\breve{P}_*\text{‘‘}\alpha$

Dem.

$\vdash . ✳205\cdot11 . \qquad \supset \vdash . \overrightarrow{\min}_P\text{‘}(\alpha \cup \breve{P}\text{‘‘}\alpha) = (\alpha \cup \breve{P}\text{‘‘}\alpha) \cap C\text{‘}P - \breve{P}\text{‘‘}(\alpha \cup \breve{P}\text{‘‘}\alpha) \quad (1)$

$\vdash . (1) . ✳201\cdot55 . \supset \vdash : \mathrm{Hp} . \supset . \overrightarrow{\min}_P\text{‘}(\alpha \cup \breve{P}\text{‘‘}\alpha) = (\alpha \cup \breve{P}\text{‘‘}\alpha) \cap C\text{‘}P - \breve{P}\text{‘‘}\alpha$

$[✳22\cdot9] \qquad = \alpha \cap C\text{‘}P - \breve{P}\text{‘‘}\alpha$

$[✳205\cdot11] \qquad = \overrightarrow{\min}_P\text{‘}\alpha \qquad (2)$

$\vdash . *201\cdot52 . *37\cdot265 . \supset \vdash : \text{Hp} . \supset . \breve{P}_*``\alpha = (\alpha \cap C`P) \cup \breve{P}``(\alpha \cap C`P) .$

[(2)] $\supset . \overrightarrow{\min}_P`\breve{P}_*``\alpha = \overrightarrow{\min}_P`(\alpha \cap C`P)$

[*205·15] $= \overrightarrow{\min}_P`\alpha$ (3)

$\vdash . (2) . (3) . \supset \vdash . \text{Prop}$

***205·191.** $\vdash : P \epsilon \text{trans} . \supset . \overrightarrow{\max}_P`\alpha = \overrightarrow{\max}_P`(\alpha \cup P``\alpha) = \overrightarrow{\max}_P`P_*``\alpha$

***205·192.** $\vdash : P \epsilon \text{trans} . \beta \subset \breve{P}``\alpha . \supset . \overrightarrow{\min}_P`(\alpha \cup \beta) = \overrightarrow{\min}_P`\alpha$

Dem.

$\vdash . *205\cdot11 . *201\cdot56 . \supset$

$\vdash : \text{Hp} . \supset . \overrightarrow{\min}_P`(\alpha \cup \beta) = (\alpha \cup \beta) \cap C`P - \breve{P}``\alpha$

[*22·68] $= (\alpha \cap C`P - \breve{P}``\alpha) \cup (\beta \cap C`P - \breve{P}``\alpha)$

[*24·3] $= \alpha \cap C`P - \breve{P}``\alpha$

[*205·11] $= \overrightarrow{\min}_P`\alpha : \supset \vdash . \text{Prop}$

***205·193.** $\vdash : P \epsilon \text{trans} . \beta \subset P``\alpha . \supset . \overrightarrow{\max}_P`(\alpha \cup \beta) = \overrightarrow{\max}_P`\alpha$

***205·194.** $\vdash : x \min_P \alpha . \supset . \sim(xPx)$

Dem.

$\vdash . *37\cdot105 . \quad \supset \vdash . x \epsilon \alpha . xPx . \supset . x \epsilon \breve{P}``\alpha$ (1)

$\vdash . (1) . \text{Transp} . \supset \vdash : x \epsilon \alpha - \breve{P}``\alpha . \supset . \sim(xPx)$ (2)

$\vdash . (2) . *205\cdot1 . \supset \vdash . \text{Prop}$

***205·195.** $\vdash : x \max_P \alpha . \supset . \sim(xPx)$

***205·196.** $\vdash :. P \epsilon \text{Rl}`J \cap \text{trans} . \supset : x \epsilon C`P . \equiv . x = \min_P`(\iota`x \cup \overleftarrow{P}`x)$

Dem.

$\vdash . *205\cdot19 . \supset \vdash :. \text{Hp} . \supset : \overrightarrow{\min}_P`(\iota`x \cup \overleftarrow{P}`x) = \overrightarrow{\min}_P`\iota`x :$

[*205·18] $\supset : x \epsilon C`P . \supset . \min_P`(\iota`x \cup \overleftarrow{P}`x) = x$ (1)

$\vdash . *205\cdot11 . \supset \vdash : \min_P`(\iota`x \cup \overleftarrow{P}`x) = x . \supset . x \epsilon C`P$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***205·197.** $\vdash :. P \epsilon \text{Rl}`J \cap \text{trans} . \supset : x \epsilon C`P . \equiv . x = \max_P`(\overrightarrow{P}`x \cup \iota`x)$

***205·2.** $\vdash :. P \epsilon \text{connex} . E ! \min_P`\alpha . y \epsilon \alpha \cap C`P . \supset : \min_P`\alpha = y . \vee . \min_P`\alpha P y$

Dem.

$\vdash . *202\cdot103 . \supset \vdash :. \text{Hp} . \supset : yP \min_P`\alpha . \vee . \min_P`\alpha = y . \vee . \min_P`\alpha P y$ (1)

$\vdash . *205\cdot14 . \supset \vdash : \text{Hp} . \supset . \sim(yP \min_P`\alpha)$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

In the remainder of the present number, when a proposition has been proved for $\min_P$, we shall not state the corresponding proposition for $\max_P$ unless it is specially important. When propositions concerning $\max_P$ are

required for reference in the sequel, we shall refer to the corresponding propositions for $\min_P$, in case no reference exists for $\max_P$.

***205·21.** $\vdash : P \epsilon \text{connex} . E ! \min_P{}^\epsilon\alpha . y \epsilon \alpha \cap C^\epsilon P - \iota^\epsilon\min_P{}^\epsilon\alpha . \supset . \min_P{}^\epsilon\alpha \, P y$ [*205·2]

***205·211.** $\vdash : P \epsilon \text{trans} \cap \text{connex} . E ! \min_P{}^\epsilon\alpha . y \epsilon \breve{P}^{\epsilon\epsilon}\alpha . \supset . \min_P{}^\epsilon\alpha \, P y$

Dem.

$\vdash . *37\cdot105 . \supset \vdash : \text{Hp} . \supset . (\exists x) . x \epsilon \alpha . x P y$ (1)

$\vdash . *13\cdot13 . \supset \vdash : x \epsilon \alpha . x P y . x = \min_P{}^\epsilon\alpha . \supset . \min_P{}^\epsilon\alpha \, P y$ (2)

$\vdash . *205\cdot21 . \supset \vdash : \text{Hp} . x \epsilon \alpha . x P y . x \neq \min_P{}^\epsilon\alpha . \supset . \min_P{}^\epsilon\alpha \, P x . x P y .$

[Hp.*201·1] $\supset . \min_P{}^\epsilon\alpha \, P y$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash . \text{Prop}$

***205·22.** $\vdash : P \epsilon \text{trans} \cap \text{connex} . E ! \min_P{}^\epsilon\alpha . \supset . \breve{P}^{\epsilon\epsilon}\alpha = \overleftarrow{P}^\epsilon\min_P{}^\epsilon\alpha$ [*205·211 . *37·181]

***205·23.** $\vdash : P \epsilon \text{connex} . x \epsilon D^\epsilon P . y \epsilon \overrightarrow{B}^\epsilon\breve{P} . \supset . x P y$

Dem.

$\vdash . *93\cdot101 . \supset \vdash : \text{Hp} . \supset . x \neq y . \sim(y P x) .$

[*202·103] $\supset . x P y : \supset \vdash . \text{Prop}$

***205·24.** $\vdash : P \epsilon \text{connex} . \supset . \overrightarrow{B}^\epsilon\breve{P} \subset p^\epsilon\overleftarrow{P}^{\epsilon\epsilon}D^\epsilon P$ [*205·23]

***205·241.** $\vdash : P \epsilon \text{connex} . \supset . \overrightarrow{B}^\epsilon P \subset p^\epsilon\overrightarrow{P}^{\epsilon\epsilon}\text{Ꮖ}^\epsilon P$ [Proof as in *205·24]

***205·25.** $\vdash . \overrightarrow{\min_P}{}^\epsilon\overleftarrow{P}^\epsilon x = (\overleftarrow{P \dot{-} P^2})^\epsilon x$

Dem.

$\vdash . *205\cdot11 . \supset \vdash . \overrightarrow{\min_P}{}^\epsilon\overleftarrow{P}^\epsilon x = \overleftarrow{P}^\epsilon x - \breve{P}^{\epsilon\epsilon}\overleftarrow{P}^\epsilon x$

[*37·301] $= \overleftarrow{P}^\epsilon x - \overleftarrow{P^2}^\epsilon x$

[*32·31·35] $= (\overleftarrow{P \dot{-} P^2})^\epsilon x . \supset \vdash . \text{Prop}$

The following proposition is used in the theory of well-ordered series (*250·2).

***205·251.** $\vdash : \exists ! \overrightarrow{\min_P}{}^\epsilon\overleftarrow{P}^\epsilon x . \equiv . x \epsilon D^\epsilon(P \dot{-} P^2)$ [*205·25]

***205·252.** $\vdash : \exists ! \overrightarrow{\max_P}{}^\epsilon\overrightarrow{P}^\epsilon x . \equiv . x \epsilon \text{Ꮖ}^\epsilon(P \dot{-} P^2)$

***205·253.** $\vdash : P \epsilon \text{connex} . E ! B^\epsilon P . \supset . \text{Ꮖ}^\epsilon P = \overleftarrow{P}^\epsilon B^\epsilon P$ [*202·524]

***205·254.** $\vdash : P \epsilon \text{connex} . E ! B^\epsilon P . \supset . \overrightarrow{\min_P}{}^\epsilon\text{Ꮖ}^\epsilon P = \overleftarrow{P \dot{-} P^2}{}^\epsilon B^\epsilon P$ [*205·253·25]

***205·255.** $\vdash : \exists ! \overrightarrow{\min_P}{}^\epsilon\text{Ꮖ}^\epsilon P . \supset . \exists ! \overrightarrow{B}^\epsilon P$

Dem.

$\vdash . *93\cdot101 . \supset \vdash : \overrightarrow{B}^\epsilon P = \Lambda . \supset . D^\epsilon P \subset \text{Ꮖ}^\epsilon P .$

[*37·271] $\supset . \text{Ꮖ}^\epsilon P = \breve{P}^{\epsilon\epsilon}\text{Ꮖ}^\epsilon P .$

[*205·122] $\supset . \overrightarrow{\min_P}{}^\epsilon\text{Ꮖ}^\epsilon P = \Lambda$ (1)

$\vdash . (1) . \text{Transp} . \supset \vdash . \text{Prop}$

***205·256.** $\vdash :. P \in \mathrm{Ser} . \supset :$

$$\mathrm{E}\,!\,\min_P{}^{\backprime}\text{Ɑ}{}^{\backprime}P . \equiv . \mathrm{E}\,!\,\breve{P}_1{}^{\backprime}B{}^{\backprime}P . \equiv . \min_P{}^{\backprime}\text{Ɑ}{}^{\backprime}P = \breve{P}_1{}^{\backprime}B{}^{\backprime}P$$

[*205·254·255 . *201·63 . *202·52·7]

***205·26.** $\vdash : Q \mathrel{\text{⋐}} P . \supset . \min_P \upharpoonright \mathrm{Cl}{}^{\backprime}C{}^{\backprime}Q \mathrel{\text{⋐}} \min_Q$

Dem.

$\vdash . *37{\cdot}201 . \supset \vdash :. \mathrm{Hp} . \alpha \subset C{}^{\backprime}Q . \supset : \breve{Q}{}^{\backprime\backprime}\alpha \subset \breve{P}{}^{\backprime\backprime}\alpha . \alpha \subset C{}^{\backprime}Q . \alpha \subset C{}^{\backprime}P :$

[Transp.*22·621] $\supset : \alpha - \breve{P}{}^{\backprime\backprime}\alpha \subset \alpha - \breve{Q}{}^{\backprime\backprime}\alpha . \alpha = \alpha \cap C{}^{\backprime}Q = \alpha \cap C{}^{\backprime}P :$

[*205·11] $\supset : \overrightarrow{\min}_P{}^{\backprime}\alpha \subset \overrightarrow{\min}_Q{}^{\backprime}\alpha :$

[*32·18] $\supset : x \min_P \alpha . \supset . x \min_Q \alpha :. \supset \vdash . \mathrm{Prop}$

***205·261.** $\vdash : P \upharpoonright \beta \in \mathrm{connex} . \beta \cap C{}^{\backprime}P \sim\in 1 . \supset . \overrightarrow{\min}\,(P \upharpoonright \beta){}^{\backprime}\alpha = \overrightarrow{\min}_P{}^{\backprime}(\alpha \cap \beta)$

Dem.

$\vdash . *205{\cdot}11 . *202{\cdot}54 . *37{\cdot}413 . *36{\cdot}34 . \supset$

$\vdash : \mathrm{Hp} . \supset . \overrightarrow{\min}\,(P \upharpoonright \beta){}^{\backprime}\alpha = \alpha \cap \beta \cap C{}^{\backprime}P - \{\beta \cap \breve{P}{}^{\backprime\backprime}(\alpha \cap \beta)\}$

[*22·93.*205·11] $= \overrightarrow{\min}_P{}^{\backprime}(\alpha \cap \beta) : \supset \vdash . \mathrm{Prop}$

***205·262.** $\vdash : P \in \mathrm{trans} \cap \mathrm{connex} . x \in \alpha \cap C{}^{\backprime}P . \beta = \overrightarrow{P}{}^{\backprime}x \cup \iota{}^{\backprime}x . \supset .$

$$\overrightarrow{\min}_P{}^{\backprime}\alpha = \overrightarrow{\min}_P{}^{\backprime}(\alpha \cap \beta)$$

Dem.

$\vdash . *32{\cdot}18 . \supset \vdash :. \mathrm{Hp} . y \in \alpha . yPx . \supset : y \in \alpha \cap \beta :$

[*37·105] $\supset : yPz . \supset . z \in \breve{P}{}^{\backprime\backprime}(\alpha \cap \beta)$ (1)

$\vdash . *51{\cdot}15 . \supset \vdash :. \mathrm{Hp} . y \in \alpha . y = x . \supset : y \in \alpha \cap \beta :$

[*37·105] $\supset : yPz . \supset . z \in \breve{P}{}^{\backprime\backprime}(\alpha \cap \beta)$ (2)

$\vdash . *51{\cdot}15 . *201{\cdot}1 . \supset \vdash : \mathrm{Hp} . y \in \alpha . xPy . yPz . \supset . x \in \alpha . xPz .$

[*37·105] $\supset . z \in \breve{P}{}^{\backprime\backprime}(\alpha \cap \beta)$ (3)

$\vdash . (1) . (2) . (3) . *202{\cdot}103 . \supset \vdash :. \mathrm{Hp} . \supset : y \in \alpha . yPz . \supset . z \in \breve{P}{}^{\backprime\backprime}(\alpha \cap \beta) :$

[*37·105·2] $\supset : \breve{P}{}^{\backprime\backprime}\alpha = \breve{P}{}^{\backprime\backprime}(\alpha \cap \beta)$ (4)

$\vdash . *37{\cdot}181 . *202{\cdot}101 . \supset \vdash : \mathrm{Hp} . \supset . \overleftarrow{P}{}^{\backprime}x \subset \breve{P}{}^{\backprime\backprime}\alpha . \overleftarrow{P}{}^{\backprime}x = C{}^{\backprime}P - \beta .$

[*22·82] $\supset . C{}^{\backprime}P - \breve{P}{}^{\backprime\backprime}\alpha \subset \beta .$

[*22·621.(4)] $\supset . C{}^{\backprime}P \cap \alpha - \breve{P}{}^{\backprime\backprime}\alpha = C{}^{\backprime}P \cap \alpha \cap \beta - \breve{P}{}^{\backprime\backprime}(\alpha \cap \beta) .$

[*205·11] $\supset . \overrightarrow{\min}_P{}^{\backprime}\alpha = \overrightarrow{\min}_P{}^{\backprime}(\alpha \cap \beta) : \supset \vdash . \mathrm{Prop}$

***205·27.** $\vdash : P \in \mathrm{trans} \cap \mathrm{connex} . x \in \alpha \cap \text{Ɑ}{}^{\backprime}P . \beta = \overrightarrow{P}{}^{\backprime}x \cup \iota{}^{\backprime}x . \supset .$

$$\overrightarrow{\min}_P{}^{\backprime}\alpha = \overrightarrow{\min}\,(P \upharpoonright \beta){}^{\backprime}\alpha = \overrightarrow{\min}_P{}^{\backprime}(\alpha \cap \beta)$$

Dem.

$\vdash . *52{\cdot}41 . \supset \vdash : \mathrm{Hp} . \overrightarrow{P}{}^{\backprime}x \neq \iota{}^{\backprime}x . \supset . \beta \sim\in 1 .$

[*205·261] $\supset . \overrightarrow{\min}\,(P \upharpoonright \beta){}^{\backprime}\alpha = \overrightarrow{\min}_P{}^{\backprime}(\alpha \cap \beta)$ (1)

$\vdash . *202\cdot101 . \supset \vdash : \text{Hp} . \overrightarrow{P}\text{‘}x = \iota\text{‘}x . \supset . C\text{‘}P - \iota\text{‘}x = \overleftarrow{P}\text{‘}x . xPx .$

$[*37\cdot105] \qquad \supset . C\text{‘}P \subset \breve{P}\text{“}(\alpha \cap \beta) .$

$[*205\cdot122 . *37\cdot413] \qquad \supset . \overrightarrow{\min}_P\text{‘}(\alpha \cap \beta) = \Lambda . \overrightarrow{\min}(P \upharpoonleft \beta)\text{‘}\alpha = \Lambda \quad (2)$

$\vdash . (1) . (2) . \supset \vdash : \text{Hp} . \supset . \overrightarrow{\min}(P \upharpoonleft \beta)\text{‘}\alpha = \overrightarrow{\min}_P\text{‘}(\alpha \cap \beta) \quad (3)$

$\vdash . (3) . *205\cdot262 . \supset \vdash . \text{Prop}$

The above proposition is used in $*250\cdot7$.

$*205\cdot3$. $\vdash : P \in \text{connex} . \supset . \overrightarrow{\min}_P\text{‘}\alpha \in 0 \cup 1 . \overrightarrow{\max}_P\text{‘}\alpha \in 0 \cup 1$

Dem.

$\vdash . *205\cdot11 . \supset \vdash :. x, y \in \overrightarrow{\min}_P\text{‘}\alpha . \supset : x, y \in \alpha \cap C\text{‘}P : z \in \alpha . \supset_z . \sim(zPx) . \sim(zPy) :$

$[*10\cdot1] \qquad \supset : x, y \in \alpha \cap C\text{‘}P . \sim(yPx) . \sim(xPy) \quad (1)$

$\vdash . (1) . *202\cdot103 . \supset \vdash :. \text{Hp} . \supset : x, y \in \overrightarrow{\min}_P\text{‘}\alpha . \supset . x = y :$

$[*52\cdot4] \qquad \supset : \overrightarrow{\min}_P\text{‘}\alpha \in 0 \cup 1 \quad (2)$

Similarly $\qquad \vdash :. \text{Hp} . \supset : \overrightarrow{\max}_P\text{‘}\alpha \in 0 \cup 1 \quad (3)$

$\vdash . (2) . (3) . \supset \vdash . \text{Prop}$

The above proposition is of great importance in the theory of maxima and minima.

$*205\cdot31$. $\vdash : P \in \text{connex} . \supset . \min_P, \max_P \in 1 \rightarrow \text{Cls} \quad [*205\cdot3 . *71\cdot12]$

$*205\cdot32$. $\vdash :. P \in \text{connex} . \supset : \exists ! \overrightarrow{\min}_P\text{‘}\alpha . \equiv . E ! \min_P\text{‘}\alpha . \equiv . \alpha \in \text{Ꮖ}\text{‘}\min_P$
$[*205\cdot31 . *71\cdot163 . *33\cdot41]$

$*205\cdot33$. $\vdash : P \in \text{connex} . \kappa \subset \text{Ꮖ}\text{‘}\min_P . \supset . \min_P \upharpoonright \kappa \in \epsilon_\Delta\text{‘}\kappa$

Dem. $\vdash . *205\cdot31 . \supset \vdash : \text{Hp} . \supset . \min_P \upharpoonright \kappa \in 1 \rightarrow \text{Cls} \quad (1)$

$\vdash . *205\cdot1 . \supset \vdash : \text{Hp} . \supset . \min_P \upharpoonright \kappa \subseteq \epsilon \quad (2)$

$\vdash . *35\cdot65 . \supset \vdash : \text{Hp} . \supset . \text{Ꮖ}\text{‘}\min_P \upharpoonright \kappa = \kappa \quad (3)$

$\vdash . (1) . (2) . (3) . *80\cdot14 . \supset \vdash . \text{Prop}$

$*205\cdot34$. $\vdash : P \in \text{connex} . \kappa \subset \text{Ꮖ}\text{‘}\min_P . \supset . \kappa \in \text{Cls}^2 \text{mult} \quad [*205\cdot33 . *88\cdot2]$

The following proposition is used in $*260\cdot17$.

$*205\cdot35$. $\vdash :: P^2 \subseteq J . P \in \text{connex} . \supset :.$

$x = \min_P\text{‘}\alpha . \equiv : x \in \alpha \cap C\text{‘}P : y \in \alpha \cap C\text{‘}P - \iota\text{‘}x . \supset_y . xPy$

Dem.

$\vdash . *205\cdot31 . *71\cdot36 . \supset \vdash :: \text{Hp} . \supset :. x = \min_P\text{‘}\alpha . \equiv : x \min_P \alpha :$

$[*205\cdot1 . *37\cdot265] \equiv : x \in \alpha \cap C\text{‘}P - \breve{P}\text{“}(\alpha \cap C\text{‘}P) :$

$[*37\cdot105] \qquad \equiv : x \in \alpha \cap C\text{‘}P : y \in \alpha \cap C\text{‘}P . \supset_y . \sim(yPx) :$

$[*51\cdot221] \equiv : x \in \alpha \cap C\text{‘}P : y = x . \supset_y . \sim(yPx) : y \in \alpha \cap C\text{‘}P - \iota\text{‘}x . \supset_x . \sim(yPx) \quad (1)$

$\vdash . *200\cdot36 . \supset \vdash :. \text{Hp} . \supset : yPz . \supset_{z,y} . y \neq z :$

$[\text{Transp} . *10\cdot1] \qquad \supset : y = x . \supset_y . \sim(yPx) \quad (2)$

$\vdash . (1) . (2) . \supset \vdash :: \text{Hp} . \supset :.$

$x = \min_P\text{‘}\alpha . \equiv : x \in \alpha \cap C\text{‘}P : y \in \alpha \cap C\text{‘}P - \iota\text{‘}x . \supset_y . \sim(yPx) :$

$[*202\cdot5] \qquad \equiv : x \in \alpha \cap C\text{‘}P : y \in \alpha \cap C\text{‘}P - \iota\text{‘}x . \supset_y . xPy :: \supset \vdash . \text{Prop}$

***205·36.** $\vdash : P \in \text{trans} \cap \text{connex} \,.\supset .\, \overrightarrow{\min}_P\text{‘}\alpha \subset p\text{‘}\overrightarrow{P_*}\text{‘‘}(\alpha \cap C\text{‘}P)$

Dem.

$\vdash . *205\cdot2 . *201\cdot18 . \supset \vdash :. \text{Hp} . x = \min_P\text{‘}\alpha . \supset : y \in (\alpha \cap C\text{‘}P) . \supset_y . x P_* y :. \supset \vdash . \text{Prop}$

The above proposition is used in *230·53.

***205·37.** $\vdash : P \in \text{trans} \,.\, \overrightarrow{\max}_P\text{‘}\alpha = \Lambda \,.\supset .\, P_*\text{‘‘}\alpha = P\text{‘‘}\alpha$ $[*201\cdot52 . *205\cdot123]$

The following proposition is used in *257·21.

***205·38.** $\vdash : P_{\text{po}} \Subset J \,.\supset .\, \mu \cap p\text{‘}\overrightarrow{P_*}\text{‘‘}\mu \subset \overrightarrow{\min}(P_{\text{po}})\text{‘}\mu$

Dem.

$\vdash . *200\cdot381 . \supset \vdash :: \text{Hp} . \supset :. x \in \mu . \supset_x . y P_* x : \supset : x \in \mu . \supset_x . \sim(x P_{\text{po}} y) :.$

$[*40\cdot51 . *37\cdot105] \quad \supset :. p\text{‘}\overrightarrow{P_*}\text{‘‘}\mu \subset -\breve{P}_{\text{po}}\text{‘‘}\mu$ (1)

$\vdash . *40\cdot62 . \supset \vdash : \exists ! \mu . \supset . p\text{‘}\overrightarrow{P_*}\text{‘‘}\mu \subset C\text{‘}P$ (2)

$\vdash . *24\cdot12 . \supset \vdash : \sim\exists ! \mu . \supset . \mu \subset C\text{‘}P$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash : \text{Hp} . \supset . \mu \cap p\text{‘}\overrightarrow{P_*}\text{‘‘}\mu \subset \mu \cap C\text{‘}P - \breve{P}_{\text{po}}\text{‘‘}\mu$

$[*205\cdot11 . *91\cdot504] \quad \subset \overrightarrow{\min}(P_{\text{po}})\text{‘}\mu : \supset \vdash . \text{Prop}$

***205·381.** $\vdash : P_{\text{po}} \Subset J \,.\, \overrightarrow{\max}_P\text{‘}\mu = \Lambda \,.\supset .\, p\text{‘}\overleftarrow{P_*}\text{‘‘}\mu = p\text{‘}\overleftarrow{P_{\text{po}}}\text{‘‘}\mu$

Dem.

$\vdash . *205\cdot38 \dfrac{\breve{P}}{P} . \supset \vdash : \text{Hp} . \supset . \mu \cap p\text{‘}\overleftarrow{P_*}\text{‘‘}\mu = \Lambda$ (1)

$\vdash . (1) . *40\cdot53 . *24\cdot37 . \supset$

$\vdash :: \text{Hp} . \supset :. x \in p\text{‘}\overleftarrow{P_*}\text{‘‘}\mu . \equiv : y \in \mu . \supset_y . y P_* x . y \neq x :$

$[*200\cdot38] \quad \equiv : y \in \mu . \supset_y . y P_{\text{po}} x :$

$[*40\cdot53] \quad \equiv : x \in p\text{‘}\overleftarrow{P_{\text{po}}}\text{‘‘}\mu :: \supset \vdash . \text{Prop}$

The three following propositions lead up to *205·42, which is used in *261·26.

***205·4.** $\vdash : C\text{‘}P \in 1 \,.\supset .\, \overrightarrow{B}\text{‘}P = \Lambda \,.\, \overrightarrow{B}\text{‘}\breve{P} = \Lambda$

Dem.

$\vdash . *56\cdot381 . *55\cdot15 . \supset \vdash : \text{Hp} . \supset . (\exists x) . \text{D}\text{‘}P = \iota\text{‘}x . \text{Q}\text{‘}P = \iota\text{‘}x .$

$[*93\cdot101] \quad \supset . \overrightarrow{B}\text{‘}P = \Lambda . \overrightarrow{B}\text{‘}\breve{P} = \Lambda : \supset \vdash . \text{Prop}$

***205·401.** $\vdash : \exists ! \overrightarrow{B}\text{‘}P \upharpoonright \alpha \,.\supset .\, \alpha \cap C\text{‘}P \sim\in 0 \cup 1 \,.\, C\text{‘}P \upharpoonright \alpha \sim\in 0 \cup 1$

Dem.

$\vdash . *205\cdot4 . \text{Transp} . \supset \vdash : \text{Hp} . \supset . C\text{‘}P \upharpoonright \alpha \sim\in 1$ (1)

$\vdash . *93\cdot103 . \quad \supset \vdash : \text{Hp} . \supset . \exists ! C\text{‘}P \upharpoonright \alpha$ (2)

$\vdash . (1) . (2) . \quad \supset \vdash : \text{Hp} . \supset . C\text{‘}P \upharpoonright \alpha \sim\in 0 \cup 1$ (3)

$\vdash . *37\cdot41 . \quad \supset \vdash . C\text{‘}P \upharpoonright \alpha \subset \alpha \cap C\text{‘}P$ (4)

$\vdash . (4) . *60\cdot32\cdot371 . \text{Transp} . \supset \vdash : C\text{‘}P \upharpoonright \alpha \sim\in 0 \cup 1 . \supset . \alpha \cap C\text{‘}P \sim\in 0 \cup 1$ (5)

$\vdash . (3) . (5) . \supset \vdash . \text{Prop}$

The following proposition, besides being required for $*205{\cdot}42$, is used in $*250{\cdot}151$.

$*205{\cdot}41$. $\vdash : P \in \text{connex} \,.\, \alpha \cap C\text{‘}P \sim\in 1 \,.\, \supset .\, \overrightarrow{\min}_P\text{‘}\alpha = \overrightarrow{B}\text{‘}P \upharpoonright \alpha$

Dem.

$\vdash .\, *202{\cdot}54 \,.\qquad \supset \vdash : \text{Hp} \,.\, \supset .\, C\text{‘}P \upharpoonright \alpha = \alpha \cap C\text{‘}P \qquad (1)$

$\vdash .\, *37{\cdot}41 \,.\qquad \supset \vdash .\, \text{Ɑ}\text{‘}P \upharpoonright \alpha = \alpha \cap \breve{P}\text{“}\alpha \qquad (2)$

$\vdash .\, (1) \,.\, (2) \,.\, *93{\cdot}103 \,.\, \supset \vdash : \text{Hp} \,.\, \supset .\, \overrightarrow{B}\text{‘}P \upharpoonright \alpha = \alpha \cap C\text{‘}P - (\alpha \cap \breve{P}\text{“}\alpha)$

$[*22{\cdot}93 . *205{\cdot}11] \qquad = \overrightarrow{\min}_P\text{‘}\alpha : \supset \vdash .\, \text{Prop}$

$*205{\cdot}42$. $\vdash : P \in \text{connex} \,.\, \text{E} ! \, B\text{‘}P \upharpoonright \alpha \,.\, \supset .\, B\text{‘}P \upharpoonright \alpha = \min_P\text{‘}\alpha$

Dem. $\vdash .\, *205{\cdot}401 \,.\, \supset \vdash : \text{Hp} \,.\, \supset .\, \alpha \cap C\text{‘}P \sim\in 1 \,.$

$[*205{\cdot}41] \qquad \supset .\, \overrightarrow{\min}_P\text{‘}\alpha = \overrightarrow{B}\text{‘}P \upharpoonright \alpha \qquad (1)$

$\vdash .\, (1) \,.\, *32{\cdot}41 \,.\, \supset \vdash .\, \text{Prop}$

The following proposition leads up to $*205{\cdot}44$.

$*205{\cdot}43$. $\vdash : P \in \text{connex} \,.\, \exists ! \, \alpha \cap \text{D}\text{‘}P \,.\, \supset .\, \overrightarrow{\min}_P\text{‘}\alpha = \overrightarrow{B}\text{‘}\alpha \urcorner P$

Dem.

$\vdash .\, *205{\cdot}11 \,.\qquad \supset \vdash .\, \overrightarrow{\min}_P\text{‘}\alpha = (\alpha \cap \text{D}\text{‘}P - \breve{P}\text{“}\alpha) \cup (\alpha \cap \overrightarrow{B}\text{‘}\breve{P} - \breve{P}\text{“}\alpha)$

$[*35{\cdot}61 . *37{\cdot}4] \qquad = \overrightarrow{B}\text{‘}\alpha \urcorner P \cup (\alpha \cap \overrightarrow{B}\text{‘}\breve{P} - \breve{P}\text{“}\alpha) \qquad (1)$

$\vdash .\, *205{\cdot}23 \,.\qquad \supset \vdash : P \in \text{connex} \,.\, x \in \alpha \cap \text{D}\text{‘}P \,.\, y \in \overrightarrow{B}\text{‘}\breve{P} \,.\, \supset .\, x \in \alpha \,.\, xPy \,.$

$[*37{\cdot}1] \qquad \supset .\, y \in \breve{P}\text{“}\alpha \qquad (2)$

$\vdash .\, (2) \,.\, *10{\cdot}23 \,.\, \supset \vdash : \text{Hp} \,.\, \supset .\, \overrightarrow{B}\text{‘}\breve{P} \subset \breve{P}\text{“}\alpha \,.$

$[*24{\cdot}3] \qquad \supset .\, \alpha \cap \overrightarrow{B}\text{‘}\breve{P} - \breve{P}\text{“}\alpha = \Lambda \qquad (3)$

$\vdash .\, (1) \,.\, (3) \,.\, \supset \vdash .\, \text{Prop}$

$*205{\cdot}44$. $\vdash : P \in \text{connex} \,.\, \text{E} ! \, B\text{‘}\alpha \urcorner P \,.\, \supset .\, \min_P\text{‘}\alpha = B\text{‘}\alpha \urcorner P \quad [*205{\cdot}43 \,.\, *32{\cdot}41]$

The following propositions deal with the circumstances under which the minimum of one class is identical with, or earlier than, that of another.

$*205{\cdot}5$. $\vdash : P \in \text{connex} \,.\, \alpha \subset \beta \,.\, \min_P\text{‘}\beta \in \alpha \,.\, \supset .\, \text{E} ! \min_P\text{‘}\alpha \,.\, \min_P\text{‘}\alpha = \min_P\text{‘}\beta$

Dem. $\vdash .\, *37{\cdot}2 \,.\, \text{Transp} \,.\, \supset \vdash : \text{Hp} \,.\, \supset .\, -P\text{“}\beta \subset -P\text{“}\alpha \,.$

$[*205{\cdot}11 . \text{Hp}] \qquad \supset .\, \min_P\text{‘}\beta \in \alpha - P\text{“}\alpha \,.$

$[*205{\cdot}1] \qquad \supset .\, \min_P\text{‘}\beta \in \overrightarrow{\min}_P\text{‘}\alpha \qquad (1)$

$\vdash .\, (1) \,.\, *205{\cdot}3 \,.\, \supset \vdash .\, \text{Prop}$

$*205{\cdot}501$. $\vdash : P \in \text{connex} \,.\, \min_P\text{‘}\alpha = \min_P\text{‘}\beta \,.\, \supset .\, \beta \subset -p\text{‘}\overrightarrow{P}\text{“}\alpha$

Dem.

$\vdash .\, *205{\cdot}11 \,.\, \supset \vdash :.\, \text{Hp} \,.\, \supset : \min_P\text{‘}\alpha \sim\in \breve{P}\text{“}\beta :$

$[*37{\cdot}105] \qquad \supset : y \in \beta \,.\, \supset_y .\, \sim(yP \min_P\text{‘}\alpha) :$

$[*205{\cdot}11] \qquad \supset : y \in \beta \,.\, \supset_y .\, (\exists x) \,.\, x \in \alpha \,.\, \sim(yPx) :$

$[*40{\cdot}51] \qquad \supset : \beta \subset -p\text{‘}\overrightarrow{P}\text{“}\alpha :.\, \supset \vdash .\, \text{Prop}$

***205·51.** $\vdash :. P \epsilon \text{connex} . \alpha \subset \beta . \text{E} ! \text{min}_P‘\alpha . \text{E} ! \text{min}_P‘\beta . \supset :$
$\text{min}_P‘\alpha = \text{min}_P‘\beta . \text{v} . \text{min}_P‘\beta \, P \, \text{min}_P‘\alpha$

Dem.

$\vdash . *22·1 . *205·1 . \supset \vdash : \text{Hp} . \supset . \text{min}_P‘\alpha \,\epsilon\, \beta \cap C‘P$ (1)

$\vdash . (1) . *205·2 . \supset \vdash . \text{Prop}$

***205·52.** $\vdash : P \epsilon \text{trans} \cap \text{connex} . \exists ! \alpha \cap p‘\overrightarrow{P}‘‘\beta .$
$\text{E} ! \text{min}_P‘\alpha . \text{E} ! \text{min}_P‘\beta . \supset . \text{min}_P‘\alpha \, P \, \text{min}_P‘\beta$

Dem.

$\vdash . *40·51 . \supset \vdash :. \text{Hp} . \supset : (\exists x) : x \epsilon \alpha : y \epsilon \beta . \supset_y . xPy$ (1)

$\vdash . *205·2 . \supset \vdash :: \text{Hp} . \supset :. x \epsilon \alpha \cap C‘P . \supset_x : \text{min}_P‘\alpha = x . \text{v} . \text{min}_P‘\alpha \, Px$ (2)

$\vdash . (1) . *205·1 . \supset \vdash :. \text{Hp} . \supset : (\exists x) . x \epsilon \alpha . xP \, \text{min}_P‘\beta :$

[*33·17] $\supset : (\exists x) . x \epsilon \alpha \cap C‘P . xP \, \text{min}_P‘\beta$ (3)

$\vdash . (2) . (3) . \quad \supset \vdash :. \text{Hp} . \supset : (\exists x) : xP \, \text{min}_P‘\beta : \text{min}_P‘\alpha = x . \text{v} . \text{min}_P‘\alpha \, Px :$

[*201·1.*13·195] $\supset : \text{min}_P‘\alpha \, P \, \text{min}_P‘\beta :. \supset \vdash . \text{Prop}$

***205·53.** $\vdash : P \epsilon \text{connex} \cap \text{Rl}‘J . x \epsilon \alpha \cap C‘P . \overrightarrow{P}‘x = P‘‘\alpha . \supset . x = \text{max}_P‘\alpha$

Dem.

$\vdash . *50·24 . \supset \vdash : \text{Hp} . \supset . x \epsilon \alpha \cap C‘P - \overrightarrow{P}‘x .$

[Hp] $\supset . x \epsilon \alpha \cap C‘P - P‘‘\alpha .$

[*205·111] $\supset . x \epsilon \overrightarrow{\text{max}_P}‘\alpha$ (1)

$\vdash . (1) . *205·3 . \supset \vdash . \text{Prop}$

***205·54.** $\vdash :. P \epsilon \text{Ser} . \supset : x \epsilon \alpha \cap C‘P . \overrightarrow{P}‘x = P‘‘\alpha . \equiv . x = \text{max}_P‘\alpha$
[*205·53·22]

***205·55.** $\vdash : P \epsilon \text{connex} . B‘P \epsilon \alpha . \supset . B‘P = \text{min}_P‘\alpha$

Dem.

$\vdash . *93·101 . *37·16 . \supset \vdash : \text{E} ! B‘P . \supset . B‘P \epsilon C‘P - \breve{P}‘‘\alpha$ (1)

$\vdash . (1) . *205·1 . \quad \supset \vdash : B‘P \epsilon \alpha . \supset . B‘P \, \text{min}_P \, \alpha$ (2)

$\vdash . (2) . *205·31 . \supset \vdash . \text{Prop}$

***205·56.** $\vdash . \overrightarrow{\text{max}_P}‘s‘\kappa \subset \text{max}_P‘‘\kappa$

Dem.

$\vdash . *205·111 . *40·38 . \supset \vdash . \overrightarrow{\text{max}_P}‘s‘\kappa \subset s‘\kappa \cap C‘P - s‘P‘‘‘\kappa$

[*40·11] $\subset \hat{y} \{(\exists \alpha) . \alpha \epsilon \kappa . y \epsilon \alpha \cap C‘P : \sim(\exists \alpha) . \alpha \epsilon \kappa . y \epsilon P‘‘\alpha\}$

[*10·56] $\subset \hat{y} \{(\exists \alpha) . \alpha \epsilon \kappa . y \epsilon \alpha \cap C‘P - P‘‘\alpha\}$

[*205·111] $\subset \hat{y} \{(\exists \alpha) . \alpha \epsilon \kappa . y \epsilon \overrightarrow{\text{max}_P}‘\alpha\}$

[*40·5] $\subset \text{max}_P‘‘\kappa . \supset \vdash . \text{Prop}$

***205·561.** $\vdash : \kappa \subset - \text{ᗡ}‘\text{max}_P . \supset . s‘\kappa \sim\epsilon \text{ᗡ}‘\text{max}_P$ [*205·56 . *37·26·29]

***205·6.** $\vdash :. P \epsilon \text{connex} . \supset : \sim \text{E} ! \text{min}_P‘\alpha . \equiv . \alpha \cap C‘P \subset \breve{P}‘‘\alpha$ [*205·32·122]

$*205 \cdot 601$. $\vdash : P \in \text{connex} \,.\, \alpha \subset C`P \,.\, \supset : \sim \text{E} \,!\, \text{min}_P`\alpha \,.\, \equiv \,.\, \alpha \subset \breve{P}``\alpha$ $[*205 \cdot 6]$

$*205 \cdot 61$. $\vdash : P \in \text{connex} \,.\, \supset \,.\, C`P = \{C`P \cap p`\overrightarrow{P}``(\alpha \cap C`P)\} \cup \overrightarrow{\text{min}}_P`\alpha \cup \breve{P}``\alpha$
$[*202 \cdot 505 \,.\, *205 \cdot 13]$

$*205 \cdot 62$. $\vdash : P \in \text{connex} \,.\, \exists \,!\, \alpha \cap C`P \,.\, \supset \,.\, C`P = p`\overrightarrow{P}``(\alpha \cap C`P) \cup \overrightarrow{\text{min}}_P`\alpha \cup \breve{P}``\alpha$
$[*40 \cdot 62 \,.\, *205 \cdot 61]$

$*205 \cdot 63$. $\vdash : P \in \text{connex} \,.\, P^2 \mathrel{\dot\subset} J \,.\, \exists \,!\, (\alpha \cap C`P) \,.\, \supset \,.$

$$p`\overrightarrow{P}``(\alpha \cap C`P) = C`P - \breve{P}``\alpha - \overrightarrow{\text{min}}_P`\alpha$$

$[*202 \cdot 502 \,.\, *205 \cdot 13]$

$*205 \cdot 64$. $\vdash : P \in \text{connex} \,.\, \exists \,!\, (\alpha \cap C`P) \,.\, \supset \,.$

$$\overrightarrow{\text{min}}_P`\alpha = C`P - \breve{P}``\alpha - p`\overrightarrow{P}``(\alpha \cap C`P)$$

Dem.

$\vdash \,.\, *205 \cdot 62 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.$

$$C`P - \breve{P}``\alpha - p`\overrightarrow{P}``(\alpha \cap C`P) = \overrightarrow{\text{min}}_P`\alpha - \breve{P}``\alpha - p`\overrightarrow{P}``(\alpha \cap C`P) \quad (1)$$

$\vdash \,.\, *205 \cdot 11 \,.\, \supset \vdash \,.\, \overrightarrow{\text{min}}_P`\alpha - \breve{P}``\alpha = \overrightarrow{\text{min}}_P`\alpha \quad (2)$

$\vdash \,.\, *205 \cdot 14 \,.\, \supset \vdash :. \, x \in \overrightarrow{\text{min}}_P`\alpha \,.\, \supset : y \in \alpha \,.\, \supset_y \,.\, \sim(yPx) :$

$[*205 \cdot 11 \,.\, *10 \cdot 1] \quad \supset : \sim(xPx) \,.\, x \in \alpha \cap C`P :$

$[*40 \cdot 51] \quad \supset : x \sim\in p`\overrightarrow{P}``(\alpha \cap C`P) \quad (3)$

$\vdash \,.\, (3) \,.\, \supset \vdash \,.\, \overrightarrow{\text{min}}_P`\alpha - p`\overrightarrow{P}``(\alpha \cap C`P) = \overrightarrow{\text{min}}_P`\alpha \quad (4)$

$\vdash \,.\, (1) \,.\, (2) \,.\, (4) \,.\, \supset \vdash \,.\, \text{Prop}$

$*205 \cdot 65$. $\vdash : P \in \text{trans} \cap \text{connex} \,.\, \text{E} \,!\, \text{min}_P`\alpha \,.\, \supset \,.\, p`\overrightarrow{P}``(\alpha \cap C`P) = \overrightarrow{P}`\text{min}_P`\alpha$

Dem.

$\vdash \,.\, *205 \cdot 2 \,.\, \supset \vdash :: \text{Hp} \,.\, \supset :. \, xP\,\text{min}_P`\alpha \,.\, \supset : y \in \alpha \cap C`P \,.\, \supset_y \,.\, xPy :$

$[*40 \cdot 51] \quad \supset : x \in p`\overrightarrow{P}``(\alpha \cap C`P) \quad (1)$

$\vdash \,.\, *205 \cdot 1 \,.\, *40 \cdot 12 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, p`\overrightarrow{P}``(\alpha \cap C`P) \subset \overrightarrow{P}`\text{min}_P`\alpha \quad (2)$

$\vdash \,.\, (1) \,.\, (2) \,.\, \supset \vdash \,.\, \text{Prop}$

$*205 \cdot 66$. $\vdash : P \in \text{trans} \cap \text{connex} \,.\, \text{E} \,!\, \text{min}_P`\alpha \,.\, \supset \,.$

$$p`\overrightarrow{P}``(\alpha \cap C`P) = \overrightarrow{P}`\text{min}_P`\alpha \,.\, \breve{P}``\alpha = \overleftarrow{P}`\text{min}_P`\alpha \,.$$
$$C`P = p`\overrightarrow{P}``(\alpha \cap C`P) \cup \iota`\text{min}_P`\alpha \cup \breve{P}``\alpha$$

$[*205 \cdot 65 \cdot 22 \,.\, *202 \cdot 101]$

$*205 \cdot 67$. $\vdash :. \, P \in \text{Ser} \,.\, \supset : x = \text{min}_P`\alpha \,.\, \equiv \,.\, \overrightarrow{P}`x = p`\overrightarrow{P}``(\alpha \cap C`P) \,.\, x \in C`P$

Dem.

$\vdash \,.\, *205 \cdot 65 \cdot 11 \,.\, \supset$

$\vdash :. \, \text{Hp} \,.\, \supset : x = \text{min}_P`\alpha \,.\, \supset \,.\, \overrightarrow{P}`x = p`\overrightarrow{P}``(\alpha \cap C`P) \,.\, x \in C`P \quad (1)$

$\vdash \,.\, *50 \cdot 24 \,.\, \supset \vdash : \text{Hp} \,.\, \overrightarrow{P}`x = p`\overrightarrow{P}``(\alpha \cap C`P) \,.\, \supset \,.\, x \sim\in p`\overrightarrow{P}``(\alpha \cap C`P) \quad (2)$

$\vdash . *200\cdot5 . \supset \vdash : \mathrm{Hp}(2) . \supset . \alpha \cap \overrightarrow{P}{}^{\backprime}x = \Lambda .$

$[*37\cdot462] \qquad \supset . x \sim\epsilon \breve{P}{}^{\backprime\backprime}\alpha \qquad (3)$

$\vdash . (2) . (3) . *202\cdot505 . \supset \vdash : \mathrm{Hp}(2) . x \epsilon C^{\backprime}P . \supset . x \epsilon \alpha \cap C^{\backprime}P - \breve{P}{}^{\backprime\backprime}\alpha .$

$[*205\cdot3\cdot11] \qquad \supset . x = \min_P{}^{\backprime}\alpha \qquad (4)$

$\vdash . (1) . (4) . \supset \vdash . \mathrm{Prop}$

***205·68.** $\vdash : \breve{P}{}^{\backprime\backprime}\alpha \subset \alpha . \supset . \overrightarrow{\min}_P{}^{\backprime}\alpha = \overrightarrow{\min}(P_{\mathrm{po}}){}^{\backprime}\alpha$

Dem.

$\vdash . *91\cdot711 . \supset \vdash : \mathrm{Hp} . \supset . \breve{P}_{\mathrm{po}}{}^{\backprime\backprime}\alpha = \breve{P}{}^{\backprime\backprime}\alpha .$

$[*205\cdot11] \qquad \supset . \overrightarrow{\min}(P_{\mathrm{po}}){}^{\backprime}\alpha = \overrightarrow{\min}_P{}^{\backprime}\alpha : \supset \vdash . \mathrm{Prop}$

***205·681.** $\vdash : P_{\mathrm{po}} \epsilon \mathrm{connex} . \breve{P}{}^{\backprime\backprime}\alpha \subset \alpha . \supset . \overrightarrow{\min}_P{}^{\backprime}\alpha \epsilon 0 \cup 1 \quad [*205\cdot68\cdot3]$

***205·7.** $\vdash : \exists ! \overrightarrow{\max}_P{}^{\backprime}P{}^{\backprime\backprime}\alpha . \supset . \exists ! \overrightarrow{\max}_P{}^{\backprime}\alpha$

Dem.

$\vdash . *37\cdot2\cdot265 . \quad \supset \vdash : \alpha \cap C^{\backprime}P \subset P{}^{\backprime\backprime}\alpha . \supset . P{}^{\backprime\backprime}\alpha \subset P{}^{\backprime\backprime}P{}^{\backprime\backprime}\alpha \qquad (1)$

$\vdash . (1) . \mathrm{Transp} . \supset \vdash : \exists ! P{}^{\backprime\backprime}\alpha - P{}^{\backprime\backprime}P{}^{\backprime\backprime}\alpha . \supset . \exists ! \alpha \cap C^{\backprime}P - P{}^{\backprime\backprime}\alpha \qquad (2)$

$\vdash . (2) . *205\cdot111 . \supset \vdash . \mathrm{Prop}$

***205·71.** $\vdash : P \epsilon \mathrm{connex} . \exists ! \overrightarrow{\max}_P{}^{\backprime}P{}^{\backprime\backprime}\alpha . \supset . \max_P{}^{\backprime}P{}^{\backprime\backprime}\alpha \, (P \dot{-} P^2) \max_P{}^{\backprime}\alpha$

Dem.

$\vdash . *205\cdot7\cdot3 . \supset \vdash : \mathrm{Hp} . \supset . E ! \max_P{}^{\backprime}P{}^{\backprime\backprime}\alpha . E ! \max_P{}^{\backprime}\alpha . \qquad (1)$

$[*205\cdot101] \qquad \supset . \max_P{}^{\backprime}P{}^{\backprime\backprime}\alpha \epsilon P{}^{\backprime\backprime}\alpha \qquad (2)$

$\vdash . (1) . *205\cdot101 . \supset \vdash :. \mathrm{Hp} . \supset : \max_P{}^{\backprime}P{}^{\backprime\backprime}\alpha \sim\epsilon P{}^{\backprime\backprime}P{}^{\backprime\backprime}\alpha :$

$[*37\cdot39] \qquad \supset : y \epsilon \alpha . \supset_y . \sim(\max_P{}^{\backprime}P{}^{\backprime\backprime}\alpha \, P^2 y) :$

$[(1)] \qquad \supset : \sim(\max_P{}^{\backprime}P{}^{\backprime\backprime}\alpha \, P^2 \max_P{}^{\backprime}\alpha) : \qquad (3)$

$[*34\cdot5 . \mathrm{Transp}] \qquad \supset : zP \max_P{}^{\backprime}\alpha . \supset . \sim(\max_P{}^{\backprime}P{}^{\backprime\backprime}\alpha \, Pz) :$

$[*205\cdot21] \qquad \supset : z \epsilon \alpha - \iota^{\backprime}\max_P{}^{\backprime}\alpha . \supset . \sim(\max_P{}^{\backprime}P{}^{\backprime\backprime}\alpha \, Pz) \qquad (4)$

$\vdash . (2) . *37\cdot1 . \supset \vdash : \mathrm{Hp} . \supset . (\exists z) . z \epsilon \alpha . \max_P{}^{\backprime}P{}^{\backprime\backprime}\alpha \, Pz \qquad (5)$

$\vdash . (4) . (5) . \quad \supset \vdash : \mathrm{Hp} . \supset . \max_P{}^{\backprime}P{}^{\backprime\backprime}\alpha \, P \max_P{}^{\backprime}\alpha \qquad (6)$

$\vdash . (3) . (6) . \supset \vdash . \mathrm{Prop}$

***205·72.** $\vdash : P \epsilon \mathrm{connex} . P \Subset P^2 . \supset . \sim\exists ! \overrightarrow{\max}_P{}^{\backprime}P{}^{\backprime\backprime}\alpha \quad [*205\cdot71 . \mathrm{Transp}]$

***205·73.** $\vdash : P \epsilon \mathrm{connex} . \min_P{}^{\backprime}\gamma = \max_P{}^{\backprime}\gamma . \supset . \gamma \cap C^{\backprime}P \epsilon 1 . \gamma \cap C^{\backprime}P = \iota^{\backprime}\min_P{}^{\backprime}\gamma$

Dem.

$\vdash . *205\cdot21 . \quad \supset \vdash :. \mathrm{Hp} . \supset : x \epsilon \gamma \cap C^{\backprime}P - \iota^{\backprime}\min_P{}^{\backprime}\gamma . \supset . \max_P{}^{\backprime}\gamma \, Px .$

$[*37\cdot1] \qquad \supset . \max_P{}^{\backprime}\gamma \epsilon P{}^{\backprime\backprime}\gamma \qquad (1)$

$\vdash . *205\cdot111 . \supset \vdash : \mathrm{Hp} . \supset . \max_P{}^{\backprime}\gamma \sim\epsilon P{}^{\backprime\backprime}\gamma \qquad (2)$

$\vdash . (2) . (1) . \mathrm{Transp} . \supset \vdash : \mathrm{Hp} . \supset . \gamma \cap C^{\backprime}P - \iota^{\backprime}\min_P{}^{\backprime}\gamma = \Lambda .$

$[*205\cdot11] \qquad \supset . \gamma \cap C^{\backprime}P = \iota^{\backprime}\min_P{}^{\backprime}\gamma : \supset \vdash . \mathrm{Prop}$

***205·731.** $\vdash :. P \in \text{connex} \cap \text{Rl}\text{‘}J . \supset : \min_P\text{‘}\gamma = \max_P\text{‘}\gamma . \equiv . \gamma \cap C\text{‘}P \in 1$ [*205·17·73]

***205·732.** $\vdash : P \in \text{connex} . \gamma \cap C\text{‘}P \sim\in 1 . E ! \min_P\text{‘}\gamma . E ! \max_P\text{‘}\gamma . \supset . \min_P\text{‘}\gamma \, P \max_P\text{‘}\gamma$

Dem.

$\vdash . *205\cdot73 . \text{Transp} . \supset \vdash : \text{Hp} . \supset . \max_P\text{‘}\alpha \neq \min_P\text{‘}\alpha .$

$[*205\cdot21] \qquad \supset . \min_P\text{‘}\alpha \, P \max_P\text{‘}\alpha : \supset \vdash . \text{Prop}$

The following propositions lead up to *205·75, which shows that the minimum of a class belongs to $D\text{‘}P$ unless the part of the class contained in $C\text{‘}P$ is $\iota\text{‘}B\text{‘}\breve{P}$.

***205·74.** $\vdash : \alpha \cap C\text{‘}P \subset \overrightarrow{B}\text{‘}\breve{P} . \supset . \overrightarrow{\min}_P\text{‘}\alpha = \alpha \cap C\text{‘}P$

Dem.

$\vdash . *93\cdot101 . \supset \vdash : \text{Hp} . \supset . \alpha \cap D\text{‘}P = \Lambda .$

$[*37\cdot261\cdot29] \qquad \supset . \breve{P}\text{‘‘}\alpha = \Lambda .$

$[*205\cdot11] \qquad \supset . \overrightarrow{\min}_P\text{‘}\alpha = \alpha \cap C\text{‘}P : \supset \vdash . \text{Prop}$

***205·741.** $\vdash : P \in \text{connex} . \alpha \cap C\text{‘}P \sim\in 1 . \supset . \overrightarrow{\min}_P\text{‘}\alpha \subset D\text{‘}P$

Dem.

$\vdash . *205\cdot21 . \supset \vdash : P \in \text{connex} . y = \min_P\text{‘}\alpha . z \in \alpha \cap C\text{‘}P - \iota\text{‘}y . \supset . yPz :$

$[*205\cdot3] \qquad \supset \vdash : P \in \text{connex} . y \in \overrightarrow{\min}_P\text{‘}\alpha . z \in \alpha \cap C\text{‘}P - \iota\text{‘}y . \supset . yPz :$

$[*33\cdot13] \qquad \supset \vdash : P \in \text{connex} . y \in \overrightarrow{\min}_P\text{‘}\alpha . \exists ! \alpha \cap C\text{‘}P - \iota\text{‘}y . \supset . y \in D\text{‘}P :$

$[*52\cdot181] \qquad \supset \vdash : P \in \text{connex} . \alpha \cap C\text{‘}P \sim\in 1 . \supset . \overrightarrow{\min}_P\text{‘}\alpha \subset D\text{‘}P : \supset \vdash . \text{Prop}$

***205·742.** $\vdash :. P \in \text{connex} . \supset : \exists ! \overrightarrow{\min}_P\text{‘}\alpha - D\text{‘}P . \equiv . \alpha \cap C\text{‘}P = \iota\text{‘}B\text{‘}\breve{P}$

Dem.

$\vdash . *205\cdot74 . \supset \vdash : \alpha \cap C\text{‘}P = \iota\text{‘}B\text{‘}\breve{P} . \supset . \overrightarrow{\min}_P\text{‘}\alpha = \iota\text{‘}B\text{‘}\breve{P} .$

$[*93\cdot101] \qquad \supset . \exists ! \min_P\text{‘}\alpha - D\text{‘}P \qquad (1)$

$\vdash . *205\cdot741 . \supset \vdash : \text{Hp} . \exists ! \overrightarrow{\min}_P\text{‘}\alpha - D\text{‘}P . \supset . \alpha \cap C\text{‘}P \in 1 \qquad (2)$

$\vdash . *205\cdot11 . \supset \vdash : \exists ! \overrightarrow{\min}_P\text{‘}\alpha - D\text{‘}P . \supset . \exists ! \alpha \cap C\text{‘}P - D\text{‘}P .$

$[*93\cdot103] \qquad \supset . \exists ! \alpha \cap \overrightarrow{B}\text{‘}\breve{P} \qquad (3)$

$\vdash . (2) . (3) . *202\cdot52 . \supset \vdash : \text{Hp} . \exists ! \overrightarrow{\min}_P\text{‘}\alpha - D\text{‘}P . \supset . \alpha \cap C\text{‘}P = \iota\text{‘}B\text{‘}\breve{P} \qquad (4)$

$\vdash . (1) . (4) . \supset \vdash . \text{Prop}$

***205·75.** $\vdash :. P \in \text{connex} . \supset : \sim(\alpha \cap C\text{‘}P = \iota\text{‘}B\text{‘}\breve{P}) . \equiv . \overrightarrow{\min}_P\text{‘}\alpha \subset D\text{‘}P$ [*205·742]

Observe that $\sim(\alpha \cap C\text{‘}P = \iota\text{‘}B\text{‘}\breve{P})$ is not in general equivalent to $\alpha \cap C\text{‘}P \neq \iota\text{‘}B\text{‘}\breve{P}$, since the latter implies $E ! B\text{‘}\breve{P}$, while the former does not.

The following proposition is important.

$*205{\cdot}8.\quad \vdash : S \epsilon P \,\overline{\text{smor}}\, Q . \supset . \overrightarrow{\min}_P{}^{\backprime}\alpha = S^{\backprime\backprime}\overrightarrow{\min}_Q{}^{\backprime}\breve{S}^{\backprime\backprime}\alpha$

Dem.

$\vdash . *205{\cdot}11 . \supset \vdash . S^{\backprime\backprime}\overrightarrow{\min}_Q{}^{\backprime}\breve{S}^{\backprime\backprime}\alpha = S^{\backprime\backprime}\{\breve{S}^{\backprime\backprime}\alpha \cap C^{\backprime}Q - \breve{Q}^{\backprime\backprime}\breve{S}^{\backprime\backprime}\alpha\}$ (1)

$\vdash . *151{\cdot}11 . \supset \vdash : \text{Hp} . \supset . \breve{S}^{\backprime\backprime}\alpha \subset C^{\backprime}Q$ (2)

$\vdash . (1) . (2) . \supset \vdash : \text{Hp} . \supset . S^{\backprime\backprime}\overrightarrow{\min}_Q{}^{\backprime}\breve{S}^{\backprime\backprime}\alpha = S^{\backprime\backprime}\{\breve{S}^{\backprime\backprime}\alpha - \breve{Q}^{\backprime\backprime}\breve{S}^{\backprime\backprime}\alpha\}$

$[*71{\cdot}381] \qquad = S^{\backprime\backprime}\breve{S}^{\backprime\backprime}\alpha - S^{\backprime\backprime}\breve{Q}^{\backprime\backprime}\breve{S}^{\backprime\backprime}\alpha$

$[*72{\cdot}5 . *150{\cdot}23] \qquad = \alpha \cap C^{\backprime}P - \breve{P}^{\backprime\backprime}\alpha$

$[*205{\cdot}11] \qquad = \overrightarrow{\min}_P{}^{\backprime}\alpha : \supset \vdash . \text{Prop}$

$*205{\cdot}81.\quad \vdash :. S \epsilon P \,\overline{\text{smor}}\, Q . \supset : \text{E} ! \min_P{}^{\backprime}\alpha . \equiv . \text{E} ! \min_Q{}^{\backprime}\breve{S}^{\backprime\backprime}\alpha$

Dem.

$\vdash . *205{\cdot}8 . *73{\cdot}22 . \supset \vdash :. \text{Hp} . \supset : \overrightarrow{\min}_P{}^{\backprime}\alpha \,\text{sm}\, \overrightarrow{\min}_Q{}^{\backprime}\breve{S}^{\backprime\backprime}\alpha :$

$[*73{\cdot}44] \qquad \supset : \overrightarrow{\min}_P{}^{\backprime}\alpha \epsilon 1 . \equiv . \overrightarrow{\min}_Q{}^{\backprime}\breve{S}^{\backprime\backprime}\alpha \epsilon 1 :$

$[*53{\cdot}3] \qquad \supset : \text{E} ! \min_P{}^{\backprime}\alpha . \equiv . \text{E} ! \min_Q{}^{\backprime}\breve{S}^{\backprime\backprime}\alpha :. \supset \vdash . \text{Prop}$

$*205{\cdot}82.\quad \vdash : S \epsilon P \,\overline{\text{smor}}\, Q . \text{E} ! \min_P{}^{\backprime}\alpha . \supset . \min_P{}^{\backprime}\alpha = S^{\backprime}\min_Q{}^{\backprime}\breve{S}^{\backprime\backprime}\alpha$

$[*53{\cdot}31 . *205{\cdot}8{\cdot}81]$

The two following propositions are used in $*251{\cdot}13$.

$*205{\cdot}83.\quad \vdash : z \sim\epsilon C^{\backprime}P . \exists ! C^{\backprime}P \cap \alpha . \supset . \overrightarrow{\min}_P{}^{\backprime}\alpha = \overrightarrow{\min}\,(P \nrightarrow z)^{\backprime}\alpha$

Dem.

$\vdash . *161{\cdot}1 . \supset \vdash : \text{Hp} . \supset . \{\text{Cnv}^{\backprime}(P \nrightarrow z)\}^{\backprime\backprime}\alpha = \breve{P}^{\backprime\backprime}\alpha \cup \iota^{\backprime}z .$

$[*161{\cdot}14{\cdot}2 . *24{\cdot}495] \quad \supset . \alpha \cap C^{\backprime}(P \nrightarrow z) - \{\text{Cnv}^{\backprime}(P \nrightarrow z)\}^{\backprime\backprime}\alpha = \alpha \cap C^{\backprime}P - \breve{P}^{\backprime\backprime}\alpha .$

$[*205{\cdot}11] \quad \supset . \overrightarrow{\min}\,(P \nrightarrow z)^{\backprime}\alpha = \overrightarrow{\min}_P{}^{\backprime}\alpha : \supset \vdash . \text{Prop}$

$*205{\cdot}831.\quad \vdash : z \sim\epsilon C^{\backprime}P . C^{\backprime}(P \nrightarrow z) \cap \alpha = \iota^{\backprime}z . \supset . \overrightarrow{\min}\,(P \nrightarrow z)^{\backprime}\alpha = \iota^{\backprime}z$

Dem.

$\vdash . *161{\cdot}11 . \quad \supset \vdash :. \text{Hp} . \supset : x \epsilon \alpha . \supset_x . \sim\{x (P \nrightarrow z) z\} :$

$[*37{\cdot}1 . \text{Transp}] \qquad \supset : z \sim\epsilon \{\text{Cnv}^{\backprime}(P \nrightarrow z)\}^{\backprime\backprime}\alpha$ (1)

$\vdash . (1) . *22{\cdot}621 . \supset \vdash : \text{Hp} . \supset . \iota^{\backprime}z = C^{\backprime}(P \nrightarrow z) \cap \alpha - \{\text{Cnv}^{\backprime}(P \nrightarrow z)\}^{\backprime\backprime}\alpha$

$[*205{\cdot}11] \qquad = \overrightarrow{\min}\,(P \nrightarrow z)^{\backprime\backprime}\alpha : \supset \vdash . \text{Prop}$

The two following propositions are used in $*251{\cdot}14$.

$*205{\cdot}832.\quad \vdash : z \sim\epsilon C^{\backprime}P . z \sim\epsilon \alpha . \supset . \overrightarrow{\max}_P{}^{\backprime}\alpha = \overrightarrow{\max}\,(P \nrightarrow z)^{\backprime}\alpha$

Dem.

$\vdash . *205{\cdot}111 . *161{\cdot}2 . \supset \vdash : P = \dot\Lambda . \supset . \overrightarrow{\max}_P{}^{\backprime}\alpha = \Lambda . \overrightarrow{\max}\,(P \nrightarrow z)^{\backprime}\alpha = \Lambda$ (1)

$\vdash . *205{\cdot}111 . *161{\cdot}11{\cdot}14 . \supset$

$\vdash : \text{Hp} . \exists ! C^{\backprime}P \cap \alpha . \supset . \overrightarrow{\max}\,(P \nrightarrow z)^{\backprime}\alpha = \alpha \cap (C^{\backprime}P \cup \iota^{\backprime}z) - (\breve{P}^{\backprime\backprime}\alpha \cup \iota^{\backprime}z)$

$[*24{\cdot}495 . *205{\cdot}111] \qquad = \overrightarrow{\max}_P{}^{\backprime}\alpha$ (2)

$\vdash . *161 \cdot 14 . *205 \cdot 151 \cdot 161 . \supset$

$\vdash : \text{Hp} . \dot{\exists} ! P . C'P \cap \alpha = \Lambda . \supset . \overrightarrow{\max}_P{}'\alpha = \Lambda . \overrightarrow{\max}(P \nrightarrow z)'\alpha = \Lambda$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash . \text{Prop}$

***205·833.** $\vdash : \dot{\exists} ! P . z \sim \epsilon C'P . z \epsilon \alpha . \supset . \overrightarrow{\max}(P \nrightarrow z)'\alpha = \iota'z$

Dem.

$\vdash . *161 \cdot 11 . \supset \vdash : \text{Hp} . \supset . (P \nrightarrow z)``\alpha = C'P .$

$[*161 \cdot 14 . *205 \cdot 111] \quad \supset . \overrightarrow{\max}(P \nrightarrow z)'\alpha = \alpha \cap (C'P \cup \iota'z) - C'P$

$[*22 \cdot 621 . \text{Hp}] \qquad = \iota'z : \supset \vdash . \text{Prop}$

The following proposition is used in *251·25.

***205·84.** $\vdash : C'P \cap C'Q = \Lambda . \exists ! C'P \cap \alpha . \supset . \overrightarrow{\min}(P \ddagger Q)'\alpha = \overrightarrow{\min}_P{}'\alpha$

Dem.

$\vdash . *160 \cdot 11 . \supset \vdash : \text{Hp} . \supset . \{\text{Cnv}'(P \ddagger Q)\}``\alpha = \breve{P}``\alpha \cup C'Q .$

$[*205 \cdot 11 . *160 \cdot 14] \quad \supset . \overrightarrow{\min}(P \ddagger Q)'\alpha = \alpha \cap (C'P \cup C'Q) - (\breve{P}``\alpha \cup C'Q)$

$[*24 \cdot 495] \qquad = \alpha \cap C'P - \breve{P}``\alpha$

$[*205 \cdot 11] \qquad = \overrightarrow{\min}_P{}'\alpha : \supset \vdash . \text{Prop}$

***205·841.** $\vdash : C'P \cap \alpha = \Lambda . \supset . \overrightarrow{\min}(P \ddagger Q)'\alpha = \overrightarrow{\min}_Q{}'\alpha$

Dem.

$\vdash . *160 \cdot 11 . \supset \vdash : \text{Hp} . \supset . \{\text{Cnv}'(P \ddagger Q)\}``\alpha = \breve{Q}``\alpha .$

$[*205 \cdot 11 . *160 \cdot 14] \quad \supset . \overrightarrow{\min}(P \ddagger Q)'\alpha = \alpha \cap (C'P \cup C'Q) - \breve{Q}``\alpha$

$[\text{Hp}] \qquad = \alpha \cap C'Q - \breve{Q}``\alpha$

$[*205 \cdot 11] \qquad = \overrightarrow{\min}_Q{}'\alpha : \supset \vdash . \text{Prop}$

The following proposition is used in *251·2.

***205·85.** $\vdash :. P \epsilon \text{Rel}^2\text{excl} . \supset : x \{\min(\Sigma'P)\} \alpha . \equiv . (\exists Q) . Q \min_P(\breve{F}``\alpha) . x \min_Q \alpha$

Dem.

$\vdash . *162 \cdot 12 \cdot 23 . *205 \cdot 1 . \supset \vdash :. x \{\min(\Sigma'P)\} \alpha . \equiv :$

$x \epsilon \alpha : (\exists Q) . Q \epsilon C'P . xFQ : \sim(\exists Q, y) . Q \epsilon C'P . y \epsilon \alpha . yQx :$

$\sim(\exists Q, R, y) . xFQ . RPQ . yFR . y \epsilon \alpha :$

$[*37 \cdot 105] \equiv : x \epsilon \alpha : (\exists Q) . Q \epsilon C'P . xFQ : xFQ . Q \epsilon C'P . \supset_Q . x \sim \epsilon \breve{Q}``\alpha :$

$xFQ . Q \epsilon C'P . \supset_Q . Q \sim \epsilon \breve{P}``\breve{F}``\alpha$ (1)

$\vdash . (1) . *163 \cdot 12 . *14 \cdot 26 . \supset \vdash :: \text{Hp} . \supset :. x \{\min(\Sigma'P)\} \alpha . \equiv :$

$(\exists Q) . xFQ . Q \epsilon C'P . x \epsilon \alpha - \breve{Q}``\alpha : xFQ . Q \epsilon C'P . \supset_Q . Q \epsilon \breve{F}``\alpha - \breve{P}``\breve{F}``\alpha :$

$[*163 \cdot 12 . *14 \cdot 26] \equiv : (\exists Q) . xFQ . Q \epsilon C'P . x \epsilon \alpha - \breve{Q}``\alpha . Q \epsilon \breve{F}``\alpha - \breve{P}``\breve{F}``\alpha :$

$[*205 \cdot 1] \qquad \equiv : (\exists Q) . Q \min_P(\breve{F}``\alpha) . x \min_Q \alpha :: \supset \vdash . \text{Prop}$

***205·9.** $\vdash : P \epsilon \text{connex} . \kappa \subset C\text{‘}P . \kappa \sim\epsilon 1 . \supset . \overrightarrow{\min} (P \upharpoonright \kappa)\text{‘}\alpha = \overrightarrow{\min}_P\text{‘}(\alpha \cap \kappa)$
[*205·261]

***205·91.** $\vdash : \breve{P}\text{“}\alpha \subset \alpha . P_{\text{po}} \upharpoonright \alpha \epsilon \text{connex} . \supset . \overrightarrow{\min}_P\text{‘}\alpha \epsilon 0 \cup 1$

Dem.

$\vdash . *205{\cdot}261 . \supset \vdash : \text{Hp} . \alpha \cap C\text{‘}P \sim\epsilon 1 . \supset . \overrightarrow{\min} (P_{\text{po}} \upharpoonright \alpha)\text{‘}\alpha = \overrightarrow{\min} (P_{\text{po}})\text{‘}\alpha$

[*205·68] $= \overrightarrow{\min}_P\text{‘}\alpha .$

[*205·3] $\supset . \overrightarrow{\min}_P\text{‘}\alpha \epsilon 0 \cup 1$ (1)

$\vdash . *93{\cdot}113 . *60{\cdot}371 . \supset \vdash : \alpha \cap C\text{‘}P \epsilon 1 . \supset . \overrightarrow{\min}_P\text{‘}\alpha \epsilon 0 \cup 1$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

*206. SEQUENT POINTS

Summary of *206.

A "sequent" of a class α is a minimum of the terms that come after the whole of $\alpha \cap C'P$; that is, we put

$$\overrightarrow{\text{seq}}_P{}'\alpha = \overrightarrow{\text{min}}_P{}'p'\overleftarrow{P}``(\alpha \cap C'P).$$

Thus the sequents of α are its immediate successors. If α has a maximum, the sequents are the immediate successors of the maximum; but if α has no maximum, there will be no one term of α which is immediately succeeded by a sequent of α; in this case, if α has a single sequent, the sequent is the "upper limit" of α. Whenever P is connected, and therefore whenever P is serial, every class has one sequent or none with respect to P, by *205·3.

It will be seen that the sequents of α are the same as the sequents of $\alpha \cap C'P$, and therefore that $\overrightarrow{\text{seq}}_P{}'\alpha$ depends only upon $\alpha \cap C'P$: if α has terms not belonging to $C'P$, they are irrelevant.

For the immediate predecessors of a class α, we put

$$\overrightarrow{\text{prec}}_P{}'\alpha = \overrightarrow{\text{max}}_P{}'p'\overrightarrow{P}``(\alpha \cap C'P).$$

We have $\text{prec}_P = \text{seq}\,(\breve{P})$, so that propositions about prec_P result from those about seq_P by merely writing $\breve{P}$ in place of P; they will therefore not be given in what follows.

Among the elementary properties of seq_P with which this number begins, the following are the most important:

*206·13. $\vdash . \overrightarrow{\text{seq}}_P{}'\alpha = \overrightarrow{\text{min}}_P{}'p'\overleftarrow{P}``(\alpha \cap C'P)$

This merely embodies the definition.

*206·131. $\vdash . \overrightarrow{\text{seq}}_P{}'\alpha = \overrightarrow{\text{seq}}_P{}'(\alpha \cap C'P)$

*206·134. $\vdash . \overrightarrow{\text{seq}}_P{}'\alpha = C'P \cap \hat{x}\,\{\alpha \cap C'P \subset \overrightarrow{P}'x \,.\, \overrightarrow{P}'x \subset -p'\overleftarrow{P}``(\alpha \cap C'P)\}$

*206·14. $\vdash : \alpha \cap C'P = \Lambda \,.\, \supset . \overrightarrow{\text{seq}}_P{}'\alpha = \overrightarrow{B}'P$

Thus if P has a first term, this is the sequent of the null class, or of any other class which has no members in common with $C'P$.

*206·16. $\vdash : P \in \text{connex} \,.\, \supset . \overrightarrow{\text{seq}}_P{}'\alpha \in 0 \cup 1$

This follows at once from *205·3. It leads to

*206·161. $\vdash : P \in \text{connex} \,.\, \supset . \text{seq}_P \in 1 \rightarrow \text{Cls}$

Thus if P is a connected relation, no class has more than one sequent. This is not in general the case with relations which are not connected, even where the idea of sequents is quite naturally applicable. Take, *e.g.*, the relation of descendant to ancestor, and let α be the class of monarchs of England. Then $\overrightarrow{\mathrm{seq}}_P{}^{\boldsymbol{\prime}}\alpha$ will be such parents of monarchs as were not themselves monarchs.

***206·171.** $\vdash : P \epsilon \mathrm{connex} \,.\, P^2 \subseteq J \,.\, \supset .$

$$\overrightarrow{\mathrm{seq}}_P{}^{\boldsymbol{\prime}}\alpha = C{}^{\boldsymbol{\prime}}P \cap \hat{x} \{\alpha \cap C{}^{\boldsymbol{\prime}}P \subset \overrightarrow{P}{}^{\boldsymbol{\prime}}x \,.\, \overrightarrow{P}{}^{\boldsymbol{\prime}}x \subset (\alpha \cap C{}^{\boldsymbol{\prime}}P) \cup P{}^{\boldsymbol{\prime\prime}}\alpha\}$$

This proposition states that x is a sequent of α if the whole of $\alpha \cap C{}^{\boldsymbol{\prime}}P$ precedes x, but every term that precedes x either belongs to α or precedes some term of α. When P is a series and α has no maximum, we have

$$\overrightarrow{\mathrm{seq}}_P{}^{\boldsymbol{\prime}}\alpha = C{}^{\boldsymbol{\prime}}P \cap \hat{x}(\overrightarrow{P}{}^{\boldsymbol{\prime}}x = P{}^{\boldsymbol{\prime\prime}}\alpha) \quad (*206{\cdot}174),$$

i.e. the sequent of α, if any, is a term whose predecessors are identical with the predecessors of members of α. This is the case of a *limit* (cf. *207).

We have next a set of propositions (*206·211·28) concerned with $\overrightarrow{P}{}^{\boldsymbol{\prime}}\mathrm{seq}_P{}^{\boldsymbol{\prime}}\alpha$ and $\overleftarrow{P}{}^{\boldsymbol{\prime}}\mathrm{seq}_P{}^{\boldsymbol{\prime}}\alpha$. When P is transitive and connected, and α is an existent class contained in $C{}^{\boldsymbol{\prime}}P$ and having a sequent, we shall have

$$\overrightarrow{P}{}^{\boldsymbol{\prime}}\mathrm{seq}_P{}^{\boldsymbol{\prime}}\alpha = \alpha \cup P{}^{\boldsymbol{\prime\prime}}\alpha \,.\, \iota{}^{\boldsymbol{\prime}}\mathrm{seq}_P{}^{\boldsymbol{\prime}}\alpha \cup \overleftarrow{P}{}^{\boldsymbol{\prime}}\mathrm{seq}_P{}^{\boldsymbol{\prime}}\alpha = p{}^{\boldsymbol{\prime}}\overleftarrow{P}{}^{\boldsymbol{\prime\prime}}\alpha.$$

That is, the predecessors of the sequent are the members of α and the predecessors of members, while the sequent and its successors are the successors of the whole of α. The various parts of this statement require various parts of the hypothesis. Thus we have

***206·211.** $\vdash : \mathrm{E}\,!\, \mathrm{seq}_P{}^{\boldsymbol{\prime}}\alpha \,.\, \supset .\, \alpha \cap C{}^{\boldsymbol{\prime}}P \subset \overrightarrow{P}{}^{\boldsymbol{\prime}}\mathrm{seq}_P{}^{\boldsymbol{\prime}}\alpha$

***206·213.** $\vdash : P \epsilon \mathrm{connex} \,.\, \mathrm{E}\,!\, \mathrm{seq}_P{}^{\boldsymbol{\prime}}\alpha \,.\, \supset .\, \overrightarrow{P}{}^{\boldsymbol{\prime}}\mathrm{seq}_P{}^{\boldsymbol{\prime}}\alpha \subset (\alpha \cap C{}^{\boldsymbol{\prime}}P) \cup P{}^{\boldsymbol{\prime\prime}}\alpha$

***206·22.** $\vdash : P \epsilon \mathrm{trans} \cap \mathrm{connex} \,.\, \mathrm{E}\,!\, \mathrm{seq}_P{}^{\boldsymbol{\prime}}\alpha \,.\, \supset .$

$$\overrightarrow{P}{}^{\boldsymbol{\prime}}\mathrm{seq}_P{}^{\boldsymbol{\prime}}\alpha = (\alpha \cap C{}^{\boldsymbol{\prime}}P) \cup P{}^{\boldsymbol{\prime\prime}}\alpha = \overrightarrow{\mathrm{max}}_P{}^{\boldsymbol{\prime}}\alpha \cup P{}^{\boldsymbol{\prime\prime}}\alpha$$

***206·23.** $\vdash : P \epsilon \mathrm{trans} \cap \mathrm{connex} \,.\, \mathrm{E}\,!\, \mathrm{seq}_P{}^{\boldsymbol{\prime}}\alpha \,.\, \supset .$

$$\iota{}^{\boldsymbol{\prime}}\mathrm{seq}_P{}^{\boldsymbol{\prime}}\alpha \cup \overleftarrow{P}{}^{\boldsymbol{\prime}}\mathrm{seq}_P{}^{\boldsymbol{\prime}}\alpha = p{}^{\boldsymbol{\prime}}\overleftarrow{P}{}^{\boldsymbol{\prime\prime}}(\alpha \cap C{}^{\boldsymbol{\prime}}P) \cap C{}^{\boldsymbol{\prime}}P$$

If P is transitive, the value of $\overrightarrow{\mathrm{seq}}_P{}^{\boldsymbol{\prime}}\alpha$ is unchanged if we add to α any set of terms contained in $P{}^{\boldsymbol{\prime\prime}}\alpha$ (*206·24); thus in particular, $\overrightarrow{\mathrm{seq}}_P{}^{\boldsymbol{\prime}}(\alpha \cup P{}^{\boldsymbol{\prime\prime}}\alpha) = \overrightarrow{\mathrm{seq}}_P{}^{\boldsymbol{\prime}}\alpha$ (*206·25). Thus we can fill up any gaps in α, and take the whole series up to the end of α, without altering the sequent.

We have next a set of propositions (*206·3—·38) on the sequent of $P{}^{\boldsymbol{\prime\prime}}\alpha$, *i.e.* of the segment defined by α. If P is a series, $\mathrm{seq}_P{}^{\boldsymbol{\prime}}P{}^{\boldsymbol{\prime\prime}}\alpha$ is the maximum of α if α has a maximum, the sequent of α if α has a sequent but no maximum, and non-existent if α has neither a maximum nor a sequent (*206·35·331·36).

Our next set of propositions (*206·4—·52) concerns the sequents of unit classes, especially of ι'$\max_P$'α, and of classes of the form $\overrightarrow{P}$'x. We have

***206·4.** $\vdash : P \mathrel{\mathsf{G}} J \,.\, x \in C\text{'}P \,.\, \supset .\, x \operatorname{seq}_P \overrightarrow{P}\text{'}x$

***206·42.** $\vdash : x \in C\text{'}P \,.\, \supset .\, \overrightarrow{\operatorname{seq}_P}\text{'}\iota\text{'}x = \overleftarrow{P \dot{-} P^2}\text{'}x = \overrightarrow{\min_P}\text{'}\overleftarrow{P}\text{'}x$

whence the three following propositions:

***206·43.** $\vdash : P \in \text{trans} \cap \text{Rl}\text{'}J \,.\, x \in C\text{'}P \,.\, \supset .\, \overrightarrow{\operatorname{seq}_P}\text{'}\iota\text{'}x = \overleftarrow{P_1}\text{'}x$

***206·45.** $\vdash :.\, P \in \text{Ser} \,.\, x \in C\text{'}P \,.\, \supset : \text{E} ! \operatorname{seq}_P\text{'}\iota\text{'}x \,.\, \equiv .\, x \in \text{D}\text{'}P_1$

***206·46.** $\vdash : P \in \text{trans} \cap \text{connex} \,.\, \text{E} ! \max_P\text{'}\alpha \,.\, \supset .\, \overrightarrow{\operatorname{seq}_P}\text{'}\alpha = \overrightarrow{\operatorname{seq}_P}\text{'}\overrightarrow{\max_P}\text{'}\alpha$

From the above propositions it results that, when P is a series, any member of C'P is the sequent of the class of its predecessors, $\breve{P}_1$'x is the sequent of ι'x if either exists, and the sequent of a class which has a maximum is the immediate successor (if any) of the maximum, *i.e.*

***206·5.** $\vdash : P \in \text{trans} \cap \text{connex} \,.\, \text{E} ! \max_P\text{'}\alpha \,.\, \text{E} ! \operatorname{seq}_P\text{'}\alpha \,.\, \supset .\, \max_P\text{'}\alpha (P \dot{-} P^2) \operatorname{seq}_P\text{'}\alpha$

We then have a set of propositions (*206·53—·57) on the sequent of p'$\overrightarrow{P}$"$(\alpha \cap C\text{'}P)$, *i.e.* the sequent of the predecessors of the whole of $\alpha \cap C$'P. These propositions are specially useful in connection with "Dedekindian" series, *i.e.* series in which every class has either a maximum or a sequent (*214). These propositions all require the full hypothesis that P is a series. In this case, $\overrightarrow{\operatorname{seq}_P}\text{'}p\text{'}\overrightarrow{P}\text{"}(\alpha \cap C\text{'}P) = \overrightarrow{\min_P}\text{'}\alpha$, *i.e.* the sequent (if any) of the predecessors of the whole of $\alpha \cap C$'P is the minimum (if any) of α. Moreover by definition the maximum of p'$\overrightarrow{P}$"$(\alpha \cap C\text{'}P)$, if any, is the precedent of α. Hence α has either a minimum or a precedent if p'$\overrightarrow{P}$"$(\alpha \cap C\text{'}P)$ has either a sequent or a maximum (*206·54). Moreover the sequent and maximum of α are respectively (if they exist) the sequent and maximum of the predecessors of all the successors of the whole of $\alpha \cap C$'P (*206·551). Hence we arrive at the conclusion that the assumption that every class of the form p'$\overrightarrow{P}$"$(\alpha \cap C\text{'}P)$ has either a maximum or a sequent is equivalent both to the assumption that every class has either a maximum or a sequent (*206·56) and to the assumption that every class has either a minimum or a precedent (*206·55). It follows that these two latter assumptions are equivalent (*206·57), *i.e.* that a series is Dedekindian when, and only when, its converse is Dedekindian (*214·14).

We deal next (*206·6—·63) with correlations, showing that if two relations are correlated, the sequents of the correlates of any class are the correlates of the sequents, *i.e.*

***206·61.** $\vdash : S \in P \,\overline{\text{smor}}\, Q \,.\, \supset .\, \overrightarrow{\operatorname{seq}_P}\text{'}\alpha = S\text{"}\overrightarrow{\operatorname{seq}_Q}\text{'}\breve{S}\text{"}\alpha$

We end with a set of propositions ($*206{\cdot}7$—$\cdot732$) showing that the sequent of a class is unchanged if we remove from the class any term other than its maximum ($*206{\cdot}72$); that if a class has terms in $C\text{‘}P$, and has both a precedent and a sequent, the precedent has the relation P^2 to the sequent ($*206{\cdot}73$), and that the precedent is not identical with the sequent ($*206{\cdot}732$). These propositions are in the nature of lemmas, whose use is chiefly in the theory of stretches ($*215$).

$*206{\cdot}01.\quad \text{seq}_P = \hat{x}\hat{\alpha}\,\{x \min_P p\text{‘}\overleftarrow{P}\text{“}(\alpha \cap C\text{‘}P)\}\quad$ Df

$*206{\cdot}02.\quad \text{prec}_P = \hat{x}\hat{\alpha}\,\{x \max_P p\text{‘}\overrightarrow{P}\text{“}(\alpha \cap C\text{‘}P)\}\quad$ Df

$*206{\cdot}1.\quad \vdash : x \,\text{seq}_P\, \alpha \,.\equiv .\, x \min_P p\text{‘}\overleftarrow{P}\text{“}(\alpha \cap C\text{‘}P)\quad [(*206{\cdot}01)]$

$*206{\cdot}101.\quad \vdash .\, \text{prec}_P = \text{seq}\,(\breve{P})\qquad [*32{\cdot}241 \,.\, *33{\cdot}22 \,.\, *205{\cdot}102]$

We shall not enunciate any other propositions on prec_P (unless for some special reason), since the above proposition enables them to be immediately deduced from the corresponding propositions on seq_P.

$*206{\cdot}11.\quad \vdash : x \,\text{seq}_P\, \alpha \,.\equiv .\, x \in p\text{‘}\overleftarrow{P}\text{“}(\alpha \cap C\text{‘}P) \cap C\text{‘}P - \breve{P}\text{“}p\text{‘}\overleftarrow{P}\text{“}(\alpha \cap C\text{‘}P)$
$[*206{\cdot}1 \,.\, *205{\cdot}1]$

Observe that when $\alpha \cap C\text{‘}P$ is not null, $p\text{‘}\overleftarrow{P}\text{“}(\alpha \cap C\text{‘}P) \subset C\text{‘}P$, so that the factor $C\text{‘}P$ on the right is unnecessary; but when $\alpha \cap C\text{‘}P = \Lambda$, we have $p\text{‘}\overleftarrow{P}\text{“}(\alpha \cap C\text{‘}P) = \text{V}$, so that the factor $C\text{‘}P$ becomes relevant. Owing to this factor, the sequents of Λ are $\overrightarrow{B}\text{‘}P$, so that if $B\text{‘}P$ exists, $B\text{‘}P$ is the sequent of Λ.

$*206{\cdot}12.\quad \vdash :: x \,\text{seq}_P\, \alpha \,. \equiv :.\, y \in \alpha \cap C\text{‘}P \,.\supset_y .\, yPx : x \in C\text{‘}P :.$
$y \in \alpha \cap C\text{‘}P \,.\supset_y .\, yPz : \supset_z .\, \sim(zPx)\quad [*206{\cdot}11 \,.\, *40{\cdot}53 \,.\, *37{\cdot}105]$

$*206{\cdot}13.\quad \vdash .\, \overrightarrow{\text{seq}}_P\text{‘}\alpha = \overrightarrow{\text{min}}_P\text{‘}p\text{‘}\overleftarrow{P}\text{“}(\alpha \cap C\text{‘}P)\qquad [*206{\cdot}1]$

$*206{\cdot}131.\quad \vdash .\, \overrightarrow{\text{seq}}_P\text{‘}\alpha = \overrightarrow{\text{seq}}_P\text{‘}(\alpha \cap C\text{‘}P)\qquad [*206{\cdot}13 \,.\, *22{\cdot}43{\cdot}621]$

$*206{\cdot}132.\quad \vdash .\, \overrightarrow{\text{seq}}_P\text{‘}\alpha = p\text{‘}\overleftarrow{P}\text{“}(\alpha \cap C\text{‘}P) \cap C\text{‘}P - \breve{P}\text{“}p\text{‘}\overleftarrow{P}\text{“}(\alpha \cap C\text{‘}P)\quad [*206{\cdot}11]$

$*206{\cdot}133.\quad \vdash : x \,\text{seq}_P\, \alpha \,.\supset .\, \sim(xPx)\quad [*205{\cdot}194 \,.\, *206{\cdot}13]$

$*206{\cdot}134.\quad \vdash .\, \overrightarrow{\text{seq}}_P\text{‘}\alpha = C\text{‘}P \cap \hat{x}\,\{\alpha \cap C\text{‘}P \subset \overrightarrow{P}\text{‘}x \,.\, \overrightarrow{P}\text{‘}x \subset -p\text{‘}\overleftarrow{P}\text{“}(\alpha \cap C\text{‘}P)\}$

Dem.

$\vdash .\, *206{\cdot}12 \,.\, *32{\cdot}18 \,.\supset$

$\vdash .\, \overrightarrow{\text{seq}}_P\text{‘}\alpha = C\text{‘}P \cap \hat{x}\,(\alpha \cap C\text{‘}P \subset \overrightarrow{P}\text{‘}x) \cap \hat{x}\,\{y \in \alpha \cap C\text{‘}P \,.\supset_y .\, yPz : \supset_z .\, \sim(zPx)\}$

$[*40{\cdot}53]\quad = C\text{‘}P \cap \hat{x}\,(\alpha \cap C\text{‘}P \subset \overrightarrow{P}\text{‘}x) \cap \hat{x}\,\{z \in p\text{‘}\overleftarrow{P}\text{“}(\alpha \cap C\text{‘}P) \,.\supset_z .\, \sim(zPx)\}$

[Transp.$*32{\cdot}18$]$\cdot$

$$= C\text{‘}P \cap \hat{x}\,(\alpha \cap C\text{‘}P \subset \overrightarrow{P}\text{‘}x) \cap \hat{x}\,\{\overrightarrow{P}\text{‘}x \subset -p\text{‘}\overleftarrow{P}\text{“}(\alpha \cap C\text{‘}P)\} \,.\supset \vdash .\,\text{Prop}$$

This formula for $\overrightarrow{\text{seq}}_P\text{‘}\alpha$ is usually more convenient than $*206{\cdot}13{\cdot}132$.

***206·14.** $\vdash : \alpha \cap C`P = \Lambda . \supset . \overrightarrow{\mathrm{seq}}_P`\alpha = \overrightarrow{B}`P$

Dem.

$\vdash . *206\cdot13 . *40\cdot2 . \supset \vdash : \mathrm{Hp} . \supset . \overrightarrow{\mathrm{seq}}_P`\alpha = \overrightarrow{\mathrm{min}}_P`V$

$[*205\cdot15 . *24\cdot26] \qquad = \overrightarrow{\mathrm{min}}_P`C`P$

$[*205\cdot12] \qquad = \overrightarrow{B}`P : \supset \vdash . \mathrm{Prop}$

***206·141.** $\vdash : \exists ! \alpha \cap C`P . \supset . \overrightarrow{\mathrm{seq}}_P`\alpha = p`\overleftarrow{P}``(\alpha \cap C`P) - \breve{P}``p`\overleftarrow{P}``(\alpha \cap C`P)$

Dem.

$\vdash . *40\cdot62 . \supset \vdash : \mathrm{Hp} . \supset . p`\overleftarrow{P}``(\alpha \cap C`P) \subset C`P \quad (1)$

$\vdash . (1) . *206\cdot132 . \supset \vdash . \mathrm{Prop}$

***206·142.** $\vdash : \exists ! \alpha \cap C`P . \supset . \overrightarrow{\mathrm{seq}}_P`\alpha \subset \breve{P}``\alpha \quad [*40\cdot61 . *206\cdot141]$

***206·143.** $\vdash : \alpha \subset C`P . \supset . \overrightarrow{\mathrm{seq}}_P`\alpha = p`\overleftarrow{P}``\alpha \cap C`P - \breve{P}``p`\overleftarrow{P}``\alpha$
$[*206\cdot132 . *22\cdot621]$

***206·144.** $\vdash : \exists ! \overrightarrow{\mathrm{seq}}_P`\alpha . \supset . \exists ! p`\overleftarrow{P}``(\alpha \cap C`P) \quad [*206\cdot132]$

***206·15.** $\vdash : \alpha \subset C`P . \exists ! \alpha . \supset . \overrightarrow{\mathrm{seq}}_P`\alpha = p`\overleftarrow{P}``\alpha - \breve{P}``p`\overleftarrow{P}``\alpha$
$[*206\cdot141 . *22\cdot621]$

***206·16.** $\vdash : P \in \mathrm{connex} . \supset . \overrightarrow{\mathrm{seq}}_P`\alpha \in 0 \cup 1 \quad [*205\cdot3 . *206\cdot13]$

***206·161.** $\vdash : P \in \mathrm{connex} . \supset . \mathrm{seq}_P \in 1 \rightarrow \mathrm{Cls} \quad [*206\cdot16 . *71\cdot12]$

Thus in a series, or in any connected relation, no class has more than one sequent.

***206·17.** $\vdash :. x \, \mathrm{seq}_P \, \alpha . \equiv : y \in \alpha \cap C`P . \supset_y . yPx : x \in C`P :$
$yPx . \supset_y . (\exists z) . z \in \alpha \cap C`P . \sim(zPy)$

Dem.

$\vdash . *37\cdot462 . *206\cdot11 . \supset$

$\vdash :. x \, \mathrm{seq}_P \, \alpha . \equiv : x \in p`\overleftarrow{P}``(\alpha \cap C`P) \cap C`P . \overrightarrow{P}`x \subset -p`\overleftarrow{P}``(\alpha \cap C`P) :$

$[*40\cdot53] \qquad \equiv : y \in \alpha \cap C`P . \supset_y . yPx : x \in C`P :$
$yPx . \supset_y . (\exists z) . z \in \alpha \cap C`P . \sim(zPy) :. \supset \vdash . \mathrm{Prop}$

The following propositions give simplified formulae for $\overrightarrow{\mathrm{seq}}_P`\alpha$ in various special cases.

***206·171.** $\vdash : P \in \mathrm{connex} . P^2 \in J . \supset .$
$\overrightarrow{\mathrm{seq}}_P`\alpha = C`P \cap \hat{x} \{\alpha \cap C`P \subset \overrightarrow{P}`x . \overrightarrow{P}`x \subset (\alpha \cap C`P) \cup P``\alpha\}$

Dem.

$\vdash . *206\cdot134 . *33\cdot152 . \supset$

$\vdash . \overrightarrow{\mathrm{seq}}_P`\alpha = C`P \cap \hat{x} \{\alpha \cap C`P \subset \overrightarrow{P}`x . \overrightarrow{P}`x \subset C`P - p`\overleftarrow{P}``(\alpha \cap C`P)\} \quad (1)$

$\vdash . (1) . *202\cdot503 . \supset \vdash . \mathrm{Prop}$

$*206\cdot172.\ \vdash : P \in \text{connex} \,.\, P^2 \Subset J \,.\, P\text{‘‘}\alpha \subset \alpha \,.\, \supset .$
$\overrightarrow{\text{seq}}_P\text{‘}\alpha = C\text{‘}P \cap \hat{x}(\alpha \cap C\text{‘}P = \overrightarrow{P}\text{‘}x)$ [$*206\cdot171 \,.\, *22\cdot62$]

$*206\cdot173.\ \vdash : P \in \text{connex} \,.\, P^2 \Subset J \,.\, \alpha \cap C\text{‘}P \subset P\text{‘‘}\alpha \,.\, \supset .$
$\overrightarrow{\text{seq}}_P\text{‘}\alpha = C\text{‘}P \cap \hat{x}\{\alpha \cap C\text{‘}P \subset \overrightarrow{P}\text{‘}x \,.\, \overrightarrow{P}\text{‘}x \subset P\text{‘‘}\alpha\}$
[$*206\cdot171 \,.\, *22\cdot62$]

$*206\cdot174.\ \vdash : P \in \text{Ser} \,.\, \alpha \cap C\text{‘}P \subset P\text{‘‘}\alpha \,.\, \supset .\, \overrightarrow{\text{seq}}_P\text{‘}\alpha = C\text{‘}P \cap \hat{x}(\overrightarrow{P}\text{‘}x = P\text{‘‘}\alpha)$

Dem.

$\vdash .\, *13\cdot12 \,.\, *22\cdot42 \,.\, \supset \vdash :.\, \text{Hp} \,.\, \supset :$
$\overrightarrow{P}\text{‘}x = P\text{‘‘}\alpha \,.\, \supset .\, \alpha \cap C\text{‘}P \subset \overrightarrow{P}\text{‘}x \,.\, \overrightarrow{P}\text{‘}x \subset P\text{‘‘}\alpha$ (1)

$\vdash .\, *37\cdot265 \,.$ $\supset \vdash : \alpha \cap C\text{‘}P \subset \overrightarrow{P}\text{‘}x \,.\, \supset .\, P\text{‘‘}\alpha \subset P\text{‘‘}\overrightarrow{P}\text{‘}x :$

[$*201\cdot501$] $\supset \vdash :.\, \text{Hp} \,.\, \supset : \alpha \cap C\text{‘}P \subset \overrightarrow{P}\text{‘}x \,.\, \supset .\, P\text{‘‘}\alpha \subset \overrightarrow{P}\text{‘}x :$

[Fact] $\supset : \alpha \cap C\text{‘}P \subset \overrightarrow{P}\text{‘}x \,.\, \overrightarrow{P}\text{‘}x \subset P\text{‘‘}\alpha \,.\, \supset .\, P\text{‘‘}\alpha = \overrightarrow{P}\text{‘}x$ (2)

$\vdash .\, (1) \,.\, (2) \,.\, *206\cdot173 \,.\, \supset \vdash .\, \text{Prop}$

The propositions $*206\cdot173\cdot174$ deal with *limits.* When a class α has no maximum, *i.e.* when $\alpha \cap C\text{‘}P \subset P\text{‘‘}\alpha$, its sequent (if any) is called its *limit.* By the above propositions, the limit is a term x such that $\alpha \cap C\text{‘}P$ precedes x, but every predecessor of x precedes some member of $\alpha \cap C\text{‘}P$ ($*206\cdot173$); it is also a term x whose predecessors are identical with the predecessors of α ($*206\cdot174$). The subject of limits will be explicitly treated in $*207$.

$*206\cdot18.\ \vdash .\, \overrightarrow{\text{seq}}_P\text{‘}\alpha \subset C\text{‘}P$ [$*206\cdot132$]

$*206\cdot181.\ \vdash : \exists ! \,\alpha \cap C\text{‘}P \,.\, \supset .\, \overrightarrow{\text{seq}}_P\text{‘}\alpha \subset \text{Cl}\text{‘}P$ [$*206\cdot142 \,.\, *37\cdot16$]

$*206\cdot2.\ \vdash .\, \overrightarrow{\text{seq}}_P\text{‘}\alpha \subset -\alpha$

Dem.

$\vdash .\, *40\cdot68 \,.\, \text{Transp} \,.\, \supset \vdash .\, p\text{‘}\overleftarrow{P}\text{‘‘}(\alpha \cap C\text{‘}P) - \breve{P}\text{‘‘}p\text{‘}\overleftarrow{P}\text{‘‘}(\alpha \cap C\text{‘}P) \subset -(\alpha \cap C\text{‘}P)$ (1)

$\vdash .\, (1) \,.\, *206\cdot132 \,.\, \supset \vdash .\, \text{Prop}$

$*206\cdot21.\ \vdash : P^2 \Subset J \,.\, \supset .\, \overrightarrow{\text{seq}}_P\text{‘}\alpha \subset -P\text{‘‘}\alpha$ [$*200\cdot53 \,.\, *206\cdot132$]

$*206\cdot211.\ \vdash : \text{E} ! \,\text{seq}_P\text{‘}\alpha \,.\, \supset .\, \alpha \cap C\text{‘}P \subset \overrightarrow{P}\text{‘}\text{seq}_P\text{‘}\alpha$

Dem.

$\vdash .\, *206\cdot17 \,.\, \supset \vdash :.\, \text{Hp} \,.\, \supset : y \in \alpha \cap C\text{‘}P \,.\, \supset_y .\, y P \,\text{seq}_P\text{‘}\alpha :.\, \supset \vdash .\, \text{Prop}$

$*206\cdot212.\ \vdash : P \in \text{trans} \,.\, \text{E} ! \,\text{seq}_P\text{‘}\alpha \,.\, \supset .\, P\text{‘‘}\alpha \subset \overrightarrow{P}\text{‘}\text{seq}_P\text{‘}\alpha$

Dem.

$\vdash .\, *206\cdot211 \,.\, \supset \vdash : \text{Hp} \,.\, \supset .\, P\text{‘‘}\alpha \subset P\text{‘‘}\overrightarrow{P}\text{‘}\text{seq}_P\text{‘}\alpha$

[$*201\cdot501$] $\subset \overrightarrow{P}\text{‘}\text{seq}_P\text{‘}\alpha : \supset \vdash .\, \text{Prop}$

***206·213.** $\vdash : P \in \text{connex} \,.\, E!\,\text{seq}_P\text{‘}\alpha \,.\, \supset .\, \overrightarrow{P}\text{‘seq}_P\text{‘}\alpha \subset (\alpha \cap C\text{‘}P) \cup P\text{“}\alpha$

Dem.

$\vdash . *206·17 . \supset \vdash :: \text{Hp} . \supset :. yP\,\text{seq}_P\text{‘}\alpha \,.\, \supset_y : (\exists z) . z \in (\alpha \cap C\text{‘}P) . \sim(zPy) :$

[*202·103] $\qquad \supset_y : (\exists z) : z \in \alpha \cap C\text{‘}P : y = z \,.\, \vee \,.\, yPz :$

[*13·195.*37·1] $\qquad \supset_y : y \in \alpha \cap C\text{‘}P \,.\, \vee \,.\, y \in P\text{“}(\alpha \cap C\text{‘}P) :$

[*37·265] $\qquad \supset_y : y \in (\alpha \cap C\text{‘}P) \cup P\text{“}\alpha :: \supset \vdash . \text{Prop}$

***206·22.** $\vdash : P \in \text{trans} \cap \text{connex} \,.\, E!\,\text{seq}_P\text{‘}\alpha \,.\, \supset .$

$$\overrightarrow{P}\text{‘seq}_P\text{‘}\alpha = (\alpha \cap C\text{‘}P) \cup P\text{“}\alpha = \overrightarrow{\text{max}}_P\text{‘}\alpha \cup P\text{“}\alpha$$

[*206·211·212·213 . *205·131]

***206·23.** $\vdash : P \in \text{trans} \cap \text{connex} \,.\, E!\,\text{seq}_P\text{‘}\alpha \,.\, \supset .$

$$\iota\text{‘seq}_P\text{‘}\alpha \cup \overleftarrow{P}\text{‘seq}_P\text{‘}\alpha = p\text{‘}\overleftarrow{P}\text{“}(\alpha \cap C\text{‘}P) \cap C\text{‘}P$$

Dem.

$\vdash . *205·22 . *206·13 . \supset$

$\vdash : \text{Hp} . \supset . \iota\text{‘seq}_P\text{‘}\alpha \cup \overleftarrow{P}\text{‘seq}_P\text{‘}\alpha = \iota\text{‘seq}_P\text{‘}\alpha \cup \breve{P}\text{“}p\text{‘}\overleftarrow{P}\text{“}(\alpha \cap C\text{‘}P)$

[*206·13.*53·31] $\qquad = \overrightarrow{\text{min}}_P\text{‘}p\text{‘}\overleftarrow{P}\text{“}(\alpha \cap C\text{‘}P) \cup \breve{P}\text{“}p\text{‘}\overleftarrow{P}\text{“}(\alpha \cap C\text{‘}P)$

[*205·13] $\qquad = p\text{‘}\overleftarrow{P}\text{“}(\alpha \cap C\text{‘}P) \cap C\text{‘}P \cup \breve{P}\text{“}p\text{‘}\overleftarrow{P}\text{“}(\alpha \cap C\text{‘}P)$

[*201·51.*37·16] $\qquad = p\text{‘}\overleftarrow{P}\text{“}(\alpha \cap C\text{‘}P) \cap C\text{‘}P : \supset \vdash . \text{Prop}$

***206·24.** $\vdash : P \in \text{trans} \,.\, \beta \subset P\text{“}\alpha \,.\, \supset .\, \overrightarrow{\text{seq}}_P\text{‘}(\alpha \cup \beta) = \overrightarrow{\text{seq}}_P\text{‘}\alpha$

Dem.

$\vdash . *201·56 . \supset \vdash : \text{Hp} . \supset . p\text{‘}\overleftarrow{P}\text{“}\{(\alpha \cup \beta) \cap C\text{‘}P\} = p\text{‘}\overleftarrow{P}\text{“}(\alpha \cap C\text{‘}P)$ (1)

$\vdash . (1) . *206·13 . \supset \vdash . \text{Prop}$

***206·25.** $\vdash : P \in \text{trans} \,.\, \supset .\, \overrightarrow{\text{seq}}_P\text{‘}(\alpha \cup P\text{“}\alpha) = \overrightarrow{\text{seq}}_P\text{‘}\alpha$ [*206·24]

***206·26.** $\vdash : P \in \text{trans} \cap \text{connex} \,.\, \exists!\,\alpha \cap C\text{‘}P \,.\, E!\,\text{seq}_P\text{‘}\alpha \,.\, \supset .$

$$p\text{‘}\overleftarrow{P}\text{“}(\alpha \cap C\text{‘}P) = \iota\text{‘seq}_P\text{‘}\alpha \cup \overleftarrow{P}\text{‘seq}_P\text{‘}\alpha$$

Dem.

$\vdash . *40·62 . \supset \vdash : \text{Hp} . \supset . p\text{‘}\overleftarrow{P}\text{“}(\alpha \cap C\text{‘}P) \subset C\text{‘}P$ (1)

$\vdash . (1) . *206·23 . \supset \vdash . \text{Prop}$

***206·27.** $\vdash : P \in \text{trans} \cap \text{connex} \,.\, E!\,\text{seq}_P\text{‘}\alpha \,.\, E!\,\text{max}_P\text{‘}\alpha \,.\, \supset .$

$$\overrightarrow{P}\text{‘seq}_P\text{‘}\alpha = \overrightarrow{P}\text{‘max}_P\text{‘}\alpha \cup \iota\text{‘max}_P\text{‘}\alpha \,.$$
$$\overleftarrow{P}\text{‘max}_P\text{‘}\alpha = \overleftarrow{P}\text{‘seq}_P\text{‘}\alpha \cup \iota\text{‘seq}_P\text{‘}\alpha$$

Dem.

$\vdash . *206·22 . \qquad \supset \vdash : \text{Hp} . \supset . \overrightarrow{P}\text{‘seq}_P\text{‘}\alpha = \overrightarrow{\text{max}}_P\text{‘}\alpha \cup P\text{“}\alpha$

[*205·22] $\qquad = \iota\text{‘max}_P\text{‘}\alpha \cup \overrightarrow{P}\text{‘max}_P\text{‘}\alpha$ (1)

$\vdash . *205·65 . \qquad \supset \vdash : \text{Hp} . \supset . \overleftarrow{P}\text{‘max}_P\text{‘}\alpha = p\text{‘}\overleftarrow{P}\text{“}(\alpha \cap C\text{‘}P)$ (2)

$\vdash . *205·151·161 . \supset \vdash : \text{Hp} . \supset . \exists!\,(\alpha \cap C\text{‘}P)$ (3)

$\vdash . (3) . *206\cdot26 . \supset \vdash : \text{Hp} . \supset . p`\overleftarrow{P}``(\alpha \cap C`P) = \iota`\text{seq}_P`\alpha \cup \overleftarrow{P}`\text{seq}_P`\alpha$ (4)

$\vdash . (2) . (4) . \quad \supset \vdash : \text{Hp} . \supset . \overleftarrow{P}`\max_P`\alpha = \iota`\text{seq}_P`\alpha \cup \overleftarrow{P}`\text{seq}_P`\alpha$ (5)

$\vdash . (1) . (5) . \supset \vdash . \text{Prop}$

$*206\cdot28$. $\vdash :. P \epsilon \text{Ser} . \supset :$

$$x \epsilon C`P - \alpha . \overrightarrow{P}`x = P``\alpha . \equiv . x = \text{seq}_P`\alpha . \sim \text{E} ! \max_P`\alpha$$

Dem.

$\vdash . *206\cdot174 . *205\cdot6 . \supset$

$\vdash :. \text{Hp} . \supset : x = \text{seq}_P`\alpha . \sim \text{E} ! \max_P`\alpha . \supset . x \epsilon C`P . \overrightarrow{P}`x = P``\alpha .$

$[*206\cdot2] \qquad \supset . x \epsilon C`P - \alpha . \overrightarrow{P}`x = P``\alpha$ (1)

$\vdash . *37\cdot1 . \qquad \supset \vdash : xPy . y \epsilon \alpha . \overrightarrow{P}`x = P``\alpha . \supset . x \epsilon \overrightarrow{P}`x$ (2)

$\vdash . (2) . \text{Transp} . \supset \vdash : P \Subset J . y \epsilon \alpha . \overrightarrow{P}`x = P``\alpha . \supset . \sim(xPy)$ (3)

$\vdash . *13\cdot14 . \qquad \supset \vdash : x \epsilon C`P - \alpha . y \epsilon \alpha . \supset . x \neq y$ (4)

$\vdash . (3) . (4) . *202\cdot103 . \supset \vdash :. \text{Hp} . \supset :$

$$x \epsilon C`P - \alpha . \overrightarrow{P}`x = P``\alpha . y \epsilon \alpha \cap C`P . \supset . yPx :$$

$[*32\cdot18] \qquad \supset : x \epsilon C`P - \alpha . \overrightarrow{P}`x = P``\alpha . \supset . \alpha \cap C`P \subset \overrightarrow{P}`x$ (5)

$\vdash . (5) . *206\cdot171\cdot16 . \supset \vdash :. \text{Hp} . \supset :$

$$x \epsilon C`P - \alpha . \overrightarrow{P}`x = P``\alpha . \supset . x = \text{seq}_P`\alpha \quad (6)$$

$\vdash . (5) . *205\cdot123 . \quad \supset \vdash :. \text{Hp} . \supset :$

$$x \epsilon C`P - \alpha . \overrightarrow{P}`x = P``\alpha . \supset . \sim \text{E} ! \max_P`\alpha \quad (7)$$

$\vdash . (1) . (6) . (7) . \supset \vdash . \text{Prop}$

$*206\cdot3$. $\vdash : P \epsilon \text{trans} \cap \text{connex} . \alpha \subset C`P . P``\alpha \subset \alpha . \text{E} ! \text{seq}_P`\alpha . \supset .$

$$\overrightarrow{P}`\text{seq}_P`\alpha = \alpha \quad [*206\cdot22]$$

$*206\cdot31$. $\vdash : P \epsilon \text{trans} \cap \text{connex} . \text{E} ! \text{seq}_P`P``\alpha . \supset . \overrightarrow{P}`\text{seq}_P`P``\alpha = P``\alpha$
$[*206\cdot3 . *201\cdot5]$

$*206\cdot32$. $\vdash : P \epsilon \text{trans} \cap \text{connex} . \text{E} ! \max_P`\alpha . \text{E} ! \text{seq}_P`P``\alpha . \supset .$

$$\max_P`\alpha = \text{seq}_P`P``\alpha$$

Dem.

$\vdash . *206\cdot31 . *205\cdot22 . \supset \vdash :. \text{Hp} . \supset : \overrightarrow{P}`\max_P`\alpha = \overrightarrow{P}`\text{seq}_P`P``\alpha :$

$[*205\cdot194 . *206\cdot133] \supset : \sim(\text{seq}_P`P``\alpha \, P \max_P`\alpha) . \sim(\max_P`\alpha \, P \, \text{seq}_P`P``\alpha) :$

$[*202\cdot103] \qquad \supset : \max_P`\alpha = \text{seq}_P`P``\alpha :. \supset \vdash . \text{Prop}$

In the hypothesis of $*206\cdot32$, we have both $\text{E} ! \max_P`\alpha$ and $\text{E} ! \text{seq}_P`P``\alpha$. So long as P is not contained in diversity, these are both necessary. For example, suppose we take

$$P = \alpha \uparrow (\alpha \cup \iota`x), \text{ where } x \sim \epsilon \alpha . \exists ! \alpha.$$

Then P is transitive and connected, but not contained in diversity. We have

$$\alpha \cup \iota`x = C`P . P``(\alpha \cup \iota`x) = \alpha = \text{D}`P.$$

Also $\quad \max_P{}^{\backprime}(\alpha \cup \iota{}^{\backprime}x) = x,$

$\overrightarrow{\mathrm{seq}}_P{}^{\backprime}P^{\backprime\backprime}(\alpha \cup \iota{}^{\backprime}x) = \overrightarrow{\min}_P{}^{\backprime}p{}^{\backprime}\overleftarrow{P}^{\backprime\backprime}\alpha = \overrightarrow{\min}_P{}^{\backprime}(\alpha \cup \iota{}^{\backprime}x) = \Lambda.$

Thus in this case $\max_P{}^{\backprime}(\alpha \cup \iota{}^{\backprime}x)$ exists, but $\mathrm{seq}_P{}^{\backprime}P^{\backprime\backprime}(\alpha \cup \iota{}^{\backprime}x)$ does not exist. When P is serial, *i.e.* when P is contained in diversity, in addition to being transitive and connected, the existence of $\max_P{}^{\backprime}\alpha$ involves that of $\mathrm{seq}_P{}^{\backprime}P^{\backprime\backprime}\alpha$, and therefore the hypothesis $\mathrm{E}\,!\,\mathrm{seq}_P{}^{\backprime}P^{\backprime\backprime}\alpha$, which appears in $*206{\cdot}32$, becomes unnecessary.

$*206{\cdot}33.\quad \vdash : P \in \mathrm{trans} \cap \mathrm{connex} \,.\, \sim\mathrm{E}\,!\,\max_P{}^{\backprime}\alpha \,.\, \supset .\, \overrightarrow{\mathrm{seq}}_P{}^{\backprime}P^{\backprime\backprime}\alpha = \overrightarrow{\mathrm{seq}}_P{}^{\backprime}\alpha$

Dem.

$\vdash .\, *205{\cdot}6 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset .\, \alpha \cap C{}^{\backprime}P \subset P^{\backprime\backprime}\alpha \,.$

$[*22{\cdot}62 . *37{\cdot}15] \quad \supset .\, (\alpha \cup P^{\backprime\backprime}\alpha) \cap C{}^{\backprime}P = P^{\backprime\backprime}\alpha \quad (1)$

$\vdash .\, *206{\cdot}25 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset .\, \overrightarrow{\mathrm{seq}}_P{}^{\backprime}\alpha = \overrightarrow{\mathrm{seq}}_P{}^{\backprime}(\alpha \cup P^{\backprime\backprime}\alpha)$

$[*206{\cdot}131] \quad = \overrightarrow{\mathrm{seq}}_P{}^{\backprime}\{(\alpha \cup P^{\backprime\backprime}\alpha) \cap C{}^{\backprime}P\}$

$[(1)] \quad = \overrightarrow{\mathrm{seq}}_P{}^{\backprime}P^{\backprime\backprime}\alpha : \supset \vdash .\, \mathrm{Prop}$

$*206{\cdot}331.\quad \vdash : P \in \mathrm{trans} \cap \mathrm{connex} \,.\, \sim\mathrm{E}\,!\,\max_P{}^{\backprime}\alpha \,.\, \mathrm{E}\,!\,\mathrm{seq}_P{}^{\backprime}\alpha \,.\, \supset .\, \mathrm{seq}_P{}^{\backprime}P^{\backprime\backprime}\alpha = \mathrm{seq}_P{}^{\backprime}\alpha$
$[*206{\cdot}33]$

$*206{\cdot}34.\quad \vdash : P \in \mathrm{Ser} \,.\, \supset .\, \overrightarrow{\max}_P{}^{\backprime}\alpha \subset \overrightarrow{\mathrm{seq}}_P{}^{\backprime}P^{\backprime\backprime}\alpha$

Dem.

$\vdash .\, *205{\cdot}101 \,.\, *37{\cdot}265 \,.\, \supset$

$\vdash :.\, y \in \overrightarrow{\max}_P{}^{\backprime}\alpha \,.\, \equiv : y \in \alpha \cap C{}^{\backprime}P : z \in \alpha \cap C{}^{\backprime}P \,.\, \supset_z .\, \sim(yPz) \quad (1)$

$\vdash .\, (1) \,.\, *202{\cdot}103 \,.\, \supset \vdash ::.\, \mathrm{Hp} \,.\, \supset ::$

$y \in \overrightarrow{\max}_P{}^{\backprime}\alpha \,.\, \supset :.\, y \in \alpha \cap C{}^{\backprime}P :.\, z \in \alpha \cap C{}^{\backprime}P \,.\, \supset_z : z = y \,.\, \vee .\, zPy \quad (2)$

$\vdash .\, (2) \,.\, *13{\cdot}195 \,.\, *201{\cdot}1 \,.\, \supset \vdash ::.\, \mathrm{Hp} \,.\, \supset ::$

$y \in \overrightarrow{\max}_P{}^{\backprime}\alpha \,.\, \supset :.\, y \in \alpha \cap C{}^{\backprime}P :.\, z \in \alpha \cap C{}^{\backprime}P \,.\, uPz \,.\, \supset_{u,z} .\, uPy :.$

$[*37{\cdot}1{\cdot}265] \quad \supset :.\, u \in P^{\backprime\backprime}\alpha \,.\, \supset_u .\, uPy :.$

$[*40{\cdot}53] \quad \supset :.\, y \in p{}^{\backprime}\overleftarrow{P}^{\backprime\backprime}P^{\backprime\backprime}\alpha \quad (3)$

$\vdash .\, (1) \,.\, *37{\cdot}1 \,.\quad \supset \vdash : y \in \overrightarrow{\max}_P{}^{\backprime}\alpha \,.\, vPy \,.\, \supset .\, v \in P^{\backprime\backprime}\alpha \quad (4)$

$\vdash .\, *50{\cdot}24 \,.\quad \supset \vdash : \mathrm{Hp} \,.\, \supset .\, \sim(vPv) \quad (5)$

$\vdash .\, (4) \,.\, (5) \,.\quad \supset \vdash :.\, \mathrm{Hp} \,.\, \supset : y \in \overrightarrow{\max}_P{}^{\backprime}\alpha \,.\, vPy \,.\, \supset .\, (\exists w) \,.\, w \in P^{\backprime\backprime}\alpha \,.\, \sim(wPv) \,.$

$[*40{\cdot}53] \quad \supset .\, v \sim\in p{}^{\backprime}\overleftarrow{P}^{\backprime\backprime}P^{\backprime\backprime}\alpha :$

$[*10{\cdot}51] \quad \supset : y \in \overrightarrow{\max}_P{}^{\backprime}\alpha \,.\, \supset .\, \sim(\exists v) \,.\, v \in p{}^{\backprime}\overleftarrow{P}^{\backprime\backprime}P^{\backprime\backprime}\alpha \,.\, vPy \,.$

$[*37{\cdot}105] \quad \supset .\, y \sim\in \breve{P}^{\backprime\backprime}p{}^{\backprime}\overleftarrow{P}^{\backprime\backprime}P^{\backprime\backprime}\alpha \quad (6)$

$\vdash .\, (3) \,.\, (6) \,.\, (1) \,.\, \supset \vdash :.\, \mathrm{Hp} \,.\, \supset :$

$y \in \overrightarrow{\max}_P{}^{\backprime}\alpha \,.\, \supset .\, y \in p{}^{\backprime}\overleftarrow{P}^{\backprime\backprime}P^{\backprime\backprime}\alpha \cap C{}^{\backprime}P - \breve{P}^{\backprime\backprime}p{}^{\backprime}\overleftarrow{P}^{\backprime\backprime}P^{\backprime\backprime}\alpha \,.$

$[*206{\cdot}143] \quad \supset .\, y \in \overrightarrow{\mathrm{seq}}_P{}^{\backprime}P^{\backprime\backprime}\alpha :.\, \supset \vdash .\, \mathrm{Prop}$

***206·35.** $\vdash : P \in \mathrm{Ser} \,.\, \mathrm{E}\,!\, \mathrm{max}_P{}^{\backprime}\alpha \,.\, \supset .\, \mathrm{max}_P{}^{\backprime}\alpha = \mathrm{seq}_P{}^{\backprime}P^{\backprime\backprime}\alpha \,.\, \mathrm{E}\,!\, \mathrm{seq}_P{}^{\backprime}P^{\backprime\backprime}\alpha$

Dem.

$\vdash .\, *206{\cdot}34 \,.\quad \supset \vdash : \mathrm{Hp} \,.\, \supset .\, \mathrm{max}_P{}^{\backprime}\alpha \in \overrightarrow{\mathrm{seq}}_P{}^{\backprime}P^{\backprime\backprime}\alpha \quad (1)$

$\vdash .\, (1) \,.\, *206{\cdot}16 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset .\, \mathrm{max}_P{}^{\backprime}\alpha = \mathrm{seq}_P{}^{\backprime}P^{\backprime\backprime}\alpha \quad (2)$

$\vdash .\, (2) \,.\, *14{\cdot}21 \,.\quad \supset \vdash : \mathrm{Hp} \,.\, \supset .\, \mathrm{E}\,!\, \mathrm{seq}_P{}^{\backprime}P^{\backprime\backprime}\alpha \quad (3)$

$\vdash .\, (2) \,.\, (3) \,.\, \supset \vdash .\, \mathrm{Prop}$

***206·36.** $\vdash :: P \in \mathrm{Ser} \,.\, \supset :.\, \mathrm{E}\,!\, \mathrm{seq}_P{}^{\backprime}P^{\backprime\backprime}\alpha \,.\, \equiv : \mathrm{E}\,!\, \mathrm{max}_P{}^{\backprime}\alpha \,.\, \mathrm{v} \,.\, \mathrm{E}\,!\, \mathrm{seq}_P{}^{\backprime}\alpha$

Dem.

$\vdash .\, *206{\cdot}35{\cdot}331 \,.\, \supset \vdash :.\, \mathrm{Hp} : \mathrm{E}\,!\, \mathrm{max}_P{}^{\backprime}\alpha \,.\, \mathrm{v} \,.\, \mathrm{E}\,!\, \mathrm{seq}_P{}^{\backprime}\alpha : \supset .\, \mathrm{E}\,!\, \mathrm{seq}_P{}^{\backprime}P^{\backprime\backprime}\alpha \quad (1)$

$\vdash .\, *206{\cdot}34 \,.\quad \supset \vdash :.\, \mathrm{Hp} \,.\, \supset : \sim \mathrm{E}\,!\, \mathrm{seq}_P{}^{\backprime}P^{\backprime\backprime}\alpha \,.\, \supset .\, \sim \mathrm{E}\,!\, \mathrm{max}_P{}^{\backprime}\alpha \,. \quad (2)$

$[*206{\cdot}33] \qquad \supset .\, \sim \mathrm{E}\,!\, \mathrm{seq}_P{}^{\backprime}\alpha \quad (3)$

$\vdash .\, (1) \,.\, (2) \,.\, (3) \,.\, \supset \vdash .\, \mathrm{Prop}$

The condition $(\alpha) : \mathrm{E}\,!\, \mathrm{max}_P{}^{\backprime}\alpha \,.\, \mathrm{v} \,.\, \mathrm{E}\,!\, \mathrm{seq}_P{}^{\backprime}\alpha$ is the definition of what may be called "Dedekindian" series, *i.e.* series in which, when any division of the field into two parts is made in such a way that the first part wholly precedes the second, then either the first part has a last term or the second part has a first term. (When these alternatives are also mutually exclusive, the series has "Dedekindian continuity.") If α is any class, $P^{\backprime\backprime}\alpha$ is the segment of $C^{\backprime}P$ defined by α. In virtue of the above proposition, every segment of a Dedekindian series has a sequent. The sequent of a class having no maximum is what is commonly called a *limit*. Thus in a series having Dedekindian continuity (in which segments never have maxima), every segment has a limit.

***206·37.** $\vdash : P \in \mathrm{Ser} \,.\, \supset .\, \overrightarrow{\mathrm{seq}}_P{}^{\backprime}P^{\backprime\backprime}\alpha = \overrightarrow{\mathrm{min}}_P{}^{\backprime}(\overrightarrow{\mathrm{max}}_P{}^{\backprime}\alpha \cup \overrightarrow{\mathrm{seq}}_P{}^{\backprime}\alpha)$

Dem.

$\vdash .\, *205{\cdot}16 \,.\, \supset \vdash : \overrightarrow{\mathrm{max}}_P{}^{\backprime}\alpha = \Lambda \,.\, \overrightarrow{\mathrm{seq}}_P{}^{\backprime}\alpha = \Lambda \,.\, \supset .$

$\qquad \overrightarrow{\mathrm{min}}_P{}^{\backprime}(\overrightarrow{\mathrm{max}}_P{}^{\backprime}\alpha \cup \overrightarrow{\mathrm{seq}}_P{}^{\backprime}\alpha) = \Lambda \quad (1)$

$\vdash .\, *206{\cdot}36 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \mathrm{Hp}(1) \,.\, \supset .\, \sim \mathrm{E}\,!\, \mathrm{seq}_P{}^{\backprime}P^{\backprime\backprime}\alpha \,.$

$[*206{\cdot}16] \qquad \supset .\, \overrightarrow{\mathrm{seq}}_P{}^{\backprime}P^{\backprime\backprime}\alpha = \Lambda \quad (2)$

$\vdash .\, *24{\cdot}24 \,.\quad \supset \vdash : \mathrm{Hp} \,.\, \overrightarrow{\mathrm{max}}_P{}^{\backprime}\alpha = \Lambda \,.\, \exists\,!\, \overrightarrow{\mathrm{seq}}_P{}^{\backprime}\alpha \,.\, \supset .$

$\qquad \overrightarrow{\mathrm{min}}_P{}^{\backprime}(\overrightarrow{\mathrm{max}}_P{}^{\backprime}\alpha \cup \overrightarrow{\mathrm{seq}}_P{}^{\backprime}\alpha) = \overrightarrow{\mathrm{min}}_P{}^{\backprime}\overrightarrow{\mathrm{seq}}_P{}^{\backprime}\alpha$

$[*205{\cdot}17 . *206{\cdot}16] \qquad = \overrightarrow{\mathrm{seq}}_P{}^{\backprime}\alpha$

$[*206{\cdot}33] \qquad = \overrightarrow{\mathrm{seq}}_P{}^{\backprime}P^{\backprime\backprime}\alpha \quad (3)$

$\vdash .\, *205{\cdot}17{\cdot}3 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \exists\,!\, \overrightarrow{\mathrm{max}}_P{}^{\backprime}\alpha \,.\, \overrightarrow{\mathrm{seq}}_P{}^{\backprime}\alpha = \Lambda \,.\, \supset .$

$\qquad \overrightarrow{\mathrm{min}}_P{}^{\backprime}(\overrightarrow{\mathrm{max}}_P{}^{\backprime}\alpha \cup \overrightarrow{\mathrm{seq}}_P{}^{\backprime}\alpha) = \overrightarrow{\mathrm{max}}_P{}^{\backprime}\alpha$

$[*206{\cdot}35] \qquad = \overrightarrow{\mathrm{seq}}_P{}^{\backprime}P^{\backprime\backprime}\alpha \quad (4)$

$\vdash . *206{\cdot}16 . *205{\cdot}3 . \supset$

$\vdash : \text{Hp} . \exists ! \overrightarrow{\text{max}}_P{}^\backprime\alpha . \exists ! \overrightarrow{\text{seq}}_P{}^\backprime\alpha . \supset .$

$\overrightarrow{\text{min}}_P{}^\backprime(\overrightarrow{\text{max}}_P{}^\backprime\alpha \cup \overrightarrow{\text{seq}}_P{}^\backprime\alpha) = \overrightarrow{\text{min}}_P{}^\backprime(\iota^\backprime\text{max}_P{}^\backprime\alpha \cup \iota^\backprime\text{seq}_P{}^\backprime\alpha)$

$[*206{\cdot}27 . *205{\cdot}182] \quad = \iota^\backprime\text{max}_P{}^\backprime\alpha$

$[*206{\cdot}35] \quad = \overrightarrow{\text{seq}}_P{}^\backprime P^{\backprime\backprime}\alpha \quad (5)$

$\vdash . (1) . (2) . (3) . (4) . (5) . \supset \vdash . \text{Prop}$

$*206{\cdot}38. \quad \vdash : P \in \text{Ser} . \supset . \overrightarrow{\text{max}}_P{}^\backprime\alpha = \alpha \cap \overrightarrow{\text{seq}}_P{}^\backprime P^{\backprime\backprime}\alpha$

Dem.

$\vdash . *206{\cdot}35 . *205{\cdot}111 . \supset$

$\vdash : \text{Hp} . \text{E} ! \,\text{max}_P{}^\backprime\alpha . \supset . \overrightarrow{\text{max}}_P{}^\backprime\alpha = \overrightarrow{\text{seq}}_P{}^\backprime P^{\backprime\backprime}\alpha . \overrightarrow{\text{max}}_P{}^\backprime\alpha \subset \alpha .$

$[*22{\cdot}621] \quad \supset . \overrightarrow{\text{max}}_P{}^\backprime\alpha = \alpha \cap \overrightarrow{\text{seq}}_P{}^\backprime P^{\backprime\backprime}\alpha \quad (1)$

$\vdash . *205{\cdot}3 . \supset \vdash : \text{Hp} . \sim \text{E} ! \,\text{max}_P{}^\backprime\alpha . \supset . \overrightarrow{\text{max}}_P{}^\backprime\alpha = \Lambda \quad (2)$

$\vdash . *206{\cdot}33 . \supset \vdash : \text{Hp} . \sim \text{E} ! \,\text{max}_P{}^\backprime\alpha . \supset . \overrightarrow{\text{seq}}_P{}^\backprime P^{\backprime\backprime}\alpha = \overrightarrow{\text{seq}}_P{}^\backprime\alpha .$

$[*206{\cdot}2] \quad \supset . \alpha \cap \overrightarrow{\text{seq}}_P{}^\backprime P^{\backprime\backprime}\alpha = \Lambda .$

$[(2)] \quad \supset . \overrightarrow{\text{max}}_P{}^\backprime\alpha = \alpha \cap \overrightarrow{\text{seq}}_P{}^\backprime P^{\backprime\backprime}\alpha \quad (3)$

$\vdash . (1) . (3) . \supset \vdash . \text{Prop}$

$*206{\cdot}4. \quad \vdash : P \Subset J . x \in C^\backprime P . \supset . x \,\text{seq}_P \overrightarrow{P}{}^\backprime x$

Dem.

$\vdash . *206{\cdot}134 . *22{\cdot}43 . \supset$

$\vdash : x \,\text{seq}_P \overrightarrow{P}{}^\backprime x . \equiv . x \in C^\backprime P . \overrightarrow{P}{}^\backprime x \subset -p^\backprime \overleftarrow{P}{}^{\backprime\backprime}\overrightarrow{P}{}^\backprime x \quad (1)$

$\vdash . *200{\cdot}5 . \supset \vdash : P \Subset J . \supset . \overrightarrow{P}{}^\backprime x \subset -p^\backprime \overleftarrow{P}{}^{\backprime\backprime}\overrightarrow{P}{}^\backprime x \quad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

$*206{\cdot}401. \quad \vdash : P \in \text{connex} \cap \text{Rl}^\backprime J . x \in C^\backprime P . \supset . x = \text{seq}_P{}^\backprime \overrightarrow{P}{}^\backprime x \quad [*206{\cdot}4{\cdot}161]$

$*206{\cdot}41. \quad \vdash . \overrightarrow{\text{min}}_P{}^\backprime \overleftarrow{P}{}^\backprime x = \overleftarrow{P \dot{-} P^2}{}^\backprime x \quad [*205{\cdot}25]$

$*206{\cdot}42. \quad \vdash : x \in C^\backprime P . \supset . \overrightarrow{\text{seq}}_P{}^\backprime \iota^\backprime x = \overleftarrow{P \dot{-} P^2}{}^\backprime x = \overrightarrow{\text{min}}_P{}^\backprime \overleftarrow{P}{}^\backprime x$

Dem.

$\vdash . *53{\cdot}01{\cdot}31 . \supset \vdash . p^\backprime \overleftarrow{P}{}^{\backprime\backprime}\iota^\backprime x = \overleftarrow{P}{}^\backprime x \quad (1)$

$\vdash . (1) . *206{\cdot}41{\cdot}143 . \supset \vdash . \text{Prop}$

$*206{\cdot}43. \quad \vdash : P \in \text{trans} \cap \text{Rl}^\backprime J . x \in C^\backprime P . \supset . \overrightarrow{\text{seq}}_P{}^\backprime \iota^\backprime x = \overleftarrow{P_1}{}^\backprime x$

$[*206{\cdot}42 . *201{\cdot}63]$

$*206{\cdot}44. \quad \vdash :. P \in \text{trans} \cap \text{Rl}^\backprime J . x \in C^\backprime P . \supset :$

$\text{E} ! \,\text{seq}_P{}^\backprime \iota^\backprime x . \equiv . \text{E} ! \,\breve{P}_1{}^\backprime x : \text{E} ! \,\text{seq}_P{}^\backprime \iota^\backprime x . \supset . \text{seq}_P{}^\backprime \iota^\backprime x = \breve{P}_1{}^\backprime x$

$[*206{\cdot}43]$

***206·45.** $\vdash :. P \in \mathrm{Ser} . x \in C‘P . \supset : E ! \mathrm{seq}_P‘\iota‘x . \equiv . x \in D‘P_1$
[*206·44 . *204·7 . *71·165]

***206·451.** $\vdash : P \in \mathrm{Ser} . E ! \mathrm{seq}_P‘\alpha . \supset . \overrightarrow{\max}_P‘\alpha = \overrightarrow{P_1}‘\mathrm{seq}_P‘\alpha$

Dem.

$\vdash . *206·41 . \supset \vdash : \mathrm{Hp} . \supset . \overrightarrow{P_1}‘\mathrm{seq}_P‘\alpha = \overrightarrow{\max}_P‘\overrightarrow{P}‘\mathrm{seq}_P‘\alpha$

[*206·22] $= \overrightarrow{\max}_P‘\{(\alpha \cap C‘P) \cup P“\alpha\}$

[*205·191] $= \overrightarrow{\max}_P‘\alpha : \supset \vdash . \mathrm{Prop}$

***206·46.** $\vdash : P \in \mathrm{trans} \cap \mathrm{connex} . E ! \max_P‘\alpha . \supset . \overrightarrow{\mathrm{seq}}_P‘\alpha = \overrightarrow{\mathrm{seq}}_P‘\overrightarrow{\max}_P‘\alpha$

Dem.

$\vdash . *206·42 . \supset \vdash : \mathrm{Hp} . \supset . \overrightarrow{\mathrm{seq}}_P‘\overrightarrow{\max}_P‘\alpha = \overrightarrow{\min}_P‘\overleftarrow{P}‘\max_P‘\alpha$

[*205·65] $= \overrightarrow{\min}_P‘p‘\overleftarrow{P}“(\alpha \cap C‘P)$

[*206·13] $= \overrightarrow{\mathrm{seq}}_P‘\alpha : \supset \vdash . \mathrm{Prop}$

***206·47.** $\vdash : P \in \mathrm{trans} . E ! \mathrm{seq}_P‘\alpha . \supset . \mathrm{seq}_P‘\alpha = \max_P‘(\alpha \cup \overrightarrow{\mathrm{seq}}_P‘\alpha)$

Dem.

$\vdash . *206·134 . \supset \vdash : \mathrm{Hp} . \supset . \alpha \cap C‘P \subset \overrightarrow{P}‘\mathrm{seq}_P‘\alpha .$

[*205·193·151] $\supset . \overrightarrow{\max}_P‘(\alpha \cup \overrightarrow{\mathrm{seq}}_P‘\alpha) = \overrightarrow{\max}_P‘\overrightarrow{\mathrm{seq}}_P‘\alpha$

[*206·133.*205·18] $= \iota‘\mathrm{seq}_P‘\alpha : \supset \vdash . \mathrm{Prop}$

***206·48.** $\vdash : P \in \mathrm{trans} \cap \mathrm{connex} . E ! \mathrm{seq}_P‘\alpha . \supset . \overrightarrow{\mathrm{seq}}_P‘\overrightarrow{\mathrm{seq}}_P‘\alpha = \overrightarrow{\mathrm{seq}}_P‘(\alpha \cup \overrightarrow{\mathrm{seq}}_P‘\alpha)$

Dem.

$\vdash . *206·47 . \supset \vdash : \mathrm{Hp} . \supset .$

$\overrightarrow{\mathrm{seq}}_P‘\overrightarrow{\mathrm{seq}}_P‘\alpha = \overrightarrow{\mathrm{seq}}_P‘\overrightarrow{\max}_P‘(\alpha \cup \overrightarrow{\mathrm{seq}}_P‘\alpha) . E ! \max_P‘(\alpha \cup \mathrm{seq}_P‘\alpha) .$

[*206·46] $\supset . \overrightarrow{\mathrm{seq}}_P‘\overrightarrow{\mathrm{seq}}_P‘\alpha = \overrightarrow{\mathrm{seq}}_P‘(\alpha \cup \overrightarrow{\mathrm{seq}}_P‘\alpha) : \supset \vdash . \mathrm{Prop}$

***206·5.** $\vdash : P \in \mathrm{trans} \cap \mathrm{connex} . E ! \max_P‘\alpha . E ! \mathrm{seq}_P‘\alpha . \supset .$
$\max_P‘\alpha \, (P \dot{-} P^2) \, \mathrm{seq}_P‘\alpha$

Dem.

$\vdash . *206·46 . \supset \vdash : \mathrm{Hp} . \supset . \overrightarrow{\mathrm{seq}}_P‘\alpha = \overrightarrow{\mathrm{seq}}_P‘\iota‘\max_P‘\alpha$

[*206·42] $= \overleftarrow{P \dot{-} P^2}‘\max_P‘\alpha : \supset \vdash . \mathrm{Prop}$

***206·51.** $\vdash : \exists ! \overrightarrow{\max}_P‘\overrightarrow{P}‘x . \supset . x \, \mathrm{seq}_P \, \overrightarrow{P}‘x$

Dem.

$\vdash . *205·161 . \quad \supset \vdash : \mathrm{Hp} . \supset . \exists ! \overrightarrow{P}‘x .$

[*33·42] $\supset . x \in C‘P$ (1)

$\vdash . (1) . *206·134 . \supset \vdash :. \mathrm{Hp} . \supset : x \, \mathrm{seq}_P \overrightarrow{P}‘x . \equiv . \overrightarrow{P}‘x \subset \overrightarrow{P}‘x . \overrightarrow{P}‘x \subset - p‘\overleftarrow{P}“\overrightarrow{P}‘x .$

[*22·42] $\equiv . \overrightarrow{P}‘x \subset - p‘\overleftarrow{P}“\overrightarrow{P}‘x$ (2)

$\vdash . *205\cdot101 . \supset \vdash :. y \in \max_P{}^\backprime \overrightarrow{P}{}^\backprime x . \supset : yPx . y \sim\in P^{\backprime\backprime}\overrightarrow{P}{}^\backprime x :$

[*37·1] $\supset : yPx : zPx . \supset_z . \sim(yPz) :$

[*32·18.*5·31] $\supset : zPx . \supset_z . y \in \overrightarrow{P}{}^\backprime x . \sim(yPz) .$

[*40·53] $\supset_z . z \sim\in p{}^\backprime\overleftarrow{P}{}^{\backprime\backprime}\overrightarrow{P}{}^\backprime x :$

[*32·18] $\supset : \overrightarrow{P}{}^\backprime x \subset - p{}^\backprime\overleftarrow{P}{}^{\backprime\backprime}\overrightarrow{P}{}^\backprime x$ (3)

$\vdash . (2) . (3) . \supset \vdash : \mathrm{Hp} . y \in \max_P{}^\backprime \overrightarrow{P}{}^\backprime x . \supset . x \,\mathrm{seq}_P\, \overrightarrow{P}{}^\backprime x : \supset \vdash . \mathrm{Prop}$

***206·52.** $\vdash : P \in \mathrm{trans} \cap \mathrm{connex} . \mathrm{E}!\max_P{}^\backprime P^{\backprime\backprime}\alpha . \supset .$

$\mathrm{E}!\,\mathrm{seq}_P{}^\backprime P^{\backprime\backprime}\alpha . \mathrm{seq}_P{}^\backprime P^{\backprime\backprime}\alpha = \max_P{}^\backprime\alpha$

Dem. $\vdash . *205\cdot7 . \quad \supset \vdash : \mathrm{Hp} . \supset . \mathrm{E}!\max_P{}^\backprime\alpha .$ (1)

[*205·22] $\supset . P^{\backprime\backprime}\alpha = \overrightarrow{P}{}^\backprime\max_P{}^\backprime\alpha$ (2)

$\vdash . (2) . *206\cdot51 . \supset \vdash : \mathrm{Hp} . \supset . \max_P{}^\backprime\alpha \,\mathrm{seq}_P\, P^{\backprime\backprime}\alpha .$

[*206·161] $\supset . \max_P{}^\backprime\alpha = \mathrm{seq}_P{}^\backprime P^{\backprime\backprime}\alpha$ (3)

$\vdash . (1) . (3) . \supset \vdash . \mathrm{Prop}$

***206·53.** $\vdash : P \in \mathrm{Ser} . \supset . \overrightarrow{\mathrm{seq}}_P{}^\backprime p{}^\backprime\overrightarrow{P}{}^{\backprime\backprime}(\alpha \cap C{}^\backprime P) = \overrightarrow{\min}_P{}^\backprime\alpha$

Dem.

$\vdash . *206\cdot13 . \supset \vdash . \overrightarrow{\mathrm{seq}}_P{}^\backprime p{}^\backprime\overrightarrow{P}{}^{\backprime\backprime}(\alpha \cap C{}^\backprime P) = \overrightarrow{\min}_P{}^\backprime p{}^\backprime\overleftarrow{P}{}^{\backprime\backprime}\{p{}^\backprime\overrightarrow{P}{}^{\backprime\backprime}(\alpha \cap C{}^\backprime P) \cap C{}^\backprime P\}$

[*205·15·16.*206·18.*200·54] $= \overrightarrow{\min}_P{}^\backprime\{C{}^\backprime P \cap p{}^\backprime\overleftarrow{P}{}^{\backprime\backprime}p{}^\backprime\overrightarrow{P}{}^{\backprime\backprime}(\alpha \cap C{}^\backprime P)\}$ (1)

$\vdash . (1) . *204\cdot62 . \supset \vdash : \mathrm{Hp} . \supset . \overrightarrow{\mathrm{seq}}_P{}^\backprime p{}^\backprime\overrightarrow{P}{}^{\backprime\backprime}(\alpha \cap C{}^\backprime P) = \overrightarrow{\min}_P{}^\backprime\{(\alpha \cap C{}^\backprime P) \cup \breve{P}{}^{\backprime\backprime}\alpha\}$

[*205·19.*201·52] $= \overrightarrow{\min}_P{}^\backprime\alpha : \supset \vdash . \mathrm{Prop}$

***206·531.** $\vdash : P \in \mathrm{Ser} . \supset .$

$$C{}^\backprime P \cap \hat{x}\{p{}^\backprime\overrightarrow{P}{}^{\backprime\backprime}(\alpha \cap C{}^\backprime P) = \overrightarrow{P}{}^\backprime x\} = \overrightarrow{\mathrm{seq}}_P{}^\backprime p{}^\backprime\overrightarrow{P}{}^{\backprime\backprime}(\alpha \cap C{}^\backprime P) = \overrightarrow{\min}_P{}^\backprime\alpha$$

Dem.

$\vdash . *206\cdot172 . *201\cdot51 . \supset$

$\vdash : \mathrm{Hp} . \supset . \overrightarrow{\mathrm{seq}}_P{}^\backprime p{}^\backprime\overleftarrow{P}{}^{\backprime\backprime}(\alpha \cap C{}^\backprime P) = C{}^\backprime P \cap \hat{x}\{p{}^\backprime\overrightarrow{P}{}^{\backprime\backprime}(\alpha \cap C{}^\backprime P) \cap C{}^\backprime P = \overrightarrow{P}{}^\backprime x\}$ (1)

$\vdash . (1) . *40\cdot62 . \supset \vdash : \mathrm{Hp} . \exists ! (\alpha \cap C{}^\backprime P) . \supset .$

$\overrightarrow{\mathrm{seq}}_P{}^\backprime p{}^\backprime\overleftarrow{P}{}^{\backprime\backprime}(\alpha \cap C{}^\backprime P) = C{}^\backprime P \cap \hat{x}\{p{}^\backprime\overrightarrow{P}{}^{\backprime\backprime}(\alpha \cap C{}^\backprime P) = \overrightarrow{P}{}^\backprime x\}$ (2)

$\vdash . *205\cdot16 . *206\cdot53 . \supset \vdash : \mathrm{Hp} . \alpha \cap C{}^\backprime P = \Lambda . \supset . \overrightarrow{\mathrm{seq}}_P{}^\backprime p{}^\backprime\overleftarrow{P}{}^{\backprime\backprime}(\alpha \cap C{}^\backprime P) = \Lambda$ (3)

$\vdash . *40\cdot2 . \supset \vdash : \alpha \cap C{}^\backprime P = \Lambda . \supset .$

$C{}^\backprime P \cap \hat{x}\{p{}^\backprime\overrightarrow{P}{}^{\backprime\backprime}(\alpha \cap C{}^\backprime P) = \overrightarrow{P}{}^\backprime x\} = C{}^\backprime P \cap \hat{x}(\mathrm{V} = \overrightarrow{P}{}^\backprime x)$ (4)

$\vdash . *50\cdot24 . \supset \vdash : \mathrm{Hp} . \supset . (x) . x \sim\in \overrightarrow{P}{}^\backprime x .$

[*24·104] $\supset . (x) . \overrightarrow{P}{}^\backprime x \neq \mathrm{V}$ (5)

$\vdash . (4) . (5) . \supset \vdash : \mathrm{Hp} . \alpha \cap C{}^\backprime P = \Lambda . \supset . C{}^\backprime P \cap \hat{x}\{p{}^\backprime\overrightarrow{P}{}^{\backprime\backprime}(\alpha \cap C{}^\backprime P) = \overrightarrow{P}{}^\backprime x\} = \Lambda$

[(3)] $= \overrightarrow{\mathrm{seq}}_P{}^\backprime p{}^\backprime\overrightarrow{P}{}^{\backprime\backprime}(\alpha \cap C{}^\backprime P)$ (6)

$\vdash . (2) . (6) . *206\cdot53 . \supset \vdash . \mathrm{Prop}$

***206·54.** $\vdash :. P \in \mathrm{Ser} . \supset : \mathrm{E} ! \,\mathrm{seq}_P\text{‘}p\text{‘}\overrightarrow{P}\text{‘‘}(\alpha \cap C\text{‘}P) . \equiv . \mathrm{E} ! \,\mathrm{min}_P\text{‘}\alpha :$

$$\mathrm{E} ! \,\mathrm{max}_P\text{‘}p\text{‘}\overrightarrow{P}\text{‘‘}(\alpha \cap C\text{‘}P) . \equiv . \mathrm{E} ! \,\mathrm{prec}_P\text{‘}\alpha$$

Dem.

$\vdash . *206{\cdot}53 . \quad \supset \vdash :. \mathrm{Hp} . \supset : \mathrm{E} ! \,\mathrm{seq}_P\text{‘}p\text{‘}\overrightarrow{P}\text{‘‘}(\alpha \cap C\text{‘}P) . \equiv . \mathrm{E} ! \,\mathrm{min}_P\text{‘}\alpha$ (1)

$\vdash . *206{\cdot}13{\cdot}101 . \supset \vdash : \mathrm{E} ! \,\mathrm{max}_P\text{‘}p\text{‘}\overrightarrow{P}\text{‘‘}(\alpha \cap C\text{‘}P) . \equiv . \mathrm{E} ! \,\mathrm{prec}_P\text{‘}\alpha$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***206·55.** $\vdash :. P \in \mathrm{Ser} . \supset : (\alpha) . \alpha \in \text{ᗡ}\text{‘}\mathrm{min}_P \cup \text{ᗡ}\text{‘}\mathrm{prec}_P . \equiv .$

$$(\alpha) . p\text{‘}\overrightarrow{P}\text{‘‘}(\alpha \cap C\text{‘}P) \in \text{ᗡ}\text{‘}\mathrm{max}_P \cup \text{ᗡ}\text{‘}\mathrm{seq}_P \quad [*206{\cdot}54{\cdot}161 . *205{\cdot}32]$$

***206·551.** $\vdash : P \in \mathrm{Ser} . \supset . \overrightarrow{\mathrm{seq}}_P\text{‘}\alpha = \overrightarrow{\mathrm{seq}}_P\text{‘}p\text{‘}\overrightarrow{P}\text{‘‘}p\text{‘}\overleftarrow{P}\text{‘‘}(\alpha \cap C\text{‘}P) .$

$$\overrightarrow{\mathrm{max}}_P\text{‘}\alpha = \overrightarrow{\mathrm{max}}_P\text{‘}p\text{‘}\overrightarrow{P}\text{‘‘}p\text{‘}\overleftarrow{P}\text{‘‘}(\alpha \cap C\text{‘}P)$$

Dem.

$\vdash . *206{\cdot}13 . \quad \supset \vdash . \overrightarrow{\mathrm{seq}}_P\text{‘}\alpha = \overrightarrow{\mathrm{min}}_P\text{‘}p\text{‘}\overleftarrow{P}\text{‘‘}(\alpha \cap C\text{‘}P)$ (1)

$\vdash . (1) . *206{\cdot}53 . \supset \vdash : \mathrm{Hp} . \supset . \overrightarrow{\mathrm{seq}}_P\text{‘}\alpha = \overrightarrow{\mathrm{seq}}_P\text{‘}p\text{‘}\overrightarrow{P}\text{‘‘}\{p\text{‘}\overleftarrow{P}\text{‘‘}(\alpha \cap C\text{‘}P) \cap C\text{‘}P\}$ (2)

$\vdash . (2) . *200{\cdot}54 . \supset \vdash : \mathrm{Hp} . \dot{\exists} ! \,P . \supset . \overrightarrow{\mathrm{seq}}_P\text{‘}\alpha = \overrightarrow{\mathrm{seq}}_P\text{‘}p\text{‘}\overrightarrow{P}\text{‘‘}p\text{‘}\overleftarrow{P}\text{‘‘}(\alpha \cap C\text{‘}P)$ (3)

$\vdash . *206{\cdot}18 . \quad \supset \vdash : P = \dot{\Lambda} . \supset . \overrightarrow{\mathrm{seq}}_P\text{‘}\alpha = \Lambda . \overrightarrow{\mathrm{seq}}_P\text{‘}p\text{‘}\overrightarrow{P}\text{‘‘}p\text{‘}\overleftarrow{P}\text{‘‘}(\alpha \cap C\text{‘}P) = \Lambda$ (4)

$\vdash . (3) . (4) . \quad \supset \vdash : \mathrm{Hp} . \supset . \overrightarrow{\mathrm{seq}}_P\text{‘}\alpha = \overrightarrow{\mathrm{seq}}_P\text{‘}p\text{‘}\overrightarrow{P}\text{‘‘}p\text{‘}\overleftarrow{P}\text{‘‘}(\alpha \cap C\text{‘}P)$ (5)

$\vdash . *206{\cdot}53 . \quad \supset \vdash : \mathrm{Hp} . \supset . \overrightarrow{\mathrm{max}}_P\text{‘}\alpha = \overrightarrow{\mathrm{prec}}_P\text{‘}p\text{‘}\overleftarrow{P}\text{‘‘}(\alpha \cap C\text{‘}P)$

$[*206{\cdot}13{\cdot}101 . *200{\cdot}54] \quad = \overrightarrow{\mathrm{max}}_P\text{‘}p\text{‘}\overrightarrow{P}\text{‘‘}p\text{‘}\overleftarrow{P}\text{‘‘}(\alpha \cap C\text{‘}P)$ (6)

$\vdash . (5) . (6) . \supset \vdash . \mathrm{Prop}$

***206·56.** $\vdash :. P \in \mathrm{Ser} . \supset : (\alpha) . p\text{‘}\overrightarrow{P}\text{‘‘}(\alpha \cap C\text{‘}P) \in \text{ᗡ}\text{‘}\mathrm{max}_P \cup \text{ᗡ}\text{‘}\mathrm{seq}_P . \equiv .$

$$(\alpha) . \alpha \in \text{ᗡ}\text{‘}\mathrm{max}_P \cup \text{ᗡ}\text{‘}\mathrm{seq}_P$$

Dem.

$\vdash . *10{\cdot}1{\cdot}11 . \supset \vdash : (\alpha) . \alpha \in \text{ᗡ}\text{‘}\mathrm{max}_P \cup \text{ᗡ}\text{‘}\mathrm{seq}_P . \supset .$

$(\alpha) . p\text{‘}\overrightarrow{P}\text{‘‘}(\alpha \cap C\text{‘}P) \in \text{ᗡ}\text{‘}\mathrm{max}_P \cup \text{ᗡ}\text{‘}\mathrm{seq}_P$ (1)

$\vdash . *10{\cdot}1 . \quad \supset \vdash : (\alpha) . p\text{‘}\overrightarrow{P}\text{‘‘}(\alpha \cap C\text{‘}P) \in \text{ᗡ}\text{‘}\mathrm{max}_P \cup \text{ᗡ}\text{‘}\mathrm{seq}_P . \supset .$

$p\text{‘}\overrightarrow{P}\text{‘‘}p\text{‘}\overleftarrow{P}\text{‘‘}(\beta \cap C\text{‘}P) \in \text{ᗡ}\text{‘}\mathrm{max}_P \cup \text{ᗡ}\text{‘}\mathrm{seq}_P .$

$[*206{\cdot}551] \quad \supset . \beta \in \text{ᗡ}\text{‘}\mathrm{max}_P \cup \text{ᗡ}\text{‘}\mathrm{seq}_P$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***206·57.** $\vdash :. P \in \mathrm{Ser} . \supset : (\alpha) . \alpha \in \text{ᗡ}\text{‘}\mathrm{min}_P \cup \text{ᗡ}\text{‘}\mathrm{prec}_P . \equiv .$

$$(\alpha) . \alpha \in \text{ᗡ}\text{‘}\mathrm{max}_P \cup \text{ᗡ}\text{‘}\mathrm{seq}_P \quad [*206{\cdot}55{\cdot}56]$$

This proposition is important, since it shows that when a serial relation satisfies Dedekind's axiom, so does its converse. Thus if all classes which have no maximum have an upper limit, then all classes which have no minimum have a lower limit, and vice versa.

***206·6.** $\vdash : S \in P \,\overline{\text{smor}}\, Q \,.\supset .\, p\text{‘}\overleftarrow{P}\text{“}(\alpha \cap C\text{‘}P) = S\text{“}p\text{‘}\overleftarrow{Q}\text{“}\breve{S}\text{“}\alpha$

Dem.

$\vdash .\, *151·11 \,.\supset \vdash : \text{Hp} \,.\supset .\, p\text{‘}\overleftarrow{P}\text{“}(\alpha \cap C\text{‘}P) = p\text{‘}S\text{“‘}\breve{Q}\text{“‘}\overleftarrow{S}\text{“}(\alpha \cap \text{D‘}S)$

$[*72·341] \qquad = S\text{“}p\text{‘}\breve{Q}\text{“‘}\overleftarrow{S}\text{“}(\alpha \cap \text{D‘}S)$

$[*71·613] \qquad = S\text{“}p\text{‘}\overleftarrow{Q}\text{“}\breve{S}\text{“}\alpha : \supset \vdash .\, \text{Prop}$

***206·61.** $\vdash : S \in P \,\overline{\text{smor}}\, Q \,.\supset .\, \overrightarrow{\text{seq}}_P\text{‘}\alpha = S\text{“}\overrightarrow{\text{seq}}_Q\text{‘}\breve{S}\text{“}\alpha$

Dem.

$\vdash .\, *205·8 \,.\, *206·6·13 \,.\supset \vdash : \text{Hp} \,.\supset .\, \overrightarrow{\text{seq}}_P\text{‘}\alpha = S\text{“}\overrightarrow{\text{min}}_Q\text{‘}\breve{S}\text{“}S\text{“}p\text{‘}\overleftarrow{Q}\text{“}\breve{S}\text{“}\alpha$

$[*72·501.*151·11] \qquad = S\text{“}\overrightarrow{\text{min}}_Q\text{‘}(p\text{‘}\overleftarrow{Q}\text{“}\breve{S}\text{“}\alpha \cap C\text{‘}Q)$

$[*206·13.*205·15] \qquad = S\text{“}\overrightarrow{\text{seq}}_Q\text{‘}\breve{S}\text{“}\alpha : \supset \vdash .\, \text{Prop}$

***206·62.** $\vdash :.\, S \in P \,\overline{\text{smor}}\, Q \,.\supset : \text{E!}\, \text{seq}_P\text{‘}\alpha \,.\equiv .\, \text{E!}\, \text{seq}_Q\text{‘}\breve{S}\text{“}\alpha$
$[*206·61 \,.\, *73·22·44 \,.\, *53·3]$

***206·63.** $\vdash : S \in P \,\overline{\text{smor}}\, Q \,.\, \text{E!}\, \text{seq}_P\text{‘}\alpha \,.\supset .\, \text{seq}_P\text{‘}\alpha = S\text{‘}\text{seq}_Q\text{‘}\breve{S}\text{“}\alpha$
$[*206·61·62 \,.\, *53·31]$

***206·7.** $\vdash : P \in \text{trans} \,.\, \beta \subset C\text{‘}P \,.\sim(yPy) \,.\, y \sim\in \overrightarrow{\text{max}}_P\text{‘}\beta \,.\supset .$
$p\text{‘}\overleftarrow{P}\text{“}\beta = p\text{‘}\overleftarrow{P}\text{“}(\beta - \iota\text{‘}y)$

Dem.

$\vdash .\, *51·222 \,.\supset \vdash : y \sim\in \beta \,.\supset .\, p\text{‘}\overleftarrow{P}\text{“}\beta = p\text{‘}\overleftarrow{P}\text{“}(\beta - \iota\text{‘}y)$ (1)

$\vdash .\, *205·111 \,.\supset \vdash :.\, \text{Hp} \,.\, y \in \beta \,.\supset : y \in P\text{“}\beta \,.\sim(yPy) :$

$[*37·1] \qquad \supset : (\exists x) \,.\, x \in \beta - \iota\text{‘}y \,.\, yPx :$

$[*10·56.\text{Hp}] \qquad \supset : z \in p\text{‘}\overleftarrow{P}\text{“}(\beta - \iota\text{‘}y) \,.\supset .\, yPz :$

$[*53·14.*51·221] \qquad \supset : p\text{‘}\overleftarrow{P}\text{“}(\beta - \iota\text{‘}y) \subset p\text{‘}\overleftarrow{P}\text{“}\beta :$

$[*40·16] \qquad \supset : p\text{‘}\overleftarrow{P}\text{“}(\beta - \iota\text{‘}y) = p\text{‘}\overleftarrow{P}\text{“}\beta$ (2)

$\vdash .\, (1) \,.\, (2) \,.\supset \vdash .\, \text{Prop}$

***206·71.** $\vdash : P \in \text{trans} \,.\, \beta \subset C\text{‘}P \,.\sim(yPy) \,.\, y \sim\in \overrightarrow{\text{max}}_P\text{‘}\beta \,.\supset .\, \overrightarrow{\text{seq}}_P\text{‘}\beta = \overrightarrow{\text{seq}}_P\text{‘}(\beta - \iota\text{‘}y)$

Dem.

$\vdash .\, *51·222 \,.\supset \vdash : y \sim\in \beta \,.\supset .\, \overrightarrow{\text{seq}}_P\text{‘}\beta = \overrightarrow{\text{seq}}_P\text{‘}(\beta - \iota\text{‘}y)$ (1)

$\vdash .\, *205·111 \,.\supset \vdash : \text{Hp} \,.\, y \in \beta \,.\supset .\, y \in P\text{“}\beta \,.\sim(yPy) \,.$

$[*37·1] \qquad \supset .\, (\exists z) \,.\, z \in \beta - \iota\text{‘}y \,.\, yPz$ (2)

$\vdash .\, (2) \,.\, *10·56 \,.\, *201·1 \,.\supset \vdash : \text{Hp} \,.\, y \in \beta \,.\, \beta - \iota\text{‘}y \subset \overrightarrow{P}\text{‘}x \,.\supset .\, yPx \,.$

$[*32·18] \qquad \supset .\, \beta \subset \overrightarrow{P}\text{‘}x$ (3)

$\vdash .\, (3) \,.\, *206·7 \,.\supset \vdash :.\, \text{Hp}(2) \,.\supset :$

$\beta \subset \overrightarrow{P}\text{‘}x \,.\, \overrightarrow{P}\text{‘}x \subset -p\text{‘}\overleftarrow{P}\text{“}\beta \,.\equiv .\, \beta - \iota\text{‘}y \subset \overrightarrow{P}\text{‘}x \,.\, \overrightarrow{P}\text{‘}x \subset -p\text{‘}\overleftarrow{P}\text{“}(\beta - \iota\text{‘}y) :$

$[*206·134] \supset : x \,\text{seq}_P\, \beta \,.\equiv .\, x \,\text{seq}_P\, (\beta - \iota\text{‘}y)$ (4)

$\vdash .\, (1) \,.\, (4) \,.\supset \vdash .\, \text{Prop}$

∗206·72. $\vdash : P \epsilon \text{trans} . \sim(yPy) . y \sim\epsilon \overrightarrow{\max}_P{}^{\boldsymbol{\cdot}}\beta . \supset . \overrightarrow{\text{seq}}_P{}^{\boldsymbol{\cdot}}\beta = \overrightarrow{\text{seq}}_P{}^{\boldsymbol{\cdot}}(\beta - \iota^{\boldsymbol{\cdot}}y)$

Dem.

$\vdash . *206{\cdot}71{\cdot}131 . *205{\cdot}151 . \supset \vdash : \text{Hp} . \supset . \overrightarrow{\text{seq}}_P{}^{\boldsymbol{\cdot}}\beta = \overrightarrow{\text{seq}}_P{}^{\boldsymbol{\cdot}}(\beta \cap C^{\boldsymbol{\cdot}}P - \iota^{\boldsymbol{\cdot}}y)$

[∗206·131] $\quad = \overrightarrow{\text{seq}}_P{}^{\boldsymbol{\cdot}}(\beta - \iota^{\boldsymbol{\cdot}}y) : \supset \vdash . \text{Prop}$

∗206·73. $\vdash : \exists ! \gamma \cap C^{\boldsymbol{\cdot}}P . \text{E} ! \text{prec}_P{}^{\boldsymbol{\cdot}}\gamma . \text{E} ! \text{seq}_P{}^{\boldsymbol{\cdot}}\gamma . \supset . \text{prec}_P{}^{\boldsymbol{\cdot}}\gamma \, P^2 \, \text{seq}_P{}^{\boldsymbol{\cdot}}\gamma$

Dem.

$\vdash . *206{\cdot}211 . \supset \vdash : \text{Hp} . \supset . \gamma \cap C^{\boldsymbol{\cdot}}P \subset \overrightarrow{P}{}^{\boldsymbol{\cdot}}\text{seq}_P{}^{\boldsymbol{\cdot}}\gamma \cap \overleftarrow{P}{}^{\boldsymbol{\cdot}}\text{prec}_P{}^{\boldsymbol{\cdot}}\gamma . \exists ! \gamma \cap C^{\boldsymbol{\cdot}}P .$

[∗34·11] $\quad \supset . \text{prec}_P{}^{\boldsymbol{\cdot}}\gamma \, P^2 \, \text{seq}_P{}^{\boldsymbol{\cdot}}\gamma : \supset \vdash . \text{Prop}$

∗206·731. $\vdash :. \exists ! \gamma \cap C^{\boldsymbol{\cdot}}P : P \epsilon \text{trans} . \lor . P^2 \in J : \supset . \sim(\text{prec}_P{}^{\boldsymbol{\cdot}}\gamma = \text{seq}_P{}^{\boldsymbol{\cdot}}\gamma)$

Dem.

$\vdash . *206{\cdot}73 . \supset$

$\vdash : \exists ! \gamma \cap C^{\boldsymbol{\cdot}}P . \text{E} ! \text{prec}_P{}^{\boldsymbol{\cdot}}\gamma . \text{E} ! \text{seq}_P{}^{\boldsymbol{\cdot}}\gamma . P \epsilon \text{trans} . \supset . \text{prec}_P{}^{\boldsymbol{\cdot}}\gamma \, P \, \text{seq}_P{}^{\boldsymbol{\cdot}}\gamma .$

[∗206·133] $\quad \supset . \text{prec}_P{}^{\boldsymbol{\cdot}}\gamma \neq \text{seq}_P{}^{\boldsymbol{\cdot}}\gamma \quad (1)$

$\vdash . *206{\cdot}73 . \supset$

$\vdash : \exists ! \gamma \cap C^{\boldsymbol{\cdot}}P . \text{E} ! \text{prec}_P{}^{\boldsymbol{\cdot}}\gamma . \text{E} ! \text{seq}_P{}^{\boldsymbol{\cdot}}\gamma . P^2 \in J . \supset . \text{prec}_P{}^{\boldsymbol{\cdot}}\gamma \neq \text{seq}_P{}^{\boldsymbol{\cdot}}\gamma \quad (2)$

$\vdash . *14{\cdot}21 . \supset \vdash : \sim(\text{E} ! \text{prec}_P{}^{\boldsymbol{\cdot}}\gamma . \text{E} ! \text{seq}_P{}^{\boldsymbol{\cdot}}\gamma) . \supset . \sim(\text{prec}_P{}^{\boldsymbol{\cdot}}\gamma = \text{seq}_P{}^{\boldsymbol{\cdot}}\gamma) \quad (3)$

$\vdash . (1) . (2) . (3) . \supset \vdash . \text{Prop}$

Note that "$\text{prec}_P{}^{\boldsymbol{\cdot}}\gamma \neq \text{seq}_P{}^{\boldsymbol{\cdot}}\gamma$" is not the same proposition as $\sim(\text{prec}_P{}^{\boldsymbol{\cdot}}\gamma = \text{seq}_P{}^{\boldsymbol{\cdot}}\gamma)$. The former involves $\text{E} ! \text{prec}_P{}^{\boldsymbol{\cdot}}\gamma . \text{E} ! \text{seq}_P{}^{\boldsymbol{\cdot}}\gamma$, while the latter does not, in virtue of the conventions as to descriptive symbols explained in ∗14.

∗206·732. $\vdash :. P \epsilon \text{trans} . \lor . P^2 \in J : \supset . \sim(\text{prec}_P{}^{\boldsymbol{\cdot}}\gamma = \text{seq}_P{}^{\boldsymbol{\cdot}}\gamma)$

Dem.

$\vdash . *206{\cdot}14 . \supset \vdash : \gamma \cap C^{\boldsymbol{\cdot}}P = \Lambda . \supset . \overrightarrow{\text{prec}}_P{}^{\boldsymbol{\cdot}}\gamma = \overrightarrow{B}{}^{\boldsymbol{\cdot}}\breve{P} . \text{seq}_P{}^{\boldsymbol{\cdot}}\gamma = \overrightarrow{B}{}^{\boldsymbol{\cdot}}P .$

[∗93·101] $\quad \supset . \overrightarrow{\text{prec}}_P{}^{\boldsymbol{\cdot}}\gamma \cap \overrightarrow{\text{seq}}_P{}^{\boldsymbol{\cdot}}\gamma = \Lambda .$

[∗53·4] $\quad \supset . \sim(\text{prec}_P{}^{\boldsymbol{\cdot}}\gamma = \text{seq}_P{}^{\boldsymbol{\cdot}}\gamma) \quad (1)$

$\vdash . (1) . *206{\cdot}731 . \supset \vdash . \text{Prop}$

*207. LIMITS

Summary of *207.

A term x is said to be the "upper limit" of α in P if α has no maximum and x is the sequent of α. In this case, x immediately follows the class α, though there is no one member of α which x immediately follows. Sequents which are limits have special importance, and it is convenient to have a special notation for them. We write "$\mathrm{lt}_P{}^{\boldsymbol{\prime}}\alpha$" for the upper limit of α; or, if it is more convenient, "$\mathrm{lt}\,(P){}^{\boldsymbol{\prime}}\alpha$." (This is more convenient when P is replaced by an expression consisting of several letters, or by a letter with a suffix.) The *lower* limit of α will be the immediate predecessor of α when α has no minimum; this we denote by $\mathrm{tl}_P{}^{\boldsymbol{\prime}}\alpha$.

The following propositions on limits for the most part follow immediately from the propositions of *206 on sequents.

Our definition is so framed that the limit of the null-class is the first member of our series (if any). This departure from usage is convenient in order that, whenever our series contains any limiting point in the ordinary sense, the *series* of limiting points may exist, *i.e.* in order that $P \upharpoonleft \mathrm{D}{}^{\boldsymbol{\prime}}\mathrm{lt}_P$ may exist whenever there are existent parts of $C{}^{\boldsymbol{\prime}}P$ which have upper limits. The series $P \upharpoonleft \mathrm{D}{}^{\boldsymbol{\prime}}\mathrm{lt}_P$ is the "first derivative" of P. The definition of a limit is

$$\mathrm{lt}_P = \mathrm{seq}_P \upharpoonright (-\mathrm{G}{}^{\boldsymbol{\prime}}\mathrm{max}_P) \quad \mathrm{Df.}$$

Besides the limit, we require, for many purposes, a single notation for the "limit or maximum." This we denote by "limax_P," putting

$$\mathrm{limax}_P = \mathrm{max}_P \cup \mathrm{lt}_P \quad \mathrm{Df.}$$

Similarly for the lower limit or minimum we use "limin_P," putting

$$\mathrm{limin}_P = \mathrm{min}_P \cup \mathrm{tl}_P \quad \mathrm{Df.}$$

We have $\mathrm{tl}_P = \mathrm{lt}\,(\breve{P})$ (*207·101) and $\mathrm{limin}_P = \mathrm{limax}\,(\breve{P})$ (*207·401). Hence it is unnecessary to prove propositions concerning lower limits, since they result immediately from propositions concerning upper limits.

In virtue of our definition of a limit, x limits α if x is a sequent of α and α has no maximum (*207·1). Thus if α has a maximum, it has no limit (*207·11), but if it has no maximum, the class of its limits is the class of its sequents (*207·12). Thus the existence of the class of limits is equivalent to the existence of the class of sequents combined with the non-existence of the class of maxima, *i.e.*

***207·13.** $\vdash : \exists ! \overrightarrow{\mathrm{lt}_P}{}^{\boldsymbol{\prime}}\alpha \,.\equiv.\, \sim\exists ! \overrightarrow{\mathrm{max}_P}{}^{\boldsymbol{\prime}}\alpha \,.\, \exists ! \overrightarrow{\mathrm{seq}_P}{}^{\boldsymbol{\prime}}\alpha$

*207·2—·232 consist of various formulae for $\overrightarrow{\mathrm{lt}}_P$‘$\alpha$. We have

***207·2.** $\vdash : P \epsilon \text{connex} . x \,\mathrm{lt}_P\, \alpha . \supset . \alpha \cap C\text{‘}P \subset \overrightarrow{P}\text{‘}x . \overrightarrow{P}\text{‘}x \subset P\text{‘‘}\alpha$

I.e. the whole of $\alpha \cap C$‘P precedes x, but any predecessor of x precedes some member of α.

***207·231.** $\vdash : P \epsilon \text{Ser} . \exists ! \overrightarrow{\mathrm{lt}}_P\text{‘}\alpha . \supset . \overrightarrow{\mathrm{lt}}_P\text{‘}\alpha = C\text{‘}P \cap \hat{x}(\overrightarrow{P}\text{‘}x = P\text{‘‘}\alpha)$

I.e. the limit of α, if it exists, is the term whose predecessors are identical with the predecessors of some part of α.

We have also

***207·232.** $\vdash :. P \epsilon \text{Ser} . \supset : x = \mathrm{lt}_P\text{‘}\alpha . \equiv . x \epsilon C\text{‘}P - \alpha . \overrightarrow{P}\text{‘}x = P\text{‘‘}\alpha$

This proposition should be compared with *205·54, which (slightly re-written) is

$$\vdash :. P \epsilon \text{Ser} . \supset : x = \max_P\text{‘}\alpha . \equiv . x \epsilon C\text{‘}P \cap \alpha . \overrightarrow{P}\text{‘}x = P\text{‘‘}\alpha$$

From the two together we arrive at

***207·51.** $\vdash :. P \epsilon \text{Ser} . \supset : x = \mathrm{limax}_P\text{‘}\alpha . \equiv . x \epsilon C\text{‘}P . \overrightarrow{P}\text{‘}x = P\text{‘‘}\alpha$

which serves to illustrate the utility of "limax_P."

We have

***207·24.** $\vdash : P \epsilon \text{connex} . \supset . \overrightarrow{\mathrm{lt}}_P\text{‘}\alpha \epsilon 0 \cup 1 . \mathrm{lt}_P \epsilon 1 \rightarrow \text{Cls}$

I.e. if P is connected, a class cannot have more than one limit; also

***207·25.** $\vdash : P \epsilon \text{trans} . \beta \subset P\text{‘‘}\alpha . \supset . \overrightarrow{\mathrm{lt}}_P\text{‘}(\alpha \cup \beta) = \overrightarrow{\mathrm{lt}}_P\text{‘}\alpha$

I.e. any terms which have some α's beyond them may be added to α without altering the limit.

We next have a set of propositions (*207·251—·27) proving that if a class has a limit, any single term of the class may be removed without altering the limit (*207·261), and that in any case, provided the class is not a unit class, its minimum (if any) may be removed without altering the limit (*207·27). We then prove (*207·291) that if P is a series, and α is a class which has a limit, the predecessors of the limit are the class P_*‘‘α.

We then have a set of propositions (*207·3—·36) on the limit of $\overrightarrow{P}$‘x and kindred matters. If x has no immediate predecessor, the limit of $\overrightarrow{P}$‘x is x, and vice versa (*207·32·33). Hence

***207·35.** $\vdash : P \epsilon \text{Rl‘}J \cap \text{connex} . \supset . \text{D‘}\mathrm{lt}_P = C\text{‘}P - \text{Ɑ‘}(P \dot{-} P^2)$

I.e. the limit-points of P are those which have no immediate predecessors.

We next turn our attention to "limax_P." This again is one-many, provided P is connected (*207·41). We have by the definition

***207·42.** $\vdash : \exists ! \overrightarrow{\max}_P\text{‘}\alpha . \supset . \overrightarrow{\mathrm{limax}}_P\text{‘}\alpha = \overrightarrow{\max}_P\text{‘}\alpha$

*207·43. $\vdash : \overrightarrow{\max}_P{}'\alpha = \Lambda . \supset . \overrightarrow{\text{limax}}_P{}'\alpha = \overrightarrow{\text{seq}}_P{}'\alpha = \overrightarrow{\text{lt}}_P{}'\alpha$

*207·44. $\vdash . \text{Cl}'\text{limax}_P = \text{Cl}'\max_P \cup \text{Cl}'\text{lt}_P = \text{Cl}'\max_P \cup \text{Cl}'\text{seq}_P$

*207·45. $\vdash . \overrightarrow{\text{limax}}_P{}'\alpha = \overrightarrow{\max}_P{}'\alpha \cup \overrightarrow{\text{lt}}_P{}'\alpha$

Also we have

*207·46. $\vdash :. x = \text{limax}_P{}'\alpha . \equiv : x = \max_P{}'\alpha . \vee . x = \text{lt}_P{}'\alpha$

which is a very useful proposition, as is also *207·51 (given above).

A useful proposition in dealing with classes of classes contained in a series is

*207·54. $\vdash : P \in \text{Ser} . \kappa \subset \text{Cl}'\text{lt}_P . \supset . \overrightarrow{\text{limax}}_P{}'\text{lt}_P{}``\kappa = \overrightarrow{\text{limax}}_P{}'s'\kappa = \overrightarrow{\text{lt}}_P{}'s'\kappa$

I.e. if every member of κ has a limit, the limit or maximum (if any) of the limits is the limit or maximum, and in fact the limit, of $s'\kappa$.

We have next a set of propositions (*207·6—·66) on correlations, proving that the limit, or the limax, of the correlates is the correlate of the limit or limax, *i.e.*

*207·6. $\vdash : S \in P \,\overline{\text{smor}}\, Q . \supset . \overrightarrow{\text{lt}}_P{}'\alpha = S``\overrightarrow{\text{lt}}_Q{}'\breve{S}``\alpha$

*207·64. $\vdash : S \in P \,\overline{\text{smor}}\, Q . \supset . \overrightarrow{\text{limax}}_P{}'\alpha = S``\overrightarrow{\text{limax}}_Q{}'\breve{S}``\alpha$

The last three propositions (*207·7—·72) are lemmas for use in the theory of stretches (*215·5·51).

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*207·01. $\text{lt}_P = \text{lt}(P) = \text{seq}_P \upharpoonright (-\text{Cl}'\max_P)$ Df

*207·02. $\text{tl}_P = \text{tl}(P) = \text{prec}_P \upharpoonright (-\text{Cl}'\min_P)$ Df

*207·03. $\text{limax}_P = \max_P \cup \text{lt}_P$ Df

*207·04. $\text{limin}_P = \min_P \cup \text{tl}_P$ Df

*207·1. $\vdash : x \,\text{lt}_P\, \alpha . \equiv . x \,\text{seq}_P\, \alpha . \sim \exists ! \overrightarrow{\max}_P{}'\alpha$ [(*207·01)]

*207·101. $\vdash . \text{tl}_P = \text{lt}(\breve{P})$ [*205·102 . *206·101 . (*207·02)]

We shall not give further propositions on lower limits, unless for some special reason, since all of them result from propositions on upper limits by means of *207·101.

*207·11. $\vdash : \exists ! \overrightarrow{\max}_P{}'\alpha . \supset . \overrightarrow{\text{lt}}_P{}'\alpha = \Lambda$ [*207·1]

*207·12. $\vdash : \overrightarrow{\max}_P{}'\alpha = \Lambda . \supset . \overrightarrow{\text{lt}}_P{}'\alpha = \overrightarrow{\text{seq}}_P{}'\alpha$ [*207·1]

*207·121. $\vdash : \alpha \cap C'P \subset P``\alpha . \supset . \overrightarrow{\text{lt}}_P{}'\alpha = \overrightarrow{\text{seq}}_P{}'\alpha$ [*207·12 . *205·123]

*207·13. $\vdash : \exists ! \overrightarrow{\text{lt}}_P{}'\alpha . \equiv . \sim \exists ! \overrightarrow{\max}_P{}'\alpha . \exists ! \overrightarrow{\text{seq}}_P{}'\alpha$ [*207·1]

*207·14. $\vdash :. \exists ! \overrightarrow{\max}_P{}'\alpha . \vee . \exists ! \overrightarrow{\text{seq}}_P{}'\alpha : \equiv : \exists ! \overrightarrow{\max}_P{}'\alpha . \vee . \exists ! \overrightarrow{\text{lt}}_P{}'\alpha$
[*207·13 . *5·63]

The above proposition is important because

$$(\alpha) : \exists ! \overrightarrow{\max_P}`\alpha \,.\vee.\, \exists ! \overrightarrow{\mathrm{lt}_P}`\alpha$$

is the characteristic of "Dedekindian" series, *i.e.* of such as fulfil Dedekind's axiom.

***207·15.** $\vdash : x \,\mathrm{lt}_P\, \alpha \,.\equiv.\, x \in C`P \,.\, \alpha \cap C`P \subset P``\alpha \cap \overrightarrow{P}`x \,.\, \overrightarrow{P}`x \subset -p`\overleftarrow{P}``(\alpha \cap C`P)$
[*207·1 . *205·123 . *206·134]

***207·16.** $\vdash . \overrightarrow{\mathrm{lt}_P}`\alpha = \overrightarrow{\mathrm{lt}_P}`(\alpha \cap C`P)$ [*207·15 . *37·265]

***207·17.** $\vdash . \overrightarrow{\mathrm{lt}_P}`\Lambda = \overrightarrow{B}`P$ [*207·12 . *205·161 . *206·14]

***207·18.** $\vdash : \text{Ɑ}`P \subset \text{D}`\mathrm{lt}_P \,.\equiv.\, C`P = \text{D}`\mathrm{lt}_P$

Dem.

$\vdash . \ast 207{\cdot}17 . \supset \vdash : \text{Ɑ}`P \subset \text{D}`\mathrm{lt}_P \,.\equiv.\, \text{Ɑ}`P \cup \overrightarrow{B}`P \subset \text{D}`\mathrm{lt}_P \,.$
[*93·103] $\equiv . C`P \subset \text{D}`\mathrm{lt}_P \,.$
[*207·15] $\equiv . C`P = \text{D}`\mathrm{lt}_P : \supset \vdash . \text{Prop}$

***207·2.** $\vdash : P \in \text{connex} \,.\, x \,\mathrm{lt}_P\, \alpha \,.\supset.\, \alpha \cap C`P \subset \overrightarrow{P}`x \,.\, \overrightarrow{P}`x \subset P``\alpha$
[*207·15 . *202·503]

***207·21.** $\vdash : P^2 \in J \,.\, x \in C`P \,.\, \alpha \cap C`P \subset \overrightarrow{P}`x \,.\, \overrightarrow{P}`x \subset P``\alpha \,.\supset.\, x \,\mathrm{lt}_P\, \alpha$

Dem.

$\vdash . \ast 200{\cdot}53 . \supset \vdash : P^2 \in J . \supset . P``\alpha \subset -p`\overleftarrow{P}``(\alpha \cap C`P)$ (1)
$\vdash . (1) . \supset \vdash : \text{Hp} . \supset . x \in C`P \,.\, \alpha \cap C`P \subset \overrightarrow{P}`x \,.\, \overrightarrow{P}`x \subset -p`\overleftarrow{P}``(\alpha \cap C`P) \,.$
[*206·134] $\supset . x \,\mathrm{seq}_P\, \alpha$ (2)
$\vdash . \ast 22{\cdot}44 . \supset \vdash : \text{Hp} . \supset . \alpha \cap C`P \subset P``\alpha \,.$
[*205·123] $\supset . \overrightarrow{\max_P}`\alpha = \Lambda$ (3)
$\vdash . (2) . (3) . \ast 207{\cdot}1 . \supset \vdash . \text{Prop}$

***207·22.** $\vdash : P \in \text{connex} \,.\, P^2 \in J \,.\supset.\, \overrightarrow{\mathrm{lt}_P}`\alpha = C`P \cap \hat{x}(\alpha \cap C`P \subset \overrightarrow{P}`x \,.\, \overrightarrow{P}`x \subset P``\alpha)$
[*207·2·21]

This is very often the most convenient form for $\overrightarrow{\mathrm{lt}_P}`\alpha$. It states that a limit of $\alpha$ is a member $x$ of $C`P$ such that $\alpha \cap C`P$ wholly precedes x, but every predecessor of x precedes some member of α.

***207·23.** $\vdash : P \in \text{Ser} . \supset . \overrightarrow{\mathrm{lt}_P}`\alpha = C`P \cap \hat{x}(\overrightarrow{P}`x = P``\alpha \,.\, \alpha \cap C`P \subset P``\alpha)$

Dem.

$\vdash . \ast 13{\cdot}12 . \ast 22{\cdot}42 . \supset$
$\vdash : \overrightarrow{P}`x = P``\alpha \,.\, \alpha \cap C`P \subset P``\alpha . \supset . \alpha \cap C`P \subset \overrightarrow{P}`x \,.\, \overrightarrow{P}`x \subset P``\alpha$ (1)
$\vdash . \ast 201{\cdot}501 . \ast 37{\cdot}265 . \supset \vdash :. P \in \text{trans} . \supset : \alpha \cap C`P \subset \overrightarrow{P}`x . \supset . P``\alpha \subset \overrightarrow{P}`x :$
[Fact] $\supset : \alpha \cap C`P \subset \overrightarrow{P}`x \,.\, \overrightarrow{P}`x \subset P``\alpha . \supset . \overrightarrow{P}`x = P``\alpha$ (2)

$\vdash . *22\cdot44 . \supset \vdash : \alpha \cap C`P \subset \overrightarrow{P}`x . \overrightarrow{P}`x \subset P``\alpha . \supset . \alpha \cap C`P \subset P``\alpha$ (3)

$\vdash . (1) . (2) . (3) . \supset$

$\vdash :. P \epsilon \text{trans} . \supset : \alpha \cap C`P \subset \overrightarrow{P}`x . \overrightarrow{P}`x \subset P``\alpha . \equiv . \overrightarrow{P}`x = P``\alpha . \alpha \cap C`P \subset P``\alpha$ (4)

$\vdash . (4) . *207\cdot22 . \supset \vdash . \text{Prop}$

***207·231.** $\vdash : P \epsilon \text{Ser} . \exists ! \overrightarrow{\text{lt}}_P`\alpha . \supset . \overrightarrow{\text{lt}}_P`\alpha = C`P \cap \hat{x}(\overrightarrow{P}`x = P``\alpha)$ [*207·23]

***207·232.** $\vdash :. P \epsilon \text{Ser} . \supset : x = \text{lt}_P`\alpha . \equiv . x \epsilon C`P - \alpha . \overrightarrow{P}`x = P``\alpha$
[*206·28 . *207·1]

***207·24.** $\vdash : P \epsilon \text{connex} . \supset . \overrightarrow{\text{lt}}_P`\alpha \epsilon 0 \cup 1 . \text{lt}_P \epsilon 1 \rightarrow \text{Cls}$

Dem.

$\vdash . *206\cdot161 . *71\cdot26 . (*207\cdot01) . \supset \vdash : \text{Hp} . \supset . \text{lt}_P \epsilon 1 \rightarrow \text{Cls} .$ (1)

[*71·12] $\supset . \overrightarrow{\text{lt}}_P`\alpha \epsilon 0 \cup 1$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***207·25.** $\vdash : P \epsilon \text{trans} . \beta \subset P``\alpha . \supset . \overrightarrow{\text{lt}}_P`(\alpha \cup \beta) = \overrightarrow{\text{lt}}_P`\alpha$

Dem.

$\vdash . *205\cdot193 . \quad \supset \vdash : \text{Hp} . \exists ! \overrightarrow{\max}_P`\alpha . \supset . \exists ! \overrightarrow{\max}_P`(\alpha \cup \beta)$ (1)

$\vdash . (1) . *207\cdot11 . \supset \vdash : \text{Hp} . \exists ! \overrightarrow{\max}_P`\alpha . \supset . \overrightarrow{\text{lt}}_P`\alpha = \Lambda . \overrightarrow{\text{lt}}_P`(\alpha \cup \beta) = \Lambda$ (2)

$\vdash . *205\cdot193 . *207\cdot12 . \supset$

$\vdash : \text{Hp} . \overrightarrow{\max}_P`\alpha = \Lambda . \supset . \overrightarrow{\text{lt}}_P`\alpha = \overrightarrow{\text{seq}}_P`\alpha . \overrightarrow{\text{lt}}_P`(\alpha \cup \beta) = \overrightarrow{\text{seq}}_P`(\alpha \cup \beta) .$

[*206·24] $\supset . \overrightarrow{\text{lt}}_P`\alpha = \overrightarrow{\text{lt}}_P`(\alpha \cup \beta)$ (3)

$\vdash . (2) . (3) . \supset \vdash . \text{Prop}$

***207·251.** $\vdash : P \epsilon \text{trans} . y \epsilon P``(\beta - \iota`y) . \supset . \overrightarrow{\text{lt}}_P`\beta = \overrightarrow{\text{lt}}_P`(\beta - \iota`y)$

Dem.

$\vdash . *51\cdot222 . \quad \supset \vdash : y \sim\epsilon \beta . \supset . \overrightarrow{\text{lt}}_P`\beta = \overrightarrow{\text{lt}}_P`(\beta - \iota`y)$ (1)

$\vdash . *207\cdot25 . \quad \supset \vdash : \text{Hp} . \supset . \overrightarrow{\text{lt}}_P`\{(\beta - \iota`y) \cup \iota`y\} = \overrightarrow{\text{lt}}_P`(\beta - \iota`y)$ (2)

$\vdash . (2) . *51\cdot221 . \supset \vdash : \text{Hp} . y \epsilon \beta . \supset . \overrightarrow{\text{lt}}_P`\beta = \overrightarrow{\text{lt}}_P`(\beta - \iota`y)$ (3)

$\vdash . (1) . (3) . \supset \vdash . \text{Prop}$

***207·26.** $\vdash : P \epsilon \text{trans} . \sim(yPy) . \exists ! \overrightarrow{\text{lt}}_P`\beta . \supset . \overrightarrow{\text{lt}}_P`\beta = \overrightarrow{\text{lt}}_P`(\beta - \iota`y)$
[*207·13·12 . *206·72]

***207·261.** $\vdash : P \epsilon \text{trans} . y \epsilon \overrightarrow{\min}_P`\beta . \exists ! \overrightarrow{\text{lt}}_P`\beta . \supset . \overrightarrow{\text{lt}}_P`\beta = \overrightarrow{\text{lt}}_P`(\beta - \iota`y)$
[*207·26 . *205·194]

***207·262.** $\vdash : P \epsilon \text{trans} \cap \text{connex} . \exists ! \overrightarrow{\text{lt}}_P`\beta . \supset . \overrightarrow{\text{lt}}_P`\beta = \overrightarrow{\text{lt}}_P`(\beta - \overrightarrow{\min}_P`\beta)$
[*207·261 . *205·3]

***207·263.** $\vdash : P \epsilon \text{trans} \cap \text{connex} . \supset . \overrightarrow{\text{lt}}_P`\beta \subset \overrightarrow{\text{lt}}_P`(\beta - \overrightarrow{\min}_P`\beta)$
[*207·262 . *24·12]

***207·27.** $\vdash : P \in \text{trans} \cap \text{connex} \,.\, \beta \cap C`P \sim\in 1 \,.\, \supset .\, \overrightarrow{\text{lt}}_P`\beta = \overrightarrow{\text{lt}}_P`(\beta - \overrightarrow{\text{min}}_P`\beta)$

Dem.

$\vdash .\, *24\cdot26\cdot101 \,.\, \supset \vdash : \overrightarrow{\text{min}}_P`\beta = \Lambda \,.\, \supset .\, \overrightarrow{\text{lt}}_P`\beta = \overrightarrow{\text{lt}}_P`(\beta - \overrightarrow{\text{min}}_P`\beta)$ (1)

$\vdash .\, *52\cdot181 \,.\, \supset$

$\vdash : \text{Hp} \,.\, \exists ! \overrightarrow{\text{min}}_P`\beta \,.\, \supset .\, (\exists y) \,.\, y \in \beta \cap C`P \,.\, y \neq \text{min}_P`\beta \,.$

[*205·2] $\supset .\, (\exists y) \,.\, y \in (\beta \cap C`P) - \iota`\text{min}_P`\beta \,.\, \text{min}_P`\beta \, P y \,.$

[*37·1] $\supset .\, \text{min}_P`\beta \in P``(\beta - \iota`\text{min}_P`\beta) \,.$

[*207·251] $\supset .\, \overrightarrow{\text{lt}}_P`\beta = \overrightarrow{\text{lt}}_P`(\beta - \iota`\text{min}_P`\beta)$ (2)

$\vdash .\, (1) \,.\, (2) \,.\, \supset \vdash .\, \text{Prop}$

***207·28.** $\vdash : P \in \text{trans} \,.\, \supset .\, \overrightarrow{\text{lt}}_P`(\alpha \cup P``\alpha) = \overrightarrow{\text{lt}}_P`\alpha$ [*207·25]

***207·281.** $\vdash : P \in \text{trans} \,.\, \sim \exists ! \overrightarrow{\text{max}}_P`\alpha \,.\, \supset .\, \overrightarrow{\text{lt}}_P`P``\alpha = \overrightarrow{\text{lt}}_P`\alpha$
[*207·28·16 . *205·123]

***207·282.** $\vdash : P \in \text{trans} \,.\, \sim \exists ! \overrightarrow{\text{max}}_P`\alpha \,.\, \sim \exists ! \overrightarrow{\text{max}}_P`\beta \,.\, P``\alpha = P``\beta \,.\, \supset .\, \overrightarrow{\text{lt}}_P`\alpha = \overrightarrow{\text{lt}}_P`\beta$
[*207·281]

***207·29.** $\vdash : P \in \text{trans} \,.\, \supset .\, \overrightarrow{\text{lt}}_P`\alpha = \overrightarrow{\text{lt}}_P`P_*``\alpha$

Dem.

$\vdash .\, *207\cdot16\cdot28 \,.\, \supset \vdash : \text{Hp} \,.\, \supset .\, \overrightarrow{\text{lt}}_P`\alpha = \overrightarrow{\text{lt}}_P`\{(\alpha \cup P``\alpha) \cap C`P\}$

[*201·52] $= \text{lt}_P`P_*``\alpha : \supset \vdash .\, \text{Prop}$

***207·291.** $\vdash : P \in \text{trans} \cap \text{connex} \,.\, E ! \,\text{lt}_P`\alpha \,.\, \supset .\, \overrightarrow{P}`\text{lt}_P`\alpha = P_*``\alpha$

Dem.

$\vdash .\, *207\cdot29 \,.\quad \supset \vdash : \text{Hp} \,.\, \supset .\, \overrightarrow{P}`\text{lt}_P`\alpha = \overrightarrow{P}`\text{lt}_P`P_*``\alpha$ (1)

$\vdash .\, *90\cdot14\cdot172 \,.\quad \supset \vdash .\, P_*``\alpha \subset C`P \,.\, P``P_*``\alpha \subset P_*``\alpha$ (2)

$\vdash .\, *207\cdot11\cdot12 \,.\quad \supset \vdash : \text{Hp} \,.\, \supset .\, \text{seq}_P`P_*``\alpha = \text{lt}_P`P_*``\alpha$ (3)

$\vdash .\, (2) \,.\, (3) \,.\, *206\cdot3 \,.\, \supset \vdash : \text{Hp} \,.\, \supset .\, \overrightarrow{P}`\text{seq}_P`P_*``\alpha = P_*``\alpha$ (4)

$\vdash .\, (1) \,.\, (3) \,.\, (4) \,.\, \supset \vdash .\, \text{Prop}$

***207·3.** $\vdash : \alpha \cap C`P = \Lambda \,.\, \supset .\, \overrightarrow{\text{lt}}_P`\alpha = \overrightarrow{B}`P$

Dem.

$\vdash .\, *205\cdot151\cdot161 \,.\, \supset \vdash : \text{Hp} \,.\, \supset .\, \overrightarrow{\text{max}}_P`\alpha = \Lambda$ (1)

$\vdash .\, *206\cdot14 \,.\quad \supset \vdash : \text{Hp} \,.\, \supset .\, \overrightarrow{\text{seq}}_P`\alpha = \overrightarrow{B}`P$ (2)

$\vdash .\, (1) \,.\, (2) \,.\, *207\cdot12 \,.\, \supset \vdash .\, \text{Prop}$

***207·31.** $\vdash : P \in J \,.\, x \in C`P - \text{ᗡ}`(P \dot{-} P^2) \,.\, \supset .\, x \,\text{lt}_P \overrightarrow{P}`x$

Dem.

$\vdash .\, *206\cdot41 \,.\, \supset \vdash : \text{Hp} \,.\, \supset .\, \overrightarrow{\text{max}}_P`\overrightarrow{P}`x = \Lambda$ (1)

$\vdash .\, *206\cdot4 \,.\quad \supset \vdash : \text{Hp} \,.\, \supset .\, x \,\text{seq}_P \overrightarrow{P}`x$ (2)

$\vdash .\, (1) \,.\, (2) \,.\, *207\cdot1 \,.\, \supset \vdash .\, \text{Prop}$

$*207{\cdot}32$. $\vdash : P \in \mathrm{Rl}‘J \cap \mathrm{connex} \,.\, x \in C‘P - \text{Ɑ}‘(P \dot{-} P^2) \,.\, \supset \,.\, x = \mathrm{lt}_P‘\overrightarrow{P}‘x$
[$*207{\cdot}31{\cdot}24$]

$*207{\cdot}33$. $\vdash : x \in \text{Ɑ}‘(P \dot{-} P^2) \,.\, \supset \,.\, \overrightarrow{\mathrm{lt}_P}‘\overrightarrow{P}‘x = \Lambda$ [$*205{\cdot}252 \,.\, *207{\cdot}11$]

$*207{\cdot}34$. $\vdash : P \in \mathrm{connex} \,.\, x \,\mathrm{lt}_P\, \alpha \,.\, \supset \,.\, x \,\mathrm{lt}_P\, \overrightarrow{P}‘x \,.\, x \sim\in \text{Ɑ}‘(P \dot{-} P^2)$

Dem.

$\vdash \,.\, *207{\cdot}15 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, x \in C‘P \,.\, \alpha \cap C‘P \subset P“\alpha \,.\, \alpha \cap C‘P \subset \overrightarrow{P}‘x \,.$
$\overrightarrow{P}‘x \subset -p‘\overleftarrow{P}“(\alpha \cap C‘P)$ (1)

$\vdash \,.\, *40{\cdot}16 \,.\, \supset \vdash : \alpha \cap C‘P \subset \overrightarrow{P}‘x \,.\, \supset \,.\, p‘\overleftarrow{P}“\overrightarrow{P}‘x \subset p‘\overleftarrow{P}“(\alpha \cap C‘P) \,.$

[$*22{\cdot}81$] $\supset \,.\, -p‘\overleftarrow{P}“(\alpha \cap C‘P) \subset -p‘\overleftarrow{P}“\overrightarrow{P}‘x$ (2)

$\vdash \,.\, (1) \,.\, (2) \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, x \in C‘P \,.\, \overrightarrow{P}‘x \subset -p‘\overleftarrow{P}“\overrightarrow{P}‘x \,.$

[$*22{\cdot}42$] $\supset \,.\, x \in C‘P \,.\, \overrightarrow{P}‘x \subset \overrightarrow{P}‘x \,.\, \overrightarrow{P}‘x \subset -p‘\overleftarrow{P}“\overrightarrow{P}‘x$ (3)

$\vdash \,.\, (1) \,.\, *202{\cdot}505 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, \overrightarrow{P}‘x \subset (\alpha \cap C‘P) \cup P“\alpha \,.\, \alpha \cap C‘P \subset P“\alpha \,.$

[$*22{\cdot}62$] $\supset \,.\, \overrightarrow{P}‘x \subset P“\alpha$ (4)

$\vdash \,.\, (1) \,.\, *37{\cdot}2{\cdot}265 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, P“\alpha \subset P“\overrightarrow{P}‘x$ (5)

$\vdash \,.\, (4) \,.\, (5) \,.$ $\supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, \overrightarrow{P}‘x \subset P“\overrightarrow{P}‘x$ (6)

$\vdash \,.\, (3) \,.\, (6) \,.\, *207{\cdot}15 \,.\, \supset \vdash \,.\, \mathrm{Prop}$

$*207{\cdot}35$. $\vdash : P \in \mathrm{Rl}‘J \cap \mathrm{connex} \,.\, \supset \,.\, \mathrm{D}‘\mathrm{lt}_P = C‘P - \text{Ɑ}‘(P \dot{-} P^2)$

Dem. $\vdash \,.\, *207{\cdot}34 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, \mathrm{D}‘\mathrm{lt}_P \subset -\text{Ɑ}‘(P \dot{-} P^2)$ (1)

$\vdash \,.\, *207{\cdot}15 \,.\, \supset \vdash \,.\, \mathrm{D}‘\mathrm{lt}_P \subset C‘P$ (2)

$\vdash \,.\, *207{\cdot}32 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, C‘P - \text{Ɑ}‘(P \dot{-} P^2) \subset \mathrm{D}‘\mathrm{lt}_P$ (3)

$\vdash \,.\, (1) \,.\, (2) \,.\, (3) \,.\, \supset \vdash \,.\, \mathrm{Prop}$

$*207{\cdot}36$. $\vdash : P \in \mathrm{Rl}‘J \cap \mathrm{connex} \,.\, \supset \,.$

$$\mathrm{D}‘\mathrm{lt}_P = \mathrm{lt}_P“\overrightarrow{P}“\{C‘P - \text{Ɑ}‘(P \dot{-} P^2)\} = \mathrm{lt}_P“\overrightarrow{P}“C‘P$$

Dem.

$\vdash \,.\, *207{\cdot}32 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, C‘P - \text{Ɑ}‘(P \dot{-} P^2) = \mathrm{lt}_P“\overrightarrow{P}“\{C‘P - \text{Ɑ}‘(P \dot{-} P^2)\}$ (1)

$\vdash \,.\, (1) \,.\, *207{\cdot}35 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, \mathrm{D}‘\mathrm{lt}_P = \mathrm{lt}_P“\overrightarrow{P}“\{C‘P - \text{Ɑ}‘(P \dot{-} P^2)\}$ (2)

$\vdash \,.\, *207{\cdot}33 \,.$ $\supset \vdash \,.\, \mathrm{lt}_P“\overrightarrow{P}“\{C‘P \cap \text{Ɑ}‘(P \dot{-} P^2)\} = \Lambda$ (3)

$\vdash \,.\, (2) \,.\, (3) \,.$ $\supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, \mathrm{D}‘\mathrm{lt}_P = \mathrm{lt}_P“\overrightarrow{P}“C‘P$ (4)

$\vdash \,.\, (2) \,.\, (4) \,.\, \supset \vdash \,.\, \mathrm{Prop}$

In virtue of this proposition, all limits are limits of classes of the form $\overrightarrow{P}‘x$. In this respect, limits (in general) differ from segments. If we call $P“\alpha$ the segment defined by α, there will in general be segments not of the form $\overrightarrow{P}‘x$. These, however, will be the segments which have no sequents, and therefore no limits; thus their existence does not introduce limits not derivable from classes of the form $\overrightarrow{P}‘x$.

***207·4.** $\vdash :. x \, \mathrm{limax}_P \, \alpha . \equiv : x \, \mathrm{max}_P \, \alpha . \mathbf{v} . x \, \mathrm{lt}_P \, \alpha :$

$\equiv : x \, \mathrm{max}_P \, \alpha . \mathbf{v} . \sim \exists ! \overrightarrow{\mathrm{max}_P} \text{‘} \alpha . x \, \mathrm{seq}_P \, \alpha$ [(*207·03)]

***207·401.** $\vdash . \mathrm{limin}_P = \mathrm{limax} (\breve{P})$ [(*207·04)]

***207·41.** $\vdash : P \epsilon \mathrm{connex} . \supset . \mathrm{limax}_P, \mathrm{limin}_P \epsilon 1 \rightarrow \mathrm{Cls}$
[*71·24 . *205·31 . *207·24 . (*207·03·04)]

***207·42.** $\vdash : \exists ! \overrightarrow{\mathrm{max}_P} \text{‘} \alpha . \supset . \overrightarrow{\mathrm{limax}_P} \text{‘} \alpha = \overrightarrow{\mathrm{max}_P} \text{‘} \alpha$ [*207·4]

***207·43.** $\vdash : \overrightarrow{\mathrm{max}_P} \text{‘} \alpha = \Lambda . \supset . \overrightarrow{\mathrm{limax}_P} \text{‘} \alpha = \overrightarrow{\mathrm{seq}_P} \text{‘} \alpha = \overrightarrow{\mathrm{lt}_P} \text{‘} \alpha$ [*207·4]

***207·44.** $\vdash . \mathrm{Cl} \text{‘} \mathrm{limax}_P = \mathrm{Cl} \text{‘} \mathrm{max}_P \cup \mathrm{Cl} \text{‘} \mathrm{lt}_P = \mathrm{Cl} \text{‘} \mathrm{max}_P \cup \mathrm{Cl} \text{‘} \mathrm{seq}_P$
[*207·14 . (*207·03)]

***207·45.** $\vdash . \overrightarrow{\mathrm{limax}_P} \text{‘} \alpha = \overrightarrow{\mathrm{max}_P} \text{‘} \alpha \cup \overrightarrow{\mathrm{lt}_P} \text{‘} \alpha$ [(*207·03)]

***207·46.** $\vdash :. x = \mathrm{limax}_P \text{‘} \alpha . \equiv : x = \mathrm{max}_P \text{‘} \alpha . \mathbf{v} . x = \mathrm{lt}_P \text{‘} \alpha$

Dem.

$\vdash . *207·45·11 . \supset \vdash :. \exists ! \overrightarrow{\mathrm{max}_P} \text{‘} \alpha . \supset : x = \mathrm{limax}_P \text{‘} \alpha . \equiv . x = \mathrm{max}_P \text{‘} \alpha$ (1)

$\vdash . *207·45·12 . \supset \vdash :. \overrightarrow{\mathrm{max}_P} \text{‘} \alpha = \Lambda . \supset : x = \mathrm{limax}_P \text{‘} \alpha . \equiv . x = \mathrm{lt}_P \text{‘} \alpha$ (2)

$\vdash . (1) . (2) . *5·32 . \supset$

$\vdash :. \exists ! \overrightarrow{\mathrm{max}_P} \text{‘} \alpha . x = \mathrm{limax}_P \text{‘} \alpha . \mathbf{v} . \overrightarrow{\mathrm{max}_P} \text{‘} \alpha = \Lambda . x = \mathrm{limax}_P \text{‘} \alpha : \equiv :$

$\exists ! \overrightarrow{\mathrm{max}_P} \text{‘} \alpha . x = \mathrm{max}_P \text{‘} \alpha . \mathbf{v} . \overrightarrow{\mathrm{max}_P} \text{‘} \alpha = \Lambda . x = \mathrm{lt}_P \text{‘} \alpha$ (3)

$\vdash . (3) . *4·42 . \supset$

$\vdash :. x = \mathrm{limax}_P \text{‘} \alpha . \equiv : \exists ! \overrightarrow{\mathrm{max}_P} \text{‘} \alpha . x = \mathrm{max}_P \text{‘} \alpha . \mathbf{v} . \overrightarrow{\mathrm{max}_P} \text{‘} \alpha = \Lambda . x = \mathrm{lt}_P \text{‘} \alpha :$

[*30·32] $\equiv : x = \mathrm{max}_P \text{‘} \alpha . \mathbf{v} . \overrightarrow{\mathrm{max}_P} \text{‘} \alpha = \Lambda . x = \mathrm{lt}_P \text{‘} \alpha :$

[*207·13] $\equiv : x = \mathrm{max}_P \text{‘} \alpha . \mathbf{v} . x = \mathrm{lt}_P \text{‘} \alpha :. \supset \vdash . \mathrm{Prop}$

***207·47.** $\vdash : \exists ! \overrightarrow{\mathrm{lt}_P} \text{‘} \alpha . \equiv . \exists ! \overrightarrow{\mathrm{limax}_P} \text{‘} \alpha . \sim \exists ! \overrightarrow{\mathrm{max}_P} \text{‘} \alpha$

Dem.

$\vdash . *207·45·11 . \supset \vdash : \exists ! \overrightarrow{\mathrm{lt}_P} \text{‘} \alpha . \supset . \exists ! \overrightarrow{\mathrm{limax}_P} \text{‘} \alpha . \sim \exists ! \overrightarrow{\mathrm{max}_P} \text{‘} \alpha$ (1)

$\vdash . *207·45 . \quad \supset \vdash : \exists ! \overrightarrow{\mathrm{limax}_P} \text{‘} \alpha . \sim \exists ! \overrightarrow{\mathrm{max}_P} \text{‘} \alpha . \supset . \exists ! \overrightarrow{\mathrm{lt}_P} \text{‘} \alpha$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***207·48.** $\vdash . \overrightarrow{\mathrm{limax}_P} \text{‘} \alpha = \overrightarrow{\mathrm{limax}_P} \text{‘} (\alpha \cap C \text{‘} P)$ [*207·45 . *205·151 . *207·16]

***207·481.** $\vdash : P \epsilon \mathrm{trans} . \supset . \overrightarrow{\mathrm{limax}_P} \text{‘} \alpha = \overrightarrow{\mathrm{limax}_P} \text{‘} P_* \text{‘‘} \alpha$
[*207·45 . *205·191 . *207·29]

***207·482.** $\vdash : P \epsilon \mathrm{Ser} . \alpha \subset C \text{‘} P . a = \mathrm{limax}_P \text{‘} \alpha . \supset . \alpha \subset \overrightarrow{P_*} \text{‘} a$

Dem.

$\vdash . *205·22 . *90·151 . \quad \supset \vdash : \mathrm{Hp} . a = \mathrm{max}_P \text{‘} \alpha . \supset . \alpha \subset \overrightarrow{P_*} \text{‘} a$ (1)

$\vdash . *207·291 . *90·151 . \supset \vdash : \mathrm{Hp} . a = \mathrm{lt}_P \text{‘} \alpha . \supset . P_* \text{‘‘} \alpha \subset \overrightarrow{P_*} \text{‘} a .$

[*90·21] $\supset . \alpha \subset \overrightarrow{P_*} \text{‘} a$ (2)

$\vdash . (1) . (2) . *207·46 . \supset \vdash . \mathrm{Prop}$

***207·5.** $\vdash : P \in \mathrm{Ser} . \supset . \overrightarrow{\mathrm{limax}}_P\text{‘}\alpha = \overrightarrow{\mathrm{seq}}_P\text{‘}P\text{“}\alpha = \overrightarrow{\mathrm{min}}_P\text{‘}(\overrightarrow{\mathrm{max}}_P\text{‘}\alpha \cup \overrightarrow{\mathrm{seq}}_P\text{‘}\alpha)$
[*206·33·35·37]

***207·51.** $\vdash :. P \in \mathrm{Ser} . \supset : x = \mathrm{limax}_P\text{‘}\alpha . \equiv . x \in C\text{‘}P . \overrightarrow{P}\text{‘}x = P\text{“}\alpha$
[*205·54 . *207·232·46]

***207·52.** $\vdash :. P \in \mathrm{Ser} . \exists ! P\text{“}\alpha . \supset : x = \mathrm{limax}_P\text{‘}\alpha . \equiv . \overrightarrow{P}\text{‘}x = P\text{“}\alpha$ [*207·51]

***207·521.** $\vdash :. P \in \mathrm{Ser} . \supset : x = \mathrm{lt}_P\text{‘}\alpha . \equiv . x \in C\text{‘}P . \overrightarrow{P}\text{‘}x = P\text{“}\alpha . \sim \mathrm{E} ! \mathrm{max}_P\text{‘}\alpha$

Dem.

$\vdash . *207\cdot51 . \supset \vdash :. \mathrm{Hp} . \supset :$

$x \in C\text{‘}P . \overrightarrow{P}\text{‘}x = P\text{“}\alpha . \sim \mathrm{E} ! \mathrm{max}_P\text{‘}\alpha . \equiv . x = \mathrm{limax}_P\text{‘}\alpha . \sim \mathrm{E} ! \mathrm{max}_P\text{‘}\alpha .$

[*207·46] $\equiv . x = \mathrm{lt}_P\text{‘}\alpha :. \supset \vdash . \mathrm{Prop}$

***207·53.** $\vdash : P \in \mathrm{Ser} . \kappa \subset \mathrm{G}\text{‘}\mathrm{limax}_P . \supset . \overrightarrow{\mathrm{limax}}_P\text{‘}\mathrm{limax}_P\text{“}\kappa = \overrightarrow{\mathrm{limax}}_P\text{‘}s\text{‘}\kappa$

Dem.

$\vdash . *207\cdot51 . \supset \vdash :. \mathrm{Hp} . \supset : \alpha \in \kappa . \supset_\alpha . \overrightarrow{P}\text{‘}\mathrm{limax}_P\text{‘}\alpha = P\text{“}\alpha :$

[*37·68] $\supset : \overrightarrow{P}\text{“}\mathrm{limax}_P\text{“}\kappa = P\text{“‘}\kappa :$

[*40·5·38] $\supset : P\text{“}\mathrm{limax}_P\text{“}\kappa = P\text{“}s\text{‘}\kappa :$

[*207·51] $\supset : x = \mathrm{limax}_P\text{‘}\mathrm{limax}_P\text{“}\kappa . \equiv . x = \mathrm{limax}_P\text{‘}s\text{‘}\kappa :. \supset \vdash . \mathrm{Prop}$

***207·54.** $\vdash : P \in \mathrm{Ser} . \kappa \subset \mathrm{G}\text{‘}\mathrm{lt}_P . \supset . \overrightarrow{\mathrm{limax}}_P\text{‘}\mathrm{lt}_P\text{“}\kappa = \overrightarrow{\mathrm{limax}}_P\text{‘}s\text{‘}\kappa = \overrightarrow{\mathrm{lt}}_P\text{‘}s\text{‘}\kappa$

Dem.

$\vdash . *205\cdot561 . *207\cdot13 . \supset \vdash : \mathrm{Hp} . \supset . s\text{‘}\kappa \sim \in \mathrm{G}\text{‘}\mathrm{max}_P .$

[*207·43] $\supset . \overrightarrow{\mathrm{limax}}_P\text{‘}s\text{‘}\kappa = \overrightarrow{\mathrm{lt}}_P\text{‘}s\text{‘}\kappa$ (1)

$\vdash . *207\cdot13\cdot43 . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{lt}_P\text{“}\kappa = \mathrm{limax}_P\text{“}\kappa .$

[*207·53] $\supset . \overrightarrow{\mathrm{limax}}_P\text{‘}\mathrm{lt}_P\text{“}\kappa = \overrightarrow{\mathrm{limax}}_P\text{‘}s\text{‘}\kappa$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***207·55.** $\vdash : P \in \mathrm{Ser} . \kappa \subset \mathrm{G}\text{‘}\mathrm{lt}_P . s\text{‘}\kappa \in \mathrm{G}\text{‘}\mathrm{lt}_P . \supset . \mathrm{limax}_P\text{‘}\mathrm{lt}_P\text{“}\kappa = \mathrm{lt}_P\text{‘}s\text{‘}\kappa$
[*207·54]

***207·6.** $\vdash : S \in P \,\overline{\mathrm{smor}}\, Q . \supset . \overrightarrow{\mathrm{lt}}_P\text{‘}\alpha = S\text{“}\overrightarrow{\mathrm{lt}}_Q\text{‘}\breve{S}\text{“}\alpha$

Dem.

$\vdash . *205\cdot8 . *37\cdot43 . \supset \vdash :. \mathrm{Hp} . \supset : \exists ! \overrightarrow{\mathrm{max}}_P\text{‘}\alpha . \equiv . \exists ! \overrightarrow{\mathrm{max}}_Q\text{‘}\breve{S}\text{“}\alpha :$ (1)

[*207·11] $\supset : \exists ! \overrightarrow{\mathrm{max}}_P\text{‘}\alpha . \supset . \overrightarrow{\mathrm{lt}}_P\text{‘}\alpha = \Lambda . \overrightarrow{\mathrm{lt}}_Q\text{‘}\breve{S}\text{“}\alpha = \Lambda$ (2)

$\vdash . (1) . \mathrm{Transp} . *207\cdot12 . \supset$

$\vdash :. \mathrm{Hp} . \overrightarrow{\mathrm{max}}_P\text{‘}\alpha = \Lambda . \supset : \overrightarrow{\mathrm{lt}}_P\text{‘}\alpha = \overrightarrow{\mathrm{seq}}_P\text{‘}\alpha . \overrightarrow{\mathrm{lt}}_Q\text{‘}\breve{S}\text{“}\alpha = \overrightarrow{\mathrm{seq}}_Q\text{‘}\breve{S}\text{“}\alpha :$

[*206·61] $\supset : \overrightarrow{\mathrm{lt}}_P\text{‘}\alpha = S\text{“}\overrightarrow{\mathrm{lt}}_Q\text{‘}\breve{S}\text{“}\alpha$ (3)

$\vdash . (2) . (3) . *37\cdot29 . \supset \vdash . \mathrm{Prop}$

***207·61.** $\vdash :. S \in P \,\overline{\mathrm{smor}}\, Q . \supset : \mathrm{E} ! \mathrm{lt}_P\text{‘}\alpha . \equiv . \mathrm{E} ! \mathrm{lt}_Q\text{‘}\breve{S}\text{“}\alpha$ [*207·6 . *53·3]

***207·62.** $\vdash : S \in P \,\overline{\text{smor}}\, Q \,.\, E\,!\,\text{lt}_P\text{‘}\alpha \,.\supset .\, \text{lt}_P\text{‘}\alpha = S\text{‘}\text{lt}_Q\text{‘}\breve{S}\text{“}\alpha$ [*207·6 . *53·31]

***207·63.** $\vdash : S \in P \,\overline{\text{smor}}\, Q \,.\supset .\, \text{lt}_P\text{“}\kappa = S\text{“}\text{lt}_Q\text{“}\breve{S}\text{“‘}\kappa$

Dem.

$\vdash . *207{\cdot}6 . *40{\cdot}5 . \supset \vdash : \text{Hp} . \supset . \text{lt}_P\text{“}\kappa = s\text{‘}S\text{“‘}\overrightarrow{\text{lt}_Q}\text{“}\breve{S}\text{“‘}\kappa$

[*40·38·5] $\qquad = S\text{“}\text{lt}_Q\text{“}\breve{S}\text{“‘}\kappa : \supset \vdash .$ Prop

***207·64.** $\vdash : S \in P \,\overline{\text{smor}}\, Q \,.\supset .\, \overrightarrow{\text{limax}}_P\text{‘}\alpha = S\text{“}\overrightarrow{\text{limax}}_Q\text{‘}\breve{S}\text{“}\alpha$
[*205·8 . *207·6·45]

***207·65.** $\vdash :. S \in P \,\overline{\text{smor}}\, Q \,.\supset : E\,!\,\text{limax}_P\text{‘}\alpha \,.\equiv .\, E\,!\,\text{limax}_Q\text{‘}\breve{S}\text{“}\alpha$
[*207·64]

***207·66.** $\vdash : S \in P \,\overline{\text{smor}}\, Q \,.\, E\,!\,\text{limax}_P\text{‘}\alpha \,.\supset .\, \text{limax}_P\text{‘}\alpha = S\text{‘}\text{limax}_Q\text{‘}\breve{S}\text{“}\alpha$
[*207·64]

***207·7.** $\vdash :. P \in \text{trans} \,.\vee .\, P^2 \mathrel{\subseteq} J : \supset :$
$\qquad \text{limin}_P\text{‘}\gamma = \text{limax}_P\text{‘}\gamma \,.\supset .\, \text{limin}_P\text{‘}\gamma = \text{min}_P\text{‘}\gamma = \text{max}_P\text{‘}\gamma$

Dem.

$\vdash . *207{\cdot}42{\cdot}43 . \supset \vdash : E\,!\,\text{min}_P\text{‘}\gamma \,.\, E\,!\,\text{limax}_P\text{‘}\gamma \,.\sim E\,!\,\text{max}_P\text{‘}\gamma \,.\supset .$
$\qquad \text{limin}_P\text{‘}\gamma = \text{min}_P\text{‘}\gamma \,.\, \text{limax}_P\text{‘}\gamma = \text{seq}_P\text{‘}\gamma \,.$

[*205·11.*206·2] $\qquad \supset .\, \text{limin}_P\text{‘}\gamma \in \gamma \,.\, \text{limax}_P\text{‘}\gamma \sim\in \gamma \,.$

[*13·14] $\qquad \supset .\, \text{limin}_P\text{‘}\gamma \neq \text{limax}_P\text{‘}\gamma \qquad (1)$

Similarly

$\vdash : E\,!\,\text{max}_P\text{‘}\gamma \,.\, E\,!\,\text{limin}_P\text{‘}\gamma \,.\sim E\,!\,\text{min}_P\text{‘}\gamma \,.\supset .\, \text{limin}_P\text{‘}\gamma \neq \text{limax}_P\text{‘}\gamma \qquad (2)$

$\vdash . *206{\cdot}732 . *207{\cdot}43{\cdot}12 . \supset$

$\vdash : \text{Hp} \,.\sim E\,!\,\text{min}_P\text{‘}\gamma \,.\sim E\,!\,\text{max}_P\text{‘}\gamma \,.\supset .\sim \{\text{limin}_P\text{‘}\gamma = \text{limax}_P\text{‘}\gamma\} \qquad (3)$

$\vdash . (1) . (2) . (3) . \supset \vdash : \text{Hp} \,.\, \text{limin}_P\text{‘}\gamma = \text{limax}_P\text{‘}\gamma \,.\supset .\, E\,!\,\text{min}_P\text{‘}\gamma \,.\, E\,!\,\text{max}_P\text{‘}\gamma \,.$

[*207·42] $\qquad \supset .\, \text{limin}_P\text{‘}\gamma = \text{min}_P\text{‘}\gamma = \text{max}_P\text{‘}\gamma : \supset \vdash .$ Prop

***207·71.** $\vdash :. P \in \text{connex} : P \in \text{trans} \,.\vee .\, P^2 \mathrel{\subseteq} J : \text{limin}_P\text{‘}\gamma = \text{limax}_P\text{‘}\gamma : \supset .$
$\qquad \gamma \cap C\text{‘}P \in 1 \,.\, \gamma \cap C\text{‘}P = \iota\text{‘}\text{limax}_P\text{‘}\gamma$
[*207·7 . *205·73]

***207·72.** $\vdash :. P \in \text{connex} \,.\, P^2 \mathrel{\subseteq} J \,.\supset : \text{limin}_P\text{‘}\gamma = \text{limax}_P\text{‘}\gamma \,.\equiv .\, \gamma \cap C\text{‘}P \in 1$
[*207·71 . *205·731·17 . *207·42]

$*208$. THE CORRELATION OF SERIES

Summary of $*208$.

The propositions of this number are chiefly important on account of their consequences in the theory of well-ordered series ($*250$ ff.) and in the theory of vector-families ($*330$ ff.). When two well-ordered series are ordinally similar, they have only one correlator; and a well-ordered series is not ordinally similar to any of its segments. Of these two propositions, the first is an immediate consequence of $*208 \cdot 41$, and the second is an immediate consequence of $*208 \cdot 47$.

Propositions concerning correlators of two relations P and Q are obtained from propositions concerning correlators of P with itself, by means of the fact that, if S, T are two correlators of P and Q, $S \mid \breve{T}$ is a correlator of P with itself. Again, correlators of P with itself are considered, in this number, as a special case of correlators of P with parts of itself. This latter is a notion which will prove important for other reasons than those for which it is used in our present context. If P is connected, and S correlates P with part of itself (so that $S ; P \subseteq P$), $C`P$ will contain terms of three kinds, (1) those for which $S`x = x$, (2) those for which $(S`x) P x$, (3) those for which $x P (S`x)$. Our propositions result from the non-existence (under certain circumstances) of maxima or minima of classes (2) and (3).

The following definition defines "correlations of P with parts (or the whole) of itself." The letters "cror" stand for "ordinal correlation." For a cardinal correlation, should occasion arise, we should use "cr," *i.e.* we should put

$$\mathrm{cr}`\alpha = s`\overline{\mathrm{sm}}\, \alpha``\mathrm{Cl}`\alpha \quad \mathrm{Df},$$

so that $\qquad S \in \mathrm{cr}`\alpha \,.\equiv.\, S \in 1 \rightarrow 1 \,.\, \mathrm{G}`S = \alpha \,.\, \mathrm{D}`S \subset \alpha.$

For the present, we are concerned with the corresponding ordinal notion; thus we require

$$S \in \mathrm{cror}`P \,.\equiv.\, S \in 1 \rightarrow 1 \,.\, \mathrm{G}`S = C`P \,.\, S ; P \subseteq P.$$

This is secured by putting

$$\mathrm{cror}`P = s`\overline{\mathrm{smor}}\, P``\mathrm{Rl}`P \quad \mathrm{Df}.$$

It will be observed that if α is what we called a "non-reflexive" class (cf. $*124$), $\mathrm{cr}`\alpha = \iota`I \upharpoonright \alpha$, and $S \in \mathrm{cr}`\alpha \,.\supset.\, \mathrm{D}`S = \alpha$. When $C`P$ is non-reflexive, the same is true of $P$; and when $C`P$ is reflexive, P is also reflexive, in the sense that it contains proper parts similar to itself, though if P is well-ordered, such proper parts cannot be *segments* of P, but must extend to the end of $C`P$.

The class of correlators of P with the whole of itself, *i.e.* $P \,\overline{\text{smor}}\, P$, is a sub-class of $\text{cror}'P$, and is specially important. This class differs widely in its properties from the corresponding cardinal class. If α has more than one member, the class $\alpha \,\overline{\text{sm}}\, \alpha$ (which is the "permutations" of α in the usual elementary sense) always has more than one member. But the class $P \,\overline{\text{smor}}\, P$ (which consists of such permutations of $C'P$ as keep the order unchanged) will consist of the single term $I \upharpoonright C'P$, unless $C'P$ contains classes which have neither a minimum nor a maximum, in which case there will be many correlators of P with itself. As a simple illustration, take the series of negative and positive integers in their natural order. Then if ν is any one of these integers, $+\nu$ is a correlator of the whole series with itself. If we take only the positive integers, $+\nu$ is no longer a correlator of the *whole* series with itself, since all integers less than ν are omitted from the correlate.

The first important use of the propositions of this number is in the beginning of the theory of well-ordered series ($*250$). The propositions there used are

$*208\cdot41$. $\vdash : P \epsilon \text{connex} \,.\, P^2 \subseteq J \,.\, \text{Cl ex}'C'P \subset \text{G}'\text{min}_P \cup \text{G}'\text{max}_P \,.$
$$P \,\text{smor}\, Q \,.\supset .\, (P \,\overline{\text{smor}}\, Q) \epsilon 1$$

I.e. if P is connected and asymmetrical, and every existent sub-class of $C'P$ has either a minimum or a maximum, P and Q cannot have more than one correlator.

$*208\cdot42$. In the same circumstances, $P \,\overline{\text{smor}}\, P = \iota'(I \upharpoonright C'P)$

$*208\cdot43$. $\vdash : \text{Cl ex}'C'P \subset \text{G}'\text{min}_P \,.\, S \epsilon \text{cror}'P \,.\supset .\, \sim(\exists x) \,.\, (S'x) \, P x$

I.e. if every existent sub-class of $C'P$ has a minimum, a correlator of P with part of itself can never move terms backwards. Thus for example, to take a simple instance, an infinite series consisting of some of the natural numbers in order of magnitude cannot have its μth term less than μ.

$*208\cdot45$. $\vdash : P \epsilon \text{connex} \,.\, \text{Cl ex}'C'P \subset \text{G}'\text{min}_P \cap \text{G}'\text{max}_P \,.\supset .\, \text{Rl}'P \cap \text{Nr}'P = \iota'P$

I.e. if P is connected and every existent sub-class of $C'P$ has both a maximum and a minimum, no proper part of P is similar to P. This proposition is important in the theory of finite series and finite ordinals.

$*208\cdot46$. $\vdash : \text{Cl ex}'C'P \subset \text{G}'\text{min}_P \,.\, S \epsilon \text{cror}'P \,.\supset .\, C'P \cap p'\overleftarrow{P}''\text{D}'S = \Lambda$

I.e. if every existent sub-class of $C'P$ has a minimum, a part of P which is similar to P must go up to the end of P, *i.e.* must not wholly precede any member of $C'P$.

$*208\cdot47$. $\vdash : \text{Cl ex}'C'P \subset \text{G}'\text{min}_P \,.\, Q \subseteq P \,.\, \exists ! \, C'P \cap p'\overleftarrow{P}''C'Q \,.\supset .\, \sim(Q \,\text{smor}\, P)$

This is an immediate consequence of $*208\cdot46$.

The proof of the above propositions proceeds simply by showing that if $S \epsilon \text{cror}'P$ and $(S'x) \, P x$, then $(S'S'x) \, P \, (S'x)$, so that x is not the earliest

term for which $(S\text{‘}x)\,Px$, since $S\text{‘}x$ is an earlier term for which the same thing holds. Hence $\hat{x}\,\{(S\text{‘}x)\,Px\}$ can have no minimum; and similarly $\hat{x}\,\{xP\,(S\text{‘}x)\}$ can have no maximum (∗208·14). So far we require no hypothesis as to P. Assuming now $P \in \text{connex} \,.\, P^2 \subseteq J$, we show similarly that if S correlates the whole of P with itself, $\hat{x}\,\{(S\text{‘}x)\,Px\}$ can have no maximum and $\hat{x}\,\{xP\,(S\text{‘}x)\}$ can have no minimum.

Propositions about correlators of P with Q follow from the above by taking two correlators S and T, and applying the above propositions to $S\,|\,\breve{T}$, which is a correlator of P with the whole of itself.

∗208·01. $\text{cror}\text{‘}P = s\text{‘}\overline{\text{smor}}\,P\text{“}\text{Rl}\text{‘}P \quad \text{Df}$

∗208·1. $\vdash : S \in \text{cror}\text{‘}P \,.\equiv .\, S \in 1 \to 1 \,.\, \text{Cl}\text{‘}S = C\text{‘}P \,.\, S ; P \subseteq P$

Dem.

$\vdash .\, ∗40{\cdot}4 \,.\, (∗208{\cdot}01) \,.\, ∗151{\cdot}11 \,.\supset$

$\vdash : S \in \text{cror}\text{‘}P \,.\equiv .\, (\exists Q) \,.\, Q \subseteq P \,.\, S \in 1 \to 1 \,.\, \text{Cl}\text{‘}S = C\text{‘}P \,.\, Q = S ; P \,.$

$[∗13{\cdot}195] \qquad \equiv .\, S \in 1 \to 1 \,.\, \text{Cl}\text{‘}S = C\text{‘}P \,.\, S ; P \subseteq P : \supset \vdash .\, \text{Prop}$

∗208·11. $\vdash : S \in \text{cror}\text{‘}P \,.\supset .\, S ; P \subseteq P \restriction \text{D}\text{‘}S$

Dem.

$\vdash .\, ∗150{\cdot}203 \,.\supset \vdash :.\, \text{Hp} \,.\supset : x\,(S;P)\,y \,.\supset .\, x, y \in \text{D}\text{‘}S \qquad (1)$

$\vdash .\, ∗208{\cdot}1 \,. \quad \supset \vdash :.\, \text{Hp} \,.\supset : x\,(S;P)\,y \,.\supset .\, xPy \qquad (2)$

$\vdash .\, (1) \,.\, (2) \,. \quad \supset \vdash .\, \text{Prop}$

∗208·111. $\vdash : S \in \text{cror}\text{‘}P \,.\supset .\, \text{D}\text{‘}S = C\text{‘}S;P = S\text{“}C\text{‘}P \,.\, \text{D}\text{‘}S \subset \text{Cl}\text{‘}S$

$[∗150{\cdot}22{\cdot}23 \,.\, ∗208{\cdot}1 \,.\, ∗33{\cdot}265]$

∗208·12. $\vdash : S \in \text{cror}\text{‘}P \,.\supset .\, \breve{S};S;P = P \,.\, P \subseteq \breve{S};P \quad [∗151{\cdot}252{\cdot}26 \,.\, ∗208{\cdot}1]$

∗208·13. $\vdash : S \in \text{cror}\text{‘}P \,.\, (S\text{‘}x)\,Px \,.\supset .\, (S\text{‘}S\text{‘}x)\,P\,(S\text{‘}x)$

Dem.

$\vdash .\, ∗208{\cdot}12 \,.\supset \vdash : \text{Hp} \,.\supset .\, (S\text{‘}x)\,(\breve{S};P)\,x \,.$

$[∗150{\cdot}41] \qquad \supset .\, (S\text{‘}S\text{‘}x)\,P\,(S\text{‘}x) : \supset \vdash .\, \text{Prop}$

∗208·131. $\vdash : S \in \text{cror}\text{‘}P \,.\, xP\,(S\text{‘}x) \,.\supset .\, (S\text{‘}x)\,P\,(S\text{‘}S\text{‘}x)$ [Proof as in ∗208·13]

∗208·14. $\vdash : S \in \text{cror}\text{‘}P \,.\supset .\, \overrightarrow{\text{min}}_P\text{‘}\hat{x}\,\{(S\text{‘}x)\,Px\} = \Lambda \,.\, \overrightarrow{\text{max}}_P\text{‘}\hat{x}\,\{xP\,(S\text{‘}x)\} = \Lambda$

Dem.

$\vdash .\, ∗208{\cdot}13 \,.\, ∗20{\cdot}3 \,.\supset \vdash :.\, \text{Hp} \,.\supset : x \in \hat{x}\{(S\text{‘}x)Px\} \,.\supset .\, S\text{‘}x \in \hat{x}\{(S\text{‘}x)Px\} \,.\, (S\text{‘}x)Px \,.$

$[∗37{\cdot}105] \qquad \supset .\, x \in \breve{P}\text{“}\hat{x}\,\{(S\text{‘}x)\,Px\} \qquad (1)$

$\vdash .\, (1) \,.\, ∗24{\cdot}3 \,. \qquad \supset \vdash : \text{Hp} \,.\supset .\, \hat{x}\,\{(S\text{‘}x)\,Px\} - \breve{P}\text{“}\hat{x}\,\{(S\text{‘}x)\,Px\} = \Lambda \,.$

$[∗205{\cdot}11] \qquad \supset .\, \overrightarrow{\text{min}}_P\text{‘}\hat{x}\,\{(S\text{‘}x)\,Px\} = \Lambda \qquad (2)$

Similarly $\qquad \vdash : \text{Hp} \,.\supset .\, \overrightarrow{\text{max}}_P\text{‘}\hat{x}\,\{xP\,(S\text{‘}x)\} = \Lambda \qquad (3)$

$\vdash .\, (2) \,.\, (3) \,.\supset \vdash .\, \text{Prop}$

Thus the proof that $\hat{x}\{(S\text{‘}x)\,Px\}$ has no minimum, and $\hat{x}\{xP\,(S\text{‘}x)\}$ no maximum, requires no hypothesis as to P. The proof that $\hat{x}\{(S\text{‘}x)\,Px\}$ has no maximum, and $\hat{x}\{xP\,(S\text{‘}x)\}$ no minimum, requires the hypothesis $P \in \text{connex} \,.\, P^2 \Subset J$. This proof results from the following propositions.

∗208·2. $\vdash : P \in \text{connex} \,.\, P^2 \Subset J \,.\, S \in \text{cror}\text{‘}P \,.\, \supset \,.\, P = \breve{S} \mathbin{;} P \,.\, S \mathbin{;} P = P \upharpoonright D\text{‘}S$

Dem.

$\vdash \,.\, *150{\cdot}41 \,.\, \supset \vdash :. \text{Hp} \,.\, \supset : x\,(\breve{S} \mathbin{;} P)\,y \,.\, \equiv \,.\, (S\text{‘}x)\,P\,(S\text{‘}y) \,.$

$[*50{\cdot}43{\cdot}45] \quad \supset \,.\, S\text{‘}x \neq S\text{‘}y \,.\, \sim\{(S\text{‘}y)\,P\,(S\text{‘}x)\} \,.$

$[*30{\cdot}37 \,.\, *150{\cdot}41] \quad \supset \,.\, x \neq y \,.\, \sim\{y\,(\breve{S} \mathbin{;} P)\,x\} \,.$

$[*208{\cdot}12 \,.\, \text{Transp}] \quad \supset \,.\, x \neq y \,.\, \sim(yPx) \quad (1)$

$\vdash \,.\, *150{\cdot}203 \,.\, \quad \supset \vdash : x\,(\breve{S} \mathbin{;} P)\,y \,.\, \supset \,.\, x, y \in \text{Ɑ}\text{‘}S \quad (2)$

$\vdash \,.\, (2) \,.\, *208{\cdot}1 \,.\, \quad \supset \vdash :. \text{Hp} \,.\, \supset : x\,(\breve{S} \mathbin{;} P)\,y \,.\, \supset \,.\, x, y \in C\text{‘}P \quad (3)$

$\vdash \,.\, (1) \,.\, (3) \,.\, *202{\cdot}103 \,.\, \supset \vdash :. \text{Hp} \,.\, \supset : x\,(\breve{S} \mathbin{;} P)\,y \,.\, \supset \,.\, xPy \quad (4)$

$\vdash \,.\, (4) \,.\, *208{\cdot}12 \,.\, \quad \supset \vdash : \text{Hp} \,.\, \supset \,.\, P = \breve{S} \mathbin{;} P \quad (5)$

$\vdash \,.\, (5) \,.\, \quad \supset \vdash : \text{Hp} \,.\, \supset \,.\, S \mathbin{;} P = S \mathbin{;} \breve{S} \mathbin{;} P$

$[*150{\cdot}38] \quad = P \upharpoonright D\text{‘}S \quad (6)$

$\vdash \,.\, (5) \,.\, (6) \,.\, \supset \vdash \,.\, \text{Prop}$

∗208·21. $\vdash : P \in \text{connex} \,.\, P^2 \Subset J \,.\, S \in \text{cror}\text{‘}P \,.\, (S\text{‘}x)\,Px \,.\, x \in D\text{‘}S \,.\, \supset \,.\, xP\,(\breve{S}\text{‘}x)$

Dem.

$\vdash \,.\, *33{\cdot}43 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, (S\text{‘}x)\,(P \upharpoonright D\text{‘}S)\,x \,.$

$[*208{\cdot}2] \quad \supset \,.\, (S\text{‘}x)\,(S \mathbin{;} P)\,x \,.$

$[*150{\cdot}41] \quad \supset \,.\, (\breve{S}\text{‘}S\text{‘}x)\,P\,(\breve{S}\text{‘}x) \,.$

$[*72{\cdot}241 \,.\, *33{\cdot}43] \quad \supset \,.\, xP\,(\breve{S}\text{‘}x) : \supset \vdash \,.\, \text{Prop}$

∗208·211. $\vdash : P \in \text{connex} \,.\, P^2 \Subset J \,.\, S \in \text{cror}\text{‘}P \,.\, xP\,(S\text{‘}x) \,.\, x \in D\text{‘}S \,.\, \supset \,.\, (\breve{S}\text{‘}x)\,Px$

[Proof as in ∗208·21]

∗208·22. $\vdash : P \in \text{connex} \,.\, P^2 \Subset J \,.\, S \in \text{cror}\text{‘}P \,.\, \text{Ɑ}\text{‘}S \subset D\text{‘}S \,.\, \supset \,.$

$$\overrightarrow{\text{max}}_P\text{‘}\hat{x}\{(S\text{‘}x)\,Px\} = \Lambda \,.\, \overrightarrow{\text{min}}_P\text{‘}\hat{x}\{xP\,(S\text{‘}x)\} = \Lambda$$

Dem.

$\vdash \,.\, *33{\cdot}43 \,.\, \quad \supset \vdash :. \text{Hp} \,.\, \supset : (S\text{‘}x)\,Px \,.\, \supset \,.\, x \in D\text{‘}S \,.\, x \in \text{Ɑ}\text{‘}S \,.$

$[*208{\cdot}21] \quad \supset \,.\, xP\,(\breve{S}\text{‘}x) \,.\, x \in \text{Ɑ}\text{‘}S \,.$

$[*72{\cdot}241] \quad \supset \,.\, xP\,(\breve{S}\text{‘}x) \,.\, \breve{S}\text{‘}x \in \hat{x}\{(S\text{‘}x)\,Px\} \,.$

$[*37{\cdot}1] \quad \supset \,.\, x \in P\text{‘‘}\hat{x}\{(S\text{‘}x)\,Px\} \quad (1)$

$\vdash \,.\, (1) \,.\, *205{\cdot}123 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, \overrightarrow{\text{max}}_P\text{‘}\hat{x}\{(S\text{‘}x)\,Px\} = \Lambda \quad (2)$

Similarly $\quad \vdash : \text{Hp} \,.\, \supset \,.\, \overrightarrow{\text{min}}_P\text{‘}\hat{x}\{xP\,(S\text{‘}x)\} = \Lambda \quad (3)$

$\vdash \,.\, (2) \,.\, (3) \,.\, \supset \vdash \,.\, \text{Prop}$

Observe that, in virtue of $*208\cdot111$, the above hypothesis gives $\mathrm{D}‘S = \mathrm{G}‘S = C‘P$, so that $S \in P \,\overline{\mathrm{smor}}\, P$. Hence we are led to $*208\cdot3$.

$*208\cdot3.$ $\vdash : P \in \mathrm{connex} \,.\, P^2 \subseteq J \,.\, S \in P \,\overline{\mathrm{smor}}\, P \,.\supset.$
$\sim\exists\,!\, \mathrm{min}_P‘\hat{x}\{(S‘x)\,Px\} \,.\sim\exists\,!\, \overrightarrow{\mathrm{max}_P}‘\hat{x}\{(S‘x)\,Px\} \,.$
$\sim\exists\,!\, \mathrm{min}_P‘\hat{x}\{xP\,(S‘x)\} \,.\sim\exists\,!\, \overrightarrow{\mathrm{max}_P}‘\hat{x}\{xP\,(S‘x)\}$

Dem.

$\vdash . *151\cdot11 . *150\cdot23 . \supset \vdash : \mathrm{Hp} . \supset . S \in 1 \to 1 . \mathrm{G}‘S = C‘P . S;P = P . \mathrm{D}‘S = C‘P .$
$[*208\cdot1] \quad \supset . S \in \mathrm{cror}‘P . \mathrm{G}‘S = \mathrm{D}‘S \qquad (1)$
$\vdash . (1) . *208\cdot14\cdot22 . \supset \vdash . \mathrm{Prop}$

$*208\cdot31.$ $\vdash : S, T \in P \,\overline{\mathrm{smor}}\, Q \,.\supset.\, S \,|\, \breve{T} \in P \,\overline{\mathrm{smor}}\, P \quad [*151\cdot131\cdot141]$

$*208\cdot32.$ $\vdash : P \in \mathrm{connex} \,.\, P^2 \subseteq J \,.\, S, T \in P \,\overline{\mathrm{smor}}\, Q \,.\supset.$
$\sim\exists\,!\, \overrightarrow{\mathrm{min}_P}‘\hat{x}\{(S‘\breve{T}‘x)\,Px\} \,.\sim\exists\,!\, \overrightarrow{\mathrm{max}_P}‘\hat{x}\{(S‘\breve{T}‘x)\,Px\} \,.$
$\sim\exists\,!\, \overrightarrow{\mathrm{min}_P}‘\hat{x}\{xP\,(S‘\breve{T}‘x)\} \,.\sim\exists\,!\, \overrightarrow{\mathrm{max}_P}‘\hat{x}\{xP\,(S‘\breve{T}‘x)\}$
$[*208\cdot3\cdot31 \,.\, *34\cdot41]$

$*208\cdot4.$ $\vdash : P \in \mathrm{connex} \,.\, P^2 \subseteq J \,.\, \mathrm{Cl\,ex}‘C‘P \subset \mathrm{G}‘\mathrm{min}_P \cup \mathrm{G}‘\mathrm{max}_P \,.$
$S, T \in P \,\overline{\mathrm{smor}}\, Q \,.\supset.\, S = T$

Dem.

$\vdash . *208\cdot32 . \quad \supset \vdash : \mathrm{Hp} . \supset . \hat{x}\{(S‘\breve{T}‘x)\,Px\} = \Lambda . \hat{x}\{xP\,(S‘\breve{T}‘x)\} = \Lambda \quad (1)$
$\vdash . *208\cdot31 . *34\cdot41 . \quad \supset \vdash :. \mathrm{Hp} . \supset : x \in C‘P . \supset . S‘\breve{T}‘x \in C‘P \qquad (2)$
$\vdash . (1) . (2) . *202\cdot103 . \supset \vdash :. \mathrm{Hp} . \supset : x \in C‘P . \supset . S‘\breve{T}‘x = x .$
$[*72\cdot241] \quad \supset . \breve{T}‘x = \breve{S}‘x :$
$[*150\cdot23] \quad \supset : x \in \mathrm{D}‘S \cup \mathrm{D}‘T . \supset . \breve{T}‘x = \breve{S}‘x :$
$[*33\cdot46] \quad \supset : S = T :. \supset \vdash . \mathrm{Prop}$

$*208\cdot41.$ $\vdash : P \in \mathrm{connex} \,.\, P^2 \subseteq J \,.\, \mathrm{Cl\,ex}‘C‘P \subset \mathrm{G}‘\mathrm{min}_P \cup \mathrm{G}‘\mathrm{max}_P \,.$
$P \,\mathrm{smor}\, Q \,.\supset.\, (P \,\overline{\mathrm{smor}}\, Q) \in 1$
$[*208\cdot4 \,.\, *151\cdot12 \,.\, *52\cdot16]$

The above proposition is of great importance in the theory of well-ordered series.

$*208\cdot42.$ $\vdash : P \in \mathrm{connex} \,.\, P^2 \subseteq J \,.\, \mathrm{Cl\,ex}‘C‘P \subset \mathrm{G}‘\mathrm{min}_P \cup \mathrm{G}‘\mathrm{max}_P \,.\supset.$
$P \,\overline{\mathrm{smor}}\, P = \iota‘(I \upharpoonright C‘P)$
$[*208\cdot4 \,.\, *51\cdot141 \,.\, *151\cdot121]$

$*208\cdot43.$ $\vdash : \mathrm{Cl\,ex}‘C‘P \subset \mathrm{G}‘\mathrm{min}_P \,.\, S \in \mathrm{cror}‘P \,.\supset.\sim(\exists x) \,.\, (S‘x)\,Px \quad [*208\cdot14]$

$*208\cdot431.$ $\vdash : \mathrm{Cl\,ex}‘C‘P \subset \mathrm{G}‘\mathrm{max}_P \,.\, S \in \mathrm{cror}‘P \,.\supset.\sim(\exists x) \,.\, xP\,(S‘x) \quad [*208\cdot14]$

***208·44** $\vdash : P \in \text{connex} . \text{Cl ex‘}C‘P \subset \text{Ɑ‘min}_P \cap \text{Ɑ‘max}_P . S \in \text{cror‘}P . \supset .$
$S = I \upharpoonright C‘P$

Dem.

$\vdash . *208·43·431 . *202·103 . \supset \vdash :. \text{Hp} . \supset : x \in C‘P . \supset . S‘x = x .$

[$*50·14 . *35·7$] $\supset . S‘x = (I \upharpoonright C‘P)‘x :$

[$*208·1 . *50·5·52$] $\supset : x \in \text{Ɑ‘}S \cup \text{Ɑ‘}(I \upharpoonright C‘P) . \supset . S‘x = (I \upharpoonright C‘P)‘x :$

[$*33·45$] $\supset : S = I \upharpoonright C‘P :. \supset \vdash . \text{Prop}$

In virtue of this proposition, if P is a finite series, no proper part of P is ordinally similar to P. (It will be shown later that a finite series is one in which every existent contained class has both a maximum and a minimum.) The following proposition gives a more explicit form of the above result.

***208·45.** $\vdash : P \in \text{connex} . \text{Cl ex‘}C‘P \subset \text{Ɑ‘min}_P \cap \text{Ɑ‘max}_P . \supset . \text{Rl‘}P \cap \text{Nr‘}P = \iota‘P$

Dem.

$\vdash . *208·44·1 . \supset \vdash :. \text{Hp} . \supset : S \in 1 \to 1 . \text{Ɑ‘}S = C‘P . S ; P \subseteq P . \supset . S = I \upharpoonright C‘P .$

[$*150·534$] $\supset . S ; P = P$ (1)

$\vdash . (1) . *13·12 . \supset \vdash :. \text{Hp} . \supset : Q \subseteq P . S \in 1 \to 1 . \text{Ɑ‘}S = C‘P . Q = S ; P . \supset . Q = P :$

[$*151·1$] $\supset : Q \subseteq P . Q \,\text{smor}\, P . \supset . Q = P :$

[$*152·1$] $\supset : \text{Rl‘}P \cap \text{Nr‘}P \subset \iota‘P$ (2)

$\vdash . *61·34 . *152·3 . \supset \vdash . \iota‘P \subset \text{Rl‘}P \cap \text{Nr‘}P$ (3)

$\vdash . (2) . (3) . \supset \vdash . \text{Prop}$

The following propositions are useful in the theory of segments of well-ordered series, since they show that a well-ordered series is never ordinally similar to any of its segments.

***208·46.** $\vdash : \text{Cl ex‘}C‘P \subset \text{Ɑ‘min}_P . S \in \text{cror‘}P . \supset . C‘P \cap p‘\overleftarrow{P}“\text{D‘}S = \Lambda$

Dem.

$\vdash . *208·1 . \supset \vdash :. S \in \text{cror‘}P . \supset : x \in C‘P \cap p‘\overleftarrow{P}“\text{D‘}S . \supset . (S‘x) P x :$

[Transp] $\supset : \sim \{(S‘x) P x\} . \supset . x \sim\in C‘P \cap p‘\overleftarrow{P}“\text{D‘}S$ (1)

$\vdash . (1) . *208·43 . \supset \vdash : \text{Hp} . \supset . (x) . x \sim\in C‘P \cap p‘\overleftarrow{P}“\text{D‘}S : \supset \vdash . \text{Prop}$

***208·461.** $\vdash : \text{Cl ex‘}C‘P \subset \text{Ɑ‘min}_P . S \in \text{cror‘}P . \dot{\exists} ! P . \supset . p‘\overleftarrow{P}“\text{D‘}S = \Lambda$
[$*208·46·1 . *40·62$]

***208·47.** $\vdash : \text{Cl ex‘}C‘P \subset \text{Ɑ‘min}_P . Q \subseteq P . \exists ! C‘P \cap p‘\overleftarrow{P}“C‘Q . \supset . \sim(Q \,\text{smor}\, P)$

Dem.

$\vdash . *208·46 . (*208·01) . \supset$

$\vdash :. \text{Hp} . \supset : Q \subseteq P . S \in Q \,\overline{\text{smor}}\, P . \supset . C‘P \cap p‘\overleftarrow{P}“\text{D‘}S = \Lambda$ (1)

$\vdash . (1) . \text{Transp} . *151·11 . *150·23 . \supset$

$\vdash :. \text{Hp} . \supset : Q \subseteq P . \exists ! C‘P \cap p‘\overleftarrow{P}“C‘Q . \supset . (S) . S \sim\in Q \,\overline{\text{smor}}\, P .$

[$*151·12$] $\supset . \sim(Q \,\text{smor}\, P) :. \supset \vdash . \text{Prop}$

SECTION B

ON SECTIONS, SEGMENTS, STRETCHES, AND DERIVATIVES

Summary of Section B.

In this section, our chief topic will be *sections* and *segments*. This topic will occupy $*211$, $*212$ and $*213$, and $*210$ will consist of propositions whose chief utility lies in their application to segments. In $*214$, we shall consider Dedekindian series, which are intimately connected with segments, owing to the fact that one of the chief propositions in the subject is that the series of segments of a series is Dedekindian. In $*215$, we shall consider "stretches," which consist of any consecutive piece of a series, and are constituted by the product of an upper and lower section. Finally, in $*216$, we shall consider the derivative of a series, or of a class α contained in a series: the former is the series of limit-points of the series, *i.e.* $P \upharpoonright \mathrm{D}\text{`}\mathrm{lt}_P$, the latter is the class of limits of existent sub-classes of $\alpha \cap C\text{`}P$, *i.e.* $\mathrm{lt}_P\text{``}\mathrm{Cl\,ex}\text{`}(\alpha \cap C\text{`}P)$.

A class is called a *section* of P when it is contained in $C\text{`}P$, and contains all the predecessors of its members, *i.e.* $\alpha$ is a section of $P$ if $\alpha \subset C\text{`}P \,.\, P\text{``}\alpha \subset \alpha$. Thus a section consists of all the field up to a certain point. It may consist of all the predecessors of x, *i.e.* it may be of the form $\overrightarrow{P}\text{`}x$; or again, it may consist of these together with $x$, in which case it is of the form $\overrightarrow{P}\text{`}x \cup \iota\text{`}x$; or again, it may be not definable by means of a single sequent or maximum, but be of the form $P\text{``}\alpha$, where α is a class without a limit or maximum. The class of sections of P is denoted by $\mathrm{sect}\text{`}P$. A section of $\breve{P}$ will be called an "upper section" of P.

The idea of a *segment* is slightly less general than that of a *section.* We define a segment of P as any class of the form $P\text{``}\alpha$, *i.e.* as any member of $\mathrm{D}\text{`}P_\epsilon$. Provided $P$ is transitive, segments are contained among sections. But even in a series sections are not, in general, contained among segments: if $P$ is a series, and if $x$ is a member of $C\text{`}P$ which has no immediate successor, $\overrightarrow{P}\text{`}x \cup \iota\text{`}x$ will be a section but not a segment.

If a segment has a maximum, it must also have a sequent. Segments which have no maximum form a specially important class of segments: these are classes α such that $\alpha = P\text{``}\alpha$; they form the class $\mathrm{D}\text{`}(P_\epsilon \dot{\cap} I)$.

The properties of sections and segments considered as classes of classes are many and various: they are considered in $*211$. In $*212$, we pass to the consideration of the *series* of sections and segments. These series are

$P_{1c} \upharpoonright \text{sect}'P$ and $P_{1c} \upharpoonright D'P_\epsilon$ (cf. $*170$). The series of such segments as have no maximum is $P_{1c} \upharpoonright D'(P_\epsilon \dot{\frown} I)$. We put

$$\varsigma'P = P_{1c} \upharpoonright D'P_\epsilon \qquad \text{Df},$$

$$\text{sgm}'P = P_{1c} \upharpoonright D'(P_\epsilon \dot{\frown} I) \quad \text{Df}.$$

It then appears that

$$\varsigma'P_* = \text{sgm}'P_* = P_{1c} \upharpoonright \text{sect}'P,$$

so that it is unnecessary to introduce a special notation for the series of sections.

Whenever P is connected and transitive, $P_{1c} \upharpoonright D'P_\epsilon$ turns out to be equivalent to logical inclusion combined with diversity (with the field limited to $D'P_\epsilon$). That is to say ($*212{\cdot}23$),

$$\vdash : P \in \text{trans} \cap \text{connex} . \supset . \varsigma'P = \hat{\alpha}\hat{\beta} \{\alpha, \beta \in D'P_\epsilon . \alpha \subset \beta . \alpha \neq \beta\}.$$

Hence it follows ($*212{\cdot}24$) that

$$\vdash : P_* \in \text{connex} . \supset . \varsigma'P_* = \hat{\alpha}\hat{\beta} \{\alpha, \beta \in \text{sect}'P . \alpha \subset \beta . \alpha \neq \beta\}.$$

We have also ($*211{\cdot}6{\cdot}17$)

$$\vdash :. P_* \in \text{connex} . \alpha, \beta \in \text{sect}'P . \supset : \alpha \subset \beta . \vee . \beta \subset \alpha.$$

Hence it easily follows that whenever P_* is connected, $\varsigma'P_*$ is a series. Similarly $\varsigma'P$ will be a series if P is transitive and connected.

The fact of connection, which is required in order that $\varsigma'P$ or $\varsigma'P_*$ may be a series, results from

$$\alpha, \beta \in \text{sect}'P . \supset : \alpha \subset \beta . \vee . \beta \subset \alpha$$

or

$$\alpha, \beta \in D'P_\epsilon . \supset : \alpha \subset \beta . \vee . \beta \subset \alpha.$$

In order to deal with such cases generally, we study, in a preliminary number ($*210$), the consequences to be deduced from the hypothesis

$$\alpha, \beta \in \kappa . \supset_{\alpha, \beta} : \alpha \subset \beta . \vee . \beta \subset \alpha.$$

We find that, with this hypothesis, putting

$$Q = \hat{\alpha}\hat{\beta} (\alpha, \beta \in \kappa . \alpha \subset \beta . \alpha \neq \beta),$$

$Q = P_{1c} \upharpoonright \kappa$ if $\kappa \subset \text{Cl}'C'P$ ($*210{\cdot}13$), and thus in the same circumstances $P_{1c} \upharpoonright \kappa$ is a series ($*210{\cdot}14$).

The interesting point about such series is their behaviour with regard to limits. Assuming that κ is not a unit class (so as to insure $\dot{\exists} ! Q$), if λ is any sub-class of κ, the logical product $p'\lambda$ is the minimum of λ if it is a member of λ ($*210{\cdot}21$), and the lower limit of λ if it is a member of κ but not of λ ($*210{\cdot}23$). Similarly $s'\lambda$ is the maximum of λ if it is a member of λ ($*210{\cdot}211$), and the upper limit of λ if it is not a member of λ but is a member of κ ($*210{\cdot}231$). Thus if κ is such that, whenever $\lambda \subset \kappa$, we have $s'\lambda \in \kappa$, it follows that every sub-class of κ has either a maximum or a limit, *i.e.* the series $P_{1c} \upharpoonright \kappa$ is Dedekindian. Now each of the three classes $\text{sect}'P$, $D'P_\epsilon$, $D'(P_\epsilon \dot{\frown} I)$ verifies this condition, *i.e.* the sum of any sub-class of any one of these classes belongs to the class in question ($*211{\cdot}63{\cdot}64{\cdot}65$). (This

holds without any hypothesis as to P.) Hence we arrive at the result that $\varsigma`P_*$ (*i.e.* the series of sections) is a Dedekindian series whenever $P_*$ is connected and $P$ is not null ($*214\cdot32$), while $\varsigma`P$ (*i.e.* the series of segments) is a Dedekindian series whenever P is transitive and connected and not null ($*214\cdot33$), and $\text{sgm}`P$ (the series of segments having no maximum) is a Dedekindian series whenever it exists and P is connected ($*214\cdot34$). These propositions are important, and are the source of much of the utility of sections and segments.

For many purposes, especially in ordinal arithmetic, it is necessary to consider sections not as classes, but as series. That is to say, if α is a member of $\text{sect}`P$, we want to deal with $P \upharpoonright \alpha$ rather than with $\alpha$. The series of all such terms as $P \upharpoonright \alpha$ might be supposed to be $P \upharpoonright ; \varsigma`P_*$. But here a limitation is necessary owing to the fact that, if $B`P$ exists, $\Lambda$ and $\iota`B`P$ are both sections, and $P \upharpoonright \Lambda$ and $P \upharpoonright \iota`B`P$ are both $\dot{\Lambda}$, so that $P \upharpoonright ; \varsigma`P_*$ will be a relation which $\dot{\Lambda}$ will have to itself. In order to avoid this, we first exclude Λ from the sections to be considered, and thus put

$$P_\varsigma = P \upharpoonright ;(\varsigma`P_*) \upharpoonright (-\iota`\Lambda) \quad \text{Df.}$$

Then P_ς is the series of sections considered as series. Provided P_{po} is a series, the relation P_ς holds between any two members Q and R of its field when, and only when, $Q \subset R \,.\, Q \neq R$. The subject of P_ς is considered in $*213$; the utility of the propositions of this number will not appear until we come to ordinal arithmetic.

The subject of Dedekindian relations is next considered ($*214$). We define a Dedekindian relation as one such that every class has either a maximum or a sequent. A Dedekindian series must have a first and a last term, since the first term must be the sequent of Λ, and the last must be the maximum of the field. A Dedekindian series may be discrete, or compact (*i.e.* such that there is a term between any two, *i.e.* such that $P^2 = P$), or partly one and partly the other. A finite series must be Dedekindian: a well-ordered series is Dedekindian if it has a last term. But the chief importance of the Dedekindian property is in connection with compact series. A compact Dedekindian series is said to possess "Dedekindian continuity"; such series have many important properties. They are a wider class than series possessing Cantorian continuity; these latter will be considered in Section F of this Part.

$*210$. ON SERIES OF CLASSES GENERATED BY THE RELATION OF INCLUSION

Summary of $*210$.

In the theory of series it frequently happens that we have to deal with a class of classes such that, of any two, one is contained in the other. *I.e.* if κ is the class of classes, we have

$$\alpha, \beta \in \kappa . \supset_{\alpha, \beta} : \alpha \subset \beta . \vee . \beta \subset \alpha.$$

Instances of this are afforded by the various classes of sections, to be considered in $*211$. When κ fulfils the above condition, the classes composing κ can be arranged in a series by the relation of inclusion (combined with inequality), *i.e.* by the relation

$$\hat{\alpha}\hat{\beta}\,(\alpha, \beta \in \kappa . \alpha \subset \beta . \alpha \neq \beta),$$

or, what comes to the same,

$$\hat{\alpha}\hat{\beta}\,(\alpha, \beta \in \kappa . \exists ! \beta - \alpha).$$

If P is any relation such that $\kappa \subset \mathrm{Cl}`C`P$, the above relation of inclusion is equal to

$$P_{1c} \upharpoonright \kappa.$$

(For the definition of P_{1c}, see $*170$.) Thus under the above circumstances, $P_{1c} \upharpoonright \kappa$ is a series, whatever P may be.

The importance of such relations of inclusion, as generators of series, is in connection with the existence of maxima and minima or limits. If we put

$$Q = \hat{\alpha}\hat{\beta}\,(\alpha, \beta \in \kappa . \exists ! \beta - \alpha),$$

where κ satisfies the above condition, then if $\lambda \subset \kappa$, and if $s`\lambda \in \kappa$, $s`\lambda$ is the maximum or the upper limit of λ with respect to Q, according as $s`\lambda$ is a member of $\lambda$ or not. Similarly if $p`\lambda \in \kappa$, $p`\lambda$ is the minimum or lower limit of $\lambda$, according as $p`\lambda$ is a member of λ or not. Hence if κ is such that the sum of any sub-class of κ is a member of κ, every sub-class of κ has either a maximum or an upper limit; and if the product of every sub-class of κ is a member of κ, every sub-class of κ has either a minimum or a lower limit.

In order that every sub-class of κ should have a minimum or a lower limit, it is sufficient that the *sum* of every sub-class of κ should be a member of κ. For, if λ is any sub-class of κ, ccnsider those members of κ which are contained in $p`\lambda$, *i.e.*

$$\kappa \cap \mathrm{Cl}`p`\lambda.$$

If $p`\lambda \in \kappa$, the sum of these classes $= p`\lambda$, and is the lower limit or minimum of κ. But if $p`\lambda \sim \in \kappa$, then every member of $\kappa$ which is not contained in $s`(\kappa \cap \mathrm{Cl}`p`\lambda)$ is also not contained in $p`\lambda$, and is therefore not contained in some member of $\lambda$. Hence $s`(\kappa \cap \mathrm{Cl}`p`\lambda)$ is the lower limit of λ.

It is owing to these propositions that segments of series are of such great importance in connection with limits.

The hypothesis that if $\lambda \subset \kappa$, $p'\lambda$ is a member of κ, will usually fail to be verified in the case when $\lambda = \Lambda$, since in this case $p'\lambda = \mathrm{V}$. But all the results desired can be obtained from the hypothesis that, if $\lambda \subset \kappa$, $(p'\lambda \cap s'\kappa) \in \kappa$. This hypothesis is equivalent to the other except in the case of Λ, in which case it requires $s'\kappa \in \kappa$, which is much more often verified than $\mathrm{V} \in \kappa$, which was required by the other hypothesis.

The principal propositions of this number are the following:

$*210{\cdot}1.\quad \vdash :: \alpha, \beta \in \kappa \,.\supset_{\alpha,\beta} : \alpha \subset \beta \,.\vee.\, \beta \subset \alpha :.\supset :.\, \alpha, \beta \in \kappa \,.\supset : \alpha \subset \beta \,.\, \alpha \neq \beta \,.\equiv.\, \exists ! \beta - \alpha$

$*210{\cdot}11.\quad \vdash : Q = \hat{\alpha}\hat{\beta}(\alpha, \beta \in \kappa \,.\, \alpha \subset \beta \,.\, \alpha \neq \beta) \,.\supset.\, Q \in \mathrm{trans} \cap \mathrm{Rl}'J$

$*210{\cdot}12.\quad \vdash : \mathrm{Hp}\, *210{\cdot}1{\cdot}11 \,.\supset.\, Q \in \mathrm{Ser}$

$*210{\cdot}13.\quad \vdash : \mathrm{Hp}\, *210{\cdot}12 \,.\, \kappa \subset \mathrm{Cl}'C'P \,.\supset.\, Q = P_{\mathrm{lc}} \upharpoonright \kappa$

$*210{\cdot}2.\quad \vdash : \mathrm{Hp}\, *210{\cdot}12 \,.\, \kappa \sim\in 1 \,.\supset.\, \overrightarrow{\min}_Q{}'\lambda = \lambda \cap \kappa \cap \iota' p'(\lambda \cap \kappa)$

$*210{\cdot}21.\quad \vdash : \mathrm{Hp}\, *210{\cdot}2 \,.\, \lambda \subset \kappa \,.\, p'\lambda \in \lambda \,.\supset.\, \min_Q{}'\lambda = p'\lambda$

$*210{\cdot}211$ gives an analogous proposition for $s'\lambda$ and $\max_Q$. We shall not here mention such analogues, unless for some special reason.

$*210{\cdot}23.\quad \vdash : \mathrm{Hp}\, *210{\cdot}2 \,.\, \lambda \subset \kappa \,.\, p'\lambda \in \kappa - \lambda \,.\supset.\, p'\lambda = \mathrm{prec}_Q{}'\lambda = \mathrm{tl}_Q{}'\lambda$

$*210{\cdot}232.\quad \vdash : \mathrm{Hp}\, *210{\cdot}2 \,.\, \lambda \subset \kappa \,.\, p'\lambda \in \kappa \,.\supset.\, p'\lambda = \mathrm{limin}_P{}'\lambda$

$*210{\cdot}251.\quad \vdash :.\, \mathrm{Hp}\, *210{\cdot}2 : \lambda \subset \kappa \,.\supset_\lambda.\, s'\lambda \in \kappa : \supset : \lambda \subset \kappa \,.\supset.\, s'\lambda \in (\overrightarrow{\max}_Q{}'\lambda \cup \overrightarrow{\mathrm{seq}}_Q{}'\lambda)$

$*210{\cdot}252.\quad \vdash :.\, \mathrm{Hp}\, *210{\cdot}2 : \lambda \subset \kappa \,.\supset_\lambda.\, p'\lambda \cap s'\kappa \in \kappa : \supset :$

$$\lambda \subset \kappa \,.\supset.\, p'\lambda \cap s'\kappa \in (\overrightarrow{\min}_Q{}'\lambda \cup \overrightarrow{\mathrm{prec}}_Q{}'\lambda) \,.\, p'\lambda \cap s'\kappa = \mathrm{limin}_Q{}'\lambda$$

$*210{\cdot}254.\quad \vdash : \mathrm{Hp}\, *210{\cdot}251 \,.\supset.\, (\lambda) \,.\, \lambda \in \text{Ɑ}'\max_Q \cup \text{Ɑ}'\mathrm{seq}_Q$

$*210{\cdot}26.\quad \vdash : \mathrm{Hp}\, *210{\cdot}2 \,.\, \lambda \subset \kappa \,.\, p'\lambda \sim\in \lambda \,.\, s'(\kappa \cap \mathrm{Cl}'p'\lambda) \in \kappa \,.\supset.$

$$s'(\kappa \cap \mathrm{Cl}'p'\lambda) = \mathrm{prec}_Q{}'\lambda$$

$*210{\cdot}28.\quad \vdash : \mathrm{Hp}\, *210{\cdot}2 \,.\, s''\mathrm{Cl}'\kappa \subset \kappa \,.\supset.$

$$(\lambda) \,.\, \lambda \in (\text{Ɑ}'\max_Q \cup \text{Ɑ}'\mathrm{seq}_Q) \cap (\text{Ɑ}'\min_Q \cup \text{Ɑ}'\mathrm{prec}_Q)$$

Thus if κ is a class of not less than two classes such that, of any two of its members, one must be contained in the other, and if Q is the relation $\alpha \subset \beta \,.\, \alpha \neq \beta$ confined to members of κ, then Q is a series ($*210{\cdot}12$) in which, provided the sums of sub-classes of κ are always members of κ, every class has either a maximum or an upper limit, and every class has either a minimum or a lower limit ($*210{\cdot}28$).

The reader will observe that, if $\alpha, \beta \in \kappa \,.\supset_{\alpha,\beta} : \alpha \subset \beta \,.\vee.\, \beta \subset \alpha$, any *finite* sub-class of κ must contain its own sum and product as members. For example, if we have two classes α and β, if $\alpha \subset \beta$, then $\alpha = p'(\iota'\alpha \cup \iota'\beta)$ and $\beta = s'(\iota'\alpha \cup \iota'\beta)$; if we have three classes α, β, γ, and $\alpha \subset \beta \,.\, \beta \subset \gamma$, then

$\alpha = p\text{‘}(\iota\text{‘}\alpha \cup \iota\text{‘}\beta \cup \iota\text{‘}\gamma)$ and $\gamma = s\text{‘}(\iota\text{‘}\alpha \cup \iota\text{‘}\beta \cup \iota\text{‘}\gamma)$; and so on. Thus the hypothesis $s\text{‘‘}\mathrm{Cl}\text{‘}\kappa \subset \kappa$ is only required in order to enable us to deal with infinite snb-classes of κ.

***210·1.** $\vdash :: \alpha, \beta \epsilon \kappa . \supset_{\alpha,\beta} : \alpha \subset \beta . \mathrm{v} . \beta \subset \alpha :. \supset :. \alpha, \beta \epsilon \kappa . \supset : \alpha \subset \beta . \alpha \neq \beta . \equiv . \exists ! \beta - \alpha$

Dem.

$\vdash . *24{\cdot}6 . \quad \supset \vdash : \alpha \subset \beta . \alpha \neq \beta . \supset . \exists ! \beta - \alpha \quad (1)$

$\vdash . *24{\cdot}55 . \quad \supset \vdash : \exists ! \beta - \alpha . \supset . \sim(\beta \subset \alpha) .$ (2)

$[*22{\cdot}42] \quad \supset . \alpha \neq \beta \quad (3)$

$\vdash . *2{\cdot}53 . \quad \supset \vdash :. \mathrm{Hp} . \alpha, \beta \epsilon \kappa . \sim(\beta \subset \alpha) . \supset . \alpha \subset \beta \quad (4)$

$\vdash . (2) . (3) . (4) . \supset \vdash :. \mathrm{Hp} . \alpha, \beta \epsilon \kappa . \supset : \exists ! \beta - \alpha . \supset . \alpha \subset \beta . \alpha \neq \beta \quad (5)$

$\vdash . (1) . (5) . \supset \vdash . \mathrm{Prop}$

***210·11.** $\vdash : Q = \hat{\alpha}\hat{\beta}(\alpha, \beta \epsilon \kappa . \alpha \subset \beta . \alpha \neq \beta) . \supset . Q \epsilon \mathrm{trans} \cap \mathrm{Rl}\text{‘}J$

Dem.

$\vdash . *50{\cdot}11 . \quad \supset \vdash : \mathrm{Hp} . \supset . Q \epsilon \mathrm{Rl}\text{‘}J \quad (1)$

$\vdash . *22{\cdot}44 . \quad \supset \vdash :. \mathrm{Hp} . \supset : \alpha Q \beta . \beta Q \gamma . \supset . \alpha \subset \gamma \quad (2)$

$\vdash . *24{\cdot}6 . *21{\cdot}33 . \supset \vdash :. \mathrm{Hp} . \supset : \alpha Q \beta . \beta Q \gamma . \supset . \exists ! \beta - \alpha . \beta \subset \gamma .$

$[*24{\cdot}58] \quad \supset . \exists ! \gamma - \alpha .$

$[*24{\cdot}21] \quad \supset . \alpha \neq \gamma \quad (3)$

$\vdash . (2) . (3) . \quad \supset \vdash :. \mathrm{Hp} . \supset : \alpha Q \beta . \beta Q \gamma . \supset . \alpha Q \gamma \quad (4)$

$\vdash . (1) . (4) . \supset \vdash . \mathrm{Prop}$

***210·12.** $\vdash : \mathrm{Hp} *210{\cdot}1{\cdot}11 . \supset . Q \epsilon \mathrm{Ser}$

Dem.

$\vdash . *10{\cdot}1 . \quad \supset \vdash :. \mathrm{Hp} . \alpha, \beta \epsilon \kappa . \supset : \alpha \subset \beta . \mathrm{v} . \beta \subset \alpha :$

$[*5{\cdot}62] \quad \supset : \alpha \subset \beta . \alpha \neq \beta . \mathrm{v} . \beta \subset \alpha . \beta \neq \alpha . \mathrm{v} . \alpha = \beta \quad (1)$

$\vdash . *21{\cdot}33 . \quad \supset \vdash :. \mathrm{Hp} . \supset : \alpha Q \beta . \supset_{\alpha,\beta} . \alpha, \beta \epsilon \kappa :$

$[*33{\cdot}352] \quad \supset : C\text{‘}Q \subset \kappa \quad (2)$

$\vdash . (1) . (2) . \supset \vdash :: \mathrm{Hp} . \supset :. \alpha, \beta \epsilon C\text{‘}Q . \supset : \alpha Q \beta . \mathrm{v} . \beta Q \alpha . \mathrm{v} . \alpha = \beta \quad (3)$

$\vdash . *210{\cdot}11 . (3) . *204{\cdot}12 . \supset \vdash . \mathrm{Prop}$

***210·121.** $\vdash : \mathrm{Hp} *210{\cdot}12 . \supset . \mathrm{D}\text{‘}Q = \kappa - \iota\text{‘}s\text{‘}\kappa . \text{Ɑ}\text{‘}Q = \kappa - \iota\text{‘}p\text{‘}\kappa$

Dem.

$\vdash . *21{\cdot}33 . \supset \vdash :: \mathrm{Hp} . \supset :. \alpha \epsilon \mathrm{D}\text{‘}Q . \equiv : \alpha \epsilon \kappa : (\exists \beta) . \beta \epsilon \kappa . \alpha \subset \beta . \alpha \neq \beta :$

$[*210{\cdot}1] \quad \equiv : \alpha \epsilon \kappa : (\exists \beta) . \beta \epsilon \kappa . \exists ! \beta - \alpha :$

$[*40{\cdot}151 . \mathrm{Transp}] \quad \equiv : \alpha \epsilon \kappa . \exists ! s\text{‘}\kappa - \alpha :$

$[*24{\cdot}55] \quad \equiv : \alpha \epsilon \kappa . \sim(s\text{‘}\kappa \subset \alpha) :$

$[*22{\cdot}41 . *40{\cdot}13] \quad \equiv : \alpha \epsilon \kappa . \alpha \neq s\text{‘}\kappa \quad (1)$

$\vdash . *21{\cdot}33 . \supset \vdash :: \mathrm{Hp} . \supset :. \alpha \epsilon \text{Ɑ}\text{‘}Q . \equiv : \alpha \epsilon \kappa : (\exists \beta) . \beta \epsilon \kappa . \beta \subset \alpha . \beta \neq \alpha :$

$[*210{\cdot}1] \quad \equiv : \alpha \epsilon \kappa : (\exists \beta) . \beta \epsilon \kappa . \exists ! \alpha - \beta :$

$[*40{\cdot}15 . \mathrm{Transp}] \quad \equiv : \alpha \epsilon \kappa . \exists ! \alpha - p\text{‘}\kappa :$

$[*24{\cdot}55] \quad \equiv : \alpha \epsilon \kappa . \sim(\alpha \subset p\text{‘}\kappa) :$

$[*22{\cdot}41 . *40{\cdot}12] \quad \equiv : \alpha \epsilon \kappa . \alpha \neq p\text{‘}\kappa \quad (2)$

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***210·122.** $\vdash : \text{Hp} \, *210{\cdot}12 \, . \, \kappa \sim\in 1 \, . \supset . \, C`Q = \kappa$

Dem.

$\vdash . *52{\cdot}181 . \supset \vdash :: \text{Hp} . \supset :. \alpha \in \kappa . \supset : (\exists \beta) . \beta \in \kappa . \beta \neq \alpha :$

$[\text{Hp}.*10{\cdot}1] \quad \supset : (\exists \beta) : \beta \in \kappa . \beta \neq \alpha : \alpha \subset \beta . \vee . \beta \subset \alpha :$

$[*21{\cdot}33] \quad \supset : (\exists \beta) : \beta \in \kappa : \alpha Q \beta . \vee . \beta Q \alpha :$

$[*33{\cdot}132] \quad \supset : \alpha \in C`Q \quad (1)$

$\vdash . *21{\cdot}33 . \supset \vdash :. \text{Hp} . \supset : \alpha Q \beta . \supset_{\alpha, \beta} . \alpha, \beta \in \kappa :$

$[*33{\cdot}352] \quad \supset : C`Q \subset \kappa \quad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***210·123.** $\vdash : \text{Hp} \, *210{\cdot}12 \, . \, \kappa \in 0 \cup 1 \, . \supset . \, Q = \dot{\Lambda}$

Dem.

$\vdash . *52{\cdot}41 . \text{Transp} . \supset \vdash : \text{Hp} . \supset . \sim (\exists \alpha, \beta) . \alpha, \beta \in \kappa . \alpha \neq \beta .$

$[*21{\cdot}33] \quad \supset . \sim (\exists \alpha, \beta) . \alpha Q \beta : \supset \vdash . \text{Prop}$

***210·124.** $\vdash :. \text{Hp} \, *210{\cdot}12 \, . \supset : \alpha Q \beta . \equiv . \alpha, \beta \in \kappa . \exists ! \beta - \alpha \quad [*210{\cdot}1]$

***210·13.** $\vdash : \text{Hp} \, *210{\cdot}12 \, . \, \kappa \subset \text{Cl}`C`P \, . \supset . \, Q = P_{1c} \upharpoonright \kappa$

Dem.

$\vdash . *170{\cdot}102 . \supset \vdash :. \text{Hp} . \supset : \alpha (P_{1c} \upharpoonright \kappa) \beta . \equiv . \alpha, \beta \in \kappa . \exists ! \beta - \alpha - P``(\alpha - \beta) . \quad (1)$

$[*210{\cdot}124] \quad \supset . \alpha Q \beta \quad (2)$

$\vdash . *210{\cdot}1{\cdot}124 . \supset \vdash :. \text{Hp} . \supset : \alpha Q \beta . \supset . \alpha, \beta \in \kappa . \exists ! \beta - \alpha . \alpha \subset \beta .$

$[*37{\cdot}29] \quad \supset . \alpha, \beta \in \kappa . \exists ! \beta - \alpha . P``(\alpha - \beta) = \Lambda .$

$[*24{\cdot}23{\cdot}313] \quad \supset . \alpha, \beta \in \kappa . \exists ! \beta - \alpha - P``(\alpha - \beta) .$

$[(1)] \quad \supset . \alpha (P_{1c} \upharpoonright \kappa) \beta \quad (3)$

$\vdash . (2) . (3) . \supset \vdash . \text{Prop}$

Thus under the hypothesis of $*210{\cdot}1$, $P_{1c} \upharpoonright \kappa$ does not depend upon P, so long as $\kappa \subset \text{Cl}`C`P$. Also we have

***210·14.** $\vdash : \text{Hp} \, *210{\cdot}1 \, . \, \kappa \subset \text{Cl}`C`P \, . \supset . \, P_{1c} \upharpoonright \kappa \in \text{Ser}$ $[*210{\cdot}12{\cdot}13]$

***210·15.** $\vdash :. \text{Hp} \, *210{\cdot}12 \, . \, \alpha, \beta \in \kappa \, . \supset : \sim (\alpha Q \beta) . \equiv . \beta \subset \alpha$ $[*210{\cdot}124 . *24{\cdot}55]$

***210·16.** $\vdash :: \text{Hp} \, *210{\cdot}1 \, . \supset :.$

$$\alpha \in \kappa . \lambda \subset \kappa . \supset : \alpha \subset p`\lambda . \vee . p`\lambda \subset \alpha : \alpha \subset s`\lambda . \vee . s`\lambda \subset \alpha$$

Dem.

$\vdash . *10{\cdot}1 . \supset \vdash :: \text{Hp} . \alpha \in \kappa . \lambda \subset \kappa . \supset :. \beta \in \lambda . \supset_\beta : \alpha \subset \beta . \vee . \beta \subset \alpha :. \quad (1)$

$[*10{\cdot}57] \quad \supset :. \beta \in \lambda . \supset_\beta . \alpha \subset \beta : \vee : (\exists \beta) . \beta \in \lambda . \beta \subset \alpha :.$

$[*40{\cdot}15{\cdot}12] \quad \supset :. \alpha \subset p`\lambda . \vee . p`\lambda \subset \alpha \quad (2)$

$\vdash . (1) . *10{\cdot}57 . \supset$

$\vdash :: \text{Hp} . \alpha \in \kappa . \lambda \subset \kappa . \supset :. \beta \in \lambda . \supset_\beta . \beta \subset \alpha : \vee : (\exists \beta) . \beta \in \kappa . \alpha \subset \beta :.$

$[*40{\cdot}151{\cdot}13] \quad \supset :. s`\lambda \subset \alpha . \vee . \alpha \subset s`\lambda \quad (3)$

$\vdash . (2) . (3) . \supset \vdash . \text{Prop}$

***210·17.** $\vdash : \mathrm{Hp} *210{\cdot}12 . \lambda \subset \kappa . \supset .$

$$\kappa - \breve{Q}``\lambda = \kappa \cap \mathrm{Cl}`p`\lambda . \kappa - Q``\lambda = \kappa \cap \hat{\gamma}(s`\lambda \subset \gamma)$$

Dem.

$\vdash . *37{\cdot}105 . \mathrm{Transp} . \supset \vdash :. \alpha \in \kappa - \breve{Q}``\lambda . \equiv : \alpha \in \kappa : \beta \in \lambda . \supset_\beta . \sim(\beta Q \alpha)$ (1)

$\vdash . (1) . *210{\cdot}15 . \supset$

$\vdash :: \mathrm{Hp} . \supset :. \alpha \in \kappa - \breve{Q}``\lambda . \equiv : \alpha \in \kappa : \beta \in \lambda . \supset_\beta . \alpha \subset \beta :$

[*40·15] $\equiv : \alpha \in \kappa \cap \mathrm{Cl}`p`\lambda$ (2)

Similarly $\vdash :. \mathrm{Hp} . \supset : \alpha \in \kappa - Q``\lambda . \equiv . \alpha \in \kappa \cap \hat{\gamma}(s`\lambda \subset \gamma)$ (3)

$\vdash . (2) . (3) . \supset \vdash . \mathrm{Prop}$

***210·2.** $\vdash : \mathrm{Hp} *210{\cdot}12 . \kappa \sim\in 1 . \supset . \overrightarrow{\min}_Q`\lambda = \lambda \cap \kappa \cap \iota`p`(\lambda \cap \kappa)$

Dem.

$\vdash . *205{\cdot}15 . *210{\cdot}122 . \supset \vdash : \mathrm{Hp} . \supset . \overrightarrow{\min}_Q`\lambda = \overrightarrow{\min}_Q`(\lambda \cap \kappa)$

[*205·11] $= \lambda \cap \kappa - \breve{Q}``(\lambda \cap \kappa)$

[*210·17] $= \lambda \cap \kappa \cap \mathrm{Cl}`p`(\lambda \cap \kappa)$ (1)

$\vdash . *40{\cdot}12 . \supset \vdash :. \alpha \in \lambda \cap \kappa . \supset : p`(\lambda \cap \kappa) \subset \alpha :$

[*22·41] $\supset : \alpha \subset p`(\lambda \cap \kappa) . \equiv . \alpha = p`(\lambda \cap \kappa)$ (2)

$\vdash . (2) . *5{\cdot}32 . \supset \vdash . \lambda \cap \kappa \cap \mathrm{Cl}`p`(\lambda \cap \kappa) = \lambda \cap \kappa \cap \iota`p`(\lambda \cap \kappa)$ (3)

$\vdash . (1) . (3) . \supset \vdash . \mathrm{Prop}$

Observe that $\lambda \cap \kappa \cap \iota`p`(\lambda \cap \kappa)$ is either $\iota`p`(\lambda \cap \kappa)$ or Λ, according as $p`(\lambda \cap \kappa)$ is or is not a member of $\lambda \cap \kappa$.

***210·201.** $\vdash : \mathrm{Hp} *210{\cdot}2 . \lambda \subset \kappa . \supset . \overrightarrow{\min}_Q`\lambda = \lambda \cap \iota`p`\lambda$
[*210·2 . *22·621]

***210·202.** $\vdash : \mathrm{Hp} *210{\cdot}2 . \supset . \overrightarrow{\max}_Q`\lambda = \lambda \cap \kappa \cap \iota`s`(\lambda \cap \kappa)$
[Proof as in *210·2]

***210·203.** $\vdash : \mathrm{Hp} *210{\cdot}2 . \lambda \subset \kappa . \supset . \overrightarrow{\max}_Q`\lambda = \lambda \cap \iota`s`\lambda$
[*210·202 . *22·621]

***210·21.** $\vdash : \mathrm{Hp} *210{\cdot}2 . \lambda \subset \kappa . p`\lambda \in \lambda . \supset . \min_Q`\lambda = p`\lambda$
[*210·201 . *51·31]

***210·211.** $\vdash : \mathrm{Hp} *210{\cdot}2 . \lambda \subset \kappa . s`\lambda \in \lambda . \supset . \max_Q`\lambda = s`\lambda$
[*210·203 . *51·31]

***210·22.** $\vdash : \mathrm{Hp} *210{\cdot}12 . \lambda \subset \kappa . p`\lambda \sim\in \lambda . \supset . \sim \exists ! \overrightarrow{\min}_Q`\lambda$
[*210·201·123 . *51·211]

***210·221.** $\vdash : \mathrm{Hp} *210{\cdot}12 . \lambda \subset \kappa . s`\lambda \sim\in \lambda . \supset . \sim \exists ! \overrightarrow{\max}_Q`\lambda$
[*210·203·123 . *51·211]

***210·222.** $\vdash :. \mathrm{Hp} *210{\cdot}2 . \lambda \subset \kappa . \supset : p`\lambda \in \lambda . \equiv . E ! \min_Q`\lambda$
[*210·21·22]

***210·223.** $\vdash :. \mathrm{Hp} * 210{\cdot}2 \,.\, \lambda \subset \kappa \,.\, \supset : s{}^{\backprime}\lambda \in \lambda \,.\, \equiv \,.\, \mathrm{E} \,!\, \mathrm{max}_Q{}^{\backprime}\lambda$ [*210·211·221]

***210·23.** $\vdash : \mathrm{Hp} * 210{\cdot}2 \,.\, \lambda \subset \kappa \,.\, p{}^{\backprime}\lambda \in \kappa - \lambda \,.\, \supset .\, p{}^{\backprime}\lambda = \mathrm{prec}_Q{}^{\backprime}\lambda = \mathrm{tl}_Q{}^{\backprime}\lambda$

Dem.

$\vdash . *210{\cdot}22 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset .\, \overrightarrow{\mathrm{min}}_Q{}^{\backprime}\lambda = \Lambda \,.$ (1)

$[*205{\cdot}122 . *210{\cdot}122] \quad \supset .\, \lambda \subset \breve{Q}{}^{\backprime\backprime}\lambda$ (2)

$\vdash . (2) . *210{\cdot}12 . *206{\cdot}174 . \supset$

$\vdash : \mathrm{Hp} \,.\, \supset .\, \overrightarrow{\mathrm{prec}}_Q{}^{\backprime}\lambda = C{}^{\backprime}Q \cap \hat{\alpha}(\overleftarrow{Q}{}^{\backprime}\alpha = \breve{Q}{}^{\backprime\backprime}\lambda)$

$[*210{\cdot}122] \qquad = \kappa \cap \hat{\alpha}(\overleftarrow{Q}{}^{\backprime}\alpha = \breve{Q}{}^{\backprime\backprime}\lambda)$ (3)

$\vdash . *37{\cdot}105 . *210{\cdot}124 . \supset$

$\vdash :. \mathrm{Hp} \,.\, \supset : \beta \in \breve{Q}{}^{\backprime\backprime}\lambda \,.\, \equiv .\, (\exists\gamma) \,.\, \gamma \in \lambda \,.\, \exists \,!\, \beta - \gamma \,.\, \beta \in \kappa \,.$

$[*40{\cdot}15.\mathrm{Transp}] \qquad \equiv .\, \exists \,!\, \beta - p{}^{\backprime}\lambda \,.\, \beta \in \kappa \,.$

$[*210{\cdot}124] \qquad \equiv .\, (p{}^{\backprime}\lambda)\, Q \beta$ (4)

$\vdash . (4) . \supset \vdash : \mathrm{Hp} \,.\, \supset .\, p{}^{\backprime}\lambda \in \kappa \,.\, \overleftarrow{Q}{}^{\backprime}p{}^{\backprime}\lambda = \breve{Q}{}^{\backprime\backprime}\lambda \,.$

$[(3)] \qquad \supset .\, p{}^{\backprime}\lambda \in \overrightarrow{\mathrm{prec}}_Q{}^{\backprime}\lambda$ (5)

$\vdash . (5) . *210{\cdot}12 . *206{\cdot}16 . \supset \vdash : \mathrm{Hp} \,.\, \supset .\, p{}^{\backprime}\lambda = \mathrm{prec}_Q{}^{\backprime}\lambda$ (6)

$\vdash . (1) . (6) . *207{\cdot}12 . \qquad \supset \vdash : \mathrm{Hp} \,.\, \supset .\, p{}^{\backprime}\lambda = \mathrm{tl}_Q{}^{\backprime}\lambda$ (7)

$\vdash . (6) . (7) . \supset \vdash . \mathrm{Prop}$

***210·231.** $\vdash : \mathrm{Hp} * 210{\cdot}2 \,.\, \lambda \subset \kappa \,.\, s{}^{\backprime}\lambda \in \kappa - \lambda \,.\, \supset .\, s{}^{\backprime}\lambda = \mathrm{seq}_Q{}^{\backprime}\lambda = \mathrm{lt}_Q{}^{\backprime}\lambda$
[Proof as in *210·23]

In virtue of *210·21·23, every class which is contained in κ, and whose product is a member of κ, has either a minimum or a lower limit; and in virtue of *210·211·231, every class which is contained in κ, and whose sum is a member of κ, has either a maximum or an upper limit.

***210·232.** $\vdash : \mathrm{Hp} * 210{\cdot}2 \,.\, \lambda \subset \kappa \,.\, p{}^{\backprime}\lambda \in \kappa \,.\, \supset .\, p{}^{\backprime}\lambda = \mathrm{limin}_Q{}^{\backprime}\lambda$ [*210·21·23]

***210·233.** $\vdash : \mathrm{Hp} * 210{\cdot}2 \,.\, \lambda \subset \kappa \,.\, s{}^{\backprime}\lambda \in \kappa \,.\, \supset .\, s{}^{\backprime}\lambda = \mathrm{limax}_Q{}^{\backprime}\lambda$ [*210·211·231]

***210·24.** $\vdash : \mathrm{Hp} * 210{\cdot}2 \,.\, \supset .\, \kappa \cap \iota{}^{\backprime}p{}^{\backprime}\kappa = \overrightarrow{B}{}^{\backprime}Q \,.\, \kappa \cap \iota{}^{\backprime}s{}^{\backprime}\kappa = \overrightarrow{B}{}^{\backprime}\breve{Q}$
[*205·12·121 . *210·201·203·122]

***210·241.** $\vdash : \mathrm{Hp} * 210{\cdot}2 \,.\, p{}^{\backprime}\kappa \in \kappa \,.\, \supset .\, p{}^{\backprime}\kappa = B{}^{\backprime}Q$ [*210·24]

***210·242.** $\vdash : \mathrm{Hp} * 210{\cdot}2 \,.\, s{}^{\backprime}\kappa \in \kappa \,.\, \supset .\, s{}^{\backprime}\kappa = B{}^{\backprime}\breve{Q}$ [*210·24]

***210·25.** $\vdash :. \mathrm{Hp} * 210{\cdot}2 : \lambda \subset \kappa \,.\, \supset_\lambda .\, p{}^{\backprime}\lambda \in \kappa : \supset :$
$\lambda \subset \kappa \,.\, \supset .\, p{}^{\backprime}\lambda \in (\overrightarrow{\mathrm{min}}_Q{}^{\backprime}\lambda \cup \overrightarrow{\mathrm{prec}}_Q{}^{\backprime}\lambda)$

Dem.

$\vdash . *210{\cdot}21 . \supset \vdash :. \mathrm{Hp} \,.\, \supset : \lambda \subset \kappa \,.\, p{}^{\backprime}\lambda \in \lambda \,.\, \supset .\, p{}^{\backprime}\lambda \in \overrightarrow{\mathrm{min}}_Q{}^{\backprime}\lambda$ (1)

$\vdash . *210{\cdot}23 . \supset \vdash :. \mathrm{Hp} \,.\, \supset : \lambda \subset \kappa \,.\, p{}^{\backprime}\lambda \sim\in \lambda \,.\, \supset .\, p{}^{\backprime}\lambda \in \overrightarrow{\mathrm{prec}}_Q{}^{\backprime}\lambda$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***210·251.** $\vdash :. \text{Hp} *210\cdot2 : \lambda \subset \kappa \,.\, \supset_\lambda .\, s`\lambda \in \kappa : \supset :$

$$\lambda \subset \kappa \,.\, \supset .\, s`\lambda \in (\overrightarrow{\max}_Q`\lambda \cup \overrightarrow{\text{seq}}_Q`\lambda)$$

[Proof as in *210·25]

***210·252.** $\vdash :. \text{Hp} *210\cdot2 : \lambda \subset \kappa \,.\, \supset_\lambda .\, p`\lambda \cap s`\kappa \in \kappa : \supset :$

$$\lambda \subset \kappa \,.\, \supset .\, p`\lambda \cap s`\kappa \in (\overrightarrow{\min}_Q`\lambda \cup \overrightarrow{\text{prec}}_Q`\lambda) \,.\, p`\lambda \cap s`\kappa = \text{limin}_Q`\lambda$$

Dem.

$\vdash .\, *40\cdot23\cdot161 \,.\, \supset \vdash : \lambda \subset \kappa \,.\, \exists ! \lambda \,.\, \supset .\, p`\lambda \subset s`\kappa \,.$

$[*22\cdot621] \qquad \supset .\, p`\lambda \cap s`\kappa = p`\lambda \qquad (1)$

$\vdash .\, (1) \,.\, *210\cdot21\cdot23 \,.\, \supset \vdash : \text{Hp} \,.\, \lambda \subset \kappa \,.\, \exists ! \lambda \,.\, \supset .\, p`\lambda \cap s`\kappa \in (\overrightarrow{\min}_Q`\lambda \cup \overrightarrow{\text{prec}}_Q`\lambda) \quad (2)$

$\vdash .\, *40\cdot2 \,. \qquad \supset \vdash : \sim \exists ! \lambda \,.\, \supset .\, p`\lambda \cap s`\kappa = s`\kappa \qquad (3)$

$\vdash .\, (3) \,.\, *24\cdot12 \,.\, \supset \vdash : \text{Hp} \,.\, \supset .\, s`\kappa \in \kappa \,.$

$[*210\cdot242] \qquad \supset .\, s`\kappa = B`\breve{Q} \,.$

$[*206\cdot14] \qquad \supset .\, s`\kappa = \text{prec}_Q`\Lambda \qquad (4)$

$\vdash .\, (3) \,.\, (4) \,.\, \supset \vdash : \text{Hp} \,.\, \sim \exists ! \lambda \,.\, \supset .\, p`\lambda \cap s`\kappa \in (\overrightarrow{\min}_Q`\lambda \cup \overrightarrow{\text{prec}}_Q`\lambda) \qquad (5)$

$\vdash .\, (2) \,.\, (5) \,.\, \supset \vdash .\, \text{Prop}$

This proposition is more useful than *210·25, because its hypothesis is much oftener verified. In order that the hypothesis of *210·25 may be verified, we must have $\text{V} \in \kappa$, since $\Lambda \subset \kappa \,.\, p`\Lambda = \text{V}$; hence we must also have $s`\kappa = \text{V}$. But the hypothesis of *210·252 only requires, as far as Λ is concerned, that we should have $s`\kappa \in \kappa$.

***210·253.** $\vdash : \text{Hp} *210\cdot252 \,.\, \supset .\, (\lambda) \,.\, \lambda \in \text{Q}`\text{min}_Q \cup \text{Q}`\text{prec}_Q$

[*210·252 . *205·15 . *206·131]

***210·254.** $\vdash : \text{Hp} *210\cdot251 \,.\, \supset .\, (\lambda) \,.\, \lambda \in \text{Q}`\text{max}_Q \cup \text{Q}`\text{seq}_Q$

[Proof as in *210·253]

***210·26.** $\vdash : \text{Hp} *210\cdot2 \,.\, \lambda \subset \kappa \,.\, p`\lambda \sim\in \lambda \,.\, s`(\kappa \cap \text{Cl}`p`\lambda) \in \kappa \,.\, \supset .$

$$s`(\kappa \cap \text{Cl}`p`\lambda) = \text{prec}_Q`\lambda$$

Dem.

$\vdash .\, *210\cdot22 \,.\, \supset \vdash : \text{Hp} \,.\, \supset .\, \sim \exists ! \overrightarrow{\min}_Q`\lambda \,.$

$[*205\cdot122] \qquad \supset .\, \lambda \subset \breve{Q}``\lambda \qquad (1)$

$\vdash .\, *60\cdot2 \,. \quad \supset \vdash : \beta \in \kappa \cap \text{Cl}`p`\lambda \,.\, \supset .\, \beta \subset p`\lambda :$

$[*40\cdot151] \quad \supset \vdash : s`(\kappa \cap \text{Cl}`p`\lambda) \subset p`\lambda \qquad (2)$

$\vdash .\, (2) \,. \quad \supset \vdash : \text{Hp} \,.\, \supset .\, s`(\kappa \cap \text{Cl}`p`\lambda) \in \kappa \cap \text{Cl}`p`\lambda \,. \qquad (3)$

$[*210\cdot211] \qquad \supset .\, s`(\kappa \cap \text{Cl}`p`\lambda) = \max_Q`(\kappa \cap \text{Cl}`p`\lambda)$

$[*210\cdot17] \qquad = \max_Q`(\kappa - \breve{Q}``\lambda)$

$[(1)] \qquad = \max_Q`(\kappa - \lambda - \breve{Q}``\lambda)$

$[*210\cdot122 . *202\cdot502 . (3)] \qquad = \max_Q`p`\overrightarrow{Q}``\lambda$

$[*206\cdot1\cdot101] \qquad = \text{prec}_Q`\lambda : \supset \vdash .\, \text{Prop}$

***210·261.** $\vdash : \text{Hp} * 210 \cdot 2 \,.\, \lambda \subset \kappa \,.\, s`\lambda \sim\epsilon \lambda \,.\, p`\hat{\alpha}(\alpha \epsilon \kappa \,.\, s`\lambda \subset \alpha) \epsilon \kappa \,.\, \supset .$
$p`\hat{\alpha}(\alpha \epsilon \kappa \,.\, s`\lambda \subset \alpha) = \text{seq}_Q`\lambda$ [Proof as in *210·26]

***210·262.** $\vdash : \text{Hp} * 210 \cdot 2 \,.\, \lambda \subset \kappa \,.\, s`\lambda \sim\epsilon \lambda \,.\, s`\kappa \cap p`\hat{\alpha}(\alpha \epsilon \kappa \,.\, s`\lambda \subset \alpha) \epsilon \kappa \,.\, \supset .$
$s`\kappa \cap p`\hat{\alpha}(\alpha \epsilon \kappa \,.\, s`\lambda \subset \alpha) = \text{seq}_Q`\lambda$

Dem.

$\vdash . * 40 \cdot 23 \cdot 161 \,.\, \supset$
$\vdash : \text{Hp} \,.\, \exists ! \hat{\alpha}(\alpha \epsilon \kappa \,.\, s`\lambda \subset \alpha) \,.\, \supset .\, p`\hat{\alpha}(\alpha \epsilon \kappa \,.\, s`\lambda \subset \alpha) \subset s`\kappa \,.$
[*22·621] $\supset .\, s`\kappa \cap p`\hat{\alpha}(\alpha \epsilon \kappa \,.\, s`\lambda \subset \alpha) = p`\hat{\alpha}(\alpha \epsilon \kappa \,.\, s`\lambda \subset \alpha) \,.$
[*210·261] $\supset .\, s`\kappa \cap p`\hat{\alpha}(\alpha \epsilon \kappa \,.\, s`\lambda \subset \alpha) = \text{seq}_Q`\lambda$ (1)
$\vdash . * 10 \cdot 51 \,.\, \supset \vdash :.\, \hat{\alpha}(\alpha \epsilon \kappa \,.\, s`\lambda \subset \alpha) = \Lambda \,.\, \supset : \alpha \epsilon \kappa \,.\, \supset_\alpha .\, \sim(s`\lambda \subset \alpha)$ (2)
$\vdash . (2) . * 210 \cdot 16 \,.\, \supset \vdash :.\, \text{Hp} \,.\, \hat{\alpha}(\alpha \epsilon \kappa \,.\, s`\lambda \subset \alpha) = \Lambda \,.\, \supset : \alpha \epsilon \kappa \,.\, \supset_\alpha .\, \alpha \subset s`\lambda :$
[*40·151] $\supset : s`\kappa \subset s`\lambda :$
[*40·161] $\supset : s`\kappa = s`\lambda$ (3)
$\vdash . * 40 \cdot 2 \,.\, \supset \vdash : \text{Hp}(3) \,.\, \supset .\, s`\kappa \cap p`\hat{\alpha}(\alpha \epsilon \kappa \,.\, s`\lambda \subset \alpha) = s`\kappa \,.$ (4)
[Hp.(3)] $\supset .\, s`\lambda \epsilon \kappa \,.$
[*210·231] $\supset .\, s`\lambda = \text{seq}_Q`\lambda \,.$
[(3).(4)] $\supset .\, s`\kappa \cap p`\hat{\alpha}(\alpha \epsilon \kappa \,.\, s`\lambda \subset \alpha) = \text{seq}_Q`\lambda$ (5)
$\vdash . (1) . (5) \,.\, \supset \vdash .$ Prop

The same remark applies to this proposition as to *210·252.

***210·27.** $\vdash :.\, \text{Hp} * 210 \cdot 2 : \lambda \subset \kappa \,.\, \supset_\lambda .\, s`\lambda \epsilon \kappa : \supset :$
$\lambda \subset \kappa \,.\, \supset_\lambda .\, \exists ! (\overrightarrow{\text{max}_Q}`\lambda \cup \overrightarrow{\text{seq}_Q}`\lambda) \,.\, \exists ! (\overrightarrow{\text{min}_Q}`\lambda \cup \overrightarrow{\text{prec}_Q}`\lambda)$

Dem.

$\vdash . * 210 \cdot 251 \,.\, \supset \vdash :.\, \text{Hp} \,.\, \supset : \lambda \subset \kappa \,.\, \supset_\lambda .\, \exists ! (\overrightarrow{\text{max}_Q}`\lambda \cup \overrightarrow{\text{seq}_Q}`\lambda)$ (1)
$\vdash . * 210 \cdot 222 \,.\, \supset \vdash : \text{Hp} \,.\, \lambda \subset \kappa \,.\, p`\lambda \epsilon \lambda \,.\, \supset .\, \exists ! \overrightarrow{\text{min}_Q}`\lambda$ (2)
$\vdash . * 10 \cdot 1 \,.\, \supset \vdash :.\, \text{Hp} \,.\, \supset : s`(\kappa \cap \text{Cl}`p`\lambda) \epsilon \kappa :$
[*210·26] $\supset : \lambda \subset \kappa \,.\, p`\lambda \sim\epsilon \lambda \,.\, \supset .\, \exists ! \overrightarrow{\text{prec}_P}`\lambda$ (3)
$\vdash . (2) . (3) \,.\, \supset \vdash :.\, \text{Hp} \,.\, \supset : \lambda \subset \kappa \,.\, \supset .\, \exists ! (\overrightarrow{\text{min}_Q}`\lambda \cup \overrightarrow{\text{prec}_Q}`\lambda)$ (4)
$\vdash . (1) . (4) \,.\, \supset \vdash .$ Prop

***210·271.** $\vdash :.\, \text{Hp} * 210 \cdot 2 : \lambda \subset \kappa \,.\, \supset_\lambda .\, p`\lambda \epsilon \kappa : \supset :$
$\lambda \subset \kappa \,.\, \supset_\lambda .\, \exists ! (\overrightarrow{\text{max}_Q}`\lambda \cup \overrightarrow{\text{seq}_Q}`\lambda) \,.\, \exists ! (\overrightarrow{\text{min}_Q}`\lambda \cup \overrightarrow{\text{prec}_Q}`\lambda)$
[Proof as in *210·27]

***210·272.** $\vdash :.\, \text{Hp} * 210 \cdot 2 : \lambda \subset \kappa \,.\, \supset_\lambda .\, p`\lambda \cap s`\kappa \epsilon \kappa : \supset :$
$\lambda \subset \kappa \,.\, \supset_\lambda .\, \exists ! (\overrightarrow{\text{max}_Q}`\lambda \cup \overrightarrow{\text{seq}_Q}`\lambda) \,.\, \exists ! (\overrightarrow{\text{min}_Q}`\lambda \cup \overrightarrow{\text{prec}_Q}`\lambda)$
[Proof as in *210·27, using *210·262]

***210·28.** $\vdash : \text{Hp} * 210 \cdot 2 \,.\, s``\text{Cl}`\kappa \subset \kappa \,.\, \supset .$
$(\lambda) \,.\, \lambda \epsilon (\text{Q}`\text{max}_Q \cup \text{Q}`\text{seq}_Q) \cap (\text{Q}`\text{min}_Q \cup \text{Q}`\text{prec}_Q)$

Dem.

$\vdash . * 37 \cdot 61 \,.\, \supset \vdash :.\, \text{Hp} \,.\, \supset : \lambda \subset \kappa \,.\, \supset_\lambda .\, s`\lambda \epsilon \kappa :$
[*210·27] $\supset : \lambda \subset \kappa \,.\, \supset_\lambda .\, \exists ! (\overrightarrow{\text{max}_Q}`\lambda \cup \overrightarrow{\text{seq}_Q}`\lambda) \,.\, \exists ! (\overrightarrow{\text{min}_Q}`\lambda \cup \overrightarrow{\text{prec}_Q}`\lambda)$ (1)

$\vdash . (1) . *22 \cdot 43 . \supset$

$\vdash : \text{Hp} . \supset . (\lambda) . \exists ! \{\overrightarrow{\text{max}}_Q`(\lambda \cap \kappa) \cup \overrightarrow{\text{seq}}_Q`\lambda \cap \kappa)\} .$

$\exists ! \{\overrightarrow{\text{min}}_Q`(\lambda \cap \kappa) \cup \overrightarrow{\text{prec}}_Q`(\lambda \cap \kappa)\} .$

[$*210 \cdot 122$] $\supset . (\lambda) . \exists ! \{\overrightarrow{\text{max}}_Q`(\lambda \cap C`Q) \cup \overrightarrow{\text{seq}}_Q`(\lambda \cap C`Q)\} .$

$\exists ! \{\overrightarrow{\text{min}}_Q`(\lambda \cap C`Q) \cup \overrightarrow{\text{prec}}_Q`(\lambda \cap C`Q)\} .$

[$*205 \cdot 15 \cdot 151 . *206 \cdot 131$] $\supset . (\lambda) . \exists ! \{\overrightarrow{\text{max}}_Q`\lambda \cup \overrightarrow{\text{seq}}_Q`\lambda\} . \exists ! \{\overrightarrow{\text{min}}_Q`\lambda \cup \overrightarrow{\text{prec}}_Q`\lambda\} .$

[$*33 \cdot 41$] $\supset . (\lambda) . \lambda \in \text{Cl}`\text{max}_Q \cup \text{Cl}`\text{seq}_Q . \lambda \in \text{Cl}`\text{min}_Q \cup \text{Cl}`\text{prec}_Q : \supset \vdash .$ Prop

$*210 \cdot 281$. $\vdash : \text{Hp} *210 \cdot 2 . p``\text{Cl}`\kappa \subset \kappa . \supset .$

$(\lambda) . \lambda \in (\text{Cl}`\text{max}_Q \cup \text{Cl}`\text{seq}_Q) \cap (\text{Cl}`\text{min}_Q \cup \text{Cl}`\text{prec}_Q)$

$*210 \cdot 282$. $\vdash :. \text{Hp} *210 \cdot 2 : \lambda \subset \kappa . \supset_\lambda . p`\lambda \cap s`\kappa \in \kappa : \supset .$

$(\lambda) . \lambda \in (\text{Cl}`\text{max}_Q \cup \text{Cl}`\text{seq}_Q) \cap (\text{Cl}`\text{min}_Q \cup \text{Cl}`\text{prec}_Q)$

Thus when either of the hypotheses of $*210 \cdot 281 \cdot 282$ is fulfilled, the series Q is Dedekindian both upwards and downwards.

$*210 \cdot 29$. $\vdash : \text{Hp} *210 \cdot 251 . \supset . (\lambda) . \lambda \in \text{Cl}`\text{limax}_P \cap \text{Cl}`\text{limin}_P$ [$*210 \cdot 28 . *207 \cdot 44$]

$*210 \cdot 291$. $\vdash : \text{Hp} *210 \cdot 252 . \supset . (\lambda) . \lambda \in \text{Cl}`\text{limax}_P \cap \text{Cl}`\text{limin}_P$
[$*210 \cdot 282 . *207 \cdot 44$]

$*$211. ON SECTIONS AND SEGMENTS

Summary of $*$211.

The theory of the modes of separation of a series into two classes, one of which wholly precedes the other, and which together make up the whole series, is of fundamental importance. When one out of a pair of such classes is given, the other is the rest of the series; we may therefore, for most purposes, confine our attention to that one of the two classes which comes first in the serial order. Any class which can be the first of such a pair we shall call a *section* of our series. If P is the series, we shall denote the class of its sections by "sect'P." If α is a section of P, we shall call $C'P - \alpha$ (which is the second class of our pair) the *complement* of α. The class of complements of sections is

$$(C'P -)``\text{sect}'P,$$

which is identical with sect'$\breve{P}$ ($*$211·75).

In order that a class may be a section of P, it is necessary and sufficient that it should be contained in $C'P$ and should contain all its own predecessors; thus we put

$$\text{sect}'P = \hat{\alpha}(\alpha \subset C'P \,.\, P``\alpha \subset \alpha) \quad \text{Df.}$$

We have also, by $*$90·23,

$$\text{sect}'P = \hat{\alpha}\,(\alpha = P_*``\alpha) \quad (*211\cdot13).$$

Among sections, a specially important class consists of classes which are composed of all the predecessors of some class, *i.e.* classes of the form $P``\beta$, *i.e.* classes which are members of $\text{D}'P_\epsilon$. Whenever $P$ is transitive, $P``P``\beta \subset P``\beta$; hence $P``\beta$ is a section according to the above definition. When $P$ is a series, the complement of $P``\beta$ (when β exists and is contained in $C'P$) is

$$\overrightarrow{\text{max}}_P{}'\beta \cup p'\overleftarrow{P}``\beta.$$

The members of $\text{D}'P_\epsilon$ are called *segments* of the series generated by P. In a series in which every sub-class has a maximum or a sequent, $\text{D}'P_\epsilon = \overrightarrow{P}``C'P$ ($*$211·38), *i.e.* the predecessors of a class are always the predecessors of a single term, namely the maximum of the class if it exists, or the sequent if no maximum exists. But if there are classes which have neither a maximum nor a sequent, the predecessors of such classes are not coextensive with the predecessors of any single term. Thus in general the series of segments will be larger than the original series. For example, if our original series is of the type of the series of rationals in order of magnitude, the series of segments is of the type of the series of real numbers, *i.e.* the type of the continuum.

Among segments, a specially important class consists of those which have no maximum. In this case, if α is such a segment, we have $\alpha \subset P``\alpha$; and since (provided $P$ is transitive) we also have, for all segments, $P``\alpha \subset \alpha$, the segments having no maximum are those for which $\alpha = P``\alpha$, *i.e.* they are the class $D`(P_\epsilon \dot{\wedge} I)$. In compact series, all segments belong to this latter class, but in general only those segments belong to it which correspond to a "Häufungsstelle." In all cases in which the existence of a limit is not known, the segment fulfils the functions of a limit; that is to say, in those places in the series where a limit might be expected, we have a segment having no limit or maximum, which takes the same place in the series of segments as would be taken by the limit in the original series if the limit existed. Segments having no limit or maximum are limiting points in the series of segments, and every class of segments which has no maximum in the series of segments has a limit in that series.

We have thus three classes to deal with, namely

(1) $\text{sect}`P$,

(2) $D`P_\epsilon$,

(3) $D`(P_\epsilon \dot{\wedge} I)$.

Of these the second is contained in the first when P is transitive (*211·15), and the third is contained in the first and second (*211·14). The second consists of those members of the first which have either a sequent or no maximum (*211·32); the third consists of those members of the first which have no maximum (*211·41). If every member of the third class has a limit, *i.e.* if

$$D`(P_\epsilon \dot{\wedge} I) \subset \text{G}`\text{seq}_P,$$

then every class has either a sequent or a maximum, *i.e.* the series is Dedekindian; and the converse also holds (*211·47).

When P is connected, of any two sections one must be contained in the other (*211·6). Moreover, if λ is contained in any one of the three classes $\text{sect}`P$, $D`P_\epsilon$, $D`(P_\epsilon \dot{\wedge} I)$, then $s`\lambda$ is a member of that class (*211·63·64·65). Hence the propositions of *210 become available. It is thus that the existence of limits in series of segments or sections is proved: the maximum or upper limit of any class λ consisting of segments or sections is $s`\lambda$, and the minimum or lower limit is the sum of the segments that are contained in every λ.

We begin, in this number, with elementary properties of $\text{sect}`P$. The sections of P are the segments of P_* (*211·13) and the sections of P_{po} (*211·17). We have

***211·26.** $\vdash . C`P \in \text{sect}`P . s`\text{sect}`P = C`P$

We then proceed to the elementary properties of segments, *i.e.* of $D`P_\epsilon$ (*211·3—·38). We have

$*211\cdot3.\quad \vdash . \overrightarrow{P}``C`P \subset \mathrm{D}`P_\epsilon$

$*211\cdot301.\quad \vdash . \mathrm{D}`P \epsilon \mathrm{D}`P_\epsilon$

$*211\cdot302.\quad \vdash : P \epsilon \mathrm{Ser} . \supset . \overrightarrow{P}``C`P = \mathrm{sect}`P \cap \mathrm{Cl}`\mathrm{seq}_P$

$*211\cdot351.\quad \vdash : P \epsilon \mathrm{Ser} . \supset . \mathrm{sect}`P - \mathrm{D}`P_\epsilon = \overrightarrow{P_*}``(C`P - \mathrm{D}`P_1)$

We then proceed to elementary properties of segments having no maximum, *i.e.* of $\mathrm{D}`(P_\epsilon \dot{\cap} I)$ ($*211\cdot4$—$\cdot47$). We have

$*211\cdot42.\quad \vdash : P \epsilon \mathrm{trans} . \supset . \mathrm{D}`(P_\epsilon \dot{\cap} I) = \mathrm{D}`P_\epsilon - \mathrm{Cl}`\mathrm{max}_P$

$*211\cdot44.\quad \vdash . \Lambda \epsilon \mathrm{D}`(P_\epsilon \dot{\cap} I) . \Lambda \epsilon \mathrm{D}`P_\epsilon . \Lambda \epsilon \mathrm{sect}`P$

$*211\cdot451.\quad \vdash : \overrightarrow{P}`x \epsilon \mathrm{D}`(P_\epsilon \dot{\cap} I) . \supset . x \sim\epsilon \mathrm{Cl}`(P \dot{-} P^2)$

Our next set of propositions ($*211\cdot5$—$\cdot553$) is concerned with compact series, *i.e.* with the hypothesis $P^2 = P$. We have

$*211\cdot51.\quad \vdash : P^2 = P . \supset . \mathrm{D}`P_\epsilon = \mathrm{D}`(P_\epsilon \dot{\cap} I)$

$*211\cdot551.\quad \vdash :. P \epsilon \mathrm{Ser} . \supset : \mathrm{Cl}`\mathrm{max}_P \cap \mathrm{Cl}`\mathrm{seq}_P = \Lambda . \equiv . P = P^2$

I.e. a series is compact when, and only when, no class has both a maximum and a sequent.

We come next to the application of the propositions of $*210$ ($*211\cdot56$—$\cdot692$). These propositions proceed from

$*211\cdot56.\quad \vdash :. P \epsilon \mathrm{connex} . \alpha, \beta \epsilon \mathrm{sect}`P . \supset : \alpha \subset \beta . \vee . \beta \subset P``\alpha$

(Here "$P_{\mathrm{po}} \epsilon \mathrm{connex}$" may be substituted in the hypothesis: cf. $*211\cdot561$.) The propositions of this set, which are very important, have been already mentioned.

Our next set of propositions ($*211\cdot7$—$\cdot762$) are concerned with the complements of sections and segments. Some of these propositions have been already mentioned; others of importance are:

$*211\cdot7.\quad \vdash : \alpha \epsilon \mathrm{sect}`P . \supset . C`P - \alpha \epsilon \mathrm{sect}`\breve{P}$

$*211\cdot703.\quad \vdash : P \epsilon \mathrm{connex} . \alpha \epsilon \mathrm{sect}`P - \iota`C`P . \supset . \exists ! p`\overleftarrow{P}``\alpha$

$*211\cdot726.\quad \vdash : P \epsilon \mathrm{connex} \cap \mathrm{Rl}`J . \alpha \epsilon \mathrm{sect}`P . \supset .$
$$\overrightarrow{\mathrm{max}_P}`\alpha = \overrightarrow{\mathrm{prec}_P}`(C`P - \alpha) . \overrightarrow{\mathrm{seq}_P}`\alpha = \overrightarrow{\mathrm{min}_P}`(C`P - \alpha)$$

$*211\cdot727.\quad \vdash :. P \epsilon \mathrm{connex} \cap \mathrm{Rl}`J . \alpha \epsilon \mathrm{sect}`P . \supset :$
$$\mathrm{E} ! \mathrm{limax}_P`\alpha . \equiv . \mathrm{E} ! \mathrm{limin}_P`(C`P - \alpha)$$

$*211\cdot728.\quad \vdash :. P \epsilon \mathrm{connex} \cap \mathrm{Rl}`J . \alpha \epsilon \mathrm{sect}`P : \sim\mathrm{E} ! \mathrm{max}_P`\alpha . \vee .$
$$\sim\mathrm{E} ! \mathrm{min}_P`(C`P - \alpha) : \supset . \overrightarrow{\mathrm{limax}_P}`\alpha = \overrightarrow{\mathrm{limin}_P}`(C`P - \alpha)$$

The remaining propositions are mainly occupied with relation-arithmetic. The most important of them is

$*211{\cdot}82.\ \ \vdash :: P \in \mathrm{Ser} \,.\, Q \in \mathrm{D}^{\backprime}P\!\upharpoonright \,.\supset :.$

$$C^{\backprime}Q \in \mathrm{sect}^{\backprime}P \,.\equiv: (\exists R) \,.\, P = Q \updownarrow R \,.\vee.\, (\exists x) \,.\, P = Q \mapsto x :$$
$$\equiv : (\exists R) \,.\, P = Q \updownarrow R \,.\vee.\, P = Q \mapsto B^{\backprime}\breve{P}$$

That is, given any series contained in P, if something can be added to make it into P, its field is a section of P, and vice versa.

$*211{\cdot}01.\ \ \mathrm{sect}^{\backprime}P = \hat{\alpha}(\alpha \subset C^{\backprime}P \,.\, P^{\backprime\backprime}\alpha \subset \alpha)$ Df

$*211{\cdot}1.\ \ \vdash : \alpha \in \mathrm{sect}^{\backprime}P \,.\equiv.\, \alpha \subset C^{\backprime}P \,.\, P^{\backprime\backprime}\alpha \subset \alpha$ [(*211·01)]

$*211{\cdot}11.\ \ \vdash : \alpha \in \mathrm{D}^{\backprime}P_\epsilon \,.\equiv.\, (\exists\beta) \,.\, \alpha = P^{\backprime\backprime}\beta$ [*37·101]

$*211{\cdot}12.\ \ \vdash : \alpha \in \mathrm{D}^{\backprime}(P_\epsilon \dot{\cap} I) \,.\equiv.\, \alpha = P^{\backprime\backprime}\alpha$

Dem.

$\vdash . *37{\cdot}101 . *50{\cdot}1 . \supset \vdash : \alpha \in \mathrm{D}^{\backprime}(P_\epsilon \dot{\cap} I) \,.\equiv.\, (\exists\beta) \,.\, \alpha = P^{\backprime\backprime}\beta \,.\, \alpha = \beta \,.$

[*13·195] $\equiv .\, \alpha = P^{\backprime\backprime}\alpha : \supset \vdash .$ Prop

$*211{\cdot}13.\ \ \vdash : \alpha \in \mathrm{sect}^{\backprime}P \,.\equiv.\, \alpha = P_*{}^{\backprime\backprime}\alpha \,.\equiv.\, \alpha \in \mathrm{D}^{\backprime}\{(P_*)_\epsilon \dot{\cap} I\} \,.\equiv.\, \alpha \in \mathrm{D}^{\backprime}(P_*)_\epsilon$

Dem.

$\vdash . *211{\cdot}1 . *90{\cdot}23 . \supset \vdash : \alpha \in \mathrm{sect}^{\backprime}P \,.\equiv.\, \alpha = P_*{}^{\backprime\backprime}\alpha$ (1)

$\vdash . *90{\cdot}17 . \supset \vdash . P_*{}^{\backprime\backprime}P_*{}^{\backprime\backprime}\beta = P_*{}^{\backprime\backprime}\beta \,.$

[*13·12] $\supset \vdash : \alpha = P_*{}^{\backprime\backprime}\beta \,.\supset.\, P_*{}^{\backprime\backprime}\alpha = \alpha :$

[*211·11] $\supset \vdash : \alpha \in \mathrm{D}^{\backprime}(P_*)_\epsilon \,.\supset.\, \alpha = P_*{}^{\backprime\backprime}\alpha$ (2)

$\vdash . *10{\cdot}24 . *211{\cdot}11 . \supset \vdash : \alpha = P_*{}^{\backprime\backprime}\alpha \,.\supset.\, \alpha \in \mathrm{D}^{\backprime}(P_*)_\epsilon$ (3)

$\vdash . (1) . (2) . (3) . *211{\cdot}12 . \supset \vdash .$ Prop

In virtue of the above proposition, the properties of sect‘P can be deduced from those of D‘P_ϵ or D‘$(P_\epsilon \dot{\cap} I)$ by substituting P_* for P.

$*211{\cdot}131.\ \ \vdash : \alpha \in \mathrm{sect}^{\backprime}P \,.\supset.\, P^{\backprime\backprime}\alpha = P_{\mathrm{po}}{}^{\backprime\backprime}\alpha$

Dem.

$\vdash . *211{\cdot}13 . \supset \vdash : \mathrm{Hp} \,.\supset.\, P^{\backprime\backprime}\alpha = P^{\backprime\backprime}P_*{}^{\backprime\backprime}\alpha$

[*91·52] $= P_{\mathrm{po}}{}^{\backprime\backprime}\alpha : \supset \vdash .$ Prop

$*211{\cdot}132.\ \ \vdash : \alpha \in \mathrm{sect}^{\backprime}P \,.\supset.\, \mathrm{D}^{\backprime}(P \upharpoonright \alpha) = \mathrm{D}^{\backprime}(P_{\mathrm{po}} \upharpoonright \alpha) \,.\, \mathrm{Cl}^{\backprime}(P \upharpoonright \alpha) = \mathrm{Cl}^{\backprime}(P_{\mathrm{po}} \upharpoonright \alpha) \,.$
$C^{\backprime}(P \upharpoonright \alpha) = C^{\backprime}(P_{\mathrm{po}} \upharpoonright \alpha)$

Dem.

$\vdash . *37{\cdot}41 . *211{\cdot}131 . \supset \vdash : \mathrm{Hp} \,.\supset.\, \mathrm{D}^{\backprime}(P_{\mathrm{po}} \upharpoonright \alpha) = \alpha \cap P^{\backprime\backprime}\alpha$

[*37·41] $= \mathrm{D}^{\backprime}(P \upharpoonright \alpha)$ (1)

$\vdash . *91{\cdot}502 . \supset \vdash . \mathrm{Cl}^{\backprime}(P \upharpoonright \alpha) \subset \mathrm{Cl}^{\backprime}(P_{\mathrm{po}} \upharpoonright \alpha)$ (2)

$\vdash . *37{\cdot}41 . \supset \vdash :. y \in \mathrm{Cl}^{\backprime}(P_{\mathrm{po}} \upharpoonright \alpha) \,.\equiv: y \in \alpha \cap \breve{P}_{\mathrm{po}}{}^{\backprime\backprime}\alpha :$

[*91·57] $\equiv : y \in (\alpha \cap \breve{P}^{\backprime\backprime}\alpha) \cup (\alpha \cap \breve{P}^{\backprime\backprime}\breve{P}_{\mathrm{po}}{}^{\backprime\backprime}\alpha)$ (3)

$\vdash . *211{\cdot}1 . \supset \vdash : \mathrm{Hp} \,.\, y \in \alpha \cap \breve{P}^{\backprime\backprime}\breve{P}_{\mathrm{po}}{}^{\backprime\backprime}\alpha \,.\supset.\, (\exists z) \,.\, zPy \,.\, z \in \alpha \,.$

[*37·105] $\supset .\, y \in \breve{P}^{\backprime\backprime}\alpha$ (4)

$\vdash . (3) . (4) . \supset \vdash : \text{Hp} . \supset . \text{Ɑ}`(P_{\text{po}} \upharpoonright \alpha) \subset \alpha \cap \breve{P}``\alpha .$

$[*37 \cdot 41] \qquad \supset . \text{Ɑ}`(P_{\text{po}} \upharpoonright \alpha) \subset \text{Ɑ}`(P \upharpoonright \alpha) \qquad (5)$

$\vdash . (2) . (5) . \supset \vdash : \text{Hp} . \supset . \text{Ɑ}`(P_{\text{po}} \upharpoonright \alpha) = \text{Ɑ}`(P \upharpoonright \alpha) \qquad (6)$

$\vdash . (1) . (6) . \supset \vdash . \text{Prop}$

***211·133.** $\vdash : P_{\text{po}} \in \text{connex} . \alpha \in \text{sect}`P - 1 . \supset . C`(P \upharpoonright \alpha) = \alpha$

Dem.

$\vdash . *202 \cdot 55 . \supset \vdash : \text{Hp} . \supset . C`(P_{\text{po}} \upharpoonright \alpha) = \alpha .$

$[*211 \cdot 132] \qquad \supset . C`(P \upharpoonright \alpha) = \alpha : \supset \vdash . \text{Prop}$

***211·14.** $\vdash . \text{D}`(P_\epsilon \dot\cap I) \subset \text{D}`P_\epsilon . \text{D}`(P_\epsilon \dot\cap I) \subset \text{sect}`P$

Dem.

$\vdash . *33 \cdot 263 . \qquad \supset \vdash . \text{D}`(P_\epsilon \dot\cap I) \subset \text{D}`P_\epsilon \qquad (1)$

$\vdash . *211 \cdot 12 . *22 \cdot 42 . \supset \vdash : \alpha \in \text{D}`(P_\epsilon \dot\cap I) . \supset . P``\alpha \subset \alpha \qquad (2)$

$\vdash . *211 \cdot 12 . *37 \cdot 15 . \supset \vdash : \alpha \in \text{D}`(P_\epsilon \dot\cap I) . \supset . \alpha \subset C`P \qquad (3)$

$\vdash . (2) . (3) . *211 \cdot 1 . \supset \vdash : \alpha \in \text{D}`(P_\epsilon \dot\cap I) . \supset . \alpha \in \text{sect}`P \qquad (4)$

$\vdash . (1) . (4) . \supset \vdash . \text{Prop}$

***211·15.** $\vdash : P \in \text{trans} . \supset . \text{D}`P_\epsilon \subset \text{sect}`P$

Dem.

$\vdash . *211 \cdot 11 . *37 \cdot 15 . \supset \vdash : \alpha \in \text{D}`P_\epsilon . \supset . \alpha \subset C`P \qquad (1)$

$\vdash . *211 \cdot 11 . *201 \cdot 5 . \supset \vdash : P \in \text{trans} . \alpha \in \text{D}`P_\epsilon . \supset . P``\alpha \subset \alpha \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***211·16.** $\vdash . P_{\text{po}}``\alpha \in \text{sect}`P$

Dem.

$\vdash . *91 \cdot 504 . *37 \cdot 15 . \supset \vdash . P_{\text{po}}``\alpha \subset C`P \qquad (1)$

$\vdash . *91 \cdot 51 \cdot 511 . \qquad \supset \vdash . P``P_{\text{po}}``\alpha \subset P_{\text{po}}``\alpha \qquad (2)$

$\vdash . (1) . (2) . *211 \cdot 1 . \supset \vdash . \text{Prop}$

***211·17.** $\vdash . \text{sect}`P = \text{sect}`P_{\text{po}} = \text{sect}`P_*$

$[*211 \cdot 13 . *90 \cdot 4 . *91 \cdot 602]$

The following propositions are useful in dealing with sectional relations, *i.e.* relations of the form $P \upharpoonright \alpha$, where $\alpha \in \text{sect}`P$. Unit sections often need special treatment, owing to the fact that for them we do not have $C`P \upharpoonright \alpha = \alpha$.

***211·18.** $\vdash : P_{\text{po}} \dot\subset J . \supset . \text{sect}`P \cap 1 = \iota``\overrightarrow{B}`P$

Dem.

$\vdash . *211 \cdot 13 . \supset \vdash : \alpha \in \text{sect}`P \cap 1 . \equiv . \alpha = P_*``\alpha . \alpha \in 1 .$

$[*52 \cdot 1 . *53 \cdot 301] \qquad \equiv . (\exists x) . \alpha = \iota`x . \overrightarrow{P_*}`x = \iota`x .$

$[*91 \cdot 54 . *90 \cdot 12] \qquad \equiv . (\exists x) . \alpha = \iota`x . \overrightarrow{P_{\text{po}}}`x \subset \iota`x . x \in C`P \qquad (1)$

$\vdash . (1) . \supset \vdash :. \text{Hp} . \supset : \alpha \in \text{sect}`P \cap 1 . \equiv . (\exists x) . \alpha = \iota`x . \overrightarrow{P_{\text{po}}}`x = \Lambda . x \in C`P .$

$[*91 \cdot 504] \qquad \equiv . (\exists x) . \alpha = \iota`x . x \sim\in \text{Ɑ}`P . x \in C`P .$

$[*93 \cdot 103] \qquad \equiv . \alpha \in \iota``\overrightarrow{B}`P :. \supset \vdash . \text{Prop}$

***211·181.** $\vdash : P_{po} \in \text{Ser} \,.\, \exists ! \overrightarrow{B}`P \,.\, \supset .\, \text{sect}`P \cap 1 = \iota`\iota`B`P$

Dem.

$\vdash .\, *202\cdot13\cdot523 \,.\, \supset \vdash : \text{Hp} \,.\, \supset .\, \overrightarrow{B}`P \in 1$ (1)

$\vdash .\, (1) \,.\, *211\cdot18 \,.\, *53\cdot3 \,.\, \supset \vdash .\, \text{Prop}$

***211·182.** $\vdash : P_{po} \in \text{Ser} \,.\, \overrightarrow{B}`P = \Lambda \,.\, \supset .\, \text{sect}`P \cap 1 = \Lambda$ [*211·18]

***211·2.** $\vdash : \alpha \in \text{sect}`P \,.\, \supset .\, \alpha = \alpha \cap C`P = \alpha \cup P``\alpha = (\alpha \cap C`P) \cup P``\alpha = P``\alpha \cup \overrightarrow{\max}_P`\alpha$

Dem.

$\vdash .\, *211\cdot1 \,.\, *22\cdot621\cdot62 \,.\, \supset \vdash : \text{Hp} \,.\, \supset .\, \alpha = \alpha \cap C`P \,.\, \alpha = \alpha \cup P``\alpha \,.$ (1)

[*13·12] $\supset .\, \alpha = (\alpha \cap C`P) \cup P``\alpha$ (2)

[*205·131] $= P``\alpha \cup \overrightarrow{\max}_P`\alpha$ (3)

$\vdash .\, (1) \,.\, (2) \,.\, (3) \,.\, \supset \vdash .\, \text{Prop}$

***211·21.** $\vdash :. \alpha \in \text{sect}`P \,.\, \supset : \sim \exists ! \overrightarrow{\max}_P`\alpha \,.\, \equiv .\, \alpha \in \text{D}`(P_\epsilon \dot{\cap} I)$

Dem.

$\vdash .\, *211\cdot2\cdot12 \,.\quad \supset \vdash : \alpha \in \text{sect}`P \,.\, \sim \exists ! \overrightarrow{\max}_P`\alpha \,.\, \supset .\, \alpha \in \text{D}`(P_\epsilon \dot{\cap} I)$ (1)

$\vdash .\, *211\cdot12 \,.\, *205\cdot111 \,.\, \supset \vdash : \alpha \in \text{D}`(P_\epsilon \dot{\cap} I) \,.\, \supset .\, \sim \exists ! \overrightarrow{\max}_P`\alpha$ (2)

$\vdash .\, (1) \,.\, (2) \,.\, \supset \vdash .\, \text{Prop}$

***211·22.** $\vdash : P \in \text{connex} \,.\, \alpha \in \text{sect}`P \,.\, \supset .\, \alpha \cup \overrightarrow{\text{seq}}_P`\alpha \in \text{sect}`P$

Dem.

$\vdash .\, *24\cdot24 \,.\, *13\cdot12 \,.\quad \supset \vdash : \text{Hp} \,.\, \overrightarrow{\text{seq}}_P`\alpha = \Lambda \,.\, \supset .\, \alpha \cup \overrightarrow{\text{seq}}_P`\alpha \in \text{sect}`P$ (1)

$\vdash .\, *206\cdot16 \,.\, *53\cdot3\cdot31 \,.\, \supset \vdash : \text{Hp} \,.\, \exists ! \overrightarrow{\text{seq}}_P`\alpha \,.\, \supset .\, P``(\alpha \cup \overrightarrow{\text{seq}}_P`\alpha) = P``\alpha \cup \overrightarrow{P}`\text{seq}_P`\alpha$

[*206·213] $\subset P``\alpha \cup (\alpha \cap C`P) \cup P``\alpha$

[*211·2] $\subset \alpha$ (2)

$\vdash .\, *211\cdot1 \,.\, *206\cdot18 \,.\quad \supset \vdash : \text{Hp} \,.\, \supset .\, \alpha \cup \overrightarrow{\text{seq}}_P`\alpha \subset C`P$ (3)

$\vdash .\, (1) \,.\, (2) \,.\, (3) \,.\, *211\cdot1 \,.\, \supset \vdash .\, \text{Prop}$

***211·23.** $\vdash : P \in \text{connex} \,.\, \alpha \in \text{sect}`P \,.\, \text{E} ! \text{seq}_P`\alpha \,.\, \supset .\, \alpha = P``(\alpha \cup \overrightarrow{\text{seq}}_P`\alpha) = \overrightarrow{P}`\text{seq}_P`\alpha$

Dem.

$\vdash .\, *206\cdot211 \,.\, *211\cdot2 \,.\, \supset \vdash : \text{Hp} \,.\, \supset .\, \alpha \subset \overrightarrow{P}`\text{seq}_P`\alpha$ (1)

$\vdash .\, *206\cdot213 \,.\, *211\cdot2 \,.\, \supset \vdash : \text{Hp} \,.\, \supset .\, \overrightarrow{P}`\text{seq}_P`\alpha \subset \alpha$ (2)

$\vdash .\, (1) \,.\, (2) \,.\quad \supset \vdash : \text{Hp} \,.\, \supset .\, \alpha = \overrightarrow{P}`\text{seq}_P`\alpha$ (3)

$\vdash .\, *53\cdot3\cdot31 \,.\quad \supset \vdash : \text{Hp} \,.\, \supset .\, P``(\alpha \cup \overrightarrow{\text{seq}}_P`\alpha) = P``\alpha \cup \overrightarrow{P}`\text{seq}_P`\alpha$

[(3)] $= P``\alpha \cup \alpha$

[*211·2] $= \alpha$ (4)

$\vdash .\, (3) \,.\, (4) \,.\, \supset \vdash .\, \text{Prop}$

***211·24.** $\vdash : P \,\epsilon\, \mathrm{connex} \,.\, \alpha \,\epsilon\, \mathrm{sect}`P \cap (\text{Ɑ}`\mathrm{seq}_P \cup -\text{Ɑ}`\max_P) \,.\supset .\, \alpha \,\epsilon\, \mathrm{D}`P_\epsilon$

Dem.

$\vdash . *211{\cdot}23{\cdot}11 \,.\supset \vdash : P \,\epsilon\, \mathrm{connex} \,.\, \alpha \,\epsilon\, \mathrm{sect}`P \cap \text{Ɑ}`\mathrm{seq}_P \,.\supset .\, \alpha \,\epsilon\, \mathrm{D}`P_\epsilon$ (1)

$\vdash . *211{\cdot}21{\cdot}14 \,.\supset \vdash : \alpha \,\epsilon\, \mathrm{sect}`P - \text{Ɑ}`\max_P \,.\supset .\, \alpha \,\epsilon\, \mathrm{D}`P_\epsilon$ (2)

$\vdash . (1) . (2) \,.\supset \vdash .$ Prop

***211·26.** $\vdash . C`P \,\epsilon\, \mathrm{sect}`P \,.\, s`\mathrm{sect}`P = C`P$

Dem.

$\vdash . *22{\cdot}42 . *37{\cdot}15 \,.\supset \vdash . C`P \subset C`P \,.\, P``C`P \subset C`P \,.$

$[*211{\cdot}1] \quad \supset \vdash . C`P \,\epsilon\, \mathrm{sect}`P$ (1)

$\vdash . (1) . *40{\cdot}13 \,.\supset \vdash . C`P \subset s`\mathrm{sect}`P$ (2)

$\vdash . *40{\cdot}151 . *211{\cdot}1 \,.\supset \vdash . s`\mathrm{sect}`P \subset C`P$ (3)

$\vdash . (2) . (3) \,.\supset \vdash . s`\mathrm{sect}`P = C`P$ (4)

$\vdash . (1) . (4) \,.\supset \vdash .$ Prop

***211·27.** $\vdash : P \,\epsilon\, \mathrm{trans} \,.\supset .\, (\alpha \cap C`P) \cup P``\alpha \,\epsilon\, \mathrm{sect}`P$

Dem.

$\vdash . *22{\cdot}43 . *37{\cdot}15 \,.\supset \vdash . (\alpha \cap C`P) \cup P``\alpha \subset C`P$ (1)

$\vdash . *37{\cdot}22{\cdot}265 \,.\supset \vdash . P``\{(\alpha \cap C`P) \cup P``\alpha\} = P``\alpha \cup P``P``\alpha$ (2)

$\vdash . (2) . *201{\cdot}5 \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, P``\{(\alpha \cap C`P) \cup P``\alpha\} = P``\alpha$ (3)

$\vdash . (1) . (3) . *211{\cdot}1 \,.\supset \vdash .$ Prop

***211·271.** $\vdash : P \,\epsilon\, \mathrm{trans} \,.\supset .\, (\exists\beta) . \beta \,\epsilon\, \mathrm{sect}`P \,.\, \overrightarrow{\max}_P`\alpha = \overrightarrow{\max}_P`\beta \,.\, \overrightarrow{\mathrm{seq}}_P`\alpha = \overrightarrow{\mathrm{seq}}_P`\beta$

Dem.

$\vdash . *205{\cdot}15{\cdot}19 \,.\supset$

$\vdash : \mathrm{Hp} \,.\supset .\, \overrightarrow{\max}_P`\alpha = \overrightarrow{\max}_P`\{(\alpha \cap C`P) \cup P``\alpha\}$ (1)

$\vdash . *206{\cdot}131{\cdot}25 \,.\supset$

$\vdash : \mathrm{Hp} \,.\supset .\, \overrightarrow{\mathrm{seq}}_P`\alpha = \overrightarrow{\mathrm{seq}}_P`\{(\alpha \cap C`P) \cup P``\alpha\}$ (2)

$\vdash . (1) . (2) . *211{\cdot}27 \,.\supset \vdash .$ Prop

***211·272.** $\vdash :. P \,\epsilon\, \mathrm{trans} \,.\supset :$

$(\alpha) . \alpha \,\epsilon\, \text{Ɑ}`\max_P \cup \text{Ɑ}`\mathrm{seq}_P \,.\equiv .\, \mathrm{sect}`P \subset \text{Ɑ}`\max_P \cup \text{Ɑ}`\mathrm{seq}_P$

Dem.

$\vdash . *24{\cdot}11{\cdot}14 \,.\supset \vdash : (\alpha) . \alpha \,\epsilon\, \text{Ɑ}`\max_P \cup \text{Ɑ}`\mathrm{seq}_P \,.\supset .\, \mathrm{sect}`P \subset \text{Ɑ}`\max_P \cup \text{Ɑ}`\mathrm{seq}_P$ (1)

$\vdash . *33{\cdot}41 \,.\supset \vdash :. \mathrm{sect}`P \subset \text{Ɑ}`\max_P \cup \text{Ɑ}`\mathrm{seq}_P \,.\supset :$

$\beta \,\epsilon\, \mathrm{sect}`P \,.\supset_\beta .\, \exists ! (\overrightarrow{\max}_P`\beta \cup \overrightarrow{\mathrm{seq}}_P`\beta) :$

$[*13{\cdot}12] \supset : \beta \,\epsilon\, \mathrm{sect}`P \,.\, \overrightarrow{\max}_P`\alpha = \overrightarrow{\max}_P`\beta \,.\, \overrightarrow{\mathrm{seq}}_P`\alpha = \overrightarrow{\mathrm{seq}}_P`\beta \,.\supset_{\alpha,\beta} .$

$\exists ! (\overrightarrow{\max}_P`\alpha \cup \overrightarrow{\mathrm{seq}}_P`\alpha) :$

$[*10{\cdot}23] \supset : (\exists\beta) . \beta \,\epsilon\, \mathrm{sect}`P \,.\, \overrightarrow{\max}_P`\alpha = \overrightarrow{\max}_P`\beta \,.\, \overrightarrow{\mathrm{seq}}_P`\alpha = \overrightarrow{\mathrm{seq}}_P`\beta \,.\supset_\alpha .$

$\exists ! (\overrightarrow{\max}_P`\alpha \cup \overrightarrow{\mathrm{seq}}_P`\alpha)$ (2)

$\vdash . (2) . *211{\cdot}271 \,.\supset$

$\vdash :. \mathrm{Hp} \,.\supset : \mathrm{sect}`P \subset \text{Ɑ}`\max_P \cup \text{Ɑ}`\mathrm{seq}_P \,.\supset .\, (\alpha) . \exists ! (\overrightarrow{\max}_P`\alpha \cup \overrightarrow{\mathrm{seq}}_P`\alpha) \,.$

$[*33{\cdot}41] \quad \supset . (\alpha) . \alpha \,\epsilon\, \text{Ɑ}`\max_P \cup \text{Ɑ}`\mathrm{seq}_P$ (3)

$\vdash . (1) . (3) \,.\supset \vdash .$ Prop

***211·28.** $\vdash :. P \epsilon \mathrm{Ser} . \alpha \subset C‘P . \alpha \sim \epsilon 1 . (C‘P - \alpha) \sim \epsilon 1 . \supset :$
$\alpha \epsilon \mathrm{sect}‘P . \equiv . P = P \upharpoonright \alpha \dotplus P \upharpoonright (C‘P - \alpha)$

Dem.

$\vdash . *204·45 . \quad \supset \vdash : \mathrm{Hp} . \alpha \epsilon \mathrm{sect}‘P . \supset . P = P \upharpoonright \alpha \dotplus P \upharpoonright (C‘P - \alpha)$ (1)

$\vdash . *160·1 . *202·55 . \supset \vdash :. \mathrm{Hp} . P = P \upharpoonright \alpha \dotplus P \upharpoonright (C‘P - \alpha) . \supset :$
$x \epsilon \alpha . y \epsilon C‘P - \alpha . \supset . xPy :$

[Transp.*204·3] $\supset : x \epsilon \alpha . yPx . \supset . y \epsilon \alpha :$

[*211·1] $\supset : \alpha \epsilon \mathrm{sect}‘P$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***211·281.** $\vdash : P \epsilon \mathrm{Ser} . C‘Q \cap C‘R = \Lambda . P = Q \dotplus R . \supset . C‘Q \epsilon \mathrm{sect}‘P$

Dem.

$\vdash . *160·1 . \supset \vdash :. \mathrm{Hp} . \supset : x \epsilon C‘Q . y \epsilon C‘R . \supset . xPy :$

[Transp.*204·3] $\supset : x \epsilon C‘Q . yPx . \supset . y \epsilon C‘Q :$

[*211·1] $\supset : C‘Q \epsilon \mathrm{sect}‘P :. \supset \vdash . \mathrm{Prop}$

***211·282.** $\vdash :. P \epsilon \mathrm{Ser} . Q \epsilon \mathrm{D}‘P \upharpoonright . C‘P - C‘Q \sim \epsilon 1 . \supset :$
$C‘Q \epsilon \mathrm{sect}‘P . \equiv . (\exists R) . C‘Q \cap C‘R = \Lambda . P = Q \dotplus R$

[*211·28·281 . *200·12]

***211·283.** $\vdash : P \Subset J . P = Q \dotplus R . \supset . C‘Q \cap C‘R = \Lambda$

Dem.

$\vdash . *160·1 . \supset \vdash : \mathrm{Hp} . \supset . C‘Q \uparrow C‘R \Subset J .$

[*200·32] $\supset . C‘Q \cap C‘R = \Lambda : \supset \vdash . \mathrm{Prop}$

The following propositions are concerned with $\mathrm{D}‘P_\epsilon$. This is to be compared with two other classes, namely $\mathrm{sect}‘P$ and $\overrightarrow{P}“C‘P$. The members of $\mathrm{sect}‘P$ which do not belong to $\mathrm{D}‘P_\epsilon$ are those which have a maximum but no sequent, *i.e.* (if P is a series), those classes which consist of a term x together with all its predecessors, where x has no immediate successor. In series in which every term except the last has an immediate successor, $C‘P$ will be the only member of $\mathrm{sect}‘P - \mathrm{D}‘P_\epsilon$, if the series has a last term; if the series has no last term, $\mathrm{sect}‘P = \mathrm{D}‘P_\epsilon$.

The members of $\mathrm{D}‘P_\epsilon$ which are not members of $\overrightarrow{P}“C‘P$ are those that have no sequent, *i.e.* those that have no upper limit (for a member of $\mathrm{D}‘P_\epsilon$ which has no sequent has also no maximum). These are the members of $\mathrm{D}‘P_\epsilon$ corresponding to a "gap," *i.e.* to a Dedekind section in which neither the earlier terms have a maximum nor the later terms a minimum. Hence in a Dedekindian series, $\mathrm{D}‘P_\epsilon = \overrightarrow{P}“C‘P$; and conversely, if $\mathrm{D}‘P_\epsilon = \overrightarrow{P}“C‘P$, the series is Dedekindian. These properties of $\mathrm{D}‘P_\epsilon$ are proved in the following propositions.

***211·3.** $\vdash . \overrightarrow{P}“C‘P \subset \mathrm{D}‘P_\epsilon$ [*53·301 . *211·11]

***211·301.** $\vdash . \mathrm{D}‘P \epsilon \mathrm{D}‘P_\epsilon$ [*37·25 . *211·11]

$*211\cdot302.\ \vdash : P \in \mathrm{Ser} . \supset . \overrightarrow{P}``C`P = \mathrm{sect}`P \cap \mathrm{Cl}`\mathrm{seq}_P$

Dem.

$\vdash . *206\cdot4 . \quad \supset \vdash : \mathrm{Hp} . \supset . \overrightarrow{P}``C`P \subset \mathrm{Cl}`\mathrm{seq}_P \quad (1)$

$\vdash . *211\cdot3\cdot15 . \supset \vdash : \mathrm{Hp} . \supset . \overrightarrow{P}``C`P \subset \mathrm{sect}`P \quad (2)$

$\vdash . *211\cdot23 . \quad \supset \vdash : \mathrm{Hp} . \supset . \mathrm{sect}`P \cap \mathrm{Cl}`\mathrm{seq}_P \subset \overrightarrow{P}``C`P \quad (3)$

$\vdash . (1) . (2) . (3) . \supset \vdash . \mathrm{Prop}$

$*211\cdot31.\ \vdash :. P \in \mathrm{trans} \cap \mathrm{connex} . \alpha \in \mathrm{D}`P_\epsilon . \supset : \mathrm{E} ! \mathrm{seq}_P`\alpha . \vee . \sim \mathrm{E} ! \max_P`\alpha$
$[*206\cdot52 . *211\cdot11]$

$*211\cdot311.\ \vdash : P \in \mathrm{trans} \cap \mathrm{connex} . \alpha \in \mathrm{D}`P_\epsilon . \mathrm{E} ! \mathrm{seq}_P`\alpha . \supset . \alpha = \overrightarrow{P}`\mathrm{seq}_P`\alpha$
$[*206\cdot31 . *211\cdot11]$

$*211\cdot312.\ \vdash : P \in \mathrm{trans} \cap \mathrm{connex} . \alpha \in \mathrm{D}`P_\epsilon . \supset . \alpha = P``(\alpha \cup \overrightarrow{\mathrm{seq}_P`\alpha})$

Dem.

$\vdash . *211\cdot15\cdot23 . \supset \vdash : \mathrm{Hp} . \mathrm{E} ! \mathrm{seq}_P`\alpha . \supset . \alpha = P``(\alpha \cup \overrightarrow{\mathrm{seq}_P`\alpha}) \quad (1)$

$\vdash . *211\cdot31 . \quad \supset \vdash : \mathrm{Hp} . \sim \mathrm{E} ! \mathrm{seq}_P`\alpha . \supset . \sim \mathrm{E} ! \max_P`\alpha .$

$[*211\cdot21\cdot15\cdot12] \quad \supset . \alpha = P``\alpha .$

$[*24\cdot24.\mathrm{Hp}] \quad \supset . \alpha = P``(\alpha \cup \overrightarrow{\mathrm{seq}_P`\alpha}) \quad (2)$

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

$*211\cdot313.\ \vdash : \alpha \in \mathrm{sect}`P \cap \mathrm{D}`P_\epsilon . \supset . (\exists \beta) . \beta \in \mathrm{sect}`P . \alpha = P``\beta$

Dem.

$\vdash . *211\cdot1\cdot11 . \supset \vdash :. \mathrm{Hp} . \supset : P``\alpha \subset \alpha : (\exists \beta) . \alpha = P``\beta :$

$[*37\cdot265] \quad \supset : P``\alpha \subset \alpha : (\exists \beta) . \beta \subset C`P . \alpha = P``\beta :$

$[*22\cdot62] \quad \supset : (\exists \beta) . \beta \subset C`P . \alpha = P``(\alpha \cup \beta) :$

$[*22\cdot58] \quad \supset : (\exists \beta) . \beta \subset C`P . P``(\alpha \cup \beta) \subset \alpha \cup \beta . \alpha = P``(\alpha \cup \beta) :$

$[*37\cdot15] \quad \supset : (\exists \beta) . \alpha \cup \beta \subset C`P . P``(\alpha \cup \beta) \subset \alpha \cup \beta . \alpha = P``(\alpha \cup \beta) :$

$[*211\cdot1] \quad \supset : (\exists \beta) . \alpha \cup \beta \in \mathrm{sect}`P . \alpha = P``(\alpha \cup \beta) :. \supset \vdash . \mathrm{Prop}$

$*211\cdot314.\ \vdash : P \in \mathrm{Rl}`J \cap \mathrm{connex} . \alpha \in \mathrm{sect}`P \cap \mathrm{D}`P_\epsilon . \mathrm{E} ! \max_P`\alpha . \supset . \mathrm{E} ! \mathrm{seq}_P`\alpha$

Dem.

$\vdash . *211\cdot313 . *205\cdot7 . \supset$

$\vdash : \mathrm{Hp} . \supset . (\exists \beta) . \beta \in \mathrm{sect}`P . \alpha = P``\beta . \mathrm{E} ! \max_P`\beta \quad (1)$

$\vdash . *37\cdot18 . \supset$

$\vdash : \beta \in \mathrm{sect}`P . \alpha = P``\beta . \mathrm{E} ! \max_P`\beta . \supset . \overrightarrow{P}`\max_P`\beta \subset \alpha \quad (2)$

$\vdash . *211\cdot1 . *205\cdot111 . \supset$

$\vdash : \mathrm{Hp}(2) . P \in \mathrm{connex} . y \in P``\beta . \supset . y \in \beta - \iota`\max_P`\beta .$

$[*205\cdot21] \quad \supset . y P \max_P`\beta \quad (3)$

$\vdash . (2) . (3) . \supset \vdash : \mathrm{Hp}(2) . P \in \mathrm{connex} . \supset . \alpha = \overrightarrow{P}`\max_P`\beta .$

$[*206\cdot4] \quad \supset . \max_P`\beta \ \mathrm{seq}_P \ \alpha \quad (4)$

$\vdash . (1) . (4) . \supset \vdash . \mathrm{Prop}$

The above proposition and the two following propositions enable us in certain cases to prove propositions concerning the relations of $\text{sect}`P$ and $\text{D}`P_\epsilon$ without assuming that P is transitive. An example of the use of these propositions occurs in $*211{\cdot}754$, where the hypothesis assumes $P \epsilon \text{Rl}`J \cap \text{connex}$. If we used $*211{\cdot}31$ and its consequences instead of $*211{\cdot}314$ and its consequences, the hypothesis of $*211{\cdot}754$ would have to assume $P \epsilon \text{Ser}$.

$*211{\cdot}315$. $\vdash :. P \epsilon \text{Rl}`J \cap \text{connex} . \alpha \epsilon \text{sect}`P . \supset :$

$$\alpha \epsilon \text{D}`P_\epsilon . \equiv . \alpha \epsilon \text{Ɑ}`\text{seq}_P \cup - \text{Ɑ}`\text{max}_P$$

Dem.

$\vdash . *211{\cdot}314 . \supset \vdash :. \text{Hp} . \supset : \alpha \epsilon \text{D}`P_\epsilon . \supset . \alpha \epsilon \text{Ɑ}`\text{seq}_P \cup - \text{Ɑ}`\text{max}_P$ (1)

$\vdash . (1) . *211{\cdot}24 . \supset \vdash . \text{Prop}$

$*211{\cdot}316$. $\vdash : P \epsilon \text{Rl}`J \cap \text{connex} . \supset . \text{sect}`P - \text{D}`P_\epsilon = \text{sect}`P \cap \text{Ɑ}`\text{max}_P - \text{Ɑ}`\text{seq}_P$

[$*211{\cdot}315$. Transp]

$*211{\cdot}317$. $\vdash : P \epsilon \text{trans} . \supset . \text{D}`P_\epsilon = P_\epsilon``\text{sect}`P$

Dem.

$\vdash . *211{\cdot}15{\cdot}313 . \supset \vdash : \text{Hp} . \supset . \text{D}`P_\epsilon \subset P_\epsilon``\text{sect}`P$ (1)

$\vdash . (1) . *37{\cdot}15 . \supset \vdash . \text{Prop}$

$*211{\cdot}32$. $\vdash : P \epsilon \text{trans} \cap \text{connex} . \supset . \text{D}`P_\epsilon = \text{sect}`P \cap (\text{Ɑ}`\text{seq}_P \cup - \text{Ɑ}`\text{max}_P)$

[$*211{\cdot}24{\cdot}15{\cdot}31$]

$*211{\cdot}321$. $\vdash : P \epsilon \text{trans} \cap \text{connex} . \supset . \text{sect}`P - \text{D}`P_\epsilon = \text{sect}`P \cap \text{Ɑ}`\text{max}_P - \text{Ɑ}`\text{seq}_P$

[$*211{\cdot}32$]

$*211{\cdot}33$. $\vdash :. P \epsilon \text{Ser} . \alpha \epsilon \text{sect}`P . \supset :$

$$\alpha \sim\epsilon \text{D}`P_\epsilon . \supset . \text{E} ! \text{seq}_P`P``\alpha . \sim \text{E} ! \text{seq}_P`\overrightarrow{\text{seq}_P}`P``\alpha$$

Dem.

$\vdash . *211{\cdot}321 . \quad \supset \vdash : \text{Hp} . \alpha \sim\epsilon \text{D}`P_\epsilon . \supset . \text{E} ! \text{max}_P`\alpha .$ (1)

[$*206{\cdot}35$] $\quad \supset . \text{E} ! \text{seq}_P`P``\alpha . \text{seq}_P`P``\alpha = \text{max}_P`\alpha$ (2)

$\vdash . (1) . *206{\cdot}46 . \supset \vdash : \text{Hp} . \alpha \sim\epsilon \text{D}`P_\epsilon . \supset . \overrightarrow{\text{seq}_P}`\alpha = \overrightarrow{\text{seq}_P}`\text{max}_P`\alpha$

[(2)] $\quad = \overrightarrow{\text{seq}_P}`\text{seq}_P`P``\alpha$ (3)

$\vdash . *211{\cdot}321 . \quad \supset \vdash : \text{Hp} . \alpha \sim\epsilon \text{D}`P_\epsilon . \supset . \sim \text{E} ! \text{seq}_P`\alpha .$

[(3)] $\quad \supset . \sim \text{E} ! \text{seq}_P`\overrightarrow{\text{seq}_P}`P``\alpha$ (4)

$\vdash . (2) . (4) . \quad \supset \vdash : \text{Hp} . \alpha \sim\epsilon \text{D}`P_\epsilon . \supset .$

$$\text{E} ! \text{seq}_P`P``\alpha . \sim \text{E} ! \text{seq}_P`\overrightarrow{\text{seq}_P}`P``\alpha : \supset \vdash . \text{Prop}$$

$*211{\cdot}34$. $\vdash :. P \epsilon \text{Ser} . \supset : \alpha \epsilon \text{sect}`P - \text{D}`P_\epsilon . \equiv .$

$$\alpha = P``\alpha \cup \iota`\text{seq}_P`P``\alpha . \sim \text{E} ! \text{seq}_P`\overrightarrow{\text{seq}_P}`P``\alpha$$

Dem.

$\vdash . *211{\cdot}321 . \supset \vdash : \text{Hp} . \alpha \epsilon \text{sect}`P - \text{D}`P_\epsilon . \supset . \text{E} ! \text{max}_P`\alpha .$

[$*206{\cdot}35$] $\quad \supset . \text{max}_P`\alpha = \text{seq}_P`P``\alpha .$ (1)

[$*211{\cdot}2$] $\quad \supset . \alpha = P``\alpha \cup \iota`\text{seq}_P`P``\alpha$ (2)

$\vdash . *211\cdot321 . \supset \vdash : \text{Hp} . \alpha \in \text{sect}'P - \text{D}'P_\epsilon . \supset . \sim \text{E} ! \, \text{seq}_P{}'\alpha .$

[$*206\cdot46.(1)$] $\supset . \sim \text{E} ! \, \text{seq}_P{}'\overrightarrow{\text{seq}_P}{}'P''\alpha$ (3)

$\vdash . *206\cdot21 . *205\cdot111 . \supset \vdash : \text{Hp} . \alpha = P''\alpha \cup \iota'\text{seq}_P{}'P''\alpha . \supset .$

$\text{seq}_P{}'P''\alpha = \text{max}_P{}'\alpha$ (4)

$\vdash . (4) . *206\cdot46 . \supset \vdash : \text{Hp} . \alpha = P''\alpha \cup \iota'\text{seq}_P{}'P''\alpha . \sim \text{E} ! \, \text{seq}_P{}'\overrightarrow{\text{seq}_P}{}'P''\alpha . \supset .$

$\sim \text{E} ! \, \text{seq}_P{}'\alpha$ (5)

$\vdash . *206\cdot18 . *22\cdot58 . \supset \vdash : \alpha = P''\alpha \cup \iota'\text{seq}_P{}'P''\alpha . \supset . \alpha \subset C'P . P''\alpha \subset \alpha$ (6)

$\vdash . (4) . (5) . (6) . *211\cdot321 . \supset$

$\vdash : \text{Hp} . \alpha = P''\alpha \cup \iota'\text{seq}_P{}'P''\alpha . \sim \text{E} ! \, \text{seq}_P{}'\overrightarrow{\text{seq}_P}{}'P''\alpha . \supset . \alpha \in \text{sect}'P - \text{D}'P_\epsilon$ (7)

$\vdash . (2) . (3) . (7) . \supset \vdash . \text{Prop}$

$*211\cdot35$. $\vdash :. P \in \text{Ser} . \supset : \alpha \in \text{sect}'P - \text{D}'P_\epsilon . \equiv .$

$(\exists x) . x \in C'P . \alpha = \overrightarrow{P}{}'x \cup \iota'x . \sim \text{E} ! \, \breve{P}_1{}'x$

Dem.

$\vdash . *211\cdot34 . \supset \vdash :. \text{Hp} . \supset :$

$\alpha \in \text{sect}'P - \text{D}'P_\epsilon . \equiv . (\exists x) . x = \text{seq}_P{}'P''\alpha . \alpha = P''\alpha \cup \iota'x . \sim \text{E} ! \, \text{seq}_P{}'\iota'x$

[$*206\cdot21 . *205\cdot111$] $\equiv . (\exists x) . x = \text{seq}_P{}'P''\alpha . x = \text{max}_P{}'\alpha .$

$\alpha = P''\alpha \cup \iota'x . \sim \text{E} ! \, \text{seq}_P{}'\iota'x .$

[$*206\cdot35$] $\equiv . (\exists x) . x = \text{max}_P{}'\alpha . \alpha = P''\alpha \cup \iota'x . \sim \text{E} ! \, \text{seq}_P{}'\iota'x .$

[$*205\cdot22$] $\equiv . (\exists x) . x = \text{max}_P{}'\alpha . \alpha = \overrightarrow{P}{}'x \cup \iota'x . \sim \text{E} ! \, \text{seq}_P{}'\iota'x .$

[$*205\cdot197$] $\equiv . (\exists x) . x \in C'P . \alpha = \overrightarrow{P}{}'x \cup \iota'x . \sim \text{E} ! \, \text{seq}_P{}'\iota'x .$

[$*206\cdot44$] $\equiv . (\exists x) . x \in C'P . \alpha = \overrightarrow{P}{}'x \cup \iota'x . \sim \text{E} ! \, \breve{P}_1{}'x :. \supset \vdash . \text{Prop}$

$*211\cdot351$. $\vdash : P \in \text{Ser} . \supset . \text{sect}'P - \text{D}'P_\epsilon = \overrightarrow{P_*}{}''(C'P - \text{D}'P_1)$

Dem.

$\vdash . *204\cdot7 . *211\cdot35 . \supset$

$\vdash :. \text{Hp} . \supset : \alpha \in \text{sect}'P - \text{D}'P_\epsilon . \equiv . (\exists x) . x \in C'P - \text{D}'P_1 . \alpha = \overrightarrow{P}{}'x \cup \iota'x .$

[$*201\cdot521$] $\equiv . (\exists x) . x \in C'P - \text{D}'P_1 . \alpha = \overrightarrow{P_*}{}'x .$

[$*37\cdot7$] $\equiv . \alpha \in \overrightarrow{P_*}{}''(C'P - \text{D}'P_1) :. \supset \vdash . \text{Prop}$

$*211\cdot36$. $\vdash :. P \in \text{Ser} . \text{D}'P_1 = \text{D}'P . \supset : \alpha \in \text{sect}'P - \text{D}'P_\epsilon . \equiv . \alpha = C'P . \text{E} ! \, B'\breve{P}$

Dem.

$\vdash . *211\cdot351 . \quad \supset \vdash : \text{Hp} . \supset . \text{sect}'P - \text{D}'P_\epsilon = \overrightarrow{P_*}{}''\overrightarrow{B}{}'\breve{P}$ (1)

$\vdash . (1) . *202\cdot52 . \supset \vdash :. \text{Hp} . \supset :$

$\alpha \in \text{sect}'P - \text{D}'P_\epsilon . \equiv . (\exists x) . x = B'\breve{P} . \alpha = \overrightarrow{P_*}{}'x .$

[$*204\cdot11 . *201\cdot521$] $\equiv . (\exists x) . x = B'\breve{P} . \alpha = C'P .$

[$*14\cdot204$] $\equiv . \alpha = C'P . \text{E} ! \, B'\breve{P} :. \supset \vdash . \text{Prop}$

***211·361.** $\vdash : P \epsilon \mathrm{Ser} . \mathrm{D}`P_1 = C`P . \supset . \mathrm{sect}`P = \mathrm{D}`P_\epsilon$

Dem.

$\vdash . *201·63 . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{D}`P_1 \subset \mathrm{D}`P .$

$[*93·103] \quad \supset . \overrightarrow{B}`\breve{P} = \Lambda \quad (1)$

$\vdash . (1) . *211·36 . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{sect}`P - \mathrm{D}`P_\epsilon = \Lambda \quad (2)$

$\vdash . (2) . *211·15 . \supset \vdash . \mathrm{Prop}$

***211·371.** $\vdash :. P \epsilon \mathrm{trans} \cap \mathrm{connex} : (\alpha) . \alpha \epsilon \mathrm{Cl}`\mathrm{max}_P \cup \mathrm{Cl}`\mathrm{seq}_P : \supset . \mathrm{D}`P_\epsilon \subset \mathrm{Cl}`\mathrm{seq}_P$
[*211·32]

***211·372.** $\vdash :. P \epsilon \mathrm{trans} \cap \mathrm{connex} : (\alpha) . \alpha \epsilon \mathrm{Cl}`\mathrm{max}_P \cup \mathrm{Cl}`\mathrm{seq}_P : \supset . \mathrm{D}`P_\epsilon = \overrightarrow{P}``C`P$

Dem.

$\vdash . *211·371 . \supset \vdash :. \mathrm{Hp} . \supset : \alpha \epsilon \mathrm{D}`P_\epsilon . \supset . \mathrm{E} ! \mathrm{seq}_P`\alpha .$

$[*206·3.*211·1·15] \quad \supset . \alpha = \overrightarrow{P}`\mathrm{seq}_P`\alpha .$

$[*206·18] \quad \supset . \alpha \epsilon \overrightarrow{P}``C`P \quad (1)$

$\vdash . (1) . *211·3 . \supset \vdash . \mathrm{Prop}$

***211·38.** $\vdash :. P \epsilon \mathrm{Ser} . \supset : (\alpha) . \alpha \epsilon \mathrm{Cl}`\mathrm{max}_P \cup \mathrm{Cl}`\mathrm{seq}_P . \equiv . \mathrm{D}`P_\epsilon = \overrightarrow{P}``C`P$

Dem.

$\vdash . *211·11 . \supset \vdash :. \mathrm{D}`P_\epsilon = \overrightarrow{P}``C`P . \equiv : (\beta) : (\exists x) . P``\beta = \overrightarrow{P}`x . x \epsilon C`P \quad (1)$

$\vdash . *206·174 . *205·111 . \supset$

$\vdash : P \epsilon \mathrm{Ser} . \sim \exists ! \overrightarrow{\mathrm{max}_P}`\beta . \supset . \overrightarrow{\mathrm{seq}_P}`\beta = C`P \cap \hat{x} (P``\beta = \overrightarrow{P}`x) \quad (2)$

$\vdash . (1) . (2) . \supset \vdash :. P \epsilon \mathrm{Ser} . \mathrm{D}`P_\epsilon = \overrightarrow{P}``C`P . \supset : \sim \exists ! \overrightarrow{\mathrm{max}_P}`\beta . \supset . \exists ! \overrightarrow{\mathrm{seq}_P}`\beta :$

$[*33·41] \quad \supset : \beta \epsilon \mathrm{Cl}`\mathrm{max}_P \cup \mathrm{Cl}`\mathrm{seq}_P \quad (3)$

$\vdash . (3) . *211·372 . \supset \vdash . \mathrm{Prop}$

The following propositions are concerned with $\mathrm{D}`(P_\epsilon \dot{\cap} I)$, *i.e.* with those sections of $P$ which have no maximum. If $P$ is compact (*i.e.* if $P^2 = P$), $\mathrm{D}`(P_\epsilon \dot{\cap} I) = \mathrm{D}`P_\epsilon$. If $P$ is also a Dedekindian series, $\mathrm{D}`(P_\epsilon \dot{\cap} I) = \overrightarrow{P}``C`P$. This is the mark of Dedekindian continuity, since it states that, if $P``\alpha$ has no maximum, there is an x for which $P``\alpha = \overrightarrow{P}`x$, and this $x$ is the upper limit of $P``\alpha$; while conversely, if x is any term of $C`P$, $\overrightarrow{P}`x$ has no maximum, so that the series is compact.

***211·4.** $\vdash . \mathrm{D}`(P_\epsilon \dot{\cap} I) \subset - \mathrm{Cl}`\mathrm{max}_P$

Dem.

$\vdash . *211·12 . \supset \vdash : \alpha \epsilon \mathrm{D}`(P_\epsilon \dot{\cap} I) . \supset . \alpha - P``\alpha = \Lambda .$

$[*205·111] \quad \supset . \overrightarrow{\mathrm{max}_P}`\alpha = \Lambda : \supset \vdash . \mathrm{Prop}$

***211·41.** $\vdash . \mathrm{D}`(P_\epsilon \dot{\cap} I) = \mathrm{sect}`P - \mathrm{Cl}`\mathrm{max}_P$

Dem.

$\vdash . *211·1 . *205·111 . \supset$

$\vdash : \alpha \epsilon \mathrm{sect}`P - \mathrm{Cl}`\mathrm{max}_P . \equiv . \alpha \subset C`P . P``\alpha \subset \alpha . \alpha \subset P``\alpha .$

$[*22·41] \quad \equiv . \alpha \subset C`P . \alpha = P``\alpha .$

$[*37·15.*211·12] \quad \equiv . \alpha \epsilon \mathrm{D}`(P_\epsilon \dot{\cap} I) : \supset \vdash . \mathrm{Prop}$

$*211\cdot411$. $\vdash : P \epsilon \text{trans} . \alpha = P``\beta . \alpha \subset P``\alpha . \supset . \alpha = P``\alpha$

Dem.

$\vdash . *30\cdot37 . \quad \supset \vdash : \text{Hp} . \supset . P``\alpha = P``P``\beta$
$[*201\cdot5] \quad \subset P``\beta$
$[\text{Hp}] \quad \subset \alpha \quad (1)$
$\vdash . (1) . *22\cdot41 . \supset \vdash : \text{Hp} . \supset . \alpha = P``\alpha : \supset \vdash . \text{Prop}$

$*211\cdot42$. $\vdash : P \epsilon \text{trans} . \supset . \text{D}`(P_\epsilon \dot{\cap} I) = \text{D}`P_\epsilon - \text{G}`\text{max}_P$

Dem.

$\vdash . *211\cdot14\cdot4 . \quad \supset \vdash . \text{D}`(P_\epsilon \dot{\cap} I) \subset \text{D}`P_\epsilon - \text{G}`\text{max}_P \quad (1)$
$\vdash . *211\cdot411\cdot11 . *205\cdot111 . \supset \vdash : \text{Hp} . \alpha \epsilon \text{D}`P_\epsilon - \text{G}`\text{max}_P . \supset . \alpha \epsilon \text{D}`(P_\epsilon \dot{\cap} I) \quad (2)$
$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

$*211\cdot43$. $\vdash : P \epsilon \text{trans} \cap \text{connex} . \supset . \text{D}`P_\epsilon - \text{G}`\text{seq}_P \subset \text{D}`(P_\epsilon \dot{\cap} I)$

Dem.

$\vdash . *211\cdot312 . \supset \vdash :. \text{Hp} . \supset : \alpha \epsilon \text{D}`P_\epsilon . \overrightarrow{\text{seq}_P}`\alpha = \Lambda . \supset . \alpha = P``\alpha .$
$[*211\cdot12] \quad \supset . \alpha \epsilon \text{D}`(P_\epsilon \dot{\cap} I) :. \supset \vdash . \text{Prop}$

$*211\cdot431$. $\vdash : P \epsilon \text{trans} \cap \text{connex} . \supset .$
$\text{D}`P_\epsilon - \text{D}`(P_\epsilon \dot{\cap} I) = \text{sect}`P \cap \text{G}`\text{max}_P \cap \text{G}`\text{seq}_P$
$[*211\cdot32\cdot41]$

$*211\cdot44$. $\vdash . \Lambda \epsilon \text{D}`(P_\epsilon \dot{\cap} I) . \Lambda \epsilon \text{D}`P_\epsilon . \Lambda \epsilon \text{sect}`P$
$[*37\cdot29 . *211\cdot12\cdot14]$

$*211\cdot45$. $\vdash : P \epsilon \text{trans} . x \sim\epsilon \text{G}`(P \dot{-} P^2) . \supset . \overrightarrow{P}`x \epsilon \text{D}`(P_\epsilon \dot{\cap} I)$

Dem.

$\vdash . *201\cdot501 . \quad \supset \vdash : \text{Hp} . \supset . P``\overrightarrow{P}`x \subset \overrightarrow{P}`x \quad (1)$
$\vdash . *33\cdot41 . *32\cdot3\cdot34 . \supset \vdash : \text{Hp} . \supset . \overrightarrow{P}`x - P``\overrightarrow{P}`x = \Lambda \quad (2)$
$\vdash . (1) . (2) . \quad \supset \vdash : \text{Hp} . \supset . \overrightarrow{P}`x = P``\overrightarrow{P}`x \quad (3)$
$\vdash . (3) . *211\cdot12 . \supset \vdash . \text{Prop}$

$*211\cdot451$. $\vdash : \overrightarrow{P}`x \epsilon \text{D}`(P_\epsilon \dot{\cap} I) . \supset . x \sim\epsilon \text{G}`(P \dot{-} P^2)$

Dem.

$\vdash . *211\cdot12 . \supset \vdash :. \text{Hp} . \supset : \overrightarrow{P}`x = P``\overrightarrow{P}`x :$
$[*37\cdot3] \quad \supset : yPx . \equiv_y . yP^2x :$
$[*10\cdot51] \quad \supset : \sim(\exists y) . yPx . \sim(yP^2x) :. \supset \vdash . \text{Prop}$

$*211\cdot452$. $\vdash :. P \epsilon \text{trans} . \supset : \overrightarrow{P}`x \epsilon \text{D}`(P_\epsilon \dot{\cap} I) . \equiv . x \sim\epsilon \text{G}`(P \dot{-} P^2)$
$[*211\cdot45\cdot451]$

$*211{\cdot}46.\quad \vdash :. P \in \text{trans} \cap \text{connex} : (\alpha) . \alpha \in \text{Ɑ‘max}_P \cup \text{Ɑ‘seq}_P : \supset .$

$$\text{D‘}(P_\epsilon \mathbin{\dot\cap} I) = \overrightarrow{P}\text{‘‘}\{C\text{‘}P - \text{Ɑ‘}(P \mathbin{\dot-} P^2)\}$$

Dem.

$\vdash . *211{\cdot}452 . \supset \vdash : \text{Hp} . \supset . \overrightarrow{P}\text{‘‘}\{C\text{‘}P - \text{Ɑ‘}(P \mathbin{\dot-} P^2)\} \subset \text{D‘}(P_\epsilon \mathbin{\dot\cap} I) \qquad (1)$

$\vdash . *211{\cdot}372{\cdot}14 . \supset$

$\vdash :. \text{Hp} . \supset : \alpha \in \text{D‘}(P_\epsilon \mathbin{\dot\cap} I) . \supset . (\exists x) . x \in C\text{‘}P . \alpha = \overrightarrow{P}\text{‘}x . \overrightarrow{P}\text{‘}x \in \text{D‘}(P_\epsilon \mathbin{\dot\cap} I) .$

$[*211{\cdot}452] \qquad \supset . (\exists x) . x \in C\text{‘}P . \alpha = \overrightarrow{P}\text{‘}x . x \sim\in \text{Ɑ‘}(P \mathbin{\dot-} P^2) .$

$[*37{\cdot}7] \qquad \supset . \alpha \in \overrightarrow{P}\text{‘‘}\{C\text{‘}P - \text{Ɑ‘}(P \mathbin{\dot-} P^2)\} \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

$*211{\cdot}47.\quad \vdash :. P \in \text{trans} . \supset : (\alpha) . \alpha \in \text{Ɑ‘max}_P \cup \text{Ɑ‘seq}_P . \equiv . \text{D‘}(P_\epsilon \mathbin{\dot\cap} I) \subset \text{Ɑ‘seq}_P$

Dem.

$\vdash . *211{\cdot}272 . *24{\cdot}43 . \supset$

$\vdash :. \text{Hp} . \supset : (\alpha) . \alpha \in \text{Ɑ‘max}_P \cup \text{Ɑ‘seq}_P . \equiv . \text{sect‘}P - \text{Ɑ‘max}_P \subset \text{Ɑ‘seq}_P .$

$[*211{\cdot}41] \qquad \equiv . \text{D‘}(P_\epsilon \mathbin{\dot\cap} I) \subset \text{Ɑ‘seq}_P :. \supset \vdash . \text{Prop}$

The following propositions are concerned with certain consequences of the hypothesis $P^2 = P$. This hypothesis is important because it is the defining characteristic of compact series.

$*211{\cdot}5.\quad \vdash : P^2 = P . \alpha = P\text{‘‘}\beta . \supset . \alpha = P\text{‘‘}\alpha$

Dem.

$\vdash . *37{\cdot}33 . \supset \vdash : \text{Hp} . \supset . P\text{‘‘}\beta = P\text{‘‘}P\text{‘‘}\beta .$

$[\text{Hp}.*13{\cdot}12] \qquad \supset . \alpha = P\text{‘‘}\alpha : \supset \vdash . \text{Prop}$

$*211{\cdot}51.\quad \vdash : P^2 = P . \supset . \text{D‘}P_\epsilon = \text{D‘}(P_\epsilon \mathbin{\dot\cap} I) \quad [*211{\cdot}5{\cdot}11{\cdot}12]$

Thus in compact series there is no distinction between the two sorts of segments.

$*211{\cdot}52.\quad \vdash :. P^2 = P . P \in \text{connex} . \supset : \text{E} ! \, \text{max}_P\text{‘}\alpha . \supset . \sim \text{E} ! \, \text{seq}_P\text{‘}\alpha$

Dem.

$\vdash . *206{\cdot}5 . \supset \vdash : P^2 \mathrel{\dot\subset} P . P \in \text{connex} . \text{E} ! \, \text{max}_P\text{‘}\alpha . \text{E} ! \, \text{seq}_P\text{‘}\alpha . \supset . \dot{\exists} ! (P \mathbin{\dot-} P^2) \quad (1)$

$\vdash . (1) . \text{Transp} . \supset \vdash : P^2 = P . P \in \text{connex} . \text{E} ! \, \text{max}_P\text{‘}\alpha . \supset . \sim \text{E} ! \, \text{seq}_P\text{‘}\alpha :$

$$\supset \vdash . \text{Prop}$$

$*211{\cdot}53.\quad \vdash :: P^2 = P . P \in \text{connex} . \supset :. \text{E} ! \, \text{max}_P\text{‘}\alpha . \vee . \text{E} ! \, \text{seq}_P\text{‘}\alpha : \equiv :$

$$\text{E} ! \, \text{max}_P\text{‘}\alpha . \equiv . \sim \text{E} ! \, \text{seq}_P\text{‘}\alpha$$

Dem.

$\vdash . *4{\cdot}64 . \supset \vdash :. \text{E} ! \, \text{max}_P\text{‘}\alpha . \vee . \text{E} ! \, \text{seq}_P\text{‘}\alpha : \equiv : \sim \text{E} ! \, \text{seq}_P\text{‘}\alpha . \supset . \text{E} ! \, \text{max}_P\text{‘}\alpha \quad (1)$

$\vdash . *4{\cdot}73 . *211{\cdot}52 . \supset$

$\vdash :: \text{Hp} . \supset :. \sim \text{E} ! \, \text{seq}_P\text{‘}\alpha . \supset . \text{E} ! \, \text{max}_P\text{‘}\alpha : \equiv : \text{E} ! \, \text{max}_P\text{‘}\alpha . \equiv . \sim \text{E} ! \, \text{seq}_P\text{‘}\alpha \quad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

The condition $(\alpha) : \text{E} ! \, \text{max}_P\text{‘}\alpha . \equiv . \sim \text{E} ! \, \text{seq}_P\text{‘}\alpha$ is the Dedekindian definition of continuity. In virtue of the above proposition, this is equivalent, in a series, to compactness combined with Dedekind's axiom, namely

$$(\alpha) : \text{E} ! \, \text{max}_P\text{‘}\alpha . \vee . \text{E} ! \, \text{seq}_P\text{‘}\alpha .$$

***211·54.** $\vdash :. P \mathrel{\ooalign{$\subset$\cr\hidewidth$\cdot$\hidewidth}} J : \exists ! \overrightarrow{\text{max}}_P{}^{\backprime}\alpha . \supset_\alpha . \sim\exists ! \overrightarrow{\text{seq}}_P{}^{\backprime}\alpha : \supset . P \mathrel{\ooalign{$\subset$\cr\hidewidth$\cdot$\hidewidth}} P^2$

Dem.

$\vdash . *10\cdot1 . \supset \vdash :. \text{Hp} . \supset : \exists ! \overrightarrow{\text{max}}_P{}^{\backprime}\iota^{\backprime}x . \supset . \sim\exists ! \text{seq}_P{}^{\backprime}\iota^{\backprime}x :$

$[*205\cdot18] \quad \supset : x \in C^{\backprime}P . \supset . \sim\exists ! \text{seq}_P{}^{\backprime}\iota^{\backprime}x .$

$[*206\cdot42] \quad \supset . \sim\exists ! \overleftarrow{P \dot- P^2}{}^{\backprime}x .$

$[*33\cdot4] \quad \supset . x \sim\in D^{\backprime}(P \dot- P^2) \quad (1)$

$\vdash . *33\cdot263 . \supset \vdash : x \sim\in C^{\backprime}P . \supset . x \sim\in D^{\backprime}(P \dot- P^2) \quad (2)$

$\vdash . (1) . (2) . \supset \vdash : \text{Hp} . \supset . D^{\backprime}(P \dot- P^2) = \Lambda .$

$[*33\cdot241 . *25\cdot3] \quad \supset . P \mathrel{\ooalign{$\subset$\cr\hidewidth$\cdot$\hidewidth}} P^2 : \supset \vdash . \text{Prop}$

***211·541.** $\vdash :. P \in \text{Rl}^{\backprime}J \cap \text{trans} : \exists ! \overrightarrow{\text{max}}_P{}^{\backprime}\alpha . \supset_\alpha . \sim\exists ! \overrightarrow{\text{seq}}_P{}^{\backprime}\alpha : \supset . P = P^2$

Dem.

$\vdash . *201\cdot1 . \supset \vdash : \text{Hp} . \supset . P^2 \mathrel{\ooalign{$\subset$\cr\hidewidth$\cdot$\hidewidth}} P \quad (1)$

$\vdash . *211\cdot54 . \supset \vdash : \text{Hp} . \supset . P \mathrel{\ooalign{$\subset$\cr\hidewidth$\cdot$\hidewidth}} P^2 \quad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***211·55.** $\vdash :: P \in \text{Ser} . \supset :. \exists ! \overrightarrow{\text{max}}_P{}^{\backprime}\alpha . \supset_\alpha . \sim\exists ! \overrightarrow{\text{seq}}_P{}^{\backprime}\alpha : \equiv . P = P^2$
[*211·52·541]

***211·551.** $\vdash :. P \in \text{Ser} . \supset : \text{ᗡ}^{\backprime}\text{max}_P \cap \text{ᗡ}^{\backprime}\text{seq}_P = \Lambda . \equiv . P = P^2$
[*211·55 . *33·41]

***211·552.** $\vdash :: P \in \text{Ser} . \supset :. E ! \text{max}_P{}^{\backprime}\alpha . \equiv_\alpha . \sim E ! \text{seq}_P{}^{\backprime}\alpha : \equiv :$
$P = P^2 : (\alpha) : E ! \text{max}_P{}^{\backprime}\alpha . \vee . E ! \text{seq}_P{}^{\backprime}\alpha$
[*211·55]

***211·553.** $\vdash :: P \in \text{Ser} . \supset :. \text{ᗡ}^{\backprime}\text{max}_P = - \text{ᗡ}^{\backprime}\text{seq}_P . \equiv :$
$P = P^2 : (\alpha) . \alpha \in \text{ᗡ}^{\backprime}\text{max}_P \cup \text{ᗡ}^{\backprime}\text{seq}_P$
[*211·552 . *71·163]

The following propositions are concerned in showing that $\text{sect}^{\backprime}P$, $D^{\backprime}P_\epsilon$, and $D^{\backprime}(P_\epsilon \dot\cap I)$ all verify the hypotheses of *210, if taken as the κ of that number.

***211·56.** $\vdash :. P \in \text{connex} . \alpha, \beta \in \text{sect}^{\backprime}P . \supset : \alpha \subset \beta . \vee . \beta \subset P^{\backprime\backprime}\alpha$

Dem.

$\vdash . *211\cdot2 . \supset \vdash :. \text{Hp} . \exists ! \alpha - \beta . \supset . \exists ! \alpha \cap C^{\backprime}P - \beta - P^{\backprime\backprime}\beta .$

$[*202\cdot501] \quad \supset . \exists ! \alpha \cap p^{\backprime}\overleftarrow{P}{}^{\backprime\backprime}\beta .$

$[*40\cdot682] \quad \supset . \beta \subset P^{\backprime\backprime}\alpha \quad (1)$

$\vdash . (1) . *24\cdot55 . \supset \vdash . \text{Prop}$

***211·561.** $\vdash :. P_{\text{po}} \in \text{connex} . \alpha, \beta \in \text{sect}^{\backprime}P . \supset : \alpha \subset \beta . \vee . \beta \subset P^{\backprime\backprime}\alpha$
[*211·56·17·131]

***211·562.** $\vdash :. P_{\text{po}} \in \text{connex} . \alpha, \beta \in \text{sect}^{\backprime}P . \supset : \alpha \subset \beta . \vee . \beta \subset \alpha$ [*211·561·1]

***211·6.** $\vdash :. P \in \text{connex} . \alpha, \beta \in \text{sect}^{\backprime}P . \supset : \alpha \subset \beta . \vee . \beta \subset \alpha$ [*211·56·1]

***211·61.** $\vdash :. P \in \text{trans} \cap \text{connex} . \alpha, \beta \in \text{D}`P_\epsilon . \supset : \alpha \subset \beta . \lor . \beta \subset \alpha$
[*211·15·6]

***211·62.** $\vdash :. P \in \text{connex} . \alpha, \beta \in \text{D}`(P_\epsilon \dot{\cap} I) . \supset : \alpha \subset \beta . \lor . \beta \subset \alpha$ [*211·14·6]

In the hypothesis of *211·61, it is necessary that P should be transitive as well as connected. Take, for example,

$$P = x \downarrow y \cup y \downarrow z \cup z \downarrow x \quad (x \neq y . x \neq z . y \neq z).$$

Then P is connected, but not transitive; also we have

$$\overrightarrow{P}`y = \iota`x . \overrightarrow{P}`z = \iota`y.$$

Hence $\qquad \iota`x, \iota`y \in \text{D}`P_\epsilon . \sim(\iota`x \subset \iota`y) . \sim(\iota`y \subset \iota`x).$

Thus connection is not sufficient in the hypothesis of *211·61.

***211·63.** $\vdash : \lambda \subset \text{sect}`P . \supset . s`\lambda \in \text{sect}`P$

Dem.

$\vdash . *211·1 . \supset \vdash :. \text{Hp} . \supset : \alpha \in \lambda . \supset_\alpha . \alpha \subset C`P :$
[*40·151] $\qquad \supset : s`\lambda \subset C`P \qquad (1)$
$\vdash . *211·1 . \supset \vdash :. \text{Hp} . \supset : \alpha \in \lambda . \supset_\alpha . P``\alpha \subset \alpha :$
[*40·8] $\qquad \supset : P``s`\lambda \subset s`\lambda \qquad (2)$
$\vdash . (1) . (2) . *211·1 . \supset \vdash . \text{Prop}$

This proposition shows that sect$`P$ verifies the hypothesis of \*210·251, with the exception of sect$`P \sim\in 1$, which requires $\dot{\exists} ! P$.

***211·631.** $\vdash : \lambda \subset \text{sect}`P . \supset . p`\lambda \cap C`P \in \text{sect}`P$

Dem.

$\vdash . *22·43 . \supset \vdash . p`\lambda \cap C`P \subset C`P \qquad (1)$
$\vdash . *211·1 . \supset \vdash :. \text{Hp} . \supset : \alpha \in \lambda . \supset_\alpha . P``\alpha \subset \alpha :$
[*40·81] $\qquad \supset : P``p`\lambda \subset p`\lambda :$
[*37·265·15] $\qquad \supset : P``(p`\lambda \cap C`P) \subset p`\lambda \cap C`P \qquad (2)$
$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***211·632.** $\vdash : \lambda \subset \text{sect}`P . \exists ! \lambda . \supset . p`\lambda \in \text{sect}`P$

Dem.

$\vdash . *40·23 . \supset \vdash : \text{Hp} . \supset . p`\lambda \subset s`\lambda .$
[*211·63·1] $\qquad \supset . p`\lambda \subset C`P \qquad (1)$
$\vdash . (1) . *211·631 . \supset \vdash . \text{Prop}$

***211·633.** $\vdash : \lambda \subset \text{sect}`P . \supset . p`\lambda \cap s`\text{sect}`P \in \text{sect}`P$ [*211·631·26]

This proposition shows that sect$`P$ verifies the hypothesis of \*210·252, with the exception of sect$`P \sim\in 1$, which requires $\dot{\exists} ! P$.

***211·64.** $\vdash : \lambda \subset \text{D}`P_\epsilon . \supset . s`\lambda \in \text{D}`P_\epsilon$

Dem.

$\vdash . *72·504 . \supset \vdash : \text{Hp} . \supset . s`\lambda = s`P_\epsilon``\breve{P}_\epsilon``\lambda$
[*40·38] $\qquad = P``s`\breve{P}_\epsilon``\lambda \qquad (1)$
$\vdash . (1) . *211·11 . \supset \vdash . \text{Prop}$

$*211{\cdot}65.\quad \vdash : \lambda \subset \mathrm{D}`(P_\epsilon \dot{\frown} I) . \supset . s`\lambda \in \mathrm{D}`(P_\epsilon \dot{\frown} I)$

Dem.

$\vdash . *211{\cdot}12 . \supset \vdash :. \mathrm{Hp} . \supset : \alpha \in \lambda . \supset_\alpha . \alpha = P_\epsilon`\alpha :$

$[*50{\cdot}17] \quad \supset : \lambda = P_\epsilon``\lambda :$

$[*40{\cdot}38] \quad \supset : s`\lambda = P``s`\lambda :$

$[*211{\cdot}12] \quad \supset : s`\lambda \in \mathrm{D}`(P_\epsilon \dot{\frown} I) :. \supset \vdash . \mathrm{Prop}$

$*211{\cdot}66.\quad \vdash : \dot{\exists} ! P . \supset . \mathrm{sect}`P, \mathrm{D}`P_\epsilon \sim\in 1$

Dem.

$\vdash . *211{\cdot}44{\cdot}26 . \quad \supset \vdash . \Lambda, C`P \in \mathrm{sect}`P \qquad (1)$

$\vdash . *33{\cdot}24 . \quad \supset \vdash : \mathrm{Hp} . \supset . \Lambda \neq C`P \qquad (2)$

$\vdash . (1) . (2) . *52{\cdot}41 . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{sect}`P \sim\in 1 \qquad (3)$

$\vdash . *211{\cdot}44{\cdot}301 . \quad \supset \vdash : \mathrm{Hp} . \supset . \Lambda, \mathrm{D}`P \in \mathrm{D}`P_\epsilon \qquad (4)$

$\vdash . *33{\cdot}24 . \quad \supset \vdash : \mathrm{Hp} . \supset . \Lambda \neq \mathrm{D}`P \qquad (5)$

$\vdash . (4) . (5) . *52{\cdot}41 . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{D}`P_\epsilon \sim\in 1 \qquad (6)$

$\vdash . (3) . (6) . \supset \vdash . \mathrm{Prop}$

$*211{\cdot}661.\quad \vdash : P \in \mathrm{trans} . \exists ! \mathrm{Cl\,ex}`C`P - \mathrm{Cl}`\mathrm{max}_P . \supset . \mathrm{D}`(P_\epsilon \dot{\frown} I) \sim\in 1$

Dem.

$\vdash . *205{\cdot}111 . \supset \vdash : \alpha \in \mathrm{Cl\,ex}`C`P - \mathrm{Cl}`\mathrm{max}_P . \supset . \exists ! \alpha . \alpha \subset C`P . \alpha \subset P``\alpha .$

$[*24{\cdot}58 . *37{\cdot}2] \quad \supset . \exists ! P``\alpha . P``\alpha \subset P``P``\alpha \qquad (1)$

$\vdash . (1) . *201{\cdot}5 . \supset$

$\vdash :. P \in \mathrm{trans} . \supset : \alpha \in \mathrm{Cl\,ex}`C`P - \mathrm{Cl}`\mathrm{max}_P . \supset . P``\alpha = P``P``\alpha . \exists ! P``\alpha .$

$[*211{\cdot}12] \quad \supset . P``\alpha \in \mathrm{D}`(P_\epsilon \dot{\frown} I) . \exists ! P``\alpha .$

$[*10{\cdot}24] \quad \supset . \exists ! \mathrm{D}`(P_\epsilon \dot{\frown} I) - \iota`\Lambda \qquad (2)$

$\vdash . (2) . *211{\cdot}44 . \supset \vdash . \mathrm{Prop}$

The following propositions sum up the above results in relation to the hypotheses of $*210$. The relation P_{1c} with its field limited to sections or segments, which occurs in the following propositions, is important, and will be considered at length in the following number.

$*211{\cdot}67.\quad \vdash : P \in \mathrm{connex} . \kappa = \mathrm{sect}`P . Q = P_{1c} \upharpoonright \kappa . \supset . \mathrm{Hp}\,*210{\cdot}12$
$[*211{\cdot}6 . *210{\cdot}13]$

$*211{\cdot}671.\quad \vdash : P \in \mathrm{connex} . \kappa = \mathrm{sect}`P . Q = P_{1c} \upharpoonright \kappa . \dot{\exists} ! P . \supset .$
$\mathrm{Hp}\,*210{\cdot}251 . \mathrm{Hp}\,*210{\cdot}252 \quad [*211{\cdot}67{\cdot}66{\cdot}63{\cdot}633]$

$*211{\cdot}68.\quad \vdash : P \in \mathrm{trans} \cap \mathrm{connex} . \kappa = \mathrm{D}`P_\epsilon . Q = P_{1c} \upharpoonright \kappa . \supset . \mathrm{Hp}\,*210{\cdot}12$
$[*211{\cdot}61 . *210{\cdot}13]$

$*211{\cdot}681.\quad \vdash : P \in \mathrm{trans} \cap \mathrm{connex} . \kappa = \mathrm{D}`P_\epsilon . Q = P_{1c} \upharpoonright \kappa . \dot{\exists} ! P . \supset . \mathrm{Hp}\,*210{\cdot}251$
$[*211{\cdot}68{\cdot}66{\cdot}64]$

$*211{\cdot}69.\quad \vdash : P \in \mathrm{connex} . \kappa = \mathrm{D}`(P_\epsilon \dot{\frown} I) . Q = P_{1c} \upharpoonright \kappa . \supset . \mathrm{Hp}\,*210{\cdot}12$
$[*211{\cdot}62 . *210{\cdot}13]$

$*211{\cdot}691.\quad \vdash : P \in \mathrm{connex} . \kappa = \mathrm{D}`(P_\epsilon \dot{\frown} I) . Q = P_{1c} \upharpoonright \kappa . \mathrm{D}`(P_\epsilon \dot{\frown} I) \sim\in 1 . \supset .$
$\mathrm{Hp}\,*210{\cdot}251 \quad [*211{\cdot}69{\cdot}65]$

***211·692.** $\vdash : P \epsilon \text{trans} \cap \text{connex} \,.\, \kappa = \text{D}'(P_\epsilon \dot{\cap} I) \,.\, Q = P_{\text{lc}} \upharpoonright \kappa \,.$
$\exists ! \,\text{Cl}\,\text{ex}'C'P - \text{Cl}'\text{max}_P \,.\supset .\, \text{Hp}\,*210{\cdot}251$ [*211·691·661]

The following propositions are concerned with the relations of sections and segments of P to sections and segments of $\breve{P}$. When $\alpha \epsilon \text{sect}'P$, $C'P - \alpha \epsilon \text{sect}'\breve{P}$, and vice versa. Also, if P is connected, the maximum of α (if any) is the precedent with respect to P (*i.e.* the sequent with respect to $\breve{P}$) of $C'P - \alpha$, and the sequent of α (if any) is the minimum with respect to P (*i.e.* the maximum with respect to $\breve{P}$) of $C'P - \alpha$. Hence the relations to be proved follow easily.

***211·7.** $\vdash : \alpha \epsilon \text{sect}'P \,.\supset .\, C'P - \alpha \epsilon \text{sect}'\breve{P}$

Dem.

$\vdash .\, *22{\cdot}43 \,. \quad \supset \vdash .\, C'P - \alpha \subset C'P$ (1)

$\vdash .\, *211{\cdot}1 \,.\, *37{\cdot}1 \,.\supset \vdash :.\, \text{Hp} \,.\supset : x \epsilon \alpha \,.\, yPx \,.\supset .\, y \epsilon \alpha :$

[Transp] $\supset : x \epsilon \alpha \,.\, y \sim\epsilon \alpha \,.\supset .\, \sim(yPx) :$

[*37·1.Transp] $\supset : x \epsilon \alpha \,.\supset .\, x \sim\epsilon \breve{P}``(-\alpha) :$

[*37·265] $\supset : \alpha \subset - \breve{P}``(C'P - \alpha) :$

[Transp] $\supset : \breve{P}``(C'P - \alpha) \subset - \alpha :$

[*37·15] $\supset : \breve{P}``(C'P - \alpha) \subset C'P - \alpha$ (2)

$\vdash .\, (1) \,.\, (2) \,.\, *211{\cdot}1 \,.\supset \vdash .\, \text{Prop}$

***211·701.** $\vdash : \alpha \epsilon \text{sect}'P \,.\, \text{E}!\, \text{max}_P{}'\alpha \,.\supset .\, p'\overleftarrow{P}``\alpha \subset \overleftarrow{P}'\text{max}_P{}'\alpha \subset C'P - \alpha$

Dem.

$\vdash .\, *40{\cdot}12 \,. \quad \supset \vdash : \text{Hp} \,.\supset .\, p'\overleftarrow{P}``\alpha \subset \overleftarrow{P}'\text{max}_P{}'\alpha$ (1)

$\vdash .\, *205{\cdot}101 \,.\supset \vdash : \text{Hp} \,.\supset .\, \text{max}_P{}'\alpha \sim\epsilon P``\alpha \,.$

[*37·1.Transp.*32·181] $\supset .\, \overleftarrow{P}'\text{max}_P{}'\alpha \subset - \alpha \,.$

[*33·152] $\supset .\, \overleftarrow{P}'\text{max}_P{}'\alpha \subset C'P - \alpha$ (2)

$\vdash .\, (1) \,.\, (2) \,.\supset \vdash .\, \text{Prop}$

***211·702.** $\vdash : P \epsilon \text{connex} \,.\, \alpha \epsilon \text{sect}'P \,.\supset .\, C'P - \alpha \subset p'\overleftarrow{P}``\alpha$ [*202·501 . *211·1]

***211·703.** $\vdash : P \epsilon \text{connex} \,.\, \alpha \epsilon \text{sect}'P - \iota'C'P \,.\supset .\, \exists !\, p'\overleftarrow{P}``\alpha$
[*211·702·1 . *24·58]

***211·71.** $\vdash : P \epsilon \text{connex} \,.\, \alpha \epsilon \text{sect}'P \,.\, \text{E}!\, \text{max}_P{}'\alpha \,.\supset .$

$$p'\overleftarrow{P}``\alpha = \overleftarrow{P}'\text{max}_P{}'\alpha = C'P - \alpha$$

Dem.

$\vdash .\, *202{\cdot}501 \,.\, *211{\cdot}2 \,.\supset \vdash : \text{Hp} \,.\supset .\, C'P - \alpha \subset p'\overleftarrow{P}``\alpha$ (1)

$\vdash .\, (1) \,.\, *211{\cdot}701 \,. \quad \supset \vdash : \text{Hp} \,.\supset .\, C'P - \alpha = p'\overleftarrow{P}``\alpha$ (2)

$\vdash .\, (2) \,.\, *211{\cdot}701 \,. \quad \supset \vdash : \text{Hp} \,.\supset .\, \overleftarrow{P}'\text{max}_P{}'\alpha \subset p'\overleftarrow{P}``\alpha \,.$

[*211·701] $\supset .\, \overleftarrow{P}'\text{max}_P{}'\alpha = p'\overleftarrow{P}``\alpha$ (3)

$\vdash .\, (2) \,.\, (3) \,.\supset \vdash .\, \text{Prop}$

If α is a section of P, we shall call $C'P - \alpha$ the *complement* of α. By the above proposition, if α is a section of P having a maximum, its complement is a section of $\breve{P}$ which is a member of $\overleftarrow{P}``C'P$.

***211·711.** $\vdash : P \in \text{connex} . P^2 \subseteq J . \alpha \in \text{sect}'P . \supset .$

$$\alpha = C'P - p'\overleftarrow{P}``\alpha . C'P \cap p'\overleftarrow{P}``\alpha = C'P - \alpha \quad [*202{\cdot}503 . *211{\cdot}2]$$

***211·712.** $\vdash : P \in \text{connex} . \alpha \in \text{sect}'P . E ! \min_P{}'(C'P - \alpha) . \supset . \alpha = \overrightarrow{P}'\min_P{}'(C'P - \alpha)$

Dem.

$\vdash . *211{\cdot}71 \dfrac{\breve{P}}{P} . \supset$

$\vdash : P \in \text{connex} . \beta \in \text{sect}'\breve{P} . E ! \min_P{}'\beta . \supset . \overrightarrow{P}'\min_P{}'\beta = C'P - \beta \quad (1)$

$\vdash . *211{\cdot}7 . *24{\cdot}492 . \supset \vdash : \alpha \in \text{sect}'P . \beta = C'P - \alpha . \supset . \beta \in \text{sect}'\breve{P} . \alpha = C'P - \beta \quad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***211·713.** $\vdash : P \in \text{connex} . \alpha \in \text{sect}'P - D'P . \supset . E ! \max_P{}'\alpha . \sim E ! \min_P{}'(C'P - \alpha)$

Dem.

$\vdash . *211{\cdot}24 . \text{Transp} . \supset \vdash : \text{Hp} . \supset . E ! \max_P{}'\alpha \quad (1)$

$\vdash . *211{\cdot}712{\cdot}3 . \supset \vdash : P \in \text{connex} . \alpha \in \text{sect}'P . E ! \min_P{}'(C'P - \alpha) . \supset . \alpha \in D'P_\epsilon \quad (2)$

$\vdash . (2) . \text{Transp} . \quad \supset \vdash : \text{Hp} . \supset . \sim E ! \min_P{}'(C'P - \alpha) \quad (3)$

$\vdash . (1) . (3) . \supset \vdash . \text{Prop}$

***211·714.** $\vdash : P \in \text{connex} . \alpha \in \text{sect}'P . \supset . \overrightarrow{\text{seq}}_P{}'\alpha \subset \overrightarrow{\min}_P{}'(C'P - \alpha)$

Dem.

$\vdash . *206{\cdot}18{\cdot}2 . \supset \vdash : x \in \overrightarrow{\text{seq}}_P{}'\alpha . \supset . x \in C'P - \alpha \quad (1)$

$\vdash . *206{\cdot}134 . \supset \vdash : x \in \overrightarrow{\text{seq}}_P{}'\alpha . \supset . \overrightarrow{P}'x \subset C'P - p'\overleftarrow{P}``\alpha \quad (2)$

$\vdash . (2) . *202{\cdot}501 . *211{\cdot}2 . \supset$

$\vdash :. \text{Hp} . \supset : x \in \overrightarrow{\text{seq}}_P{}'\alpha . \supset . \overrightarrow{P}'x \subset \alpha .$

$[*37{\cdot}462] \quad \supset . x \sim\in \breve{P}``(C'P - \alpha) \quad (3)$

$\vdash . (1) . (3) . *205{\cdot}11 . \supset \vdash . \text{Prop}$

The above hypothesis is not sufficient to secure $\overrightarrow{\text{seq}}_P{}'\alpha = \overrightarrow{\min}_P{}'(C'P - \alpha)$, as may be seen by putting

$$P = \alpha \uparrow (\alpha \cup \iota'x), \text{ where } \exists ! \alpha . x \sim\in \alpha.$$

We then have $P \in \text{connex} . P``\alpha = \alpha . C'P - \alpha = \iota'x . p'\overleftarrow{P}``\alpha = \alpha \cup \iota'x$. Thus $\overrightarrow{\min}_P{}'(C'P - \alpha) = \iota'x . \overrightarrow{\text{seq}}_P{}'\alpha = \Lambda$. It will be seen that $\alpha \uparrow (\alpha \cup \iota'x) \in \text{trans}$, so that it is useless to add $P \in \text{trans}$ to the hypothesis of *211·714. A sufficient addition is $P \subseteq J$, as is proved in the following proposition.

***211·715.** $\vdash : P \in \text{connex} \cap \text{Rl}'J . \alpha \in \text{sect}'P . \supset . \overrightarrow{\text{seq}}_P{}'\alpha = \overrightarrow{\min}_P{}'(C'P - \alpha)$

Dem.

$\vdash . *205{\cdot}14 . \supset \vdash : x \min_P (C'P - \alpha) . \supset . x \in C'P - \alpha . \overrightarrow{P}'x \cap (C'P - \alpha) = \Lambda \quad (1)$

$\vdash . (1) . *33\cdot152 . *211\cdot2 . \supset$

$\vdash :. \text{Hp} . \supset : x \min_P (C'P - \alpha) . \supset . x \in C'P - \alpha - P``\alpha . \overrightarrow{P}'x \subset \alpha .$

[*202·501] $\supset . x \in C'P \cap p'\overleftarrow{P}``\alpha . \overrightarrow{P}'x \subset \alpha .$

[*200·5] $\supset . x \in C'P \cap p'\overleftarrow{P}``\alpha . \overrightarrow{P}'x \subset - p'\overleftarrow{P}``\alpha .$

[*37·1.Transp] $\supset . x \in C'P \cap p'\overleftarrow{P}``\alpha - \breve{P}``p'\overleftarrow{P}``\alpha .$

[*206·11] $\supset . x \,\text{seq}_P\, \alpha$ (2)

$\vdash . (2) . *211\cdot714 . \supset \vdash . \text{Prop}$

***211·72.** $\vdash : P \in \text{connex} . \alpha \in \text{sect}'P - \text{D}'P_\epsilon . \supset .$

$$C'P - \alpha = \breve{P}``(C'P - \alpha) . C'P - \alpha \in \text{D}'\{(\breve{P})_\epsilon \dot{\cap} I\} \quad [*211\cdot21\cdot7\cdot713]$$

***211·721.** $\vdash : P \in \text{connex} . \alpha \in \text{sect}'P \cap (\text{Cl}'\text{max}_P \cup \text{Cl}'\text{seq}_P) . \supset .$

$$\overrightarrow{\text{seq}}_P{}'\alpha = \overrightarrow{\text{min}}_P{}'(C'P - \alpha)$$

Dem.

$\vdash . *211\cdot71 . \supset \vdash : P \in \text{connex} . \alpha \in \text{sect}'P \cap \text{Cl}'\text{max}_P . \supset . p'\overleftarrow{P}``\alpha = C'P - \alpha .$

[*206·13] $\supset . \overrightarrow{\text{seq}}_P{}'\alpha = \overrightarrow{\text{min}}_P{}'(C'P - \alpha)$ (1)

$\vdash . *211\cdot714 . \supset$

$\vdash :. P \in \text{connex} . \alpha \in \text{sect}'P . \supset : \overrightarrow{\text{seq}}_P{}'\alpha \subset \overrightarrow{\text{min}}_P{}'(C'P - \alpha) :$

[*205·3.*206·16] $\supset : \exists ! \overrightarrow{\text{seq}}_P{}'\alpha . \supset . \overrightarrow{\text{seq}}_P{}'\alpha = \overrightarrow{\text{min}}_P{}'(C'P - \alpha)$ (2)

$\vdash . (2) . \supset \vdash : P \in \text{connex} . \alpha \in \text{sect}'P \cap \text{Cl}'\text{seq}_P . \supset . \overrightarrow{\text{seq}}_P{}'\alpha = \overrightarrow{\text{min}}_P{}'(C'P - \alpha)$ (3)

$\vdash . (1) . (3) . \supset \vdash . \text{Prop}$

***211·722.** $\vdash : P \in \text{connex} . \alpha \in \text{sect}'P . \text{E} ! \text{max}_P{}'\alpha . \text{E} ! \text{seq}_P{}'\alpha . \supset .$

$$\text{max}_P{}'\alpha = \text{prec}_P{}'(C'P - \alpha)$$

Dem.

$\vdash . *211\cdot721\cdot7 . \supset \vdash : \text{Hp} . \supset . C'P - \alpha \in \text{sect}'\breve{P} . \text{E} ! \text{min}_P{}'(C'P - \alpha) .$

$\left[*211\cdot721 \dfrac{\breve{P}, C'P - \alpha}{P, \quad \alpha}\right] \supset . \overrightarrow{\text{prec}}_P{}'(C'P - \alpha) = \overrightarrow{\text{max}}_P{}'\{C'P - (C'P - \alpha)\}$

[*24·492] $= \overrightarrow{\text{max}}_P{}'\alpha$

[Hp] $= \iota'\text{max}_P{}'\alpha : \supset \vdash . \text{Prop}$

We have always, if $P \in \text{connex} . \alpha \in \text{sect}'P$,

$$\overrightarrow{\text{prec}}_P{}'(C'P - \alpha) \subset \overrightarrow{\text{max}}_P{}'\alpha.$$

The converse inclusion does not always hold, as appears (on writing $\breve{P}$ in place of P) from the note to *211·714. To secure the converse implication, it is sufficient to assume $P \Subset J$ or $\text{E} ! \text{seq}_P{}'\alpha$ or $\sim \text{E} ! \text{max}_P{}'\alpha$.

***211·723.** $\vdash : P \in \text{connex} . \alpha \in \text{sect}'P . \supset . \overrightarrow{\text{prec}}_P{}'(C'P - \alpha) \subset \overrightarrow{\text{max}}_P{}'\alpha$

Dem.

$\vdash . *202\cdot11 . *211\cdot7 . \supset \vdash : \text{Hp} . \supset . \breve{P} \in \text{connex} . C'P - \alpha \in \text{sect}'\breve{P} .$

$\left[*211\cdot714 \dfrac{\breve{P}}{P} . *205\cdot102 . *206\cdot101\right] \supset . \overrightarrow{\text{prec}}_P{}'(C'P - \alpha) \subset \overrightarrow{\text{max}}_P{}'\alpha : \supset \vdash . \text{Prop}$

∗211·724. $\vdash : P \in \text{connex} \,.\, \alpha \in \text{sect}`P \cap (\text{Ꭰ}`\text{seq}_P \cup -\text{Ꭰ}`\text{max}_P) \,.\, \supset \,.$
$$\overrightarrow{\text{max}}_P`\alpha = \overrightarrow{\text{prec}}_P`(C`P - \alpha)$$

Dem.

$\vdash \,.\, ∗211·722 \,.\, \supset \vdash : P \in \text{connex} \,.\, \alpha \in \text{sect}`P \cap \text{Ꭰ}`\text{seq}_P \cap \text{Ꭰ}`\text{max}_P \,.\, \supset \,.$
$$\overrightarrow{\text{max}}_P`\alpha = \overrightarrow{\text{prec}}_P`(C`P - \alpha) \quad (1)$$

$\vdash \,.\, ∗211·723 \,.\, ∗24·13 \,.\, \supset \vdash : P \in \text{connex} \,.\, \alpha \in \text{sect}`P - \text{Ꭰ}`\text{max}_P \,.\, \supset \,.$
$$\overrightarrow{\text{max}}_P`\alpha = \overrightarrow{\text{prec}}_P`(C`P - \alpha) \quad (2)$$

$\vdash \,.\, (1) \,.\, (2) \,.\, ∗22·91 \,.\, \supset \vdash \,.\, \text{Prop}$

∗211·725. $\vdash : P \in \text{connex} \,.\, \alpha \in \text{sect}`P \cap \text{Ꭰ}`\text{seq}_P \,.\, \supset \,.$
$$\overrightarrow{\text{max}}_P`\alpha = \overrightarrow{\text{prec}}_P`(C`P - \alpha) \,.\, \overrightarrow{\text{seq}}_P`\alpha = \overrightarrow{\text{min}}_P`(C`P - \alpha) \quad [∗211·721·724]$$

∗211·726. $\vdash : P \in \text{connex} \cap \text{Rl}`J \,.\, \alpha \in \text{sect}`P \,.\, \supset \,.$
$$\overrightarrow{\text{max}}_P`\alpha = \overrightarrow{\text{prec}}_P`(C`P - \alpha) \,.\, \overrightarrow{\text{seq}}_P`\alpha = \overrightarrow{\text{min}}_P`(C`P - \alpha)$$

Dem.

$\vdash \,.\, ∗200·11 \,.\, ∗202·11 \,.\, ∗211·7 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, \breve{P} \in \text{connex} \cap \text{Rl}`J \,.\, C`P - \alpha \in \text{sect}`\breve{P} \,.$

$[∗211·715 \,.\, ∗205·102 \,.\, ∗206·101] \quad \supset \,.\, \overrightarrow{\text{prec}}_P`(C`P - \alpha) = \overrightarrow{\text{max}}_P`\alpha \quad (1)$

$\vdash \,.\, (1) \,.\, ∗211·715 \,.\, \supset \vdash \,.\, \text{Prop}$

∗211·727. $\vdash :. P \in \text{connex} \cap \text{Rl}`J \,.\, \alpha \in \text{sect}`P \,.\, \supset :$
$$E \,!\, \text{limax}_P`\alpha \,.\, \equiv \,.\, E \,!\, \text{limin}_P`(C`P - \alpha) \quad [∗211·726 \,.\, ∗207·44]$$

∗211·728. $\vdash :. P \in \text{connex} \cap \text{Rl}`J \,.\, \alpha \in \text{sect}`P : \sim E \,!\, \text{max}_P`\alpha \,.\, \vee \,.$
$$\sim E \,!\, \text{min}_P`(C`P - \alpha) : \supset \,.\, \overrightarrow{\text{limax}}_P`\alpha = \overrightarrow{\text{limin}}_P`(C`P - \alpha)$$

Dem.

$\vdash \,.\, ∗211·726 \,.\, ∗207·43·12 \,.\, \supset \vdash : \text{Hp} \,.\, \sim E \,!\, \text{max}_P`\alpha \,.\, \supset \,.$
$$\overrightarrow{\text{limax}}_P`\alpha = \overrightarrow{\text{min}}_P`(C`P - \alpha)$$

$[∗207·46 \,.\, ∗211·726] \quad = \overrightarrow{\text{limin}}_P`(C`P - \alpha) \quad (1)$

Similarly $\vdash : \text{Hp} \,.\, \sim E \,!\, \text{min}_P`(C`P - \alpha) \,.\, \supset \,.\, \overrightarrow{\text{limax}}_P`\alpha = \overrightarrow{\text{limin}}_P`(C`P - \alpha) \quad (2)$

$\vdash \,.\, (1) \,.\, (2) \,.\, \supset \vdash \,.\, \text{Prop}$

∗211·729. $\vdash : P \in \text{connex} \cap \text{Rl}`J \,.\, \alpha \in \text{sect}`P - (\text{Ꭰ}`\text{max}_P \cap \text{Ꭰ}`\text{seq}_P) \,.\, \supset \,.$
$$\overrightarrow{\text{limax}}_P`\alpha = \overrightarrow{\text{limin}}_P`(C`P - \alpha) \quad [∗211·728·726]$$

∗211·73. $\vdash : P \in \text{connex} \,.\, \alpha \in \text{sect}`P - \text{D}`(P_\epsilon \dot{\cap} I) \,.\, \supset \,.$
$$C`P - \alpha \in \text{D}`\{(\breve{P})_\epsilon \dot{\cap} I\} - \text{Ꭰ}`\text{prec}_P \cup \{\text{sect}`P - \text{D}`(\breve{P})_\epsilon\}$$

Dem.

$\vdash \,.\, ∗211·21 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, \alpha \in \text{sect}`P - \text{Ꭰ}`\text{max}_P \,.$

$[∗211·7·723] \quad \supset \,.\, C`P - \alpha \in \text{sect}`\breve{P} - \text{Ꭰ}`\text{prec}_P \,.$

$[∗24·41] \quad \supset \,.\, C`P - \alpha \in (\text{sect}`\breve{P} - \text{Ꭰ}`\text{max}_P - \text{Ꭰ}`\text{prec}_P) \cup$
$(\text{sect}`\breve{P} \cap \text{Ꭰ}`\text{max}_P - \text{Ꭰ}`\text{prec}_P) \,.$

$[∗211·31·21] \quad \supset \,.\, C`P - \alpha \in \{\text{D}`(\breve{P})_\epsilon \dot{\cap} I\} - \text{Ꭰ}`\text{prec}_P \cup \{\text{sect}`P - \text{D}`(\breve{P})_\epsilon\} :$
$\supset \vdash \,.\, \text{Prop}$

***211·74.** $\vdash : P \in \text{trans} \cap \text{connex} \,.\, \alpha \in \mathrm{D}\text{‘}P_\epsilon - \mathrm{D}\text{‘}(P_\epsilon \dot{\cap} I) \,.\supset .$

$$C\text{‘}P - \alpha \in \mathrm{D}\text{‘}(\breve{P})_\epsilon - \mathrm{D}\text{‘}\{(\breve{P})_\epsilon \dot{\cap} I\}$$

Dem.

$\vdash . *211{\cdot}431 . \supset \vdash : \text{Hp} . \supset . \alpha \in \text{sect}\text{‘}P \cap \text{ᗡ}\text{‘}\text{max}_P \cap \text{ᗡ}\text{‘}\text{seq}_P .$

$[*211{\cdot}7{\cdot}725] \qquad \supset . C\text{‘}P - \alpha \in \text{sect}\text{‘}\breve{P} \cap \text{ᗡ}\text{‘}\text{prec}_P \cap \text{ᗡ}\text{‘}\text{min}_P .$

$\left[*211{\cdot}431 \dfrac{\breve{P}}{P}\right] \qquad \supset . C\text{‘}P - \alpha \in \mathrm{D}\text{‘}(\breve{P})_\epsilon - \mathrm{D}\text{‘}\{(\breve{P})_\epsilon \dot{\cap} I\} : \supset \vdash . \text{Prop}$

The following propositions sum up our previous results.

***211·75.** $\vdash :. \alpha \subset C\text{‘}P \,.\, Q = \breve{P} \,.\supset: \alpha \in \text{sect}\text{‘}P \,.\equiv.\, C\text{‘}P - \alpha \in \text{sect}\text{‘}Q \quad [*211{\cdot}7]$

***211·751.** $\vdash :. P \in \text{Ser} \,.\, \alpha \subset C\text{‘}P \,.\, Q = \breve{P} \,.\supset:$

$$\alpha \in \mathrm{D}\text{‘}P_\epsilon \,.\equiv.\, C\text{‘}P - \alpha \in \text{sect}\text{‘}Q \cap (\text{ᗡ}\text{‘}\text{max}_Q \cup - \text{ᗡ}\text{‘}\text{seq}_Q)$$

Dem.

$\vdash . *211{\cdot}32 . \supset \vdash :. \text{Hp} . \supset : \alpha \in \mathrm{D}\text{‘}P_\epsilon \,.\equiv.\, \alpha \in \text{sect}\text{‘}P \cap (\text{ᗡ}\text{‘}\text{seq}_P \cup - \text{ᗡ}\text{‘}\text{max}_P) .$

$[*211{\cdot}75{\cdot}726] \qquad \equiv . C\text{‘}P - \alpha \in \text{sect}\text{‘}Q \cap (\text{ᗡ}\text{‘}\text{max}_Q \cup - \text{ᗡ}\text{‘}\text{seq}_Q) :. \supset \vdash . \text{Prop}$

In the above proposition, "$P \in \text{trans}$" is necessary in order that $\mathrm{D}\text{‘}P_\epsilon$ may be contained in $\text{sect}\text{‘}P$, and "$P \in \text{Rl}\text{‘}J$" is necessary in order that "$(C\text{‘}P - \alpha) \sim\in \text{ᗡ}\text{‘}\text{seq}_Q$" may imply "$\alpha \sim\in \text{ᗡ}\text{‘}\text{max}_P$." Hence the full hypothesis "$P \in \text{Ser}$" becomes necessary.

***211·752.** $\vdash :. P \in \text{connex} \,.\, \alpha \subset C\text{‘}P \,.\, Q = \breve{P} \,.\supset:$

$$\alpha \in \mathrm{D}\text{‘}(P_\epsilon \dot{\cap} I) \,.\supset.\, C\text{‘}P - \alpha \in \text{sect}\text{‘}Q - \text{ᗡ}\text{‘}\text{seq}_Q$$

Dem.

$\vdash . *211{\cdot}41 . \qquad \supset \vdash : \alpha \in \mathrm{D}\text{‘}(P_\epsilon \dot{\cap} I) \,.\equiv.\, \alpha \in \text{sect}\text{‘}P - \text{ᗡ}\text{‘}\text{max}_P \qquad (1)$

$\vdash . (1) . *211{\cdot}7{\cdot}723 . \supset \vdash : \text{Hp} \,.\, \alpha \in \mathrm{D}\text{‘}(P_\epsilon \dot{\cap} I) \,.\supset .$

$$C\text{‘}P - \alpha \in \text{sect}\text{‘}Q - \text{ᗡ}\text{‘}\text{seq}_Q : \supset \vdash . \text{Prop}$$

***211·753.** $\vdash :. P \in \text{Rl}\text{‘}J \cap \text{connex} \,.\, \alpha \subset C\text{‘}P \,.\, Q = \breve{P} \,.\supset:$

$$\alpha \in \mathrm{D}\text{‘}(P_\epsilon \dot{\cap} I) \,.\equiv.\, C\text{‘}P - \alpha \in \text{sect}\text{‘}Q - \text{ᗡ}\text{‘}\text{seq}_Q \quad [*211{\cdot}41{\cdot}7{\cdot}726]$$

***211·754.** $\vdash :. P \in \text{Rl}\text{‘}J \cap \text{connex} \,.\, \alpha \subset C\text{‘}P \,.\, Q = \breve{P} \,.\supset:$

$$\alpha \in \text{sect}\text{‘}P - \mathrm{D}\text{‘}P_\epsilon \,.\equiv.\, C\text{‘}P - \alpha \in \mathrm{D}\text{‘}(Q_\epsilon \dot{\cap} I) \cap \text{ᗡ}\text{‘}\text{seq}_Q$$

Dem.

$\vdash . *211{\cdot}316 . \supset$

$\vdash :. \text{Hp} . \supset : \alpha \in \text{sect}\text{‘}P - \mathrm{D}\text{‘}P_\epsilon \,.\equiv.\, \alpha \in \text{sect}\text{‘}P \cap (\text{ᗡ}\text{‘}\text{max}_P \cap - \text{ᗡ}\text{‘}\text{seq}_P) .$

$[*211{\cdot}7{\cdot}726] \qquad \equiv . C\text{‘}P - \alpha \in \text{sect}\text{‘}Q \cap (\text{ᗡ}\text{‘}\text{seq}_Q \cap - \text{ᗡ}\text{‘}\text{max}_Q) .$

$[*211{\cdot}41] \qquad \equiv . C\text{‘}P - \alpha \in \mathrm{D}\text{‘}(Q_\epsilon \dot{\cap} I) \cap \text{ᗡ}\text{‘}\text{seq}_Q :. \supset \vdash . \text{Prop}$

***211·755.** $\vdash :. P \in \text{trans} \cap \text{connex} \,.\, \alpha \subset C\text{‘}P \,.\, Q = \breve{P} \,.\supset:$

$$\alpha \in \mathrm{D}\text{‘}P_\epsilon - \mathrm{D}\text{‘}(P_\epsilon \dot{\cap} I) \,.\equiv.\, C\text{‘}P - \alpha \in \mathrm{D}\text{‘}Q_\epsilon - \mathrm{D}\text{‘}(Q_\epsilon \dot{\cap} I) \quad [*211{\cdot}74]$$

***211·756.** $\vdash :. P \in \text{Rl}\text{‘}J \cap \text{connex} \,.\, \alpha \subset C\text{‘}P \,.\, Q = \breve{P} \,.\supset:$

$$\alpha \in \text{sect}\text{‘}P - \mathrm{D}\text{‘}(P_\epsilon \dot{\cap} I) \,.\equiv.\, C\text{‘}P - \alpha \in \text{sect}\text{‘}Q \cap \text{ᗡ}\text{‘}\text{seq}_Q \quad [*211{\cdot}41{\cdot}7{\cdot}726]$$

$*211\cdot757$. $\vdash :. P \in \mathrm{Ser} . \alpha \subset C`P . \supset :$
$$\alpha \in \mathrm{sect}`P - \mathrm{D}`(P_\epsilon \dot{\frown} I) . \equiv . C`P - \alpha \in \overleftarrow{P}``C`P \quad [*211\cdot756\cdot302]$$

$*211\cdot76$. $\vdash : P \in \mathrm{Ser} . \supset . \mathrm{D}`P_\epsilon = (C`P -)``(\mathrm{sect}`\breve{P} - \mathrm{G}`\mathrm{tl}_P)$

Dem.

$\vdash . *207\cdot13 . \mathrm{Transp} . \supset \vdash . - \mathrm{G}`\mathrm{tl}_P = \mathrm{G}`\mathrm{min}_P \cup - \mathrm{G}`\mathrm{seq}_P \quad (1)$

$\vdash . (1) . *211\cdot751 . \supset \vdash :. \mathrm{Hp} . \supset : \alpha \in \mathrm{D}`P_\epsilon . \equiv . \alpha \subset C`P . C`P - \alpha \in \mathrm{sect}`\breve{P} - \mathrm{G}`\mathrm{tl}_P .$

$[*24\cdot492] \qquad \equiv . (\exists \beta) . \beta \in \mathrm{sect}`\breve{P} - \mathrm{G}`\mathrm{tl}_P . \alpha = C`P - \beta .$

$[*38\cdot13] \qquad \equiv . \alpha \in (C`P -)``(\mathrm{sect}`\breve{P} - \mathrm{G}`\mathrm{tl}_P) :. \supset \vdash . \mathrm{Prop}$

$*211\cdot761$. $\vdash : P \in \mathrm{Ser} . \supset . \mathrm{sect}`P \cap \mathrm{G}`\mathrm{lt}_P = (C`P -)``\{\mathrm{sect}`\breve{P} - \mathrm{D}`(\breve{P})_\epsilon\}$
[Proof as in $*211\cdot76$]

$*211\cdot762$. $\vdash : P \in \mathrm{Ser} . \supset . \mathrm{D}`(P_\epsilon \dot{\frown} I) = (C`P -)``(\mathrm{sect}`\breve{P} - \overleftarrow{P}``C`P)$

Dem.

$\vdash . *211\cdot757 . \mathrm{Transp} . \supset$

$\vdash :. \mathrm{Hp} . \supset : \alpha \in \mathrm{sect}`P . C`P - \alpha \sim\in \overleftarrow{P}``C`P . \equiv . \alpha \in \mathrm{D}`(P_\epsilon \dot{\frown} I) \quad (1)$

$\vdash . (1) . *24\cdot492 . *38\cdot13 . \supset \vdash . \mathrm{Prop}$

$*211\cdot8$. $\vdash : P_{\mathrm{po}} \in \mathrm{Ser} . \alpha \in \mathrm{sect}`P . \supset .$
$$\overrightarrow{\mathrm{max}}_P`\alpha = \overrightarrow{\mathrm{max}}\,(P_{\mathrm{po}})`\alpha . \overrightarrow{\mathrm{min}}_P`(C`P - \alpha) = \overrightarrow{\mathrm{min}}\,(P_{\mathrm{po}})`(C`P - \alpha) = \overrightarrow{\mathrm{seq}}\,(P_{\mathrm{po}})`\alpha$$

Dem.

$\vdash . *211\cdot13 . *91\cdot602 . \quad \supset \vdash : \mathrm{Hp} . \supset . \alpha \in \mathrm{sect}`P_{\mathrm{po}} \quad (1)$

$\vdash . *211\cdot131 . *205\cdot111 . \supset \vdash : \mathrm{Hp} . \supset . \overrightarrow{\mathrm{max}}_P`\alpha = \overrightarrow{\mathrm{max}}\,(P_{\mathrm{po}})`\alpha \quad (2)$

$\vdash . (2) \frac{\breve{P}}{P} . *211\cdot7 . (1) . \quad \supset \vdash : \mathrm{Hp} . \supset . \overrightarrow{\mathrm{min}}_P`(C`P - \alpha) = \overrightarrow{\mathrm{min}}\,(P_{\mathrm{po}})`(C`P - \alpha) \quad (3)$

$[*211\cdot726] \qquad = \overrightarrow{\mathrm{seq}}\,(P_{\mathrm{po}})`\alpha \quad (4)$

$\vdash . (2) . (3) . (4) . \supset \vdash . \mathrm{Prop}$

The above proposition is used in $*232\cdot352$ and $*234\cdot242$.

The following propositions lead up to $*211\cdot82$, which is used in $*213\cdot4$. $*211\cdot83\cdot841\cdot9$ are also used in $*213$.

$*211\cdot81$. $\vdash : P \in \mathrm{Ser} . \alpha \in \mathrm{sect}`P . \alpha \sim\in 1 . C`P - \alpha \in 1 . \supset .$
$$C`P - \alpha = \iota`B`\breve{P} . P = P \upharpoonright \alpha \mathrel{+\!\!\to} B`\breve{P} . \alpha = \mathrm{D}`P$$

Dem.

$\vdash . *211\cdot7\cdot181\cdot182 \frac{\breve{P}}{P} . \supset \vdash : \mathrm{Hp} . \supset . C`P - \alpha = \iota`B`\breve{P} \quad (1)$

$\vdash . *204\cdot461 . \qquad \supset \vdash : \mathrm{Hp} . \supset . P = P \upharpoonright \mathrm{D}`P \mathrel{+\!\!\to} B`\breve{P} \quad (2)$

$\vdash . (1) . *211\cdot1 . \qquad \supset \vdash : \mathrm{Hp} . \supset . \alpha = \mathrm{D}`P \quad (3)$

$\vdash . (1) . (2) . (3) . \supset \vdash . \mathrm{Prop}$

$*211{\cdot}811.\ \vdash : P \epsilon \mathrm{Ser} - \iota{}'\Lambda \,.\, P = Q \mathbin{+\!\!\!\!\rightarrow} x \,.\supset .\, C{}'Q \,\epsilon\, \mathrm{sect}{}'P \,.\, x = B{}'\breve{P} \,.\, C{}'Q = \mathrm{D}{}'P$

Dem.

$\vdash .*161{\cdot}11 .\supset \vdash :. \mathrm{Hp} .\supset : y \,\epsilon\, C{}'Q .\supset_y . yPx :$

$[*204{\cdot}1] \qquad \supset : x \sim\epsilon\, C{}'Q :$

$[*161{\cdot}15] \qquad \supset : x = B{}'\breve{P} \qquad (1)$

$\vdash .*161{\cdot}13 .\supset \vdash : \mathrm{Hp} .\supset . C{}'Q = \mathrm{D}{}'P \qquad (2)$

$\vdash . (1) . (2) . *211{\cdot}1 .\supset \vdash . \mathrm{Prop}$

$*211{\cdot}812.\ \vdash :. P \epsilon \mathrm{Ser} - \iota{}'\Lambda \,.\, Q \,\epsilon\, \mathrm{D}{}'P{\upharpoonright} .\supset :$

$C{}'Q \,\epsilon\, \mathrm{sect}{}'P \,.\, C{}'P - C{}'Q \,\epsilon\, 1 . \equiv . (\exists x) . P = Q \mathbin{+\!\!\!\!\rightarrow} x . \equiv . P = Q \mathbin{+\!\!\!\!\rightarrow} B{}'\breve{P}$

Dem.

$\vdash .*204{\cdot}4 .*201{\cdot}12 .\supset \vdash : \mathrm{Hp} .\supset . C{}'Q \sim\epsilon\, 1 \qquad (1)$

$\vdash .*204{\cdot}41 . \qquad \supset \vdash : \mathrm{Hp} .\supset . Q = P \upharpoonright C{}'Q \qquad (2)$

$\vdash . (1) . (2) . *211{\cdot}81 .\supset \vdash : \mathrm{Hp} . C{}'Q \,\epsilon\, \mathrm{sect}{}'P . C{}'P - C{}'Q \,\epsilon\, 1 .\supset .$

$C{}'P - C{}'Q = \iota{}'B{}'\breve{P} . P = Q \mathbin{+\!\!\!\!\rightarrow} B{}'\breve{P} \qquad (3)$

$\vdash .*211{\cdot}811 .\supset \vdash :. \mathrm{Hp} .\supset : (\exists x) . P = Q \mathbin{+\!\!\!\!\rightarrow} x . \equiv . P = Q \mathbin{+\!\!\!\!\rightarrow} B{}'\breve{P} \qquad (4)$

$\vdash .*211{\cdot}811 .\supset \vdash : \mathrm{Hp} . P = Q \mathbin{+\!\!\!\!\rightarrow} x .\supset . C{}'Q \,\epsilon\, \mathrm{sect}{}'P . C{}'P - C{}'Q \,\epsilon\, 1 \qquad (5)$

$\vdash . (3) . (4) . (5) .\supset \vdash . \mathrm{Prop}$

$*211{\cdot}82.\ \vdash :: P \epsilon \mathrm{Ser} \,.\, Q \,\epsilon\, \mathrm{D}{}'P{\upharpoonright} .\supset :.$

$C{}'Q \,\epsilon\, \mathrm{sect}{}'P . \equiv : (\exists R) . P = Q \ddagger R . \vee . (\exists x) . P = Q \mathbin{+\!\!\!\!\rightarrow} x :$

$\equiv : (\exists R) . P = Q \ddagger R . \vee . P = Q \mathbin{+\!\!\!\!\rightarrow} B{}'\breve{P}$

$[*211{\cdot}282{\cdot}283{\cdot}812 . *160{\cdot}22 . *161{\cdot}2]$

$*211{\cdot}83.\ \vdash : \dot{\exists} ! P . x \sim\epsilon\, C{}'P .\supset . \mathrm{sect}{}'(P \mathbin{+\!\!\!\!\rightarrow} x) = \mathrm{sect}{}'P \cup \iota{}'(C{}'P \cup \iota{}'x)$

Dem.

$\vdash .*211{\cdot}1 .\supset$

$\vdash :. \mathrm{Hp} .\supset : \alpha \,\epsilon\, \mathrm{sect}{}'(P \mathbin{+\!\!\!\!\rightarrow} x) . \equiv . \alpha \subset C{}'P \cup \iota{}'x . (P \mathbin{+\!\!\!\!\rightarrow} x){}''\alpha \subset \alpha \qquad (1)$

$\vdash . (1) . *161{\cdot}11 .\supset$

$\vdash :. \mathrm{Hp} .\supset : \alpha \,\epsilon\, \mathrm{sect}{}'(P \mathbin{+\!\!\!\!\rightarrow} x) . x \,\epsilon\, \alpha . \equiv . \alpha \subset C{}'P \cup \iota{}'x . P{}''\alpha \cup C{}'P \subset \alpha . x \,\epsilon\, \alpha .$

$[*22{\cdot}41] \qquad \equiv . \alpha = C{}'P \cup \iota{}'x \qquad (2)$

$\vdash . (1) . *161{\cdot}11 .\supset$

$\vdash :. \mathrm{Hp} .\supset : \alpha \,\epsilon\, \mathrm{sect}{}'(P \mathbin{+\!\!\!\!\rightarrow} x) . x \sim\epsilon\, \alpha . \equiv . \alpha \subset C{}'P \cup \iota{}'x . P{}''\alpha \subset \alpha . x \sim\epsilon\, \alpha .$

$[*51{\cdot}25] \qquad \equiv . \alpha \subset C{}'P . P{}''\alpha \subset \alpha .$

$[*211{\cdot}1] \qquad \equiv . \alpha \,\epsilon\, \mathrm{sect}{}'P \qquad (3)$

$\vdash . (2) . (3) .\supset \vdash . \mathrm{Prop}$

$*211{\cdot}84.\ \vdash : C{}'P \cap C{}'Q = \Lambda .\supset . \mathrm{sect}{}'(P \ddagger Q) = \mathrm{sect}{}'P \cup (C{}'P \cup){}''\mathrm{sect}{}'Q$

$= \mathrm{sect}{}'P \cup (C{}'P \cup){}''(\mathrm{sect}{}'Q - \iota{}'\Lambda)$

Dem.

$\vdash .*211{\cdot}1 .\supset \vdash : \alpha \,\epsilon\, \mathrm{sect}{}'(P \ddagger Q) . \equiv . \alpha \subset C{}'P \cup C{}'Q . (P \ddagger Q){}''\alpha \subset \alpha \qquad (1)$

$\vdash . (1) . *160{\cdot}11 . \supset$

$\vdash :. \text{Hp} . \supset : \alpha \in \text{sect}`(P \mathbin{\ddagger} Q) . \alpha \subset C`P . \equiv . \alpha \subset C`P . P``\alpha \subset \alpha .$

$[*211{\cdot}1] \qquad \equiv . \alpha \in \text{sect}`P \qquad (2)$

$\vdash . (1) . *160{\cdot}11 . \supset$

$\vdash :. \text{Hp} . \supset : \alpha \in \text{sect}`(P \mathbin{\ddagger} Q) . \exists ! \alpha \cap C`Q . \equiv .$

$\alpha \subset C`P \cup C`Q . C`P \cup Q``\alpha \subset \alpha . \exists ! \alpha \cap C`Q .$

$[*24{\cdot}43{\cdot}491] \equiv . \alpha - C`P \subset C`Q . C`P \subset \alpha . Q``\alpha \subset \alpha - C`P . \exists ! \alpha - C`P .$

$[*24{\cdot}491 . *37{\cdot}265]$

$\equiv . \alpha - C`P \subset C`Q . C`P \subset \alpha . Q``(\alpha - C`P) \subset \alpha - C`P . \exists ! \alpha - C`P .$

$[*211{\cdot}1] \qquad \equiv . \alpha - C`P \in \text{sect}`Q - \iota`\Lambda . C`P \subset \alpha .$

$[*22{\cdot}92] \qquad \equiv . \alpha \in (C`P \cup)``(\text{sect}`Q - \iota`\Lambda) \qquad (3)$

$\vdash . *211{\cdot}26{\cdot}44 . \supset \vdash . C`P \in \text{sect}`P . C`P \in (C`P \cup)``(\text{sect}`Q \cap \iota`\Lambda) \qquad (4)$

$\vdash . (2) . (3) . \supset \vdash : \text{Hp} . \supset . \text{sect}`(P \mathbin{\ddagger} Q) = \text{sect}`P \cup (C`P \cup)``(\text{sect}`Q - \iota`\Lambda)$

$[(4)] \qquad = \text{sect}`P \cup (C`P \cup)``\text{sect}`Q : \supset \vdash . \text{Prop}$

$*211{\cdot}841$. $\vdash : C`P \cap C`Q = \Lambda . \supset .$

$\text{sect}`(P \mathbin{\ddagger} Q) - \iota`\Lambda = (\text{sect}`P - \iota`\Lambda) \cup (C`P \cup)``(\text{sect}`Q - \iota`\Lambda) \quad [*211{\cdot}84]$

$*211{\cdot}9$. $\vdash . \text{sect}`(x \downarrow y) = \iota`\Lambda \cup \iota`\iota`x \cup \iota`(\iota`x \cup \iota`y)$

Dem.

$\vdash . *211{\cdot}1{\cdot}26 . \supset \vdash . \Lambda \in \text{sect}`(x \downarrow y) . \iota`x \cup \iota`y \in \text{sect}`(x \downarrow y) \qquad (1)$

$\vdash . *55{\cdot}13 . \quad \supset \vdash : x \neq y . \supset . (x \downarrow y)``\iota`x = \Lambda \qquad (2)$

$\vdash . *55{\cdot}13 . \quad \supset \vdash : x = y . \supset . (x \downarrow y)``\iota`x = \iota`x \qquad (3)$

$\vdash . (2) . (3) . \quad \supset \vdash . (x \downarrow y)``\iota`x \subset \iota`x .$

$[*211{\cdot}1] \qquad \supset \vdash . \iota`x \in \text{sect}`(x \downarrow y) \qquad (4)$

$\vdash . *211{\cdot}1 . *54{\cdot}4 . \supset$

$\vdash :. \beta \in \text{sect}`(x \downarrow y) . \supset : \beta = \Lambda . \vee . \beta = \iota`x . \vee . \beta = \iota`y . \vee . \beta = \iota`x \cup \iota`y \qquad (5)$

$\vdash . *55{\cdot}13 . \quad \supset \vdash : x \neq y . \supset . x \in (x \downarrow y)``\iota`y - \iota`y .$

$[*211{\cdot}1] \qquad \supset . \iota`y \sim\in \text{sect}`P \qquad (6)$

$\vdash . *51{\cdot}23 . \quad \supset \vdash : x = y . \supset . \iota`y = \iota`x \qquad (7)$

$\vdash . (5) . (6) . (7) . \supset \vdash :. \beta \in \text{sect}`(x \downarrow y) . \supset : \beta = \Lambda . \vee . \beta = \iota`x . \vee . \beta = \iota`x \cup \iota`y \quad (8)$

$\vdash . (1) . (4) . (8) . \supset \vdash . \text{Prop}$

$*$212. THE SERIES OF SEGMENTS

Summary of $*$212.

The series of segments or sections of a series may be ordered by the relation of inclusion, after the manner considered in $*$210. Since, as was shown in $*$211, sections and segments have the properties assigned to κ in the hypothesis of $*$210, the resulting series are such that every class has either a maximum or a sequent, and either a minimum or a precedent; *i.e.* the series of segments or sections are Dedekindian. Most of the properties of the series of sections and of the series of segments which have no maximum, only require that the original relation should be connected. The properties of the series of segments in general ($\mathrm{D}`P_\epsilon$) require also that the original relation should be transitive.

We denote the series of segments by $\varsigma`P$, putting

$$\varsigma`P = P_{\mathrm{lc}} \upharpoonright \mathrm{D}`P_\epsilon \quad \mathrm{Df.}$$

We then have, in virtue of $*$210·13 and $*$211·61,

$*$**212·23.** $\vdash : P \epsilon \mathrm{trans} \cap \mathrm{connex} . \supset . \varsigma`P = \hat{\alpha}\hat{\beta}\{\alpha, \beta \epsilon \mathrm{D}`P_\epsilon . \alpha \subset \beta . \alpha \neq \beta\}$

In like manner, for the series of segments which have no maximum, we put

$$\mathrm{sgm}`P = P_{\mathrm{lc}} \upharpoonright \mathrm{D}`(P_\epsilon \dot{\cap} I) \quad \mathrm{Df,}$$

and we have

$*$**212·22.** $\vdash : P \epsilon \mathrm{connex} . \supset . \mathrm{sgm}`P = \hat{\alpha}\hat{\beta}\{\alpha, \beta \epsilon \mathrm{D}`(P_\epsilon \dot{\cap} I) . \alpha \subset \beta . \alpha \neq \beta\}$

We do not need a special notation for the series of sections, since, in virtue of $*$211·13, it is $\varsigma`P_*$ or $\mathrm{sgm}`P_*$. Thus, by $*$212·23,

$*$**212·24.** $\vdash : P_* \epsilon \mathrm{connex} . \supset . \varsigma`P_* = \hat{\alpha}\hat{\beta}\{\alpha, \beta \epsilon \mathrm{sect}`P . \alpha \subset \beta . \alpha \neq \beta\}$

We begin the number with various propositions on the fields, etc. of these relations, and on the conditions for their existence. We have

$*$**212·132.** $\vdash . \mathrm{D}`\varsigma`P = \mathrm{D}`P_\epsilon - \iota`\mathrm{D}`P . \mathrm{G}`\varsigma`P = \mathrm{D}`P_\epsilon - \iota`\Lambda$

$*$**212·133.** $\vdash : \dot{\exists} ! P . \supset . C`\varsigma`P = \mathrm{D}`P_\epsilon . B`\varsigma`P = \Lambda . B`\mathrm{Cnv}`\varsigma`P = \mathrm{D}`P$

$*$**212·14.** $\vdash : \dot{\exists} ! P . \equiv . \dot{\exists} ! \varsigma`P$

$*$**212·152.** $\vdash . \mathrm{G}`\mathrm{sgm}`P = \mathrm{D}`(P_\epsilon \dot{\cap} I) - \iota`\Lambda$

$*$**212·17.** $\vdash : \dot{\exists} ! \varsigma`P_* . \equiv . \exists ! \mathrm{sect}`P - \iota`\Lambda . \equiv . \mathrm{sect}`P \sim\epsilon 1 . \equiv . \dot{\exists} ! P$

$*$**212·172.** $\vdash : \dot{\exists} ! P . \supset . C`\varsigma`P_* = \mathrm{sect}`P . B`\varsigma`P_* = \Lambda . B`\mathrm{Cnv}`\varsigma`P_* = C`P$

Of the next set of propositions ($*$212·2—·25), several have already been mentioned. An important proposition is

$*$**212·25.** $\vdash : P \epsilon \mathrm{Ser} . \supset . \overrightarrow{P} ; P = (\varsigma`P) \upharpoonright \overrightarrow{P}``C`P$

for this shows that the series of segments contains a series similar to P.

We take up next the application of the propositions of $*210$ to the series of sections and segments. We show that if $P \epsilon \text{connex}$, $\text{sgm}`P$ and $\varsigma`P_*$ are series ($*212{\cdot}3$), and that if P is also transitive, $\varsigma`P$ is a series ($*212{\cdot}31$). We have

$*212{\cdot}322$. $\vdash : P \epsilon \text{connex} . \dot{\exists} ! P . \lambda \subset \text{sect}`P . \supset . s`\lambda = \text{limax}(\varsigma`P_*)`\lambda$

$*212{\cdot}34$. $\vdash : P \epsilon \text{connex} . \dot{\exists} ! P . \lambda \subset \text{sect}`P . \supset . p`\lambda \cap C`P = \text{limin}(\varsigma`P_*)`\lambda$

so that every class of sections has both an upper limit or maximum and a lower limit or minimum ($*212{\cdot}35$).

We then prove similar propositions for $\varsigma`P$ and $\text{sgm}`P$, except that in place of $*212{\cdot}34$ we have

$*212{\cdot}431$. $\vdash : P \epsilon \text{trans} \cap \text{connex} . \dot{\exists} ! P . \lambda \subset \text{D}`P_\epsilon . \supset .$
$$s`(\text{D}`P_\epsilon \cap \text{Cl}`p`\lambda) = \text{limin}(\varsigma`P)`\lambda$$

$*212{\cdot}53$. $\vdash : P \epsilon \text{connex} . \dot{\exists} ! \text{sgm}`P . \lambda \subset \text{D}`(P_\epsilon \dot{\cap} I) . \supset .$
$$s`\{\text{D}`(P_\epsilon \dot{\cap} I) \cap \text{Cl}`p`\lambda\} = \text{limin}(\text{sgm}`P)`\lambda$$

The reason of the difference from $*212{\cdot}34$ is that the product of an existent class of segments may not be a segment. Suppose, for example, the segments are all those that contain a given term x, where x has no immediate successor; then their logical product is $\overrightarrow{P}`x \cup \iota`x$, which is a section but not a segment.

We have next ($*212{\cdot}6$—$\cdot667$) a number of propositions on the limits and maxima of sub-classes of $\overrightarrow{P}``C`P$ in the series $\varsigma`P$. The interest of this subject lies in its relation to irrationals. If $\alpha$ is a class contained in $C`P$ and having no limit or maximum, $\overrightarrow{P}``\alpha$ is contained in $C`\varsigma`P$, and has a limit in $\varsigma`P$. We may call this limit an *irrational* segment. There is no irrational term in $C`P$, because in P there is no limit to α; but the limit, in $\varsigma`P$, of $\overrightarrow{P}``\alpha$ may be called irrational, because it corresponds to no term in $C`P$. It should be observed that (as will be proved in Section F) if $P$ is similar to the series of rationals, $\varsigma`P$ is similar to the series of real numbers.

The most useful propositions in this subject are:

$*212{\cdot}6$. $\vdash : P \epsilon \text{Ser} . \alpha \subset C`P . \supset .$
$$\overrightarrow{\max}(\varsigma`P)`\overrightarrow{P}``\alpha = \overrightarrow{\max}(\overrightarrow{P} ; P)`\overrightarrow{P}``\alpha = \overrightarrow{P}``\overrightarrow{\max}_P`\alpha$$

$*212{\cdot}601$. $\vdash :. P \epsilon \text{Ser} . \alpha \subset C`P . \supset :$
$$\text{E} ! \max_P`\alpha . \equiv . \text{E} ! \max(\overrightarrow{P} ; P)`\overrightarrow{P}``\alpha . \equiv . \text{E} ! \max(\varsigma`P)`\overrightarrow{P}``\alpha$$

$*212{\cdot}602$. $\vdash :. P \epsilon \text{Ser} . \dot{\exists} ! P . \alpha \subset C`P . \supset : \text{E} ! \max_P`\alpha . \equiv . P``\alpha \epsilon \overrightarrow{P}``\alpha$

$*212{\cdot}61$. $\vdash : P \epsilon \text{trans} \cap \text{connex} . \dot{\exists} ! P . \supset . \text{limax}(\varsigma`P)`\overrightarrow{P}``\alpha = P``\alpha$

$*212{\cdot}632$. $\vdash : P \epsilon \text{Ser} . \dot{\exists} ! P . \alpha \subset C`P . P``\alpha \sim \epsilon \overrightarrow{P}``C`P . \supset . P``\alpha = \text{lt}(\varsigma`P)`\overrightarrow{P}``\alpha$

***212·661.** $\vdash : P \in \mathrm{Ser} \,.\, \kappa \subset \mathrm{D}`P_\epsilon \,.\, \mathrm{E}\,!\,\mathrm{lt}(\varsigma`P)`\kappa \,.\, \supset \,.$

$$\mathrm{lt}(\varsigma`P)`\kappa = \mathrm{lt}(\varsigma`P)`\overrightarrow{P}``s`\kappa = s`\kappa$$

This shows that every limit in the series of segments is a limit of a class of what we may call *rational* segments (*i.e.* segments of the form $\overrightarrow{P}`x$), namely it is the limit of $\overrightarrow{P}``s`\kappa$.

***212·667.** $\vdash : P \in \mathrm{Ser} \,.\, \supset \,.\, \mathrm{D}`\mathrm{lt}(\varsigma`P) - \iota`\Lambda = \text{Cl}`\mathrm{sgm}`P$

This shows that the segments (other than Λ) which are limits of classes of segments are the segments (other than Λ) which have no maximum in P.

The number ends with a set of propositions (*212·7—·72) on the relations of the sections and segments of two correlated series. If S is a correlator of P with Q, then S_ϵ (with its converse domain limited) is a correlator of $\varsigma`P_*$ with $\varsigma`Q_*$, $\varsigma`P$ with $\varsigma`Q$ and $\mathrm{sgm}`P$ with $\mathrm{sgm}`Q$ (*212·71·711·712). Hence

***212·72.** $\vdash : P \,\mathrm{smor}\, Q \,.\, \supset \,.\, \varsigma`P_* \,\mathrm{smor}\, \varsigma`Q_* \,.\, \varsigma`P \,\mathrm{smor}\, \varsigma`Q \,.\, \mathrm{sgm}`P \,\mathrm{smor}\, \mathrm{sgm}`Q$

This proposition is used in the next number, and also in *271.

***212·01.** $\varsigma`P = P_{1c} \upharpoonright \mathrm{D}`P_\epsilon$ Df

***212·02.** $\mathrm{sgm}`P = P_{1c} \upharpoonright \mathrm{D}`(P_\epsilon \dot{\frown} I)$ Df

***212·1.** $\vdash : \alpha(\varsigma`P)\beta \,.\equiv .\, \alpha, \beta \in \mathrm{D}`P_\epsilon \,.\, \exists\,!\,\beta - \alpha - P``(\alpha - \beta)$
[*170·102 . *37·15]

***212·11.** $\vdash : \alpha(\mathrm{sgm}`P)\beta \,.\equiv .\, \alpha, \beta \in \mathrm{D}`(P_\epsilon \dot{\frown} I) \,.\, \exists\,!\,\beta - \alpha$

Dem.

$\vdash .\, *170·102 \,.\, *37·15 \,.\, \supset$

$\vdash : \alpha(\mathrm{sgm}`P)\beta \,.\equiv .\, \alpha, \beta \in \mathrm{D}`(P_\epsilon \dot{\frown} I) \,.\, \exists\,!\,\beta - \alpha - P``(\alpha - \beta)$ (1)

$\vdash .\, *211·12 \,.\, \supset \vdash : \alpha \in \mathrm{D}`(P_\epsilon \dot{\frown} I) \,.\, \supset \,.\, -\alpha = -P``\alpha \,.$

[*37·2.Transp] $\supset \,.\, -\alpha \subset -P``(\alpha - \beta) \,.$

[*22·621] $\supset \,.\, -\alpha - P``(\alpha - \beta) = -\alpha$ (2)

$\vdash .\, (1) \,.\, (2) \,.\, \supset \vdash .\, \mathrm{Prop}$

***212·12.** $\vdash : \alpha(\mathrm{sgm}`P_*)\beta \,.\equiv .\, \alpha, \beta \in \mathrm{sect}`P \,.\, \exists\,!\,\beta - \alpha$ [*211·13 . *212·11]

Thus $\mathrm{sgm}`P_*$ has the same connection with $\mathrm{sect}`P$ as $\mathrm{sgm}`P$ has with $\mathrm{D}`(P_\epsilon \dot{\frown} I)$. When P is transitive, $\mathrm{sgm}`P_*$ also has the same connection with $\mathrm{sect}`P$ as $\varsigma`P$ has with $\mathrm{D}`P_\epsilon$. The following proposition makes these facts more explicit.

***212·121.** $\vdash .\, \mathrm{sgm}`P_* = \varsigma`P_* = P_{1c} \upharpoonright \mathrm{sect}`P$

Dem.

$\vdash .\, *211·13 \,.\, \supset \vdash .\, \mathrm{sgm}`P_* = P_{1c} \upharpoonright \mathrm{sect}`P$ (1)

$\vdash .\, *212·1 \,.\, *211·13 \,.\, \supset \vdash : \alpha(\varsigma`P_*)\beta \,.\equiv .\, \alpha, \beta \in \mathrm{sect}`P \,.\, \exists\,!\,\beta - \alpha - P_*``(\alpha - \beta)$ (2)

$\vdash . *211\cdot13 . \supset \vdash : \alpha \in \text{sect}‘P . \supset . \alpha = P_*‘‘\alpha .$

[*37·2] $\supset . P_*‘‘(\alpha - \beta) \subset \alpha .$

[Transp.*22·621] $\supset . -\alpha - P_*‘‘(\alpha - \beta) = -\alpha$ (3)

$\vdash . (2) . (3) . \supset \vdash : \alpha (\varsigma‘P_*) \beta . \equiv . \alpha, \beta \in \text{sect}‘P . \exists ! \beta - \alpha .$

[*211·12] $\equiv . \alpha (\text{sgm}‘P_*) \beta$ (4)

$\vdash . (1) . (4) . \supset \vdash .$ Prop

***212·122.** $\vdash . \varsigma‘P, \text{sgm}‘P \in \text{Rl}‘J$ [*170·17]

***212·123.** $\vdash . C‘\varsigma‘P, C‘\text{sgm}‘P \sim\in 1$ [*200·12 . *212·122]

***212·13.** $\vdash : \Lambda (\varsigma‘P) \beta . \equiv . \beta \in \text{D}‘P_\epsilon - \iota‘\Lambda$ [*170·6]

***212·131.** $\vdash : \alpha (\varsigma‘P)(\text{D}‘P) . \equiv . \alpha \in \text{D}‘P_\epsilon - \iota‘\text{D}‘P$

Dem.

$\vdash . *212\cdot1 . \supset \vdash : \alpha (\varsigma‘P)(\text{D}‘P) . \equiv . \alpha, \text{D}‘P \in \text{D}‘P_\epsilon . \exists ! \text{D}‘P - \alpha - P‘‘(\alpha - \text{D}‘P) .$

[*211·301.*37·15] $\equiv . \alpha \in \text{D}‘P_\epsilon . \alpha \subset \text{D}‘P . \exists ! \text{D}‘P - \alpha .$

[*24·55.*22·41] $\equiv . \alpha \in \text{D}‘P_\epsilon . \alpha \subset \text{D}‘P . \alpha \neq \text{D}‘P .$

[*37·15] $\equiv . \alpha \in \text{D}‘P_\epsilon . \alpha \neq \text{D}‘P : \supset \vdash .$ Prop

***212·132.** $\vdash . \text{D}‘\varsigma‘P = \text{D}‘P_\epsilon - \iota‘\text{D}‘P . \text{ᗡ}‘\varsigma‘P = \text{D}‘P_\epsilon - \iota‘\Lambda$

Dem.

$\vdash . *212\cdot13\cdot131 . \supset \vdash . \text{D}‘P_\epsilon - \iota‘\text{D}‘P \subset \text{D}‘\varsigma‘P . \text{D}‘P_\epsilon - \iota‘\Lambda \subset \text{ᗡ}‘\varsigma‘P$ (1)

$\vdash . *212\cdot1 . \supset \vdash . \text{D}‘\varsigma‘P \subset \text{D}‘P_\epsilon . \text{ᗡ}‘\varsigma‘P \subset \text{D}‘P_\epsilon$ (2)

$\vdash . *212\cdot1 . \supset \vdash : \alpha \in \text{D}‘\varsigma‘P . \supset . (\exists\beta) . \beta \in \text{D}‘P_\epsilon . \exists ! \beta - \alpha .$

[*37·15] $\supset . \exists ! \text{D}‘P - \alpha$ (3)

$\vdash . *212\cdot1 . \supset \vdash : \beta \in \text{ᗡ}‘\varsigma‘P . \supset . (\exists\alpha) . \exists ! \beta - \alpha .$

[*24·561] $\supset . \exists ! \beta$ (4)

$\vdash . (3) . (4) . \supset \vdash . \text{D}‘\varsigma‘P \subset - \iota‘\text{D}‘P . \text{ᗡ}‘\varsigma‘P \subset - \iota‘\Lambda$ (5)

$\vdash . (1) . (2) . (5) . \supset \vdash .$ Prop

***212·133.** $\vdash : \dot{\exists} ! P . \supset . C‘\varsigma‘P = \text{D}‘P_\epsilon . B‘\varsigma‘P = \Lambda . B‘\text{Cnv}‘\varsigma‘P = \text{D}‘P$

Dem.

$\vdash . *33\cdot24 . \supset \vdash : \text{Hp} . \supset . \Lambda \neq \text{D}‘P .$

[*212·132] $\supset . \Lambda \in \text{D}‘\varsigma‘P . \text{D}‘P \in \text{ᗡ}‘\varsigma‘P .$

[*51·221] $\supset . \text{D}‘\varsigma‘P = \{(\text{D}‘P_\epsilon - \iota‘\text{D}‘P) - \iota‘\Lambda\} \cup \iota‘\Lambda .$

[*212·132] $\supset . C‘\varsigma‘P = \{(\text{D}‘P_\epsilon - \iota‘\Lambda) - \iota‘\text{D}‘P\} \cup \iota‘\Lambda \cup (\text{D}‘P_\epsilon - \iota‘\Lambda)$

[*22·63] $= (\text{D}‘P_\epsilon - \iota‘\Lambda) \cup \iota‘\Lambda$

[*51·221] $= \text{D}‘P_\epsilon$ (1)

$\vdash . (1) . *93\cdot103 . *212\cdot132 . \supset \vdash : \text{Hp} . \supset . \overrightarrow{B}‘\varsigma‘P = \text{D}‘P_\epsilon - (\text{D}‘P_\epsilon - \iota‘\Lambda)$

[*211·44] $= \iota‘\Lambda$ (2)

$\vdash . (1) . *93\cdot103 . *212\cdot132 . \supset \vdash : \text{Hp} . \supset . \overrightarrow{B}‘\text{Cnv}‘\varsigma‘P = \text{D}‘P_\epsilon - (\text{D}‘P_\epsilon - \iota‘\text{D}‘P)$

[*211·301] $= \iota‘\text{D}‘P$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash .$ Prop

***212·134.** $\vdash : P = \dot{\Lambda} . \supset . \varsigma‘P = \dot{\Lambda}$ [*170·35]

***212·14.** $\vdash : \dot{\exists} ! P . \equiv . \dot{\exists} ! \varsigma‘P$

Dem.

$\vdash . *212·133 . *211·301 . \supset \vdash : \dot{\exists} ! P . \supset . \exists ! C‘\varsigma‘P .$

[*33·24] $\supset . \dot{\exists} ! \varsigma‘P$ (1)

$\vdash . (1) . *212·134 . \supset \vdash .$ Prop

***212·141.** $\vdash : \alpha \epsilon C‘\varsigma‘P . \equiv . \alpha \epsilon \mathrm{D}‘P_\epsilon . \dot{\exists} ! P$

Dem.

$\vdash . *10·24 . \quad \supset \vdash : \alpha \epsilon C‘\varsigma‘P . \supset . \exists ! C‘\varsigma‘P .$

[*33·24.*212·14] $\supset . \dot{\exists} ! P .$ (1)

[*212·133.Hp] $\supset . \alpha \epsilon \mathrm{D}‘P_\epsilon$ (2)

$\vdash . *212·133 . \supset \vdash : \alpha \epsilon \mathrm{D}‘P_\epsilon . \dot{\exists} ! P . \supset . \alpha \epsilon C‘\varsigma‘P$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash .$ Prop

***212·142.** $\vdash : \dot{\exists} ! \varsigma‘P . \equiv . \mathrm{D}‘P_\epsilon \sim \epsilon 1$

Dem.

$\vdash . *211·66 . *212·14 . \quad \supset \vdash : \dot{\exists} ! \varsigma‘P . \supset . \mathrm{D}‘P_\epsilon \sim \epsilon 1$ (1)

$\vdash . *212·132 . *211·44 . \supset \vdash : \mathrm{D}‘P_\epsilon \sim \epsilon 1 . \supset . \exists ! \mathrm{Cl}‘\varsigma‘P .$

[*33·24] $\supset . \dot{\exists} ! \varsigma‘P$ (2)

$\vdash . (1) . (2) . \supset \vdash .$ Prop

***212·15.** $\vdash : \Lambda (\mathrm{sgm}‘P) \beta . \equiv . \beta \epsilon \mathrm{D}‘(P_\epsilon \dot{\frown} I) - \iota‘\Lambda$ [Proof as in *212·13]

***212·151.** $\vdash : P = \dot{\Lambda} . \supset . \mathrm{sgm}‘P = \dot{\Lambda}$ [*170·35]

The converse implication does not hold in this case. For the existence of sgm‘P, it is necessary that $C‘P$ should contain existent classes having no maximum.

***212·152.** $\vdash . \mathrm{Cl}‘\mathrm{sgm}‘P = \mathrm{D}‘(P_\epsilon \dot{\frown} I) - \iota‘\Lambda$ [Proof as in *212·132]

***212·153.** $\vdash : \dot{\exists} ! \mathrm{sgm}‘P . \equiv . \exists ! \mathrm{D}‘(P_\epsilon \dot{\frown} I) - \iota‘\Lambda . \equiv . \mathrm{D}‘(P_\epsilon \dot{\frown} I) \sim \epsilon 1$

Dem.

$\vdash . *212·15 . \quad \supset \vdash : \exists ! \mathrm{D}‘(P_\epsilon \dot{\frown} I) - \iota‘\Lambda . \supset . \dot{\exists} ! \mathrm{sgm}‘P$ (1)

$\vdash . *212·152 . \supset \vdash : \dot{\exists} ! \mathrm{sgm}‘P . \supset . \exists ! \mathrm{D}‘(P_\epsilon \dot{\frown} I) - \iota‘\Lambda$ (2)

$\vdash . *212·11 . \quad \supset \vdash : \dot{\exists} ! \mathrm{sgm}‘P . \supset . (\exists \alpha, \beta) . \alpha, \beta \epsilon \mathrm{D}‘(P_\epsilon \dot{\frown} I) . \alpha \neq \beta .$

[*52·16.Transp] $\supset . \mathrm{D}‘(P_\epsilon \dot{\frown} I) \sim \epsilon 1$ (3)

$\vdash . *211·44 . *52·181 . \supset$

$\vdash : \mathrm{D}‘(P_\epsilon \dot{\frown} I) \sim \epsilon 1 . \supset . (\exists \beta) . \beta \epsilon \mathrm{D}‘(P_\epsilon \dot{\frown} I) . \beta \neq \Lambda .$

[*212·15] $\supset . \dot{\exists} ! \mathrm{sgm}‘P$ (4)

$\vdash . (1) . (2) . (3) . (4) . \supset \vdash .$ Prop

***212·154.** $\vdash : \dot{\exists} ! \mathrm{sgm}‘P . \supset . C‘\mathrm{sgm}‘P = \mathrm{D}‘(P_\epsilon \dot{\frown} I)$

Dem.

$\vdash . *212·153·15 . \supset \vdash : \mathrm{Hp} . \supset . \Lambda \epsilon \mathrm{D}‘\mathrm{sgm}‘P .$

[*212·152] $\supset . \mathrm{D}‘(P_\epsilon \dot{\frown} I) \subset C‘\mathrm{sgm}‘P$ (1)

$\vdash . *212·11 . \quad \supset \vdash . C‘\mathrm{sgm}‘P \subset \mathrm{D}‘(P_\epsilon \dot{\frown} I)$ (2)

$\vdash . (1) . (2) . \supset \vdash .$ Prop

***212·155.** $\vdash : \dot{\exists} ! \text{sgm}'P . \supset . \Lambda = B'\text{sgm}'P$ [*212·152·154 . *93·103]

***212·156.** $\vdash : \alpha \in C'\text{sgm}'P . \equiv . \alpha \in \text{D}'(P_\epsilon \dot{\frown} I) . \dot{\exists} ! \text{sgm}'P .$

$\equiv . \alpha \in \text{D}'(P_\epsilon \dot{\frown} I) . \text{D}'(P_\epsilon \dot{\frown} I) \sim\in 1$

Dem.

$\vdash . *212{\cdot}154 . \supset \vdash : \alpha \in \text{D}'(P_\epsilon \dot{\frown} I) . \dot{\exists} ! \text{sgm}'P . \supset . \alpha \in C'\text{sgm}'P$ (1)

$\vdash . *10{\cdot}24 . *33{\cdot}24 . \supset \vdash : \alpha \in C'\text{sgm}'P . \supset . \dot{\exists} ! \text{sgm}'P .$ (2)

[*212·154] $\supset . \alpha \in \text{D}'(P_\epsilon \dot{\frown} I)$ (3)

$\vdash . (1) . (2) . (3) . *212{\cdot}153 . \supset \vdash . \text{Prop}$

***212·16.** $\vdash : \text{G}'P \subset \text{D}'P . \supset . \text{D}'P \in \text{D}'(P_\epsilon \dot{\frown} I)$

Dem.

$\vdash . *37{\cdot}27 . \supset \vdash : \text{Hp} . \supset . P``\text{D}'P = \text{D}'P$ (1)

$\vdash . (1) . *211{\cdot}12 . \supset \vdash . \text{Prop}$

***212·161.** $\vdash : \text{G}'P \subset \text{D}'P . \dot{\exists} ! P . \supset . \dot{\exists} ! \text{sgm}'P$

Dem.

$\vdash . *33{\cdot}24 . *212{\cdot}16 . \supset \vdash : \text{Hp} . \supset . \text{D}'P \in \text{D}'(P_\epsilon \dot{\frown} I) - \iota'\Lambda .$

[*212·15] $\supset . \Lambda \, (\text{sgm}'P) \, (\text{D}'P) .$

[*11·36] $\supset . \dot{\exists} ! \text{sgm}'P : \supset \vdash . \text{Prop}$

***212·162.** $\vdash : \text{G}'P \subset \text{D}'P . \dot{\exists} ! P . \supset .$

$\text{D}'P = B'\text{Cnv}'\text{sgm}'P . \text{D}'\text{sgm}'P = \text{D}'(P_\epsilon \dot{\frown} I) - \iota'\text{D}'P$

Dem.

$\vdash . *212{\cdot}16{\cdot}152 . *33{\cdot}24 . \supset \vdash : \text{Hp} . \supset . \text{D}'P \in \text{G}'\text{sgm}'P$ (1)

$\vdash . *212{\cdot}11 . *37{\cdot}24 . \supset \vdash : \text{Hp} . \alpha \in \text{D}'(P_\epsilon \dot{\frown} I) - \iota'\text{D}'P . \supset . \alpha \, (\text{sgm}'P) \, (\text{D}'P)$ (2)

$\vdash . *37{\cdot}24 . \quad \supset \vdash : \alpha \in \text{D}'(P_\epsilon \dot{\frown} I) . \supset . \sim\dot{\exists} ! (\alpha - \text{D}'P) .$

[*212·11] $\supset . \sim\{(\text{D}'P) \, (\text{sgm}'P) \, \alpha\}$ (3)

$\vdash . (2) . (3) . \supset \vdash : \text{Hp} . \supset . \text{D}'(P_\epsilon \dot{\frown} I) - \iota'\text{D}'P \subset \text{D}'\text{sgm}'P . \text{D}'P \sim\in \text{D}'\text{sgm}'P$ (4)

$\vdash . (1) . (4) . *212{\cdot}154 . \supset \vdash . \text{Prop}$

***212·17.** $\vdash : \dot{\exists} ! \varsigma'P_* . \equiv . \dot{\exists} ! \text{sect}'P - \iota'\Lambda . \equiv . \text{sect}'P \sim\in 1 . \equiv . \dot{\exists} ! P$

Dem.

$\vdash . *212{\cdot}132 . *211{\cdot}13 . \supset \vdash : \dot{\exists} ! \varsigma'P_* . \equiv . \dot{\exists} ! \text{sect}'P - \iota'\Lambda$ (1)

$\vdash . *212{\cdot}142 . *211{\cdot}13 . \supset \vdash : \dot{\exists} ! \varsigma'P_* . \equiv . \text{sect}'P \sim\in 1$ (2)

$\vdash . *212{\cdot}14 . \quad \supset \vdash : \dot{\exists} ! \varsigma'P_* . \equiv . \dot{\exists} ! P_* .$

[*90·141] $\equiv . \dot{\exists} ! P$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash . \text{Prop}$

***212·171.** $\vdash . \text{D}'\varsigma'P_* = \text{sect}'P - \iota'C'P . \text{G}'\varsigma'P_* = \text{sect}'P - \iota'\Lambda$

[*212·132 . *211·13 . *90·14]

***212·172.** $\vdash : \dot{\exists} ! P . \supset . C'\varsigma'P_* = \text{sect}'P . B'\varsigma'P_* = \Lambda . B'\text{Cnv}'\varsigma'P_* = C'P$

[*212·133 . *211·13 . *90·141]

***212·173.** $\vdash : \alpha \in C'\varsigma'P_* . \equiv . \alpha \in \text{sect}'P . \dot{\exists} ! P . \equiv . \alpha \in \text{sect}'P . \text{sect}'P \sim\in 1$

[*212·141·142·14 . *211·13]

***212·18.** $\vdash . \varsigma\text{‘}\breve{P}_* = (C\text{‘}P -)\,;\,\mathrm{Cnv}\text{‘}\varsigma\text{‘}P_*$

Dem.

$\vdash . *212\cdot12\cdot121 . \supset \vdash : \alpha(\varsigma\text{‘}\breve{P}_*)\beta . \equiv . \alpha, \beta \in \mathrm{sect}\text{‘}\breve{P} . \exists ! \beta - \alpha .$

[*211·7] $\equiv . (\exists\gamma, \delta) . \gamma, \delta \in \mathrm{sect}\text{‘}P . \alpha = C\text{‘}P - \gamma . \beta = C\text{‘}P - \delta . \exists ! \beta - \alpha .$

[*24·55] $\equiv . (\exists\gamma, \delta) . \gamma, \delta \in \mathrm{sect}\text{‘}P . \alpha = C\text{‘}P - \gamma . \beta = C\text{‘}P - \delta . \sim(\beta \subset \alpha) .$

[*211·1.*24·492] $\equiv . (\exists\gamma, \delta) . \gamma, \delta \in \mathrm{sect}\text{‘}P . \alpha = C\text{‘}P - \gamma . \beta = C\text{‘}P - \delta . \sim(\gamma \subset \delta) .$

[*212·12.*24·55] $\equiv . \alpha \{(C\text{‘}P -)\,;\,\mathrm{Cnv}\text{‘}\varsigma\text{‘}P_*\} \beta : \supset \vdash . \mathrm{Prop}$

***212·181.** $\vdash . (\varsigma\text{‘}\breve{P}_*) \,\mathrm{smor}\, (\mathrm{Cnv}\text{‘}\varsigma\text{‘}P_*)$ [*212·18]

The above proposition is used in *252·43.

***212·2.** $\vdash . \mathrm{sgm}\text{‘}P \subseteq \varsigma\text{‘}P . \mathrm{sgm}\text{‘}P \subseteq \varsigma\text{‘}P_*$ [*211·14 . *212·1·11·12]

***212·21.** $\vdash : P \in \mathrm{trans} . \supset . \varsigma\text{‘}P \subseteq \varsigma\text{‘}P_*$ [*211·15 . *212·12]

***212·22.** $\vdash : P \in \mathrm{connex} . \supset . \mathrm{sgm}\text{‘}P = \hat{\alpha}\hat{\beta} \{\alpha, \beta \in \mathrm{D}\text{‘}(P_\epsilon \dot{\cap} I) . \alpha \subset \beta . \alpha \neq \beta\}$
[*211·62 . *210·1 . *212·11]

***212·23.** $\vdash : P \in \mathrm{trans} \cap \mathrm{connex} . \supset . \varsigma\text{‘}P = \hat{\alpha}\hat{\beta} \{\alpha, \beta \in \mathrm{D}\text{‘}P_\epsilon . \alpha \subset \beta . \alpha \neq \beta\}$
[*210·13 . *211·61 . (*212·01)]

***212·24.** $\vdash : P_* \in \mathrm{connex} . \supset . \varsigma\text{‘}P_* = \hat{\alpha}\hat{\beta} \{\alpha, \beta \in \mathrm{sect}\text{‘}P . \alpha \subset \beta . \alpha \neq \beta\}$
[*212·121·22 . *211·13]

***212·25.** $\vdash : P \in \mathrm{Ser} . \supset . \overrightarrow{P}\,;\,P = (\varsigma\text{‘}P) \upharpoonright \overrightarrow{P}\text{“}C\text{‘}P$

Dem.

$\vdash . *204\cdot33\cdot331 . \supset$

$\vdash :. \mathrm{Hp} . \supset : \alpha(\overrightarrow{P}\,;\,P)\beta . \equiv . \alpha, \beta \in \overrightarrow{P}\text{“}C\text{‘}P . \alpha \subset \beta . \alpha \neq \beta .$

[*212·23.*211·3] $\equiv . \alpha, \beta \in \overrightarrow{P}\text{“}C\text{‘}P . \alpha (\varsigma\text{‘}P) \beta :. \supset \vdash . \mathrm{Prop}$

The following propositions, down to *212·55, consist of applications of the propositions of *210, where the κ of that number is replaced by sect‘P, D‘P_ϵ, or D‘$(P_\epsilon \dot{\cap} I)$, and the Q is replaced by $P_{1c} \upharpoonright \kappa$, *i.e.* by ς‘P_*, ς‘P, or sgm‘P. The propositions which follow are important, since the use of segments, especially in connection with continuity, depends largely upon them.

***212·3.** $\vdash : P \in \mathrm{connex} . \supset . \mathrm{sgm}\text{‘}P, \varsigma\text{‘}P_* \in \mathrm{Ser}$
[*211·67 . *210·14 . *212·121]

***212·31.** $\vdash : P \in \mathrm{trans} \cap \mathrm{connex} . \supset . \varsigma\text{‘}P \in \mathrm{Ser}$
[*211·68 . *210·14 . (*212·01)]

***212·32.** $\vdash : P \in \mathrm{connex} . \dot{\exists} ! P . \lambda \subset \mathrm{sect}\text{‘}P . s\text{‘}\lambda \in \lambda . \supset . s\text{‘}\lambda = \max (\varsigma\text{‘}P_*)\text{‘}\lambda$
[*210·211 . *211·67 . *212·17]

We write max $(\varsigma$‘$P_*)$‘λ, instead of putting ς‘P_* below the line, because, when we have to deal with an expression not consisting of a single letter, it is inconvenient to write it as a suffix, especially when it contains a suffix itself, as in this case.

$*212\cdot321.\ \vdash : P \in \text{connex} \,.\, \dot{\exists} ! P \,.\, \lambda \subset \text{sect}`P \,.\, s`\lambda \sim\in \lambda \,.\, \supset \,.\, s`\lambda = \text{seq}\,(\varsigma`P_*)`\lambda$
$\qquad = \text{lt}\,(\varsigma`P_*)`\lambda$

[$*210\cdot231 \,.\, *211\cdot67 \,.\, *212\cdot17 \,.\, *211\cdot63$]

$*212\cdot322.\ \vdash : P \in \text{connex} \,.\, \dot{\exists} ! P \,.\, \lambda \subset \text{sect}`P \,.\, \supset \,.\, s`\lambda = \text{limax}\,(\varsigma`P_*)`\lambda$

[$*212\cdot32\cdot321 \,.\, *207\cdot46$]

$*212\cdot33.\ \vdash : P \in \text{connex} \,.\, \dot{\exists} ! P \,.\, \lambda \subset \text{sect}`P \,.\, p`\lambda \cap C`P \in \lambda \,.\, \supset \,.$
$\qquad p`\lambda \cap C`P = \min\,(\varsigma`P_*)`\lambda$

Dem.

$\vdash \,.\, *211\cdot671 \,.\, *210\cdot252 \,.\, *211\cdot26 \,.\, \supset$

$\vdash : \text{Hp} \,.\, \supset \,.\, p`\lambda \cap C`P \in \overrightarrow{\min}\,(\varsigma`P_*)`\lambda \cup \overrightarrow{\text{prec}}\,(\varsigma`P_*)`\lambda \quad (1)$

$\vdash \,.\, *206\cdot2 \,.\, \supset \vdash : p`\lambda \cap C`P \in \lambda \,.\, \supset \,.\, p`\lambda \cap C`P \sim\in \overrightarrow{\text{prec}}\,(\varsigma`P_*)`\lambda \quad (2)$

$\vdash \,.\, (1) \,.\, (2) \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, p`\lambda \cap C`P \in \overrightarrow{\min}\,(\varsigma`P_*)`\lambda \quad (3)$

$\vdash \,.\, (3) \,.\, *205\cdot31 \,.\, \supset \vdash \,.\, \text{Prop}$

$*212\cdot331.\ \vdash : P \in \text{connex} \,.\, \dot{\exists} ! P \,.\, \lambda \subset \text{sect}`P \,.\, p`\lambda \cap C`P \sim\in \lambda \,.\, \supset \,.$
$\qquad p`\lambda \cap C`P = \text{prec}\,(\varsigma`P_*)`\lambda = \text{tl}\,(\varsigma`P_*)`\lambda$

Dem.

$\vdash \,.\, *211\cdot671 \,.\, *210\cdot252 \,.\, *211\cdot26 \,.\, \supset$

$\vdash : \text{Hp} \,.\, \supset \,.\, p`\lambda \cap C`P = \text{limin}\,(\varsigma`P_*)`\lambda \quad (1)$

$\vdash \,.\, *205\cdot1 \,.\, \text{Transp} \,.\, \supset \vdash : p`\lambda \cap C`P \sim\in \lambda \,.\, \supset \,.\, p`\lambda \cap C`P \sim\in \overrightarrow{\min}\,(\varsigma`P_*)`\lambda \quad (2)$

$\vdash \,.\, (1) \,.\, (2) \,.\, *206\cdot161 \,.\, \supset \vdash \,.\, \text{Prop}$

$*212\cdot34.\ \vdash : P \in \text{connex} \,.\, \dot{\exists} ! P \,.\, \lambda \subset \text{sect}`P \,.\, \supset \,.\, p`\lambda \cap C`P = \text{limin}\,(\varsigma`P_*)`\lambda$

[$*212\cdot33\cdot331 \,.\, *207\cdot46$]

$*212\cdot35.\ \vdash : P \in \text{connex} \,.\, \dot{\exists} ! P \,.\, \supset \,.$

$(\lambda) \,.\, \lambda \in \{\text{Cl}`\max\,(\varsigma`P_*) \cup \text{Cl}`\text{seq}\,(\varsigma`P_*)\} \cap \{\text{Cl}`\min\,(\varsigma`P_*) \cup \text{Cl}`\text{prec}\,(\varsigma`P_*)\}$

[$*210\cdot28 \,.\, *211\cdot671 \,.\, *212\cdot121$]

$*212\cdot36.\ \vdash :.\, P \in \text{connex} \,.\, \supset : \lambda \in C`\text{sgm}`\varsigma`P_* \,.\, \supset \,.\, E!\,\text{seq}\,(\varsigma`P_*)`\lambda$

Dem.

$\vdash \,.\, *211\cdot47 \,.\, *212\cdot35\cdot3 \,.\, \supset$

$\vdash :.\, \text{Hp} \,.\, \dot{\exists} ! P \,.\, \supset : \lambda \in C`\text{sgm}`\varsigma`P_* \,.\, \supset \,.\, E!\,\text{seq}\,(\varsigma`P_*)`\lambda \quad (1)$

$\vdash \,.\, *33\cdot24 \,.\, \supset \vdash : \lambda \in C`\text{sgm}`\varsigma`P_* \,.\, \supset \,.\, \dot{\exists} !\, \text{sgm}`\varsigma`P_* \,.$

[$*212\cdot151.\text{Transp}$] $\qquad \supset \,.\, \dot{\exists} !\, \varsigma`P_* \,.$

[$*212\cdot17$] $\qquad \supset \,.\, \dot{\exists} ! P \quad (2)$

$\vdash \,.\, (1) \,.\, (2) \,.\, \supset \vdash \,.\, \text{Prop}$

$*212\cdot4.\ \vdash : P \in \text{trans} \cap \text{connex} \,.\, \dot{\exists} ! P \,.\, \lambda \subset \text{D}`P_\epsilon \,.\, s`\lambda \in \lambda \,.\, \supset \,.\, s`\lambda = \max\,(\varsigma`P)`\lambda$

[$*211\cdot68\cdot66 \,.\, *210\cdot211$]

$*212\cdot401.\ \vdash : P \in \text{trans} \cap \text{connex} \,.\, \dot{\exists} ! P \,.\, \lambda \subset \text{D}`P_\epsilon \,.\, s`\lambda \sim\in \lambda \,.\, \supset \,.$
$\qquad s`\lambda = \text{seq}\,(\varsigma`P)`\lambda = \text{lt}\,(\varsigma`P)`\lambda$

[$*211\cdot68\cdot66\cdot64 \,.\, *210\cdot231$]

$*212{\cdot}402$. $\vdash : P \epsilon \text{trans} \cap \text{connex} . \dot{\exists} ! P . \lambda \subset \text{D}\text{‘}P_\epsilon . \supset . s\text{‘}\lambda = \text{limax}(\varsigma\text{‘}P)\text{‘}\lambda$
[$*212{\cdot}4{\cdot}401 . *207{\cdot}46$]

$*212{\cdot}41$. $\vdash : P \epsilon \text{trans} \cap \text{connex} . \dot{\exists} ! P . \lambda \subset \text{D}\text{‘}P_\epsilon . p\text{‘}\lambda \epsilon \lambda . \supset . p\text{‘}\lambda = \min(\varsigma\text{‘}P)\text{‘}\lambda$
[$*211{\cdot}68{\cdot}66 . *210{\cdot}21$]

$*212{\cdot}411$. $\vdash : P \epsilon \text{trans} \cap \text{connex} . \dot{\exists} ! P . \lambda \subset \text{D}\text{‘}P_\epsilon . p\text{‘}\lambda \epsilon \text{D}\text{‘}P_\epsilon - \lambda . \supset .$
$p\text{‘}\lambda = \text{prec}(\varsigma\text{‘}P)\text{‘}\lambda = \text{tl}(\varsigma\text{‘}P)\text{‘}\lambda$
[$*211{\cdot}68{\cdot}66 . *210{\cdot}23$]

$*212{\cdot}42$. $\vdash : P \epsilon \text{trans} \cap \text{connex} . \dot{\exists} ! P . \lambda \subset \text{D}\text{‘}P_\epsilon . p\text{‘}\lambda \sim\epsilon \lambda . \supset .$
$s\text{‘}(\text{D}\text{‘}P_\epsilon \cap \text{Cl}\text{‘}p\text{‘}\lambda) = \text{prec}(\varsigma\text{‘}P)\text{‘}\lambda = \text{tl}(\varsigma\text{‘}P)\text{‘}\lambda$
[$*210{\cdot}26{\cdot}22 . *211{\cdot}68{\cdot}66{\cdot}64$]

The cases considered in $*212{\cdot}411$ and $*212{\cdot}42$ are not mutually exclusive, since if $p\text{‘}\lambda \epsilon \text{D}\text{‘}P_\epsilon$, we have $s\text{‘}(\text{D}\text{‘}P_\epsilon \cap \text{Cl}\text{‘}p\text{‘}\lambda) = p\text{‘}\lambda$.

$*212{\cdot}421$. $\vdash : P \epsilon \text{trans} \cap \text{connex} . \dot{\exists} ! P . \lambda \subset \text{D}\text{‘}P_\epsilon . p\text{‘}\lambda \sim\epsilon \text{D}\text{‘}P_\epsilon . \supset .$
$s\text{‘}(\text{D}\text{‘}P_\epsilon \cap \text{Cl}\text{‘}p\text{‘}\lambda) = P\text{“}p\text{‘}\lambda$

Dem.

$\vdash . *211{\cdot}15{\cdot}1 . \quad \supset \vdash :. \text{Hp} . \supset : \alpha \epsilon \lambda . \supset_\alpha . P\text{“}\alpha \subset \alpha :$
[$*40{\cdot}81$] $\quad \supset : P\text{“}p\text{‘}\lambda \subset p\text{‘}\lambda$ (1)
$\vdash . (1) . *211{\cdot}11 . \quad \supset \vdash : \text{Hp} . \supset . P\text{“}p\text{‘}\lambda \epsilon \text{D}\text{‘}P_\epsilon \cap \text{Cl}\text{‘}p\text{‘}\lambda .$
[$*40{\cdot}13$] $\quad \supset . P\text{“}p\text{‘}\lambda \subset s\text{‘}(\text{D}\text{‘}P_\epsilon \cap \text{Cl}\text{‘}p\text{‘}\lambda)$ (2)
$\vdash . *13{\cdot}196 . *60{\cdot}2 . \supset \vdash :. \text{Hp} . \supset : \alpha \epsilon \text{D}\text{‘}P_\epsilon \cap \text{Cl}\text{‘}p\text{‘}\lambda . \supset_\alpha . \alpha \subset p\text{‘}\lambda . \alpha \neq p\text{‘}\lambda :$
[$*211{\cdot}56{\cdot}15{\cdot}632$] $\quad \supset : \exists ! \lambda . \alpha \epsilon \text{D}\text{‘}P_\epsilon \cap \text{Cl}\text{‘}p\text{‘}\lambda . \supset_\alpha . \alpha \subset P\text{“}p\text{‘}\lambda :$
[$*40{\cdot}151$] $\quad \supset : \exists ! \lambda . \supset . s\text{‘}(\text{D}\text{‘}P_\epsilon \cap \text{Cl}\text{‘}p\text{‘}\lambda) \subset P\text{“}p\text{‘}\lambda$ (3)
$\vdash . *40{\cdot}2 . *37{\cdot}24 . \supset \vdash : \lambda = \Lambda . \supset . s\text{‘}(\text{D}\text{‘}P_\epsilon \cap \text{Cl}\text{‘}p\text{‘}\lambda) \subset \text{D}\text{‘}P . P\text{“}p\text{‘}\lambda = \text{D}\text{‘}P$ (4)
$\vdash . (3) . (4) . \quad \supset \vdash : \text{Hp} . \supset . s\text{‘}(\text{D}\text{‘}P_\epsilon \cap \text{Cl}\text{‘}p\text{‘}\lambda) \subset P\text{“}p\text{‘}\lambda$ (5)
$\vdash . (2) . (5) . \supset \vdash . \text{Prop}$

$*212{\cdot}43$. $\vdash : P \epsilon \text{trans} \cap \text{connex} . \dot{\exists} ! P . \lambda \subset \text{D}\text{‘}P_\epsilon . p\text{‘}\lambda \sim\epsilon \text{D}\text{‘}P_\epsilon . \supset .$
$P\text{“}p\text{‘}\lambda = \text{prec}(\varsigma\text{‘}P)\text{‘}\lambda = \text{tl}(\varsigma\text{‘}P)\text{‘}\lambda$ [$*212{\cdot}42{\cdot}421$]

Thus with regard to the lower end of a class chosen out of $C\text{‘}\varsigma\text{‘}P$, we have three cases to distinguish: (1) if $p\text{‘}\lambda \epsilon \lambda$, $p\text{‘}\lambda$ is the minimum; (2) if $p\text{‘}\lambda \epsilon \text{D}\text{‘}P_\epsilon - \lambda$, $p\text{‘}\lambda$ is the lower limit; (3) if $p\text{‘}\lambda \sim\epsilon \text{D}\text{‘}P_\epsilon$, $P\text{“}p\text{‘}\lambda$ is the lower limit.

$*212{\cdot}431$. $\vdash : P \epsilon \text{trans} \cap \text{connex} . \dot{\exists} ! P . \lambda \subset \text{D}\text{‘}P_\epsilon . \supset .$
$s\text{‘}(\text{D}\text{‘}P_\epsilon \cap \text{Cl}\text{‘}p\text{‘}\lambda) = \text{limin}(\varsigma\text{‘}P)\text{‘}\lambda$

Dem.

$\vdash . *212{\cdot}42 . \supset \vdash : \text{Hp} . p\text{‘}\lambda \sim\epsilon \lambda . \supset . s\text{‘}(\text{D}\text{‘}P_\epsilon \cap \text{Cl}\text{‘}p\text{‘}\lambda) = \text{tl}(\varsigma\text{‘}P)\text{‘}\lambda$ (1)
$\vdash . *22{\cdot}441 . \supset \vdash : \text{Hp} . p\text{‘}\lambda \epsilon \lambda . \supset . p\text{‘}\lambda \epsilon (\text{D}\text{‘}P_\epsilon \cap \text{Cl}\text{‘}p\text{‘}\lambda) .$
[$*40{\cdot}13$] $\quad \supset . p\text{‘}\lambda \subset s\text{‘}(\text{D}\text{‘}P_\epsilon \cap \text{Cl}\text{‘}p\text{‘}\lambda)$ (2)
$\vdash . *60{\cdot}2 . \quad \supset \vdash : \alpha \epsilon \text{D}\text{‘}P_\epsilon \cap \text{Cl}\text{‘}p\text{‘}\lambda . \supset . \alpha \subset p\text{‘}\lambda :$
[$*40{\cdot}151$] $\quad \supset \vdash . s\text{‘}(\text{D}\text{‘}P_\epsilon \cap \text{Cl}\text{‘}p\text{‘}\lambda) \subset p\text{‘}\lambda$ (3)
$\vdash . (2) . (3) . \supset \vdash : \text{Hp} . p\text{‘}\lambda \epsilon \lambda . \supset . s\text{‘}(\text{D}\text{‘}P_\epsilon \cap \text{Cl}\text{‘}p\text{‘}\lambda) = p\text{‘}\lambda .$
[$*212{\cdot}41$] $\quad \supset . s\text{‘}(\text{D}\text{‘}P_\epsilon \cap \text{Cl}\text{‘}p\text{‘}\lambda) = \min(\varsigma\text{‘}P)\text{‘}\lambda$ (4)
$\vdash . (1) . (4) . *207{\cdot}46 . \supset \vdash . \text{Prop}$

***212·44.** $\vdash : P \in \text{trans} \cap \text{connex} \,.\, \dot{\exists} \,!\, P \,.\, \supset .$

$(\lambda) \,.\, \lambda \in \{\text{ᗡ}\text{'max}\,(\varsigma\text{'}P) \cup \text{ᗡ}\text{'seq}\,(\varsigma\text{'}P)\} \cap \{\text{ᗡ}\text{'min}\,(\varsigma\text{'}P) \cup \text{ᗡ}\text{'prec}\,(\varsigma\text{'}P)\}$

[*211·681 . *210·28]

***212·45.** $\vdash :. P \in \text{trans} \cap \text{connex} \,.\, \supset : \lambda \in C\text{'sgm'}\varsigma\text{'}P \,.\, \supset .\, \text{E} \,!\, \text{seq}\,(\varsigma\text{'}P)\text{'}\lambda$

Dem.

$\vdash . \ast 211\cdot 47 \,.\, \ast 212\cdot 44\cdot 31 \,.\, \supset$

$\vdash :. \text{Hp} \,.\, \dot{\exists} \,!\, P \,.\, \supset : \lambda \in C\text{'sgm'}\varsigma\text{'}P \,.\, \supset .\, \text{E} \,!\, \text{seq}\,(\varsigma\text{'}P)\text{'}\lambda \qquad (1)$

$\vdash . \ast 33\cdot 24 \,.\, \supset \vdash : \lambda \in C\text{'sgm'}\varsigma\text{'}P \,.\, \supset .\, \dot{\exists} \,!\, \text{sgm'}\varsigma\text{'}P \,.$

[*212·151.Transp] $\supset .\, \dot{\exists} \,!\, \varsigma\text{'}P \,.$

[*212·14] $\supset .\, \dot{\exists} \,!\, P \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

The proofs of the following propositions are exactly analogous to those of the corresponding propositions on $\varsigma\text{'}P$.

***212·5.** $\vdash : P \in \text{connex} \,.\, \dot{\exists} \,!\, \text{sgm'}P \,.\, \lambda \subset \text{D'}(P_\epsilon \dot{\cap} I) \,.\, s\text{'}\lambda \in \lambda \,.\, \supset .$

$s\text{'}\lambda = \max\,(\text{sgm'}P)\text{'}\lambda$

***212·501.** $\vdash : P \in \text{connex} \,.\, \dot{\exists} \,!\, \text{sgm'}P \,.\, \lambda \subset \text{D'}(P_\epsilon \dot{\cap} I) \,.\, s\text{'}\lambda \sim\in \lambda \,.\, \supset .$

$s\text{'}\lambda = \text{seq}\,(\text{sgm'}P)\text{'}\lambda = \text{lt}\,(\text{sgm'}P)\text{'}\lambda$

***212·502.** $\vdash : P \in \text{connex} \,.\, \dot{\exists} \,!\, \text{sgm'}P \,.\, \lambda \subset \text{D'}(P_\epsilon \dot{\cap} I) \,.\, \supset .\, s\text{'}\lambda = \text{limax}(\text{sgm'}P)\text{'}\lambda$

[*212·5·501]

***212·51.** $\vdash : P \in \text{connex} \,.\, \dot{\exists} \,!\, \text{sgm'}P \,.\, \lambda \subset \text{D'}(P_\epsilon \dot{\cap} I) \,.\, p\text{'}\lambda \in \lambda \,.\, \supset .$

$p\text{'}\lambda = \min\,(\text{sgm'}P)\text{'}\lambda$

***212·511.** $\vdash : P \in \text{connex} \,.\, \dot{\exists} \,!\, \text{sgm'}P \,.\, \lambda \subset \text{D'}(P_\epsilon \dot{\cap} I) \,.\, p\text{'}\lambda \in \text{D'}(P_\epsilon \dot{\cap} I) - \lambda \,.\, \supset .$

$p\text{'}\lambda = \text{prec}\,(\text{sgm'}P)\text{'}\lambda = \text{tl}\,(\text{sgm'}P)\text{'}\lambda$

***212·52.** $\vdash : P \in \text{connex} \,.\, \dot{\exists} \,!\, \text{sgm'}P \,.\, \lambda \subset \text{D'}(P_\epsilon \dot{\cap} I) \,.\, p\text{'}\lambda \sim\in \lambda \,.\, \supset .$

$s\text{'}\{\text{D'}(P_\epsilon \dot{\cap} I) \cap \text{Cl'}p\text{'}\lambda\} = \text{prec}\,(\text{sgm'}P)\text{'}\lambda = \text{tl}\,(\text{sgm'}P)\text{'}\lambda$

This proposition includes *212·511, since, if $p\text{'}\lambda \in \text{D'}(P_\epsilon \dot{\cap} I)$, we have

$$s\text{'}\{\text{D'}(P_\epsilon \dot{\cap} I) \cap \text{Cl'}p\text{'}\lambda\} = p\text{'}\lambda.$$

***212·53.** $\vdash : P \in \text{connex} \,.\, \dot{\exists} \,!\, \text{sgm'}P \,.\, \lambda \subset \text{D'}(P_\epsilon \dot{\cap} I) \,.\, \supset .$

$s\text{'}\{\text{D'}(P_\epsilon \dot{\cap} I) \cap \text{Cl'}p\text{'}\lambda\} = \text{limin}\,(\text{sgm'}P)\text{'}\lambda$ [*212·51·52]

The proof proceeds as in *212·431.

***212·54.** $\vdash : P \in \text{connex} \,.\, \dot{\exists} \,!\, \text{sgm'}P \,.\, \supset .$

$(\lambda) \,.\, \lambda \in \{\text{ᗡ}\text{'max}\,(\text{sgm'}P) \cup \text{ᗡ}\text{'seq}\,(\text{sgm'}P)\} \cap \{\text{ᗡ}\text{'min}\,(\text{sgm'}P) \cup \text{ᗡ}\text{'prec}\,(\text{sgm'}P)\}$

***212·55.** $\vdash :. P \in \text{connex} \,.\, \supset : \lambda \in C\text{'sgm'sgm'}P \,.\, \supset .\, \text{E} \,!\, \text{seq}\,(\text{sgm'}P)\text{'}\lambda$

The following propositions are concerned with the relations of maxima, limits and sequents in P and $\varsigma\text{'}P$ respectively. The series $\overrightarrow{P};P$, which is ordinally similar to P, is contained in $\varsigma\text{'}P$; and if α has a maximum or limit in P, the maximum or limit of $\overrightarrow{P}\text{''}\alpha$ in $\varsigma\text{'}P$ is $\overrightarrow{P}\text{'max}_P\text{'}\alpha$ or $\overrightarrow{P}\text{'lt}_P\text{'}\alpha$.

In this way, a series (namely $\overrightarrow{P};P$) which has the same ordinal properties as P can be placed in a certain Dedekindian series (namely $\varsigma`P$) in such a way that the classes which have limits in P are those whose correlates have limits which are members of $\overrightarrow{P}``C`P$, while those whose correlates have limits which are not members of $\overrightarrow{P}``C`P$ are those which have neither a maximum nor a limit in $P$. These relations are important in many connections. For example, if $P$ is of the type of the rationals, $\varsigma`P$ is of the type of the real numbers: $C`\varsigma`P - \overrightarrow{P}``C`P$ corresponds to the irrationals, and classes contained in $\overrightarrow{P}``C`P$ but having a limit not belonging to $\overrightarrow{P}``C`P$ correspond to series of rationals having an irrational limit. In the original series $P$, there are no irrational limits; but if $\alpha$ is a class in $C`P$ and having no limit, $\overrightarrow{P}``\alpha$ has an irrational limit in $\varsigma`P$.

***212·6.** $\vdash : P \in \mathrm{Ser} . \alpha \subset C`P . \supset .$

$$\overrightarrow{\max}(\varsigma`P)`\overrightarrow{P}``\alpha = \overrightarrow{\max}(\overrightarrow{P};P)`\overrightarrow{P}``\alpha = \overrightarrow{P}``\mathrm{max}_P`\alpha$$

Dem.

$\vdash . *205·9 . *200·12 . \supset$

$\vdash : \mathrm{Hp} . \supset . \overrightarrow{\max}(\varsigma`P)`\overrightarrow{P}``\alpha = \overrightarrow{\max}(\overrightarrow{P};P)`\overrightarrow{P}``\alpha$ (1)

$\vdash . *204·35 . *205·8 . \supset$

$\vdash : \mathrm{Hp} . \supset . \overrightarrow{\max}(\overrightarrow{P};P)`\overrightarrow{P}``\alpha = \overrightarrow{P}``\mathrm{max}_P`\alpha$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***212·601.** $\vdash :. P \in \mathrm{Ser} . \alpha \subset C`P . \supset :$

$$E ! \max_P`\alpha . \equiv . E ! \max(\overrightarrow{P};P)`\overrightarrow{P}``\alpha . \equiv . E ! \max(\varsigma`P)`\overrightarrow{P}``\alpha$$

[*212·6]

***212·602.** $\vdash :. P \in \mathrm{Ser} . \dot{\exists} ! P . \alpha \subset C`P . \supset : E ! \max_P`\alpha . \equiv . P``\alpha \in \overrightarrow{P}``\alpha$

Dem.

$\vdash . *212·601 . *210·223 . \supset$

$\vdash :. \mathrm{Hp} . \supset : E ! \max_P`\alpha . \equiv . s`\overrightarrow{P}``\alpha \in \overrightarrow{P}``\alpha .$

[*40·5] $\equiv . P``\alpha \in \overrightarrow{P}``\alpha :. \supset \vdash . \mathrm{Prop}$

***212·61.** $\vdash : P \in \mathrm{trans} \cap \mathrm{connex} . \dot{\exists} ! P . \supset . \mathrm{limax}(\varsigma`P)`\overrightarrow{P}``\alpha = P``\alpha$

[*212·402 . *40·5]

***212·62.** $\vdash :. P \in \mathrm{Ser} . \dot{\exists} ! P . \supset :$

$$E ! \mathrm{limax}_P`\alpha . \equiv . E ! \mathrm{limax}(\overrightarrow{P};P)`\overrightarrow{P}``\alpha .$$
$$\equiv . \mathrm{limax}(\varsigma`P)`\overrightarrow{P}``\alpha = \overrightarrow{P}`\mathrm{limax}_P`\alpha .$$
$$\equiv . \mathrm{limax}(\varsigma`P)`\overrightarrow{P}``\alpha \in \overrightarrow{P}``C`P$$

Dem.

$\vdash . *204·35 . *207·65 . \supset \vdash :. \mathrm{Hp} . \supset : E ! \mathrm{limax}_P`\alpha . \equiv . E ! \mathrm{limax}(\overrightarrow{P};P)`\overrightarrow{P}``\alpha$ (1)

$\vdash . *207 \cdot 51 . \supset \vdash :. \text{Hp} . \supset : \overrightarrow{P}\text{‘}\text{limax}_P\text{‘}\alpha = P\text{“}\alpha . \equiv . \text{limax}_P\text{‘}\alpha = \text{limax}_P\text{‘}\alpha .$

$[*14 \cdot 28] \qquad \equiv . \text{E} ! \text{limax}_P\text{‘}\alpha \qquad (2)$

$\vdash . (2) . *212 \cdot 61 . \supset$

$\vdash :. \text{Hp} . \supset : \text{E} ! \text{limax}_P\text{‘}\alpha . \equiv . \text{limax} (\varsigma\text{‘}P)\text{‘}\overrightarrow{P}\text{“}\alpha = \overrightarrow{P}\text{‘}\text{limax}_P\text{‘}\alpha \qquad (3)$

$\vdash . *207 \cdot 51 . *14 \cdot 204 . \supset$

$\vdash :. \text{Hp} . \supset : \text{E} ! \text{limax}_P\text{‘}\alpha . \equiv . (\exists x) . x \in C\text{‘}P . \overrightarrow{P}\text{‘}x = P\text{“}\alpha .$

$[*37 \cdot 7] \qquad \equiv . P\text{“}\alpha \in \overrightarrow{P}\text{“}C\text{‘}P \qquad (4)$

$\vdash . (1) . (3) . (4) . \supset \vdash . \text{Prop}$

$*212 \cdot 621$. $\vdash :. P \in \text{Ser} . \beta \subset C\text{‘}P . \supset : \text{limax}_P\text{‘}\alpha \in \beta . \equiv . \text{limax} (\varsigma\text{‘}P)\text{‘}\overrightarrow{P}\text{“}\alpha \in \overrightarrow{P}\text{“}\beta$

Dem.

$\vdash . *33 \cdot 24 . \supset \vdash : \text{Hp} . \text{limax}_P\text{‘}\alpha \in \beta . \supset . \dot{\exists} ! P . \text{limax}_P\text{‘}\alpha \in \beta .$

$[*14 \cdot 21] \qquad \supset . \dot{\exists} ! P . \text{E} ! \text{limax}_P\text{‘}\alpha . \text{limax}_P\text{‘}\alpha \in \beta .$

$[*212 \cdot 62] \qquad \supset . \text{limax} (\varsigma\text{‘}P)\text{‘}\overrightarrow{P}\text{“}\alpha \in \overrightarrow{P}\text{“}\beta \qquad (1)$

$\vdash . *33 \cdot 24 . *22 \cdot 621 . \supset \vdash : \text{Hp} . \text{limax} (\varsigma\text{‘}P)\text{‘}\overrightarrow{P}\text{“}\alpha \in \overrightarrow{P}\text{“}\beta . \supset .$

$\dot{\exists} ! P . \text{limax} (\varsigma\text{‘}P)\text{‘}\overrightarrow{P}\text{“}\alpha \in \overrightarrow{P}\text{“}\beta \cap \overrightarrow{P}\text{“}C\text{‘}P .$

$[*212 \cdot 62] \qquad \supset . \text{limax} (\varsigma\text{‘}P)\text{‘}\overrightarrow{P}\text{“}\alpha \in \overrightarrow{P}\text{“}\beta . \text{limax} (\varsigma\text{‘}P)\text{‘}\overrightarrow{P}\text{“}\beta = \overrightarrow{P}\text{‘}\text{limax}_P\text{‘}\alpha .$

$[*72 \cdot 512 . *204 \cdot 34] \supset . \text{limax}_P\text{‘}\alpha \in \beta \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

$*212 \cdot 63$. $\vdash : P \in \text{Ser} . \dot{\exists} ! P . \alpha \subset C\text{‘}P . \sim \text{E} ! \max_P\text{‘}\alpha . \supset . \text{lt} (\varsigma\text{‘}P)\text{‘}\overrightarrow{P}\text{“}\alpha = P\text{“}\alpha$ $[*212 \cdot 61 \cdot 601 . *207 \cdot 43]$

$*212 \cdot 631$. $\vdash :. P \in \text{Ser} . \dot{\exists} ! P . \alpha \subset C\text{‘}P . \supset : \text{E} ! \text{lt}_P\text{‘}\alpha . \equiv . \text{lt} (\varsigma\text{‘}P)\text{‘}\overrightarrow{P}\text{“}\alpha = \overrightarrow{P}\text{‘}\text{lt}_P\text{‘}\alpha .$

$\equiv . \text{lt} (\varsigma\text{‘}P)\text{‘}\overrightarrow{P}\text{“}\alpha \in \overrightarrow{P}\text{“}C\text{‘}P$

Dem.

$\vdash . *207 \cdot 47 . \supset \vdash : \text{E} ! \text{lt}_P\text{‘}\alpha . \equiv . \text{E} ! \text{limax}_P\text{‘}\alpha . \sim \text{E} ! \max_P\text{‘}\alpha \qquad (1)$

$\vdash . (1) . *212 \cdot 62 \cdot 601 . \supset$

$\vdash :. \text{Hp} . \supset : \text{E} ! \text{lt}_P\text{‘}\alpha . \equiv . \text{limax} (\varsigma\text{‘}P)\text{‘}\overrightarrow{P}\text{“}\alpha = \overrightarrow{P}\text{‘}\text{limax}_P\text{‘}\alpha . \sim \text{E} ! \max (\varsigma\text{‘}P)\text{‘}\overrightarrow{P}\text{“}\alpha .$

$[*207 \cdot 43 \cdot 11] \qquad \equiv . \text{lt} (\varsigma\text{‘}P)\text{‘}\overrightarrow{P}\text{“}\alpha = \overrightarrow{P}\text{‘}\text{lt}_P\text{‘}\alpha \qquad (2)$

$\vdash . (1) . *212 \cdot 62 \cdot 601 . \supset$

$\vdash :. \text{Hp} . \supset : \text{E} ! \text{lt}_P\text{‘}\alpha . \equiv . \text{limax} (\varsigma\text{‘}P)\text{‘}\overrightarrow{P}\text{“}\alpha \in \overrightarrow{P}\text{“}C\text{‘}P . \sim \text{E} ! \max (\varsigma\text{‘}P)\text{‘}\overrightarrow{P}\text{“}\alpha .$

$[*207 \cdot 43 \cdot 11] \qquad \equiv . \text{lt} (\varsigma\text{‘}P)\text{‘}\overrightarrow{P}\text{“}\alpha \in \overrightarrow{P}\text{“}C\text{‘}P \qquad (3)$

$\vdash . (2) . (3) . \supset \vdash . \text{Prop}$

$*212 \cdot 632$. $\vdash : P \in \text{Ser} . \dot{\exists} ! P . \alpha \subset C\text{‘}P . P\text{“}\alpha \sim\in \overrightarrow{P}\text{“}C\text{‘}P . \supset . P\text{“}\alpha = \text{lt} (\varsigma\text{‘}P)\text{‘}\overrightarrow{P}\text{“}\alpha$

Dem.

$\vdash . *212 \cdot 602 . \supset \vdash : \text{Hp} . \supset . \sim \text{E} ! \max_P\text{‘}\alpha .$

$[*212 \cdot 601] \qquad \supset . \sim \text{E} ! \max (\varsigma\text{‘}P)\text{‘}\overrightarrow{P}\text{“}\alpha .$

$[*212 \cdot 61] \qquad \supset . \text{lt} (\varsigma\text{‘}P)\text{‘}\overrightarrow{P}\text{“}\alpha = P\text{“}\alpha : \supset \vdash . \text{Prop}$

***212·633.** $\vdash :. P \in \mathrm{Ser} . \dot{\exists} ! P . x \in C\text{‘}P . \beta \subset C\text{‘}P . \supset :$

$$x = \mathrm{lt}_P\text{‘}\beta . \equiv . \overrightarrow{P}\text{‘}x = \mathrm{lt}\,(\varsigma\text{‘}P)\text{‘}\overrightarrow{P}\text{“}\beta$$

Dem.

$\vdash . *212\cdot631 . *14\cdot21 . \supset$

$\vdash :. \mathrm{Hp} . \supset : x = \mathrm{lt}_P\text{‘}\beta . \supset . \overrightarrow{P}\text{‘}x = \mathrm{lt}\,(\varsigma\text{‘}P)\text{‘}\overrightarrow{P}\text{“}\beta$ (1)

$\vdash . *212\cdot402 . \supset \vdash : \mathrm{Hp} . \overrightarrow{P}\text{‘}x = \mathrm{lt}\,(\varsigma\text{‘}P)\text{‘}\overrightarrow{P}\text{“}\beta . \supset . \overrightarrow{P}\text{‘}x = P\text{“}\beta$ (2)

$\vdash . *206\cdot2 . \quad \supset \vdash : \mathrm{Hp} . \overrightarrow{P}\text{‘}x = \mathrm{lt}\,(\varsigma\text{‘}P)\text{‘}\overrightarrow{P}\text{“}\beta . \supset . \overrightarrow{P}\text{‘}x \sim\in \overrightarrow{P}\text{“}\beta .$

[*72·512.*204·34] $\supset . x \sim\in \beta$ (3)

$\vdash . (2) . (3) . *207\cdot232 . \supset \vdash : \mathrm{Hp} . \overrightarrow{P}\text{‘}x = \mathrm{lt}\,(\varsigma\text{‘}P)\text{‘}\overrightarrow{P}\text{“}\beta . \supset . x = \mathrm{lt}_P\text{‘}\beta$ (4)

$\vdash . (1) . (4) . \supset \vdash . \mathrm{Prop}$

***212·65.** $\vdash :. P \in \mathrm{Ser} . \alpha \subset C\text{‘}P . \supset : E ! \mathrm{seq}_P\text{‘}\alpha . \equiv . \overrightarrow{P}\text{‘}\mathrm{seq}_P\text{‘}\alpha = \mathrm{seq}\,(\varsigma\text{‘}P)\text{‘}\overrightarrow{P}\text{“}\alpha$

Dem.

$\vdash . *206\cdot17 . *210\cdot15 . *211\cdot3 . \supset$

$\vdash :: \mathrm{Hp} . \supset :. \overrightarrow{P}\text{‘}\mathrm{seq}_P\text{‘}\alpha = \mathrm{seq}\,(\varsigma\text{‘}P)\text{‘}\overrightarrow{P}\text{“}\alpha . \equiv :$

$$y \in \alpha \cap C\text{‘}P . \supset_y . \overrightarrow{P}\text{‘}y \subset \overrightarrow{P}\text{‘}\mathrm{seq}_P\text{‘}\alpha . \overrightarrow{P}\text{‘}y \neq \overrightarrow{P}\text{‘}\mathrm{seq}_P\text{‘}\alpha :$$

$$\gamma \in \mathrm{D}\text{‘}P_\epsilon . \gamma \subset \overrightarrow{P}\text{‘}\mathrm{seq}_P\text{‘}\alpha . \gamma \neq \overrightarrow{P}\text{‘}\mathrm{seq}_P\text{‘}\alpha . \supset_\gamma . (\exists z) . z \in \alpha . \gamma \subset \overrightarrow{P}\text{‘}z :$$

[*204·33.*206·22] $\equiv : y \in \alpha \cap C\text{‘}P . \supset_y . yP\,\mathrm{seq}_P\text{‘}\alpha :$

$\gamma \in \mathrm{D}\text{‘}P_\epsilon . \gamma \subset (\alpha \cap C\text{‘}P) \cup P\text{“}\alpha . \gamma \neq (\alpha \cap C\text{‘}P) \cup P\text{“}\alpha . \supset_\gamma . (\exists z) . z \in \alpha . \gamma \subset \overrightarrow{P}\text{‘}z$ (1)

$\vdash . *211\cdot56 . \supset \vdash : \mathrm{Hp} . \gamma \in \mathrm{D}\text{‘}P_\epsilon . z \in C\text{‘}P - \gamma . \supset . \gamma \subset \overrightarrow{P}\text{‘}z$ (2)

$\vdash . (2) . \supset \vdash : \mathrm{Hp} . \gamma \in \mathrm{D}\text{‘}P_\epsilon . \gamma \subset (\alpha \cap C\text{‘}P) \cup P\text{“}\alpha . \gamma \neq (\alpha \cap C\text{‘}P) \cup P\text{“}\alpha . \supset .$

$$(\exists z) . z \in \alpha . \gamma \subset \overrightarrow{P}\text{‘}z \quad (3)$$

$\vdash . (1) . (3) . \supset$

$\vdash :: \mathrm{Hp} . \supset :. \overrightarrow{P}\text{‘}\mathrm{seq}_P\text{‘}\alpha = \mathrm{seq}\,(\varsigma\text{‘}P)\text{‘}\overrightarrow{P}\text{“}\alpha . \equiv : y \in \alpha \cap C\text{‘}P . \supset_y . yP\,\mathrm{seq}_P\text{‘}\alpha :$

[*206·211.*14·21] $\equiv : E ! \mathrm{seq}_P\text{‘}\alpha :: \supset \vdash . \mathrm{Prop}$

***212·651.** $\vdash :. P \in \mathrm{Ser} . \alpha \subset C\text{‘}P . \supset :$

$$E ! \mathrm{seq}_P\text{‘}\alpha . \equiv . \mathrm{seq}\,(\varsigma\text{‘}P)\text{‘}\overrightarrow{P}\text{“}\alpha \in \overrightarrow{P}\text{“}C\text{‘}P . \equiv . E ! \mathrm{seq}\,(\overrightarrow{P} ; P)\text{‘}\overrightarrow{P}\text{“}\alpha$$

Dem.

$\vdash . *212\cdot65 . \supset \vdash :. \mathrm{Hp} . \supset : E ! \mathrm{seq}_P\text{‘}\alpha . \supset . \mathrm{seq}\,(\varsigma\text{‘}P)\text{‘}\overrightarrow{P}\text{“}\alpha \in \overrightarrow{P}\text{“}C\text{‘}P$ (1)

$\vdash . *206\cdot17 . *210\cdot15 . *211\cdot3 . \supset$

$\vdash :: \mathrm{Hp} . \supset :. \mathrm{seq}\,(\varsigma\text{‘}P)\text{‘}\overrightarrow{P}\text{“}\alpha = \overrightarrow{P}\text{‘}w . w \in C\text{‘}P . \equiv :$

$$y \in \alpha \cap C\text{‘}P . \supset_y . \overrightarrow{P}\text{‘}y \subset \overrightarrow{P}\text{‘}w . \overrightarrow{P}\text{‘}y \neq \overrightarrow{P}\text{‘}w :$$

$$\gamma \in \mathrm{D}\text{‘}P_\epsilon . \gamma \subset \overrightarrow{P}\text{‘}w . \gamma \neq \overrightarrow{P}\text{‘}w . \supset_\gamma . (\exists z) . z \in \alpha . \gamma \subset \overrightarrow{P}\text{‘}z : w \in C\text{‘}P :$$

[*204·33.*211·3] $\supset : \alpha \cap C\text{‘}P \subset \overrightarrow{P}\text{‘}w : yPw . \supset_y . (\exists z) . z \in \alpha . \overrightarrow{P}\text{‘}y \subset \overrightarrow{P}\text{‘}z : w \in C\text{‘}P :$

[*204·32] $\supset : \alpha \cap C\text{‘}P \subset \overrightarrow{P}\text{‘}w . \overrightarrow{P}\text{‘}w \subset \alpha \cup P\text{“}\alpha . w \in C\text{‘}P :$

[*206·171.*33·15] $\supset : w = \mathrm{seq}_P\text{‘}\alpha$ (2)

$\vdash . (2) . *37 \cdot 7 . *14 \cdot 204 . \supset$

$\vdash :. \mathrm{Hp} . \supset : \mathrm{seq}\,(\varsigma‘P)‘\overrightarrow{P}“\alpha \in \overrightarrow{P}“C‘P . \supset . \mathrm{E}\,!\,\mathrm{seq}_P‘\alpha$ (3)

$\vdash . (1) . (3) . *206 \cdot 62 . \supset \vdash . \mathrm{Prop}$

***212·652.** $\vdash : P \in \mathrm{Ser} . \alpha \subset C‘P . \mathrm{E}\,!\,\mathrm{max}_P‘\alpha . \mathrm{E}\,!\,\mathrm{seq}\,(\varsigma‘P)‘\overrightarrow{P}“\alpha . \supset .$
$\mathrm{seq}\,(\varsigma‘P)‘\overrightarrow{P}“\alpha = \alpha \cup P“\alpha$

Dem.

$\vdash . *212 \cdot 6 \cdot 601 . *206 \cdot 46 . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{seq}(\varsigma‘P)‘\overrightarrow{P}“\alpha = \mathrm{seq}(\varsigma‘P)‘\iota‘\overrightarrow{P}‘\mathrm{max}_P‘\alpha$ (1)

$\vdash . *206 \cdot 17 . *210 \cdot 15 . *211 \cdot 3 . \supset$

$\vdash :: \mathrm{Hp} . \supset :. \beta = \mathrm{seq}\,(\varsigma‘P)‘\iota‘\overrightarrow{P}‘\mathrm{max}_P‘\alpha . \equiv :$
$\beta \in \mathrm{D}‘P_\epsilon . \overrightarrow{P}‘\mathrm{max}_P‘\alpha \subset \beta . \overrightarrow{P}‘\mathrm{max}_P‘\alpha \neq \beta :$
$\gamma \in \mathrm{D}‘P_\epsilon . \gamma \subset \beta . \gamma \neq \beta . \supset_\gamma . \gamma \subset \overrightarrow{P}‘\mathrm{max}_P‘\alpha :$

[*201·55.*210·1] $\supset : \beta \in \mathrm{D}‘P_\epsilon . \exists\,!\,\beta - P“(\overrightarrow{P}‘\mathrm{max}_P‘\alpha \cup \iota‘\mathrm{max}_P‘\alpha) :$
$\gamma \in \mathrm{D}‘P_\epsilon . \gamma \subset \beta . \gamma \neq \beta . \supset_\gamma . \gamma \subset \overrightarrow{P}‘\mathrm{max}_P‘\alpha :$

[*211·56] $\supset : \beta \in \mathrm{D}‘P_\epsilon . \overrightarrow{P}‘\mathrm{max}_P‘\alpha \cup \iota‘\mathrm{max}_P‘\alpha \subset \beta :$
$\gamma \in \mathrm{D}‘P_\epsilon . \gamma \subset \beta . \gamma \neq \beta . \supset_\gamma . \gamma \subset \overrightarrow{P}‘\mathrm{max}_P‘\alpha :$

[*211·3] $\supset : \beta \in \mathrm{D}‘P_\epsilon . \overrightarrow{P}‘\mathrm{max}_P‘\alpha \cup \iota‘\mathrm{max}_P‘\alpha \subset \beta : x \in \beta . \supset_x . \overrightarrow{P}‘x \subset \overrightarrow{P}‘\mathrm{max}_P‘\alpha :$

[*40·5] $\supset : \beta \in \mathrm{D}‘P_\epsilon . \overrightarrow{P}‘\mathrm{max}_P‘\alpha \cup \iota‘\mathrm{max}_P‘\alpha \subset \beta . P“\beta \subset \overrightarrow{P}‘\mathrm{max}_P‘\alpha :$

[*202·56] $\supset : \beta \in \mathrm{D}‘P_\epsilon . \overrightarrow{P}‘\mathrm{max}_P‘\alpha \cup \iota‘\mathrm{max}_P‘\alpha = \beta :$

[*205·131·22] $\supset : \beta = \alpha \cup P“\alpha$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***212·653.** $\vdash :. P \in \mathrm{Ser} . \mathrm{E}\,!\,\mathrm{max}_P‘\alpha . \alpha \subset C‘P . \supset : \mathrm{E}\,!\,\mathrm{seq}_P‘\alpha . \equiv . \mathrm{E}\,!\,\mathrm{seq}(\varsigma‘P)‘\overrightarrow{P}“\alpha$

Dem.

$\vdash . *212 \cdot 652 . \quad \supset \vdash :. \mathrm{Hp} . \mathrm{E}\,!\,\mathrm{seq}\,(\varsigma‘P)‘\overrightarrow{P}“\alpha . \supset . \alpha \cup P“\alpha \in \mathrm{D}‘P_\epsilon$ (1)

$\vdash . *205 \cdot 191 . \quad \supset \vdash : \mathrm{Hp} . \supset . \mathrm{E}\,!\,\mathrm{max}_P‘(\alpha \cup P“\alpha)$ (2)

$\vdash . (1) . (2) . *211 \cdot 31 . \supset \vdash : \mathrm{Hp}(1) . \supset . \mathrm{E}\,!\,\mathrm{seq}_P‘(\alpha \cup P“\alpha) .$

[*206·25] $\supset . \mathrm{E}\,!\,\mathrm{seq}_P‘\alpha$ (3)

$\vdash . *212 \cdot 65 . \quad \supset \vdash : \mathrm{Hp} . \mathrm{E}\,!\,\mathrm{seq}_P‘\alpha . \supset . \mathrm{E}\,!\,\mathrm{seq}\,(\varsigma‘P)‘\overrightarrow{P}“\alpha$ (4)

$\vdash . (3) . (4) . \supset \vdash . \mathrm{Prop}$

***212·66.** $\vdash : P \in \mathrm{trans} \cap \mathrm{connex} . \kappa \subset \mathrm{D}‘P_\epsilon . \sim \mathrm{E}\,!\,\mathrm{max}\,(\varsigma‘P)‘\kappa . \supset . \sim \mathrm{E}\,!\,\mathrm{max}_P‘s‘\kappa$

Dem.

$\vdash . *210 \cdot 1 . *212 \cdot 23 . \supset$

$\vdash :. \mathrm{Hp} . \supset : \beta \in \kappa . \supset_\beta . (\exists\gamma) . \gamma \in \kappa . \beta \subset \gamma . \exists\,!\,\gamma - \beta :$

[*201·5] $\supset : \beta \in \kappa . x \in \beta . \supset_{\beta, x} . (\exists\gamma) . \gamma \in \kappa . \exists\,!\,\gamma - \overrightarrow{P}‘x - \iota‘x :$

[*202·101] $\supset : x \in s‘\kappa . \supset_x . (\exists\gamma) . \gamma \in \kappa . \exists\,!\,\gamma \cap \overleftarrow{P}‘x .$

[*37·46] $\supset_x . x \in P“s‘\kappa :. \supset \vdash . \mathrm{Prop}$

***212·661.** $\vdash : P \epsilon \mathrm{Ser} . \kappa \subset \mathrm{D}`P_\epsilon . \mathrm{E} ! \mathrm{lt}(\varsigma`P)`\kappa . \supset . \mathrm{lt}(\varsigma`P)`\kappa = \mathrm{lt}(\varsigma`P)`\overrightarrow{P}``s`\kappa = s`\kappa$

Dem.

$\vdash . *212·402 . \quad \supset \vdash : \mathrm{Hp} . \supset . \mathrm{lt}(\varsigma`P)`\kappa = s`\kappa \quad (1)$

$\vdash . *212·402 . \quad \supset \vdash : \mathrm{Hp} . \supset . \mathrm{limax}(\varsigma`P)`\overrightarrow{P}``s`\kappa = P``s`\kappa$

$[*212·66] \qquad = s`\kappa \quad (2)$

$\vdash . *212·601·66 . \supset \vdash : \mathrm{Hp} . \supset . \sim \mathrm{E} ! \max(\varsigma`P)`\overrightarrow{P}``s`\kappa \quad (3)$

$\vdash . (2) . (3) . \quad \supset \vdash : \mathrm{Hp} . \supset . \mathrm{lt}(\varsigma`P)`\overrightarrow{P}``s`\kappa = s`\kappa \quad (4)$

$\vdash . (1) . (4) . \supset \vdash . \mathrm{Prop}$

***212·662.** $\vdash : P \epsilon \mathrm{Ser} . \kappa \subset \mathrm{D}`P_\epsilon . \mathrm{E} ! \mathrm{lt}(\varsigma`P)`\kappa . \supset .$

$(\exists \lambda) . \lambda \subset \overrightarrow{P}``C`P . \mathrm{lt}(\varsigma`P)`\kappa = \mathrm{lt}(\varsigma`P)`\lambda$

[*212·661]

***212·663.** $\vdash : P \epsilon \mathrm{Ser} . x \epsilon C`P . \overrightarrow{P}`x \epsilon \mathrm{D}`\mathrm{lt}(\varsigma`P) . \supset .$

$\overrightarrow{P}`x = \mathrm{lt}(\varsigma`P)`\overrightarrow{P}``\overrightarrow{P}`x . x = \mathrm{lt}_P`\overrightarrow{P}`x$

Dem.

$\vdash . *212·661 . \supset \vdash : P \epsilon \mathrm{Ser} . x \epsilon C`P . \overrightarrow{P}`x = \mathrm{lt}(\varsigma`P)`\kappa . \supset .$

$\overrightarrow{P}`x = s`\kappa . \overrightarrow{P}`x = \mathrm{lt}(\varsigma`P)`\overrightarrow{P}``s`\kappa .$

$[*13·12.*212·66] \quad \supset . \overrightarrow{P}`x = \mathrm{lt}(\varsigma`P)`\overrightarrow{P}``\overrightarrow{P}`x . \sim \mathrm{E} ! \max_P`\overrightarrow{P}`x .$

$[*206·4] \quad \supset . \overrightarrow{P}`x = \mathrm{lt}(\varsigma`P)`\overrightarrow{P}``\overrightarrow{P}`x . x = \mathrm{lt}_P`\overrightarrow{P}`x : \supset \vdash . \mathrm{Prop}$

***212·664.** $\vdash :. P \epsilon \mathrm{Ser} . x \epsilon C`P . \supset : x \epsilon \mathrm{D}`\mathrm{lt}_P . \equiv . \overrightarrow{P}`x \epsilon \mathrm{D}`\mathrm{lt}(\varsigma`P)$

Dem.

$\vdash . *212·631 . \supset \vdash : \mathrm{Hp} . x = \mathrm{lt}_P`\alpha . \supset . \overrightarrow{P}`x = \mathrm{lt}_P`\overrightarrow{P}``\alpha \quad (1)$

$\vdash . (1) . *212·663 . \supset \vdash . \mathrm{Prop}$

***212·665.** $\vdash : P \epsilon \mathrm{Ser} . \dot{\exists} ! P . \alpha \epsilon \mathrm{D}`(P_\epsilon \dot{\wedge} I) . \supset . \mathrm{lt}(\varsigma`P)`\overrightarrow{P}``\alpha = \alpha$

Dem.

$\vdash . *211·4 . \supset \vdash : \mathrm{Hp} . \supset . \sim \mathrm{E} ! \max_P`\alpha .$

$[*212·601·44] \quad \supset . \mathrm{lt}(\varsigma`P)`\overrightarrow{P}``\alpha = \mathrm{limax}(\varsigma`P)`\overrightarrow{P}``\alpha$

$[*212·402.*40·5] \qquad = P``\alpha$

$[*211·12] \qquad = \alpha : \supset \vdash . \mathrm{Prop}$

***212·666.** $\vdash : P \epsilon \mathrm{Ser} . \dot{\exists} ! P . \supset . \mathrm{D}`\mathrm{lt}(\varsigma`P) = \mathrm{D}`(P_\epsilon \dot{\wedge} I)$

Dem.

$\vdash . *212·66·661 . \supset \vdash : \mathrm{Hp} . \kappa \subset \mathrm{D}`P_\epsilon . \gamma = \mathrm{lt}(\varsigma`P)`\kappa . \supset . \gamma = s`\kappa . \sim \mathrm{E} ! \max_P`\gamma .$

$[*211·64·42] \qquad \supset . \gamma \epsilon \mathrm{D}`(P_\epsilon \dot{\wedge} I) \quad (1)$

$\vdash . (1) . *212·665 . \supset \vdash . \mathrm{Prop}$

***212·667.** $\vdash : P \epsilon \mathrm{Ser} . \supset . \mathrm{D}`\mathrm{lt}(\varsigma`P) - \iota`\Lambda = \mathrm{C\!l}`\mathrm{sgm}`P \quad$ [*212·152·151·666]

***212·7.** $\vdash : S \in P \overline{\text{smor}}\, Q . \supset . \text{sect}`P = S_\epsilon``\text{sect}`Q . C`\text{s}`P_* = S_\epsilon``C`\text{s}`Q_*$

Dem.

$\vdash . *151·11·131 . \quad \supset \vdash : \text{Hp} . \beta \subset C`Q . \supset . S``\beta \subset C`P \quad (1)$

$\vdash . *37·2 . \quad \supset \vdash : Q``\beta \subset \beta . \supset . S``Q``\beta \subset S``\beta \quad (2)$

$\vdash . (2) . *72·503 . \quad \supset \vdash : \text{Hp} . \beta \subset C`Q . Q``\beta \subset \beta . \supset . S``Q``\breve{S}``S``\beta \subset S``\beta .$

$[*151·11] \quad \supset . P``S``\beta \subset S``\beta \quad (3)$

$\vdash . (1) . (3) . *211·1 . \supset \vdash : \text{Hp} . \beta \in \text{sect}`Q . \supset . S``\beta \in \text{sect}`P \quad (4)$

$\vdash . (4) . *151·131 . \quad \supset \vdash : \text{Hp} . \alpha \in \text{sect}`P . \supset . \breve{S}``\alpha \in \text{sect}`Q .$

$[*72·502] \quad \supset . \alpha \in S_\epsilon``\text{sect}`Q \quad (5)$

$\vdash . (4) . (5) . \quad \supset \vdash : \text{Hp} . \supset . \text{sect}`P = S_\epsilon``\text{sect}`Q \quad (6)$

$\vdash . (6) . *212·17·172 . \supset \vdash . \text{Prop}$

***212·701.** $\vdash : S \in P \overline{\text{smor}}\, Q . \supset . \text{D}`P_\epsilon = S_\epsilon``\text{D}`Q_\epsilon . C`\text{s}`P = S_\epsilon``C`\text{s}`Q$

[Proof as in *212·7]

***212·702.** $\vdash : S \in P \overline{\text{smor}}\, Q . \supset .$

$\text{D}`(P_\epsilon \dot{\cap} I) = S_\epsilon``\text{D}`(Q_\epsilon \dot{\cap} I) . C`\text{sgm}`P = S_\epsilon``C`\text{sgm}`Q$

[Proof as in *212·7]

***212·71.** $\vdash : S \in P \overline{\text{smor}}\, Q . \supset . S_\epsilon \upharpoonright C`\text{s}`Q_* \in (\text{s}`P_*) \overline{\text{smor}} (\text{s}`Q_*)$

Dem.

$\vdash . *71·381 . \supset \vdash :: \text{Hp} . \supset :. \alpha, \beta \in \text{sect}`Q . \supset : \exists ! \beta - \alpha . \equiv . \exists ! S``\beta - S``\alpha \quad (1)$

$\vdash . (1) . *212·7 . \supset \vdash :. \text{Hp} . \alpha, \beta \in \text{sect}`Q . \supset : \alpha (\text{s}`Q_*) \beta . \equiv . S``\alpha (\text{s}`P_*) S``\beta .$

$[*150·41] \quad \equiv . \alpha \{\breve{S}_\epsilon ; (\text{s}`P_*)\} \beta \quad (2)$

$\vdash . (2) . *212·172 . \supset \vdash : \text{Hp} . \supset . \text{s}`Q_* \subseteq \breve{S}_\epsilon ; (\text{s}`P_*) \quad (3)$

Similarly $\quad \vdash : \text{Hp} . \supset . \text{s}`P_* \subseteq S_\epsilon ; (\text{s}`Q_*) \quad (4)$

$\vdash . *72·451 . \quad \supset \vdash : \text{Hp} . \supset . S_\epsilon \upharpoonright C`\text{s}`Q_* \in 1 \to 1 \quad (5)$

$\vdash . (3) . (4) . (5) . *151·27 . \supset \vdash . \text{Prop}$

***212·711.** $\vdash : S \in P \overline{\text{smor}}\, Q . \supset . S_\epsilon \upharpoonright C`\text{s}`Q \in (\text{s}`P) \overline{\text{smor}} (\text{s}`Q)$

[Proof as in *212·71]

***212·712.** $\vdash : S \in P \overline{\text{smor}}\, Q . \supset . S_\epsilon \upharpoonright C`\text{sgm}`P \in (\text{sgm}`P) \overline{\text{smor}} (\text{sgm}`Q)$

[Proof as in *212·7]

***212·72.** $\vdash : P \text{ smor } Q . \supset . \text{s}`P_* \text{ smor } \text{s}`Q_* . \text{s}`P \text{ smor } \text{s}`Q . \text{sgm}`P \text{ smor } \text{sgm}`Q$

[*212·71·711·712]

$*213$. SECTIONAL RELATIONS

Summary of $*213$.

If α is a section of P, $P \upharpoonright \alpha$ is called a *sectional relation* of P; and if α is a segment of P, $P \upharpoonright \alpha$ is called a *segmental relation* of P. If P_{po} is serial, sectional relations may be arranged in a series by the relation of inclusion ($*213\cdot153$). That is, if we call the series of sectional relations P_s, we shall so define P_s as to secure that if P_{po} is serial,

$$QP_sR \,.\equiv.\, Q, R \in P\upharpoonright``(\text{sect}`P - \iota`\Lambda) \,.\, Q \subseteq R \,.\, Q \neq R \quad (*213\cdot21).$$

The natural definition to take would be

$$P_s = P\upharpoonright ; \varsigma`P_*.$$

But this has the disadvantage that if xBP,

$$P\upharpoonright \iota`x = P\upharpoonright \Lambda \,.\, \Lambda, \iota`x \in \text{sect}`P.$$

Thus $P\upharpoonright \alpha = P\upharpoonright \beta$ does not imply $\alpha = \beta$; and when P is serial, $P\upharpoonright ; \varsigma`P_*$ is not serial, because $\dot\Lambda\,(P\upharpoonright ; \varsigma`P_*)\,\dot\Lambda$. In order to obviate this inconvenience, we confine ourselves to sections which are not null, putting

$$P_s = P\upharpoonright ; (\varsigma`P_*)\upharpoonright(-\iota`\Lambda) \quad \text{Df.}$$

With the above definition, we have ($*213\cdot151\cdot152$), if $P_{po} \in \text{Ser}$,

$$(P\upharpoonright) \upharpoonright C`\{(\varsigma`P_*)\upharpoonright - \iota`\Lambda\} \in 1 \to 1$$

and $$P_s \,\text{smor}\, (\varsigma`P_*)\upharpoonright(-\iota`\Lambda).$$

The relation P_s is very useful in dealing with well-ordered series; in this case, we have (as will be shown later)

$$P_s = P\upharpoonright ; \overrightarrow{P} ; P\upharpoonright \text{Cl}`P \dotplus\!\!\to P.$$

It will be seen that, if $P_{po} \in \text{Ser}$, whenever P exists, $P = B`\breve{P}_s$ ($*213\cdot158$); and whenever $\overrightarrow{B}`P$ exists, $\dot\Lambda = B`P_s$ ($*213\cdot155$).

We have, if $P_{po} \in \text{Ser}$,

$$QP_sR \,.\equiv.\, R \in C`P_s \,.\, Q \in \text{D}`R_s \quad (*213\cdot245).$$

Hence $\quad R \in C`P_s \,.\supset.\, \overrightarrow{P_s}`R = \text{D}`R_s \,.\, R_s = P_s \upharpoonright C`R_s \quad (*213\cdot246\cdot242)$.

If P is serial, the sectional relations of P are all relations such that by adding something to them they become P, *i.e.* they are

$$\hat{Q}\{(\exists R) \,.\, P = Q \,\text{\textcircled{+}}\, R \,.\vee.\, (\exists x) \,.\, P = Q \dotplus\!\!\to x\} \quad (*213\cdot4).$$

Hence their relation-numbers are those that can be made equal to that of P by being added to. This fact is important in connection with the theory of greater and less among relation-numbers.

The propositions of this number are rendered complicated by the necessity of taking account of the possibility of a section being a unit class. This

necessitates a good many propositions which are merely lemmas; but in the end the complications mostly disappear.

We begin with propositions on the field, etc., of P_s. We have

$*213\cdot141.\ \vdash . \mathrm{D}`P_s = P\upharpoonright``(\mathrm{sect}`P - \iota`\Lambda - \iota`C`P)$

$*213\cdot142.\ \vdash : P_{\mathrm{po}} \subseteq J . \supset . C`P_s = P\upharpoonright``(\mathrm{sect}`P - \iota`\Lambda)$

$*213\cdot16.\ \vdash . \mathrm{D}`P_s = P\upharpoonright``(\mathrm{sect}`P - \iota`\Lambda) - \iota`P$

$*213\cdot161.\ \vdash : P_{\mathrm{po}} \subseteq J . \exists ! \overrightarrow{B}`P . \supset . P\upharpoonright``\mathrm{sect}`P = P\upharpoonright``(\mathrm{sect}`P - \iota`\Lambda) = C`P_s$

$*213\cdot162.\ \vdash : P_{\mathrm{po}} \in \mathrm{Ser} . \supset . \mathrm{G}`P_s = P\upharpoonright``\mathrm{sect}`P - \iota`\dot{\Lambda}$

We then prove:

$*213\cdot17.\ \vdash : P_{\mathrm{po}} \in \mathrm{Ser} . \supset . \mathrm{Nr}`\varsigma`P_* = \dot{1} \dotplus \mathrm{Nr}`P_s .$
$\mathrm{Nr}`(\varsigma`P_*)\upharpoonright(\mathrm{G}`\varsigma`P_*) = \mathrm{Nr}`P_s$

If P is finite, it follows from the above that $\varsigma`P_*$ is not similar to P_s; but if P is infinite and has a beginning and is well-ordered, we find

$$\mathrm{Nr}`\varsigma`P_* = \mathrm{Nr}`P_s .$$

$*213\cdot172.\ \vdash : P_{\mathrm{po}}, Q_{\mathrm{po}} \in \mathrm{Ser} . P \,\mathrm{smor}\, Q . \supset . P_s \,\mathrm{smor}\, Q_s$

We then have a set of propositions ($*213\cdot2$—$\cdot251$) chiefly concerned with the sections of R, where $R \in C`P_s$. Besides those already mentioned, the following are important:

$*213\cdot24.\ \vdash : \beta \in \mathrm{sect}`P . R = P\upharpoonright\beta . \supset . \mathrm{sect}`R = \mathrm{sect}`P \cap \mathrm{Cl}`C`R$

$*213\cdot243.\ \vdash . \overrightarrow{P_s}`P = \mathrm{D}`P_s$

$*213\cdot25.\ \vdash :. P_{\mathrm{po}} \in \mathrm{Ser} . Q, R \in C`P_s . \supset : Q \in \mathrm{D}`R_s . \vee . R \in \mathrm{D}`Q_s . \vee . Q = R$

Our next set ($*213\cdot3$—$\cdot32$) is concerned with $\dot{\Lambda}$ and $x \downarrow y$. We have

$*213\cdot3.\ \vdash : P = \dot{\Lambda} . \supset . P_s = \dot{\Lambda}$

$*213\cdot32.\ \vdash : P \in 2_r . \supset . P_s = \dot{\Lambda} \downarrow P . P_s \in 2_r$

We then have three propositions ($*213\cdot4\cdot41\cdot42$) showing that a sectional relation of P is one which becomes P by being added to. We proceed to a set of propositions ($*213\cdot5$—$\cdot58$) on $(P \rightarrow\!\!\!+ x)_s$ and $(P \,\dot{+}\, Q)_s$, leading to

$*213\cdot57.\ \vdash : P_{\mathrm{po}} \subseteq J . \mathrm{Nr}`Q = \mathrm{Nr}`P \dotplus \dot{1} . \supset . \mathrm{Nr}`Q_s = \mathrm{Nr}`P_s \dotplus \dot{1}$

$*213\cdot58.\ \vdash : P_{\mathrm{po}} \subseteq J . Q_{\mathrm{po}} \in \mathrm{Ser} . C`P \cap C`Q = \Lambda . \supset .$
$\mathrm{Nr}`(P \,\dot{+}\, Q)_s = \mathrm{Nr}`P_s \dotplus \mathrm{Nr}`Q_s$

$*213\cdot01.\ P_s = P\upharpoonright;(\varsigma`P_*)\upharpoonright(-\iota`\Lambda)$ Df

$*213\cdot1.\ \vdash : QP_sR . \equiv .$
$(\exists \alpha, \beta) . \alpha, \beta \in \mathrm{sect}`P - \iota`\Lambda . \exists ! \beta - \alpha . Q = P\upharpoonright\alpha . R = P\upharpoonright\beta$
[$*212\cdot12\cdot121 . (*213\cdot01)$]

***213·11.** $\vdash :. P_{\text{po}} \epsilon \text{connex} . \supset : QP_{\varsigma}R . \equiv .$
$(\exists \alpha, \beta) . \alpha, \beta \epsilon \text{sect}`P - \iota`\Lambda . \alpha \subset \beta . \alpha \neq \beta . Q = P \upharpoonright \alpha . R = P \upharpoonright \beta$
[*213·1 . *211·6·17 . *210·1]

***213·12.** $\vdash . \text{D}`(\text{s}`P_{*}) \upharpoonright (- \iota`\Lambda) = \text{sect}`P - \iota`\Lambda - \iota`C`P$

Dem.

$\vdash . *212{\cdot}12 . \supset \vdash : \alpha \epsilon \text{D}`(\text{s}`P_{*}) \upharpoonright (- \iota`\Lambda) . \equiv . (\exists \beta) . \beta \epsilon \text{sect}`P . \exists ! \beta - \alpha . \alpha \neq \Lambda .$
[*212·12] $\equiv . \alpha \neq \Lambda . \alpha \epsilon \text{D}`\text{s}`P_{*} .$
[*212·171] $\equiv . \alpha \epsilon \text{sect}`P - \iota`\Lambda - \iota`C`P : \supset \vdash . \text{Prop}$

***213·121.** $\vdash : P_{\text{po}} \epsilon \text{Ser} . \supset . \overrightarrow{B}`(\text{s}`P_{*}) \upharpoonright (- \iota`\Lambda) = \text{sect}`P \cap 1 = \iota``\overrightarrow{B}`P$

Dem.

$\vdash . *212{\cdot}12 . *213{\cdot}12 . \supset \vdash :. \beta \epsilon \overrightarrow{B}`(\text{s}`P_{*}) \upharpoonright (- \iota`\Lambda) . \equiv :$
$\beta \epsilon \text{sect}`P - \iota`\Lambda - \iota`C`P : \alpha \epsilon \text{sect}`P . \exists ! \beta - \alpha . \supset_{\alpha} . \alpha = \Lambda$ (1)

$\vdash . *211{\cdot}3{\cdot}13{\cdot}1 . *37{\cdot}18 . \supset$
$\vdash : \beta \epsilon \text{sect}`P . x \epsilon \beta . \supset . \overrightarrow{P_{*}}`x \epsilon \text{sect}`P . \overrightarrow{P_{*}}`x \subset \beta . \exists ! \overrightarrow{P_{*}}`x$ (2)

$\vdash . (1) . \text{Transp} . (2) . \supset \vdash :. \beta \epsilon \overrightarrow{B}`(\text{s}`P_{*}) \upharpoonright (- \iota`\Lambda) . \supset :$
$\beta \epsilon \text{sect}`P - \iota`\Lambda - \iota`C`P : x \epsilon \beta . \supset_{x} . \overrightarrow{P_{*}}`x = \beta$ (3)

$\vdash . *200{\cdot}391 . \supset \vdash :. \text{Hp} . \beta \epsilon \text{sect}`P - \iota`\Lambda : x \epsilon \beta . \supset_{x} . \overrightarrow{P_{*}}`x = \beta . \supset :$
$\beta \epsilon \text{sect}`P - \iota`\Lambda : x, y \epsilon \beta . \supset_{x,y} . x = y :$
[*52·16] $\supset : \beta \epsilon \text{sect}`P \cap 1$ (4)

$\vdash . (3) . (4) . \quad \supset \vdash : \text{Hp} . \supset . \overrightarrow{B}`(\text{s}`P_{*}) \upharpoonright (- \iota`\Lambda) \subset \text{sect}`P \cap 1$ (5)

$\vdash . *213{\cdot}12 . *200{\cdot}12 . \supset \vdash : \text{Hp} . \supset . \text{sect}`P \cap 1 \subset \text{D}`(\text{s}`P_{*}) \upharpoonright (- \iota`\Lambda)$ (6)

$\vdash . *51{\cdot}401 . \quad \supset \vdash :. \beta \epsilon \text{sect}`P \cap 1 . \supset : \alpha \subset \beta . \alpha \neq \beta . \supset . \alpha = \Lambda$ (7)

$\vdash . (7) . *212{\cdot}22{\cdot}121 . \quad \supset \vdash : \text{Hp} . \supset . \text{sect}`P \cap 1 \subset - \text{Ɑ}`(\text{s}`P_{*}) \upharpoonright (- \iota`\Lambda)$ (8)

$\vdash . (5) . (6) . (8) . *211{\cdot}18 . \supset \vdash . \text{Prop}$

***213·122.** $\vdash : P_{\text{po}} \epsilon \text{Ser} . \exists ! \overrightarrow{B}`P . \supset . B`(\text{s}`P) \upharpoonright (- \iota`\Lambda) = \iota`B`P$
[*213·121 . *211·181]

***213·123.** $\vdash : P_{\text{po}} \epsilon \text{Ser} . \overrightarrow{B}`P = \Lambda . \supset . \overrightarrow{B}`(\text{s}`P) \upharpoonright (- \iota`\Lambda) = \Lambda$
[*213·121]

***213·124.** $\vdash :. P_{\text{po}} \epsilon \text{Ser} . \supset : \text{E} ! B`(\text{s}`P) \upharpoonright (- \iota`\Lambda) . \equiv . \text{E} ! B`P$
[*213·122·123]

***213·125.** $\vdash : P_{\text{po}} \mathrel{\dot{\subset}} J . \supset . C`\text{s}`P_{*} - \iota`\Lambda \sim\epsilon 1$

Dem.

$\vdash . *212{\cdot}17 . \quad \supset \vdash : P = \dot{\Lambda} . \supset . C`\text{s}`P_{*} = \Lambda .$
[*52·21] $\supset . C`\text{s}`P_{*} - \iota`\Lambda \sim\epsilon 1$ (1)

$\vdash . *212{\cdot}172 . \supset \vdash : \dot{\exists} ! P . \supset . C`\text{s}`P_{*} = \text{sect}`P . C`P \epsilon C`\text{s}`P_{*} - \iota`\Lambda$ (2)

$\vdash . *211{\cdot}13{\cdot}3 . *200{\cdot}39 . \supset \vdash : \text{Hp} . x \epsilon \text{D}`P . \supset . \overrightarrow{P_{*}}`x \epsilon \text{sect}`P . \exists ! C`P - \overrightarrow{P_{*}}`x$ (3)

$\vdash . (2) . (3) . \quad \supset \vdash : \text{Hp} . \dot{\exists} ! P . \supset . C`\text{s}`P_{*} - \iota`\Lambda \sim\epsilon 1$ (4)

$\vdash . (1) . (4) . \supset \vdash . \text{Prop}$

The hypothesis $P_{po} \in J$, in the above proposition, restricts P more than is necessary for the truth of the conclusion. What we really require is $P = \dot{\Lambda} . \mathbf{v} . (\exists x) . x \epsilon C'P . \overrightarrow{P_*}'x \neq C'P$, *i.e.* $\overrightarrow{P_*}``C'P \neq \iota'C'P$. This holds if either (1) the field of $P$ does not consist of a single family, or (2) there is a member of $C'P$ which does not have the relation $P_{po}$ to itself. Thus the only case excluded is that of a single cyclic family. The hypothesis $\overrightarrow{P_*}``C'P \neq \iota'C'P$ may be substituted for $P_{po} \in J$ in most of the subsequent propositions of this number in which $P_{po} \in J$ occurs in the hypothesis. We have, however, preferred the hypothesis $P_{po} \in J$, as it gives a more immediate application to the case of $P \epsilon \mathrm{Ser}$, which is the case in which the propositions of the present number are important.

***213·126.** $\vdash : P_{po} \in J . \dot{\exists} ! P . \supset . \exists ! \mathrm{sect}'P - \iota'\Lambda - \iota'C'P$

Dem.

$\vdash . *213·125 . *212·172 . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{sect}'P - \iota'\Lambda \sim\epsilon 1$ (1)

$\vdash . *211·26 . *33·24 . \quad \supset \vdash : \mathrm{Hp} . \supset . C'P \epsilon \mathrm{sect}'P - \iota'\Lambda$ (2)

$\vdash . (1) . (2) . *52·181 . \supset \vdash . \mathrm{Prop}$

***213·13.** $\vdash : P_{po} \in J . \supset . C`(\varsigma'P_*) \upharpoonright (-\iota'\Lambda) = \mathrm{sect}'P - \iota'\Lambda$

Dem.

$\vdash . *213·125 . \supset$

$\vdash :: \mathrm{Hp} . \supset :. \alpha \epsilon \mathrm{sect}'P - \iota'\Lambda . \supset : (\exists\beta) : \beta \epsilon \mathrm{sect}'P - \iota'\Lambda : \exists ! \alpha - \beta . \mathbf{v} . \exists ! \beta - \alpha :$

$[*212·12] \quad \supset : (\exists\beta) : \alpha \{(\varsigma'P_*) \upharpoonright (-\iota'\Lambda)\} \beta . \mathbf{v} . \beta \{(\varsigma'P_*) \upharpoonright (-\iota'\Lambda)\} \alpha :$

$[*33·132] \quad \supset : \alpha \epsilon C`(\varsigma'P_*) \upharpoonright (-\iota'\Lambda)$ (1)

$\vdash . (1) . *212·172 . \quad \supset \vdash : \mathrm{Hp} . \dot{\exists} ! P . \supset . C`(\varsigma'P_*) \upharpoonright (-\iota'\Lambda) = \mathrm{sect}'P - \iota'\Lambda$ (2)

$\vdash . *212·17 . *211·1 . \supset \vdash : P = \dot{\Lambda} . \supset . C`(\varsigma'P_*) \upharpoonright (-\iota'\Lambda) = \mathrm{sect}'P - \iota'\Lambda$ (3)

$\vdash . (2) . (3) . \supset \vdash . \mathrm{Prop}$

***213·131.** $\vdash : P_{po} \epsilon \mathrm{Ser} . \supset . \mathrm{\Pi}`(\varsigma'P_*) \upharpoonright (-\iota'\Lambda) = \mathrm{sect}'P - \iota'\Lambda - \iota``\overrightarrow{B}'P$
[*213·13·121]

***213·132.** $\vdash : P_{po} \epsilon \mathrm{Ser} . \exists ! \overrightarrow{B}'P . \supset . \mathrm{\Pi}`(\varsigma'P_*) \upharpoonright (-\iota'\Lambda) = \mathrm{sect}'P - \iota'\Lambda - \iota'\iota'B'P$
[*213·13·122]

***213·133.** $\vdash : P_{po} \epsilon \mathrm{Ser} . \overrightarrow{B}'P = \Lambda . \supset . \mathrm{\Pi}`(\varsigma'P_*) \upharpoonright (-\iota'\Lambda) = \mathrm{sect}'P - \iota'\Lambda$
[*213·13·123]

***213·134.** $\vdash : P_{po} \in J . \dot{\exists} ! P . \supset . B`\mathrm{Cnv}`(\varsigma'P_*) \upharpoonright (-\iota'\Lambda) = C'P$ [*213·12·13]

***213·14.** $\vdash . \mathrm{D}'P_\varsigma = P \upharpoonright ``\mathrm{D}`(\varsigma'P_*) \upharpoonright (-\iota'\Lambda) . \mathrm{\Pi}'P_\varsigma = P \upharpoonright ``\mathrm{\Pi}`(\varsigma'P_*) \upharpoonright (-\iota'\Lambda) .$
$C'P_\varsigma = P \upharpoonright ``C`(\varsigma'P_*) \upharpoonright (-\iota'\Lambda)$
[*150·21·211·22]

***213·141.** $\vdash . \mathrm{D}'P_\varsigma = P \upharpoonright ``(\mathrm{sect}'P - \iota'\Lambda - \iota'C'P)$
[*213·12·14]

***213·142.** $\vdash : P_{\text{po}} \in J . \supset . C`P_s = P \upharpoonright ``(\text{sect}`P - \iota`\Lambda)$
[*213·13·14]

***213·143.** $\vdash : P_{\text{po}} \in \text{Ser} . \supset . \text{Ɑ}`P_s = P \upharpoonright ``(\text{sect}`P - \iota`\Lambda - \iota``\overrightarrow{B}`P)$
[*213·131·14]

***213·144.** $\vdash : P_{\text{po}} \in \text{Ser} . \exists ! \overrightarrow{B}`P . \supset . \text{Ɑ}`P_s = P \upharpoonright ``(\text{sect}`P - \iota`\Lambda - \iota`\iota`B`P)$
[*213·132·14]

***213·145.** $\vdash : P_{\text{po}} \in \text{Ser} . \overrightarrow{B}`P = \Lambda . \supset . \text{Ɑ}`P_s = P \upharpoonright ``(\text{sect}`P - \iota`\Lambda)$
[*213·143]

***213·146.** $\vdash : P \in J . \supset . P \upharpoonright ``\text{sect}`P = P \upharpoonright ``(\text{sect}`P - 1)$

Dem.

$\vdash . *37·22 . \supset \vdash . P \upharpoonright ``\text{sect}`P = P \upharpoonright ``(\text{sect}`P - 1) \cup P \upharpoonright ``(\text{sect}`P \cap 1)$ (1)
$\vdash . *200·35 . \supset \vdash : Q \in P \upharpoonright ``(\text{sect}`P \cap 1) . \supset . Q = \dot{\Lambda} .$
[*36·27] $\supset . Q = P \upharpoonright \Lambda .$
[*211·44] $\supset . Q \in P \upharpoonright ``(\text{sect}`P - 1)$ (2)
$\vdash . (1) . (2) . \supset \vdash .$ Prop

***213·15.** $\vdash :. P_{\text{po}} \in \text{Ser} . \alpha \in \text{sect}`P - \iota`\Lambda . \supset : P \upharpoonright \alpha = \dot{\Lambda} . \equiv . \alpha \in 1$

Dem.

$\vdash . *200·35 . \supset \vdash : \text{Hp} . \alpha \in 1 . \supset . P \upharpoonright \alpha = \dot{\Lambda}$ (1)
$\vdash . *52·41 . \supset \vdash :. \text{Hp} . \alpha \sim\in 1 . \supset : (\exists x, y) . x, y \in \alpha . x \neq y :$
[*211·1.*202·103] $\supset : (\exists x, y) : x (P_{\text{po}} \upharpoonright \alpha) y . \vee . y (P_{\text{po}} \upharpoonright \alpha) x :$
[*11·7] $\supset : \dot{\exists} ! P_{\text{po}} \upharpoonright \alpha :$
[*37·41] $\supset : \exists ! \alpha \cap P_{\text{po}}``\alpha :$
[*211·131] $\supset : \exists ! \alpha \cap P``\alpha :$
[*37·41] $\supset : \dot{\exists} ! P \upharpoonright \alpha$ (2)
$\vdash . (1) . (2) . \supset \vdash .$ Prop

***213·151.** $\vdash : P_{\text{po}} \in \text{Ser} . \supset . (P \upharpoonright) \upharpoonright (\text{sect}`P - \iota`\Lambda) \in 1 \rightarrow 1$

Dem.

$\vdash . *213·15 . \supset$
$\vdash : \text{Hp} . \alpha \in \text{sect}`P - \iota`\Lambda - 1 . \beta \in \text{sect}`P - \iota`\Lambda . P \upharpoonright \alpha = P \upharpoonright \beta . \supset . \beta \sim\in 1 .$
[*211·133] $\supset . C`P \upharpoonright \beta = \beta . C`P \upharpoonright \alpha = \alpha .$
[Hp] $\supset . \alpha = \beta$ (1)
$\vdash . *213·15 . \supset$
$\vdash : \text{Hp} . \alpha \in \text{sect}`P \cap 1 . \beta \in \text{sect}`P - \iota`\Lambda . P \upharpoonright \alpha = P \upharpoonright \beta . \supset . \beta \in 1$ (2)
$\vdash . (2) . *211·18 . \supset \vdash : \text{Hp}(2) . \supset . \alpha, \beta \in \iota``\overrightarrow{B}`P .$
[*202·523·13] $\supset . \alpha = \beta$ (3)
$\vdash . (1) . (3) . \supset \vdash :. \text{Hp} . \supset : \alpha, \beta \in \text{sect}`P - \iota`\Lambda . P \upharpoonright \alpha = P \upharpoonright \beta . \supset . \alpha = \beta :.$
$\supset \vdash .$ Prop

***213·152.** $\vdash : P_{\text{po}} \in \text{Ser} . \supset . P_s \,\text{smor}\, (\varsigma`P_*) \upharpoonright (-\iota`\Lambda)$ [*213·151·13]

***213·153.** $\vdash : P_{po} \in \mathrm{Ser} . \supset . P_s \in \mathrm{Ser}$ [*213·152 . *212·3 . *204·4·21]

***213·154.** $\vdash : P_{po} \in \mathrm{Ser} . \supset . \overrightarrow{B}{}^{\backprime}P_s = P \upharpoonleft {}^{\backprime\backprime}\iota^{\backprime\backprime}\overrightarrow{B}{}^{\backprime}P$ [*213·151·121 . *151·5]

***213·155.** $\vdash : P_{po} \in \mathrm{Ser} . \exists ! \overrightarrow{B}{}^{\backprime}P . \supset . B^{\backprime}P_s = \dot{\Lambda}$

Dem.

$\vdash . *213{\cdot}151{\cdot}122 . *151{\cdot}5 . \supset \vdash : \mathrm{Hp} . \supset . B^{\backprime}P_s = P \upharpoonleft (\iota^{\backprime}B^{\backprime}P)$
[*200·35] $= \dot{\Lambda} : \supset \vdash . \mathrm{Prop}$

***213·156.** $\vdash : P_{po} \in \mathrm{Ser} . \overrightarrow{B}{}^{\backprime}P = \Lambda . \supset . \overrightarrow{B}{}^{\backprime}P_s = \Lambda$ [*213·154]

***213·157.** $\vdash :. P_{po} \in \mathrm{Ser} . \supset : \mathrm{E} ! B^{\backprime}P . \equiv . \mathrm{E} ! B^{\backprime}P_s$ [*213·155·156]

***213·158.** $\vdash : P_{po} \in \mathrm{Ser} . \dot{\exists} ! P . \supset . B^{\backprime}\breve{P}_s = P$

Dem.

$\vdash . *213{\cdot}151{\cdot}134 . *151{\cdot}5 . \supset \vdash : \mathrm{Hp} . \supset . B^{\backprime}\breve{P}_s = P \upharpoonleft C^{\backprime}P : \supset \vdash . \mathrm{Prop}$

***213·16.** $\vdash . \mathrm{D}^{\backprime}P_s = P \upharpoonleft {}^{\backprime\backprime}(\mathrm{sect}^{\backprime}P - \iota^{\backprime}\Lambda) - \iota^{\backprime}P$

Dem.

$\vdash . *213{\cdot}141 . \supset$
$\vdash : Q \in \mathrm{D}^{\backprime}P_s . \equiv . (\exists \alpha) . \alpha \in \mathrm{sect}^{\backprime}P - \iota^{\backprime}\Lambda . Q = P \upharpoonleft \alpha . \alpha \neq C^{\backprime}P$ (1)
$\vdash . *211{\cdot}1 . \supset \vdash :. \alpha \in \mathrm{sect}^{\backprime}P . Q = P \upharpoonleft \alpha . \supset : \alpha \neq C^{\backprime}P . \equiv . \exists ! C^{\backprime}P - \alpha .$
[*36·25.Transp] $\equiv . Q \neq P$ (2)
$\vdash . (1) . (2) . \supset \vdash : Q \in \mathrm{D}^{\backprime}P_s . \equiv . (\exists \alpha) . \alpha \in \mathrm{sect}^{\backprime}P - \iota^{\backprime}\Lambda . Q = P \upharpoonleft \alpha . Q \neq P :$
$\supset \vdash . \mathrm{Prop}$

***213·161.** $\vdash : P_{po} \in J . \exists ! \overrightarrow{B}{}^{\backprime}P . \supset . P \upharpoonleft {}^{\backprime\backprime}\mathrm{sect}^{\backprime}P = P \upharpoonleft {}^{\backprime\backprime}(\mathrm{sect}^{\backprime}P - \iota^{\backprime}\Lambda) = C^{\backprime}P_s$

Dem.

$\vdash . *211{\cdot}18 . \supset \vdash : \mathrm{Hp} . \supset . \iota^{\backprime\backprime}\overrightarrow{B}{}^{\backprime}P \subset \mathrm{sect}^{\backprime}P \cap 1 . \exists ! \iota^{\backprime\backprime}\overrightarrow{B}{}^{\backprime}P .$
[*37·2·45] $\supset . P \upharpoonleft {}^{\backprime\backprime}\iota^{\backprime\backprime}\overrightarrow{B}{}^{\backprime}P \subset P \upharpoonleft {}^{\backprime\backprime}(\mathrm{sect}^{\backprime}P - \iota^{\backprime}\Lambda) . \exists ! P \upharpoonleft {}^{\backprime\backprime}\iota^{\backprime\backprime}\overrightarrow{B}{}^{\backprime}P .$
[*200·35] $\supset . \dot{\Lambda} \in P \upharpoonleft {}^{\backprime\backprime}(\mathrm{sect}^{\backprime}P - \iota^{\backprime}\Lambda) .$
[*36·27] $\supset . P \upharpoonleft \Lambda \in P \upharpoonleft {}^{\backprime\backprime}(\mathrm{sect}^{\backprime}P - \iota^{\backprime}\Lambda) .$
[*37·22] $\supset . P \upharpoonleft {}^{\backprime\backprime}\mathrm{sect}^{\backprime}P = P \upharpoonleft {}^{\backprime\backprime}(\mathrm{sect}^{\backprime}P - \iota^{\backprime}\Lambda)$
[*213·142] $= C^{\backprime}P_s : \supset \vdash . \mathrm{Prop}$

***213·162.** $\vdash : P_{po} \in \mathrm{Ser} . \supset . \mathrm{G}^{\backprime}P_s = P \upharpoonleft {}^{\backprime\backprime}\mathrm{sect}^{\backprime}P - \iota^{\backprime}\dot{\Lambda}$

Dem.

$\vdash . *213{\cdot}143 . \supset \vdash :. \mathrm{Hp} . \supset : Q \in \mathrm{G}^{\backprime}P_s . \equiv .$
$(\exists \alpha) . \alpha \in \mathrm{sect}^{\backprime}P - \iota^{\backprime}\Lambda - \iota^{\backprime\backprime}\overrightarrow{B}{}^{\backprime}P . Q = P \upharpoonleft \alpha .$ (1)
[*213·15.*211·18] $\supset . Q \in P \upharpoonleft {}^{\backprime\backprime}\mathrm{sect}^{\backprime}P - \iota^{\backprime}\dot{\Lambda}$ (2)
$\vdash . *213{\cdot}15 . \supset \vdash :. \mathrm{Hp} . \supset : Q \in P \upharpoonleft {}^{\backprime\backprime}\mathrm{sect}^{\backprime}P - \iota^{\backprime}\dot{\Lambda} . \supset .$
$(\exists \alpha) . \alpha \in \mathrm{sect}^{\backprime}P - \iota^{\backprime}\Lambda - \iota^{\backprime\backprime}\overrightarrow{B}{}^{\backprime}P . Q = P \upharpoonleft \alpha .$
[(1)] $\supset . Q \in \mathrm{G}^{\backprime}P_s$ (3)
$\vdash . (2) . (3) . \supset \vdash . \mathrm{Prop}$

***213·163.** $\vdash : P_{\text{po}} \in \text{Ser} . \overrightarrow{B}`P = \Lambda . \supset . C`P_s = P \upharpoonright ``\text{sect}`P - \iota`\dot{\Lambda}$

Dem.

$\vdash . *213·156 . \supset \vdash : \text{Hp} . \supset . C`P = \text{Cl}`P_s$ (1)

$\vdash . (1) . *213·162 . \supset \vdash . \text{Prop}$

***213·164.** $\vdash : P_{\text{po}} \in \text{Ser} . \overrightarrow{B}`P = \Lambda . \supset . \text{D}`P_s = P \upharpoonright ``\text{sect}`P - \iota`\dot{\Lambda} - \iota`P$

[*213·142·163·16]

***213·17.** $\vdash : P_{\text{po}} \in \text{Ser} . \supset . \text{Nr}`\varsigma`P_* = \dot{1} \dotplus \text{Nr}`P_s .$

$\text{Nr}`(\varsigma`P_*) \upharpoonright (\text{Cl}`\varsigma`P_*) = \text{Nr}`P_s$

Dem.

$\vdash . *212·171·172 . \supset \vdash : \dot{\exists} ! P . \supset . B`\varsigma`P_* = \Lambda . \text{Cl}`\varsigma`P_* = \text{sect}`P - \iota`\Lambda .$

$(\varsigma`P_*) \upharpoonright (-\iota`\Lambda) = (\varsigma`P_*) \upharpoonright (\text{Cl}`\varsigma`P_*)$ (1)

$\vdash . (1) . *213·125 . \supset \vdash : \text{Hp} . \dot{\exists} ! P . \supset . \text{Cl}`\varsigma`P_* \sim \in 1$ (2)

$\vdash . *212·3 . *91·602 . \supset \vdash : \text{Hp} . \supset . \varsigma`P_* \in \text{connex}$ (3)

$\vdash . (1) . *213·152 . \supset \vdash : \text{Hp} . \dot{\exists} ! P . \supset . \text{Nr}`(\varsigma`P_*) \upharpoonright (\text{Cl}`\varsigma`P_*) = \text{Nr}`P_s$ (4)

$\vdash . (1) . (2) . (3) . *204·46 . \supset$

$\vdash : \text{Hp} . \dot{\exists} ! P . \supset . \text{Nr}`\varsigma`P_* = \dot{1} \dotplus \text{Nr}`(\varsigma`P_*) \upharpoonright (\text{Cl}`\varsigma`P_*)$

[(4)] $= \dot{1} \dotplus \text{Nr}`P_s$ (5)

$\vdash . *212·17 . *150·42 . \supset \vdash : P = \dot{\Lambda} . \supset . \varsigma`P_* = \dot{\Lambda} . P_s = \dot{\Lambda} .$

[*161·201] $\supset . \text{Nr}`\varsigma`P_* = \dot{1} \dotplus \text{Nr}`P_s . \text{Nr}`(\varsigma`P_*) \upharpoonright (\text{Cl}`\varsigma`P_*) = \text{Nr}`P_s$ (6)

$\vdash . (4) . (5) . (6) . \supset \vdash . \text{Prop}$

***213·171.** $\vdash :. P_{\text{po}}, Q_{\text{po}} \in \text{Ser} . \supset : P_s \,\text{smor}\, Q_s . \equiv . \varsigma`P_* \,\text{smor}\, \varsigma`Q_*$

Dem.

$\vdash . *212·172 . \supset \vdash :. \text{Hp} . \dot{\exists} ! P . \dot{\exists} ! Q . \supset : \text{E} ! B`\varsigma`P_* . \text{E} ! B`\varsigma`Q_* :$

[*204·47.*91·602.*212·3]

$\supset : \varsigma`P_* \,\text{smor}\, \varsigma`Q_* . \equiv . (\varsigma`P_*) \upharpoonright (\text{Cl}`\varsigma`P_*) \,\text{smor}\, (\varsigma`Q_*) \upharpoonright (\text{Cl}`\varsigma`Q_*) .$

[*213·17] $\equiv . P_s \,\text{smor}\, Q_s$ (1)

$\vdash . *213·158 . \supset \vdash : \text{Hp} . \dot{\exists} ! P . P_s \,\text{smor}\, Q_s . \supset . \dot{\exists} ! Q_s .$

[*212·17.*150·42] $\supset . \dot{\exists} ! Q$ (2)

$\vdash . *212·17 . \supset \vdash : \text{Hp} . \dot{\exists} ! P . \varsigma`P_* \,\text{smor}\, \varsigma`Q_* . \supset . \dot{\exists} ! Q$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash :. \text{Hp} . \dot{\exists} ! P . \supset : \varsigma`P_* \,\text{smor}\, \varsigma`Q_* . \equiv . P_s \,\text{smor}\, Q_s$ (4)

$\vdash . *212·17 . \supset \vdash : \text{Hp} . P = \dot{\Lambda} . \varsigma`P_* \,\text{smor}\, \varsigma`Q_* . \supset . \varsigma`P_* = \dot{\Lambda} . \varsigma`Q_* = \dot{\Lambda} .$

[*150·42] $\supset . P_s = \dot{\Lambda} . Q_s = \dot{\Lambda}$ (5)

$\vdash . *213·17 . \supset \vdash : \text{Hp} . P_s \,\text{smor}\, Q_s . \supset . \varsigma`P_* \,\text{smor}\, \varsigma`Q_*$ (6)

$\vdash . (5) . (6) . \supset \vdash :. \text{Hp} . P = \dot{\Lambda} . \supset : \varsigma`P_* \,\text{smor}\, \varsigma`Q_* . \equiv . P_s \,\text{smor}\, Q_s$ (7)

$\vdash . (4) . (7) . \supset \vdash . \text{Prop}$

***213·172.** $\vdash : P_{\text{po}}, Q_{\text{po}} \in \text{Ser} . P \,\text{smor}\, Q . \supset . P_s \,\text{smor}\, Q_s$ [*212·72 . *213·171]

***213·18.** $\vdash : P \in \text{connex} . R \in \text{D}'P_s . \supset . \exists ! C'P \cap p'\overleftarrow{P}``C'R$

Dem.

$\vdash . *213\cdot 1 . \supset \vdash : R \in \text{D}'P_s . \supset . (\exists \alpha) . \alpha \in \text{sect}'P - \iota'C'P . R = P \upharpoonright \alpha .$

$[*37\cdot 41] \qquad \supset . (\exists \alpha) . \alpha \in \text{sect}'P - \iota'C'P . C'R \subset \alpha .$

$[*40\cdot 16] \qquad \supset . (\exists \alpha) . \alpha \in \text{sect}'P - \iota'C'P . p'\overleftarrow{P}``\alpha \subset p'\overleftarrow{P}``C'R \quad (1)$

$\vdash . *211\cdot 703 . \supset \vdash : \text{Hp} . \alpha \in \text{sect}'P - \iota'C'P . \supset . \exists ! p'\overleftarrow{P}``\alpha \quad (2)$

$\vdash . *211\cdot 1 . \quad \supset \vdash : \alpha \in \text{sect}'P - \iota'C'P . \supset . \exists ! C'P - \alpha .$

$[*33\cdot 24] \qquad \supset . \dot{\exists} ! P \quad (3)$

$\vdash . (2) . (3) . *40\cdot 69 . \supset \vdash : \text{Hp} . \alpha \in \text{sect}'P - \iota'C'P . \supset . \exists ! C'P \cap p'\overleftarrow{P}``\alpha \quad (4)$

$\vdash . (1) . (4) . \supset$

$\vdash : \text{Hp} . R \in \text{D}'P_s . \supset . (\exists \alpha) . \exists ! C'P \cap p'\overleftarrow{P}``\alpha . p'\overleftarrow{P}``\alpha \subset p'\overleftarrow{P}``C'R : \supset \vdash . \text{Prop}$

***213·2.** $\vdash :. P_{\text{po}} \in \text{Ser} . \alpha, \beta \in \text{sect}'P - \iota'\Lambda . Q = P \upharpoonright \alpha . R = P \upharpoonright \beta . \supset :$

$\exists ! \beta - \alpha . \equiv . \dot{\exists} ! R \dot{-} Q . \equiv . Q \Subset R . Q \neq R . \equiv . \alpha \subset \beta . \alpha \neq \beta$

Dem.

$\vdash . *36\cdot 24 . \qquad \supset \vdash : \alpha \subset \beta . \supset . P \upharpoonright \alpha \Subset P \upharpoonright \beta \quad (1)$

$\vdash . *211\cdot 133 . \qquad \supset \vdash : \text{Hp} . \alpha, \beta \sim \in 1 . P \upharpoonright \alpha \Subset P \upharpoonright \beta . \supset . \alpha \subset \beta \quad (2)$

$\vdash . *211\cdot 181\cdot 182 . \supset \vdash : \text{Hp} . \alpha \in 1 . \supset . \alpha = \iota'B'P .$

$[*202\cdot 521] \qquad \supset . \alpha \subset \beta \quad (3)$

$\vdash . *213\cdot 15 . \qquad \supset \vdash : \text{Hp} . \beta \in 1 . \alpha \sim \in 1 . \supset . \sim (P \upharpoonright \alpha \Subset P \upharpoonright \beta) \quad (4)$

$\vdash . (2) . (3) . (4) . \quad \supset \vdash : \text{Hp} . P \upharpoonright \alpha \Subset P \upharpoonright \beta . \supset . \alpha \subset \beta \quad (5)$

$\vdash . (1) . (5) . \qquad \supset \vdash :. \text{Hp} . \supset : \alpha \subset \beta . \equiv . Q \Subset R : \quad (6)$

$[\text{Transp}] \qquad \supset : \exists ! \alpha - \beta . \equiv . \dot{\exists} ! Q \dot{-} R \quad (7)$

$\vdash . (6) . *213\cdot 151 . \supset$

$\vdash :. \text{Hp} . \supset : \alpha \subset \beta . \alpha \neq \beta . \equiv . Q \Subset R . Q \neq R \quad (8)$

$\vdash . (7) . (8) . *210\cdot 1 . *211\cdot 562 . \supset \vdash . \text{Prop}$

***213·21.** $\vdash :. P_{\text{po}} \in \text{Ser} . \supset : Q P_s R . \equiv . Q, R \in P \upharpoonright``(\text{sect}'P - \iota'\Lambda) . \dot{\exists} ! R \dot{-} Q .$

$\equiv . Q, R \in P \upharpoonright``(\text{sect}'P - \iota'\Lambda) . Q \Subset R . Q \neq R$

Dem.

$\vdash . *213\cdot 1\cdot 2 . \supset \vdash :. \text{Hp} . \supset :$

$Q P_s R . \equiv . (\exists \alpha, \beta) . \alpha, \beta \in \text{sect}'P - \iota'\Lambda . Q = P \upharpoonright \alpha . R = P \upharpoonright \beta . \dot{\exists} ! R \dot{-} Q .$

$\equiv . (\exists \alpha, \beta) . \alpha, \beta \in \text{sect}'P - \iota'\Lambda . Q = P \upharpoonright \alpha . R = P \upharpoonright \beta . Q \Subset R . Q \neq R \quad (1)$

$\vdash . (1) . *37\cdot 6 . \supset \vdash . \text{Prop}$

***213·22.** $\vdash :. P_{\text{po}} \in \text{Ser} . \exists ! \overrightarrow{B}'P . \supset :$

$Q P_s R . \equiv . Q, R \in P \upharpoonright``\text{sect}'P . \dot{\exists} ! R \dot{-} Q . \equiv . Q, R \in P \upharpoonright``\text{sect}'P . Q \Subset R . Q \neq R$

$[*213\cdot 21\cdot 161]$

***213·23.** $\vdash :. P_{\text{po}} \in \text{connex} . Q, R \in C'P_s . \supset : Q \Subset R . \text{v} . R \Subset Q$

$[*213\cdot 1 . *211\cdot 6\cdot 17 . *36\cdot 24]$

∗213·24. $\vdash : \beta \in \mathrm{sect}\text{‘}P \,.\, R = P \upharpoonright \beta \,.\, \supset .\, \mathrm{sect}\text{‘}R = \mathrm{sect}\text{‘}P \cap \mathrm{Cl}\text{‘}C\text{‘}R$

Dem.

$\vdash .\, {*}36{\cdot}29 \,.\, \supset \vdash :.\, \mathrm{Hp} \,.\, \supset : R \mathrel{\dot\subset} P :$ (1)

$[{*}211{\cdot}1] \qquad \supset : \alpha \in \mathrm{sect}\text{‘}P \cap \mathrm{Cl}\text{‘}C\text{‘}R \,.\, \supset .\, \alpha \subset C\text{‘}R \,.\, R\text{“}\alpha \subset \alpha \,.$

$\supset .\, \alpha \in \mathrm{sect}\text{‘}R$ (2)

$\vdash .\, (1) \,.\, {*}211{\cdot}1 \,.\, \supset$

$\vdash :.\, \mathrm{Hp} \,.\, \supset : \alpha \in \mathrm{sect}\text{‘}R \,.\, \supset .\, \alpha \subset C\text{‘}R \,.\, \alpha \subset C\text{‘}P \,.\, (P \upharpoonright \beta)\text{“}\alpha \subset \alpha$ (3)

$\vdash .\, (3) \,.\, {*}37{\cdot}41{\cdot}413 \,.\, \supset$

$\vdash : \mathrm{Hp} \,.\, \alpha \in \mathrm{sect}\text{‘}R \,.\, \supset .\, \alpha \subset \beta \,.\, \beta \cap P\text{“}(\alpha \cap \beta) \subset \alpha \,.$

$[{*}22{\cdot}621 . {*}37{\cdot}2] \qquad \supset .\, \beta \cap P\text{“}\alpha \subset \alpha \,.\, P\text{“}\alpha \subset P\text{“}\beta \,.$

$[{*}211{\cdot}1] \qquad \supset .\, \beta \cap P\text{“}\alpha \subset \alpha \,.\, P\text{“}\alpha \subset \beta \,.$

$[{*}22{\cdot}621] \qquad \supset .\, P\text{“}\alpha \subset \alpha$ (4)

$\vdash .\, (3) \,.\, (4) \,.\, \supset \vdash :.\, \mathrm{Hp} \,.\, \supset : \alpha \in \mathrm{sect}\text{‘}R \,.\, \supset .\, \alpha \subset C\text{‘}R \,.\, \alpha \in \mathrm{sect}\text{‘}P$ (5)

$\vdash .\, (2) \,.\, (5) \,.\, \supset \vdash .\, \mathrm{Prop}$

∗213·241. $\vdash : R \in P \upharpoonright\text{“}\mathrm{sect}\text{‘}P \,.\, \supset .\, R_s \mathrel{\dot\subset} P_s \upharpoonright C\text{‘}R_s$

Dem.

$\vdash .\, {*}213{\cdot}1 \,.\, \supset$

$\vdash :.\, \mathrm{Hp} \,.\, \supset : QR_sQ' \,.\, \equiv .\, (\exists \alpha, \alpha') \,.\, \alpha, \alpha' \in \mathrm{sect}\text{‘}R - \iota\text{‘}\Lambda \,.$

$Q = R \upharpoonright \alpha \,.\, Q' = R \upharpoonright \alpha' \,.\, \exists ! \alpha' - \alpha \,.$

$[{*}213{\cdot}24] \qquad \equiv .\, (\exists \alpha, \alpha') \,.\, \alpha, \alpha' \in \mathrm{sect}\text{‘}P \cap \mathrm{Cl}\text{‘}C\text{‘}R - \iota\text{‘}\Lambda \,.$

$Q = R \upharpoonright \alpha \,.\, Q' = R \upharpoonright \alpha' \,.\, \exists ! \alpha' - \alpha \,.$

$[{*}213{\cdot}1] \qquad \supset .\, QP_sQ'$ (1)

$\vdash .\, (1) \,.\, {*}33{\cdot}17 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset .\, R_s \mathrel{\dot\subset} P_s \upharpoonright C\text{‘}R_s : \supset \vdash .\, \mathrm{Prop}$

∗213·242. $\vdash : P_{\mathrm{po}} \in \mathrm{Ser} \,.\, R \in P \upharpoonright\text{“}\mathrm{sect}\text{‘}P \,.\, \supset .\, R_s = P_s \upharpoonright C\text{‘}R_s$

Dem.

$\vdash .\, {*}213{\cdot}1 \,.\, {*}211{\cdot}1 \,.\, \supset \vdash :.\, Q\,(P_s \upharpoonright C\text{‘}R_s)\, Q' \,.\, \supset :$

$(\exists \alpha, \alpha') \,.\, \alpha, \alpha' \in \mathrm{sect}\text{‘}P - \iota\text{‘}\Lambda \,.\, Q = P \upharpoonright \alpha \,.\, Q' = P \upharpoonright \alpha' \,.\, \exists ! \alpha' - \alpha :$

$(\exists \gamma, \gamma') \,.\, \gamma, \gamma' \in \mathrm{sect}\text{‘}R - \iota\text{‘}\Lambda \,.\, Q = R \upharpoonright \gamma \,.\, Q' = R \upharpoonright \gamma'$ (1)

$\vdash .\, {*}213{\cdot}24{\cdot}151 \,.\, \supset$

$\vdash :.\, \mathrm{Hp} \,.\, \supset : \alpha \in \mathrm{sect}\text{‘}P \,.\, \gamma \in \mathrm{sect}\text{‘}R \,.\, Q = P \upharpoonright \alpha = R \upharpoonright \gamma \,.\, \supset .\, \alpha = \gamma$ (2)

$\vdash .\, (1) \,.\, (2) \,.\, \supset \vdash :.\, \mathrm{Hp} \,.\, \supset : Q\,(P_s \upharpoonright C\text{‘}R_s)\, Q' \,.\, \supset .$

$(\exists \gamma, \gamma') \,.\, \gamma, \gamma' \in \mathrm{sect}\text{‘}R - \iota\text{‘}\Lambda \,.\, Q = R \upharpoonright \gamma \,.\, Q' = R \upharpoonright \gamma' \,.\, \exists ! \gamma' - \gamma \,.$

$[{*}213{\cdot}1] \supset .\, QR_sQ'$ (3)

$\vdash .\, (3) \,.\, {*}213{\cdot}241 \,.\, \supset \vdash .\, \mathrm{Prop}$

∗213·243. $\vdash .\, \overrightarrow{P_s}\text{‘}P = \mathrm{D}\text{‘}P_s$

Dem.

$\vdash .\, {*}213{\cdot}1 \,.\, \supset \vdash : R \in \overrightarrow{P_s}\text{‘}P \,.\, \equiv .\, (\exists \alpha, \beta) \,.\, \alpha, \beta \in \mathrm{sect}\text{‘}P - \iota\text{‘}\Lambda \,.$

$R = P \upharpoonright \alpha \,.\, P = P \upharpoonright \beta \,.\, \exists ! \beta - \alpha$ (1)

$\vdash .\, {*}37{\cdot}41 \,.\, \supset \vdash .\, C\text{‘}(P \upharpoonright \beta) \subset \beta$ (2)

$\vdash . (2) . \quad \supset \vdash : \exists ! C'P - \beta . \supset . \exists ! C'P - C'(P \upharpoonright \beta) .$

$[*13\cdot14] \quad \supset . P \neq P \upharpoonright \beta \quad (3)$

$\vdash . (3) . \text{Transp} . \supset \vdash : \beta \in \text{sect}'P . P = P \upharpoonright \beta . \supset . C'P = \beta \quad (4)$

$\vdash . (1) . (4) . \supset \vdash : R \in \overrightarrow{P_s}'P . \equiv . (\exists \alpha) . \alpha \in \text{sect}'P - \iota'\Lambda . R = P \upharpoonright \alpha . \exists ! C'P - \alpha .$

$[*211\cdot1] \quad \equiv . (\exists \alpha) . \alpha \in \text{sect}'P - \iota'\Lambda - \iota'C'P . R = P \upharpoonright \alpha .$

$[*213\cdot141] \quad \equiv . R \in D'P_s : \supset \vdash . \text{Prop}$

***213·244.** $\vdash : R \in C'P_s . Q \in D'R_s . \supset . QP_sR$

Dem.

$\vdash . *213\cdot243 . \supset \vdash :. R \in C'P_s . \supset : Q \in D'R_s . \supset . QR_sR .$

$[*213\cdot241] \quad \supset . QP_sR : \supset \vdash . \text{Prop}$

***213·245.** $\vdash :. P_{\text{po}} \in \text{Ser} . \supset : QP_sR . \equiv . R \in C'P_s . Q \in D'R_s$

Dem.

$\vdash . *213\cdot11 . \supset \vdash :. \text{Hp} . \supset :$

$QP_sR . \equiv . (\exists \alpha, \beta) . \alpha, \beta \in \text{sect}'P - \iota'\Lambda . Q = P \upharpoonright \alpha . R = P \upharpoonright \beta . \alpha \subset \beta . \alpha \neq \beta .$

$[*213\cdot24] \equiv . (\exists \alpha, \beta) . \beta \in \text{sect}'P - \iota'\Lambda . R = P \upharpoonright \beta . \alpha \in \text{sect}'R - \iota'\Lambda .$

$\alpha \subset \beta . \alpha \neq \beta . Q = P \upharpoonright \alpha .$

$[*213\cdot142 . *211\cdot133 . *36\cdot21]$

$\equiv . (\exists \alpha) . R \in C'P_s . \alpha \in \text{sect}'R - \iota'\Lambda . \alpha \subset C'R . \alpha \neq C'R . Q = R \upharpoonright \alpha .$

$[*213\cdot141] \equiv . R \in C'P_s . Q \in D'R_s :. \supset \vdash . \text{Prop}$

***213·246.** $\vdash : P_{\text{po}} \in \text{Ser} . R \in C'P_s . \supset . \overrightarrow{P_s}'R = D'R_s \quad [*213\cdot245]$

***213·247.** $\vdash :. P_{\text{po}} \in \text{Ser} . \supset : Q (P_s \upharpoonright D'P_s) R . \equiv . R \in D'P_s . Q \in D'R_s$
$[*213\cdot245]$

***213·25.** $\vdash :. P_{\text{po}} \in \text{Ser} . Q, R \in C'P_s . \supset : Q \in D'R_s . \vee . R \in D'Q_s . \vee . Q = R$

Dem.

$\vdash . *213\cdot153 . \supset \vdash :. \text{Hp} . \supset : QP_sR . \vee . RP_sQ . \vee . Q = R :$

$[*213\cdot245] \quad \supset : Q \in D'R_s . \vee . R \in D'Q_s . \vee . Q = R :. \supset \vdash . \text{Prop}$

***213·251.** $\vdash :. P_{\text{po}} \in \text{Ser} . Q, R \in C'P_s . \sim(Q = \dot{\Lambda} . R = \dot{\Lambda}) . \supset :$

$Q \in C'R_s . \vee . R \in D'Q_s$

Dem.

$\vdash . *213\cdot158 . \quad \supset \vdash : \text{Hp} . \dot{\exists} ! R . Q = R . \supset . Q \in C'R_s \quad (1)$

$\vdash . (1) . *13\cdot12 . \supset \vdash : \text{Hp} . \dot{\exists} ! Q . Q = R . \supset . Q \in C'R_s \quad (2)$

$\vdash . (1) . (2) . \quad \supset \vdash : \text{Hp} . Q = R . \supset . Q \in C'R_s \quad (3)$

$\vdash . (3) . *213\cdot25 . \supset \vdash . \text{Prop}$

***213·3.** $\vdash : P = \dot{\Lambda} . \supset . P_s = \dot{\Lambda}$

Dem.

$\vdash . *212\cdot17 . \supset \vdash : \text{Hp} . \supset . \mathfrak{s}'P_* = \dot{\Lambda} .$

$[*150\cdot42] \quad \supset . P_s = \dot{\Lambda} : \supset \vdash . \text{Prop}$

***213·301.** $\vdash : \exists ! \text{sect}'P - \iota'\Lambda - \iota'C'P . \supset . \dot{\exists} ! P_s \quad [*213\cdot141]$

***213·302.** $\vdash :. P_{po} \in J . \supset : \dot{\exists} ! P . \equiv . \dot{\exists} ! P_s$

Dem.

$\vdash . *213\cdot126\cdot301 . \supset \vdash : \mathrm{Hp} . \dot{\exists} ! P . \supset . \dot{\exists} ! P_s$ (1)

$\vdash . (1) . *213\cdot3 . \supset \vdash . \mathrm{Prop}$

***213·31.** $\vdash : x \neq y . \supset . (x \downarrow y)_s = \dot{\Lambda} \downarrow (x \downarrow y)$

Dem.

$\vdash . *211\cdot9 . \supset$

$\vdash : \mathrm{Hp} . \supset . \mathrm{sect}`(x \downarrow y) - \iota`\Lambda = \iota`\iota`x \cup \iota`(\iota`x \cup \iota`y) . \exists ! (\iota`x \cup \iota`y) - \iota`x .$

$[*213\cdot1\cdot141] \quad \supset . \{(x \downarrow y) \upharpoonright \iota`x\} (x \downarrow y)_s \{(x \downarrow y) \upharpoonright (\iota`x \cup \iota`y)\} .$

$\mathrm{D}`(x \downarrow y)_s = \iota`(x \downarrow y) \upharpoonright \iota`x .$

$[*200\cdot35 . *55\cdot15] \supset . \dot{\Lambda} (x \downarrow y)_s (x \downarrow y) . \mathrm{D}`(x \downarrow y)_s = \iota`\dot{\Lambda}$ (1)

$\vdash . *213\cdot153 . *204\cdot25 . \supset \vdash : \mathrm{Hp} . \supset . (x \downarrow y)_s \in \mathrm{Ser}$ (2)

$\vdash . (1) . (2) . *204\cdot27 . \supset \vdash . \mathrm{Prop}$

***213·32.** $\vdash : P \in 2_r . \supset . P_s = \dot{\Lambda} \downarrow P . P_s \in 2_r$ [*213·31]

***213·4.** $\vdash : P \in \mathrm{Ser} . \supset .$

$P \upharpoonright ``\mathrm{sect}`P = \hat{Q} \{(\exists R) . P = Q \,\text{⧺}\, R . \vee . (\exists x) . P = Q \rightarrow\!\!\!\!+ x\}$

Dem.

$\vdash . *211\cdot82 . *5\cdot32 . \supset$

$\vdash :: \mathrm{Hp} . \supset :. Q \in P \upharpoonright ``\mathrm{sect}`P . \equiv :$

$Q \in \mathrm{D}`P \upharpoonright : (\exists R) . P = Q \,\text{⧺}\, R . \vee . (\exists x) . P = Q \rightarrow\!\!\!\!+ x$ (1)

$\vdash . *211\cdot283 . *160\cdot5 . \supset \vdash : \mathrm{Hp} . P = Q \,\text{⧺}\, R . \supset . Q \in \mathrm{D}`P \upharpoonright$ (2)

$\vdash . *161\cdot11 . \supset \vdash : \mathrm{Hp} . P = Q \rightarrow\!\!\!\!+ x . \supset . Q = P \upharpoonright C`P$ (3)

$\vdash . (2) . (3) . \supset \vdash :. \mathrm{Hp} : (\exists R) . P = Q \,\text{⧺}\, R . \vee . (\exists x) . P = Q \rightarrow\!\!\!\!+ x : \supset .$

$Q \in \mathrm{D}`P \upharpoonright$ (4)

$\vdash . (1) . (4) . \supset \vdash . \mathrm{Prop}$

***213·41.** $\vdash : P \in \mathrm{Ser} . \exists ! \overrightarrow{B}`P . \supset .$

$C`P_s = \hat{Q} \{(\exists R) . P = Q \,\text{⧺}\, R . \vee . (\exists x) . P = Q \rightarrow\!\!\!\!+ x\}$ [*213·4·161]

***213·42.** $\vdash : P \in \mathrm{Ser} . \overrightarrow{B}`P = \Lambda . \supset .$

$C`P_s = \hat{Q} \{(\exists R) . P = Q \,\text{⧺}\, R . \vee . (\exists x) . P = Q \rightarrow\!\!\!\!+ x\} - \iota`\dot{\Lambda}$ [*213·4·163]

***213·5.** $\vdash : P_{po} \in J . x \sim\in C`P . \supset . \mathrm{D}`(P \rightarrow\!\!\!\!+ x)_s = C`P_s$

Dem.

$\vdash . *213\cdot141 . *211\cdot83 . \supset$

$\vdash : \mathrm{Hp} . \dot{\exists} ! P . \supset . \mathrm{D}`(P \rightarrow\!\!\!\!+ x)_s = (P \rightarrow\!\!\!\!+ x) \upharpoonright ``(\mathrm{sect}`P - \iota`\Lambda)$

$[*36\cdot4 . *161\cdot1] \quad = P \upharpoonright ``(\mathrm{sect}`P - \iota`\Lambda)$

$[*213\cdot142] \quad = C`P_s$ (1)

$\vdash . *213\cdot3 . *161\cdot2 . \supset \vdash : P = \dot{\Lambda} . \supset . \mathrm{D}`(P \rightarrow\!\!\!\!+ x)_s = \Lambda . C`P_s = \Lambda$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

$*213{\cdot}51$. $\vdash : P_{\text{po}} \subseteq J \,.\, x \sim\epsilon\, C'P \,.\, \supset \,.\, (P \nrightarrow x)_s = P_s \nrightarrow (P \nrightarrow x)$

Dem.

$\vdash \,.\, *213{\cdot}1 \,.\, *211{\cdot}83 \,.\, \supset \vdash :: \text{Hp} \,.\, \dot{\exists}\,!\, P \,.\, \supset :. \, Q\,(P \nrightarrow x)_s\, R \,.\, \equiv :$

$\qquad (\exists \alpha, \beta) \,.\, \alpha, \beta \,\epsilon\, \text{sect}'P - \iota'\Lambda \cup \iota'(C'P \cup \iota'x) \,.$

$\qquad \exists\,!\, \beta - \alpha \,.\, Q = (P \nrightarrow x) \upharpoonright \alpha \,.\, R = (P \nrightarrow x) \upharpoonright \beta \,.$

$[*211{\cdot}1 . *36{\cdot}4] \equiv : (\exists \alpha, \beta) \,.\, \alpha, \beta \,\epsilon\, \text{sect}'P - \iota'\Lambda \,.\, \exists\,!\, \beta - \alpha \,.\, Q = P \upharpoonright \alpha \,.\, R = P \upharpoonright \beta \,.\, \mathsf{v} \,.$

$\qquad (\exists \alpha) \,.\, \alpha \,\epsilon\, \text{sect}'P - \iota'\Lambda \,.\, Q = P \upharpoonright \alpha \,.\, R = P \nrightarrow x :$

$[*213{\cdot}1{\cdot}142] \quad \equiv : QP_sR \,.\, \mathsf{v} \,.\, Q \,\epsilon\, C'P_s \,.\, R = P \nrightarrow x :$

$[*161{\cdot}11] \quad \equiv : Q\,\{P_s \nrightarrow (P \nrightarrow x)\}\, R$ (1)

$\vdash \,.\, *213{\cdot}3 \,.\, *161{\cdot}2 \,.\, \supset \vdash : P = \dot{\Lambda} \,.\, \supset \,.\, (P \nrightarrow x)_s = \dot{\Lambda} \,.\, P_s \nrightarrow (P \nrightarrow x) = \dot{\Lambda}$ (2)

$\vdash \,.\, (1) \,.\, (2) \,.\, \supset \vdash \,.\, \text{Prop}$

$*213{\cdot}52$. $\vdash :. \, Q_{\text{po}} \,\epsilon\, \text{connex} \,.\, C'P \cap C'Q = \Lambda \,.\, \supset :$

$\qquad (\exists \beta) \,.\, \beta \cap C'Q \sim\epsilon\, 1 \,.\, \beta \,\epsilon\, (C'P \,\cup)``(\text{sect}'Q - \iota'\Lambda) \,.\, S = (P \ddagger Q) \upharpoonright \beta \,.\, \equiv \,.$

$\qquad (\exists \gamma) \,.\, \gamma \,\epsilon\, \text{sect}'Q - \iota'\Lambda - 1 \,.\, S = P \ddagger Q \upharpoonright \gamma$

Dem.

$\vdash \,.\, *37{\cdot}6 \,.\, \supset \vdash : \beta \,\epsilon\, (C'P \,\cup)``(\text{sect}'Q - \iota'\Lambda) \,.\, S = (P \ddagger Q) \upharpoonright \beta \,.\, \equiv \,.$

$\qquad (\exists \gamma) \,.\, \gamma \,\epsilon\, \text{sect}'Q - \iota'\Lambda \,.\, \beta = C'P \cup \gamma \,.\, S = (P \ddagger Q) \upharpoonright (C'P \cup \gamma)$ (1)

$\vdash \,.\, *160{\cdot}11 \,.\, \supset \vdash :: \text{Hp} \,.\, \gamma \,\epsilon\, \text{sect}'Q \,.\, \supset :. \, x\,\{(P \ddagger Q) \upharpoonright (C'P \cup \gamma)\}\, y \,.\, \equiv :$

$\qquad xPy \,.\, \mathsf{v} \,.\, x \,\epsilon\, C'P \,.\, y \,\epsilon\, \gamma \,.\, \mathsf{v} \,.\, x\,(Q \upharpoonright \gamma)\, y :.$

$[*211{\cdot}133 . *160{\cdot}11] \supset :. \, \gamma \sim\epsilon\, 1 \,.\, \supset \,.\, (P \ddagger Q) \upharpoonright (C'P \cup \gamma) = P \ddagger Q \upharpoonright \gamma$ (2)

$\vdash \,.\, *24{\cdot}24 \,.\, \supset \vdash : \text{Hp} \,.\, \beta = C'P \cup \gamma \,.\, \supset \,.\, \beta \cap C'Q = \gamma \cap C'Q$ (3)

$\vdash \,.\, (1) \,.\, (2) \,.\, (3) \,.\, \supset$

$\vdash :. \, \text{Hp} \,.\, \supset : \beta \cap C'Q \sim\epsilon\, 1 \,.\, \beta \,\epsilon\, (C'P \,\cup)``(\text{sect}'Q - \iota'\Lambda) \,.\, S = (P \ddagger Q) \upharpoonright \beta \,.\, \equiv \,.$

$\qquad (\exists \gamma) \,.\, \gamma \,\epsilon\, \text{sect}'Q - \iota'\Lambda - 1 \,.\, S = P \ddagger Q \upharpoonright \gamma \,.\, \beta = C'P \cup \gamma$ (4)

$\vdash \,.\, (4) \,.\, *10{\cdot}281 \,.\, *13{\cdot}19 \,.\, \supset \vdash \,.\, \text{Prop}$

$*213{\cdot}53$. $\vdash : P_{\text{po}} \subseteq J \,.\, Q_{\text{po}} \,\epsilon\, \text{Ser} \,.\, \overrightarrow{B}'Q = \Lambda \,.\, C'P \cap C'Q = \Lambda \,.\, \supset \,.$

$\qquad (P \ddagger Q)_s = P_s \ddagger (P \ddagger ; Q_s)$

Dem.

$\vdash \,.\, *213{\cdot}1 \,.\, *211{\cdot}841 \,.\, \supset \vdash :: \text{Hp} \,.\, \supset :. \, R\,(P \ddagger Q)_s\, S \,.\, \equiv :$

$\qquad (\exists \alpha, \beta) \,.\, \alpha, \beta \,\epsilon\, \text{sect}'P - \iota'\Lambda \cup (C'P \,\cup)``(\text{sect}'Q - \iota'\Lambda) \,.$

$\qquad \exists\,!\, \beta - \alpha \,.\, R = (P \ddagger Q) \upharpoonright \alpha \,.\, S = (P \ddagger Q) \upharpoonright \beta :$

$[*211{\cdot}182] \equiv : (\exists \alpha, \beta) \,.\, \alpha, \beta \,\epsilon\, \text{sect}'P - \iota'\Lambda \cup (C'P \,\cup)``(\text{sect}'Q - 1 - \iota'\Lambda) \,.$

$\qquad \exists\,!\, \beta - \alpha \,.\, R = (P \ddagger Q) \upharpoonright \alpha \,.\, S = (P \ddagger Q) \upharpoonright \beta :$

$[*160{\cdot}1 . *213{\cdot}52]$

$\quad \equiv : (\exists \alpha, \beta) \,.\, \alpha, \beta \,\epsilon\, \text{sect}'P - \iota'\Lambda \,.\, \exists\,!\, \beta - \alpha \,.\, R = P \upharpoonright \alpha \,.\, S = P \upharpoonright \beta \,.\, \mathsf{v} \,.$

$\qquad (\exists \alpha, \gamma) \,.\, \alpha \,\epsilon\, \text{sect}'P - \iota'\Lambda \,.\, \gamma \,\epsilon\, \text{sect}'Q - \iota'\Lambda \,.\, R = P \upharpoonright \alpha \,.\, S = P \ddagger Q \upharpoonright \gamma \,.\, \mathsf{v} \,.$

$\qquad (\exists \gamma, \delta) \,.\, \gamma, \delta \,\epsilon\, \text{sect}'Q - \iota'\Lambda \,.\, \exists\,!\, \delta - \gamma \,.\, R = P \ddagger Q \upharpoonright \gamma \,.\, S = P \ddagger Q \upharpoonright \delta :$

$[*213{\cdot}1{\cdot}142] \equiv : RP_sS \,.\, \mathsf{v} \,.\, R \,\epsilon\, C'P_s \,.\, S \,\epsilon\, C'P \ddagger ; Q_s \,.\, \mathsf{v} \,.\, R\,(P \ddagger ; Q_s)\, S :$

$[*160{\cdot}11] \quad \equiv : R\,\{P_s \ddagger (P \ddagger ; Q_s)\}\, S :: \supset \vdash \,.\, \text{Prop}$

***213·531.** $\vdash :: Q_{\text{po}} \in \text{Ser} \,.\, \exists ! \overrightarrow{B}`Q \,.\, C`P \cap C`Q = \Lambda \,.\, \supset :.$

$(\exists\beta) \,.\, \beta \in (C`P \cup)``(\text{sect}`Q - \iota`\Lambda) \,.\, S = (P ⤉ Q) \upharpoonright \beta \,.\, \equiv :$

$S = P +\!\!\!\rightarrow B`Q \,.\, \mathbf{v} \,.\, (\exists\gamma) \,.\, \gamma \in \text{sect}`Q - \iota`\Lambda - \iota`\iota`B`Q \,.\, S = P ⤉ Q \upharpoonright \gamma$

Dem.

$\vdash \,.\, *213\cdot52 \,.\, \supset$

$\vdash :: \text{Hp} \,.\, \supset :. (\exists\beta) \,.\, \beta \in (C`P \cup)``(\text{sect}`Q - \iota`\Lambda) \,.\, S = (P ⤉ Q) \upharpoonright \beta \,.\, \equiv :$

$(\exists\beta) \,.\, \beta \in (C`P \cup)``(\text{sect}`Q \cap 1) \,.\, S = (P ⤉ Q) \upharpoonright \beta \,.\, \mathbf{v} \,.$

$(\exists\gamma) \,.\, \gamma \in \text{sect}`Q - \iota`\Lambda - 1 \,.\, S = P ⤉ Q \upharpoonright \gamma :$

$[*211\cdot181] \equiv : (\exists\beta) \,.\, \beta = C`P \cup \iota`B`Q \,.\, S = (P ⤉ Q) \upharpoonright \beta \,.\, \mathbf{v} \,.$

$(\exists\gamma) \,.\, \gamma \in \text{sect}`Q - \iota`\Lambda - \iota`\iota`B`Q \,.\, S = P ⤉ Q \upharpoonright \gamma$ (1)

$\vdash \,.\, *160\cdot11 \,.\, \supset \vdash :: \text{Hp} \,.\, \supset :. x \{(P ⤉ Q) \upharpoonright (C`P \cup \iota`B`Q)\} y \,.\, \equiv :$

$xPy \,.\, \mathbf{v} \,.\, x \in C`P \,.\, y = B`Q :$

$[*161\cdot11] \quad \equiv : x (P +\!\!\!\rightarrow B`Q) y$ (2)

$\vdash \,.\, (1) \,.\, (2) \,.\, \supset \vdash \,.\, \text{Prop}$

***213·54.** $\vdash : \dot{\exists} ! P \,.\, P_{\text{po}} \subseteq J \,.\, Q_{\text{po}} \in \text{Ser} \,.\, \exists ! \overrightarrow{B}`Q \,.\, C`P \cap C`Q = \Lambda \,.\, \text{ᗡ}`Q_s \sim\in 1 \,.\, \supset \,.$

$(P ⤉ Q)_s = P_s +\!\!\!\rightarrow (P +\!\!\!\rightarrow B`Q) ⤉ \{P ⤉ ; (Q_s \upharpoonright \text{ᗡ}`Q_s)\}$

Dem.

$\vdash \,.\, *213\cdot1 \,.\, *211\cdot841 \,.\, \supset \vdash :: \text{Hp} \,.\, \supset :. R (P ⤉ Q)_s S \,.\, \equiv :$

$(\exists\alpha, \beta) \,.\, \alpha, \beta \in \text{sect}`P - \iota`\Lambda \cup (C`P \cup)``(\text{sect}`Q - \iota`\Lambda) \,.\, \exists ! \beta - \alpha \,.$

$R = (P ⤉ Q) \upharpoonright \alpha \,.\, S = (P ⤉ Q) \upharpoonright \beta :$

$[*213\cdot531]$

$\equiv : (\exists\alpha, \beta) \,.\, \alpha, \beta \in \text{sect}`P - \iota`\Lambda \,.\, \exists ! \beta - \alpha \,.\, R = P \upharpoonright \alpha \,.\, Q = P \upharpoonright \beta \,.\, \mathbf{v} \,.$

$(\exists\alpha) \,.\, \alpha \in \text{sect}`P - \iota`\Lambda \,.\, R = P \upharpoonright \alpha \,.\, S = P +\!\!\!\rightarrow B`Q \,.\, \mathbf{v} \,.$

$(\exists\alpha, \gamma) \,.\, \alpha \in \text{sect}`P - \iota`\Lambda \,.\, R = P \upharpoonright \alpha \,.\, \beta \in \text{sect}`Q - \iota`\Lambda - \iota`\iota`B`Q \,.$

$S = P ⤉ Q \upharpoonright \gamma \,.\, \mathbf{v} \,.$

$(\exists\gamma) \,.\, R = P +\!\!\!\rightarrow B`Q \,.\, \beta \in \text{sect}`Q - \iota`\Lambda - \iota`\iota`B`Q \,.\, S = P ⤉ Q \upharpoonright \gamma \,.\, \mathbf{v} \,.$

$(\exists\gamma, \delta) \,.\, \gamma, \delta \in \text{sect}`Q - \iota`\Lambda - \iota`\iota`B`Q \,.\, \exists ! \delta - \gamma \,.\, R = P ⤉ Q \upharpoonright \gamma \,.$

$S = P ⤉ Q \upharpoonright \delta :$

$[*213\cdot1\cdot142\cdot132]$

$\equiv : RP_sS \,.\, \mathbf{v} \,.\, R \in C`P_s \,.\, S = P +\!\!\!\rightarrow B`Q \,.\, \mathbf{v} \,.\, R = P +\!\!\!\rightarrow B`Q \,.\, S \in P ⤉ ``\text{ᗡ}`Q_s \,.$

$\mathbf{v} \,.\, R, S \in (P ⤉ ``) \text{ᗡ}`Q_s \,.\, R (P ⤉ ; Q) S \,.\, \mathbf{v} \,.\, R \in C`P_s \,.\, S \in P ⤉ ``\text{ᗡ}`Q_s :$

$[*161\cdot11 \,.\, *211\cdot133 \,.\, *160\cdot11]$

$\equiv : R \{P_s +\!\!\!\rightarrow (P +\!\!\!\rightarrow B`Q) ⤉ (P ⤉ ; Q_s \upharpoonright \text{ᗡ}`Q_s)\} S :: \supset \vdash \,.\, \text{Prop}$

***213·541.** $\vdash : P_{po} \in \mathrm{Ser} \,.\, \exists ! \overrightarrow{B}\text{‘}P \,.\, \text{Ɑ}\text{‘}P_s \in 1 \,.\, \supset .\, P \in 2_r$

Dem.

$\vdash . *213\cdot144 . *211\cdot26 . \supset \vdash :. \mathrm{Hp} . \supset : P \upharpoonright \text{‘‘}(\mathrm{sect}\text{‘}P - \iota\text{‘}\Lambda - \iota\text{‘}\iota\text{‘}B\text{‘}P) = \iota\text{‘}P :$

$[*211\cdot3\cdot13] \qquad \supset : x \in \text{Ɑ}\text{‘}P . \supset . P \upharpoonright \overrightarrow{P_*}\text{‘}x = P .$

$[*202\cdot55] \qquad \supset . \overrightarrow{P_*}\text{‘}x = C\text{‘}P .$

$[*200\cdot39] \qquad \supset . \overleftarrow{P}_{po}\text{‘}x = \Lambda .$

$[*202\cdot522\cdot523] \qquad \supset . x = B\text{‘}\breve{P} :$

$[*204\cdot271] \qquad \supset : P_{po} \in 2_r :$

$[*56\cdot111 . *91\cdot504] \qquad \supset : P \in 2_r :. \supset \vdash . \mathrm{Prop}$

***213·55.** $\vdash : \dot\exists ! P . P_{po} \mathrel{\dot\subset} J . Q \in 2_r . C\text{‘}P \cap C\text{‘}Q = \Lambda . \supset .$
$$(P \mathbin{\uparrow\!\!\!\!+} Q)_s = P_s \mathbin{+\!\!\!\rightarrow} (P \mathbin{+\!\!\!\rightarrow} B\text{‘}Q) \mathbin{+\!\!\!\rightarrow} (P \mathbin{\uparrow\!\!\!\!+} Q)$$

Dem. As in *213·54,

$\vdash :: \mathrm{Hp} . \supset :. R (P \mathbin{\uparrow\!\!\!\!+} Q)_s S .$

$\equiv : R P_s S . \vee . R \in C\text{‘}P_s . S = P \mathbin{+\!\!\!\rightarrow} B\text{‘}Q . \vee . R \in C\text{‘}P_s . S \in P \mathbin{\uparrow\!\!\!\!+} \text{‘‘}\text{Ɑ}\text{‘}Q_s .$
$\vee . R = P \mathbin{+\!\!\!\rightarrow} B\text{‘}Q . S \in P \mathbin{\uparrow\!\!\!\!+} \text{‘‘}\text{Ɑ}\text{‘}Q_s . \vee . R, S \in P \mathbin{\uparrow\!\!\!\!+} \text{‘‘}\text{Ɑ}\text{‘}Q_s .$
$R (P \mathbin{\uparrow\!\!\!\!+} ; Q_s) S :$

$[*213\cdot32] \equiv : R P_s S . \vee . R \in C\text{‘}P_s . S = P \mathbin{+\!\!\!\rightarrow} B\text{‘}Q . \vee . R \in C\text{‘}P_s . S = P \mathbin{\uparrow\!\!\!\!+} Q .$
$\vee . R = P \mathbin{+\!\!\!\rightarrow} B\text{‘}Q . S = P \mathbin{\uparrow\!\!\!\!+} Q . \vee . R = P \mathbin{\uparrow\!\!\!\!+} Q . S = P \mathbin{\uparrow\!\!\!\!+} Q .$
$R (P \mathbin{\uparrow\!\!\!\!+} ; Q_s) S :$

$[*213\cdot32] \equiv : R P_s S . \vee . R \in C\text{‘}P_s . S = P \mathbin{+\!\!\!\rightarrow} B\text{‘}Q . \vee . R \in C\text{‘}P_s . S = P \mathbin{\uparrow\!\!\!\!+} Q .$
$\vee . R = P \mathbin{+\!\!\!\rightarrow} B\text{‘}Q . S = P \mathbin{\uparrow\!\!\!\!+} Q :$

$[*161\cdot11] \equiv : R \{P_s \mathbin{+\!\!\!\rightarrow} (P \mathbin{+\!\!\!\rightarrow} B\text{‘}Q) \mathbin{+\!\!\!\rightarrow} (P \mathbin{\uparrow\!\!\!\!+} Q)\} S :: \supset \vdash . \mathrm{Prop}$

***213·56.** $\vdash :. P_{po} \mathrel{\dot\subset} J . Q_{po} \in \mathrm{Ser} . C\text{‘}P \cap C\text{‘}Q = \Lambda . \supset :$

$\overrightarrow{B}\text{‘}Q = \Lambda . \supset . (P \mathbin{\uparrow\!\!\!\!+} Q)_s = P_s \mathbin{\uparrow\!\!\!\!+} (P \mathbin{\uparrow\!\!\!\!+} ; Q_s) :$

$\dot\exists ! P . \exists ! \overrightarrow{B}\text{‘}Q . Q \sim\in 2_r . \supset .$
$$(P \mathbin{\uparrow\!\!\!\!+} Q)_s = P_s \mathbin{+\!\!\!\rightarrow} (P \mathbin{+\!\!\!\rightarrow} B\text{‘}Q) \mathbin{\uparrow\!\!\!\!+} \{P \mathbin{\uparrow\!\!\!\!+} ; (Q_s \upharpoonright \text{Ɑ}\text{‘}Q_s)\} :$$

$\dot\exists ! P . Q \in 2_r . \supset . (P \mathbin{\uparrow\!\!\!\!+} Q)_s = P_s \mathbin{+\!\!\!\rightarrow} (P \mathbin{+\!\!\!\rightarrow} B\text{‘}Q) \mathbin{+\!\!\!\rightarrow} (P \mathbin{\uparrow\!\!\!\!+} Q) :$

$P = \dot\Lambda . \supset . (P \mathbin{\uparrow\!\!\!\!+} Q)_s = Q_s \quad [*213\cdot53\cdot54\cdot541\cdot55 . *160\cdot22]$

***213·561.** $\vdash : C\text{‘}P \cap C\text{‘}Q = \Lambda . \supset . (P \mathbin{\uparrow\!\!\!\!+}) \upharpoonright C\text{‘}Q_s \in 1 \rightarrow 1$

Dem.

$\vdash . *213\cdot1 . \supset \vdash : R \in C\text{‘}Q_s . \supset . C\text{‘}R \subset C\text{‘}Q \qquad (1)$

$\vdash . (1) . \supset \vdash :. \mathrm{Hp} . R, S \in C\text{‘}Q_s . \supset : C\text{‘}P \cap C\text{‘}R = \Lambda . C\text{‘}P \cap C\text{‘}S = \Lambda :$

$[*160\cdot52] \qquad \supset : P \mathbin{\uparrow\!\!\!\!+} R = P \mathbin{\uparrow\!\!\!\!+} S . \supset . R = S :. \supset \vdash . \mathrm{Prop}$

***213·57.** $\vdash : P_{po} \mathrel{\dot\subset} J . \mathrm{Nr}\text{‘}Q = \mathrm{Nr}\text{‘}P \dot{+} \dot{1} . \supset . \mathrm{Nr}\text{‘}Q_s = \mathrm{Nr}\text{‘}P_s \dot{+} \dot{1}$

Dem.

$\vdash . *181\cdot2\cdot12 . (*181\cdot01) . \supset$

$\vdash : \mathrm{Hp} . \supset . (\exists R, x) . R \,\mathrm{smor}\, P . x \sim\in C\text{‘}R . Q = R \mathbin{+\!\!\!\rightarrow} x .$

$[*213\cdot51] \quad \supset . (\exists R, x) . R \,\mathrm{smor}\, P . x \sim\in C\text{‘}R . Q_s = R_s \mathbin{+\!\!\!\rightarrow} (R \mathbin{+\!\!\!\rightarrow} x) .$

$[*181\cdot32] \quad \supset . (\exists R) . R \,\mathrm{smor}\, P . \mathrm{Nr}\text{‘}Q_s = \mathrm{Nr}\text{‘}R_s \dot{+} \dot{1} .$

$[*213\cdot172] \supset . \mathrm{Nr}\text{‘}Q_s = \mathrm{Nr}\text{‘}P_s \dot{+} \dot{1} : \supset \vdash . \mathrm{Prop}$

***213·58.** $\vdash : P_{po} \subseteq J . Q_{po} \in \mathrm{Ser} . C'P \cap C'Q = \Lambda . \supset . \mathrm{Nr}'(P \mathbin{\ddagger} Q)_s = \mathrm{Nr}'P_s \dotplus \mathrm{Nr}'Q_s$

Dem.

$\vdash . *213·53·561 . \supset$

$\vdash : \mathrm{Hp} . \overrightarrow{B}'Q = \Lambda . \supset . (P \mathbin{\ddagger} Q)_s = P_s \mathbin{\ddagger} (P \mathbin{\ddagger} {}^{;} Q_s) . \mathrm{Nr}'P \mathbin{\ddagger} {}^{;} Q_s = \mathrm{Nr}'Q_s .$

[*180·32] $\supset . \mathrm{Nr}'(P \mathbin{\ddagger} Q)_s = \mathrm{Nr}'P_s \dotplus \mathrm{Nr}'Q^s$ (1)

$\vdash . *213·54·561 . *181·32 . \supset$

$\vdash : \mathrm{Hp} . \dot{\exists} ! P . \exists ! \overrightarrow{B}'Q . \text{C}'Q_s \sim \in 1 . \supset . \mathrm{Nr}'(P \mathbin{\ddagger} Q)_s = \mathrm{Nr}'P_s \dotplus \dot{1} \dotplus \mathrm{Nr}'Q_s \upharpoonright \text{C}'Q_s$

[*204·46.*213·157] $= \mathrm{Nr}'P_s \dotplus \mathrm{Nr}'Q_s$ (2)

$\vdash . *213·541·55 . *181·32 . \supset$

$\vdash : \mathrm{Hp} . \dot{\exists} ! P . \text{C}'Q_s \in 1 . \supset . Q \in 2_r . \mathrm{Nr}'(P \mathbin{\ddagger} Q)_s = \mathrm{Nr}'P_s \dotplus \dot{1} \dotplus \dot{1} .$

[*181·56] $\supset . Q \in 2_r . \mathrm{Nr}'(P \mathbin{\ddagger} Q)_s = \mathrm{Nr}'P_s \dotplus 2_r .$

[*213·32] $\supset . \mathrm{Nr}'(P \mathbin{\ddagger} Q)_s = \mathrm{Nr}'P_s \dotplus \mathrm{Nr}'Q_s$ (3)

$\vdash . *160·22 . *213·3 . \supset \vdash : P = \dot{\Lambda} . \supset . \mathrm{Nr}'(P \mathbin{\ddagger} Q)_s = \mathrm{Nr}'P_s \dotplus \mathrm{Nr}'Q_s$ (4)

$\vdash . (1) . (2) . (3) . (4) . \supset \vdash .$ Prop

$*214$. DEDEKINDIAN RELATIONS

Summary of $*214$.

We call a relation "Dedekindian" when it is such that every class has either a maximum or a sequent with respect to it. As a rule, the hypothesis that a relation is Dedekindian is only important in the case of serial relations. Dedekindian series have considerable importance, especially in connection with limits.

When P is transitive, the hypothesis that P is Dedekindian is equivalent to the hypothesis that every section of P has a maximum or a sequent ($*214\cdot13$); it is also equivalent to the assumption that every segment of P has a maximum or a sequent ($*214\cdot131$), *i.e.* to the assumption that every segment of P which has no maximum has a limit, *i.e.* to

$$\text{D}`(P_\epsilon \dot{\cap} I) \subset \text{Ɑ}`\text{lt}_P.$$

When P is a series, the hypothesis that it is Dedekindian is equivalent to the hypothesis that every segment has a sequent ($*214\cdot15$), *i.e.* to the hypothesis that the class of segments is the class $\overrightarrow{P}``C`P$ ($*214\cdot151$). If $P$ is a Dedekindian series, so is $\breve{P}$, and vice versa ($*214\cdot14$). Whenever $P$ is connected and not null, $s`P_*$ is a Dedekindian series ($*214\cdot32$), and so is $\text{sgm}`P$ if it exists ($*214\cdot34$); whenever $P$ is transitive and connected and not null, $s`P$ is a Dedekindian series ($*214\cdot33$). All these propositions have been virtually proved already: almost the only thing new in the present number is the definition, which is

$$\text{Ded} = \hat{P}\{(\alpha) . \alpha \in \text{Ɑ}`\text{max}_P \cup \text{Ɑ}`\text{seq}_P\} \quad \text{Df.}$$

$*214\cdot4$—$\cdot43$ give properties of series which have Dedekindian continuity. We have

$*214\cdot4$. $\vdash :. P^2 = P . P \in \text{connex} . \supset : P \in \text{Ded} . \equiv . \text{Ɑ}`\text{max}_P = - \text{Ɑ}`\text{seq}_P$

$*214\cdot41$. $\vdash :. P \in \text{Ser} . \supset : P^2 = P . P \in \text{Ded} . \equiv . \text{Ɑ}`\text{max}_P = - \text{Ɑ}`\text{seq}_P$

I.e. in a series, Dedekindian continuity is equivalent to the assumption that the classes which have a maximum are the same as the classes which have no sequent.

$*214\cdot42$. $\vdash : P \in \text{Ser} \cap \text{Ded} . P^2 = P . \alpha \in \text{sect}`P . \supset . \text{limax}_P`\alpha = \text{limin}_P`(C`P - \alpha)$

This proposition is important in dealing with Dedekind "cuts."

$*214\cdot43$. $\vdash :. P \in \text{Ser} \cap \text{Ded} . \alpha \in \text{sect}`P . \supset :$

$$\text{limax}_P`\alpha = \text{limin}_P`(C`P - \alpha) . \vee . \text{max}_P`\alpha \, P_1 \, \text{min}_P`(C`P - \alpha)$$

$*214\cdot5$ shows that a Dedekindian relation has a beginning and an end; the following propositions deal with $P \dot{\cap} J$ when P is Dedekindian.

$*214\cdot6$ shows that a relation which is similar to a Dedekindian relation is Dedekindian.

We call a relation "semi-Dedekindian" if it becomes Dedekindian by the addition of one term at the end; the definition is

$*214\cdot02$. semi Ded $= \hat{P}\,(\text{sect}`P - \iota`C`P \subset \text{Cl}`\text{max}_P \cup \text{Cl}`\text{seq}_P)$ Df

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$*214\cdot01$. Ded $= \hat{P}\,\{(\alpha) . \alpha \in \text{Cl}`\text{max}_P \cup \text{Cl}`\text{seq}_P\}$ Df

$*214\cdot02$. semi Ded $= \hat{P}\,(\text{sect}`P - \iota`C`P \subset \text{Cl}`\text{max}_P \cup \text{Cl}`\text{seq}_P)$ Df

$*214\cdot1$. $\vdash : P \in \text{Ded} . \equiv . (\alpha) . \alpha \in \text{Cl}`\text{max}_P \cup \text{Cl}`\text{seq}_P$ [$(*214\cdot01)$]

$*214\cdot101$. $\vdash : P \in \text{Ded} . \equiv . - \text{Cl}`\text{max}_P \subset \text{Cl}`\text{seq}_P . \equiv . - \text{Cl}`\text{max}_P \subset \text{Cl}`\text{lt}_P$
[$*214\cdot1 . *24\cdot312 . *207\cdot12$]

$*214\cdot11$. $\vdash : P \in \text{Ded} . \equiv . (\alpha) . \alpha \in \text{Cl}`\text{max}_P \cup \text{Cl}`\text{lt}_P . \equiv . (\alpha) . \alpha \in \text{Cl}`\text{limax}_P$
[$*214\cdot1 . *207\cdot14\cdot44$]

$*214\cdot12$. $\vdash :. P \in \text{Ded} . \equiv : \alpha \subset C`P . \supset_\alpha . \alpha \in \text{Cl}`\text{max}_P \cup \text{Cl}`\text{seq}_P$
[$*214\cdot1 . *205\cdot151 . *206\cdot131$]

$*214\cdot13$. $\vdash :. P \in \text{trans} . \supset : P \in \text{Ded} . \equiv . \text{sect}`P \subset \text{Cl}`\text{max}_P \cup \text{Cl}`\text{seq}_P$
[$*211\cdot272 . *214\cdot1$]

$*214\cdot131$. $\vdash :. P \in \text{trans} . \supset : P \in \text{Ded} . \equiv . \text{D}`(P_\epsilon \dot{\cap} I) \subset \text{Cl}`\text{seq}_P$ [$*211\cdot47 . *214\cdot1$]

$*214\cdot132$. $\vdash :. P \in \text{trans} . \supset : P \in \text{Ded} . \equiv . \text{D}`P_\epsilon \subset \text{Cl}`\text{max}_P \cup \text{Cl}`\text{seq}_P$
[$*214\cdot131 . *211\cdot42$]

$*214\cdot14$. $\vdash :. P \in \text{Ser} . \supset : P \in \text{Ded} . \equiv . \breve{P} \in \text{Ded}$ [$*206\cdot57 . *214\cdot1$]

$*214\cdot141$. $\vdash :. P \in \text{Ser} . \supset : P \in \text{Ded} . \equiv . (\alpha) . p`\overrightarrow{P}``(\alpha \cap C`P) \in \text{Cl}`\text{max}_P \cup \text{Cl}`\text{seq}_P$
[$*206\cdot56 . *214\cdot1$]

$*214\cdot15$. $\vdash :. P \in \text{Ser} . \supset : P \in \text{Ded} . \equiv . \text{D}`P_\epsilon \subset \text{Cl}`\text{seq}_P$
[$*206\cdot36 . *214\cdot1 . *211\cdot11$]

$*214\cdot151$. $\vdash :. P \in \text{Ser} . \supset : P \in \text{Ded} . \equiv . \text{D}`P_\epsilon = \overrightarrow{P}``C`P$ [$*211\cdot38 . *214\cdot1$]

$*214\cdot2$. $\vdash : P \in \text{trans} \cap \text{connex} \cap \text{Ded} . \supset . \text{D}`P_\epsilon \subset \text{Cl}`\text{seq}_P$ [$*211\cdot371$]

$*214\cdot21$. $\vdash : P \in \text{trans} \cap \text{connex} \cap \text{Ded} . \supset . \text{D}`P_\epsilon = \overrightarrow{P}``C`P$ [$*211\cdot372$]

$*214\cdot22$. $\vdash : P \in \text{trans} \cap \text{connex} \cap \text{Ded} . \supset . \text{D}`(P_\epsilon \dot{\cap} I) = \overrightarrow{P}``\{C`P - \text{Cl}`(P \dot{-} P^2)\}$
[$*211\cdot46$]

$*214\cdot23$. $\vdash : P \in \text{trans} \cap \text{connex} \cap \text{Ded} . \sim \text{E}!\, \text{max}_P`\alpha . \supset .$
$\text{seq}_P`\alpha = \text{max}_P`(\alpha \cup \iota`\text{seq}_P`\alpha) . \text{E}!\, \text{max}_P`(\alpha \cup \iota`\text{seq}_P`\alpha)$

Dem.

$\vdash . *214\cdot101 . \supset \vdash : \text{Hp} . \supset . \text{E}!\, \text{seq}_P`\alpha .$
[$*206\cdot47$] $\supset . \text{seq}_P`\alpha = \text{max}_P`(\alpha \cup \iota`\text{seq}_P`\alpha) .$ (1)
[$*14\cdot21$] $\supset . \text{E}!\, \text{max}_P`(\alpha \cup \iota`\text{seq}_P`\alpha)$ (2)
$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***214·24.** $\vdash : P \epsilon \text{connex} \cap \text{Ded} . \alpha \epsilon \text{sect}'P . \supset . \overrightarrow{\text{seq}}_P{}'\alpha = \overrightarrow{\text{min}}_P{}'(C'P - \alpha)$ [*211·721]

***214·241.** $\vdash : P \epsilon \text{connex} . \breve{P} \epsilon \text{Ded} . \alpha \epsilon \text{sect}'P . \supset . \overrightarrow{\text{max}}_P{}'\alpha = \overrightarrow{\text{prec}}_P{}'(C'P - \alpha)$ $\left[*214{\cdot}24 \frac{\breve{P}}{P} . *211{\cdot}7\right]$

***214·3.** $\vdash :: \alpha, \beta \epsilon \kappa . \supset_{\alpha,\beta} : \alpha \subset \beta . \text{v} . \beta \subset \alpha :.$
$\kappa \sim\epsilon 1 . Q = \hat{\alpha}\hat{\beta}(\alpha, \beta \epsilon \kappa . \alpha \subset \beta . \alpha \neq \beta) :. \supset :.$
$\lambda \subset \kappa . \supset_\lambda . s'\lambda \epsilon \kappa : \supset . Q \epsilon \text{Ser} \cap \text{Ded}$
[*210·12·253]

***214·31.** $\vdash :. \text{Hp} *214{\cdot}3 : \lambda \subset \kappa . \supset_\lambda . p'\lambda \cap s'\kappa \epsilon \kappa : \supset . Q \epsilon \text{Ser} \cap \text{Ded}$ [*210·12·254]

***214·32.** $\vdash : P \epsilon \text{connex} . \dot{\exists} ! P . \supset . \varsigma'P_* \epsilon \text{Ser} \cap \text{Ded}$ [*212·3·35]

***214·33.** $\vdash : P \epsilon \text{trans} \cap \text{connex} . \dot{\exists} ! P . \supset . \varsigma'P \epsilon \text{Ser} \cap \text{Ded}$ [*212·31·44]

***214·34.** $\vdash : P \epsilon \text{connex} . \dot{\exists} ! \text{sgm}'P . \supset . \text{sgm}'P \epsilon \text{Ser} \cap \text{Ded}$ [*212·3·54]

***214·4.** $\vdash :. P^2 = P . P \epsilon \text{connex} . \supset : P \epsilon \text{Ded} . \equiv . \text{Cl}'\text{max}_P = -\text{Cl}'\text{seq}_P$ [*211·53]

***214·41.** $\vdash :. P \epsilon \text{Ser} . \supset : P^2 = P . P \epsilon \text{Ded} . \equiv . \text{Cl}'\text{max}_P = -\text{Cl}'\text{seq}_P$ [*211·552]

***214·42.** $\vdash : P \epsilon \text{Ser} \cap \text{Ded} . P^2 = P . \alpha \epsilon \text{sect}'P . \supset . \text{limax}_P{}'\alpha = \text{limin}_P{}'(C'P - \alpha)$

Dem.

$\vdash . *211{\cdot}721 . \supset \vdash :. \text{Hp} . \supset : \overrightarrow{\text{seq}}_P{}'\alpha = \overrightarrow{\text{min}}_P{}'(C'P - \alpha) :$
[*214·101] $\supset : \sim \text{E} ! \text{max}_P{}'\alpha . \supset . \text{lt}_P{}'\alpha = \text{min}_P{}'(C'P - \alpha)$ (1)
$\vdash . *211{\cdot}726 . \supset \vdash : \text{Hp} . \text{E} ! \text{max}_P{}'\alpha . \supset . \text{max}_P{}'\alpha = \text{prec}_P{}'(C'P - \alpha)$ (2)
$\vdash . *214{\cdot}14{\cdot}41 . \supset \vdash : \text{Hp} . \text{E} ! \text{prec}_P{}'(C'P - \alpha) . \supset . \sim \text{E} ! \text{max}_P{}'(C'P - \alpha) .$
[*207·12] $\supset . \text{prec}_P{}'(C'P - \alpha) = \text{tl}_P{}'(C'P - \alpha)$ (3)
$\vdash . (2) . (3) . \supset \vdash : \text{Hp} . \text{E} ! \text{max}_P{}'\alpha . \supset . \text{max}_P{}'\alpha = \text{tl}_P{}'(C'P - \alpha)$ (4)
$\vdash . (1) . (4) . *207{\cdot}46 . \supset$
$\vdash :. \text{Hp} . \supset : \text{limax}_P{}'\alpha = \text{min}_P{}'(C'P - \alpha) . \text{v} . \text{limax}_P{}'\alpha = \text{tl}_P{}'(C'P - \alpha) :$
[*207·46] $\supset : \text{limax}_P{}'\alpha = \text{limin}_P{}'(C'P - \alpha) :. \supset \vdash . \text{Prop}$

***214·43.** $\vdash :. P \epsilon \text{Ser} \cap \text{Ded} . \alpha \epsilon \text{sect}'P . \supset :$
$\text{limax}_P{}'\alpha = \text{limin}_P{}'(C'P - \alpha) . \text{v} . \text{max}_P{}'\alpha \, P_1 \, \text{min}_P{}'(C'P - \alpha)$

Dem.

$\vdash . *214{\cdot}11 . \supset \vdash :. \text{Hp} . \supset : \sim \text{E} ! \text{max}_P{}'\alpha . \supset . \text{limax}_P{}'\alpha = \text{seq}_P{}'\alpha$
[*211·715] $= \text{min}_P{}'(C'P - \alpha)$ (1)
$\vdash . *211{\cdot}726 . \supset \vdash : \text{E} ! \text{max}_P{}'\alpha . \sim \text{E} ! \text{min}_P{}'(C'P - \alpha) . \supset .$
$\text{limax}_P{}'\alpha = \text{tl}_P{}'(C'P - \alpha)$ (2)
$\vdash . (1) . (2) . *207{\cdot}46 . \supset \vdash :. \text{Hp} : \sim \text{E} ! \text{max}_P{}'\alpha . \text{v} . \sim \text{E} ! \text{min}_P{}'(C'P - \alpha) : \supset .$
$\text{limax}_P{}'\alpha = \text{limin}_P{}'(C'P - \alpha)$ (3)

$\vdash . *211{\cdot}726 . \supset \vdash : \text{Hp} . \text{E}!\,\max_P{}^{\backprime}\alpha . \text{E}!\,\min_P{}^{\backprime}(C{}^{\backprime}P - \alpha) . \supset .$

$\text{E}!\,\max_P{}^{\backprime}\alpha . \text{E}!\,\text{seq}_P{}^{\backprime}\alpha . \text{seq}_P{}^{\backprime}\alpha = \min_P{}^{\backprime}(C{}^{\backprime}P - \alpha) .$

$[*206{\cdot}5] \quad \supset . \max_P{}^{\backprime}\alpha \, P_1 \min_P{}^{\backprime}(C{}^{\backprime}P - \alpha) \quad (4)$

$\vdash . (3) . (4) . \supset \vdash . \text{Prop}$

The following propositions are no longer mere restatements of previous results.

***214·5.** $\vdash : P \in \text{Ded} . \supset . \exists !\,\overrightarrow{B}{}^{\backprime}P . \exists !\,\overrightarrow{B}{}^{\backprime}\breve{P} . \overrightarrow{B}{}^{\backprime}P = \overrightarrow{\text{seq}_P}{}^{\backprime}\Lambda . \overrightarrow{B}{}^{\backprime}\breve{P} = \overrightarrow{\max_P}{}^{\backprime}C{}^{\backprime}P$

Dem.

$\vdash . *205{\cdot}161 . *214{\cdot}101 . \supset \vdash : \text{Hp} . \supset . \exists !\,\overrightarrow{\text{seq}_P}{}^{\backprime}\Lambda .$

$[*206{\cdot}14] \quad \supset . \exists !\,\overrightarrow{B}{}^{\backprime}P . \overrightarrow{B}{}^{\backprime}P = \overrightarrow{\text{seq}_P}{}^{\backprime}\Lambda \quad (1)$

$\vdash . *206{\cdot}18{\cdot}2 . \quad \supset \vdash . \overrightarrow{\text{seq}_P}{}^{\backprime}C{}^{\backprime}P = \Lambda \quad (2)$

$\vdash . (2) . *214{\cdot}1 . \supset \vdash : \text{Hp} . \supset . \exists !\,\overrightarrow{\max_P}{}^{\backprime}C{}^{\backprime}P .$

$[*93{\cdot}117] \quad \supset . \exists !\,\overrightarrow{B}{}^{\backprime}\breve{P} . \overrightarrow{B}{}^{\backprime}\breve{P} = \overrightarrow{\max_P}{}^{\backprime}C{}^{\backprime}P \quad (3)$

$\vdash . (1) . (3) . \supset \vdash . \text{Prop}$

***214·51.** $\vdash :. P \in \text{Ded} . \supset : \sim(xPx) . \vee . x \in \text{D}{}^{\backprime}(P \dot{-} P^2)$

Dem.

$\vdash . *214{\cdot}1 . \supset \vdash :. \text{Hp} . \supset : \exists !\,\overrightarrow{\max_P}{}^{\backprime}\iota{}^{\backprime}x . \vee . \exists !\,\overrightarrow{\text{seq}_P}{}^{\backprime}\iota{}^{\backprime}x :$

$[*53{\cdot}301 . *206{\cdot}42] \quad \supset : \exists !\,\iota{}^{\backprime}x - \overrightarrow{P}{}^{\backprime}x . \vee . \exists !\,\overleftarrow{P \dot{-} P^2}{}^{\backprime}x :$

$[*51{\cdot}31 . *33{\cdot}4] \quad \supset : \sim(xPx) . \vee . x \in \text{D}{}^{\backprime}(P \dot{-} P^2) :. \supset \vdash . \text{Prop}$

***214·52.** $\vdash : P \in \text{Ded} . P \mathrel{\text{G}} P^2 . \supset . P \mathrel{\text{G}} J \quad [*214{\cdot}51]$

***214·53.** $\vdash : P \in \text{Ded} . \supset . \text{D}{}^{\backprime}P = \text{D}{}^{\backprime}(P \mathbin{\dot{\cap}} J)$

Dem.

$\vdash . *214{\cdot}51 . \supset \vdash : \text{Hp} . xPx . \supset . x \in \text{D}{}^{\backprime}(P \dot{-} P^2) .$

$[*33{\cdot}13] \quad \supset . (\exists y) . xPy . x \dot{-} P^2 y .$

$[*34{\cdot}54 . \text{Transp}] \quad \supset . (\exists y) . xPy . x \neq y \quad (1)$

$\vdash . (1) . *13{\cdot}195 . \supset \vdash :. \text{Hp} . \supset : (\exists y) . xPy . \supset . (\exists y) . xPy . x \neq y :$

$[*33{\cdot}13] \quad \supset : \text{D}{}^{\backprime}P \subset \text{D}{}^{\backprime}(P \mathbin{\dot{\cap}} J) :$

$[*33{\cdot}25] \quad \supset : \text{D}{}^{\backprime}P = \text{D}{}^{\backprime}(P \mathbin{\dot{\cap}} J) :. \supset \vdash . \text{Prop}$

***214·531.** $\vdash : P \in \text{Ded} . \supset . C{}^{\backprime}P = C{}^{\backprime}(P \mathbin{\dot{\cap}} J)$

Dem.

$\vdash . *93{\cdot}12 . \supset \vdash :. x \in \overrightarrow{B}{}^{\backprime}\breve{P} . \supset : x \sim\in \text{D}{}^{\backprime}P : (\exists y) . yPx :$

$[*13{\cdot}14] \quad \supset : (\exists y) . yPx . x \neq y :$

$[*33{\cdot}13] \quad \supset : x \in \text{Q}{}^{\backprime}(P \mathbin{\dot{\cap}} J) \quad (1)$

$\vdash . (1) . *214{\cdot}53 . \supset \vdash : \text{Hp} . \supset . \text{D}{}^{\backprime}P \cup \overrightarrow{B}{}^{\backprime}\breve{P} \subset C{}^{\backprime}(P \mathbin{\dot{\cap}} J) .$

$[*93{\cdot}12] \quad \supset . C{}^{\backprime}P \subset C{}^{\backprime}(P \mathbin{\dot{\cap}} J) .$

$[*33{\cdot}252] \quad \supset . C{}^{\backprime}P = C{}^{\backprime}(P \mathbin{\dot{\cap}} J) : \supset \vdash . \text{Prop}$

$*214{\cdot}532.\ \vdash : P \in \mathrm{Ded} \,.\supset .\, \text{Ɑ}\text{‘}P = \text{Ɑ}\text{‘}(P \mathbin{\dot\cap} J)$

Dem.

$\vdash .\, *34{\cdot}54 \,.\ \supset \vdash : \overrightarrow{P}\text{‘}x = \iota\text{‘}x \,.\supset .\, \overrightarrow{P}\text{‘}x = P\text{‘‘}\overrightarrow{P}\text{‘}x \,.$

$[*205{\cdot}123] \qquad \supset .\, \overrightarrow{\max}_P\text{‘}\iota\text{‘}x = \Lambda \qquad (1)$

$\vdash .\, *206{\cdot}134 \,.\supset \vdash : \overrightarrow{P}\text{‘}x = \iota\text{‘}x \,.\supset .$

$\overrightarrow{\mathrm{seq}}_P\text{‘}\overrightarrow{P}\text{‘}x = C\text{‘}P \cap \hat{y}\,\{\iota\text{‘}x \subset \overrightarrow{P}\text{‘}y \,.\, \overrightarrow{P}\text{‘}y \subset -\,p\text{‘}\overleftarrow{P}\text{‘‘}\iota\text{‘}x\}$

$[*53{\cdot}301{\cdot}01] \qquad = C\text{‘}P \cap \hat{y}\,\{\iota\text{‘}x \subset \overrightarrow{P}\text{‘}y \,.\, \overrightarrow{P}\text{‘}y \subset -\,\overleftarrow{P}\text{‘}x\}$

$[\mathrm{Hp}] \qquad = C\text{‘}P \cap \hat{y}\,\{\iota\text{‘}x \subset \overrightarrow{P}\text{‘}y \,.\, \overrightarrow{P}\text{‘}y \subset -\,\iota\text{‘}x\}$

$[*51{\cdot}161] \qquad = \Lambda \qquad (2)$

$\vdash .\, (1) \,.\, (2) \,.\ \supset \vdash : \overrightarrow{P}\text{‘}x = \iota\text{‘}x \,.\supset .\, \overrightarrow{\max}_P\text{‘}\overrightarrow{P}\text{‘}x = \Lambda \,.\, \overrightarrow{\mathrm{seq}}_P\text{‘}\overrightarrow{P}\text{‘}x = \Lambda \qquad (3)$

$\vdash .\, (3) \,.\, \mathrm{Transp} \,.\supset \vdash :.\, \mathrm{Hp} \,.\supset : (x) \,.\, \overrightarrow{P}\text{‘}x \neq \iota\text{‘}x :$

$[*51{\cdot}401 . \mathrm{Transp}] \qquad \supset : (x) : \exists !\, \overrightarrow{P}\text{‘}x \,.\supset .\, \exists !\, \overrightarrow{P}\text{‘}x - \iota\text{‘}x :$

$[*33{\cdot}41] \qquad \supset : x \in \text{Ɑ}\text{‘}P \,.\supset .\, x \in \text{Ɑ}\text{‘}(P \mathbin{\dot\cap} J) \qquad (4)$

$\vdash .\, (4) \,.\, *33{\cdot}251 \,.\supset \vdash .\, \mathrm{Prop}$

$*214{\cdot}54.\ \vdash : P \in \mathrm{Ded} \,.\supset .\, P \mathbin{\dot\cap} J \in \mathrm{Ded}$

Dem.

$\vdash .\, *205{\cdot}111{\cdot}195 \,.\supset \vdash .\, \overrightarrow{\max}_P\text{‘}\alpha \subset \alpha \cap C\text{‘}(P \mathbin{\dot\cap} J) - P\text{‘‘}\alpha$

$[*37{\cdot}201] \qquad \subset \alpha \cap C\text{‘}(P \mathbin{\dot\cap} J) - (P \mathbin{\dot\cap} J)\text{‘‘}\alpha$

$[*205{\cdot}111] \qquad \subset \overrightarrow{\max}\,(P \mathbin{\dot\cap} J)\text{‘}\alpha \,.$

$[*24{\cdot}59] \qquad \supset \vdash : \sim \exists !\, \overrightarrow{\max}\,(P \mathbin{\dot\cap} J)\text{‘}\alpha \,.\supset .\, \sim \exists !\, \overrightarrow{\max}_P\text{‘}\alpha \qquad (1)$

$\vdash .\, (1) \,.\, *214{\cdot}1 \,.\ \supset \vdash : \mathrm{Hp} \,.\, \sim \exists !\, \overrightarrow{\max}\,(P \mathbin{\dot\cap} J)\text{‘}\alpha \,.\supset .\, \exists !\, \mathrm{seq}_P\text{‘}\alpha \qquad (2)$

$\vdash .\, *206{\cdot}2{\cdot}17 \,.\supset$

$\vdash :.\, x \,\mathrm{seq}_P\, \alpha \,.\equiv : y \in \alpha \cap C\text{‘}P \,.\supset_y .\, yPx \,.\, y \neq x : x \in C\text{‘}P :$

$yPx \,.\supset_y .\, (\exists z) \,.\, z \in \alpha \,.\, \sim(zPy) :$

$[*214{\cdot}531] \quad \equiv : y \in \alpha \cap C\text{‘}(P \mathbin{\dot\cap} J) \,.\supset_y .\, y (P \mathbin{\dot\cap} J)\, x : x \in C\text{‘}(P \mathbin{\dot\cap} J) :$

$yPx \,.\supset_y .\, (\exists z) \,.\, z \in \alpha \,.\, \sim(zPy) :$

$[*23{\cdot}43 . *3{\cdot}14] \supset : y \in \alpha \cap C\text{‘}(P \mathbin{\dot\cap} J) \,.\supset_y .\, y\,(P \mathbin{\dot\cap} J)\, x : x \in C\text{‘}(P \mathbin{\dot\cap} J) :$

$y\,(P \mathbin{\dot\cap} J)\, x \,.\supset_y .\, (\exists z) \,.\, z \in \alpha \,.\, \sim\{z\,(P \mathbin{\dot\cap} J)\, y\} :$

$[*206{\cdot}17] \qquad \supset : x \,\mathrm{seq}\,(P \mathbin{\dot\cap} J)\, \alpha \qquad (3)$

$\vdash .\, (2) \,.\, (3) \,.\supset \vdash : \mathrm{Hp} \,.\, \sim \exists !\, \overrightarrow{\max}\,(P \mathbin{\dot\cap} J)\text{‘}\alpha \,.\supset .\, \exists !\, \mathrm{seq}\,(P \mathbin{\dot\cap} J)\text{‘}\alpha \qquad (4)$

$\vdash .\, (4) \,.\, *214{\cdot}1 \,.\supset \vdash .\, \mathrm{Prop}$

$*214{\cdot}6.\ \vdash : P \in \mathrm{Ded} \,.\, P \,\mathrm{smor}\, Q \,.\supset .\, Q \in \mathrm{Ded}$

Dem.

$\vdash .\, *207{\cdot}65 \,.\, *214{\cdot}11 \,.\supset$

$\vdash : P \in \mathrm{Ded} \,.\, S \in P \,\overline{\mathrm{smor}}\, Q \,.\supset .\, (\alpha) \,.\, \breve{S}\text{‘‘}\alpha \in \text{Ɑ}\text{‘}\mathrm{limax}_Q \,.$

$[*71{\cdot}481] \qquad \supset .\, \mathrm{Cl}\text{‘}\text{Ɑ}\text{‘}S \subset \text{Ɑ}\text{‘}\mathrm{limax}_Q \,.$

$[*151{\cdot}11 . *214{\cdot}12] \qquad \supset .\, Q \in \mathrm{Ded} : \supset \vdash .\, \mathrm{Prop}$

***214·7.** $\vdash :. P \in \text{semi Ded} . \equiv : \alpha \in \text{sect}`P . \alpha \neq C`P . \supset_\alpha . \exists ! (\overrightarrow{\text{max}_P}`\alpha \cup \overrightarrow{\text{seq}_P}`\alpha)$
[(*214·02)]

***214·71.** $\vdash . \text{Ded} \subset \text{semi Ded}$ [*214·1·7]

***214·72.** $\vdash :. P \in \text{trans} . \supset : P \in \text{Ded} . \equiv . P \in \text{semi Ded} . \exists ! \overrightarrow{B}`\breve{P}$
[*214·7·13 . *205·121]

***214·73.** $\vdash . \text{semi Ded} - \iota`\dot{\Lambda} \subset \text{Cl}`B$ [*206·14 . *211·44 . *214·7]

The proof of the following proposition is given in a somewhat compressed form, since, if given with the usual fullness, it would require various lemmas not required elsewhere.

***214·74.** $\vdash : P \in \text{Ser} \cap \text{semi Ded} . \supset . P \upharpoonright \overleftarrow{P_*}`x \in \text{semi Ded}$

Dem.

$\vdash . *214·7 . \supset \vdash : \text{Hp} . \alpha \in \text{sect}`P . \alpha \neq C`P . \supset . \exists ! (\overrightarrow{\text{max}_P}`\alpha \cup \overrightarrow{\text{seq}_P}`\alpha)$ (1)

$\vdash . *205·261 . \supset$

$\vdash : \text{Hp}(1) . \overleftarrow{P_*}`x \sim\in 1 . x \in \alpha . \supset . \overrightarrow{\text{max}}(P \upharpoonright \overleftarrow{P_*}`x)`\alpha = \overrightarrow{\text{max}_P}`(\alpha \cap \overleftarrow{P_*}`x)$

[*205·262] $= \overrightarrow{\text{max}_P}`\alpha$ (2)

$\vdash . *211·75·56 . \supset \vdash : \text{Hp}(2) . \supset . C`P - \alpha \subset \overleftarrow{P_*}`x$ (3)

$\vdash . (3) . *211·715 . \supset \vdash : \text{Hp}(2) . Q = P \upharpoonright \overleftarrow{P_*}`x . \supset . \overrightarrow{\text{seq}_P}`\alpha = \overrightarrow{\text{min}_P}`(\overleftarrow{P_*}`x - \alpha)$

[*205·261] $= \overrightarrow{\text{min}_Q}`(-\alpha)$

[*211·715.*206·25] $= \overrightarrow{\text{seq}_Q}`(\alpha \cap \overleftarrow{P_*}`x)$ (4)

$\vdash . (2) . (4) . \supset$

$\vdash : \text{Hp}(4) . \supset . \overrightarrow{\text{max}_P}`\alpha \cup \overrightarrow{\text{seq}_P}`\alpha = \overrightarrow{\text{max}_Q}`(\alpha \cap \overleftarrow{P_*}`x) \cup \overrightarrow{\text{seq}_Q}`(\alpha \cap \overleftarrow{P_*}`x)$ (5)

$\vdash . (1) . (5) . \supset \vdash : \text{Hp} . \alpha \in \text{sect}`P . \alpha \neq C`P . \overleftarrow{P_*}`x \sim\in 1 . x \in \alpha . Q = P \upharpoonright \overleftarrow{P_*}`x . \supset .$
$\exists ! \{\overrightarrow{\text{max}_Q}`(\alpha \cap \overleftarrow{P_*}`x) \cup \overrightarrow{\text{seq}_Q}`(\alpha \cap \overleftarrow{P_*}`x)\}$ (6)

$\vdash . *211·715 . \supset \vdash : \text{Hp} . \overleftarrow{P_*}`x \sim\in 1 . \alpha = \overrightarrow{P}`x . \supset . \overrightarrow{\text{seq}_P}`\alpha = \overrightarrow{\text{min}_P}`\overleftarrow{P_*}`x$

[*205·261] $= \overrightarrow{\text{min}}(P \upharpoonright \overleftarrow{P_*}`x)`\overleftarrow{P_*}`x$

[*206·14] $= \overrightarrow{\text{seq}}(P \upharpoonright \overleftarrow{P_*}`x)`\Lambda$ (7)

$\vdash . (7) . *206·401 . \supset \vdash : \text{Hp} . \overleftarrow{P_*}`x \sim\in 1 . \supset . \exists ! \overrightarrow{\text{seq}}(P \upharpoonright \overleftarrow{P_*}`x)`\Lambda$ (8)

$\vdash . (6) . (8) . \supset \vdash :. \text{Hp} . \overleftarrow{P_*}`x \sim\in 1 . Q = P \upharpoonright \overleftarrow{P_*}`x . \supset :$
$\beta \in \text{sect}`Q - \iota`C`Q . \supset_\beta . \exists ! (\overrightarrow{\text{max}_Q}`\beta \cup \overrightarrow{\text{seq}_Q}`\beta) :$

[*214·7] $\supset : Q \in \text{semi Ded}$ (9)

$\vdash . *214·7 . *200·35 . \supset \vdash : \text{Hp} . \overleftarrow{P_*}`x \in 1 . \supset . P \upharpoonright \overleftarrow{P_*}`x \in \text{semi Ded}$ (10)

$\vdash . (9) . (10) . \supset \vdash . \text{Prop}$

***214·75.** $\vdash : P \in \text{semi Ded} . P \,\text{smor}\, Q . \supset . Q \in \text{semi Ded}$
[*205·8 . *206·61 . *212·7]

*215. STRETCHES

Summary of *215.

A *stretch* of a series is any piece taken out of it, and not having any gaps; that is, it is a class contained in the series, and containing all terms which come between any two of its terms. Thus it is defined as

$$\hat{\alpha}(\alpha \subset C`P \,.\, P``\alpha \cap \breve{P}``\alpha \subset \alpha).$$

We denote the class of stretches by "str$`P$," where "str" stands for "stretch" or "Strecke." A stretch which has no predecessors is a section of P; one which has no successors is a section of $\breve{P}$. The properties of stretches are chiefly important in connection with compact series. In discrete series, stretches are the same as intervals.

If P is transitive, stretches of P are the products of sections of P and sections of $\breve{P}$, *i.e.* of upper and lower sections of P (*215·16). If P is connected, and α is a lower section, β an upper section, then if the two have a stretch $\alpha \cap \beta$ in common, we have

$$\alpha = P``(\alpha \cap \beta) \cup (\alpha \cap \beta) \,.\, \beta = \breve{P}``(\alpha \cap \beta) \cup (\alpha \cap \beta) \quad (*215·161).$$

A slightly more general form of this proposition is

***215·165.** $\vdash : P_{po} \in \text{connex} \,.\, \alpha \in \text{sect}`P \,.\, \beta \in \text{sect}`\breve{P} \,.\, \exists ! \alpha \cap \beta \,.\, \supset .$

$$\alpha = P_*``(\alpha \cap \beta) \,.\, \beta = \breve{P}_*``(\alpha \cap \beta) \,.\, P``\alpha = P_{po}``(\alpha \cap \beta) \,.\, \breve{P}``\beta = \breve{P}_{po}``(\alpha \cap \beta)$$

A specially important case is when α and β have just one term in common. In this case we have

***215·166.** $\vdash : P_{po} \in \text{Ser} \,.\, \alpha \in \text{sect}`P \,.\, \beta \in \text{sect}`\breve{P} \,.\, \alpha \cap \beta \in 1 \,.\, \supset .$

$$\alpha \cap \beta = \iota`\text{max}_P`\alpha = \iota`\text{min}_P`\beta$$

When $\alpha \cap \beta$ has more than one term, if the upper limit or maximum of α and the lower limit or minimum of β both exist, the latter precedes the former (*215·52); if α and β have no common part, but together exhaust the field of P, we have either $\text{limax}_P`\alpha = \text{limin}_P`\beta$ or $\text{limax}_P`\alpha P_1 \text{limin}_P`\beta$, assuming $E ! \text{limax}_P`\alpha \,.\, E ! \text{limin}_P`\beta$ (*215·54). Hence if $\text{limax}_P`\alpha$ has no immediate successor, it must be identical with $\text{limin}_P`\beta$. Thus we have

***215·543.** $\vdash : P \in \text{Ser} \,.\, \alpha \in \text{sect}`P \,.\, \beta \in \text{sect}`\breve{P} \,.\, \alpha \cup \beta = C`P \,.\, \alpha \cap \beta \in 0 \cup 1 \,.$

$$E ! \text{limax}_P`\alpha \,.\, \text{limax}_P`\alpha \sim\in D`P_1 \,.\, \supset .\, \text{limax}_P`\alpha = \text{limin}_P`\beta$$

The above propositions will be useful in Section C (*231 and *233).

***215·01.** $\text{str}`P = \hat{\alpha}(\alpha \subset C`P \,.\, P``\alpha \cap \breve{P}``\alpha \subset \alpha)$ Df

***215·1.** $\vdash : \alpha \in \text{str}`P \,.\equiv .\, \alpha \subset C`P \,.\, P``\alpha \cap \breve{P}``\alpha \subset \alpha$ [(*215·01)]

***215·11.** $\vdash . \mathrm{str}'P = \mathrm{str}'\breve{P}$ $[(*215{\cdot}01) . *33{\cdot}22]$

***215·13.** $\vdash . \mathrm{sect}'P \subset \mathrm{str}'P . \mathrm{sect}'\breve{P} \subset \mathrm{str}'P$ $[*215{\cdot}1 . *211{\cdot}1]$

***215·14.** $\vdash : \alpha \epsilon \mathrm{sect}'P . \beta \epsilon \mathrm{sect}'\breve{P} . \supset . \alpha \cap \beta \epsilon \mathrm{str}'P$

Dem.

$\vdash . *211{\cdot}1 . \supset \vdash : \mathrm{Hp} . \supset . \alpha \subset C'P . P``\alpha \subset \alpha . \breve{P}``\beta \subset \beta .$

$[*22{\cdot}43 . *37{\cdot}21]$ $\supset . \alpha \cap \beta \subset C'P . P``(\alpha \cap \beta) \subset \alpha . \breve{P}``(\alpha \cap \beta) \subset \beta .$

$[*22{\cdot}49]$ $\supset . \alpha \cap \beta \subset C'P . P``(\alpha \cap \beta) \cap \breve{P}``(\alpha \cap \beta) \subset \alpha \cap \beta .$

$[*215{\cdot}1]$ $\supset . \alpha \cap \beta \epsilon \mathrm{str}'P : \supset \vdash . \mathrm{Prop}$

***215·15.** $\vdash : P \epsilon \mathrm{trans} . \alpha \epsilon \mathrm{str}'P . \supset . \alpha \cup P``\alpha \epsilon \mathrm{sect}'P . \alpha \cup \breve{P}``\alpha \epsilon \mathrm{sect}'\breve{P} .$
$\alpha = (\alpha \cup P``\alpha) \cap (\alpha \cup \breve{P}``\alpha)$

Dem.

$\vdash . *211{\cdot}27 . *215{\cdot}1 . \supset \vdash : \mathrm{Hp} . \supset . \alpha \cup P``\alpha \epsilon \mathrm{sect}'P . \alpha \cup \breve{P}``\alpha \epsilon \mathrm{sect}'\breve{P}$ (1)

$\vdash . *215{\cdot}1 . *22{\cdot}62 . \supset \vdash : \mathrm{Hp} . \supset . \alpha = \alpha \cup (P``\alpha \cap \breve{P}``\alpha)$

$[*22{\cdot}69]$ $= (\alpha \cup P``\alpha) \cap (\alpha \cup \breve{P}``\alpha)$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***215·16.** $\vdash : P \epsilon \mathrm{trans} . \supset . \mathrm{str}'P = \hat{\gamma} \{(\exists \alpha, \beta) . \alpha \epsilon \mathrm{sect}'P . \beta \epsilon \mathrm{sect}'\breve{P} . \gamma = \alpha \cap \beta\}$
$= s'\{(\mathrm{sect}'P) \underset{,,}{\cap} ``\mathrm{sect}'\breve{P}\}$

$[*215{\cdot}14{\cdot}15 . *40{\cdot}7]$

***215·161.** $\vdash : P \epsilon \mathrm{connex} . \alpha \epsilon \mathrm{sect}'P . \beta \epsilon \mathrm{sect}'\breve{P} . \exists ! \alpha \cap \beta . \supset .$
$\alpha = P``(\alpha \cap \beta) \cup (\alpha \cap \beta) . \beta = \breve{P}``(\alpha \cap \beta) \cup (\alpha \cap \beta)$

Dem.

$\vdash . *211{\cdot}1 . *37{\cdot}2 . \supset \vdash : \mathrm{Hp} . \supset . P``(\alpha \cap \beta) \cup (\alpha \cap \beta) \subset \alpha$ (1)

$\vdash . *211{\cdot}702 . \supset \vdash :. \mathrm{Hp} . x \epsilon \alpha - \beta . \supset : y \epsilon \beta . \supset . xPy :$

$[*37{\cdot}1]$ $\supset : \exists ! (\alpha \cap \beta) . \supset . x \epsilon P``(\alpha \cap \beta)$ (2)

$\vdash . (2) . \supset \vdash : \mathrm{Hp} . x \epsilon \alpha . \supset . x \epsilon P``(\alpha \cap \beta) \cup (\alpha \cap \beta)$ (3)

$\vdash . (1) . (3) . \supset \vdash : \mathrm{Hp} . \supset . \alpha = P``(\alpha \cap \beta) \cup (\alpha \cap \beta)$ (4)

$\vdash . (4) \frac{\breve{P}}{P} . \supset \vdash : \mathrm{Hp} . \supset . \beta = \breve{P}``(\alpha \cap \beta) \cup (\alpha \cap \beta)$ (5)

$\vdash . (4) . (5) . \supset \vdash . \mathrm{Prop}$

***215·162.** $\vdash : P \epsilon \mathrm{trans} \cap \mathrm{connex} . \alpha \epsilon \mathrm{sect}'P . \beta \epsilon \mathrm{sect}'\breve{P} . \exists ! \alpha \cap \beta . \supset .$
$P``\alpha = P``(\alpha \cap \beta) . \breve{P}``\beta = \breve{P}``(\alpha \cap \beta)$

Dem.

$\vdash . *215{\cdot}161 . \supset \vdash : \mathrm{Hp} . \supset . P``\alpha = P``P``(\alpha \cap \beta) \cup P``(\alpha \cap \beta)$

$[*201{\cdot}5]$ $= P``(\alpha \cap \beta)$ (1)

Similarly $\vdash : \mathrm{Hp} . \supset . \breve{P}``\beta = \breve{P}``(\alpha \cap \beta)$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

$*215\cdot163$. $\vdash : P \,\epsilon\, \text{trans} \cap \text{connex} \,.\, \alpha \,\epsilon\, \text{sect}`P \,.\, \beta \,\epsilon\, \text{sect}`\breve{P} \,.\, \exists\,!\, \alpha \cap \beta \,.\, \supset \,.$
$$p`\overleftarrow{P}``\alpha = p`\overleftarrow{P}``(\alpha \cap \beta)$$

Dem.

$\vdash \,.\, *40\cdot16 \,.\quad \supset \vdash \,.\, p`\overleftarrow{P}``\alpha \subset p`\overleftarrow{P}``(\alpha \cap \beta)$ (1)

$\vdash \,.\, *10\cdot56 \,.\, *37\cdot1 \,.\, \supset \vdash :.\, \text{Hp} : y \,\epsilon\, \alpha \cap \beta \,.\, \supset_y \,.\, yPx : z \,\epsilon\, P``(\alpha \cap \beta) : \supset \,.\, zPx$ (2)

$\vdash \,.\, (2) \,.\, *215\cdot161 \,.\, \supset \vdash :.\, \text{Hp} : y \,\epsilon\, \alpha \cap \beta \,.\, \supset_y \,.\, yPx : \supset : z \,\epsilon\, \alpha \,.\, \supset_z \,.\, zPx$ (3)

$\vdash \,.\, (1) \,.\, (3) \,.\, \supset \vdash \,.\, \text{Prop}$

$*215\cdot164$. $\vdash : \text{Hp}\, *215\cdot162 \,.\, \supset \,.\, \overrightarrow{\text{min}}_P`\beta = \overrightarrow{\text{min}}_P`(\alpha \cap \beta) \,.\, \overrightarrow{\text{max}}_P`\alpha = \overrightarrow{\text{max}}_P`(\alpha \cap \beta) \,.$
$$\overrightarrow{\text{seq}}_P`\alpha = \overrightarrow{\text{seq}}_P`(\alpha \cap \beta) \,.\, \overrightarrow{\text{prec}}_P`\beta = \overrightarrow{\text{prec}}_P`(\alpha \cap \beta) \,.$$
$$\overrightarrow{\text{lt}}_P`\alpha = \overrightarrow{\text{lt}}_P`(\alpha \cap \beta) \,.\, \overrightarrow{\text{limax}}_P`\alpha = \overrightarrow{\text{limax}}_P`(\alpha \cap \beta)$$

Dem.

$\vdash \,.\, *215\cdot162 \,.\quad \supset \vdash : \text{Hp} \,.\, \supset \,.\, \overrightarrow{\text{max}}_P`\alpha = \alpha - P``(\alpha \cap \beta)$

$[*215\cdot161]\quad = \alpha \cap \beta - P``(\alpha \cap \beta)$

$[*205\cdot111]\quad = \overrightarrow{\text{max}}_P`(\alpha \cap \beta)$ (1)

Similarly $\vdash : \text{Hp} \,.\, \supset \,.\, \overrightarrow{\text{min}}_P`\beta = \overrightarrow{\text{min}}_P`(\alpha \cap \beta)$ (2)

$\vdash \,.\, *215\cdot163 \,.\, *206\cdot13 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, \overrightarrow{\text{seq}}_P`\alpha = \overrightarrow{\text{seq}}_P`(\alpha \cap \beta)$ (3)

Similarly $\vdash : \text{Hp} \,.\, \supset \,.\, \overrightarrow{\text{prec}}_P`\beta = \overrightarrow{\text{prec}}_P`(\alpha \cap \beta)$ (4)

$\vdash \,.\, (1) \,.\, (3) \,.\, *207\cdot11\cdot12 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, \overrightarrow{\text{lt}}_P`\alpha = \overrightarrow{\text{lt}}_P`(\alpha \cap \beta)$ (5)

$\vdash \,.\, (1) \,.\, (5) \,.\, *207\cdot45 \,.\quad \supset \vdash : \text{Hp} \,.\, \supset \,.\, \overrightarrow{\text{limax}}_P`\alpha = \overrightarrow{\text{limax}}_P`(\alpha \cap \beta)$ (6)

$\vdash \,.\, (1) \,.\, (2) \,.\, (3) \,.\, (4) \,.\, (5) \,.\, (6) \,.\, \supset \vdash \,.\, \text{Prop}$

$*215\cdot165$. $\vdash : P_{\text{po}} \,\epsilon\, \text{connex} \,.\, \alpha \,\epsilon\, \text{sect}`P \,.\, \beta \,\epsilon\, \text{sect}`\breve{P} \,.\, \exists\,!\, \alpha \cap \beta \,.\, \supset \,.$
$$\alpha = P_*``(\alpha \cap \beta) \,.\, \beta = \breve{P}_*``(\alpha \cap \beta) \,.\, P``\alpha = P_{\text{po}}``(\alpha \cap \beta) \,.\, \breve{P}``\beta = \breve{P}_{\text{po}}``(\alpha \cap \beta)$$

Dem.

$\vdash \,.\, *211\cdot17 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, \alpha \,\epsilon\, \text{sect}`P_{\text{po}} \,.\, \beta \,\epsilon\, \text{sect}`\breve{P}_{\text{po}} \,.\, \exists\,!\, \alpha \cap \beta \,.$

$[*215\cdot161]\quad \supset \,.\, \alpha = P_*``(\alpha \cap \beta) \,.\, \beta = \breve{P}_*``(\alpha \cap \beta) \,.$ (1)

$[*91\cdot52]\quad \supset \,.\, P``\alpha = P_{\text{po}}``(\alpha \cap \beta) \,.\, \breve{P}``\beta = \breve{P}_{\text{po}}``(\alpha \cap \beta)$ (2)

$\vdash \,.\, (1) \,.\, (2) \,.\, \supset \vdash \,.\, \text{Prop}$

$*215\cdot166$. $\vdash : P_{\text{po}} \,\epsilon\, \text{Ser} \,.\, \alpha \,\epsilon\, \text{sect}`P \,.\, \beta \,\epsilon\, \text{sect}`\breve{P} \,.\, \alpha \cap \beta \,\epsilon\, 1 \,.\, \supset \,.$
$$\alpha \cap \beta = \iota`\text{max}_P`\alpha = \iota`\text{min}_P`\beta$$

Dem.

$\vdash \,.\, *215\cdot161 \,.\, *211\cdot17 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, \alpha = (\alpha \cap \beta) \cup P_{\text{po}}``(\alpha \cap \beta) \,.$

$[*215\cdot165]\quad \supset \,.\, \alpha - P``\alpha = (\alpha \cap \beta) - P_{\text{po}}``(\alpha \cap \beta) \,.$

$[*205\cdot11]\quad \supset \,.\, \overrightarrow{\text{max}}_P`\alpha = \overrightarrow{\text{max}}\,(P_{\text{po}})`(\alpha \cap \beta)$

$[*205\cdot17]\quad = \alpha \cap \beta$ (1)

Similarly $\vdash : \text{Hp} \,.\, \supset \,.\, \overrightarrow{\text{min}}_P`\beta = \alpha \cap \beta$ (2)

$\vdash \,.\, (1) \,.\, (2) \,.\, \supset \vdash \,.\, \text{Prop}$

***215·17.** $\vdash : P \epsilon \text{trans} \,.\, \supset \,.\, \breve{P}``\alpha \cap P``\beta \,\epsilon\, \text{str}`P$

Dem.

$\vdash \,.\, *211{\cdot}15{\cdot}11 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, P``\beta \,\epsilon\, \text{sect}`P \,.\, \breve{P}``\alpha \,\epsilon\, \text{sect}`\breve{P}$ (1)

$\vdash \,.\, (1) \,.\, *215{\cdot}14 \,.\, \supset \vdash \,.\, \text{Prop}$

***215·18.** $\vdash \,.\, P(x \mapsto y),\ P(x \vdash y),\ P(x \dashv y),\ P(x - y) \,\epsilon\, \text{str}`P$

Dem.

$\vdash \,.\, *211{\cdot}13{\cdot}3 \,.\, \supset \vdash \,.\, \overrightarrow{P_*}`y \,\epsilon\, \text{sect}`P \,.\, \overleftarrow{P_*}`x \,\epsilon\, \text{sect}`\breve{P}$ (1)

$\vdash \,.\, *211{\cdot}16 \,.\quad \supset \vdash \,.\, \overrightarrow{P}_{\text{po}}`y \,\epsilon\, \text{sect}`P \,.\, \overleftarrow{P}_{\text{po}}`x \,\epsilon\, \text{sect}`\breve{P}$ (2)

$\vdash \,.\, (1) \,.\, (2) \,.\, *215{\cdot}14 \,.\, \supset \vdash \,.\, \text{Prop}$

***215·19.** $\vdash : P^2 \mathrel{G} J \,.\, x \,\epsilon\, C`P \,.\, \supset \,.\, \iota`x \,\epsilon\, \text{str}`P$

Dem.

$\vdash \,.\, *53{\cdot}301 \,.\quad \supset \vdash \,.\, P``\iota`x \cap \breve{P}``\iota`x = \overrightarrow{P}`x \cap \overleftarrow{P}`x$ (1)

$\vdash \,.\, (1) \,.\, *50{\cdot}43 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, P``\iota`x \cap \breve{P}``\iota`x = \Lambda$ (2)

$\vdash \,.\, (2) \,.\, *215{\cdot}1 \,.\, \supset \vdash \,.\, \text{Prop}$

***215·2.** $\vdash : P \,\epsilon\, \text{connex} \,.\, \alpha \,\epsilon\, \text{str}`P \,.\, x \,\epsilon\, \alpha \,.\, \supset \,.\, P``\alpha = \alpha - \overrightarrow{\text{max}_P}`\alpha \cup \overrightarrow{P}`x \,.$
$\breve{P}``\alpha = \alpha - \overrightarrow{\text{min}_P}`\alpha \cup \overleftarrow{P}`x$

Dem.

$\vdash \,.\, *205{\cdot}111 \,.\, \supset \vdash \,.\, \alpha - \overrightarrow{\text{max}_P}`\alpha \subset P``\alpha$ (1)

$\vdash \,.\, *37{\cdot}18 \,.\quad \supset \vdash : \text{Hp} \,.\, \supset \,.\, \overrightarrow{P}`x \subset P``\alpha$ (2)

$\vdash \,.\, (1) \,.\, (2) \,.\quad \supset \vdash : \text{Hp} \,.\, \supset \,.\, \alpha - \overrightarrow{\text{max}_P}`\alpha \cup \overrightarrow{P}`x \subset P``\alpha$ (3)

$\vdash \,.\, *202{\cdot}103 \,.\, \supset \vdash :. \text{Hp} \,.\, y \,\epsilon\, P``\alpha \,.\, \supset : y \,\epsilon\, \overrightarrow{P}`x \cup \iota`x \cup \overleftarrow{P}`x :$

[*37·181] $\supset : y \,\epsilon\, \overrightarrow{P}`x \cup \iota`x \,.\, \vee \,.\, y \,\epsilon\, \breve{P}``\alpha :$

[*4·73] $\supset : y \,\epsilon\, \overrightarrow{P}`x \cup \iota`x \,.\, \vee \,.\, y \,\epsilon\, P``\alpha \cap \breve{P}``\alpha :$

[*215·1] $\supset : y \,\epsilon\, \overrightarrow{P}`x \cup \iota`x \cup \alpha :$

[Hp] $\supset : y \,\epsilon\, \overrightarrow{P}`x \cup \alpha$ (4)

$\vdash \,.\, *205{\cdot}111 \,.\, \supset \vdash \,.\, y \,\epsilon\, P``\alpha \,.\, \supset \,.\, y \sim\epsilon\, \overrightarrow{\text{max}_P}`\alpha$ (5)

$\vdash \,.\, (4) \,.\, (5) \,.\quad \supset \vdash : \text{Hp} \,.\, y \,\epsilon\, P``\alpha \,.\, \supset \,.\, y \,\epsilon\, \alpha - \overrightarrow{\text{max}_P}`\alpha \cup \overrightarrow{P}`x$ (6)

$\vdash \,.\, (3) \,.\, (6) \,.\quad \supset \vdash : \text{Hp} \,.\, \supset \,.\, P``\alpha = \alpha - \overrightarrow{\text{max}_P}`\alpha \cup \overrightarrow{P}`x$ (7)

Similarly $\vdash : \text{Hp} \,.\, \supset \,.\, \breve{P}``\alpha = \alpha - \overrightarrow{\text{min}_P}`\alpha \cup \overleftarrow{P}`x$ (8)

$\vdash \,.\, (7) \,.\, (8) \,.\, \supset \vdash \,.\, \text{Prop}$

***215·21.** $\vdash : P \,\epsilon\, \text{connex} \,.\, \alpha, \beta \,\epsilon\, \text{str}`P \,.\, \exists ! \,\alpha \cap \beta \,.\, \supset \,.\, \alpha \cap \beta \,\epsilon\, \text{str}`P$

Dem.

$\vdash \,.\, *215{\cdot}2 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, (\exists x) \,.\, x \,\epsilon\, \alpha \cap \beta \,.\, P``\alpha \subset \alpha \cup \overrightarrow{P}`x \,.\, P``\beta \subset \beta \cup \overrightarrow{P}`x \,.$
$\breve{P}``\alpha \subset \alpha \cup \overleftarrow{P}`x \,.\, \breve{P}``\beta \subset \beta \cup \overleftarrow{P}`x \,.$

$[*22\cdot68] \supset . (\exists x) . x \in \alpha \cap \beta . P``\alpha \cap P``\beta \subset (\alpha \cap \beta) \cup \overrightarrow{P}`x .$
$\breve{P}``\alpha \cap \breve{P}``\beta \subset (\alpha \cap \beta) \cup \overleftarrow{P}`x .$

$[*37\cdot21] \supset . (\exists x) . x \in \alpha \cap \beta . P``(\alpha \cap \beta) \subset (\alpha \cap \beta) \cup \overrightarrow{P}`x . \breve{P}``(\alpha \cap \beta) \subset (\alpha \cap \beta) \cup \overleftarrow{P}`x .$

$[*22\cdot69] \supset . (\exists x) . x \in \alpha \cap \beta . P``(\alpha \cap \beta) \cap \breve{P}``(\alpha \cap \beta) \subset (\alpha \cap \beta) \cup (\overrightarrow{P}`x \cap \overleftarrow{P}`x)$ (1)

$\vdash . *37\cdot18 . \supset \vdash : x \in \alpha \cap \beta . \supset . \overrightarrow{P}`x \subset P``\alpha \cap P``\beta . \overleftarrow{P}`x \subset \breve{P}``\alpha \cap \breve{P}``\beta .$

$[*22\cdot49] \qquad \supset . \overrightarrow{P}`x \cap \overleftarrow{P}`x \subset P``\alpha \cap \breve{P}``\alpha \cap P``\beta \cap \breve{P}``\beta$ (2)

$\vdash . (2) . *215\cdot1 . \supset \vdash :. \mathrm{Hp} . \supset : x \in \alpha \cap \beta . \supset . \overrightarrow{P}`x \cap \overleftarrow{P}`x \subset \alpha \cap \beta$ (3)

$\vdash . (1) . (3) . \quad \supset \vdash : \mathrm{Hp} . \supset . (\exists x) . x \in \alpha \cap \beta . P``(\alpha \cap \beta) \cap \breve{P}``(\alpha \cap \beta) \subset \alpha \cap \beta .$

$[*215\cdot1] \qquad \supset . \alpha \cap \beta \in \mathrm{str}`P : \supset \vdash . \mathrm{Prop}$

$*215\cdot22$. $\vdash : \alpha, \beta \in \mathrm{str}`P . \supset . \alpha \cap \beta \in \mathrm{str}`P$

Dem.

$\vdash . *215\cdot1 . \supset \vdash : \mathrm{Hp} . \supset . \alpha \subset C`P . \beta \subset C`P . P``\alpha \cap \breve{P}``\alpha \subset \alpha . P``\beta \cap \breve{P}``\beta \subset \beta .$

$[*22\cdot47\cdot49] \qquad \supset . \alpha \cap \beta \subset C`P . P``\alpha \cap P``\beta \cap \breve{P}``\alpha \cap \breve{P}``\beta \subset \alpha \cap \beta .$

$[*37\cdot21] \qquad \supset . \alpha \cap \beta \subset C`P . P``(\alpha \cap \beta) \cap \breve{P}``(\alpha \cap \beta) \subset \alpha \cap \beta .$

$[*215\cdot1] \qquad \supset . \alpha \cap \beta \in \mathrm{str}`P : \supset \vdash . \mathrm{Prop}$

$*215\cdot23$. $\vdash : P \in \mathrm{connex} . \mu \subset \mathrm{str}`P . \exists ! p`\mu . \supset . s`\mu \in \mathrm{str}`P$

Dem.

$\vdash . *215\cdot2 . \supset \vdash :. \mathrm{Hp} . x \in p`\mu . \supset : \alpha \in \mu . \supset_\alpha . P``\alpha \subset \alpha \cup \overrightarrow{P}`x . \breve{P}``\alpha \subset \alpha \cup \overleftarrow{P}`x :$

$[*40\cdot13] \qquad \supset : \alpha \in \mu . \supset_\alpha . P``\alpha \subset s`\mu \cup \overrightarrow{P}`x . \breve{P}``\alpha \subset s`\mu \cup \overleftarrow{P}`x :$

$[*40\cdot43\cdot38] \qquad \supset : P``s`\mu \subset s`\mu \cup \overrightarrow{P}`x . \breve{P}``s`\mu \subset s`\mu \cup \overleftarrow{P}`x :$

$[*22\cdot49\cdot69] \qquad \supset : P``s`\mu \cap \breve{P}``s`\mu \subset s`\mu \cup (\overrightarrow{P}`x \cap \overleftarrow{P}`x)$ (1)

$\vdash . *40\cdot14 . \supset \vdash : \mathrm{Hp} . x \in p`\mu . \alpha \in \mu . \supset . x \in \alpha . \alpha \in \mathrm{str}`P .$

$[*37\cdot18] \qquad \supset . \overrightarrow{P}`x \cap \overleftarrow{P}`x \subset P``\alpha \cap \breve{P}``\alpha . \alpha \in \mathrm{str}`P .$

$[*215\cdot1] \qquad \supset . \overrightarrow{P}`x \cap \overleftarrow{P}`x \subset \alpha .$

$[*40\cdot13] \qquad \supset . \overrightarrow{P}`x \cap \overleftarrow{P}`x \subset s`\mu$ (2)

$\vdash . (1) . (2) . \supset \vdash : \mathrm{Hp} . \exists ! \mu . \supset . P``s`\mu \cap \breve{P}``s`\mu \subset s`\mu$ (3)

$\vdash . *37\cdot29 . \quad \supset \vdash : \mu = \Lambda . \supset . P``s`\mu \cap \breve{P}``s`\mu \subset s`\mu$ (4)

$\vdash . (3) . (4) . \supset \vdash . \mathrm{Prop}$

$*215\cdot24$. $\vdash : \mu \subset \mathrm{str}`P . \supset . C`P \cap p`\mu \in \mathrm{str}`P$

Dem.

$\vdash . *37\cdot265 . \supset \vdash . P``(p`\mu \cap C`P) \cap \breve{P}``(p`\mu \cap C`P) = P``p`\mu \cap \breve{P}``p`\mu$ (1)

$\vdash . *37\cdot2 . \quad \supset \vdash : \alpha \in \mu . \supset . P``p`\mu \cap \breve{P}``p`\mu \subset P``\alpha \cap \breve{P}``\alpha$ (2)

$\vdash.(2).*215\cdot1.\supset\vdash:.\mathrm{Hp}.\supset:\alpha\in\mu.\supset.P``p`\mu\cap\breve{P}``p`\mu\subset\alpha:$

$[*40\cdot15]\quad\supset:P``p`\mu\cap\breve{P}``p`\mu\subset p`\mu\quad(3)$

$\vdash.(1).(3).*215\cdot1.\supset\vdash.\mathrm{Prop}$

***215·25.** $\vdash:\mu\subset\mathrm{str}`P.\exists!\mu.\supset.p`\mu\in\mathrm{str}`P$

Dem.

$\vdash.*40\cdot24.*215\cdot1.\supset\vdash:\mathrm{Hp}.\supset.p`\mu\subset C`P\quad(1)$

$\vdash.(1).*215\cdot24.\supset\vdash.\mathrm{Prop}$

***215·3.** $\vdash:.P\in\mathrm{connex}.\alpha,\beta\in\mathrm{str}`P-\iota`\Lambda.\alpha\cap\beta=\Lambda.\supset:$

$\alpha\subset P``\beta.\equiv.\alpha\subset p`\overrightarrow{P}``\beta.\equiv.\beta\subset p`\overleftarrow{P}``\alpha.\equiv.\beta\subset\breve{P}``\alpha$

Dem.

$\vdash.*215\cdot1.\supset\vdash:\mathrm{Hp}.\supset.\alpha\subset C`P-\beta\quad(1)$

$\vdash.*22\cdot48.\supset\vdash:\alpha\subset P``\beta.\supset.\alpha\cap\breve{P}``\beta\subset P``\beta\cap\breve{P}``\beta:$

$[*215\cdot1]\quad\supset\vdash:\mathrm{Hp}.\alpha\subset P``\beta.\supset.\alpha\cap\breve{P}``\beta\subset\beta.$

$[*22\cdot621.\mathrm{Hp}]\quad\supset.\alpha\cap\breve{P}``\beta=\Lambda\quad(2)$

$\vdash.(1).(2).\supset\vdash:\mathrm{Hp}.\alpha\subset P``\beta.\supset.\alpha\subset C`P-\beta-\breve{P}``\beta.$

$[*202\cdot501]\quad\supset.\alpha\subset p`\overrightarrow{P}``\beta\quad(3)$

$\vdash.*40\cdot61.\supset\vdash:.\mathrm{Hp}.\alpha\subset p`\overrightarrow{P}``\beta.\supset.\alpha\subset P``\beta\quad(4)$

$\vdash.(3).(4).\supset\vdash:.\mathrm{Hp}.\supset:\alpha\subset P``\beta.\equiv.\alpha\subset p`\overrightarrow{P}``\beta.\quad(5)$

$[*40\cdot67]\quad\equiv.\beta\subset p`\overleftarrow{P}``\alpha.\quad(6)$

$\left[(5)\frac{\breve{P}}{P}\right]\quad\equiv.\beta\subset\breve{P}``\alpha\quad(7)$

$\vdash.(5).(6).(7).\supset\vdash.\mathrm{Prop}$

***215·31.** $\vdash:P\in\mathrm{trans}\cap\mathrm{connex}.\alpha\in\mathrm{str}`P.\mathrm{E}!\min_P`\alpha.\mathrm{E}!\max_P`\alpha.\supset.$

$\alpha=P(\min_P`\alpha\mapsto\max_P`\alpha)$

Dem.

$\vdash.*205\cdot2.*90\cdot15\cdot151.\supset\vdash:\mathrm{Hp}.y\in\alpha.\supset.\min_P`\alpha\,P_*y\quad(1)$

$\vdash.(1)\frac{\breve{P}}{P}.*205\cdot102.\quad\supset\vdash:\mathrm{Hp}.y\in\alpha.\supset.yP_*\max_P`\alpha\quad(2)$

$\vdash.(1).(2).*121\cdot103.\supset\vdash:\mathrm{Hp}.\supset.\alpha\subset P(\min_P`\alpha\mapsto\max_P`\alpha)\quad(3)$

$\vdash.*121\cdot242.*201\cdot19.*205\cdot2.\supset\vdash:\mathrm{Hp}.\supset.$

$P(\min_P`\alpha\mapsto\max_P`\alpha)=\iota`\min_P`\alpha\cup(\overleftarrow{P}`\min_P`\alpha\cap\overrightarrow{P}`\max_P`\alpha)\cup\iota`\max_P`\alpha$

$[*37\cdot18]\quad\subset\iota`\min_P`\alpha\cup(\breve{P}``\alpha\cap P``\alpha)\cup\iota`\max_P`\alpha$

$[*205\cdot11\cdot111.*215\cdot1]\quad\subset\alpha\quad(4)$

$\vdash.(3).(4).\supset\vdash.\mathrm{Prop}$

***215·32.** $\vdash : P \epsilon \text{trans} \cap \text{connex} . \alpha \epsilon \text{str}`P . \text{E} ! \text{min}_P`\alpha . \text{E} ! \text{seq}_P`\alpha . \supset .$
$\alpha = P(\text{min}_P`\alpha \mapsto \text{seq}_P`\alpha)$

Dem.

$\vdash . *206·211 . *205·2 . \supset \vdash : \text{Hp} . \supset . \alpha \subset \overleftarrow{P_*}`\text{min}_P`\alpha \cap \overrightarrow{P}`\text{seq}_P`\alpha$ (1)

$\vdash . *206·22 . *205·22 . \supset \vdash : \text{Hp} . \supset . \overleftarrow{P}`\text{min}_P`\alpha \cap \overrightarrow{P}`\text{seq}_P`\alpha = \breve{P}``\alpha \cap (\alpha \cup P``\alpha)$

[*215·1] $\subset \alpha .$

[*201·19.*121·241] $\supset . P(\text{min}_P`\alpha \mapsto \text{seq}_P`\alpha) \subset \alpha$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***215·33.** $\vdash : P \epsilon \text{trans} \cap \text{connex} . \alpha \epsilon \text{str}`P . \text{E} ! \text{prec}_P`\alpha . \text{E} ! \text{seq}_P`\alpha . \supset .$
$\alpha = P(\text{prec}_P`\alpha - \text{seq}_P`\alpha)$ [*206·22 . *215·1]

***215·4.** $\vdash : P \epsilon \text{connex} . \mu \epsilon \text{Cl excl}`(\text{str}`P - \iota`\Lambda) . \supset . P_{\text{cl}} \upharpoonright \mu = P_{\text{lc}} \upharpoonright \mu$

Dem.

$\vdash . *84·12 . \supset \vdash :. \text{Hp} . \supset : \alpha, \beta \epsilon \mu . \alpha \neq \beta . \supset . \alpha \cap \beta = \Lambda :$ (1)

[*170·1] $\supset : \alpha (P_{\text{cl}} \upharpoonright \mu) \beta . \equiv . \alpha, \beta \epsilon \mu . \exists ! \alpha - \breve{P}``\beta .$

[*215·3.Transp] $\equiv . \alpha, \beta \epsilon \mu . \exists ! \beta - P``\alpha .$

[(1).*170·102] $\equiv . \alpha, \beta \epsilon \mu . \alpha P_{\text{lc}} \mu :. \supset \vdash . \text{Prop}$

***215·41.** $\vdash : P \epsilon \text{trans} \cap \text{connex} . \mu \epsilon \text{Cl excl}`(\text{str}`P - \iota`\Lambda) . \supset . P_{\text{lc}} \upharpoonright \mu \epsilon \text{Ser}$

Dem.

$\vdash . *84·12 . *170·102 . \supset \vdash :. \text{Hp} . \supset : \alpha (P_{\text{lc}} \upharpoonright \mu) \beta . \equiv . \exists ! \beta - P``\alpha$ (1)

$\vdash . *215·3 . \supset$

$\vdash : \text{Hp} . \alpha, \beta \epsilon \mu . \alpha \subset P``\beta . \beta \subset P``\alpha . \supset . \alpha \subset P``\beta . \alpha \subset \breve{P}``\beta .$

[*215·1] $\supset . \alpha \subset \beta$ (2)

Similarly $\vdash : \text{Hp}(2) . \supset . \beta \subset \alpha$ (3)

$\vdash . (2) . (3) . \supset \vdash :: \text{Hp} . \alpha, \beta \epsilon \mu . \supset :. \alpha \subset P``\beta . \beta \subset P``\alpha . \supset . \alpha = \beta :.$

[Transp.(1)] $\supset :. \alpha \neq \beta . \supset : \alpha (P_{\text{lc}} \upharpoonright \mu) \beta . \vee . \beta (P_{\text{lc}} \upharpoonright \mu) \alpha$ (4)

$\vdash . *37·1 . \supset$

$\vdash :: \text{Hp} . \beta \cap \gamma = \Lambda . \sim(\gamma \subset P``\beta) . \supset :. (\exists z) : z \epsilon \gamma : y \epsilon \beta . \supset_y . \sim(zPy) . z \neq y :.$

[*202·103] $\supset :. (\exists z) : z \epsilon \gamma : y \epsilon \beta . \supset_y . yPz :.$

[*11·61] $\supset :. y \epsilon \beta . \supset_y . (\exists z) . z \epsilon \gamma . yPz :.$

[*37·1] $\supset :. \beta \subset P``\gamma :.$

[*201·5.*37·2] $\supset :. \gamma \subset P``\alpha . \supset . \beta \subset P``\alpha :.$

[Transp] $\supset :. \exists ! \beta - P``\alpha . \supset . \exists ! \gamma - P``\alpha$ (5)

$\vdash . (5) . (1) . \supset \vdash :. \text{Hp} . \supset : \alpha (P_{\text{lc}} \upharpoonright \mu) \beta . \beta (P_{\text{lc}} \upharpoonright \mu) \gamma . \supset . \alpha (P_{\text{lc}} \upharpoonright \mu) \gamma$ (6)

$\vdash . (4) . (6) . *170·17 . \supset \vdash . \text{Prop}$

***215·42.** $\vdash : P \epsilon \text{trans} \cap \text{connex} . \mu \epsilon \text{Cl excl}`(\text{str}`P - \iota`\Lambda) . \mu \sim \epsilon 1 . \supset . C`P_{\text{lc}} \upharpoonright \mu = \mu$
[*202·55 . *215·41]

***215·5.** $\vdash :. P \epsilon \text{trans} \cap \text{connex} . \alpha \epsilon \text{sect}`P . \beta \epsilon \text{sect}`\breve{P} . \supset :$
$\exists ! \alpha \cap \beta . \text{limax}_P`\alpha = \text{limin}_P`\beta . \supset . \alpha \cap \beta \epsilon 1$ [*207·71 . *215·164]

***215·51.** $\vdash : P \epsilon \mathrm{Ser} . \alpha \epsilon \mathrm{sect}‘P . \beta \epsilon \mathrm{sect}‘\breve{P} . \alpha \cap \beta \epsilon 1 . \supset .$

$\mathrm{limax}_P‘\alpha = \mathrm{limin}_P‘\beta = \breve{\iota}‘(\alpha \cap \beta)$ [*207·72 . *215·164]

***215·52.** $\vdash : \mathrm{Hp}$ *215·5 $. \alpha \cap \beta \sim \epsilon 0 \cup 1 . \mathrm{E} ! \mathrm{limax}_P‘\alpha . \mathrm{E} ! \mathrm{limin}_P‘\beta . \supset .$

$\mathrm{limin}_P‘\beta \, P \, \mathrm{limax}_P‘\alpha$

Dem.

$\vdash .$ *215·164 $. \supset \vdash :. \mathrm{Hp} . \supset : \mathrm{limax}_P‘\alpha = \mathrm{max}_P‘(\alpha \cap \beta) . \lor . \mathrm{limax}_P‘\alpha = \mathrm{seq}_P‘(\alpha \cap \beta) :$

$\mathrm{limin}_P‘\beta = \mathrm{min}_P‘(\alpha \cap \beta) . \lor . \mathrm{limin}_P‘\beta = \mathrm{prec}_P‘(\alpha \cap \beta)$ (1)

$\vdash .$ *205·732 $. \supset \vdash : \mathrm{Hp} . \mathrm{limax}_P‘\alpha = \mathrm{max}_P‘(\alpha \cap \beta) . \mathrm{limin}_P‘\beta = \mathrm{min}_P‘(\alpha \cap \beta) . \supset .$

$\mathrm{limin}_P‘\beta \, P \, \mathrm{limax}_P‘\alpha$ (2)

$\vdash .$ *206·15 $. \supset \vdash : \mathrm{Hp} . \mathrm{limax}_P‘\alpha = \mathrm{seq}_P‘(\alpha \cap \beta) . \mathrm{limin}_P‘\beta = \mathrm{min}_P‘(\alpha \cap \beta) . \supset .$

$\mathrm{limin}_P‘\beta \, P \, \mathrm{limax}_P‘\alpha$ (3)

$\vdash . (3) \dfrac{\breve{P}, \beta, \alpha}{P, \alpha, \beta} . \supset \vdash : \mathrm{Hp} . \mathrm{limax}_P‘\alpha = \mathrm{max}_P‘(\alpha \cap \beta) . \mathrm{limin}_P‘\beta = \mathrm{prec}_P‘(\alpha \cap \beta) . \supset .$

$\mathrm{limin}_P‘\beta \, P \, \mathrm{limax}_P‘\alpha$ (4)

$\vdash .$ *206·73 $. \supset \vdash : \mathrm{Hp} . \mathrm{limax}_P‘\alpha = \mathrm{seq}_P‘(\alpha \cap \beta) . \mathrm{limin}_P‘\beta = \mathrm{prec}_P‘(\alpha \cap \beta) . \supset .$

$\mathrm{limin}_P‘\beta \, P \, \mathrm{limax}_P‘\alpha$ (5)

$\vdash . (1) . (2) . (3) . (4) . (5) . \supset \vdash . \mathrm{Prop}$

***215·53.** $\vdash : \mathrm{Hp}$ *215·5 $. \alpha \cap \beta = \Lambda . \mathrm{E} ! \mathrm{limax}_P‘\alpha . \mathrm{E} ! \mathrm{limin}_P‘\beta . \supset .$

$\mathrm{limax}_P‘\alpha \, P_* \, \mathrm{limin}_P‘\beta$

Dem.

$\vdash .$ *207·2 . *205·22 $. \supset \vdash : \mathrm{Hp} . \supset . \overrightarrow{P}‘\mathrm{limax}_P‘\alpha \subset P‘‘\alpha . \overleftarrow{P}‘\mathrm{limin}_P‘\beta \subset \breve{P}‘‘\beta .$

[*211·1] $\supset . \overrightarrow{P}‘\mathrm{limax}_P‘\alpha \subset P‘‘\alpha . \overleftarrow{P}‘\mathrm{limin}_P‘\beta \subset \beta$ (1)

$\vdash . (1) .$ *37·1 $. \supset \vdash : \mathrm{Hp} . \mathrm{limin}_P‘\beta \, P \, \mathrm{limax}_P‘\alpha . \supset . (\exists x) . x \epsilon \alpha . \mathrm{limin}_P‘\beta \, P x .$

[(1)] $\supset . \exists ! \alpha \cap \beta$ (2)

$\vdash . (2) . \mathrm{Transp} . \supset \vdash . \mathrm{Prop}$

***215·54.** $\vdash : P \epsilon \mathrm{Ser} . \alpha \epsilon \mathrm{sect}‘P . \beta \epsilon \mathrm{sect}‘\breve{P} . \alpha \cap \beta = \Lambda . \alpha \cup \beta = C‘P .$

$\mathrm{E} ! \mathrm{limax}_P‘\alpha . \mathrm{E} ! \mathrm{limin}_P‘\beta . \supset : \mathrm{limax}_P‘\alpha = \mathrm{limin}_P‘\beta . \lor .$

$\mathrm{limax}_P‘\alpha \, P_1 \, \mathrm{limin}_P‘\beta$

Dem.

$\vdash .$ *211·726 $. \supset \vdash : \mathrm{Hp} . \mathrm{E} ! \mathrm{max}_P‘\alpha . \mathrm{E} ! \mathrm{min}_P‘\beta . \supset .$

$\mathrm{limax}_P‘\alpha = \mathrm{max}_P‘\alpha . \mathrm{limin}_P‘\beta = \mathrm{seq}_P‘\alpha .$

[*206·5] $\supset . \mathrm{limax}_P‘\alpha \, P_1 \, \mathrm{limin}_P‘\beta$ (1)

$\vdash .$ *211·726 $. \supset \vdash : \mathrm{Hp} . \sim \mathrm{E} ! \mathrm{max}_P‘\alpha . \supset .$

$\mathrm{limax}_P‘\alpha = \mathrm{min}_P‘\beta . \mathrm{limin}_P‘\beta = \mathrm{min}_P‘\beta$ (2)

$\vdash .$ *211·726 $. \supset \vdash : \mathrm{Hp} . \sim \mathrm{E} ! \mathrm{min}_P‘\beta . \supset .$

$\mathrm{limax}_P‘\alpha = \mathrm{max}_P‘\alpha . \mathrm{limin}_P‘\beta = \mathrm{max}_P‘\alpha$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash . \mathrm{Prop}$

***215·541.** $\vdash :: P \epsilon \mathrm{Ser} \,.\, \alpha \epsilon \mathrm{sect}`P \,.\, \beta \epsilon \mathrm{sect}`\breve{P} \,.\, \alpha \cup \beta = C`P \,.\, \supset :.$
$\alpha \cap \beta \epsilon 0 \cup 1 \,.\, \supset : \mathrm{E}\,!\, \mathrm{limax}_P`\alpha \,.\, \equiv .\, \mathrm{E}\,!\, \mathrm{limin}_P`\beta$
[*211·727 . *215·51]

***215·542.** $\vdash : \mathrm{Hp}\, *215{\cdot}541 \,.\, \alpha \cap \beta = \Lambda \,.\, \mathrm{E}\,!\, \mathrm{limax}_P`\alpha \,.\, \mathrm{limax}_P`\alpha \sim\epsilon \mathrm{D}`P_1 \,.\, \supset .$
$\mathrm{limax}_P`\alpha = \mathrm{limin}_P`\beta$
[*215·54 . *211·727]

***215·543.** $\vdash : P \epsilon \mathrm{Ser} \,.\, \alpha \epsilon \mathrm{sect}`P \,.\, \beta \epsilon \mathrm{sect}`\breve{P} \,.\, \alpha \cup \beta = C`P \,.\, \alpha \cap \beta \epsilon 0 \cup 1 \,.$
$\mathrm{E}\,!\, \mathrm{limax}_P`\alpha \,.\, \mathrm{limax}_P`\alpha \sim\epsilon \mathrm{D}`P_1 \,.\, \supset .\, \mathrm{limax}_P`\alpha = \mathrm{limin}_P`\beta$
[*215·542·51]

$*216$. DERIVATIVES

Summary of $*216$.

If α is any class, and P is any series, the *derivative* (or *first derivative*) of α with respect to P is the class of limits of existent sub-classes of $\alpha \cap C'P$, *i.e.* $\mathrm{lt}_P``\mathrm{Cl\,ex}`(\alpha \cap C`P)$. That is, a term x belongs to the derivative of α if a set of terms exists which is contained both in α and in $C`P$, and has $x$ for its limit. The derivative of $\alpha$ with respect to $P$ will be denoted by $\delta_P`\alpha$.

In general, there will be members of α not contained in $\delta_P`\alpha$, and members of $\delta_P`\alpha$ not contained in α. α is said to be *dense* in P if all its terms except the first (if there is a first) belong to $\delta_P`\alpha$, that is, if all its terms except the first are limits of existent classes contained in α. α is said to be *closed* in P if every existent sub-class of α which has no maximum has a limit which belongs to α, *i.e.* if every existent sub-class of α has a limit or a maximum, and the derivative of α is contained in α. If α is both dense and closed, it is called *perfect*. In this case, all its terms are limits of classes chosen out of α, and every class chosen out of α has a limit or maximum in α.

The second derivative of α is $\delta_P`\delta_P`\alpha$, *i.e.* $\delta_P{}^2`\alpha$, and so on. (Derivatives of infinite order cannot be dealt with till a later stage.) If P is serial, the second derivative of α is always contained in the first ($*216{\cdot}14$).

If P is a Dedekindian series, α is closed whenever $\delta_P`\alpha \subset \alpha$. In order to secure a Dedekindian series, it is sometimes convenient to replace $P$ by the ordinally similar series $\overrightarrow{P};P$, which is contained in the Dedekindian series $\varsigma`P$. Then α is replaced by $\overrightarrow{P}``\alpha$, and $\alpha$ is closed if the derivative of $\overrightarrow{P}``\alpha$ with respect to $\varsigma`P$ is contained in $\overrightarrow{P}``\alpha$. The relation of the derivative of $\alpha$ in $P$ to the derivative of $\overrightarrow{P}``\alpha$ in $\varsigma`P$ has been treated in $*212{\cdot}6$ and following propositions. This subject is resumed below ($*216{\cdot}5$ ff.).

The derivative of the series P will be defined as the series of its limit-points, and denoted by $\nabla`P$. Thus we put

$$\nabla`P = P \upharpoonright \mathrm{D}`\mathrm{lt}_P.$$

If P is a series, the derivative of a class α consists of those members x of $\mathrm{C\!\!\!I}`P$ which are such that members of α exist in every interval which ends in x, *i.e.*

$*216{\cdot}13$. $\vdash :: P \in \mathrm{Ser} . \supset :. x \in \delta_P`\alpha . \equiv : x \in \mathrm{C\!\!\!I}`P : yPx . \supset_y . \exists ! \alpha \cap \overleftarrow{P}`y \cap \overrightarrow{P}`x$

We have

$*216{\cdot}2$. $\vdash . \delta_P`C`P = \mathrm{D}`\mathrm{lt}_P - \overrightarrow{B}`P$

$*216{\cdot}3$. $\vdash : \alpha \in \mathrm{dense}`P . \equiv . \alpha - \overrightarrow{\min}_P`\alpha \subset \delta_P`\alpha$

***216·32.** $\vdash : \alpha \epsilon \text{closed}'P . \equiv . \text{Cl ex}'(\alpha \cap C'P) \subset \text{Ɑ}'\text{limax}_P . \delta_P'\alpha \subset \alpha$

We prove (*216·4—·412) that the properties of α with respect to P, as regards being dense, closed, or perfect, belong to $\breve{S}``\alpha$ with respect to Q if S is a correlator of P with Q.

We next consider the relation of α in P to $\overrightarrow{P}``\alpha$ in $\varsigma'P$ (*216·5—·56). The point of these propositions is that $\varsigma'P$ is Dedekindian, so that a class is closed in $\varsigma'P$ if it contains its first derivative. (It is usual to *define* a class as closed whenever it contains its first derivative; but this involves the tacit assumption that the series P is Dedekindian. If P is the series of real numbers, this assumption is of course verified.) We prove (*216·52) that the derivative of $\overrightarrow{P}``\alpha$ in $\varsigma'P$ is $P```(\text{Cl ex}'\alpha - \text{Ɑ}'\text{max}_P)$, *i.e.* is the class of segments defined by such existent sub-classes of α as have no maximum; we show that α is dense, closed, or perfect in P according as $\overrightarrow{P}``\alpha$ is dense, closed, or perfect in $\varsigma'P$ (*216·53·54·56), and that $\alpha$ and $\overrightarrow{P}``\alpha$ are closed if $\overrightarrow{P}``\alpha$ contains its first derivative (*216·54).

We end with various propositions on $\nabla'P$ (*216·6—·621), of which the chief is

***216·611.** $\vdash : P \epsilon \text{Ser} . \dot{\exists} ! \nabla'P . \supset . C'\nabla'P = C'P - \text{Ɑ}'P_1 = \delta_P'C'P \cup \overrightarrow{B}'P$

This subject will be resumed in connection with well-ordered series in *264.

***216·01.** $\delta_P'\alpha = \text{lt}_P``\text{Cl ex}'(\alpha \cap C'P)$ Df

***216·02.** $\text{dense}'P = \hat{\alpha}(\alpha - \overrightarrow{\text{min}}_P'\alpha \subset \delta_P'\alpha)$ Df

***216·03.** $\text{closed}'P = \hat{\alpha}\{\text{Cl ex}'(\alpha \cap C'P) \subset \text{Ɑ}'\text{limax}_P . \delta_P'\alpha \subset \alpha\}$ Df

***216·04.** $\text{perf}'P = \text{dense}'P \cap \text{closed}'P$ Df

***216·05.** $\nabla'P = P \upharpoonright \text{D}'\text{lt}_P$ Df

***216·1.** $\vdash : x \epsilon \delta_P'\alpha . \equiv . (\exists\beta) . \beta \subset \alpha \cap C'P . \exists ! \beta . x \,\text{lt}_P\, \beta$ [(*216·01)]

***216·101.** $\vdash : x \epsilon \delta_P'\alpha . \equiv . (\exists\beta) . \beta \subset \alpha . \exists ! \beta . \beta \subset P``\beta . x \,\text{seq}_P\, \beta$

Dem.

$\vdash . *216·1 . *207·1 . \supset$

$\vdash : x \epsilon \delta_P'\alpha . \equiv . (\exists\beta) . \beta \subset \alpha \cap C'P . \exists ! \beta . \beta \cap C'P \subset P``\beta . x \,\text{seq}_P\, \beta .$

[*37·15] $\equiv . (\exists\beta) . \beta \subset \alpha . \exists ! \beta . \beta \subset P``\beta . x \,\text{seq}_P\, \beta : \supset \vdash . \text{Prop}$

***216·11.** $\vdash . \delta_P'\alpha \subset \breve{P}``\alpha$

Dem.

$\vdash . *216·101 . *206·142 . \supset \vdash : x \epsilon \delta_P'\alpha . \supset . (\exists\beta) . \beta \subset \alpha . \exists ! \beta . x \epsilon \breve{P}``\beta .$

[*37·2] $\supset . x \epsilon \breve{P}``\alpha : \supset \vdash . \text{Prop}$

***216·111.** $\vdash . \delta_P`\alpha \subset \text{G}`P$ [*216·11 . *37·16]

***216·12.** $\vdash . \delta_P`\alpha = \delta_P`(\alpha \cap C`P)$ [*22·5 . (*216·01)]

***216·13.** $\vdash :: P \epsilon \text{Ser} . \supset :. x \epsilon \delta_P`\alpha . \equiv : x \epsilon \text{G}`P : yPx . \supset_y . \exists ! \alpha \cap \overleftarrow{P}`y \cap \overrightarrow{P}`x$

Dem.

$\vdash . *206·173 . *216·101 . \supset \vdash :: P \epsilon \text{connex} . P^2 \in J . \supset :.$

$x \epsilon \delta_P`\alpha . \equiv : (\exists \beta) . \beta \subset \alpha . \exists ! \beta . \beta \subset \overrightarrow{P}`x . \overrightarrow{P}`x \subset P``\beta :$

$[*37·46] \equiv : (\exists \beta) : \beta \subset \alpha . \exists ! \beta . \beta \subset \overrightarrow{P}`x : yPx . \supset_y . \exists ! \beta \cap \overleftarrow{P}`y :$

$[*24·58] \supset : yPx . \supset_y . \exists ! \alpha \cap \overrightarrow{P}`x \cap \overleftarrow{P}`y$ (1)

$\vdash . *33·41·152 . \supset$

$\vdash :. x \epsilon \text{G}`P : yPx . \supset_y . \exists ! \alpha \cap \overrightarrow{P}`x \cap \overleftarrow{P}`y : \supset : \exists ! \alpha \cap \overrightarrow{P}`x . \alpha \cap \overrightarrow{P}`x \subset \alpha \cap C`P :$

$[*216·1] \qquad \supset : x \, \text{lt}_P (\alpha \cap \overrightarrow{P}`x) . \supset . x \epsilon \delta_P`\alpha$ (2)

$\vdash . *37·2 . *201·501 . \supset \vdash : \text{Hp} . \supset . P``(\alpha \cap \overrightarrow{P}`x) \subset \overrightarrow{P}`x$ (3)

$\vdash . *50·24 . \qquad \supset \vdash : \text{Hp} . \supset . x \sim\epsilon (\alpha \cap \overrightarrow{P}`x)$ (4)

$\vdash . (3) . (4) . *207·232 . \supset$

$\vdash :: \text{Hp} . x \epsilon \text{G}`P . \supset :. x \, \text{lt}_P (\alpha \cap \overrightarrow{P}`x) . \equiv : \overrightarrow{P}`x \subset P``(\alpha \cap \overrightarrow{P}`x) :$

$[*37·46] \qquad \equiv : yPx . \supset_y . \exists ! \alpha \cap \overrightarrow{P}`x \cap \overleftarrow{P}`y$ (5)

$\vdash . (2) . (5) . \supset \vdash :. \text{Hp} . x \epsilon \text{G}`P : yPx . \supset_y . \exists ! \alpha \cap \overrightarrow{P}`x \cap \overleftarrow{P}`y : \supset . x \epsilon \delta_P`\alpha$ (6)

$\vdash . (1) . (6) . *216·111 . \supset \vdash . \text{Prop}$

***216·14.** $\vdash : P \epsilon \text{Ser} . \supset . \delta_{P^2}`\alpha \subset \delta_P`\alpha$

Dem.

$\vdash . *71·47 . \supset \vdash :. \text{Hp} . \supset : \beta \subset \text{lt}_P``\text{Cl ex}`\alpha . \exists ! \beta . \supset .$

$(\exists \kappa) . \kappa \subset \text{Cl ex}`\alpha . \beta = \text{lt}_P``\kappa . \exists ! \beta .$

$[*37·26] \qquad \supset . (\exists \lambda) . \lambda \subset \text{Cl ex}`\alpha \cap \text{G}`\text{lt}_P . \beta = \text{lt}_P``\lambda . \exists ! \beta .$

$[*207·54] \supset . (\exists \lambda) . \lambda \subset \text{Cl ex}`\alpha \cap \text{G}`\text{lt}_P . \beta = \text{lt}_P``\lambda . \exists ! \beta . \overrightarrow{\text{limax}}_P`\beta = \overrightarrow{\text{lt}}_P`s`\lambda .$

$[*216·1.*37·29.*53·24.\text{Transp}] \supset . \overrightarrow{\text{limax}}_P`\beta \subset \delta_P`\alpha .$

$[*207·45] \qquad \supset . \overrightarrow{\text{lt}}_P`\beta \subset \delta_P`\alpha$ (1)

$\vdash . (1) . (*216·01) . \supset \vdash :. \text{Hp} . \supset : \beta \epsilon \text{Cl ex}`\delta_P`\alpha . \supset . \overrightarrow{\text{lt}}_P`\beta \subset \delta_P`\alpha :$

$[*40·43·5] \qquad \supset : \text{lt}_P``\text{Cl ex}`\delta_P`\alpha \subset \delta_P`\alpha :. \supset \vdash . \text{Prop}$

***216·15.** $\vdash : \alpha \subset \beta . \supset . \delta_P`\alpha \subset \delta_P`\beta$ [*37·2 . (*216·01)]

***216·16.** $\vdash : P \epsilon \text{trans} \cap \text{connex} . \supset . \delta_P`\alpha = \delta_P`(\alpha - \overrightarrow{\text{min}}_P`\alpha)$

Dem.

$\vdash . *24·26·101 . \supset \vdash : \overrightarrow{\text{min}}_P`\alpha = \Lambda . \supset . \delta_P`\alpha = \delta_P`(\alpha - \overrightarrow{\text{min}}_P`\alpha)$ (1)

$\vdash . *51·36 . \supset \vdash : \beta \subset \alpha . \exists ! \beta . E ! \text{min}_P`\alpha . \text{min}_P`\alpha \sim\epsilon \beta . \supset .$

$\beta \epsilon \text{Cl ex}`(\alpha - \iota`\text{min}_P`\alpha) .$

$[*37·18] \qquad \supset . \overrightarrow{\text{lt}}_P`\beta \subset \delta_P`(\alpha - \iota`\text{min}_P`\alpha)$ (2)

$\vdash . *205 \cdot 5 . \supset \vdash : \text{Hp} . \beta \subset \alpha . \exists ! \beta . \text{min}_P{}'\alpha \in \beta . \supset . \text{min}_P{}'\alpha = \text{min}_P{}'\beta .$

[*207·262] $\supset . \overrightarrow{\text{lt}_P}{}'\beta \subset \overrightarrow{\text{lt}_P}{}'(\beta - \iota'\text{min}_P{}'\alpha)$ (3)

$\vdash . (3) . *37 \cdot 18 . \supset \vdash : \text{Hp}(3) . \beta \neq \iota'\text{min}_P{}'\alpha . \supset . \overrightarrow{\text{lt}_P}{}'\beta \subset \delta_P{}'(\alpha - \iota'\text{min}_P{}'\alpha)$ (4)

$\vdash . *205 \cdot 194 \cdot 8 . \supset \vdash : \text{E} ! \text{min}_P{}'\alpha . \supset . \text{min}_P{}'\alpha = \text{max}_P{}'\iota'\text{min}_P{}'\alpha .$

[*207·11] $\supset . \overrightarrow{\text{lt}_P}{}'\iota'\text{min}_P{}'\alpha = \Lambda$ (5)

$\vdash . (5) . *24 \cdot 12 . \supset \vdash : \text{E} ! \text{min}_P{}'\alpha . \beta = \iota'\text{min}_P{}'\alpha . \supset . \overrightarrow{\text{lt}_P}{}'\beta \subset \delta_P{}'(\alpha - \iota'\text{min}_P{}'\alpha)$ (6)

$\vdash . (4) . (6) . \supset \vdash : \text{Hp}(3) . \supset . \overrightarrow{\text{lt}_P}{}'\beta \subset \delta_P{}'(\alpha - \iota'\text{min}_P{}'\alpha)$ (7)

$\vdash . (2) . (7) . \supset \vdash : \text{Hp} . \beta \subset \alpha . \exists ! \beta . \text{E} ! \text{min}_P{}'\alpha . \supset . \overrightarrow{\text{lt}_P}{}'\beta \subset \delta_P{}'(\alpha - \iota'\text{min}_P{}'\alpha)$ (8)

$\vdash . (8) . *40 \cdot 5 \cdot 43 . \supset \vdash : \text{Hp} . \text{E} ! \text{min}_P{}'\alpha . \supset . \delta_P{}'\alpha \subset \delta_P{}'(\alpha - \iota'\text{min}_P{}'\alpha)$ (9)

$\vdash . (9) . *216 \cdot 15 . \supset \vdash : \text{Hp} . \text{E} ! \text{min}_P{}'\alpha . \supset . \delta_P{}'\alpha = \delta_P{}'(\alpha - \iota'\text{min}_P{}'\alpha)$ (10)

$\vdash . (1) . (10) . \supset \vdash . \text{Prop}$

***216·2.** $\vdash . \delta_P{}'C'P = \text{D}'\text{lt}_P - \overrightarrow{B}'P$

Dem.

$\vdash . *37 \cdot 15 . *216 \cdot 111 . \supset \vdash . \delta_P{}'C'P \subset \text{D}'\text{lt}_P - \overrightarrow{B}'P$ (1)

$\vdash . *216 \cdot 1 . \qquad \supset \vdash : x \in \text{D}'\text{lt}_P - \delta_P{}'C'P . \supset . x \, \text{lt}_P \Lambda .$

[*207·3] $\supset . x \in \overrightarrow{B}'P$ (2)

$\vdash . (2) . \text{Transp} . \qquad \supset \vdash . \text{D}'\text{lt}_P - \overrightarrow{B}'P \subset \delta_P{}'C'P$ (3)

$\vdash . (1) . (3) . \supset \vdash . \text{Prop}$

***216·21.** $\vdash : P \in \text{Rl}'J \cap \text{connex} . \supset . \delta_P{}'C'P = \text{Cl}'P - \text{Cl}'(P \dot{-} P^2)$
[*207·35 . *216·2]

***216·22.** $\vdash : P \in \text{Rl}'J \cap \text{connex} . P \Subset P^2 . \supset . \delta_P{}'C'P = \text{Cl}'P$ [*216·21]

***216·23.** $\vdash : P \in \text{trans} . \supset . \delta_P{}'C'P = \text{seq}_P{}''\text{Cl}'\text{sgm}'P = \text{lt}_P{}''\text{Cl}'\text{sgm}'P$

Dem.

$\vdash . *206 \cdot 25 . *216 \cdot 101 . \supset$

$\vdash :. \text{Hp} . \supset : x \in \delta_P{}'C'P . \equiv . (\exists\beta) . \exists ! \beta . \beta \subset P''\beta . x \, \text{seq}_P (P''\beta) .$

[*24·58.*37·29] $\equiv . (\exists\beta) . \beta \subset P''\beta . \exists ! P''\beta . x \, \text{seq}_P (P''\beta) .$

[*201·55] $\supset . (\exists\beta) . P''P''\beta = P''\beta . \exists ! P''\beta . x \, \text{seq}_P (P''\beta) .$

[*212·152] $\supset . x \in \text{seq}_P{}''\text{Cl}'\text{sgm}'P$ (1)

$\vdash . *211 \cdot 4 . \supset \vdash . \text{seq}_P{}''\text{Cl}'\text{sgm}'P = \text{lt}_P{}''\text{Cl}'\text{sgm}'P$ (2)

$\vdash . *212 \cdot 152 . (*216 \cdot 01) . \supset \vdash . \text{lt}_P{}''\text{Cl}'\text{sgm}'P \subset \delta_P{}'C'P$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash . \text{Prop}$

***216·3.** $\vdash : \alpha \in \text{dense}'P . \equiv . \alpha - \overrightarrow{\text{min}_P}{}'\alpha \subset \delta_P{}'\alpha$ [(*216·02)]

***216·31.** $\vdash : \alpha \in \text{dense}'P . \equiv . \alpha \subset C'P . \alpha \cap \breve{P}''\alpha \subset \delta_P{}'\alpha$

Dem.

$\vdash . *216 \cdot 3 \cdot 111 . \supset \vdash : \alpha \in \text{dense}'P . \supset . \alpha - \overrightarrow{\text{min}_P}{}'\alpha \subset \text{Cl}'P .$

[*205·11] $\supset . \alpha \subset C'P .$ (1)

[*205·11] $\supset . \alpha - \overrightarrow{\text{min}_P}{}'\alpha = \alpha \cap \breve{P}''\alpha$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***216·32.** $\vdash : \alpha \epsilon \text{closed}`P . \equiv . \text{Cl ex}`(\alpha \cap C`P) \subset \text{Cl}`\text{limax}_P . \delta_P`\alpha \subset \alpha$
[(*216·03)]

***216·33.** $\vdash :. \alpha \epsilon \text{closed}`P . \equiv : \beta \subset \alpha . \exists ! \beta . \beta \subset P``\beta . \supset_\beta . \exists ! \overrightarrow{\text{lt}}_P`\beta . \overrightarrow{\text{lt}}_P`\beta \subset \alpha$

Dem.

$\vdash . *207·45 . *205·123 . \supset \vdash :. \text{Cl ex}`(\alpha \cap C`P) \subset \text{Cl}`\text{limax}_P . \equiv :$

$\beta \subset \alpha . \exists ! \beta . \beta \subset C`P . \beta \subset P``\beta . \supset_\beta . \exists ! \overrightarrow{\text{lt}}_P`\beta :$

[*37·15] $\equiv : \beta \subset \alpha . \exists ! \beta . \beta \subset P``\beta . \supset_\beta . \exists ! \overrightarrow{\text{lt}}_P`\beta$ (1)

$\vdash . *40·43·5 . \supset \vdash :. \delta_P`\alpha \subset \alpha . \equiv : \beta \subset \alpha . \exists ! \beta . \beta \subset C`P . \supset_\beta . \overrightarrow{\text{lt}}_P`\beta \subset \alpha :$

[*207·11.*24·12] $\equiv : \beta \subset \alpha . \exists ! \beta . \beta \subset C`P . \overrightarrow{\text{max}}_P`\beta = \Lambda . \supset_\beta . \overrightarrow{\text{lt}}_P`\beta \subset \alpha :$

[*205·123.*37·15] $\equiv : \beta \subset \alpha . \exists ! \beta . \beta \subset P``\beta . \supset_\beta . \overrightarrow{\text{lt}}_P`\beta \subset \alpha$ (2)

$\vdash . (1) . (2) . *216·32 . \supset \vdash . \text{Prop}$

***216·34.** $\vdash :: P \epsilon \text{connex} . \supset :. \alpha \epsilon \text{closed}`P . \equiv :$
$\beta \subset \alpha . \exists ! \beta . \beta \subset P``\beta . \supset_\beta . \text{lt}_P`\beta \epsilon \alpha$ [*216·33 . *71·332 . *207·24]

***216·35.** $\vdash : P \epsilon \text{Ser} . \text{Cl ex}`\alpha \subset \text{Cl}`\text{limax}_P . \supset . \text{Cl ex}`\delta_P`\alpha \subset \text{Cl}`\text{limax}_P$

Dem.

$\vdash . *71·47 . *37·26 . \supset$

$\vdash :. \text{Hp} . \supset : \beta \epsilon \text{Cl ex}`\delta_P`\alpha . \supset . (\exists \lambda) . \lambda \subset \text{Cl ex}`\alpha \cap \text{Cl}`\text{lt}_P . \beta = \text{lt}_P``\lambda . \exists ! \beta .$

[*207·54]

$\supset . (\exists \lambda) . \lambda \subset \text{Cl ex}`\alpha \cap \text{Cl}`\text{lt}_P . \beta = \text{lt}_P``\lambda . \exists ! \beta . \overrightarrow{\text{limax}}_P`\beta = \overrightarrow{\text{limax}}_P`s`\lambda .$

[*37·29.Transp]

$\supset . (\exists \lambda) . \lambda \subset \text{Cl ex}`\alpha \cap \text{Cl}`\text{lt}_P . \beta = \text{lt}_P``\lambda . \exists ! \lambda . \overrightarrow{\text{limax}}_P`\beta = \overrightarrow{\text{limax}}_P`s`\lambda .$

[*53·24.Transp] $\supset . (\exists \lambda) . s`\lambda \epsilon \text{Cl ex}`\alpha . \overrightarrow{\text{limax}}_P`\beta = \overrightarrow{\text{limax}}_P`s`\lambda .$

[Hp] $\supset . \exists ! \overrightarrow{\text{limax}}_P`\beta :. \supset \vdash . \text{Prop}$

***216·36.** $\vdash : \alpha \epsilon \text{perf}`P . \equiv . \alpha \epsilon \text{dense}`P \cap \text{closed}`P$ [(*216·04)]

***216·37.** $\vdash : \alpha \epsilon \text{perf}`P . \equiv . \text{Cl ex}`\alpha \subset \text{Cl}`\text{limax}_P . \delta_P`\alpha = \alpha - \overrightarrow{\text{min}}_P`\alpha$
[*216·3·32·36]

***216·371.** $\vdash : \alpha \epsilon \text{perf}`P . \equiv . \text{Cl ex}`\alpha \subset \text{Cl}`\text{limax}_P . \alpha \subset C`P . \delta_P`\alpha = \alpha \cap \breve{P}``\alpha$
[*216·31·32·11·36]

***216·38.** $\vdash : P \epsilon \text{trans} \cap \text{connex} . \alpha \epsilon \text{dense}`P . \supset . \delta_P`\alpha \epsilon \text{dense}`P . \delta_P`\alpha \subset \delta_P`\delta_P`\alpha$

Dem.

$\vdash . *216·3·15 . \supset \vdash : \text{Hp} . \supset . \delta_P`(\alpha - \overrightarrow{\text{min}}_P`\alpha) \subset \delta_P`\delta_P`\alpha .$
[*216·16] $\supset . \delta_P`\alpha \subset \delta_P`\delta_P`\alpha .$
[*216·3] $\supset . \delta_P`\alpha \epsilon \text{dense}`P : \supset \vdash . \text{Prop}$

***216·381.** $\vdash : P \epsilon \text{Ser} . \alpha \epsilon \text{dense}`P . \supset . \delta_P`\alpha = \delta_P`\delta_P`\alpha . \overrightarrow{\text{min}}_P`\delta_P`\alpha = \Lambda$
[*216·38·14·11]

***216·382.** $\vdash : P \epsilon \mathrm{Ser} \,.\, \alpha \epsilon \mathrm{dense}\text{‘}P \,.\, \mathrm{Cl\,ex}\text{‘}\alpha \subset \text{Ɑ}\text{‘}\mathrm{limax}_P \,.\, \supset .\, \delta_P\text{‘}\alpha \epsilon \mathrm{perf}\text{‘}P$
[*216·35·381·37]

***216·4.** $\vdash : S \epsilon P \,\overline{\mathrm{smor}}\, Q \,.\, \supset .\, \delta_P\text{‘}\alpha = S\text{‘‘}\delta_Q\text{‘}\breve{S}\text{‘‘}\alpha \,.\, \breve{S}\text{‘‘}\delta_P\text{‘}\alpha = \delta_Q\text{‘}\breve{S}\text{‘‘}\alpha$

Dem.

$\vdash .\, *207\cdot63 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset .\, \delta_P\text{‘}\alpha = S\text{‘‘}\mathrm{lt}_Q\text{‘‘}\breve{S}\text{‘‘‘}\mathrm{Cl\,ex}\text{‘}\alpha$

[*71·491] $= S\text{‘‘}\mathrm{lt}_Q\text{‘‘}\mathrm{Cl\,ex}\text{‘}\breve{S}\text{‘‘}\alpha$

[(*216·01)] $= S\text{‘‘}\delta_Q\text{‘}\breve{S}\text{‘‘}\alpha$ (1)

$\vdash .\, (1) \,.\, *72\cdot52 \,.\, *216\cdot111 \,.\, \supset \vdash .\, \mathrm{Prop}$

***216·401.** $\vdash : S \epsilon P \,\overline{\mathrm{smor}}\, Q \,.\, \supset .\, P \upharpoonright \delta_P\text{‘}\alpha = S ; (Q \upharpoonright \delta_Q\text{‘}\breve{S}\text{‘‘}\alpha)$

Dem.

$\vdash .\, *150\cdot37 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset .\, S ; (Q \upharpoonright \delta_Q\text{‘}\breve{S}\text{‘‘}\alpha) = (S ; Q) \upharpoonright S\text{‘‘}\delta_Q\text{‘}\breve{S}\text{‘‘}\alpha$

[*216·4.*151·11] $= P \upharpoonright \delta_P\text{‘}\alpha : \supset \vdash .\, \mathrm{Prop}$

***216·41.** $\vdash :.\, S \epsilon P \,\overline{\mathrm{smor}}\, Q \,.\, \alpha \subset C\text{‘}P \,.\, \supset : \alpha \epsilon \mathrm{dense}\text{‘}P \,.\, \equiv .\, \breve{S}\text{‘‘}\alpha \epsilon \mathrm{dense}\text{‘}Q$

Dem.

$\vdash .\, *216\cdot3 \,.\, *37\cdot2 \,.\, \supset$

$\vdash :.\, \mathrm{Hp} \,.\, \supset : \alpha \epsilon \mathrm{dense}\text{‘}P \,.\, \supset .\, \breve{S}\text{‘‘}(\alpha - \overrightarrow{\mathrm{min}}_P\text{‘}\alpha) \subset \breve{S}\text{‘‘}\delta_P\text{‘}\alpha \,.$

[*71·38.*205·8] $\supset .\, \breve{S}\text{‘‘}\alpha - \overrightarrow{\mathrm{min}}_Q\text{‘}\breve{S}\text{‘‘}\alpha \subset \breve{S}\text{‘‘}\delta_P\text{‘}\alpha \,.$

[*216·4] $\supset .\, \breve{S}\text{‘‘}\alpha - \overrightarrow{\mathrm{min}}_Q\text{‘}\breve{S}\text{‘‘}\alpha \subset \delta_Q\text{‘}\breve{S}\text{‘‘}\alpha \,.$

[*216·3] $\supset .\, \breve{S}\text{‘‘}\alpha \epsilon \mathrm{dense}\text{‘}Q$ (1)

$\vdash .\, (1) \dfrac{Q, P, \breve{S}\text{‘‘}\alpha}{P, Q, \alpha} \,.\, \supset$

$\vdash :.\, \mathrm{Hp} \,.\, \supset : \breve{S}\text{‘‘}\alpha \epsilon \mathrm{dense}\text{‘}Q \,.\, \supset .\, S\text{‘‘}\breve{S}\text{‘‘}\alpha \epsilon \mathrm{dense}\text{‘}P \,.$

[*72·502] $\supset .\, \alpha \epsilon \mathrm{dense}\text{‘}P$ (2)

$\vdash .\, (1) \,.\, (2) \,.\, \supset \vdash .\, \mathrm{Prop}$

***216·411.** $\vdash :.\, S \epsilon P \,\overline{\mathrm{smor}}\, Q \,.\, \alpha \subset C\text{‘}P \,.\, \supset : \alpha \epsilon \mathrm{closed}\text{‘}P \,.\, \equiv .\, \breve{S}\text{‘‘}\alpha \epsilon \mathrm{closed}\text{‘}Q$

Dem.

$\vdash .\, *207\cdot64 \,.\, *37\cdot431 \,.\, \supset$

$\vdash :.\, \mathrm{Hp} \,.\, \supset : \beta \subset C\text{‘}P \,.\, \exists ! \beta \,.\, \beta \epsilon \text{Ɑ}\text{‘}\mathrm{limax}_P \,.\, \supset .$

$\breve{S}\text{‘‘}\beta \subset C\text{‘}Q \,.\, \exists ! \breve{S}\text{‘‘}\beta \,.\, \breve{S}\text{‘‘}\beta \epsilon \text{Ɑ}\text{‘}\mathrm{limax}_Q :$

[*71·49] $\supset : \mathrm{Cl\,ex}\text{‘}\alpha \subset \text{Ɑ}\text{‘}\mathrm{limax}_P \,.\, \supset .\, \mathrm{Cl\,ex}\text{‘}\breve{S}\text{‘‘}\alpha \subset \text{Ɑ}\text{‘}\mathrm{limax}_Q$ (1)

$\vdash .\, *37\cdot2 \,.\, *216\cdot4 \,.\quad \supset \vdash :.\, \mathrm{Hp} \,.\, \supset : \delta_P\text{‘}\alpha \subset \alpha \,.\, \supset .\, \delta_Q\text{‘}\breve{S}\text{‘‘}\alpha \subset \breve{S}\text{‘‘}\alpha$ (2)

$\vdash .\, (1) \,.\, (2) \,.\, *216\cdot32 \,.\, \supset \vdash :.\, \mathrm{Hp} \,.\, \supset : \alpha \epsilon \mathrm{closed}\text{‘}P \,.\, \supset .\, \breve{S}\text{‘‘}\alpha \epsilon \mathrm{closed}\text{‘}Q$ (3)

$\vdash .\, (3) \dfrac{Q, P, \breve{S}\text{‘‘}\alpha}{P, Q, \alpha} \,.\quad \supset \vdash :.\, \mathrm{Hp} \,.\, \supset : \breve{S}\text{‘‘}\alpha \epsilon \mathrm{closed}\text{‘}Q \,.\, \supset .\, S\text{‘‘}\breve{S}\text{‘‘}\alpha \epsilon \mathrm{closed}\text{‘}P \,.$

[*72·502] $\supset .\, \alpha \epsilon \mathrm{closed}\text{‘}P$ (4)

$\vdash .\, (3) \,.\, (4) \,.\, \supset \vdash .\, \mathrm{Prop}$

***216·412.** $\vdash :. S \in P \,\overline{\mathrm{smor}}\, Q \,.\, \alpha \subset C\text{‘}P \,.\supset : \alpha \in \mathrm{perf}\text{‘}P \,.\equiv .\, \breve{S}\text{“}\alpha \in \mathrm{perf}\text{‘}Q$
[*216·41·411·36]

***216·5.** $\vdash : P \in \mathrm{Ser} \,.\supset .\, \text{ᗡ}\text{‘}\varsigma\text{‘}P - \overrightarrow{P}\text{“}C\text{‘}P \subset \delta\,(\varsigma\text{‘}P)\text{‘}\overrightarrow{P}\text{“}C\text{‘}P$

Dem.

$\vdash .\, *212{\cdot}134 \,.\, *216{\cdot}111 \,.\supset$

$\vdash : \mathrm{Hp} \,.\, P = \dot{\Lambda} \,.\supset .\, \text{ᗡ}\text{‘}\varsigma\text{‘}P - \overrightarrow{P}\text{“}C\text{‘}P = \Lambda \,.\, \delta\,(\varsigma\text{‘}P)\text{‘}\overrightarrow{P}\text{“}C\text{‘}P = \Lambda$ (1)

$\vdash .\, *212{\cdot}632 \,.\supset$

$\vdash :. \mathrm{Hp} \,.\, \dot{\exists} ! P \,.\, P\text{“}\alpha \sim\in \overrightarrow{P}\text{“}C\text{‘}P \,.\supset : P\text{“}\alpha = \mathrm{lt}\,(\varsigma\text{‘}P)\text{‘}\overrightarrow{P}\text{“}\alpha :$

$[*216{\cdot}1] \supset : \exists ! \alpha \,.\, \alpha \subset C\text{‘}P \,.\supset .\, P\text{“}\alpha \in \delta\,(\varsigma\text{‘}P)\text{‘}\overrightarrow{P}\text{“}C\text{‘}P$ (2)

$\vdash .\, (2) \,.\, *212{\cdot}132 \,.\, *37{\cdot}265 \,.\supset$

$\vdash : \mathrm{Hp} \,.\, \dot{\exists} ! P \,.\, \beta \in \text{ᗡ}\text{‘}\varsigma\text{‘}P - \overrightarrow{P}\text{“}C\text{‘}P \,.\supset .\, \beta \in \delta\,(\varsigma\text{‘}P)\text{‘}\overrightarrow{P}\text{“}C\text{‘}P$ (3)

$\vdash .\, (1) \,.\, (3) \,.\supset \vdash .$ Prop

***216·51.** $\vdash : P \in \mathrm{Ser} \,.\supset .$

$$\delta\,(\varsigma\text{‘}P)\text{‘}\overrightarrow{P}\text{“}C\text{‘}P = \delta\,(\varsigma\text{‘}P)\text{‘}C\text{‘}\varsigma\text{‘}P = \mathrm{D}\text{‘}\mathrm{lt}\,(\varsigma\text{‘}P) - \iota\text{‘}\Lambda = \text{ᗡ}\text{‘}\mathrm{sgm}\text{‘}P$$

Dem.

$\vdash .\, *212{\cdot}661 \,.\supset \vdash : \mathrm{Hp} \,.\, \kappa \subset \mathrm{D}\text{‘}P_\epsilon \,.\, x = \mathrm{lt}\,(\varsigma\text{‘}P)\text{‘}\kappa \,.\supset .\, x = \mathrm{lt}\,(\varsigma\text{‘}P)\text{‘}\overrightarrow{P}\text{“}s\text{‘}\kappa$ (1)

$\vdash .\, *207{\cdot}13 \,.\, *212{\cdot}133 \,.\supset \vdash : \mathrm{Hp} \,.\, \kappa \subset \mathrm{D}\text{‘}P_\epsilon \,.\, x = \mathrm{lt}\,(\varsigma\text{‘}P)\text{‘}\kappa \,.\supset .\, \kappa \neq \iota\text{‘}\Lambda$ (2)

$\vdash .\, (1) \,.\, (2) \,.\, *40{\cdot}26 \,.\supset$

$\vdash : \mathrm{Hp} \,.\, \kappa \subset \mathrm{D}\text{‘}P_\epsilon \,.\, \exists ! \kappa \,.\, x = \mathrm{lt}\,(\varsigma\text{‘}P)\text{‘}\kappa \,.\supset .\, \exists ! s\text{‘}\kappa \,.\, x = \mathrm{lt}\,(\varsigma\text{‘}P)\text{‘}\overrightarrow{P}\text{“}s\text{‘}\kappa \,.$

$[*216{\cdot}1] \quad \supset .\, x \in \delta\,(\varsigma\text{‘}P)\text{‘}\overrightarrow{P}\text{“}C\text{‘}P$ (3)

$\vdash .\, (3) \,.\, *216{\cdot}1 \,. \quad \supset \vdash : \mathrm{Hp} \,.\supset .\, \delta\,(\varsigma\text{‘}P)\text{‘}C\text{‘}\varsigma\text{‘}P \subset \delta\,(\varsigma\text{‘}P)\text{‘}\overrightarrow{P}\text{“}C\text{‘}P$ (4)

$\vdash .\, *211{\cdot}3 \,.\, *216{\cdot}15 \,.\supset \vdash .\, \delta\,(\varsigma\text{‘}P)\text{‘}\overrightarrow{P}\text{“}C\text{‘}P \subset \delta\,(\varsigma\text{‘}P)\text{‘}C\text{‘}\varsigma\text{‘}P$ (5)

$\vdash .\, (4) \,.\, (5) \,. \quad \supset \vdash : \mathrm{Hp} \,.\supset .\, \delta\,(\varsigma\text{‘}P)\text{‘}\overrightarrow{P}\text{“}C\text{‘}P = \delta\,(\varsigma\text{‘}P)\text{‘}C\text{‘}\varsigma\text{‘}P$ (6)

$[*216{\cdot}2 . *212{\cdot}133] \quad = \mathrm{D}\text{‘}\mathrm{lt}\,(\varsigma\text{‘}P) - \iota\text{‘}\Lambda$ (7)

$\vdash .\, (6) \,.\, (7) \,.\, *212{\cdot}667 \,.\supset \vdash .$ Prop

***216·52.** $\vdash : P \in \mathrm{Ser} \,.\, \dot{\exists} ! P \,.\, \alpha \subset C\text{‘}P \,.\supset .\, \delta\,(\varsigma\text{‘}P)\text{‘}\overrightarrow{P}\text{“}\alpha = P\text{“}\text{“}(\mathrm{Cl\,ex}\text{‘}\alpha - \text{ᗡ}\text{‘}\mathrm{max}_P)$

Dem.

$\vdash .\, *216{\cdot}1 \,.\supset \vdash :. \mathrm{Hp} \,.\supset : \gamma \in \delta(\varsigma\text{‘}P)\text{‘}\overrightarrow{P}\text{“}\alpha \,.\equiv .\, (\exists \kappa) \,.\, \kappa \subset \overrightarrow{P}\text{“}\alpha \,.\, \exists ! \kappa \,.\, \gamma = \mathrm{lt}\,(\varsigma\text{‘}P)\text{‘}\kappa \,.$

$[*212{\cdot}402] \quad \equiv .\, (\exists \kappa) \,.\, \kappa \subset \overrightarrow{P}\text{“}\alpha \,.\, \exists ! \kappa \,.\sim \mathrm{E} ! \max\,(\varsigma\text{‘}P)\text{‘}\kappa \,.\, \gamma = s\text{‘}\kappa \,.$

$[*71{\cdot}47 . *37{\cdot}2] \quad \equiv .\, (\exists \beta) \,.\, \beta \subset \alpha \,.\, \exists ! \beta \,.\sim \mathrm{E} ! \max\,(\varsigma\text{‘}P)\text{‘}\overrightarrow{P}\text{“}\beta \,.\, \gamma = s\text{‘}\overrightarrow{P}\text{“}\beta \,.$

$[*40{\cdot}5 . *212{\cdot}601] \equiv .\, (\exists \beta) \,.\, \beta \subset \alpha \,.\, \exists ! \beta \,.\sim \mathrm{E} ! \max_P\text{‘}\beta \,.\, \gamma = P\text{“}\beta \,.$

$[*37{\cdot}6] \quad \equiv .\, \gamma \in P\text{“}\text{“}(\mathrm{Cl\,ex}\text{‘}\alpha - \text{ᗡ}\text{‘}\mathrm{max}_P) :. \supset \vdash .$ Prop

***216·521.** $\vdash : P \in \mathrm{Ser} \,.\, \alpha \subset C\text{‘}P \,.\supset .\, \overrightarrow{P}\text{“}(\alpha - \overrightarrow{\min}_P\text{‘}\alpha) = \overrightarrow{P}\text{“}\alpha - \overrightarrow{\min}\,(\varsigma\text{‘}P)\text{‘}\overrightarrow{P}\text{“}\alpha$

Dem.

$\vdash .\, *71{\cdot}381 \,.\, *204{\cdot}34 \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, \overrightarrow{P}\text{“}(\alpha - \overrightarrow{\min}_P\text{‘}\alpha) = \overrightarrow{P}\text{“}\alpha - \overrightarrow{P}\text{“}\overrightarrow{\min}_P\text{‘}\alpha$

$[*212{\cdot}6] \quad = \overrightarrow{P}\text{“}\alpha - \overrightarrow{\min}\,(\varsigma\text{‘}P)\text{‘}\overrightarrow{P}\text{“}\alpha : \supset \vdash .$ Prop

***216·53.** $\vdash :. P \in \mathrm{Ser} . \dot{\exists} ! P . \alpha \subset C\text{‘}P . \supset : \alpha \in \mathrm{dense}\text{‘}P . \equiv . \overrightarrow{P}\text{‘‘}\alpha \in \mathrm{dense}\text{‘}\varsigma\text{‘}P$

Dem.

$\vdash . *216\cdot52\cdot3 . \supset$

$\vdash :: \mathrm{Hp} . \supset :. \overrightarrow{P}\text{‘‘}\alpha \in \mathrm{dense}\text{‘}\varsigma\text{‘}P . \equiv :$

$\qquad \overrightarrow{P}\text{‘‘}\alpha - \overrightarrow{\min}(\varsigma\text{‘}P)\text{‘}\overrightarrow{P}\text{‘‘}\alpha \subset P\text{‘‘‘}(\mathrm{Cl\,ex}\text{‘}\alpha - \text{Ɑ}\text{‘}\max_P) :$

$[*216\cdot521] \equiv : \overrightarrow{P}\text{‘‘}(\alpha - \overrightarrow{\min}_P\text{‘}\alpha) \subset P\text{‘‘‘}(\mathrm{Cl\,ex}\text{‘}\alpha - \text{Ɑ}\text{‘}\max_P) :$

$[*37\cdot6] \quad \equiv : x \in \alpha - \overrightarrow{\min}_P\text{‘}\alpha . \supset_x . (\exists\beta) . \beta \subset \alpha . \exists ! \beta . \sim E ! \max_P\text{‘}\beta . \overrightarrow{P}\text{‘}x = P\text{‘‘}\beta :$

$[*207\cdot521] \equiv : x \in \alpha - \overrightarrow{\min}_P\text{‘}\alpha . \supset_x . (\exists\beta) . \beta \subset \alpha . \exists ! \beta . x = \mathrm{lt}_P\text{‘}\beta :$

$[*216\cdot1] \quad \equiv : x \in \alpha - \overrightarrow{\min}_P\text{‘}\alpha . \supset_x . x \in \delta_P\text{‘}\alpha :$

$[*216\cdot3] \quad \equiv : \alpha \in \mathrm{dense}\text{‘}P :: \supset \vdash . \mathrm{Prop}$

***216·54.** $\vdash :. P \in \mathrm{Ser} . \dot{\exists} ! P . \alpha \subset C\text{‘}P . \supset : \alpha \in \mathrm{closed}\text{‘}P . \equiv . \delta(\varsigma\text{‘}P)\text{‘}\overrightarrow{P}\text{‘‘}\alpha \subset \overrightarrow{P}\text{‘‘}\alpha$

Dem.

$\vdash . *216\cdot52 . \supset$

$\vdash :: \mathrm{Hp} . \supset :. \delta(\varsigma\text{‘}P)\text{‘}\overrightarrow{P}\text{‘‘}\alpha \subset \overrightarrow{P}\text{‘‘}\alpha . \equiv : P\text{‘‘‘}(\mathrm{Cl\,ex}\text{‘}\alpha - \text{Ɑ}\text{‘}\max_P) \subset \overrightarrow{P}\text{‘‘}\alpha :$

$[*37\cdot6] \quad \equiv : \beta \subset \alpha . \exists ! \beta . \sim E ! \max_P\text{‘}\beta . \supset_\beta . (\exists x) . x \in \alpha . P\text{‘‘}\beta = \overrightarrow{P}\text{‘}x :$

$[*207\cdot521] \equiv : \beta \subset \alpha . \exists ! \beta . \sim E ! \max_P\text{‘}\beta . \supset_\beta . \mathrm{lt}_P\text{‘}\beta \in \alpha :$

$[*216\cdot34] \equiv : \alpha \in \mathrm{closed}\text{‘}P :: \supset \vdash . \mathrm{Prop}$

***216·55.** $\vdash :. P \in \mathrm{Ser} . \dot{\exists} ! P . \alpha \subset C\text{‘}P . \supset : \alpha \in \mathrm{closed}\text{‘}P . \equiv . \overrightarrow{P}\text{‘‘}\alpha \in \mathrm{closed}\text{‘}\varsigma\text{‘}P$

Dem.

$\vdash . *212\cdot44 . \supset \vdash : \mathrm{Hp} . \supset . \overrightarrow{P}\text{‘‘}\alpha \subset \text{Ɑ}\text{‘}\mathrm{limax}(\varsigma\text{‘}P) \qquad (1)$

$\vdash . (1) . *212\cdot54\cdot32 . \supset \vdash . \mathrm{Prop}$

***216·56.** $\vdash :. P \in \mathrm{Ser} . \dot{\exists} ! P . \alpha \subset C\text{‘}P . \supset : \alpha \in \mathrm{perf}\text{‘}P . \equiv . \overrightarrow{P}\text{‘‘}\alpha \in \mathrm{perf}\text{‘}\varsigma\text{‘}P .$

$\qquad \equiv . \delta(\varsigma\text{‘}P)\text{‘}\overrightarrow{P}\text{‘‘}\alpha = \overrightarrow{P}\text{‘‘}\alpha - \overrightarrow{\min}(\varsigma\text{‘}P)\text{‘}\overrightarrow{P}\text{‘‘}\alpha$

[*216·53·54·55·36·37 . *212·44]

***216·6.** $\vdash : x(\nabla\text{‘}P)y . \equiv . x, y \in \mathrm{D}\text{‘}\mathrm{lt}_P . xPy \quad [(*216\cdot05)]$

***216·601.** $\vdash : x \in \mathrm{D}\text{‘}\mathrm{lt}_P \cap \text{Ɑ}\text{‘}P . P \in \mathrm{connex} . E ! B\text{‘}P . \supset . (B\text{‘}P)(\nabla\text{‘}P)x$

Dem.

$\vdash . *206\cdot14 . \supset \vdash : \mathrm{Hp} . \supset . B\text{‘}P \in \mathrm{D}\text{‘}\mathrm{lt}_P \qquad (1)$

$\vdash . *202\cdot524 . \supset \vdash : \mathrm{Hp} . \supset . (B\text{‘}P)Px \qquad (2)$

$\vdash . (1) . (2) . *216\cdot6 . \supset \vdash . \mathrm{Prop}$

***216·602.** $\vdash : P \in \mathrm{connex} . E ! B\text{‘}P . \supset . \text{Ɑ}\text{‘}\nabla\text{‘}P = \mathrm{D}\text{‘}\mathrm{lt}_P - \overrightarrow{B}\text{‘}P = \delta_P\text{‘}C\text{‘}P$

Dem.

$\vdash . *216\cdot601 . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{D}\text{‘}\mathrm{lt}_P - \overrightarrow{B}\text{‘}P \subset \text{Ɑ}\text{‘}\nabla\text{‘}P \qquad (1)$

$\vdash . *216\cdot6 . \supset \vdash . \text{Ɑ}\text{‘}\nabla\text{‘}P \subset \mathrm{D}\text{‘}\mathrm{lt}_P - \overrightarrow{B}\text{‘}P \qquad (2)$

$\vdash . (1) . (2) . *216\cdot2 . \supset \vdash . \mathrm{Prop}$

***216·603.** $\vdash : P \in \text{connex} . \dot{\exists} ! \nabla\text{'}P . \supset . C\text{'}\nabla\text{'}P = \text{D'lt}_P$

Dem.

$\vdash . *200\cdot35 . \supset \vdash : \text{Hp} . \supset . \text{D'lt}_P \sim\in 1 .$

$[*202\cdot55] \qquad \supset . C\text{'}\nabla\text{'}P = \text{D'lt}_P : \supset \vdash . \text{Prop}$

***216·61.** $\vdash : P \in \text{Ser} . \text{E} ! B\text{'}P . \supset . \text{ΙΙ'}\nabla\text{'}P = \text{ΙΙ'}P - \text{ΙΙ'}P_1$ [*216·602·21]

***216·611.** $\vdash : P \in \text{Ser} . \dot{\exists} ! \nabla\text{'}P . \supset . C\text{'}\nabla\text{'}P = C\text{'}P - \text{ΙΙ'}P_1 = \delta_P\text{'}C\text{'}P \cup \overrightarrow{B}\text{'}P$

Dem.

$\vdash . *216\cdot603 . *206\cdot14 . \supset \vdash : \text{Hp} . \supset . C\text{'}\nabla\text{'}P = (\text{D'lt}_P - \overrightarrow{B}\text{'}P) \cup \overrightarrow{B}\text{'}P$

$[*216\cdot2] \qquad = \delta_P\text{'}C\text{'}P \cup \overrightarrow{B}\text{'}P \qquad (1)$

$[*216\cdot21] \qquad = (\text{ΙΙ'}P - \text{ΙΙ'}P_1) \cup \overrightarrow{B}\text{'}P$

$[*93\cdot103 . *24\cdot412] \qquad = C\text{'}P - \text{ΙΙ'}P_1 \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***216·612.** $\vdash : P \in \text{Ser} . \supset . \text{ΙΙ'}\nabla\text{'}P \subset \text{ΙΙ'}P - \text{ΙΙ'}P_1$

Dem.

$\vdash . *216\cdot6 . \quad \supset \vdash . \text{ΙΙ'}\nabla\text{'}P \subset \text{D'lt}_P - \overrightarrow{B}\text{'}P \qquad (1)$

$\vdash . *216\cdot2\cdot21 . \supset \vdash : \text{Hp} . \supset . \text{D'lt}_P - \overrightarrow{B}\text{'}P = \text{ΙΙ'}P - \text{ΙΙ'}P_1 \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***216·62.** $\vdash : P \in \text{Ser} . \dot{\exists} ! \nabla\text{'}P . \supset . C\text{'}\nabla\text{'}P = \text{seq}_P\text{''}C\text{'sgm'}P = \text{lt}_P\text{''}C\text{'sgm'}P$

Dem.

$\vdash . *216\cdot611 . \supset \vdash : \text{Hp} . \supset . C\text{'}\nabla\text{'}P = \delta_P\text{'}C\text{'}P \cup \overrightarrow{B}\text{'}P$

$[*216\cdot23] \qquad = \text{seq}_P\text{''ΙΙ'sgm'}P \cup \overrightarrow{B}\text{'}P \qquad (1)$

$[*206\cdot14] \qquad = \text{seq}_P\text{''}(\text{ΙΙ'sgm'}P \cup \iota\text{'}\Lambda) \qquad (2)$

$\vdash . *211\cdot45 . \supset \vdash : \text{Hp} . \exists ! \text{ΙΙ'}P - \text{ΙΙ'}P_1 . \supset . \exists ! \text{D'}(P_\epsilon \dot{\wedge} I) - \iota\text{'}\Lambda .$

$[*212\cdot153] \qquad \supset . \dot{\exists} ! \text{sgm'}P .$

$[*212\cdot155] \qquad \supset . \text{ΙΙ'sgm'}P \cup \iota\text{'}\Lambda = C\text{'sgm'}P \qquad (3)$

$\vdash . (1) . *216\cdot23 . *207\cdot17 . \supset \vdash : \text{Hp} . \supset . C\text{'}\nabla\text{'}P = \text{lt}_P\text{''}(\text{ΙΙ'sgm'}P \cup \iota\text{'}\Lambda) \qquad (4)$

$\vdash . (2) . (3) . (4) . \supset \vdash . \text{Prop}$

***216·621.** $\vdash : P \in \text{Ser} . \dot{\exists} ! \nabla\text{'}P . \supset . \dot{\exists} ! \text{sgm'}P . \exists ! \text{ΙΙ'}P - \text{ΙΙ'}P_1$ [*216·62·612]

*217. ON SEGMENTS OF SUMS AND CONVERSES

Summary of *217.

The purpose of the present number is to prove *217·43, which is required in the theory of real numbers (Part VI, Section A), where Q will be the series of positive ratios including zero, P will be the series of negative ratios in the order from zero to $-\infty$ (both excluded), α the real number zero, and Z and W two different series either of which may be taken as the series of negative and positive real numbers. In virtue of *217·43, these two series are ordinally similar.

***217·1.** $\vdash : \alpha \cap C'Q = \Lambda . \supset . (P \ddagger Q)``\alpha = P``\alpha$ [*160·1]

***217·11.** $\vdash : \exists ! \alpha \cap C'Q . \supset . (P \ddagger Q)``\alpha = C'P \cup Q``\alpha$ [*160·1]

***217·12.** $\vdash . \text{D}'(P \ddagger Q)_\epsilon \subset \text{D}'P_\epsilon \cup (C'P \cup)``\text{D}'Q_\epsilon$ [*217·1·11 . *211·11]

***217·13.** $\vdash : C'P \cap C'Q = \Lambda . \supset . P``\alpha = (P \ddagger Q)``(\alpha - C'Q)$ [*217·1]

***217·14.** $\vdash : \exists ! Q``\alpha . \supset . C'P \cup Q``\alpha = (P \ddagger Q)``\alpha$ [*217·11]

***217·15.** $\vdash : C'P \cap C'Q = \Lambda . \supset . \text{D}'P_\epsilon \cup (C'P \cup)``(\text{D}'Q_\epsilon - \iota'\Lambda) \subset \text{D}'(P \ddagger Q)_\epsilon$ [*217·13·14]

***217·16.** $\vdash :. C'P \cap C'Q = \Lambda : \sim \exists ! \overrightarrow{B}'\breve{P} . \vee . \exists ! \overrightarrow{B}'Q : \supset . C'P \in \text{D}'(P \ddagger Q)_\epsilon$

Dem.

$\vdash . *211·301 . \quad \supset \vdash : \sim \exists ! \overrightarrow{B}'\breve{P} . \supset . C'P \in \text{D}'P_\epsilon$ (1)

$\vdash . (1) . *217·15 . \supset \vdash : \text{Hp} . \sim \exists ! \overrightarrow{B}'\breve{P} . \supset . C'P \in \text{D}'(P \ddagger Q)_\epsilon$ (2)

$\vdash . *217·11 . \quad \supset \vdash : \exists ! \overrightarrow{B}'Q . \supset . (P \ddagger Q)``\overrightarrow{B}'Q = C'P$ (3)

$\vdash . (2) . (3) . \supset \vdash . \text{Prop}$

***217·17.** $\vdash :. C'P \cap C'Q = \Lambda : \sim \exists ! \overrightarrow{B}'\breve{P} . \vee . \exists ! \overrightarrow{B}'Q : \supset .$
$\text{D}'(P \ddagger Q)_\epsilon = \text{D}'P_\epsilon \cup (C'P \cup)``\text{D}'Q_\epsilon$ [*217·12·15·16]

***217·18.** $\vdash :. C'P \cap C'Q = \Lambda : \sim \exists ! \overrightarrow{B}'\breve{P} . \vee . \sim \exists ! \overrightarrow{B}'Q : \supset .$
$\text{D}'(P \ddagger Q)_\epsilon = \text{D}'P_\epsilon \cup (C'P \cup)``(\text{D}'Q_\epsilon - \iota'\Lambda)$

Dem.

$\vdash . *211·301 . \supset \vdash : \sim \exists ! \overrightarrow{B}'\breve{P} . \supset . (C'P \cup)``\iota'\Lambda \subset \text{D}'P_\epsilon$ (1)

$\vdash . (1) . *217·17 . \supset \vdash : \text{Hp} . \sim \exists ! \overrightarrow{B}'\breve{P} . \supset .$
$\text{D}'(P \ddagger Q)_\epsilon = \text{D}'P_\epsilon \cup (C'P \cup)``(\text{D}'Q_\epsilon - \iota'\Lambda)$ (2)

$\vdash . *217·11 . \quad \supset \vdash : \sim \exists ! \overrightarrow{B}'Q . \exists ! \alpha \cap C'Q . \supset . \exists ! (P \ddagger Q)``\alpha \cap C'Q$ (3)

$\vdash . *217·1 . \quad \supset \vdash : \sim \exists ! \overrightarrow{B}'Q . \exists ! \overrightarrow{B}'\breve{P} . \alpha \cap C'Q = \Lambda . \supset . (P \ddagger Q)``\alpha \neq C'P$ (4)

$\vdash . (3) . (4) . \quad \supset \vdash : \text{Hp} . \sim \exists ! \overrightarrow{B}'Q . \exists ! \overrightarrow{B}'\breve{P} . \supset . C'P \sim \in \text{D}'(P \ddagger Q)_\epsilon$ (5)

$\vdash . (5) . *217·12·15 . \supset$

$\vdash : \text{Hp}(5) . \supset . \text{D}'(P \ddagger Q)_\epsilon = \text{D}'P_\epsilon \cup (C'P \cup)``(\text{D}'Q_\epsilon - \iota'\Lambda)$ (6)

$\vdash . (2) . (6) . \supset \vdash . \text{Prop}$

***217·2.** $\vdash : C'P \cap C'Q = \Lambda . \supset . D'P_\epsilon \cap (C'P \cup)``(D'Q_\epsilon - \iota'\Lambda) = \Lambda$

Dem.

$\vdash . *211·11 . \supset$

$\vdash : D'P_\epsilon \subset Cl'C'P : \alpha \in (C'P \cup)``(D'Q_\epsilon - \iota'\Lambda) . \supset . \exists ! \alpha \cap C'Q : \supset \vdash .$ Prop

***217·21.** $\vdash : \exists ! \overrightarrow{B}'\breve{P} . \supset . D'P_\epsilon \cap (C'P \cup)``D'Q_\epsilon = \Lambda$

Dem.

$\vdash . *211·11 . \supset \vdash : Hp . \alpha \in D'P_\epsilon . \supset . \exists ! C'P - \alpha : \supset \vdash .$ Prop

***217·22.** $\vdash : P, Q \in \text{trans} \cap \text{connex} . C'P \cap C'Q = \Lambda . \exists ! \overrightarrow{B}'\breve{P} . \exists ! \overrightarrow{B}'Q . \supset .$
$s'(P \ddagger Q) = s'P \ddagger (C'P \cup);s'Q$

Dem.

$\vdash . *201·401 . *202·401 . \supset \vdash : Hp . \supset . P \ddagger Q \in \text{trans} \cap \text{connex}$ (1)

$\vdash . (1) . *212·23 . \supset$

$\vdash :: Hp . \supset :. \alpha \{s'(P \ddagger Q)\} \beta . \equiv : \alpha, \beta \in D'(P \ddagger Q)_\epsilon . \alpha \subset \beta . \alpha \neq \beta :$

$[*217·17·21] \equiv : \alpha, \beta \in D'P_\epsilon . \alpha \subset \beta . \alpha \neq \beta . v . \alpha \in D'P_\epsilon . \beta \in (C'P \cup)``D'Q_\epsilon .$
$v . \alpha, \beta \in (C'P \cup)``D'Q_\epsilon . \alpha \subset \beta . \alpha \neq \beta :$

$[*212·23] \quad \equiv : \alpha (s'P) \beta . v . \alpha \in C's'P . \beta \in C'(C'P \cup);s'Q .$
$v . \alpha \{(C'P \cup);s'Q\} \beta :$

$[*160·11] \quad \equiv : \alpha \{s'P \ddagger (C'P \cup);s'Q\} \beta :: \supset \vdash .$ Prop

***217·23.** $\vdash :. P, Q \in \text{trans} \cap \text{connex} . C'P \cap C'Q = \Lambda : \sim\exists ! \overrightarrow{B}'\breve{P} . v . \sim\exists ! \overrightarrow{B}'Q : \supset .$
$s'(P \ddagger Q) = s'P \ddagger (C'P \cup);(s'Q) \upharpoonright (-\iota'\Lambda)$

Dem.

$\vdash . *201·401 . *202·401 . *212·23 . \supset$

$\vdash :: Hp . \supset :. \alpha \{s'(P \ddagger Q)\} \beta . \equiv : \alpha, \beta \in D'(P \ddagger Q)_\epsilon . \alpha \subset \beta . \alpha \neq \beta :$

$[*217·18·2] \equiv : \alpha, \beta \in D'P_\epsilon . \alpha \subset \beta . \alpha \neq \beta . v . \alpha \in D'P_\epsilon . \beta \in (C'P \cup)``(D'Q_\epsilon - \iota'\Lambda) .$
$v . \alpha, \beta \in (C'P \cup)``(D'Q_\epsilon - \iota'\Lambda) . \alpha \subset \beta . \alpha \neq \beta :$

$[*212·23.*160·11] \equiv : \alpha \{s'P \ddagger (C'P \cup);(s'Q) \upharpoonright (-\iota'\Lambda)\} \beta :: \supset \vdash .$ Prop

***217·24.** $\vdash : \alpha \cap \beta = \Lambda . \supset . (\alpha \cup) \upharpoonright Cl'\beta \in 1 \to 1$ [*24·481]

***217·25.** $\vdash : C'P \cap C'Q = \Lambda . \supset . (C'P \cup) \upharpoonright C's'Q \in \{(C'P \cup);s'Q\} \overline{smor} (s'Q)$
[*217·24]

***217·3.** $\vdash : P \in Ser . \supset . D'P_\epsilon = \overrightarrow{P}``C'P \cup D'(P_\epsilon \dot{\cap} I) - \text{Cl}'seq_P$
[*211·32·302·41]

***217·301.** $\vdash : P \in Ser . \gamma \in D'(P_\epsilon \dot{\cap} I) - \text{Cl}'seq_P . \supset . \gamma = C'P - \breve{P}``(C'P - \gamma)$

Dem.

$\vdash . *211·727 . \supset \vdash : Hp . \supset . \sim E ! \text{limin}_P'(C'P - \gamma) .$

$[*207·44.*211·7] \qquad \supset . C'P - \gamma \in \text{sect}'\breve{P} - \text{Cl}'min_P .$

$[*211·41·12] \qquad \supset . C'P - \gamma = \breve{P}``(C'P - \gamma) : \supset \vdash .$ Prop

***217·31.** $\vdash : P \epsilon \mathrm{Ser} . \gamma \epsilon \mathrm{D}‘P_\epsilon . \supset . (\exists \beta) . \gamma = P``(C‘P - \breve{P}``\beta)$

Dem.

$\vdash . *201·53 . \supset$

$\vdash : \mathrm{Hp} . \gamma = \overrightarrow{P}‘x . \beta = \overleftarrow{P_*}‘x . \supset . C‘P - \breve{P}``\beta = \overrightarrow{P_*}‘x .$

$[*201·53] \qquad \supset . P``(C‘P - \breve{P}``\beta) = \gamma \qquad (1)$

$\vdash . (1) . *217·3·301 . \supset \vdash . \mathrm{Prop}$

***217·32.** $\vdash : P \epsilon \mathrm{Ser} . \supset . \mathrm{D}‘(\breve{P})_\epsilon = (\breve{P})_\epsilon``(C‘P -)``\mathrm{D}‘P_\epsilon$

Dem.

$\vdash . *217·31 \frac{\breve{P}}{P} . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{D}‘(\breve{P})_\epsilon \subset (\breve{P})_\epsilon ``(C‘P -)``\mathrm{D}‘P_\epsilon \qquad (1)$

$\vdash . (1) . *37·16 . \supset \vdash . \mathrm{Prop}$

***217·33.** $\vdash . (\alpha -) \upharpoonright \mathrm{Cl}‘\alpha \epsilon 1 \rightarrow 1$

Dem.

$\vdash . *24·492 . \supset \vdash : \beta \subset \alpha . \gamma \subset \alpha . \alpha - \beta = \alpha - \gamma . \supset . \beta = \gamma : \supset \vdash . \mathrm{Prop}$

***217·34.** $\vdash : P \epsilon \mathrm{Ser} . \supset . P_\epsilon \upharpoonright (\mathrm{sect}‘P - \mathrm{G}‘\mathrm{lt}_P) \epsilon 1 \rightarrow 1$

Dem.

$\vdash . *211·1 . \supset \vdash : \alpha, \beta \epsilon \mathrm{sect}‘P . P``\alpha = P``\beta . \exists ! \beta - \alpha . \supset . \exists ! \beta - P``\beta \qquad (1)$

$\vdash . (1) . *205·111 . \qquad \supset \vdash : \mathrm{Hp} . \mathrm{Hp}(1) . \supset . \mathrm{E} ! \max_P‘\beta \qquad (2)$

$\vdash . *211·56 . \qquad \supset \vdash : \mathrm{Hp}(2) . \supset . \alpha \subset P``\beta . \qquad (3)$

$[*205·111.(2)] \qquad \supset . \max_P‘\beta \sim \epsilon \alpha \qquad (4)$

$\vdash . (3) . \qquad \supset \vdash : \mathrm{Hp}(2) . \supset . \alpha = P``\beta .$

$[*205·22.(2).\mathrm{Hp}] \qquad \supset . \alpha = \overrightarrow{P}‘\max_P‘\beta = P``\alpha \qquad (5)$

$\vdash . (4) . (5) . *207·232 . \supset \vdash : \mathrm{Hp}(2) . \supset . \max_P‘\beta = \mathrm{lt}_P‘\alpha \qquad (6)$

$\vdash . (6) . \mathrm{Transp} . \supset \vdash : \mathrm{Hp} . \alpha, \beta \epsilon \mathrm{sect}‘P . P``\alpha = P``\beta . \sim \mathrm{E} ! \mathrm{lt}_P‘\alpha . \supset . \beta \subset \alpha \quad (7)$

Similarly $\qquad \vdash : \mathrm{Hp} . \alpha, \beta \epsilon \mathrm{sect}‘P . P``\alpha = P``\beta . \sim \mathrm{E} ! \mathrm{lt}_P‘\beta . \supset . \alpha \subset \beta \quad (8)$

$\vdash . (7) . (8) . \supset \vdash : \mathrm{Hp} . \alpha, \beta \epsilon \mathrm{sect}‘P - \mathrm{G}‘\mathrm{lt}_P . P``\alpha = P``\beta . \supset . \alpha = \beta : \supset \vdash . \mathrm{Prop}$

***217·35.** $\vdash : P \epsilon \mathrm{Ser} . \supset . (\breve{P})_\epsilon \,|\, (C‘P -) \upharpoonright \mathrm{D}‘P_\epsilon \epsilon 1 \rightarrow 1$

Dem.

$\vdash . *217·33 . \supset \vdash . (C‘P -) \upharpoonright \mathrm{D}‘P_\epsilon \epsilon 1 \rightarrow 1 \qquad (1)$

$\vdash . *211·76 . \supset \vdash : \mathrm{Hp} . \supset . (C‘P -)``\mathrm{D}‘P_\epsilon = \mathrm{sect}‘\breve{P} - \mathrm{G}‘\mathrm{tl}_P .$

$[*217·34] \qquad \supset . (\breve{P})_\epsilon \upharpoonright (C‘P -)``\mathrm{D}‘P_\epsilon \epsilon 1 \rightarrow 1 \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***217·36.** $\vdash : P \epsilon \mathrm{Ser} . \supset . \mathfrak{s}‘\breve{P} = (\breve{P})_\epsilon ; (C‘P -) ; \mathrm{Cnv}‘\mathfrak{s}‘P$

Dem.

$\vdash . *212·23 . \supset$

$\vdash :. \mathrm{Hp} . \supset : \beta (\mathfrak{s}‘P) \alpha . \gamma = \breve{P}``(C‘P - \alpha) . \delta = \breve{P}``(C‘P - \beta) . \supset .$

$\beta \subset \alpha . \alpha \neq \beta . C‘P - \alpha \subset C‘P - \beta .$

$[*37·2.*217·35] \supset . \gamma \subset \delta . \gamma \neq \delta .$

$[*212·23] \qquad \supset . \gamma (\mathfrak{s}‘\breve{P}) \delta \qquad (1)$

$\vdash . (1) . \supset \vdash : \mathrm{Hp} . \supset . (\breve{P})_\epsilon ; (C‘P -) ; \mathrm{Cnv}‘\mathrm{s}‘P \Subset \mathrm{s}‘\breve{P}$ (2)

$\vdash . (1) . \mathrm{Transp} . \supset$

$\vdash : \mathrm{Hp} . \delta (\mathrm{s}‘\breve{P}) \gamma . \gamma = \breve{P}‘‘(C‘P - \alpha) . \delta = \breve{P}‘‘(C‘P - \beta) . \alpha, \beta \in \mathrm{D}‘P_\epsilon . \supset . \alpha \subset \beta$ (3)

$\vdash . *217{\cdot}35 . \qquad \supset \vdash : \mathrm{Hp}(3) . \supset . \alpha \neq \beta$ (4)

$\vdash . (3) . (4) . *212{\cdot}23 . \supset \vdash : \mathrm{Hp}(3) . \supset . \delta \{(\breve{P})_\epsilon ; (C‘P -) ; \mathrm{Cnv}‘\mathrm{s}‘P\} \gamma$ (5)

$\vdash . *217{\cdot}31 . \supset \vdash : \mathrm{Hp} . \delta (\mathrm{s}‘\breve{P}) \gamma . \supset .$

$(\exists \alpha, \beta) . \gamma = \breve{P}‘‘(C‘P - \alpha) . \delta = \breve{P}‘‘(C‘P - \beta) . \alpha, \beta \in \mathrm{D}‘P_\epsilon$ (6)

$\vdash . (5) . (6) . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{s}‘\breve{P} \Subset (\breve{P})_\epsilon ; (C‘P -) ; \mathrm{Cnv}‘\mathrm{s}‘P$ (7)

$\vdash . (2) . (7) . \supset \vdash . \mathrm{Prop}$

***217·37.** $\vdash : P \in \mathrm{Ser} . \supset . (\breve{P})_\epsilon \mid (C‘P -) \upharpoonright \mathrm{D}‘P_\epsilon \in (\mathrm{s}‘\breve{P}) \overline{\mathrm{smor}} (\mathrm{Cnv}‘\mathrm{s}‘P)$
[*217·35·36]

***217·38.** $\vdash : P \in \mathrm{Ser} . \supset . (\mathrm{s}‘\breve{P}) \mathrm{smor} (\mathrm{Cnv}‘\mathrm{s}‘P)$ [*217·37]

***217·4.** $\vdash : P, Q \in \mathrm{Ser} . C‘P \cap C‘Q = \Lambda . \mathrm{E}!\, B‘P . \mathrm{E}!\, B‘Q . \supset .$

$\mathrm{s}‘(\breve{P} \ddagger Q) = (\breve{P})_\epsilon ; (C‘P -) ; \mathrm{Cnv}‘\mathrm{s}‘P \ddagger (C‘P \cup) ; \mathrm{s}‘Q$ [*217·22·36]

***217·41.** $\vdash :. P, Q \in \mathrm{Ser} . C‘P \cap C‘Q = \Lambda : \sim \mathrm{E}!\, B‘P . \vee . \sim \mathrm{E}!\, B‘Q : \supset .$

$\mathrm{s}‘(\breve{P} \ddagger Q) = (\breve{P})_\epsilon ; (C‘P -) ; \mathrm{Cnv}‘\mathrm{s}‘P \ddagger (C‘P \cup) ; (\mathrm{s}‘Q) \upharpoonleft (- \iota‘\Lambda)$
[*217·23·36]

***217·411.** $\vdash : \mathrm{Hp} *217{\cdot}41 . \supset . \{\mathrm{s}‘(\breve{P} \ddagger Q)\} \upharpoonleft (- \iota‘\Lambda) =$

$(\breve{P})_\epsilon ; (C‘P -) ; \mathrm{Cnv}‘(\mathrm{s}‘P) \upharpoonleft (- \iota‘\mathrm{D}‘P) \ddagger (C‘P \cup) ; (\mathrm{s}‘Q) \upharpoonleft (- \iota‘\Lambda)$
[*217·41]

***217·42.** $\vdash : \mathrm{Hp} *217{\cdot}41 . \supset . \{\mathrm{s}‘(\breve{P} \ddagger Q)\} \upharpoonleft \{- \iota‘\Lambda - \iota‘\mathrm{D}‘(P \ddagger Q)\} =$

$(\breve{P})_\epsilon ; (C‘P -) ; \mathrm{Cnv}‘(\mathrm{s}‘P) \upharpoonleft (- \iota‘\Lambda - \iota‘\mathrm{D}‘P) \mapsto \text{Œ}‘P$

$\ddagger (C‘P \cup) ; (\mathrm{s}‘Q) \upharpoonleft (- \iota‘\Lambda - \iota‘\mathrm{D}‘Q)$ [*217·411]

***217·43.** $\vdash :. P, Q \in \mathrm{Ser} . C‘P \cap C‘Q = \Lambda : \sim \mathrm{E}!\, B‘P . \vee . \sim \mathrm{E}!\, B‘Q :$

$X = (\mathrm{s}‘P) \upharpoonleft (- \iota‘\Lambda - \iota‘\mathrm{D}‘P) . Y = (\mathrm{s}‘Q) \upharpoonleft (- \iota‘\Lambda - \iota‘\mathrm{D}‘Q) .$

$Z = \{\mathrm{s}‘(\breve{P} \ddagger Q)\} \upharpoonleft \{- \iota‘\Lambda - \iota‘\mathrm{D}‘(\breve{P} \ddagger Q)\} .$

$W = \breve{X} \mapsto \alpha \ddagger Y . \alpha \sim \in C‘X \cup C‘Y . \supset .$

$(\breve{P})_\epsilon \mid (C‘P -) \upharpoonright (\mathrm{D}‘P_\epsilon - \iota‘\Lambda - \iota‘\mathrm{D}‘P) \uplus (\mathrm{D}‘P) \downarrow \alpha$

$\uplus (C‘P \cup) \upharpoonright (\mathrm{D}‘Q_\epsilon - \iota‘\Lambda - \iota‘\mathrm{D}‘Q) \in Z \overline{\mathrm{smor}}\, W$ [*217·37·25·42]

SECTION C

ON CONVERGENCE, AND THE LIMITS OF FUNCTIONS

Summary of Section C.

The purpose of this section is to express in a general form the definitions of convergence, the limits of functions, the continuity of functions, and kindred notions, and to give such elementary consequences of these definitions as may seem illustrative.

In the definitions usually given in treatises on analysis, it is assumed that both the arguments and the values of the function are numbers of some kind, generally real numbers, and limits are taken with respect to the order of magnitude. There is, however, nothing essential in the definitions to demand so narrow a hypothesis. What is essential is that the arguments should be given as belonging to a series, and that the values should also be given as belonging to a series, which need not be the same series as that to which the arguments belong. In what follows, therefore, we assume that all the possible arguments to our function, or at any rate all the arguments which we consider, belong to the field of a certain relation Q, which, in cases where our definitions are useful, will be a serial relation; we assume similarly that the values of our function, at least for arguments belonging to $C‘Q$, belong to the field of a relation P, which, in all important cases, will be a serial relation. The function itself we represent by the relation of the value to the argument; that is, the relation of $f(x)$ to x is to be R, so that, if the function is one-valued, $f(x) = R‘x$. (If the function is not one-valued, $f(x)$ is any member of $\overrightarrow{R}‘x$.) Thus we may speak of R as the function, Q as the argument-series, and P as the value-series.

To take an illustration: Suppose we are given a set of real numbers $x_1, x_2, \ldots x_\nu, \ldots$, where ν may be any finite integer. Here x_ν is a function of ν; the argument-series is that of the finite integers in order of magnitude, the value-series is that of the real numbers (or any part of this series which contains all the values $x_1, x_2, \ldots x_\nu, \ldots$). The function R is the relation of x_ν to ν, so that $x_\nu = R‘\nu$. In this case, calling the argument-series Q and the value-series P (as will be done throughout this section), we have $\text{Ɑ}‘R = C‘Q =$ the finite integers, $R``C‘Q = \text{D}‘R =$ the *class* $x_1, x_2, \ldots x_\nu, \ldots$, and $R;Q =$ the *series* $x_1, x_2, \ldots x_\nu, \ldots$. The series which arranges $x_1, x_2, \ldots x_\nu, \ldots$ in the order of their own magnitudes, instead of the order of magnitude of their suffixes, is $P \upharpoonright \text{D}‘R$ or $P \upharpoonright R``C‘Q$. This will not be equal to $R;Q$ unless the function is one which continually increases, *i.e.* one for which $\mu < \nu \,.\supset.\, x_\mu < x_\nu$.

In general, the propositions of the present section are only important when P and Q are series. If our assertions are not to be trivial, we must have $\exists ! C\text{‘}Q \cap \text{G‘}R$ and $\exists ! C\text{‘}P \cap R\text{“}C\text{‘}Q$, *i.e.* there must be arguments in $C\text{‘}Q$ which lead to values in $C\text{‘}P$. It will also generally happen that the function is one-valued, *i.e.* that $R \epsilon 1 \rightarrow \text{Cls}$. But the above conditions, though necessary to the *importance* of our propositions, are in general much narrower than the hypotheses that are necessary for the *truth* of our propositions.

The present section is wholly self-contained, that is to say, its propositions are not referred to in the sequel. We have, in this section, carried the subject as far as seemed suitable for the present work; its further development belongs to treatises on analysis.

We begin (*230) with a general conception which is involved in the notion of convergency. We shall say that the values of a function converge (or, simply, that the function itself converges) into the class α, if for late enough arguments the values always belong to the class α, *i.e.* if there is a term y such that, if yQ_*z, $R\text{‘}z \epsilon \alpha$, or, to avoid assuming that R is one-valued, $\overrightarrow{R}\text{‘}z \subset \alpha$. Thus the values of the function converge into the class α if

$$(\exists y) . y \epsilon C\text{‘}Q \cap \text{G‘}R . R\text{“}\overleftarrow{Q_*}\text{‘}y \subset \alpha.$$

If a term y is one such that, from y onward, all values belong to α, we write $y \epsilon R\overline{Q}_{\text{cn}}\,\alpha$ (where "cn" stands for "convergent"), *i.e.* we put

$$R\overline{Q}_{\text{cn}}\,\alpha = \hat{y}\,\{y \epsilon C\text{‘}Q \cap \text{G‘}R . R\text{“}\overleftarrow{Q_*}\text{‘}y \subset \alpha\} \quad \text{Df.}$$

When there is such a y, *i.e.* when the function converges into the class α, we write "$RQ_{\text{cn}}\,\alpha$," *i.e.* we put

$$Q_{\text{cn}} = \hat{R}\hat{\alpha}\,(\exists ! R\overline{Q}_{\text{cn}}\,\alpha) \quad \text{Df.}$$

"$RQ_{\text{cn}}\,\alpha$" may be read "R is Q-convergent into α." This means that for arguments sufficiently late in the Q-series, the value of the function is always a member of α. Thus *e.g.* if $R\text{‘}x = 1/x$, and $\alpha = \hat{y}\,(y < 1)$, $RQ_{\text{cn}}\,\alpha$, and if $z > 1$, $z \epsilon R\overline{Q}_{\text{cn}}\,\alpha$.

We next consider (*231) *limiting sections* and *ultimate oscillations* of functions. For this purpose, we proceed as follows. If $RQ_{\text{cn}}\,\alpha$, then $P_*\text{“}\alpha$ is a section of the P-series such that, for sufficiently late arguments, the values of the function must belong to $P_*\text{“}\alpha$. Hence if we take all possible values of α for which $RQ_{\text{cn}}\alpha$, and take the logical product of all the resulting sections $P_*\text{“}\alpha$, we get a section containing all the "ultimate" values of the function; moreover this is obviously the smallest section which has this property, because, if we take any section β which contains all the "ultimate" values, we have $RQ_{\text{cn}}\,\beta$, and $P_*\text{“}\beta = \beta$, and therefore the logical product in question is contained in β. The logical product in question is

$$p\text{‘}P_*\text{“‘}\overleftarrow{Q_{\text{cn}}}\text{‘}R.$$

In order to avoid trivial exceptions which arise when $C\text{‘}Q \cap \text{Ɑ}\text{‘}R = \Lambda$, we define the "limiting section" as

$$p\text{‘}P_{*}\text{“‘}\overleftarrow{Q}_{\mathrm{cn}}\text{‘}R \cap C\text{‘}P.$$

This "limiting section" we denote by $P\overline{R}_{\mathrm{sc}}Q$, where the letters "sc" stand for "section." Thus we put

$$P\overline{R}_{\mathrm{sc}}Q = p\text{‘}P_{*}\text{“‘}\overleftarrow{Q}_{\mathrm{cn}}\text{‘}R \cap C\text{‘}P \quad \text{Df.}$$

$P\overline{R}_{\mathrm{sc}}Q$ is the class of those members x of the series P which are such that, given any argument however late, there are still arguments as late or later for which the value of the function is not less than x. In like manner, $\breve{P}\overline{R}_{\mathrm{sc}}Q$, which we will call the "limiting upper section," consists of those members x of the series P which are such that, given any argument however late, there are still arguments as late or later for which the value of the function is not greater than x. Thus the product of $P\overline{R}_{\mathrm{sc}}Q$ and $\breve{P}\overline{R}_{\mathrm{sc}}Q$ is the smallest stretch which contains all the "ultimate" values of the function, *i.e.* it is the stretch consisting of those terms x which are such that, however late an argument we take, there are arguments as late or later for which the value of the function is not greater than x, and also arguments for which it is not less than x. Thus the product of $P\overline{R}_{\mathrm{sc}}Q$ and $\breve{P}\overline{R}_{\mathrm{sc}}Q$ represents what we may call the "ultimate oscillation" of the function. We shall denote it by $P\overline{R}_{\mathrm{os}}Q$, putting

$$P\overline{R}_{\mathrm{os}}Q = P\overline{R}_{\mathrm{sc}}Q \cap \breve{P}\overline{R}_{\mathrm{sc}}Q \quad \text{Df.}$$

We may express $P\overline{R}_{\mathrm{sc}}Q$ in a form not involving Q_{cn}, namely (*231·12)

$$P\overline{R}_{\mathrm{sc}}Q = p\text{‘}P_{*}\text{“‘}R\text{“‘}\overleftarrow{Q_{*}}\text{“}(C\text{‘}Q \cap \text{Ɑ}\text{‘}R) \cap C\text{‘}P.$$

This formula for $P\overline{R}_{\mathrm{sc}}Q$ may be elucidated by the following considerations. If y is any member of $C\text{‘}Q$, then $\text{Ɑ}\text{‘}R \cap \overleftarrow{Q_{*}}\text{‘}y$ consists of all arguments from y onwards. Hence $R\text{“}(\text{Ɑ}\text{‘}R \cap \overleftarrow{Q_{*}}\text{‘}y)$, *i.e.* $R\text{“}\overleftarrow{Q_{*}}\text{‘}y$, consists of all values of the function for arguments from y onwards. Hence $P_{*}\text{“}R\text{“}\overleftarrow{Q_{*}}\text{‘}y$ consists of all members of the P-series which are equalled or surpassed by values of the function for arguments equal to or later than y. Now if a term x belongs to the class $P_{*}\text{“}R\text{“}\overleftarrow{Q_{*}}\text{‘}y$ for every argument y, it is a term such that, however far up the argument-series Q we go, we shall still find values as great as or greater than x. When this is the case, we may say that x is P-persistent. In this case, x may be regarded as not greater than the "ultimate" values of the function. Now the class of arguments concerned is $C\text{‘}Q \cap \text{Ɑ}\text{‘}R$. Hence the class of P-persistent terms is

$$p\text{‘}P_{*}\text{“‘}R\text{“‘}\overleftarrow{Q_{*}}\text{“}(C\text{‘}Q \cap \text{Ɑ}\text{‘}R),$$

where the factor $C\text{‘}P$ may be added in order to accommodate the formula to the trivial case where $C\text{‘}Q \cap \text{Ɑ}\text{‘}R = \Lambda$ (the only case in which the factor $C\text{‘}P$

makes any difference). Thus the class of P-persistent terms is the limiting section. Similarly the $\breve{P}$-persistent terms are the limiting upper section. These are the terms which are not less than the "ultimate" values of the function. Thus the product $P\overline{R}_{os}Q$ is the terms which are neither greater than all ultimate values, nor less; hence it is the class of ultimate values, which may be appropriately called the "ultimate oscillation."

It will be seen that $P\overline{R}_{os}Q$, being the product of an upper and lower section, is itself a stretch: we may call it (alternatively) the "limiting stretch." It consists of all members x of the P-series such that the function does not, however great we make the argument, become and remain less than x, nor yet become and remain greater than x. If $P\overline{R}_{os}Q$ consists of a single term, that term is the limit of the function as the argument travels up the series Q. (This is, of course, in general different from the limit of the values of the function considered simply as a class of members of $C'P$, *i.e.* it is different from $\mathrm{lt}_P{}'R``C'Q$.) If $P\overline{R}_{os}Q$ does not consist of a single term or none, we shall have two limits to consider, namely $\mathrm{limax}_P{}'P\overline{R}_{os}Q$ and $\mathrm{limin}_P{}'P\overline{R}_{os}Q$, which give the two boundaries of the ultimate values of the function. When the class $P\overline{R}_{os}Q$ is null, the function may be regarded as having a definite limit: in this case, $P\overline{R}_{sc}Q$ and $\breve{P}\overline{R}_{sc}Q$ are the two parts of an "irrational" Dedekind cut, *i.e.* a cut in which the first portion has no maximum and the second no minimum. Thus $P\overline{R}_{os}Q \in 0 \cup 1$ is the condition for a definite limit of the function as the argument grows indefinitely.

The above gives the generalization of the limit of a function when the argument may be any member of $C'Q \cap \text{Cl}'R$. In order to obtain limits for other classes of arguments, it is only necessary, as a rule, to limit the field of Q to the class of arguments in question, *i.e.* to replace Q by $Q \upharpoonright \alpha$ (cf. *232). In order, however, to avoid vexatious and trivial exceptions arising when $\alpha \in 1$, it is more convenient to replace Q by $Q_* \upharpoonright \alpha$. Thus the section of P defined by the class of arguments α is $P\overline{R}_{sc}(Q_* \upharpoonright \alpha)$. We put

$$(P\overline{R}Q)_{sc}{}'\alpha = P\overline{R}_{sc}(Q_* \upharpoonright \alpha) \quad \text{Df.}$$

This definition is useful because we very often wish to be able to exhibit the limiting section defined by α as a function of α. The section $(P\overline{R}Q)_{sc}{}'\alpha$ is such that, if x is any member of it, and y is any argument belonging to α, there is in α an argument equal to or later than y, for which the function has a value equal to or later than x. Thus x is such that the function does not ultimately become less than x as the argument increases in the class α. The limit or maximum of such terms as x is the limit or maximum of the ultimate values of the function as the argument approaches the top of α. The class of ultimate values is

$$(P\overline{R}Q)_{sc}{}'\alpha \cap (\breve{P}\overline{R}Q)_{sc}{}'\alpha, \text{ which we call } (P\overline{R}Q)_{os}{}'\alpha.$$

If the function has a definite limit as the argument increases in α, the class of ultimate values must not contain more than one term.

Our next number (*233) deals with the limit of a function for a given argument. The limit or maximum of the class of ultimate values is not necessarily the value for the limit of α. It will be found, however, that, with a suitable hypothesis, the limiting section $(P\bar{R}Q)_{sc}`\alpha$ depends only upon $Q_*``(\alpha \cap \text{Cl}`R)$, and if $\alpha \cap \text{Cl}`R$ has no maximum, it depends only upon $Q``(\alpha \cap \text{Cl}`R)$. Thus if $\alpha \cap \text{Cl}`R$ and $\beta \cap \text{Cl}`R$ both have the same limit, they define the same limiting section. Hence if $a$ is the limit of $\alpha$, the limiting section of $\alpha$ is $(P\bar{R}Q)_{sc}`\overrightarrow{Q}`a$. The upper limit of this is the upper limit of the ultimate values as the argument approaches a from below. We put

$$R\,(PQ)`a = \text{limax}_P`(P\bar{R}Q)_{sc}`\overrightarrow{Q}`a \quad \text{Df.}$$

We have thus four limits of the function as the argument approaches a, namely

$$R\,(PQ)`a,\ \ R\,(\breve{P}Q)`a,\ \ R\,(P\breve{Q})`a,\ \ R\,(\breve{P}\breve{Q})`a.$$

If R is a continuous function, these four are all equal to $R`a$; but in general they are different from each other and from $R`a$. The subject of the continuity of functions is dealt with in *234. When $R\,(PQ)`a = R\,(\breve{P}Q)`a$, each is the limit of the function for the argument a for approaches from below. It should be observed that if R is defined for a set of arguments which are dense in Q, *i.e.* if $\delta_Q`\text{Cl}`R = C`Q$, then $R\,(PQ)`a$ and $R\,(\breve{P}Q)`a$ are defined for all arguments in $C`Q$.

$*230$. ON CONVERGENTS

Summary of $*230$.

In the present number, we have to consider the notion of a function converging into a given class, or, as we may express it, the notion that the value of the function "ultimately" belongs to the given class. If R is the function in question, α the given class, and Q a series to which the arguments belong, we say that "R is Q-convergent into α" if there is an argument y such that, for all arguments from y onward (in the Q-order), the value of the function is an α. That is, R is Q-convergent into α if

$$(\exists y) \,.\, y \in C\text{‘}Q \cap \text{ᗡ}\text{‘}R \,.\, R\text{“}\overleftarrow{Q_*}\text{‘}y \subset \alpha.$$

A term y which is of this nature is said to belong to the class $R\overline{Q}_{\mathrm{cn}}\alpha$. Thus R is Q-convergent into α if the class $R\overline{Q}_{\mathrm{cn}}\alpha$ is not null. Hence we have the following pair of definitions:

$$R\overline{Q}_{\mathrm{cn}}\alpha = C\text{‘}Q \cap \text{ᗡ}\text{‘}R \cap \hat{y}\,(R\text{“}\overleftarrow{Q_*}\text{‘}y \subset \alpha) \quad \mathrm{Df},$$

$$Q_{\mathrm{cn}} = \hat{R}\hat{\alpha}\,(\exists \,!\, R\overline{Q}_{\mathrm{cn}}\alpha) \qquad \mathrm{Df}.$$

In all the cases that have any importance, R will be a one-valued function (*i.e.* a one-many relation), Q will be a series, and $C\text{‘}Q \cap \text{ᗡ}\text{‘}R$ will be a class having no maximum in Q. For, if $C\text{‘}Q \cap \text{ᗡ}\text{‘}R$ has a maximum in Q, then the classes into which R converges are simply those to which the value for this maximum belongs. The following propositions, though only *important* under the above circumstances, are in general *true* under much wider hypotheses.

It is possible to generalize still further the notion of convergence, so as to apply to any property which belongs to R when confined to sufficiently late arguments. For this purpose, we have to consider $R \upharpoonright \overleftarrow{Q_*}\text{‘}z$, where z is to be confined to terms later than or equal to some term y. If, under these circumstances, $R \upharpoonright \overleftarrow{Q_*}\text{‘}z$ always belongs to the class λ, we may say that R ultimately becomes a λ. We may put

$$R\overline{Q}_{\mathrm{cng}}\lambda = \hat{y}\,\{y \in C\text{‘}Q \cap \text{ᗡ}\text{‘}R : yQ_*z \,.\, \supset_z .\, R \upharpoonright \overleftarrow{Q_*}\text{‘}z \in \lambda\} \quad \mathrm{Df},$$

$$Q_{\mathrm{cng}} = \hat{R}\hat{\lambda}\,(\exists \,!\, R\overline{Q}_{\mathrm{cng}}\lambda) \qquad \mathrm{Df}.$$

This is the general conception of which Q_{cn} is a particular case; in fact,

$$\vdash : RQ_{\mathrm{cn}}\alpha \,.\equiv .\, RQ_{\mathrm{cng}}(\breve{\mathrm{D}}\text{“}\mathrm{Cl}\text{‘}\alpha).$$

Q_{cng} will have to be used when the ultimate properties of the function with which we are concerned are not properties of its values; but when they are properties of its values, Q_{cn} enables us to deal with them more easily than Q_{cng}

In this number, we prove the following propositions among others:

$*230{\cdot}171$. $\vdash : y \in R\overline{Q}_{\mathrm{cn}}(\overleftarrow{P_*}\text{‘}x) \,.\supset .\, x \in P_*\text{“}R\text{“}\overleftarrow{Q_*}\text{‘}y$

$*230\cdot211.\ \vdash :. \alpha \subset \beta . \supset : RQ_{cn}\alpha . \supset . RQ_{cn}\beta$

$*230\cdot253.\ \vdash :. R``C`Q \subset \alpha . \supset : RQ_{cn}\alpha . \equiv . \exists ! C`Q \cap \text{Ꮖ}`R .$
$\equiv . \exists ! R``C`Q . \equiv . \dot{\exists} ! (R \upharpoonright C`Q)$

$*230\cdot4.\ \ \vdash . R\bar{Q}_{cn}\alpha = \text{Ꮖ}`R \cap \breve{Q}_*``(R\bar{Q}_{cn}\alpha)$

$*230\cdot42.\ \ \vdash :. Q_* \epsilon \text{ connex} . \supset : RQ_{cn}\alpha . RQ_{cn}\beta . \equiv . RQ_{cn}(\alpha \cap \beta)$

$*230\cdot53.\ \ \vdash :. Q \epsilon \text{ trans} \cap \text{connex} . E ! \max_Q`\text{Ꮖ}`R . \supset : RQ_{cn}\alpha . \equiv . \overrightarrow{R}`\max_Q`\text{Ꮖ}`R \subset \alpha$

In virtue of this proposition, the case when $E ! \max_Q`\text{Ꮖ}`R$ is uninteresting, and in order to obtain interesting interpretations of our propositions, it is necessary to suppose that $\text{Ꮖ}`R$ has no maximum. Similarly when, in later numbers, we consider $\text{Ꮖ}`R \cap \overrightarrow{Q}`x$, we shall only obtain interesting results when this has no maximum, which requires that $Q$ should be a compact series ($Q^2 = Q$) and $\text{Ꮖ}`R$ should be dense in Q. These assumptions are, however, not usually required for the *truth* of our propositions.

$*230\cdot01.\ \ R\bar{Q}_{cn}\alpha = C`Q \cap \text{Ꮖ}`R \cap \hat{y}(R``\overleftarrow{Q_*}`y \subset \alpha)$ Df

$*230\cdot02.\ \ Q_{cn} = \hat{R}\hat{\alpha}(\exists ! R\bar{Q}_{cn}\alpha)$ Df

$*230\cdot1.\ \ \vdash : y \epsilon R\bar{Q}_{cn}\alpha . \equiv . y \epsilon C`Q \cap \text{Ꮖ}`R . R``\overleftarrow{Q_*}`y \subset \alpha$ [(*230·01)]

$*230\cdot11.\ \ \vdash : RQ_{cn}\alpha . \equiv . \exists ! R\bar{Q}_{cn}\alpha . \equiv . (\exists y) . y \epsilon C`Q \cap \text{Ꮖ}`R . R``\overleftarrow{Q_*}`y \subset \alpha$
[(*230·02)]

$*230\cdot12.\ \ \vdash : y \epsilon R\bar{Q}_{cn}\alpha . \supset . \overleftarrow{Q_*}`y \cap \text{Ꮖ}`R \subset R\bar{Q}_{cn}\alpha$

Dem.

$\vdash . *230\cdot1 . *201\cdot14\cdot15 . \supset$

$\vdash : y \epsilon R\bar{Q}_{cn}\alpha . yQ_* z . z \epsilon \text{Ꮖ}`R . \supset . R``\overleftarrow{Q_*}`y \subset \alpha . z \epsilon C`Q \cap \text{Ꮖ}`R . \overleftarrow{Q_*}`z \subset \overleftarrow{Q_*}`y .$

[*37·2] $\supset . z \epsilon C`Q \cap \text{Ꮖ}`R . R``\overleftarrow{Q_*}`z \subset \alpha .$

[*230·1] $\supset . z \epsilon R\bar{Q}_{cn}\alpha : \supset \vdash . \text{Prop}$

$*230\cdot13.\ \ \vdash . R\bar{Q}_{cn}\alpha = (R \upharpoonright C`Q)\bar{Q}_{cn}\alpha$

Dem.

$\vdash . *35\cdot64 . \supset \vdash . C`Q \cap \text{Ꮖ}`R = C`Q \cap \text{Ꮖ}`(R \upharpoonright C`Q)$ (1)

$\vdash . *37\cdot421 . \supset \vdash . R``\overleftarrow{Q_*}`y = (R \upharpoonright C`Q)``\overleftarrow{Q_*}`y$ (2)

$\vdash . (1) . (2) . *230\cdot1 . \supset \vdash . \text{Prop}$

$*230\cdot131.\ \ \vdash : R \upharpoonright C`Q = T \upharpoonright C`Q . \supset . R\bar{Q}_{cn}\alpha = T\bar{Q}_{cn}\alpha$ [*230·13]

$*230\cdot14.\ \ \vdash : y \epsilon R\bar{Q}_{cn}\alpha . \supset . \exists ! C`Q \cap \text{Ꮖ}`R . \exists ! \alpha \cap D`R$

Dem.

$\vdash . *230\cdot1 . \supset \vdash : \text{Hp} . \supset . y \epsilon C`Q \cap \text{Ꮖ}`R . \overrightarrow{R}`y \subset \alpha .$

[*33·41] $\supset . y \epsilon C`Q \cap \text{Ꮖ}`R . \exists ! \overrightarrow{R}`y . \overrightarrow{R}`y \subset \alpha .$

[*22·621.*33·15] $\supset . \exists ! C`Q \cap \text{Ꮖ}`R . \exists ! \alpha \cap D`R : \supset \vdash . \text{Prop}$

$*230\cdot141.\ \vdash . R\bar{Q}_{\text{cn}}\Lambda = \Lambda$ [$*230\cdot14$. Transp]

$*230\cdot142.\ \vdash :. R = \dot{\Lambda} . \vee . Q = \dot{\Lambda} : \supset . R\bar{Q}_{\text{cn}}\alpha = \Lambda$ [$*230\cdot14$. Transp . $*33\cdot24$]

$*230\cdot15.\ \vdash : RQ_{\text{cn}}\alpha . \supset . \exists ! C`Q \cap \text{Ɑ}`R . \exists ! \alpha \cap \text{D}`R$ [$*230\cdot14\cdot11$]

$*230\cdot151.\ \vdash : RQ_{\text{cn}}\alpha . \supset . \dot{\exists} ! R . \dot{\exists} ! Q . \exists ! \alpha$ [$*230\cdot15$]

$*230\cdot152.\ \vdash :. R = \dot{\Lambda} . \vee . Q = \dot{\Lambda} . \vee . \alpha = \Lambda : \supset . \sim(RQ_{\text{cn}}\alpha)$ [$*230\cdot151$. Transp]

$*230\cdot16.\ \vdash . R\bar{Q}_{\text{cn}}\alpha = R\,(\overline{Q_* \upharpoonright \text{Ɑ}`R})_{\text{cn}}\,\alpha$

Dem.

$\vdash . *230\cdot14 . \supset \vdash : C`Q \cap \text{Ɑ}`R = \Lambda . \supset . R\bar{Q}_{\text{cn}}\alpha = \Lambda . R\,(\overline{Q_* \upharpoonright \text{Ɑ}`R})_{\text{cn}}\,\alpha = \Lambda$ (1)

$\vdash . *90\cdot41 . \supset \vdash : \exists ! C`Q \cap \text{Ɑ}`R . \supset . C`(Q_* \upharpoonright \text{Ɑ}`R) = C`Q \cap \text{Ɑ}`R$ (2)

$\vdash . *37\cdot26 . \supset \vdash . R``\overleftarrow{Q_*}`y = R``(\overleftarrow{Q_*}`y \cap \text{Ɑ}`R)$ (3)

$\vdash . (3) . *35\cdot102 . \supset \vdash : y \in \text{Ɑ}`R . \supset . R``\overleftarrow{Q_*}`y = R``\overleftarrow{Q_* \upharpoonright \text{Ɑ}`R}`y$ (4)

$\vdash . (2) . (4) . *230\cdot1 . \supset$

$\vdash :. \exists ! C`Q \cap \text{Ɑ}`R . \supset : y \in R\bar{Q}_{\text{cn}}\alpha . \equiv . y \in C`(Q_* \upharpoonright \text{Ɑ}`R) \cap \text{Ɑ}`R . R``\overleftarrow{Q_* \upharpoonright \text{Ɑ}`R}`y \subset \alpha .$

[$*230\cdot1$] $\equiv . y \in R\,(\overline{Q_* \upharpoonright \text{Ɑ}`R})_{\text{cn}}\,\alpha$ (5)

$\vdash . (1) . (5) . \supset \vdash$. Prop

$*230\cdot161.\ \vdash : Q_* \upharpoonright \text{Ɑ}`R = S_* \upharpoonright \text{Ɑ}`R . \supset . R\bar{Q}_{\text{cn}}\alpha = R\bar{S}_{\text{cn}}\alpha$ [$*230\cdot16$]

$*230\cdot17.\ \vdash : y \in C`Q \cap \text{Ɑ}`R . \supset . \exists ! R``\overleftarrow{Q_*}`y$

Dem.

$\vdash . *90\cdot12 . *33\cdot41 . \supset \vdash : \text{Hp} . \supset . y \in \overleftarrow{Q_*}`y . \exists ! \overrightarrow{R}`y .$

[$*37\cdot18$] $\supset . \exists ! R``\overleftarrow{Q_*}`y : \supset \vdash$. Prop

$*230\cdot171.\ \vdash : y \in R\bar{Q}_{\text{cn}}(\overleftarrow{P_*}`x) . \supset . x \in P_*``R``\overleftarrow{Q_*}`y$

Dem.

$\vdash . *230\cdot1\cdot17 . \supset \vdash : \text{Hp} . \supset . \exists ! R``\overleftarrow{Q_*}`y . R``\overleftarrow{Q_*}`y \subset \overleftarrow{P_*}`x .$

[$*22\cdot621$] $\supset . \exists ! R``\overleftarrow{Q_*}`y \cap \overleftarrow{P_*}`x .$

[$*37\cdot46$] $\supset . x \in P_*``R``\overleftarrow{Q_*}`y : \supset \vdash$. Prop

$*230\cdot21.\ \vdash : \alpha \subset \beta . \supset . R\bar{Q}_{\text{cn}}\alpha \subset R\bar{Q}_{\text{cn}}\beta$ [$*230\cdot1 . *22\cdot44$]

$*230\cdot211.\ \vdash :. \alpha \subset \beta . \supset : RQ_{\text{cn}}\alpha . \supset . RQ_{\text{cn}}\beta$ [$*230\cdot21\cdot11$]

$*230\cdot22.\ \vdash . R\bar{Q}_{\text{cn}}\alpha \cup R\bar{Q}_{\text{cn}}\beta \subset R\bar{Q}_{\text{cn}}(\alpha \cup \beta)$ [$*230\cdot21$]

$*230\cdot221.\ \vdash :. RQ_{\text{cn}}\alpha . \vee . RQ_{\text{cn}}\beta : \supset . RQ_{\text{cn}}(\alpha \cup \beta)$ [$*230\cdot211$]

$*230\cdot23.\ \vdash . R\bar{Q}_{\text{cn}}\alpha \cap R\bar{Q}_{\text{cn}}\beta = R\bar{Q}_{\text{cn}}(\alpha \cap \beta)$

Dem.

$\vdash . *230\cdot1 . \supset \vdash : y \in R\bar{Q}_{\text{cn}}\alpha \cap R\bar{Q}_{\text{cn}}\beta . \equiv . y \in C`Q \cap \text{Ɑ}`R . R``\overleftarrow{Q_*}`y \subset \alpha . R``\overleftarrow{Q_*}`y \subset \beta .$

[Comp. $*230\cdot1$] $\equiv . y \in R\bar{Q}_{\text{cn}}(\alpha \cap \beta) : \supset \vdash$. Prop

$*230\cdot231.\ \vdash : RQ_{\text{cn}}(\alpha \cap \beta) . \supset . RQ_{\text{cn}}\alpha . RQ_{\text{cn}}\beta$ [$*230\cdot211$]

***230·24.** $\vdash . R\overline{Q}_{cn}\alpha \cap R\overline{Q}_{cn}(\beta - \alpha) = \Lambda$ [*230·23·141]

***230·25.** $\vdash . R\overline{Q}_{cn}\alpha = R\overline{Q}_{cn}(\alpha \cap \mathrm{D}`R) = R\overline{Q}_{cn}(\alpha \cap R``\breve{Q}_*``\text{Ɑ}`R) = R\overline{Q}_{cn}(\alpha \cap R``C`Q)$

Dem.

$\vdash . *37{\cdot}15 . \supset \vdash : R``\overleftarrow{Q_*}`y \subset \alpha . \equiv . R``\overleftarrow{Q_*}`y \subset \alpha \cap \mathrm{D}`R$ (1)

$\vdash . *37{\cdot}18 . \supset \vdash :. y \in \text{Ɑ}`R . \supset : R``\overleftarrow{Q_*}`y \subset R``\breve{Q}_*``\text{Ɑ}`R :$

[Comp] $\supset : R``\overleftarrow{Q_*}`y \subset \alpha . \equiv . R``\overleftarrow{Q_*}`y \subset \alpha \cap R``\breve{Q}_*``\text{Ɑ}`R$ (2)

$\vdash . *37{\cdot}2{\cdot}18 . \supset \vdash . R``\overleftarrow{Q_*}`y \subset R``C`Q .$

[Comp] $\supset \vdash : R``\overleftarrow{Q_*}`y \subset \alpha . \equiv . R``\overleftarrow{Q_*}`y \subset \alpha \cap R``C`Q$ (3)

$\vdash . (1) . (2) . (3) . *230{\cdot}1 . \supset \vdash .$ Prop

***230·251.** $\vdash . R\overline{Q}_{cn}(R``C`Q) = C`Q \cap \text{Ɑ}`R$

Dem.

$\vdash . *33{\cdot}15 . *37{\cdot}2 . \supset \vdash . (y) . R``\overleftarrow{Q_*}`y \subset R``C`Q$ (1)

$\vdash . (1) . *230{\cdot}1 . \supset \vdash .$ Prop

***230·252.** $\vdash : R``C`Q \subset \alpha . \supset . R\overline{Q}_{cn}\alpha = C`Q \cap \text{Ɑ}`R$ [*230·25·251]

***230·253.** $\vdash :. R``C`Q \subset \alpha . \supset : RQ_{cn}\alpha . \equiv . \exists ! C`Q \cap \text{Ɑ}`R . \equiv .$
$\exists ! R``C`Q . \equiv . \dot{\exists} ! (R \upharpoonright C`Q)$ [*230·11·252 . *37·401 . *35·64]

***230·31.** $\vdash . s`R\overline{Q}_{cn}``\kappa \subset R\overline{Q}_{cn}(s`\kappa)$

Dem.

$\vdash . *230{\cdot}21 . \supset \vdash : \alpha \in \kappa . \supset . R\overline{Q}_{cn}\alpha \subset R\overline{Q}_{cn}(s`\kappa) : \supset \vdash .$ Prop

***230·311.** $\vdash . Q_{cn}``\kappa \subset \overrightarrow{Q_{cn}}`s`\kappa$

Dem.

$\vdash . *230{\cdot}211 . \supset \vdash : \alpha \in \kappa . RQ_{cn}\alpha . \supset . RQ_{cn}(s`\kappa) : \supset \vdash .$ Prop

***230·32.** $\vdash . R\overline{Q}_{cn}(p`\kappa) = C`Q \cap \text{Ɑ}`R \cap p`R\overline{Q}_{cn}``\kappa$

Dem.

$\vdash . *230{\cdot}1 . \supset$

$\vdash :. y \in R\overline{Q}_{cn}(p`\kappa) . \equiv : y \in C`Q \cap \text{Ɑ}`R . R``\overleftarrow{Q_*}`y \subset p`\kappa :$

[*40·15] $\equiv : y \in C`Q \cap \text{Ɑ}`R : \alpha \in \kappa . \supset_\alpha . R``\overleftarrow{Q_*}`y \subset \alpha :$

[*4·73] $\equiv : y \in C`Q \cap \text{Ɑ}`R : \alpha \in \kappa . \supset_\alpha . y \in C`Q \cap \text{Ɑ}`R . R``\overleftarrow{Q_*}`y \subset \alpha :$

[*230·1] $\equiv : y \in C`Q \cap \text{Ɑ}`R \cap p`R\overline{Q}_{cn}``\kappa :. \supset \vdash .$ Prop

***230·321.** $\vdash : \exists ! \kappa . \supset . R\overline{Q}_{cn}(p`\kappa) = p`R\overline{Q}_{cn}``\kappa . p`R\overline{Q}_{cn}``\kappa \subset C`Q \cap \text{Ɑ}`R$

Dem.

$\vdash . *230{\cdot}1 . \quad \supset \vdash : \alpha \in \kappa . \supset_\alpha . R\overline{Q}_{cn}\alpha \subset C`Q \cap \text{Ɑ}`R$ (1)

$\vdash . (1) . *40{\cdot}23{\cdot}151 . \supset \vdash : \mathrm{Hp} . \supset . p`R\overline{Q}_{cn}``\kappa \subset C`Q \cap \text{Ɑ}`R$ (2)

$\vdash . (2) . *230{\cdot}32 . \supset \vdash .$ Prop

***230·4.** $\vdash . R\bar{Q}_{\mathrm{cn}}\alpha = \text{Ɑ}`R \cap \breve{Q}_*``(R\bar{Q}_{\mathrm{cn}}\alpha)$

Dem.

$\vdash . *230·11 . *90·21 . \supset \vdash . R\bar{Q}_{\mathrm{cn}}\alpha \subset \text{Ɑ}`R . R\bar{Q}_{\mathrm{cn}}\alpha \subset \breve{Q}_*``(R\bar{Q}_{\mathrm{cn}}\alpha)$ (1)

$\vdash . *201·14·15 . \quad \supset \vdash : R``\overleftarrow{Q_*}`y \subset \alpha . yQ_*z . \supset . R``\overleftarrow{Q_*}`z \subset \alpha$ (2)

$\vdash . (2) . *230·1 . \quad \supset \vdash : y \in R\bar{Q}_{\mathrm{cn}}\alpha . yQ_*z . z \in \text{Ɑ}`R . \supset . z \in R\bar{Q}_{\mathrm{cn}}\alpha :$

$[*37·105] \quad \supset \vdash . \text{Ɑ}`R \cap \breve{Q}_*``(R\bar{Q}_{\mathrm{cn}}\alpha) \subset R\bar{Q}_{\mathrm{cn}}\alpha$ (3)

$\vdash . (1) . (3) . \supset \vdash .$ Prop

***230·41.** $\vdash :. Q_* \in \mathrm{connex} . \supset : R\bar{Q}_{\mathrm{cn}}\alpha \subset R\bar{Q}_{\mathrm{cn}}\beta . \mathrm{v} . R\bar{Q}_{\mathrm{cn}}\beta \subset R\bar{Q}_{\mathrm{cn}}\alpha$

Dem.

$\vdash . *211·61 . *201·15 . \supset$

$\vdash :. \mathrm{Hp} . \supset : \breve{Q}_*``(R\bar{Q}_{\mathrm{cn}}\alpha) \subset \breve{Q}_*``(R\bar{Q}_{\mathrm{cn}}\beta) . \mathrm{v} . \breve{Q}_*``(R\bar{Q}_{\mathrm{cn}}\beta) \subset \breve{Q}_*``(R\bar{Q}_{\mathrm{cn}}\alpha) :$

$[\mathrm{Fact}.*230·4] \supset : R\bar{Q}_{\mathrm{cn}}\alpha \subset R\bar{Q}_{\mathrm{cn}}\beta . \mathrm{v} . R\bar{Q}_{\mathrm{cn}}\beta \subset R\bar{Q}_{\mathrm{cn}}\alpha :. \supset \vdash .$ Prop

***230·42.** $\vdash :. Q_* \in \mathrm{connex} . \supset : RQ_{\mathrm{cn}}\alpha . RQ_{\mathrm{cn}}\beta . \equiv . RQ_{\mathrm{cn}}(\alpha \cap \beta)$

Dem.

$\vdash . *230·41 . \supset \vdash :. \mathrm{Hp} . \supset : R\bar{Q}_{\mathrm{cn}}\alpha \cap R\bar{Q}_{\mathrm{cn}}\beta = R\bar{Q}_{\mathrm{cn}}\alpha . \mathrm{v} . R\bar{Q}_{\mathrm{cn}}\alpha \cap R\bar{Q}_{\mathrm{cn}}\beta = R\bar{Q}_{\mathrm{cn}}\beta :$

$[*230·23] \quad \supset : R\bar{Q}_{\mathrm{cn}}(\alpha \cap \beta) = R\bar{Q}_{\mathrm{cn}}\alpha . \mathrm{v} . R\bar{Q}_{\mathrm{cn}}(\alpha \cap \beta) = R\bar{Q}_{\mathrm{cn}}\beta :$

$[*230·11] \quad \supset : RQ_{\mathrm{cn}}\alpha . RQ_{\mathrm{cn}}\beta . \supset . RQ_{\mathrm{cn}}(\alpha \cap \beta)$ (1)

$\vdash . (1) . *230·231 . \supset \vdash .$ Prop

***230·421.** $\vdash : Q_* \in \mathrm{connex} . \alpha \cap \beta = \Lambda . \supset . \sim\{RQ_{\mathrm{cn}}\alpha . RQ_{\mathrm{cn}}\beta\}$ [*230·42·141]

***230·51.** $\vdash : RQ_{\mathrm{cn}}\alpha . \supset . p`\overleftarrow{Q_*}``C`Q \cap \text{Ɑ}`R \subset R\bar{Q}_{\mathrm{cn}}\alpha$

Dem.

$\vdash . *201·14 . \supset \vdash : y \in C`Q \cap \text{Ɑ}`R . R``\overleftarrow{Q_*}`y \subset \alpha . z \in p`\overleftarrow{Q_*}``C`Q . \supset . R``\overleftarrow{Q_*}`z \subset \alpha$ (1)

$\vdash . *230·151 . *40·62 . \supset \vdash : \mathrm{Hp} . \supset . p`\overleftarrow{Q_*}``C`Q \subset C`Q$ (2)

$\vdash . (1) . (2) . *230·1 . \quad \supset \vdash : \mathrm{Hp} . y \in R\bar{Q}_{\mathrm{cn}}\alpha . z \in p`\overleftarrow{Q_*}``C`Q \cap \text{Ɑ}`R . \supset .$

$z \in C`Q \cap \text{Ɑ}`R . R``\overleftarrow{Q_*}`z \subset \alpha$ (3)

$\vdash . (3) . *230·1·11 . \supset \vdash .$ Prop

***230·511.** $\vdash : y \in p`\overleftarrow{Q_*}``C`Q . \supset . \overleftarrow{Q_*}`y = p`\overleftarrow{Q_*}``C`Q$

Dem.

$\vdash . *40·12 . \supset \vdash : \mathrm{Hp} . \supset . p`\overleftarrow{Q_*}``C`Q \subset \overleftarrow{Q_*}`y$ (1)

$\vdash . *40·53 . \supset \vdash :. \mathrm{Hp} . z \in \overleftarrow{Q_*}`y . \supset : x \in C`Q . \supset_x . xQ_*y : yQ_*z :$

$[*201·15] \quad \supset : x \in C`Q . \supset_x . xQ_*z :$

$[*40·53] \quad \supset : z \in p`\overleftarrow{Q_*}``C`Q$ (2)

$\vdash . (1) . (2) . \supset \vdash .$ Prop

$$*230{\cdot}512.\ \vdash : \text{Ɑ}‘R \cap p‘\overleftarrow{Q_*}‘‘C‘Q \subset R\overline{Q}_{\text{cn}}\alpha \,.\supset .\, R‘‘p‘\overleftarrow{Q_*}‘‘C‘Q \subset \alpha$$

Dem.

$$\vdash . *230{\cdot}1 \,.\supset \vdash :.\ \text{Hp} \,.\supset : y \in \text{Ɑ}‘R \cap p‘\overleftarrow{Q_*}‘‘C‘Q \,.\supset .\, R‘‘\overleftarrow{Q_*}‘y \subset \alpha \,.$$

$$[*230{\cdot}511] \qquad \supset .\, R‘‘p‘\overleftarrow{Q_*}‘‘C‘Q \subset \alpha \qquad (1)$$

$$\vdash . (1) . *10{\cdot}23 \,.\supset \vdash : \text{Hp} \,.\, \exists ! \,\text{Ɑ}‘R \cap p‘\overleftarrow{Q_*}‘‘C‘Q \,.\supset .\, R‘‘p‘\overleftarrow{Q_*}‘‘C‘Q \subset \alpha \qquad (2)$$

$$\vdash . *37{\cdot}26{\cdot}29 \,.\ \supset \vdash : \text{Ɑ}‘R \cap p‘\overleftarrow{Q_*}‘‘C‘Q = \Lambda \,.\supset .\, R‘‘p‘\overleftarrow{Q_*}‘‘C‘Q = \Lambda \,.$$

$$[*24{\cdot}12] \qquad \supset .\, R‘‘p‘\overleftarrow{Q_*}‘‘C‘Q \subset \alpha \qquad (3)$$

$$\vdash . (2) . (3) \,.\supset \vdash .\, \text{Prop}$$

$$*230{\cdot}513.\ \vdash :.\ \dot{\exists} ! \, Q \,.\supset : R‘‘p‘\overleftarrow{Q_*}‘‘C‘Q \subset \alpha \,.\equiv .\, \text{Ɑ}‘R \cap p‘\overleftarrow{Q_*}‘‘C‘Q \subset R\overline{Q}_{\text{cn}}\alpha$$

Dem.

$$\vdash . *230{\cdot}511 \,.\supset \vdash :.\ y \in \text{Ɑ}‘R \cap p‘\overleftarrow{Q_*}‘‘C‘Q \,.\supset :$$

$$R‘‘p‘\overleftarrow{Q_*}‘‘C‘Q \subset \alpha \,.\supset .\, y \in \text{Ɑ}‘R \,.\, R‘‘\overleftarrow{Q_*}‘y \subset \alpha \qquad (1)$$

$$\vdash . (1) . *40{\cdot}62 . *230{\cdot}1 \,.\supset$$

$$\vdash : \dot{\exists} ! \, Q \,.\, y \in \text{Ɑ}‘R \cap p‘\overleftarrow{Q_*}‘‘C‘Q \,.\, R‘‘p‘\overleftarrow{Q_*}‘‘C‘Q \subset \alpha \,.\supset .\, y \in R\overline{Q}_{\text{cn}}\alpha \qquad (2)$$

$$\vdash . (2) . \text{Comm} \,.\supset \vdash :.\ \dot{\exists} ! \, Q \,.\supset :$$

$$R‘‘p‘\overleftarrow{Q_*}‘‘C‘Q \subset \alpha \,.\supset .\, \text{Ɑ}‘R \cap p‘\overleftarrow{Q_*}‘‘C‘Q \subset R\overline{Q}_{\text{cn}}\alpha \qquad (3)$$

$$\vdash . (3) . *230{\cdot}512 \,.\supset \vdash .\, \text{Prop}$$

$$*230{\cdot}514.\ \vdash : \dot{\exists} ! \, Q \,.\, \exists ! \, p‘\overleftarrow{Q_*}‘‘C‘Q \cap \text{Ɑ}‘R \,.\, R‘‘p‘\overleftarrow{Q_*}‘‘C‘Q \subset \alpha \,.\supset .\, RQ_{\text{cn}}\alpha$$

Dem.

$$\vdash . *230{\cdot}513 \,.\supset \vdash : \text{Hp} \,.\supset .\, \exists ! \, p‘\overleftarrow{Q_*}‘‘C‘Q \cap \text{Ɑ}‘R \,.\, p‘\overleftarrow{Q_*}‘‘C‘Q \cap \text{Ɑ}‘R \subset R\overline{Q}_{\text{cn}}\alpha \,.$$

$$[*24{\cdot}58 . *230{\cdot}11] \qquad \supset .\, RQ_{\text{cn}}\alpha : \supset \vdash .\, \text{Prop}$$

$$*230{\cdot}52.\ \vdash : \exists ! \, C‘Q \cap \text{Ɑ}‘R \,.\, \exists ! \,\text{Ɑ}‘R \cap p‘\overleftarrow{Q_*}‘‘s‘R\overline{Q}_{\text{cn}}‘‘\kappa \,.\, \kappa \subset \overleftarrow{Q_{\text{cn}}}‘R \,.\supset .\, p‘\kappa \in \overleftarrow{Q_{\text{cn}}}‘R$$

Dem.

$$\vdash . *40{\cdot}16 \,.\supset \vdash : \alpha \in \kappa \,.\supset .\, p‘\overleftarrow{Q_*}‘‘s‘R\overline{Q}_{\text{cn}}‘‘\kappa \subset p‘\overleftarrow{Q_*}‘‘R\overline{Q}_{\text{cn}}\alpha \qquad (1)$$

$$\vdash . (1) . *40{\cdot}61 \,.\supset$$

$$\vdash :.\ \text{Hp} \,.\supset : \alpha \in \kappa \,.\supset .\, p‘\overleftarrow{Q_*}‘‘s‘R\overline{Q}_{\text{cn}}‘‘\kappa \subset \check{Q}_*‘‘R\overline{Q}_{\text{cn}}\alpha \,.$$

$$[\text{Fact} . *230{\cdot}4] \qquad \supset .\, \text{Ɑ}‘R \cap p‘\overleftarrow{Q_*}‘‘s‘R\overline{Q}_{\text{cn}}‘‘\kappa \subset R\overline{Q}_{\text{cn}}\alpha :$$

$$[*40{\cdot}44] \qquad \supset : \text{Ɑ}‘R \cap p‘\overleftarrow{Q_*}‘‘s‘R\overline{Q}_{\text{cn}}‘‘\kappa \subset p‘R\overline{Q}_{\text{cn}}‘‘\kappa :$$

$$[*230{\cdot}321] \qquad \supset : \exists ! \, \kappa \,.\supset .\, \text{Ɑ}‘R \cap p‘\overleftarrow{Q_*}‘‘s‘R\overline{Q}_{\text{cn}}‘‘\kappa \subset R\overline{Q}_{\text{cn}}(p‘\kappa) \,.$$

$$[\text{Hp} . *24{\cdot}58] \qquad \supset .\, \exists ! \, R\overline{Q}_{\text{cn}}(p‘\kappa) \,.$$

$$[*230{\cdot}11] \qquad \supset .\, p‘\kappa \in \overleftarrow{Q_{\text{cn}}}‘R \qquad (2)$$

$$\vdash . *230{\cdot}253 . *40{\cdot}2 \,.\supset \vdash : \exists ! \, C‘Q \cap \text{Ɑ}‘R \,.\, \kappa = \Lambda \,.\supset .\, p‘\kappa \in \overleftarrow{Q_{\text{cn}}}‘R \qquad (3)$$

$$\vdash . (2) . (3) \,.\supset \vdash .\, \text{Prop}$$

$*230 \cdot 53$. $\vdash :. Q \in \text{trans} \cap \text{connex} . E ! \max_Q{}^{\backprime}\text{Ɑ}{}^{\backprime}R . \supset : R Q_{\text{cn}} \alpha . \equiv . \overrightarrow{R}{}^{\backprime}\max_Q{}^{\backprime}\text{Ɑ}{}^{\backprime}R \subset \alpha$

Dem.

$\vdash . *205 \cdot 111 . \quad \supset \vdash : \text{Hp} . \supset . \max_Q{}^{\backprime}\text{Ɑ}{}^{\backprime}R \in C{}^{\backprime}Q \cap \text{Ɑ}{}^{\backprime}R \qquad (1)$

$\vdash . *205 \cdot 141 . *201 \cdot 18 . \supset \vdash : \text{Hp} . \supset . \overleftarrow{Q_*}{}^{\backprime}\max_Q{}^{\backprime}\text{Ɑ}{}^{\backprime}R \cap \text{Ɑ}{}^{\backprime}R = \iota{}^{\backprime}\max_Q{}^{\backprime}\text{Ɑ}{}^{\backprime}R .$

$[*37 \cdot 26 . *53 \cdot 301] \quad \supset . R{}^{\backprime\backprime}(\overleftarrow{Q_*}{}^{\backprime}\max_Q{}^{\backprime}\text{Ɑ}{}^{\backprime}R) = \overrightarrow{R}{}^{\backprime}\max_Q{}^{\backprime}\text{Ɑ}{}^{\backprime}R \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash :. \text{Hp} . \supset : \overrightarrow{R}{}^{\backprime}\max_Q{}^{\backprime}\text{Ɑ}{}^{\backprime}R \subset \alpha . \supset . \max_Q{}^{\backprime}\text{Ɑ}{}^{\backprime}R \in (R\overline{Q}_{\text{cn}}\alpha) .$

$[*230 \cdot 11] \quad \supset . R Q_{\text{cn}} \alpha \qquad (3)$

$\vdash . *205 \cdot 36 . \supset \vdash :. \text{Hp} . \supset : y \in C{}^{\backprime}Q \cap \text{Ɑ}{}^{\backprime}R . R{}^{\backprime\backprime}\overleftarrow{Q_*}{}^{\backprime}y \subset \alpha . \supset . R{}^{\backprime\backprime}\overleftarrow{Q_*}{}^{\backprime}\max_Q{}^{\backprime}\text{Ɑ}{}^{\backprime}R \subset \alpha :$

$[*230 \cdot 11] \quad \supset : R Q_{\text{cn}} \alpha . \supset . R{}^{\backprime\backprime}\overleftarrow{Q_*}{}^{\backprime}\max_Q{}^{\backprime}\text{Ɑ}{}^{\backprime}R \subset \alpha .$

$[(2)] \quad \supset . \overrightarrow{R}{}^{\backprime}\max_Q{}^{\backprime}\text{Ɑ}{}^{\backprime}R \subset \alpha \qquad (4)$

$\vdash . (3) . (4) . \supset \vdash . \text{Prop}$

$*230 \cdot 54$. $\vdash :. Q \in \text{trans} \cap \text{connex} . E ! \max_Q{}^{\backprime}\text{Ɑ}{}^{\backprime}R . \supset : \kappa \subset \overleftarrow{Q_{\text{cn}}}{}^{\backprime}R . \equiv . p{}^{\backprime}\kappa \in \overleftarrow{Q_{\text{cn}}}{}^{\backprime}R$

Dem.

$\vdash . *230 \cdot 53 . \supset \vdash :: \text{Hp} . \supset :. \kappa \subset \overleftarrow{Q_{\text{cn}}}{}^{\backprime}R . \equiv : \alpha \in \kappa . \supset_\alpha . \overrightarrow{R}{}^{\backprime}\max_Q{}^{\backprime}\text{Ɑ}{}^{\backprime}R \subset \alpha :$

$[*40 \cdot 15] \quad \equiv : \overrightarrow{R}{}^{\backprime}\max_Q{}^{\backprime}\text{Ɑ}{}^{\backprime}R \subset p{}^{\backprime}\kappa :$

$[*230 \cdot 53] \quad \equiv : p{}^{\backprime}\kappa \in \overleftarrow{Q_{\text{cn}}}{}^{\backprime}R :: \supset \vdash . \text{Prop}$

$*$231. LIMITING SECTIONS AND ULTIMATE OSCILLATION OF A FUNCTION

Summary of $*$231.

In the present number we are concerned with the limiting section defined in a series P, to which the values of a function R belong, as the arguments to the function increase in the argument-series Q. That is, we are concerned with the section consisting of those terms x of $C\text{‘}P$ which are such that, however great the argument to R becomes, there are still values at least as great as x. Such terms as x may be said to be P-persistent; x is P-persistent if the function does not ultimately become and remain less than x. The class of persistent terms is called the *limiting section.* The limiting section may be defined as follows. If α is any class into which R is Q-convergent, then the section $P_*\text{“}\alpha$ is such that the values of the function are ultimately contained in it. The product of such terms as $P_*\text{“}\alpha$ is the smallest section having this property. Hence if x be any member of this section, then ultimately (*i.e.* for arguments far enough along the Q series) the values of the function R do not persistently remain less than x in the P series. Thus the product of such terms as $P_*\text{“}\alpha$ is the limiting section, and we may therefore put

$$P\overline{R}_{sc}Q = p\text{‘}P_*\text{“‘}\overleftarrow{Q}_{cn}\text{‘}R \cap C\text{‘}P \quad \text{Df},$$

where the letters "sc" are intended to suggest "section." (The factor $C\text{‘}P$ on the right is superfluous except when $\overleftarrow{Q}_{cn}\text{‘}R = \Lambda$, *i.e.* when $C\text{‘}Q \cap \text{ᗡ}\text{‘}R = \Lambda$.)

We will call the limiting section of $\breve{P}$, *i.e.* $\breve{P}\overline{R}_{sc}Q$, the "limiting upper section." It will be seen that if x is a member of $\breve{P}\overline{R}_{sc}Q$, then the function does not ultimately become and remain, as far as some of its arguments are concerned, greater than x, that is, however great we make the argument, we still find values not greater than x. Hence if x belongs to both $P\overline{R}_{sc}Q$ and $\breve{P}\overline{R}_{sc}Q$, we find values not less than x and values not greater than x however great we make the argument. This class, $P\overline{R}_{sc}Q \cap \breve{P}\overline{R}_{sc}Q$, may therefore be regarded as the class of ultimate values of the function. We will call it the "ultimate oscillation" of the function, since, as the argument approaches ∞, the value of the function ultimately oscillates in this stretch of P, and no smaller stretch has the same property. We will denote this class by "$P\overline{R}_{os}Q$," where "os" is intended to suggest "oscillation." $P\overline{R}_{os}Q$ is a stretch in $C\text{‘}P$, because it is the product of two sections. Hence we shall also call it the "limiting stretch." When the function has a definite limit as the argument approaches ∞, the limiting stretch must not contain more than one term.

Limits of functions for arguments x in the middle of $C\text{‘}Q \cap \text{ᗡ}\text{‘}R$, which will be considered later, are derived from the limits considered in the present number by limiting the field of Q to predecessors of x.

In this number we prove the following propositions among others:

$*231{\cdot}103.\ \vdash . P\overline{R}_{os}Q = P_{po}\overline{R}_{os}Q = P_*\overline{R}_{os}Q$

$*231{\cdot}12.\ \vdash . P\overline{R}_{sc}Q = p‘P_*‘‘‘R‘‘‘\overleftarrow{Q_*}‘‘(C‘Q \cap \text{Ꭺ}‘R) \cap C‘P$

$*231{\cdot}13.\ \vdash . P\overline{R}_{sc}Q \in \text{sect}‘P$

$*231{\cdot}141.\ \vdash : Q_* \in \text{connex} . RQ_{cn}(\overleftarrow{P_*}‘x) . \supset . x \in P\overline{R}_{sc}Q$

$*231{\cdot}191.\ \vdash : P_{po} \in \text{connex} . \exists ! P\overline{R}_{os}Q . \supset .$

$$P\overline{R}_{sc}Q = P_*‘‘(P\overline{R}_{os}Q) . P‘‘(P\overline{R}_{sc}Q) = P_{po}‘‘(P\overline{R}_{os}Q)$$

$*231{\cdot}192.\ \vdash :. P_{po} \in \text{connex} . \exists ! P\overline{R}_{os}Q . \exists ! P\overline{R}_{os}Q' . \supset :$

$$P\overline{R}_{os}Q = P\overline{R}_{os}Q' . \equiv . P\overline{R}_{sc}Q = P\overline{R}_{sc}Q' . \breve{P}\overline{R}_{sc}Q = \breve{P}\overline{R}_{sc}Q'$$

$*231{\cdot}193.\ \vdash : P_{po} \in \text{Ser} . P\overline{R}_{os}Q \in 1 . \supset .$

$$P\overline{R}_{os}Q = \iota‘\max_P‘(P\overline{R}_{sc}Q) = \iota‘\min_P‘(\breve{P}\overline{R}_{sc}Q)$$

This proposition is frequently used in the present section.

In all ordinary circumstances, we shall have $C‘P = P\overline{R}_{sc}Q \cup \breve{P}\overline{R}_{sc}Q$, so that if the upper and lower limiting sections do not have more than one term in common (*i.e.* if $P\overline{R}_{os}Q \in 1$), they define a Dedekind cut in P. The following propositions are concerned with this fact:

$*231{\cdot}202.\ \vdash : P_*, Q_* \in \text{connex} . \exists ! P\overline{R}_{sc}Q . \supset . C‘P - (P\overline{R}_{sc}Q) \subset \breve{P}\overline{R}_{sc}Q$

$*231{\cdot}21.\ \vdash : P_*, Q_* \in \text{connex} . C‘Q \cap \text{Ꭺ}‘R \subset Q_*‘‘\breve{R}‘‘C‘P . \supset .$

$$C‘P = P\overline{R}_{sc}Q \cup \breve{P}\overline{R}_{sc}Q$$

$*231{\cdot}22.\ \vdash : P_*, Q_* \in \text{connex} . R‘‘C‘Q \subset C‘P . \supset . C‘P = P\overline{R}_{sc}Q \cup \breve{P}\overline{R}_{sc}Q$

Note that "$R‘‘C‘Q \subset C‘P$" is the hypothesis that for arguments belonging to $C‘Q$, the values belong to $C‘P$.

$*231{\cdot}24.\ \vdash : P_* \in \text{connex} . R‘‘C‘Q \subset C‘P . \sim\{RQ_{cn}(\overrightarrow{P_*}‘x)\} . \supset . \overrightarrow{P_*}‘x \subset P\overline{R}_{sc}Q$

$*231{\cdot}01.\quad P\overline{R}_{sc}Q = p‘P_*‘‘‘\overleftarrow{Q}_{cn}‘R \cap C‘P \quad$ Df

$*231{\cdot}02.\quad P\overline{R}_{os}Q = P\overline{R}_{sc}Q \cap \breve{P}\overline{R}_{sc}Q \quad$ Df

$*231{\cdot}1.\quad \vdash . P\overline{R}_{sc}Q = p‘P_*‘‘‘\overleftarrow{Q}_{cn}‘R \cap C‘P \quad [(*231{\cdot}01)]$

$*231{\cdot}101.\ \vdash . P\overline{R}_{os}Q = P\overline{R}_{sc}Q \cap \breve{P}\overline{R}_{sc}Q \quad [(*231{\cdot}02)]$

$*231{\cdot}102.\ \vdash . P\overline{R}_{sc}Q = P_{po}\overline{R}_{sc}Q = P_*\overline{R}_{sc}Q \quad [*231{\cdot}1 . *91{\cdot}602 . *90{\cdot}4]$

$*231{\cdot}103.\ \vdash . P\overline{R}_{os}Q = P_{po}\overline{R}_{os}Q = P_*\overline{R}_{os}Q \quad [*231{\cdot}102{\cdot}101]$

$*231{\cdot}11.\ \vdash :. x \in P\overline{R}_{sc}Q . \equiv : RQ_{cn}\alpha . \supset_\alpha . x \in P_*‘‘\alpha : x \in C‘P \quad [*231{\cdot}1]$

$*231{\cdot}111.\ \vdash :. x \in P\overline{R}_{sc}Q . \equiv : y \in C‘Q \cap \text{Ꭺ}‘R . R‘‘\overleftarrow{Q_*}‘y \subset \alpha . \supset_{y,\alpha} . x \in P_*‘‘\alpha : x \in C‘P$
$[*231{\cdot}11 . *230{\cdot}11]$

***231·112.** $\vdash :. x \in P\overline{R}_{sc}Q . \equiv : y \in C`Q \cap \text{Ɑ}`R . \supset_y . x \in P_*``R``\overleftarrow{Q_*}`y : x \in C`P$

Dem.

$\vdash . *231·111 . *22·42 . \supset$

$\vdash :. x \in P\overline{R}_{sc}Q . \supset : y \in C`Q \cap \text{Ɑ}`R . \supset_y . x \in P_*``R``\overleftarrow{Q_*}`y : x \in C`P$ (1)

$\vdash . *37·2 . \supset$

$\vdash :. y \in C`Q \cap \text{Ɑ}`R . \supset_y . x \in P_*``R``\overleftarrow{Q_*}`y : x \in C`P : \supset :$

$y \in C`Q \cap \text{Ɑ}`R . R``\overleftarrow{Q_*}`y \subset \alpha . \supset_{y,\alpha} . x \in P_*``\alpha : x \in C`P :$

[*231·111] $\supset : x \in P\overline{R}_{sc}Q$ (2)

$\vdash . (1) . (2) . \supset \vdash .$ Prop

***231·113.** $\vdash :. x \in P\overline{R}_{sc}Q . \equiv : y \in C`Q \cap \text{Ɑ}`R . \supset_y . x (P_* \mid R \mid \breve{Q}_*) y : x \in C`P$
[*231·112 . *37·3]

If R is a one-valued function (*i.e.* a one-many relation), and if we write $x \leqslant x'$ for xP_*x', and $y \leqslant y'$ for yQ_*y', we have

$$x \in P\overline{R}_{sc}Q . \equiv : y \in C`Q \cap \text{Ɑ}`R . \supset_y . (\exists y') . y \leqslant y' . x \leqslant R`y' : x \in C`P.$$

That is, x belongs to $P\overline{R}_{sc}Q$ if, for any argument y in $C`Q$, we can find an argument y', greater than or equal to y, for which the value is greater than or equal to x.

***231·12.** $\vdash . P\overline{R}_{sc}Q = p`P_*```R```\overleftarrow{Q_*}``(C`Q \cap \text{Ɑ}`R) \cap C`P$ [*231·112]

This is usually the most convenient formula for $P\overline{R}_{sc}Q$.

***231·121.** $\vdash : \exists ! C`Q \cap \text{Ɑ}`R . \supset .$

$$P\overline{R}_{sc}Q = p`P_*```\overleftarrow{Q_{cn}}`R = p`P_*```R```\overleftarrow{Q_*}``(C`Q \cap \text{Ɑ}`R)$$

Dem.

$\vdash . *230·253 . \supset \vdash : \text{Hp} . \supset . \exists ! \overleftarrow{Q_{cn}}`R .$

[*40·23.*37·47] $\supset . p`P_*```\overleftarrow{Q_{cn}}`R \subset s`P_*```\overleftarrow{Q_{cn}}`R .$

[*40·38.*37·16] $\supset . p`P_*```\overleftarrow{Q_{cn}}`R \subset C`P$ (1)

$\vdash . *40·23 . \supset \vdash : \text{Hp} . \supset . p`P_*```R```\overleftarrow{Q_*}``(C`Q \cap \text{Ɑ}`R) \subset s`P_*```R```\overleftarrow{Q_*}``(C`Q \cap \text{Ɑ}`R)$

[*40·38.*37·16] $\subset C`P$ (2)

$\vdash . (1) . (2) . *231·1·12 . \supset \vdash .$ Prop

***231·13.** $\vdash . P\overline{R}_{sc}Q \in \text{sect}`P$ [*211·631·13 . *231·12]

***231·131.** $\vdash . P\overline{R}_{sc}Q \subset C`P$ [*231·1]

***231·132.** $\vdash : \exists ! C`Q \cap \text{Ɑ}`R . \supset . P\overline{R}_{sc}Q \subset P_*``R``\breve{Q}_*``\text{Ɑ}`R$

Dem.

$\vdash . *40·23 . *231·121 . \supset \vdash : \text{Hp} . \supset . P\overline{R}_{sc}Q \subset s`P_*```R```\overleftarrow{Q_*}``(C`Q \cap \text{Ɑ}`R)$

[*40·38] $\subset P_*``R``s`\overleftarrow{Q_*}``(C`Q \cap \text{Ɑ}`R)$

[*40·52.*37·265] $\subset P_*``R``\breve{Q}_*``\text{Ɑ}`R : \supset \vdash .$ Prop

$*231\cdot133$. $\vdash : C`P \cap \text{Ꭰ}`R = \Lambda . \supset . P\overline{R}_{sc}Q = C`P$ [$*231\cdot12 . *37\cdot29 . *40\cdot2$]

$*231\cdot134$. $\vdash . P_{po}``(P\overline{R}_{sc}Q) = P``(P\overline{R}_{sc}Q)$ [$*211\cdot131 . *231\cdot13$]

$*231\cdot14$. $\vdash :: R \in 1 \rightarrow \text{Cls} . \supset :. x \in P\overline{R}_{sc}Q . \equiv :$
$y \in C`Q \cap \text{Ꭰ}`R . \supset_y . (\exists z) . yQ_*z . xP_*(R`z) : x \in C`P$ [$*71\cdot7 . *231\cdot113$]

$*231\cdot141$. $\vdash : Q_* \in \text{connex} . RQ_{cn}(\overleftarrow{P_*}`x) . \supset . x \in P\overline{R}_{sc}Q$

Dem.

$\vdash . *230\cdot4 . \supset \vdash : y \in R\overline{Q}_{cn}(\overleftarrow{P_*}`x) . z \in \text{Ꭰ}`R . yQ_*z . \supset . z \in R\overline{Q}_{cn}(\overleftarrow{P_*}`x) .$
[$*230\cdot171$] $\supset . x \in P_*``R``\overleftarrow{Q_*}`z$ (1)
$\vdash . *230\cdot171 . *96\cdot3 . \supset$
$\vdash : y \in R\overline{Q}_{cn}(\overleftarrow{P_*}`x) . zQ_*y . \supset . x \in P_*``R``\overleftarrow{Q_*}`y . \overleftarrow{Q_*}`y \subset \overleftarrow{Q_*}`z .$
[$*37\cdot2$] $\supset . x \in P_*``R``\overleftarrow{Q_*}`z$ (2)
$\vdash . (1) . (2) . \supset \vdash : \text{Hp} . y \in R\overline{Q}_{cn}(\overleftarrow{P_*}`x) . z \in C`Q \cap \text{Ꭰ}`R . \supset . x \in P_*``R``\overleftarrow{Q_*}`z$ (3)
$\vdash . (3) . *230\cdot11 . \supset \vdash :. \text{Hp} . \supset : z \in C`Q \cap \text{Ꭰ}`R . \supset_z . x \in P_*``R``\overleftarrow{Q_*}`z$ (4)
$\vdash . *230\cdot151 . \quad \supset \vdash : \text{Hp} . \supset . x \in C`P$ (5)
$\vdash . (4) . (5) . *231\cdot112 . \supset \vdash . \text{Prop}$

$*231\cdot142$. $\vdash : \alpha \in \text{sect}`P . \dot{R}Q_{cn}\alpha . \supset . P\overline{R}_{sc}Q \subset \alpha$

Dem.

$\vdash . *231\cdot1 . *40\cdot12 . \supset \vdash : \text{Hp} . \supset . P\overline{R}_{sc}Q \subset P_*``\alpha .$
[$*211\cdot13$] $\supset . P\overline{R}_{sc}Q \subset \alpha : \supset \vdash . \text{Prop}$

$*231\cdot143$. $\vdash : RQ_{cn}(\overrightarrow{P_*}`x) . \supset . P\overline{R}_{sc}Q \subset \overrightarrow{P_*}`x$ [$*231\cdot142 . *211\cdot13$]

$*231\cdot144$. $\vdash : RQ_{cn}(\overrightarrow{P}_{po}`x) . \supset . P\overline{R}_{sc}Q \subset \overrightarrow{P}_{po}`x$ [$*231\cdot142 . *211\cdot16$]

$*231\cdot15$. $\vdash : R``C`Q \subset C`P . \supset . C`P \cap p`\overrightarrow{P_*}``C`P \subset P\overline{R}_{sc}Q$

Dem.

$\vdash . *37\cdot2 . \supset \vdash :. \text{Hp} . \supset : R``\overleftarrow{Q_*}`y \subset C`P :$
[$*40\cdot16$] $\supset : p`\overrightarrow{P_*}``C`P \subset p`\overrightarrow{P_*}``R``\overleftarrow{Q_*}`y :$
[$*40\cdot23$] $\supset : y \in C`Q \cap \text{Ꭰ}`R . \supset_y . p`\overrightarrow{P_*}``C`P \subset P_*``R``\overleftarrow{Q_*}`y :$
[$*231\cdot12$] $\supset : C`P \cap p`\overrightarrow{P_*}``C`P \subset P\overline{R}_{sc}Q :. \supset \vdash . \text{Prop}$

$*231\cdot151$. $\vdash : P_* \in \text{connex} . C`P \cap p`\overrightarrow{P_*}``C`P \subset P\overline{R}_{sc}Q . \supset . \overrightarrow{B}`P \subset P\overline{R}_{sc}Q$
[$*202\cdot521$]

$*231\cdot152$. $\vdash : P_* \in \text{connex} . C`P \cap p`\overrightarrow{P_*}``C`P \subset P\overline{R}_{sc}Q . \exists ! \overrightarrow{B}`P . \supset . B`P \in P\overline{R}_{sc}Q$
[$*231\cdot151 . *202\cdot523$]

The hypothesis $C\text{‘}P \cap p\text{‘}\overrightarrow{P_*}\text{“}C\text{‘}P \subset P\overline{R}_{sc}Q$ is verified not only when $R\text{“}C\text{‘}Q \subset C\text{‘}P$, but also under certain more general hypotheses. Two such hypotheses, namely

$$C\text{‘}Q \cap \text{Ɑ}\text{‘}R \subset \breve{R}\text{“}C\text{‘}P$$

and
$$C\text{‘}Q \cap \text{Ɑ}\text{‘}R \subset Q_*\text{“}\breve{R}\text{“}C\text{‘}P,$$

are considered in the following propositions.

***231·153.** $\vdash : C\text{‘}Q \cap \text{Ɑ}\text{‘}R \subset Q_*\text{“}\breve{R}\text{“}C\text{‘}P \,.\supset.\, C\text{‘}P \cap p\text{‘}\overrightarrow{P_*}\text{“}C\text{‘}P \subset P\overline{R}_{sc}Q$

Dem.

$\vdash . *37\cdot1 . \supset \vdash :: \text{Hp} . \supset :. y \in C\text{‘}Q \cap \text{Ɑ}\text{‘}R . \supset_y : (\exists z) . z \in C\text{‘}P . z (R \mid \breve{Q}_*) y :$

[*40·51] $\qquad \supset_y : x \in p\text{‘}\overrightarrow{P_*}\text{“}C\text{‘}P . \supset_x . (\exists z) . z \in C\text{‘}P . z (R \mid \breve{Q}_*) y . x P_* z .$

[*34·1] $\qquad \supset_x . x (P_* \mid R \mid \breve{Q}_*) y \qquad (1)$

$\vdash . (1) . \text{Comm} . \supset$

$\vdash :. \text{Hp} . x \in C\text{‘}P \cap p\text{‘}\overrightarrow{P_*}\text{“}C\text{‘}P . \supset : y \in C\text{‘}Q \cap \text{Ɑ}\text{‘}R . \supset_y . x (P_* \mid R \mid \breve{Q}_*) y : x \in C\text{‘}P :$

[*231·113] $\qquad \supset : x \in P\overline{R}_{sc}Q :. \supset \vdash . \text{Prop}$

***231·154.** $\vdash : R\text{“}C\text{‘}Q \subset C\text{‘}P . \supset . C\text{‘}Q \cap \text{Ɑ}\text{‘}R \subset \breve{R}\text{“}C\text{‘}P$

Dem.

$\vdash . *37\cdot2 . \supset \vdash : \text{Hp} . \supset . \breve{R}\text{“}R\text{“}C\text{‘}Q \subset \breve{R}\text{“}C\text{‘}P .$

[*37·501] $\qquad \supset . C\text{‘}Q \cap \text{Ɑ}\text{‘}R \subset \breve{R}\text{“}C\text{‘}P : \supset \vdash . \text{Prop}$

***231·155.** $\vdash : C\text{‘}Q \cap \text{Ɑ}\text{‘}R \subset \breve{R}\text{“}C\text{‘}P . \supset . C\text{‘}Q \cap \text{Ɑ}\text{‘}R \subset Q_*\text{“}\breve{R}\text{“}C\text{‘}P$

Dem.

$\vdash . *22\cdot43\cdot45 . \supset \vdash : \text{Hp} . \supset . C\text{‘}Q \cap \text{Ɑ}\text{‘}R \subset C\text{‘}Q \cap \breve{R}\text{“}C\text{‘}P$

[*90·33] $\qquad \subset Q_*\text{“}\breve{R}\text{“}C\text{‘}P : \supset \vdash . \text{Prop}$

***231·156.** $\vdash :. C\text{‘}Q \cap \text{Ɑ}\text{‘}R \subset Q_*\text{“}\breve{R}\text{“}C\text{‘}P . \equiv : \Lambda \sim\in P_*\text{“‘}R\text{“‘}\overleftarrow{Q_*}\text{“}(C\text{‘}Q \cap \text{Ɑ}\text{‘}R) :$

$\qquad \equiv : z \in C\text{‘}Q \cap \text{Ɑ}\text{‘}R . \supset_z . \exists ! C\text{‘}P \cap R\text{“}\overleftarrow{Q_*}\text{‘}z$

Dem.

$\vdash . *37\cdot1 . \supset \vdash :. C\text{‘}Q \cap \text{Ɑ}\text{‘}R \subset Q_*\text{“}\breve{R}\text{“}C\text{‘}P . \equiv :$

$\qquad z \in C\text{‘}Q \cap \text{Ɑ}\text{‘}R . \supset_z . (\exists x) . x \in C\text{‘}P . z (Q_* \mid \breve{R}) x :$

[*37·3] $\qquad \equiv : z \in C\text{‘}Q \cap \text{Ɑ}\text{‘}R . \supset_z . (\exists x) . x \in C\text{‘}P . x \in R\text{“}\overleftarrow{Q_*}\text{‘}z :$

[*22·33] $\qquad \equiv : z \in C\text{‘}Q \cap \text{Ɑ}\text{‘}R . \supset_z . \exists ! C\text{‘}P \cap R\text{“}\overleftarrow{Q_*}\text{‘}z : \qquad (1)$

[*37·265·43] $\equiv : z \in C\text{‘}Q \cap \text{Ɑ}\text{‘}R . \supset_z . \exists ! P_*\text{“}R\text{“}\overleftarrow{Q_*}\text{‘}z \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***231·16.** $\vdash : P_* \in \text{connex} . \exists ! \overrightarrow{B}\text{‘}P . C\text{‘}Q \cap \text{Ɑ}\text{‘}R \subset Q_*\text{“}\breve{R}\text{“}C\text{‘}P . \supset .$

$\qquad \exists ! P\overline{R}_{sc}Q . B\text{‘}P \in P\overline{R}_{sc}Q$ [*231·152·153]

$*231{\cdot}161.\ \vdash : P_* \in \text{connex} \,.\, \exists ! \overrightarrow{B}{}'P \,.\, R``C'Q \subset C'P \,.\, \supset .\, \exists ! P\overline{R}_{\text{sc}}Q \,.\, B'P \in P\overline{R}_{\text{sc}}Q$
$[*231{\cdot}154{\cdot}155{\cdot}16]$

$*231{\cdot}17.\ \vdash : R``C'Q \subset C'P \,.\, \supset .\, R\overline{Q}_{\text{cn}}\alpha \subset R\overline{Q}_{\text{cn}}(P_*``\alpha)$

Dem.

$\vdash .\, *90{\cdot}13 \,.\, \supset \vdash :.\, \text{Hp} \,.\, \supset : y \in C'Q \,.\, \supset .\, R``\overleftarrow{Q_*}{}'y \subset C'P :$

$[*230{\cdot}1] \quad \supset : y \in R\overline{Q}_{\text{cn}}\alpha \,.\, \supset .\, y \in C'Q \cap \text{Ꭰ}'R \,.\, R``\overleftarrow{Q_*}{}'y \subset \alpha \cap C'P \,.$

$[*90{\cdot}33] \quad \supset .\, y \in C'Q \cap \text{Ꭰ}'R \,.\, R``\overleftarrow{Q_*}{}'y \subset P_*``\alpha \,.$

$[*230{\cdot}1] \quad \supset .\, y \in R\overline{Q}_{\text{cn}}(P_*``\alpha) :. \supset \vdash .\, \text{Prop}$

$*231{\cdot}171.\ \vdash : R``C'Q \subset C'P \,.\, RQ_{\text{cn}}\alpha \,.\, \supset .\, RQ_{\text{cn}}(P_*``\alpha) \quad [*231{\cdot}17 \,.\, *230{\cdot}11]$

$*231{\cdot}18.\ \vdash : R``C'Q \subset C'P \,.\, \supset .\, P\overline{R}_{\text{sc}}Q = p'(\text{sect}'P \cap \overleftarrow{Q_{\text{cn}}}{}'R) \cap C'P$

Dem.

$\vdash .\, *231{\cdot}11 \,.\, *211{\cdot}13 \,.\, \supset$

$\vdash :.\, x \in P\overline{R}_{\text{sc}}Q \,.\, \supset : \beta Q_{\text{cn}}R \,.\, \beta \in \text{sect}'P \,.\, \supset_\beta .\, x \in \beta : x \in C'P \qquad (1)$

$\vdash .\, *231{\cdot}171 \,.\, \supset$

$\vdash :: \text{Hp} \,.\, \supset :.\, RQ_{\text{cn}}(P_*``\alpha) \,.\, \supset_\alpha .\, x \in P_*``\alpha : \supset : RQ_{\text{cn}}\alpha \,.\, \supset_\alpha .\, x \in P_*``\alpha :.$

$[*13{\cdot}195 . *231{\cdot}11]$

$\supset :. (\exists\alpha) \,.\, \beta = P_*``\alpha \,.\, RQ_{\text{cn}}\beta \,.\, \supset_\beta .\, x \in \beta : x \in C'P : \supset : x \in P\overline{R}_{\text{sc}}Q :.$

$[*211{\cdot}13] \supset :. \beta \in \text{sect}'P \,.\, RQ_{\text{cn}}\beta \,.\, \supset_\beta .\, x \in \beta : x \in C'P : \supset : x \in P\overline{R}_{\text{sc}}Q \qquad (2)$

$\vdash .\, (1) \,.\, (2) \,.\, \supset$

$\vdash :: \text{Hp} \,.\, \supset :.\, x \in P\overline{R}_{\text{sc}}Q \,.\, \equiv : \beta \in \text{sect}'P \,.\, RQ_{\text{cn}}\beta \,.\, \supset_\beta .\, x \in \beta : x \in C'P :: \supset \vdash .\, \text{Prop}$

$*231{\cdot}181.\ \vdash : P \in \text{Ser} \,.\, R``C'Q \subset C'P \,.\, \supset .\, P\overline{R}_{\text{sc}}Q = C'P \cap p'(\overrightarrow{P}``C'P \cap \overleftarrow{Q_{\text{cn}}}{}'R)$

Dem.

$\vdash .\, *231{\cdot}18 \,.\, *211{\cdot}302 \,.\, *40{\cdot}16 \,.\, \supset$

$\vdash : \text{Hp} \,.\, \supset .\, P\overline{R}_{\text{sc}}Q \subset C'P \cap p'(\overrightarrow{P}``C'P \cap \overleftarrow{Q_{\text{cn}}}{}'R) \qquad (1)$

$\vdash .\, *40{\cdot}55 \,.\, *230{\cdot}211 \,.\, \supset \vdash :.\, \alpha \in \text{sect}'P \cap \overleftarrow{Q_{\text{cn}}}{}'R \,.\, z \in C'P \cap p'\overleftarrow{P}``\alpha \,.\, \supset :$

$\overrightarrow{P}{}'z \in \overleftarrow{Q_{\text{cn}}}{}'R :$

$[*40{\cdot}12] \quad \supset : x \in p'(\overrightarrow{P}``C'P \cap \overleftarrow{Q_{\text{cn}}}{}'R) \,.\, \supset .\, x \in \overrightarrow{P}{}'z \qquad (2)$

$\vdash .\, (2) \,.\, \text{Comm} \,.\, \supset \vdash :.\, x \in C'P \cap p'(\overrightarrow{P}``C'P \cap \overleftarrow{Q_{\text{cn}}}{}'R) \,.\, \alpha \in \text{sect}'P \cap \overleftarrow{Q_{\text{cn}}}{}'R \,.\, \supset :$

$x \in C'P : z \in C'P \cap p'\overleftarrow{P}``\alpha \,.\, \supset_z .\, x \in \overrightarrow{P}{}'z :$

$[*40{\cdot}41] \quad \supset : x \in C'P \cap p'\overrightarrow{P}``(C'P \cap p'\overleftarrow{P}``\alpha) \qquad (3)$

$\vdash .\, *211{\cdot}711 \,.\, \supset$

$\vdash : \text{Hp} \,.\, \alpha \in \text{sect}'P \,.\, \supset .\, C'P \cap p'\overrightarrow{P}``(C'P \cap p'\overleftarrow{P}``\alpha) = C'P \cap p'\overrightarrow{P}``(C'P - \alpha)$

$[*211{\cdot}7{\cdot}711] \quad = C'P - (C'P - \alpha) \qquad (4)$

$\vdash .\, (3) \,.\, (4) \,.\, \supset$

$\vdash :.\, \text{Hp} \,.\, x \in C'P \cap p'(\overrightarrow{P}``C'P \cap \overleftarrow{Q_{\text{cn}}}{}'R) \,.\, \supset : \alpha \in \text{sect}'P \cap \overleftarrow{Q_{\text{cn}}}{}'R \,.\, \supset_\alpha .\, x \in \alpha :$

$[*231{\cdot}18] \quad \supset : x \in P\overline{R}_{\text{sc}}Q \qquad (5)$

$\vdash .\, (1) \,.\, (5) \,.\, \supset \vdash .\, \text{Prop}$

***231·182.** $\vdash : P \epsilon \text{Ser} . R\text{“}C\text{‘}Q \subset C\text{‘}P . \exists ! C\text{‘}P - (P\overline{R}_{\text{sc}}Q) . \supset .$

$$P\overline{R}_{\text{sc}}Q = p\text{‘}(\overrightarrow{P}\text{“}C\text{‘}P \cap \overleftarrow{Q_{\text{cn}}}\text{‘}R) . \exists ! (\overrightarrow{P}\text{“}C\text{‘}P \cap \overleftarrow{Q_{\text{cn}}}\text{‘}R)$$

Dem.

$\vdash . *231\cdot181 . \supset$

$\vdash : \text{Hp} . \exists ! (\overrightarrow{P}\text{“}C\text{‘}P \cap \overleftarrow{Q_{\text{cn}}}\text{‘}R) . \supset . P\overline{R}_{\text{sc}}Q = p\text{‘}(\overrightarrow{P}\text{“}C\text{‘}P \cap \overleftarrow{Q_{\text{cn}}}\text{‘}R)$ (1)

$\vdash . *230\cdot11 . \supset \vdash :. \text{Hp} . \overrightarrow{P}\text{“}C\text{‘}P \cap \overleftarrow{Q_{\text{cn}}}\text{‘}R = \Lambda . \supset :$

$x \epsilon C\text{‘}P . y \epsilon C\text{‘}Q \cap \text{Ɑ}\text{‘}R . \supset_{x, y} . \sim(R\text{“}\overleftarrow{Q_*}\text{‘}y \subset \overrightarrow{P}\text{‘}x) .$

[*90·33.Hp] $\supset_{x, y} . \sim(P_*\text{“}R\text{“}\overleftarrow{Q_*}\text{‘}y \subset \overrightarrow{P}\text{‘}x) .$

[*211·56] $\supset_{x, y} . \overrightarrow{P_*}\text{‘}x \subset P_*\text{“}R\text{“}\overleftarrow{Q_*}\text{‘}y .$

[*90·13] $\supset_{x, y} . x \epsilon P_*\text{“}R\text{“}\overleftarrow{Q_*}\text{‘}y$ (2)

$\vdash . (2) . *231\cdot12 . \supset \vdash : \text{Hp} . \overrightarrow{P}\text{“}C\text{‘}P \cap \overleftarrow{Q_{\text{cn}}}\text{‘}R = \Lambda . \supset . C\text{‘}P \subset P\overline{R}_{\text{sc}}Q$ (3)

$\vdash . (3) . \text{Transp} . \supset \vdash . \text{Hp} . \supset . \exists ! \overrightarrow{P}\text{“}C\text{‘}P \cap \overleftarrow{Q_{\text{cn}}}\text{‘}R$ (4)

$\vdash . (1) . (4) . \supset \vdash . \text{Prop}$

***231·19.** $\vdash : P \epsilon \text{trans} . Q_* \epsilon \text{connex} . R\text{“}C\text{‘}Q \subset C\text{‘}P . \supset .$

$$P\overline{R}_{\text{os}}Q = p\text{‘}(\text{str}\text{‘}P \cap \overleftarrow{Q_{\text{cn}}}\text{‘}R) \cap C\text{‘}P$$

Dem.

$\vdash . *231\cdot18\cdot101 . \supset \vdash :: \text{Hp} . \supset :.$

$x \epsilon P\overline{R}_{\text{os}}Q . \equiv : \alpha \epsilon \text{sect}\text{‘}P . \beta \epsilon \text{sect}\text{‘}\breve{P} . \alpha, \beta \epsilon \overleftarrow{Q_{\text{cn}}}\text{‘}R . \supset_{\alpha, \beta} . x \epsilon \alpha \cap \beta : x \epsilon C\text{‘}P :$

[*13·191.*11·35] $\equiv : (\exists \alpha, \beta) . \alpha \epsilon \text{sect}\text{‘}P . \beta \epsilon \text{sect}\text{‘}\breve{P} . \alpha, \beta \epsilon \overleftarrow{Q_{\text{cn}}}\text{‘}R . \gamma = \alpha \cap \beta . \supset_\gamma .$

$x \epsilon \gamma : x \epsilon C\text{‘}P$ (1)

$\vdash . *230\cdot42 . \supset \vdash :. \text{Hp} . \supset : \alpha, \beta \epsilon \overleftarrow{Q_{\text{cn}}}\text{‘}R . \equiv . \alpha \cap \beta \epsilon \overleftarrow{Q_{\text{cn}}}\text{‘}R$ (2)

$\vdash . *215\cdot16 . \supset$

$\vdash :. \text{Hp} . \supset : (\exists \alpha, \beta) . \alpha \epsilon \text{sect}\text{‘}P . \beta \epsilon \text{sect}\text{‘}\breve{P} . \gamma = \alpha \cap \beta . \equiv . \gamma \epsilon \text{str}\text{‘}P$ (3)

$\vdash . (1) . (2) . (3) . \supset$

$\vdash :: \text{Hp} . \supset :. x \epsilon P\overline{R}_{\text{os}}Q . \equiv : \gamma \epsilon \text{str}\text{‘}P . RQ_{\text{cn}}\gamma . \supset_\gamma . x \epsilon \gamma :: \supset \vdash . \text{Prop}$

***231·191.** $\vdash : P_{\text{po}} \epsilon \text{connex} . \exists ! P\overline{R}_{\text{os}}Q . \supset .$

$$P\overline{R}_{\text{sc}}Q = P_*\text{“}(P\overline{R}_{\text{os}}Q) . P\text{“}(P\overline{R}_{\text{sc}}Q) = P_{\text{po}}\text{“}(P\overline{R}_{\text{os}}Q)$$

[*215·165 . *231·13·101]

***231·192.** $\vdash :. P_{\text{po}} \epsilon \text{connex} . \exists ! P\overline{R}_{\text{os}}Q . \exists ! P\overline{R}_{\text{os}}Q' . \supset :$

$$P\overline{R}_{\text{os}}Q = P\overline{R}_{\text{os}}Q' . \equiv . P\overline{R}_{\text{sc}}Q = P\overline{R}_{\text{sc}}Q' . \breve{P}\overline{R}_{\text{sc}}Q = \breve{P}\overline{R}_{\text{sc}}Q'$$

[*231·191·101]

***231·193.** $\vdash : P_{\text{po}} \epsilon \text{Ser} . P\overline{R}_{\text{os}}Q \epsilon 1 . \supset .$

$$P\overline{R}_{\text{os}}Q = \iota\text{‘}\text{max}_P\text{‘}(P\overline{R}_{\text{sc}}Q) = \iota\text{‘}\text{min}_P\text{‘}(\breve{P}\overline{R}_{\text{sc}}Q)$$

[*215·166 . *231·13·101]

This proposition is of fundamental importance.

***231·2.** $\vdash : P_*, Q_* \in \text{connex} \,.\, C\text{'}Q \cap \text{ᗡ}\text{'}R \subset Q_*\text{''}\breve{R}\text{''}C\text{'}P \,.\supset .$
$$C\text{'}P - (P\overline{R}_{\text{sc}}Q) \subset \breve{P}\overline{R}_{\text{sc}}Q$$

Dem.

$\vdash . *231\cdot112 . \supset$

$\vdash : x \in C\text{'}P - (P\overline{R}_{\text{sc}}Q) . \supset . (\exists y) . y \in C\text{'}Q \cap \text{ᗡ}\text{'}R . x \in C\text{'}P - P_*\text{''}R\text{''}\overleftarrow{Q_*}\text{'}y$ (1)

$\vdash . *202\cdot501 . *90\cdot33 . \supset$

$\vdash :. \text{Hp} . x \in C\text{'}P - P_*\text{''}R\text{''}\overleftarrow{Q_*}\text{'}y . \supset : x \in p\text{'}\overleftarrow{P_*}\text{''}(R\text{''}\overleftarrow{Q_*}\text{'}y \cap C\text{'}P) :$

[*96·3] $\supset : yQ_*z . \supset_z . x \in p\text{'}\overleftarrow{P_*}\text{''}(R\text{''}\overleftarrow{Q_*}\text{'}z \cap C\text{'}P) :$

[*40·61] $\supset : yQ_*z . z \in \text{ᗡ}\text{'}R . \supset_z . x \in \breve{P}_*\text{''}(R\text{''}\overleftarrow{Q_*}\text{'}z \cap C\text{'}P) .$

[*37·265] $\supset_z . x \in \breve{P}_*\text{''}R\text{''}\overleftarrow{Q_*}\text{'}z :$ (2)

[*90·12] $\supset : y \in C\text{'}Q \cap \text{ᗡ}\text{'}R . \supset . x \in \breve{P}_*\text{''}R\text{''}\overleftarrow{Q_*}\text{'}y :$

[*96·3.*37·2] $\supset : y \in C\text{'}Q \cap \text{ᗡ}\text{'}R . zQ_*y . \supset_z . x \in \breve{P}_*\text{''}R\text{''}\overleftarrow{Q_*}\text{'}z$ (3)

$\vdash . (2) . (3) . *202\cdot137 . \supset \vdash :. \text{Hp} . y \in C\text{'}Q \cap \text{ᗡ}\text{'}R . x \in C\text{'}P - P_*\text{''}R\text{''}\overleftarrow{Q_*}\text{'}y . \supset :$
$$z \in C\text{'}Q \cap \text{ᗡ}\text{'}R . \supset_z . x \in \breve{P}_*\text{''}R\text{''}\overleftarrow{Q_*}\text{'}z : x \in C\text{'}P :$$

[*231·112] $\supset : x \in \breve{P}\overline{R}_{\text{sc}}Q$ (4)

$\vdash . (1) . (4) . \supset \vdash :. \text{Hp} . \supset : x \in C\text{'}P - (P\overline{R}_{\text{sc}}Q) . \supset . x \in \breve{P}\overline{R}_{\text{sc}}Q :. \supset \vdash . \text{Prop}$

This proposition is fundamental in the theory of limiting segments.

***231·201.** $\vdash : P_*, Q_* \in \text{connex} . R\text{''}C\text{'}Q \subset C\text{'}P . \supset . C\text{'}P - (P\overline{R}_{\text{sc}}Q) \subset \breve{P}\overline{R}_{\text{sc}}Q$
[*231·2·154·155]

***231·202.** $\vdash : P_*, Q_* \in \text{connex} . \exists ! P\overline{R}_{\text{sc}}Q . \supset . C\text{'}P - (P\overline{R}_{\text{sc}}Q) \subset \breve{P}\overline{R}_{\text{sc}}Q$

Dem.

$\vdash . *40\cdot22 . \text{Transp} . *231\cdot12 . \supset$

$\vdash : \text{Hp} . \supset . \Lambda \sim\in P_*\text{'''}R\text{'''}\overleftarrow{Q_*}\text{''}(C\text{'}Q \cap \text{ᗡ}\text{'}R) .$

[*231·156] $\supset . C\text{'}Q \cap \text{ᗡ}\text{'}R \subset Q_*\text{''}\breve{R}\text{''}C\text{'}P .$

[*231·2] $\supset . C\text{'}P - (P\overline{R}_{\text{sc}}Q) \subset \breve{P}\overline{R}_{\text{sc}}Q : \supset \vdash . \text{Prop}$

***231·21.** $\vdash : P_*, Q_* \in \text{connex} . C\text{'}Q \cap \text{ᗡ}\text{'}R \subset Q_*\text{''}\breve{R}\text{''}C\text{'}P . \supset .$
$$C\text{'}P = P\overline{R}_{\text{sc}}Q \cup \breve{P}\overline{R}_{\text{sc}}Q$$

Dem.

$\vdash . *231\cdot13 . \supset \vdash : P\overline{R}_{\text{sc}}Q \subset C\text{'}P . \breve{P}\overline{R}_{\text{sc}}Q \subset C\text{'}P$ (1)

$\vdash . *231\cdot2 . \supset \vdash : \text{Hp} . \supset . C\text{'}P \subset P\overline{R}_{\text{sc}}Q \cup \breve{P}\overline{R}_{\text{sc}}Q$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***231·22.** $\vdash : P_*, Q_* \in \text{connex} . R\text{''}C\text{'}Q \subset C\text{'}P . \supset . C\text{'}P = P\overline{R}_{\text{sc}}Q \cup \breve{P}\overline{R}_{\text{sc}}Q$
[*231·201·13]

***231·23.** $\vdash :. P_* \in \text{connex} . R``C`Q \subset C`P . \supset :$

$$\exists ! R``\overleftarrow{Q_*}`y - \overrightarrow{P_*}`x . \supset . \overrightarrow{P_*}`x \subset P_{\text{po}}``R``\overleftarrow{Q_*}`y$$

Dem.

$\vdash . *90·14 . \quad \supset \vdash : \text{Hp} . \supset . R``\overleftarrow{Q_*}`y \subset C`P \qquad (1)$

$\vdash . (1) . *90·21 . \supset \vdash : \text{Hp} . \exists ! R``\overleftarrow{Q_*}`y - \overrightarrow{P_*}`x . \supset . \exists ! P_*``R``\overleftarrow{Q_*}`y - \overrightarrow{P_*}`x .$

$[*211·56 . *202·13] \qquad \supset . \overrightarrow{P_*}`x \subset P_{\text{po}}``R``\overleftarrow{Q_*}`y : \supset \vdash . \text{Prop}$

***231·24.** $\vdash : P_* \in \text{connex} . R``C`Q \subset C`P . \sim \{RQ_{\text{cn}}(\overrightarrow{P_*}`x)\} . \supset . \overrightarrow{P_*}`x \subset P\overline{R}_{\text{sc}}Q$

Dem.

$\vdash . *230·11 . \supset \vdash :. \text{Hp} . \supset : y \in C`Q \cap \text{Ꭰ}`R . \supset_y . \exists ! R``\overleftarrow{Q_*}`y - \overrightarrow{P_*}`x .$

$[*231·23] \qquad \supset_y . \overrightarrow{P_*}`x \subset P_{\text{po}}``R``\overleftarrow{Q_*}`y :$

$[*91·54 . *40·44] \qquad \supset : \overrightarrow{P_*}`x \subset p`P_*```R```\overleftarrow{Q_*}``(C`Q \cap \text{Ꭰ}`R) \qquad (1)$

$\vdash . (1) . *231·12 . *90·14 . \supset \vdash . \text{Prop}$

***231·25.** $\vdash :. P \in \text{Ser} . Q_* \in \text{connex} . R``C`Q \subset C`P . P\overline{R}_{\text{os}}Q = \Lambda .$

$\text{E} ! \text{limax}_P`(P\overline{R}_{\text{sc}}Q) . \supset : \text{limax}_P`(P\overline{R}_{\text{sc}}Q) = \text{limin}_P`(\breve{P}\overline{R}_{\text{sc}}Q) . \vee .$

$\text{limax}_P`(P\overline{R}_{\text{sc}}Q) P_1 \text{limin}_P`(\breve{P}\overline{R}_{\text{sc}}Q)$

$[*215·54·541 . *231·13·22]$

***231·251.** $\vdash : \text{Hp} *231·25 . \text{limax}_P`(P\overline{R}_{\text{sc}}Q) \sim \in \text{D}`P_1 . \supset .$

$\text{limax}_P`(P\overline{R}_{\text{sc}}Q) = \text{limin}_P`(\breve{P}\overline{R}_{\text{sc}}Q) \quad [*231·25]$

***231·252.** $\vdash : P \in \text{Ser} . Q_* \in \text{connex} . R``C`Q \subset C`P . P\overline{R}_{\text{os}}Q \in 0 \cup 1 .$

$\text{E} ! \text{limax}_P`(P\overline{R}_{\text{sc}}Q) . \text{limax}_P`(P\overline{R}_{\text{sc}}Q) \sim \in \text{D}`P_1 . \supset .$

$\text{limax}_P`(P\overline{R}_{\text{sc}}Q) = \text{limin}_P`(\breve{P}\overline{R}_{\text{sc}}Q)$

$[*215·543 . *231·13·22]$

***231·4.** $\vdash : Q \in \text{trans} \cap \text{connex} . \text{E} ! \max_Q`\text{Ꭰ}`R . \supset . P\overline{R}_{\text{sc}}Q = P_*``\overrightarrow{R}`\max_Q`\text{Ꭰ}`R$

Dem.

$\vdash . *230·53 . *231·121 . \supset$

$\vdash :: \text{Hp} . \supset :. x \in P\overline{R}_{\text{sc}}Q . \equiv : \overrightarrow{R}`\max_Q`\text{Ꭰ}`R \subset \alpha . \supset_\alpha . x \in P_*``\alpha :$

$[*37·2 . *22·42] \qquad \equiv : x \in P_*``\overrightarrow{R}`\max_Q`\text{Ꭰ}`R :: \supset \vdash . \text{Prop}$

***231·41.** $\vdash : Q \in \text{trans} \cap \text{connex} . \text{E} ! R`\max_Q`\text{Ꭰ}`R . \supset . P\overline{R}_{\text{sc}}Q = \overrightarrow{P_*}`R`\max_Q`\text{Ꭰ}`R$

Dem.

$\vdash . *30·5 . *231·4 . *53·31 . \supset$

$\vdash : \text{Hp} . \supset . P\overline{R}_{\text{sc}}Q = P_*``\iota`R`\max_Q`\text{Ꭰ}`R$

$[*53·301] \qquad = \overrightarrow{P_*}`R`\max_Q`\text{Ꭰ}`R : \supset \vdash . \text{Prop}$

$*232$. ON THE OSCILLATION OF A FUNCTION AS THE ARGUMENT APPROACHES A GIVEN LIMIT

Summary of $*232$.

In the preceding number, we considered the ultimate oscillation of a function when the argument grows without limit. If, in the propositions of the last number, we confine the field of Q to $\overrightarrow{Q}\text{‘}x$, where $x \in \text{Ɑ}\text{‘}Q$, the ultimate oscillation becomes the ultimate oscillation as the argument approaches x from below. If the ultimate oscillation consists of a single term, this is the limit of the function as the argument approaches x from below. If, instead of confining the argument to $\overrightarrow{Q}\text{‘}x$, we confine it to any other class whose limit is x, we shall, under a very usual hypothesis, obtain the same value for the ultimate oscillation as if we confined it to $\overrightarrow{Q}\text{‘}x$. And more generally, under a similar hypothesis, if α and β are two classes of arguments which define the same section (*i.e.* such that $Q_*\text{“}\alpha = Q_*\text{“}\beta$), then, whether or not this section has a limit, the ultimate sections and the ultimate oscillation are the same for α as they are for β. Hence we are led to consider first the result of confining the field of Q, not to $\overrightarrow{Q}\text{‘}x$, but to any class α. In order not to have to exclude explicitly the case in which $\alpha \in 1$, we deal with $Q_* \upharpoonright \alpha$, not $Q \upharpoonright \alpha$. Hence we are led to the following definitions:

$*232{\cdot}01$. $(P\overline{R}Q)_{sc}\text{‘}\alpha = P\overline{R}_{sc}(Q_* \upharpoonright \alpha)$ Df

$*232{\cdot}02$. $(P\overline{R}Q)_{os}\text{‘}\alpha = P\overline{R}_{os}(Q_* \upharpoonright \alpha)$ Df

Most of the propositions of the present number are immediate consequences of corresponding propositions in $*231$. The most important application of the propositions of the present number is to the case where α is of the form $\overrightarrow{Q}\text{‘}x$, x being a member of $\delta_Q\text{‘}\text{Ɑ}\text{‘}R$. We may, in this case, take in place of $\overrightarrow{Q}\text{‘}x$ any other class of arguments (*e.g.* a progression of arguments $x_1, x_2, \ldots x_\nu, \ldots$) having x for its limit, without altering the limiting sections or the ultimate oscillation. Hence the limit of the function for a given argument (if it exists) may be determined by choosing any selection of arguments having the given argument as their limit (cf. $*233{\cdot}142$, below).

From the definition of $(P\overline{R}Q)_{sc}\text{‘}\alpha$ we obtain immediately

$*232{\cdot}11$. $\vdash :. \, x \in (P\overline{R}Q)_{sc}\text{‘}\alpha \,.\equiv :$

$$y \in \alpha \cap C\text{‘}Q \cap \text{Ɑ}\text{‘}R \,.\supset_y .\, x \in P_*\text{“}R\text{“}(\alpha \cap \overleftarrow{Q_*}\text{‘}y) : x \in C\text{‘}P$$

We prove that $(P\overline{R}Q)_{sc}\text{‘}\alpha = (P\overline{R}Q)_{sc}\text{‘}(\alpha \cap C\text{‘}Q \cap \text{Ɑ}\text{‘}R)$ ($*232{\cdot}131$), and that if $\alpha \cap C\text{‘}Q \cap \text{Ɑ}\text{‘}R = \Lambda$, the two limiting sections and the ultimate oscillation are all equal to $C\text{‘}P$ ($*232{\cdot}15$). Also we have

$*232{\cdot}14$. $\vdash : Q \in \text{trans} \cap \text{connex} \,.\, \alpha \cap C\text{‘}Q \sim\in 1 \,.\supset .\, (P\overline{R}Q)_{sc}\text{‘}\alpha = P\overline{R}_{sc}(Q \upharpoonright \alpha)$

Thus the substitution of Q_* for Q in our definitions has the effect of making them applicable to unit classes, and of enabling us to substitute the hypothesis $Q_* \epsilon \text{connex}$ for $Q \epsilon \text{trans} \cap \text{connex}$. But when Q is transitive and connected (and therefore when Q is a series), the substitution of Q_* for Q in the definitions makes no difference unless α is a unit class. This case is trivial, since the only interest of our definitions is when α has no maximum in Q.

From $*231{\cdot}22$ we obtain

$*232{\cdot}22$. $\vdash : P_*, Q_* \upharpoonright \alpha \epsilon \text{connex} \,.\, R``(\alpha \cap C`Q) \subset C`P \,.\, \supset .$
$$C`P = (P\bar{R}Q)_{sc}`\alpha \cup (\breve{P}\bar{R}Q)_{sc}`\alpha$$

We have next a set of propositions concerned in discovering circumstances under which two classes α and β which determine the same section in Q (and therefore have the same limit, if any) give the same values for the two limiting sections. For this purpose, it is only necessary to discover circumstances under which we may substitute $Q_*``(\alpha \cap \text{Ɑ}`R)$ for α. When this can be done, the ultimate oscillation of the function as the argument approaches the limit of α can be determined by taking any set of arguments having this limit. We have

$*232{\cdot}301$. $\vdash \,.\, (P\bar{R}Q)_{sc}`\alpha \subset (P\bar{R}Q)_{sc}`Q_*``(\alpha \cap \text{Ɑ}`R)$

$*232{\cdot}32$. $\vdash : (P\bar{R}Q)_{os}`Q_*``(\alpha \cap \text{Ɑ}`R) \epsilon 0 \cup 1 \,.\supset .\, (P\bar{R}Q)_{os}`\alpha \epsilon 0 \cup 1$

Thus if the function has a limit as the argument approaches the limit of $Q_*``(\alpha \cap \text{Ɑ}`R)$, it also has a limit as the argument approaches the limit of α.

$*232{\cdot}33$. $\vdash : P_*, Q_* \upharpoonright \alpha \epsilon \text{connex} \,.\, R``(\alpha \cap C`Q) \subset C`P \,.\, \supset .$
$$(P\bar{R}Q)_{sc}`\alpha \cup (\breve{P}\bar{R}Q)_{sc}`\alpha = (P\bar{R}Q)_{sc}`Q_*``(\alpha \cap \text{Ɑ}`R) \cup (\breve{P}\bar{R}Q)_{sc}`Q_*``(\alpha \cap \text{Ɑ}`R) = C`P$$
whence

$*232{\cdot}34$. $\vdash : \text{Hp}\, *232{\cdot}33 \,.\, (P\bar{R}Q)_{os}`Q_*``(\alpha \cap \text{Ɑ}`R) = \Lambda \,.\supset .$
$$(P\bar{R}Q)_{sc}`\alpha = (P\bar{R}Q)_{sc}`Q_*``(\alpha \cap \text{Ɑ}`R) \,.\, (\breve{P}\bar{R}Q)_{sc}`\alpha = (\breve{P}\bar{R}Q)_{sc}`Q_*``(\alpha \cap \text{Ɑ}`R)$$

We have also

$*232{\cdot}341$. $\vdash : P_* \epsilon \text{connex} \,.\, \exists ! (P\bar{R}Q)_{os}`\alpha \,.\, (P\bar{R}Q)_{os}`Q_*``(\alpha \cap \text{Ɑ}`R) \epsilon 1 \,.\supset .$
$$(P\bar{R}Q)_{sc}`\alpha = (P\bar{R}Q)_{sc}`Q_*``(\alpha \cap \text{Ɑ}`R) \,.\, (\breve{P}\bar{R}Q)_{sc}`\alpha = (\breve{P}\bar{R}Q)_{sc}`Q_*``(\alpha \cap \text{Ɑ}`R)$$

Hence we arrive at the conclusion that, if P_{po} is a series, and x is the limit of the function for the class $Q_*``(\alpha \cap \text{Ɑ}`R)$, if $x$ is a member of $(P\bar{R}Q)_{sc}`\alpha$, it is its maximum ($*232{\cdot}352$), while if x is not a member of $(P\bar{R}Q)_{sc}`\alpha$, it is its sequent ($*232{\cdot}356$), assuming $(P\bar{R}Q)_{sc}`\alpha \cup (\breve{P}\bar{R}Q)_{sc}`\alpha = C`P$, which, as we saw ($*233{\cdot}22$), is generally the case, and assuming also $P \epsilon \text{Ser}$. On the other hand, if $(P\bar{R}Q)_{sc}`\alpha$ has no maximum, $x$ is the minimum of $(\breve{P}\bar{R}Q)_{sc}`\alpha$; and if $(P\bar{R}Q)_{sc}`\alpha$ has a maximum other than $x$, this is $P_1`x$

(*232·357·358). This latter case is impossible unless x has an immediate predecessor. Hence we arrive at the following proposition:

***232·38.** $\vdash : P \epsilon \text{Ser} . Q_* \upharpoonright \alpha \epsilon \text{connex} . R``(\alpha \cap C`Q) \subset C`P .$
$(P\bar{R}Q)_{os}`Q_*``(\alpha \cap \text{G}`R) \epsilon 0 \cup (1 - \text{Cl}`C`P_1) . \supset .$
$\overrightarrow{\text{limax}}_P`(P\bar{R}Q)_{sc}`\alpha = \overrightarrow{\text{limax}}_P`(P\bar{R}Q)_{sc}`Q_*``(\alpha \cap \text{G}`R) .$
$\overrightarrow{\text{limin}}_P`(\breve{P}\bar{R}Q)_{sc}`\alpha = \overrightarrow{\text{limin}}_P`(\breve{P}\bar{R}Q)_{sc}`Q_*``(\alpha \cap \text{G}`R)$

Applying this to a series having Dedekindian continuity, we know that $P_1 = \Lambda$, and that the limax and limin always exist. Hence

***232·39.** $\vdash :. P \epsilon \text{Ser} \cap \text{Ded} . P^2 = P . Q_* \epsilon \text{connex} . R``C`Q \subset C`P . \supset :$
$(P\bar{R}Q)_{os}`Q_*``(\alpha \cap \text{G}`R) \epsilon 0 \cup 1 . \supset_\alpha .$
$\text{limax}_P`(P\bar{R}Q)_{sc}`\alpha = \text{limax}_P`(P\bar{R}Q)_{sc}`Q_*``(\alpha \cap \text{G}`R) =$
$\text{limin}_P`(\breve{P}\bar{R}Q)_{sc}`\alpha = \text{limin}_P`(\breve{P}\bar{R}Q)_{sc}`Q_*``(\alpha \cap \text{G}`R)$

That is to say, if the value-series P has Dedekindian continuity, and contains all values for arguments in $C`Q$, then, provided the function has a definite limit for the class $Q_*``(\alpha \cap \text{G}`R)$, this is its limit also for the class α; that is to say, any collection of arguments having the same limit or maximum as a given section will give the same limit for the function.

***232·01.** $(P\bar{R}Q)_{sc}`\alpha = P\bar{R}_{sc}(Q_* \upharpoonright \alpha)$ Df

***232·02.** $(P\bar{R}Q)_{os}`\alpha = P\bar{R}_{os}(Q_* \upharpoonright \alpha)$ Df

***232·1.** $\vdash . (P\bar{R}Q)_{sc}`\alpha = P\bar{R}_{sc}(Q_* \upharpoonright \alpha)$ [(*232·01)]

***232·101.** $\vdash . (P\bar{R}Q)_{os}`\alpha = P\bar{R}_{os}(Q_* \upharpoonright \alpha) = (P\bar{R}Q)_{sc}`\alpha \cap (\breve{P}\bar{R}Q)_{sc}`\alpha$ [(*232·02)]

***232·11.** $\vdash :. x \epsilon (P\bar{R}Q)_{sc}`\alpha . \equiv :$
$y \epsilon \alpha \cap C`Q \cap \text{G}`R . \supset_y . x \epsilon P_*``R``(\alpha \cap \overleftarrow{Q_*}`y) : x \epsilon C`P$

Dem.

$\vdash . *90·41·42 . *231·112 . \supset$
$\vdash :. x \epsilon (P\bar{R}Q)_{sc}`\alpha . \equiv : y \epsilon \alpha \cap C`Q \cap \text{G}`R . \supset_y . x \epsilon P_*``R``(\overleftarrow{Q_* \upharpoonright \alpha})`y : x \epsilon C`P :$
$[*35·102] \equiv : y \epsilon \alpha \cap C`Q \cap \text{G}`R . \supset_y . x \epsilon P_*``R``(\alpha \cap \overleftarrow{Q_*}`y) : x \epsilon C`P :. \supset \vdash . \text{Prop}$

***232·12.** $\vdash . (P\bar{R}Q)_{sc}`\alpha = p`P_*```R```(\alpha \cap)``\overleftarrow{Q_*}``(\alpha \cap C`Q \cap \text{G}`R) \cap C`P$
[*232·11]

***232·121.** $\vdash : \gamma = \alpha \cap C`Q \cap \text{G}`R . \supset . (P\bar{R}Q)_{sc}`\alpha = p`P_*```R```(\gamma \cap)``\overleftarrow{Q_*}``\gamma \cap C`P$

Dem.

$\vdash . *90·13 . \supset \vdash . \alpha \cap \overleftarrow{Q_*}`y = \alpha \cap C`Q \cap \overleftarrow{Q_*}`y .$
$[*37·26] \quad \supset \vdash . R``(\alpha \cap \overleftarrow{Q_*}`y) = R``(\alpha \cap C`Q \cap \text{G}`R \cap \overleftarrow{Q_*}`y)$ (1)
$\vdash . (1) . *232·11 . \supset \vdash . \text{Prop}$

***232·13.** $\vdash : \alpha \cap C`Q \cap \text{G}`R = \beta \cap C`Q \cap \text{G}`R . \supset . (P\bar{R}Q)_{sc}`\alpha = (P\bar{R}Q)_{sc}`\beta$
[*232·121]

***232·131.** $\vdash . (P\bar{R}Q)_{sc}`\alpha = (P\bar{R}Q)_{sc}`(\alpha \cap C`Q \cap \text{Œ}`R)$ [*232·13]

From the above propositions it follows that the values of $(P\bar{R}Q)_{sc}`\alpha$, $(\breve{P}\bar{R}Q)_{sc}`\alpha$, and $(P\bar{R}Q)_{os}`\alpha$ depend only upon $\alpha \cap C`Q \cap \text{Œ}`R$; thus if $\alpha$ is not contained in $C`Q \cap \text{Œ}`R$, the part not contained in $C`Q \cap \text{Œ}`R$ is irrelevant.

***232·14.** $\vdash : Q \epsilon \text{trans} \cap \text{connex} . \alpha \cap C`Q \sim\epsilon 1 . \supset . (P\bar{R}Q)_{sc}`\alpha = P\bar{R}_{sc}(Q \upharpoonright \alpha)$
[*232·1 . *202·54·541]

***232·15.** $\vdash : \alpha \cap C`Q \cap \text{Œ}`R = \Lambda . \supset . (P\bar{R}Q)_{sc}`\alpha = (\breve{P}\bar{R}Q)_{sc}`\alpha = (P\bar{R}Q)_{os}`\alpha = C`P$
[*232·12·101 . *37·29 . *40·2]

***232·151.** $\vdash : \dot{\exists} ! P . (P\bar{R}Q)_{os}`\alpha = \Lambda . \supset . \exists ! \alpha \cap C`Q \cap \text{Œ}`R$
[*232·15 . Transp . *33·24]

***232·2.** $\vdash : C`Q \cap \text{Œ}`R \subset \alpha . \supset . (P\bar{R}Q)_{sc}`\alpha = P\bar{R}_{sc}Q$

Dem.

$\vdash . *22·621 . *232·11 . \supset$

$\vdash :: \text{Hp} . \supset :. x \epsilon (P\bar{R}Q)_{sc}`\alpha . \equiv : y \epsilon C`Q \cap \text{Œ}`R . \supset_y . x \epsilon P_*``R``\overleftarrow{Q_*}`y :$

[*231·112] $\equiv : x \epsilon P\bar{R}_{sc}Q :: \supset \vdash . \text{Prop}$

***232·21.** $\vdash : P_*, Q_* \upharpoonright \alpha \epsilon \text{connex} . \alpha \cap C`Q \cap \text{Œ}`R \subset Q_*``(\alpha \cap \breve{R}``C`P) . \supset .$

$C`P = (P\bar{R}Q)_{sc}`\alpha \cup (\breve{P}\bar{R}Q)_{sc}`\alpha$

$\left[*231·21 \frac{Q_* \upharpoonright \alpha}{Q}\right]$

***232·22.** $\vdash : P_*, Q_* \upharpoonright \alpha \epsilon \text{connex} . R``(\alpha \cap C`Q) \subset C`P . \supset .$

$C`P = (P\bar{R}Q)_{sc}`\alpha \cup (\breve{P}\bar{R}Q)_{sc}`\alpha$ [*231·22]

***232·23.** $\vdash : y \epsilon C`Q \cap \text{Œ}`R . \supset . (P\bar{R}Q)_{sc}`\iota`y = P_*``\overrightarrow{R}`y$

Dem.

$\vdash . *232·11 . *13·191 . \supset$

$\vdash :. \text{Hp} . \supset : x \epsilon (P\bar{R}Q)_{sc}`\iota`y . \equiv . x \epsilon P_*``R``(\iota`y \cap \overleftarrow{Q_*}`y) :. \supset \vdash . \text{Prop}$

***232·24.** $\vdash : Q \epsilon \text{trans} \cap \text{connex} . \text{E} ! \max_Q`(\alpha \cap \text{Œ}`R) . \supset .$

$(P\bar{R}Q)_{sc}`\alpha = P_*``\overrightarrow{R}`\max_Q`(\alpha \cap \text{Œ}`R)$

Dem.

$\vdash . *232·14 . \supset \vdash : \text{Hp} . \alpha \cap C`Q \cap \text{Œ}`R \sim\epsilon 1 . \supset .$

$(P\bar{R}Q)_{sc}`\alpha = P\bar{R}_{sc}\{Q \upharpoonright (\alpha \cap \text{Œ}`R)\}$

[*231·4.*205·9] $= P_*``\overrightarrow{R}`\max_Q`(\alpha \cap \text{Œ}`R)$ (1)

$\vdash . *205·17 . *232·23·131 . \supset$

$\vdash : \alpha \cap C`Q \cap \text{Œ}`R \epsilon 1 . \supset . (P\bar{R}Q)_{sc}`\alpha = P_*``\overrightarrow{R}`\max_Q`(\alpha \cap \text{Œ}`R)$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

∗232·3. $\vdash : \alpha \subset \mathrm{D}`R . \supset . (P\bar{R}Q)_{sc}`\alpha \subset (P\bar{R}Q)_{sc}`Q_*``\alpha$

Dem.

$\vdash . *96{\cdot}3 . \supset \vdash : y \,\epsilon\, Q_*``\alpha . \supset . (\exists z) . z \,\epsilon\, \alpha \cap C`Q . \overleftarrow{Q_*}`z \subset \overleftarrow{Q_*}`y .$

$[\mathrm{Fact}.*37{\cdot}2] \supset . (\exists z) . z \,\epsilon\, \alpha \cap C`Q . P_*``R``(\alpha \cap \overleftarrow{Q_*}`z) \subset P_*``R``(\alpha \cap \overleftarrow{Q_*}`y)$ (1)

$\vdash . (1) . *232{\cdot}11 . \supset \vdash :. \mathrm{Hp} . x \,\epsilon\, (P\bar{R}Q)_{sc}`\alpha . \supset : y \,\epsilon\, Q_*``\alpha . \supset . x \,\epsilon\, P_*``R``(\alpha \cap \overleftarrow{Q_*}`y) .$

$[*90{\cdot}33]$ $\supset . x \,\epsilon\, P_*``R``(Q_*``\alpha \cap \overleftarrow{Q_*}`y) :$

$[*232{\cdot}11]$ $\supset : x \,\epsilon\, (P\bar{R}Q)_{sc}`Q_*``\alpha :. \supset \vdash .$ Prop

∗232·301. $\vdash . (P\bar{R}Q)_{sc}`\alpha \subset (P\bar{R}Q)_{sc}`Q_*``(\alpha \cap \mathrm{D}`R)$

Dem.

$\vdash . *232{\cdot}13 . \supset \vdash . (P\bar{R}Q)_{sc}`\alpha = (P\bar{R}Q)_{sc}`(\alpha \cap \mathrm{D}`R)$

$[*232{\cdot}3]$ $\subset (P\bar{R}Q)_{sc}`Q_*``(\alpha \cap \mathrm{D}`R) . \supset \vdash .$ Prop

∗232·31. $\vdash . (P\bar{R}Q)_{os}`\alpha \subset (P\bar{R}Q)_{os}`Q_*``(\alpha \cap \mathrm{D}`R)$

Dem.

$\vdash . *232{\cdot}301 \dfrac{\breve{P}}{P} . \supset \vdash . (\breve{P}\bar{R}Q)_{sc}`\alpha \subset (\breve{P}\bar{R}Q)_{sc}`Q_*``(\alpha \cap \mathrm{D}`R)$ (1)

$\vdash . *232{\cdot}301 . (1) . *232{\cdot}101 . \supset \vdash .$ Prop

∗232·32. $\vdash : (P\bar{R}Q)_{os}`Q_*``(\alpha \cap \mathrm{D}`R) \,\epsilon\, 0 \cup 1 . \supset . (P\bar{R}Q)_{os}`\alpha \,\epsilon\, 0 \cup 1$ $[*232{\cdot}31]$

∗232·33. $\vdash : P_*, Q_* \upharpoonright \alpha \,\epsilon\, \mathrm{connex} . R``(\alpha \cap C`Q) \subset C`P . \supset .$

$(P\bar{R}Q)_{sc}`\alpha \cup (\breve{P}\bar{R}Q)_{sc}`\alpha = (P\bar{R}Q)_{sc}`Q_*``(\alpha \cap \mathrm{D}`R) \cup (\breve{P}\bar{R}Q)_{sc}`Q_*``(\alpha \cap \mathrm{D}`R) = C`P$

Dem.

$\vdash . *232{\cdot}22{\cdot}301 . \supset$

$\vdash : \mathrm{Hp} . \supset . C`P \subset (P\bar{R}Q)_{sc}`Q_*``(\alpha \cap \mathrm{D}`R) \cup (\breve{P}\bar{R}Q)_{sc}`Q_*``(\alpha \cap \mathrm{D}`R)$ (1)

$\vdash . (1) . *231{\cdot}131 . \supset \vdash .$ Prop

∗232·34. $\vdash : \mathrm{Hp}\, *232{\cdot}33 . (P\bar{R}Q)_{os}`Q_*``(\alpha \cap \mathrm{D}`R) = \Lambda . \supset .$

$(P\bar{R}Q)_{sc}`\alpha = (P\bar{R}Q)_{sc}`Q_*``(\alpha \cap \mathrm{D}`R) . (\breve{P}\bar{R}Q)_{sc}`\alpha = (\breve{P}\bar{R}Q)_{sc}`Q_*``(\alpha \cap \mathrm{D}`R)$

$[*232{\cdot}33{\cdot}301 . *24{\cdot}482]$

∗232·341. $\vdash : P_* \,\epsilon\, \mathrm{connex} . \exists ! (P\bar{R}Q)_{os}`\alpha . (P\bar{R}Q)_{os}`Q_*``(\alpha \cap \mathrm{D}`R) \,\epsilon\, 1 . \supset .$

$(P\bar{R}Q)_{sc}`\alpha = (P\bar{R}Q)_{sc}`Q_*``(\alpha \cap \mathrm{D}`R) . (\breve{P}RQ)_{sc}`\alpha = (\breve{P}RQ)_{sc}`Q_*``(\alpha \cap \mathrm{D}`R)$

$[*231{\cdot}192 . *232{\cdot}31 . *60{\cdot}38]$

∗232·35. $\vdash : P_* \,\epsilon\, \mathrm{connex} . (P\bar{R}Q)_{os}`Q_*``(\alpha \cap \mathrm{D}`R) = \iota`x . \supset .$

$(P\bar{R}Q)_{sc}`\alpha \subset \overrightarrow{P_*}`x . (\breve{P}\bar{R}Q)_{sc}`\alpha \subset \overleftarrow{P_*}`x$

$[*232{\cdot}301 . *231{\cdot}191]$

∗232·351. $\vdash : \mathrm{Hp}\, *232{\cdot}35 . x \,\epsilon\, (P\bar{R}Q)_{sc}`\alpha . \supset . (P\bar{R}Q)_{sc}`\alpha = \overrightarrow{P_*}`x$

Dem.

$\vdash . *231{\cdot}13 . \supset \vdash : \mathrm{Hp} . \supset . \overrightarrow{P_*}`x \subset (P\bar{R}Q)_{sc}`\alpha$ (1)

$\vdash . (1) . *232{\cdot}35 . \supset \vdash .$ Prop

***232·352.** $\vdash : \text{Hp} *232\cdot351 . P_{po} \in J . \supset . x = \max_P{}^{\backprime}(P\overline{R}Q)_{sc}{}^{\backprime}\alpha$
[*211·8 . *205·197 . *232·351]

***232·353.** $\vdash : \text{Hp} *232\cdot35 . (P\overline{R}Q)_{sc}{}^{\backprime}\alpha \cup (\breve{P}\overline{R}Q)_{sc}{}^{\backprime}\alpha = C{}^{\backprime}P . x \sim\in (P\overline{R}Q)_{sc}{}^{\backprime}\alpha . \supset .$
$(\breve{P}\overline{R}Q)_{sc}{}^{\backprime}\alpha = \overleftarrow{P_*}{}^{\backprime}x$

Dem.

$\vdash . *231\cdot13 . \supset \vdash : \text{Hp} . \supset . x \in (\breve{P}\overline{R}Q)_{sc}{}^{\backprime}\alpha .$
[*232·351] $\supset . (\breve{P}\overline{R}Q)_{sc}{}^{\backprime}\alpha = \overleftarrow{P_*}{}^{\backprime}x : \supset \vdash . \text{Prop}$

***232·354.** $\vdash : \text{Hp} *232\cdot353 . P_{po} \in J . \supset . x = \min_P{}^{\backprime}(\breve{P}\overline{R}Q)_{sc}{}^{\backprime}\alpha$ $\left[*232\cdot352 \frac{\breve{P}}{P}\right]$

***232·355.** $\vdash : \text{Hp} *232\cdot353 . \supset . (P\overline{R}Q)_{sc}{}^{\backprime}\alpha = \overrightarrow{P_{po}}{}^{\backprime}x$

Dem.

$\vdash . *232\cdot35 . \supset \vdash : \text{Hp} . \supset . (P\overline{R}Q)_{sc}{}^{\backprime}\alpha \subset \overrightarrow{P_*}{}^{\backprime}x - \iota{}^{\backprime}x$
[*91·542] $\subset \overrightarrow{P_{po}}{}^{\backprime}x$ (1)
$\vdash . *232\cdot353 . \supset \vdash : \text{Hp} . \supset . C{}^{\backprime}P - \overleftarrow{P_*}{}^{\backprime}x \subset C{}^{\backprime}P - (\breve{P}\overline{R}Q)_{sc}{}^{\backprime}\alpha .$
[*202·101.Hp] $\supset . \overrightarrow{P_{po}}{}^{\backprime}x \subset (P\overline{R}Q)_{sc}{}^{\backprime}\alpha$ (2)
$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***232·356.** $\vdash :. P \in \text{Ser} . (P\overline{R}Q)_{os}{}^{\backprime}Q_*{}^{\backprime\backprime}(\alpha \cap \text{Ɑ}{}^{\backprime}R) = \iota{}^{\backprime}x .$
$(P\overline{R}Q)_{sc}{}^{\backprime}\alpha \cup (\breve{P}\overline{R}Q)_{sc}{}^{\backprime}\alpha = C{}^{\backprime}P . \supset : x \sim\in (P\overline{R}Q)_{sc}{}^{\backprime}\alpha . \supset . x = \text{seq}_P{}^{\backprime}(P\overline{R}Q)_{sc}{}^{\backprime}\alpha$
[*206·172 . *231·13 . *232·355]

***232·357.** $\vdash : \text{Hp} *232\cdot35 . P_{po} \in J . \sim\text{E}!\max_P{}^{\backprime}(P\overline{R}Q)_{sc}{}^{\backprime}\alpha . \supset . x = \min_P{}^{\backprime}(\breve{P}\overline{R}Q)_{sc}{}^{\backprime}\alpha$

Dem.

$\vdash . *232\cdot352 . \text{Transp} . \supset \vdash : \text{Hp} . \supset . x \sim\in (P\overline{R}Q)_{sc}{}^{\backprime}\alpha .$
[*232·354] $\supset . x = \min_P{}^{\backprime}(\breve{P}\overline{R}Q)_{sc}{}^{\backprime}\alpha : \supset \vdash . \text{Prop}$

***232·358.** $\vdash : \text{Hp} *232\cdot35 . P_{po} \in J . (P\overline{R}Q)_{sc}{}^{\backprime}\alpha \cup (\breve{P}\overline{R}Q)_{sc}{}^{\backprime}\alpha = C{}^{\backprime}P .$
$\text{E}!\max_P{}^{\backprime}(P\overline{R}Q)_{sc}{}^{\backprime}\alpha . \max_P{}^{\backprime}(P\overline{R}Q)_{sc}{}^{\backprime}\alpha \neq x . \supset . \max_P{}^{\backprime}(P\overline{R}Q)_{sc}{}^{\backprime}\alpha P_1 x$

Dem.

$\vdash . *232\cdot352 . \text{Transp} . \supset \vdash : \text{Hp} . \supset . x \sim\in (P\overline{R}Q)_{sc}{}^{\backprime}\alpha .$
[*232·356] $\supset . x = \text{seq}_P{}^{\backprime}(P\overline{R}Q)_{sc}{}^{\backprime}\alpha .$
[*206·5] $\supset . \max_P{}^{\backprime}(P\overline{R}Q)_{sc}{}^{\backprime}\alpha P_1 x : \supset \vdash . \text{Prop}$

***232·36.** $\vdash :. P \in \text{Ser} . (P\overline{R}Q)_{os}{}^{\backprime}Q_*{}^{\backprime\backprime}(\alpha \cap \text{Ɑ}{}^{\backprime}R) = \iota{}^{\backprime}x .$
$(P\overline{R}Q)_{sc}{}^{\backprime}\alpha \cup (\breve{P}\overline{R}Q)_{sc}{}^{\backprime}\alpha = C{}^{\backprime}P . \supset :$
$x \in (P\overline{R}Q)_{os}{}^{\backprime}\alpha . \supset . x = \max_P{}^{\backprime}(P\overline{R}Q)_{sc}{}^{\backprime}\alpha = \min_P{}^{\backprime}(\breve{P}\overline{R}Q)_{sc}{}^{\backprime}\alpha :$
$x \in (P\overline{R}Q)_{sc}{}^{\backprime}\alpha - (\breve{P}\overline{R}Q)_{sc}{}^{\backprime}\alpha . \supset . x = \max_P{}^{\backprime}(\breve{P}\overline{R}Q)_{sc}{}^{\backprime}\alpha = \text{prec}_P{}^{\backprime}(P\overline{R}Q)_{sc}{}^{\backprime}\alpha :$
$x \in (\breve{P}\overline{R}Q)_{sc}{}^{\backprime}\alpha - (P\overline{R}Q)_{sc}{}^{\backprime}\alpha . \supset . x = \text{seq}_P{}^{\backprime}(P\overline{R}Q)_{sc}{}^{\backprime}\alpha = \min_P{}^{\backprime}(\breve{P}\overline{R}Q)_{sc}{}^{\backprime}\alpha$
[*232·352·354·356]

***232·361.** $\vdash : \mathrm{Hp} \ast 232{\cdot}36 \,.\, x \sim\in \text{Ɑ}\text{‘}P_1 \,.\, \supset .\, x = \mathrm{limax}_P\text{‘}(P\overline{R}Q)_{\mathrm{sc}}\text{‘}\alpha$

Dem.

$\vdash . \ast 232{\cdot}358 \,.\, \mathrm{Transp} \,.\, \supset$

$\vdash :. \mathrm{Hp} \,.\, \supset : \mathrm{E}\,!\, \mathrm{max}_P\text{‘}(P\overline{R}Q)_{\mathrm{sc}}\text{‘}\alpha \,.\, \supset .\, \mathrm{max}_P\text{‘}(P\overline{R}Q)_{\mathrm{sc}}\text{‘}\alpha = x$ (1)

$\vdash . \ast 232{\cdot}352 \,.\, \mathrm{Transp} \,.\, \supset$

$\vdash :. \mathrm{Hp} \,.\, \supset : \sim\mathrm{E}\,!\, \mathrm{max}_P\text{‘}(P\overline{R}Q)_{\mathrm{sc}}\text{‘}\alpha \,.\, \supset .\, x \sim\in (P\overline{R}Q)_{\mathrm{sc}}\text{‘}\alpha \,.$

$[\ast 232{\cdot}356] \qquad \supset .\, x = \mathrm{seq}_P\text{‘}(P\overline{R}Q)\text{‘}\alpha$ (2)

$\vdash . (1) . (2) . \ast 207{\cdot}46 \,.\, \supset \vdash . \mathrm{Prop}$

***232·37.** $\vdash : P \in \mathrm{Ser} \,.\, (P\overline{R}Q)_{\mathrm{os}}\text{‘}Q_{\ast}\text{‘‘}(\alpha \cap \text{Ɑ}\text{‘}R) \in 1 - \mathrm{Cl}\text{‘}\text{Ɑ}\text{‘}P_1 \,.$

$(P\overline{R}Q)_{\mathrm{sc}}\text{‘}\alpha \cup \breve{(P\overline{R}Q)}_{\mathrm{sc}}\text{‘}\alpha = C\text{‘}P \,.\, \supset .$

$\mathrm{limax}_P\text{‘}(P\overline{R}Q)_{\mathrm{sc}}\text{‘}\alpha = \mathrm{max}_P\text{‘}(P\overline{R}Q)_{\mathrm{sc}}\text{‘}Q_{\ast}\text{‘‘}(\alpha \cap \text{Ɑ}\text{‘}R)$

$= \breve{\iota}\text{‘}(P\overline{R}Q)_{\mathrm{os}}\text{‘}Q_{\ast}\text{‘‘}(\alpha \cap \text{Ɑ}\text{‘}R)$

$[\ast 232{\cdot}361 \,.\, \ast 231{\cdot}193]$

***232·38.** $\vdash : P \in \mathrm{Ser} \,.\, Q_{\ast} \restriction \alpha \in \mathrm{connex} \,.\, R\text{‘‘}(\alpha \cap C\text{‘}Q) \subset C\text{‘}P \,.$

$(P\overline{R}Q)_{\mathrm{os}}\text{‘}Q_{\ast}\text{‘‘}(\alpha \cap \text{Ɑ}\text{‘}R) \in 0 \cup (1 - \mathrm{Cl}\text{‘}C\text{‘}P_1) \,.\, \supset .$

$\overrightarrow{\mathrm{limax}}_P\text{‘}(P\overline{R}Q)_{\mathrm{sc}}\text{‘}\alpha = \overrightarrow{\mathrm{limax}}_P\text{‘}(P\overline{R}Q)_{\mathrm{sc}}\text{‘}Q_{\ast}\text{‘‘}(\alpha \cap \text{Ɑ}\text{‘}R) \,.$

$\overrightarrow{\mathrm{limin}}_P\text{‘}\breve{(P\overline{R}Q)}_{\mathrm{sc}}\text{‘}\alpha = \overrightarrow{\mathrm{limin}}_P\text{‘}\breve{(P\overline{R}Q)}_{\mathrm{sc}}\text{‘}Q_{\ast}\text{‘‘}(\alpha \cap \text{Ɑ}\text{‘}R)$

$[\ast 232{\cdot}33{\cdot}34{\cdot}37]$

***232·39.** $\vdash :. P \in \mathrm{Ser} \cap \mathrm{Ded} \,.\, P^2 = P \,.\, Q_{\ast} \in \mathrm{connex} \,.\, R\text{‘‘}C\text{‘}Q \subset C\text{‘}P \,.\, \supset :$

$(P\overline{R}Q)_{\mathrm{os}}\text{‘}Q_{\ast}\text{‘‘}(\alpha \cap \text{Ɑ}\text{‘}R) \in 0 \cup 1 \,.\, \supset_{\alpha} .$

$\mathrm{limax}_P\text{‘}(P\overline{R}Q)_{\mathrm{sc}}\text{‘}\alpha = \mathrm{limax}_P\text{‘}(P\overline{R}Q)_{\mathrm{sc}}\text{‘}Q_{\ast}\text{‘‘}(\alpha \cap \text{Ɑ}\text{‘}R)$

$= \mathrm{limin}_P\text{‘}\breve{(P\overline{R}Q)}_{\mathrm{sc}}\text{‘}\alpha = \mathrm{limin}_P\text{‘}\breve{(P\overline{R}Q)}_{\mathrm{sc}}\text{‘}Q_{\ast}\text{‘‘}(\alpha \cap \text{Ɑ}\text{‘}R)$

Dem.

$\vdash . \ast 201{\cdot}63 \,.\, \ast 232{\cdot}38 \,.\, \supset \vdash : \mathrm{Hp} \,.\, (P\overline{R}Q)_{\mathrm{os}}\text{‘}Q_{\ast}\text{‘‘}(\alpha \cap \text{Ɑ}\text{‘}R) \in 0 \cup 1 \,.\, \supset .$

$\mathrm{limax}_P\text{‘}(P\overline{R}Q)_{\mathrm{sc}}\text{‘}\alpha = \mathrm{limax}_P\text{‘}(P\overline{R}Q)_{\mathrm{sc}}\text{‘}Q_{\ast}\text{‘‘}(\alpha \cap \text{Ɑ}\text{‘}R) \,.$

$\mathrm{limin}_P\text{‘}\breve{(P\overline{R}Q)}_{\mathrm{sc}}\text{‘}\alpha = \mathrm{limin}_P\text{‘}\breve{(P\overline{R}Q)}_{\mathrm{sc}}\text{‘}Q_{\ast}\text{‘‘}(\alpha \cap \text{Ɑ}\text{‘}R)$ (1)

$\vdash . \ast 231{\cdot}193 \,.\, \supset \vdash : \mathrm{Hp}(1) \,.\, (P\overline{R}Q)_{\mathrm{os}}\text{‘}Q_{\ast}\text{‘‘}(\alpha \cap \text{Ɑ}\text{‘}R) \in 1 \,.\, \supset .$

$\mathrm{limax}_P\text{‘}(P\overline{R}Q)_{\mathrm{sc}}\text{‘}Q_{\ast}\text{‘‘}(\alpha \cap \text{Ɑ}\text{‘}R) = \mathrm{limin}_P\text{‘}\breve{(P\overline{R}Q)}_{\mathrm{sc}}\text{‘}Q_{\ast}\text{‘‘}(\alpha \cap \text{Ɑ}\text{‘}R)$ (2)

$\vdash . \ast 214{\cdot}42 . \ast 232{\cdot}33 \,.\, \supset \vdash : \mathrm{Hp}(1) \,.\, (P\overline{R}Q)_{\mathrm{os}}\text{‘}Q_{\ast}\text{‘‘}(\alpha \cap \text{Ɑ}\text{‘}R) = \Lambda \,.\, \supset .$

$\mathrm{limax}_P\text{‘}(P\overline{R}Q)_{\mathrm{sc}}\text{‘}Q_{\ast}\text{‘‘}(\alpha \cap \text{Ɑ}\text{‘}R) = \mathrm{limin}_P\text{‘}\breve{(P\overline{R}Q)}_{\mathrm{sc}}\text{‘}Q_{\ast}\text{‘‘}(\alpha \cap \text{Ɑ}\text{‘}R)$ (3)

$\vdash . (1) . (2) . (3) \,.\, \supset \vdash . \mathrm{Prop}$

***232·5.** $\vdash . (P\overline{R}Q)_{\mathrm{sc}}\text{‘}\overrightarrow{Q}\text{‘}x = C\text{‘}P \cap \hat{y}\,\{z \in \overrightarrow{Q}\text{‘}x \cap \text{Ɑ}\text{‘}R \,.\, \supset_z .\, y \in P_{\ast}\text{‘‘}R\text{‘‘}(\overrightarrow{Q}\text{‘}x \cap \overleftarrow{Q_{\ast}}\text{‘}z)\}$

$[\ast 232{\cdot}11]$

***232·51.** $\vdash : Q \in \mathrm{trans} \cap \mathrm{connex} \,.\, \mathrm{E}\,!\, \mathrm{max}_Q\text{‘}(\overrightarrow{Q}\text{‘}x \cap \text{Ɑ}\text{‘}R) \,.\, \supset .$

$(P\overline{R}Q)_{\mathrm{sc}}\text{‘}\overrightarrow{Q}\text{‘}x = P_{\ast}\text{‘‘}\overrightarrow{R}\text{‘}\mathrm{max}_Q\text{‘}(\overrightarrow{Q}\text{‘}x \cap \text{Ɑ}\text{‘}R)$ $[\ast 232{\cdot}24]$

***232·511.** $\vdash : Q \in \text{trans} \cap \text{connex} \,.\, \text{E} \,!\, R`\text{max}_Q`(\overrightarrow{Q}`x \cap \text{Œ}`R) \,.\, \supset .$

$$(P\overline{R}Q)_{\text{sc}}`\overrightarrow{Q}`x = \overrightarrow{P_*}`R`\text{max}_Q`(\overrightarrow{Q}`x \cap \text{Œ}`R) \quad [*232·51]$$

***232·52.** $\vdash : Q \in \text{connex} \,.\, yQx \,.\, \overrightarrow{Q}`x \cap (\overleftarrow{Q}`y \cup \iota`y) \cap \text{Œ}`R = \Lambda \,.\, \supset .$

$$(P\overline{R}Q)_{\text{sc}}`\overrightarrow{Q}`x = (P\overline{R}Q)_{\text{sc}}`\overrightarrow{Q}`y \quad [*232·13]$$

***232·53.** $\vdash : Q \in \text{connex} \,.\, z \in \overrightarrow{Q}`x \cap \text{Œ}`R \,.\, \supset .\, (P\overline{R}Q)_{\text{sc}}`\overrightarrow{Q}`x = (P\overline{R}Q)_{\text{sc}}`(\overrightarrow{Q}`x \cap \overleftarrow{Q_*}`z)$

Dem.

$\vdash . *232·5 . *96·3 . \supset \vdash :. \text{Hp} \,.\, y \in (P\overline{R}Q)_{\text{sc}}`\overrightarrow{Q}`x \,.\, \supset :$

$$u \in \overrightarrow{Q}`x \cap \overleftarrow{Q_*}`z \cap \text{Œ}`R \,.\, \supset_u .\, y \in P_*``R``(\overrightarrow{Q}`x \cap \overleftarrow{Q_*}`u) \,.\, \overleftarrow{Q_*}`u \subset \overleftarrow{Q_*}`z :$$

$[*22·621.*232·11] \supset : y \in (P\overline{R}Q)_{\text{sc}}`(\overrightarrow{Q}`x \cap \overleftarrow{Q_*}`z)$ (1)

$\vdash . *232·11 . *37·2 . \supset \vdash :. \text{Hp} \,.\, y \in (P\overline{R}Q)_{\text{sc}}`(\overrightarrow{Q}`x \cap \overleftarrow{Q_*}`z) \,.\, \supset :$

$$u \in \overrightarrow{Q}`x \cap \overleftarrow{Q_*}`z \cap \text{Œ}`R \,.\, \supset_u .\, y \in P_*``R``(\overrightarrow{Q}`x \cap \overleftarrow{Q_*}`u) \quad (2)$$

$[*96·3] \supset : u \in \overrightarrow{Q}`x \cap \overrightarrow{Q_*}`z \cap \text{Œ}`R \,.\, \supset_u .\, y \in P_*``R``(\overrightarrow{Q}`x \cap \overleftarrow{Q_*}`u)$ (3)

$\vdash . (2) . (3) . \supset \vdash :. \text{Hp}(2) \,.\, \supset : u \in \overrightarrow{Q}`x \cap \text{Œ}`R \,.\, \supset_u .\, y \in P_*``R``(\overrightarrow{Q}`x \cap \overleftarrow{Q_*}`u) :$

$[*232·5] \qquad \supset : y \in (P\overline{R}Q)_{\text{sc}}`\overrightarrow{Q}`x$ (4)

$\vdash . (1) . (4) . \supset \vdash . \text{Prop}$

$*$233. ON THE LIMITS OF FUNCTIONS

Summary of $*$233.

There are four limits of a function as the argument approaches some term a in the argument-series, namely the upper and lower limits of the ultimate oscillation for approaches from below and above respectively. If the ultimate oscillation for approaches to a from below reduces to a single term, *i.e.* if $(P\bar{R}Q)_{os}\text{‘}\overrightarrow{Q}\text{‘}a \in 1$, that one term is *the* limit of the function for approaches to a from below. If this one term is also the ultimate oscillation for approaches from above, we may call it simply *the* limit of the function for the argument a. This may or may not (when it exists) be equal to the value for the argument a. It is characteristic of *continuous* functions that *the* limit exists for every argument, and is always equal to the value for that argument. Continuous functions will be considered in $*$234.

The upper limit or maximum of the ultimate oscillation as the argument approaches a is the upper limit or maximum of the ultimate section. Hence if we put

$$R(PQ)\text{‘}a = \text{limax}_P\text{‘}(P\bar{R}Q)_{sc}\text{‘}\overrightarrow{Q}\text{‘}a \quad \text{Df},$$

the four limits of the function as the argument approaches a will be

$$R(PQ)\text{‘}a, \quad R(\breve{P}Q)\text{‘}a, \quad R(P\breve{Q})\text{‘}a, \quad R(\breve{P}\breve{Q})\text{‘}a.$$

It will be seen that $R(PQ)\text{‘}a$ is a function of $\overrightarrow{Q}\text{‘}a$. It may happen that, if we put α in place of $\overrightarrow{Q}\text{‘}a$, the function will have a definite limit as the argument increases in α, although α has no limit or maximum. Thus if, for example, Q consists of the series of rationals, and P of the series of real numbers, if α is a class of rationals not having a rational limit, we may regard the limit of the function (if it exists), as the argument increases in α, as the value of the function for the irrational limit of α. In this way we can extend the domain of definition of a function.

In order to be able to deal with the cases in which α has no limit, we put

$$(P\bar{R}Q)_{lmx}\text{‘}\alpha = \text{limax}_P\text{‘}(P\bar{R}Q)_{sc}\text{‘}\alpha \quad \text{Df.}$$

If P is a Dedekindian series, $(P\bar{R}Q)_{lmx}\text{‘}\alpha$ always exists. If we take α to be any segment of Q, we thus get a new function, derived from R, but having segments of Q instead of members of $C\text{‘}Q$ as its arguments. Thus if R had rationals for its arguments, this new function will have real numbers for its arguments. (Real numbers may be regarded as segments of the series of rationals.)

The function $R(PQ)`a$ is a particular case of the above; thus we take as our definition

$$R(PQ)`a = (P\bar{R}Q)_{lmx}`\overrightarrow{Q}`a \quad \text{Df},$$

or, what comes to the same thing,

$$R(PQ) = (P\bar{R}Q)_{lmx} \,|\, \overrightarrow{Q} \quad \text{Df}.$$

The following propositions of this number are important:

$*233{\cdot}15.\ \vdash :. P \in \text{Ser} \cap \text{Ded} . (P\bar{R}Q)_{sc}`\alpha \cup (\breve{P}\bar{R}Q)_{sc}`\alpha = C`P . (P\bar{R}Q)_{os}`\alpha = \Lambda . \supset :$
$(P\bar{R}Q)_{lmx}`\alpha = (\breve{P}\bar{R}Q)_{lmx}`\alpha . \vee . \{(P\bar{R}Q)_{lmx}`\alpha\} P_1 \{(\breve{P}\bar{R}Q)_{lmx}`\alpha\}$

$*233{\cdot}16.\ \vdash :. P \in \text{Ser} \cap \text{Ded} . P^2 = P . Q_* \in \text{connex} . R``C`Q \subset C`P . \supset :$
$(P\bar{R}Q)_{os}`\alpha \in 0 \cup 1 . \supset_\alpha . (P\bar{R}Q)_{lmx}`\alpha = (\breve{P}\bar{R}Q)_{lmx}`\alpha$

$*233{\cdot}2$—$\cdot25$ are applications of the more important of the propositions $*232{\cdot}34$—$\cdot39$, showing circumstances under which the limit of the function for the class α is the same as for the class $Q_*``(\alpha \cap \text{Ɑ}`R)$.

$*233{\cdot}4$ and following propositions apply the earlier propositions of $*233$ to the case where α is replaced by $\overrightarrow{Q}`a$, and therefore $(P\bar{R}Q)_{lmx}`\alpha$ is replaced by $R(PQ)`a$. We have

$*233{\cdot}43.\ \vdash : P_{po} \in \text{Ser} . (P\bar{R}Q)_{os}`\overrightarrow{Q}`a \in 1 . \supset .$
$R(PQ)`a = R(\breve{P}Q)`a = \breve{\iota}`(P\bar{R}Q)_{os}`\overrightarrow{Q}`a$

$*233{\cdot}433.\ \vdash :. P \in \text{Ser} . Q_* \upharpoonright \overrightarrow{Q}`a \in \text{connex} . R``\overrightarrow{Q}`a \subset C`P . (P\bar{R}Q)_{os}`\overrightarrow{Q}`a = \Lambda .$
$\text{E} ! \, R(PQ)`a . \text{E} ! \, R(\breve{P}Q)`a . \supset :$
$R(PQ)`a = R(\breve{P}Q)`a . \vee . \{R(PQ)`a\} P_1 \{R(\breve{P}Q)`a\}$

$*233{\cdot}45.\ \vdash :. P \in \text{Ser} \cap \text{Ded} . P^2 = P . Q_* \in \text{connex} . R``C`Q \subset C`P . \supset :$
$R(PQ)`a = R(\breve{P}Q)`a . \equiv_a . (P\bar{R}Q)_{os}`\overrightarrow{Q}`a \in 0 \cup 1$

I.e. in a series having Dedekindian continuity, the necessary and sufficient condition that the two limits of the function as the argument approaches a from below should be equal is that the ultimate oscillation should not have more than one term.

We have next a set of propositions ($*233{\cdot}5$—$\cdot53$) on the possibility of replacing $\overrightarrow{Q}`a$ by a class α having a for its limit, without altering the limits of the function. We have to begin with

$*233{\cdot}5.\ \vdash : Q \in \text{Ser} . a = \text{lt}_Q`(\alpha \cap \text{Ɑ}`R) . \supset . \overrightarrow{Q}`a = Q_*``(\alpha \cap \text{Ɑ}`R)$

in virtue of $*207{\cdot}291$. Thence by earlier propositions of this number,

$*233{\cdot}512.\ \vdash :. \text{Hp}\,*233{\cdot}5 . P \in \text{Ser} . R``(\alpha \cap C`Q) \subset C`P . (P\bar{R}Q)_{os}`\overrightarrow{Q}`a = \iota`x . \supset :$
$x = R(PQ)`a = R(\breve{P}Q)`a : x = (P\bar{R}Q)_{lmx}`\alpha . \vee . (P\bar{R}Q)_{lmx}`\alpha P_1 x$

whence we obtain

$*233{\cdot}514.\ \vdash : \mathrm{Hp}\,*233{\cdot}512\,.\,x \sim\in C`P_1\,.\,\supset\,.\,x = (P\bar{R}Q)_{\mathrm{lmx}}`\alpha = (\breve{P}\bar{R}Q)_{\mathrm{lmx}}`\alpha$

Thus if P, Q are series, and x is the limit of the function for the argument a (x being a term which has no immediate successor or predecessor), x is the limit of the function for any class of arguments whose limit is a. Hence we arrive at the proposition

$*233{\cdot}53.\ \vdash : Q \in \mathrm{Ser}\,.\,P \in \mathrm{Ser} \cap \mathrm{Ded}\,.\,P^2 = P\,.\,R``C`Q \subset C`P\,.\,\alpha \subset \text{ฯ}`R\,.\,\mathrm{E}\,!\,\mathrm{lt}_Q`\alpha\,.$
$(P\bar{R}Q)_{\mathrm{os}}`Q_*``\alpha \in 0 \cup 1\,.\,\supset\,.$
$(P\bar{R}Q)_{\mathrm{lmx}}`\alpha = (\breve{P}\bar{R}Q)_{\mathrm{lmx}}`\alpha = R\,(PQ)`\mathrm{lt}_Q`\alpha = R\,(\breve{P}Q)`\mathrm{lt}_Q`\alpha$

Thus if P has Dedekindian continuity, and α is a class of arguments having a limit, and if the ultimate oscillation as the argument approaches this limit has not more than one term, the limit of the function for the class α exists, and is equal to the limit of the function for the argument $\mathrm{lt}_Q`\alpha$.

$*233{\cdot}01.\ (P\bar{R}Q)_{\mathrm{lmx}} = \mathrm{limax}_P \,|\, (P\bar{R}Q)_{\mathrm{sc}}$ Df

$*233{\cdot}02.\ R\,(PQ) = (P\bar{R}Q)_{\mathrm{lmx}} \,|\, \overrightarrow{Q}$ Df

$*233{\cdot}1.\ \vdash : y\,\{(P\bar{R}Q)_{\mathrm{lmx}}\}\,\alpha\,.\equiv.\,y\,(\mathrm{limax}_P)\,\{(P\bar{R}Q)_{\mathrm{sc}}`\alpha\}$ [(*233·01)]

$*233{\cdot}101.\ \vdash : y = (P\bar{R}Q)_{\mathrm{lmx}}`\alpha\,.\equiv.\,y = \mathrm{limax}_P`(P\bar{R}Q)_{\mathrm{sc}}`\alpha$ [*233·1]

$*233{\cdot}102.\ \vdash : \mathrm{E}\,!\,\mathrm{limax}_P`(P\bar{R}Q)_{\mathrm{sc}}`\alpha\,.\equiv.\,(P\bar{R}Q)_{\mathrm{lmx}}`\alpha = \mathrm{limax}_P`(P\bar{R}Q)_{\mathrm{sc}}`\alpha\,.$
$\equiv.\,\mathrm{E}\,!\,(P\bar{R}Q)_{\mathrm{lmx}}`\alpha$ [*233·101 . *14·28]

$*233{\cdot}103.\ \vdash : P \in \mathrm{connex}\,.\,\supset\,.\,(P\bar{R}Q)_{\mathrm{lmx}} \in 1 \to \mathrm{Cls}$ [*207·41 . *233·1]

$*233{\cdot}11.\ \vdash :.\,P \in \mathrm{Ser}\,.\,\supset : y = (P\bar{R}Q)_{\mathrm{lmx}}`\alpha\,.\equiv.\,y \in C`P\,.\,\overrightarrow{P}`y = P``(P\bar{R}Q)_{\mathrm{sc}}`\alpha$
[*207·51 . *233·101]

$*233{\cdot}111.\ \vdash :.\,P \in \mathrm{Ser}\,.\,\exists\,!\,P``(P\bar{R}Q)_{\mathrm{sc}}`\alpha\,.\,\supset :$
$y = (P\bar{R}Q)_{\mathrm{lmx}}`\alpha\,.\equiv.\,\overrightarrow{P}`y = P``(P\bar{R}Q)_{\mathrm{sc}}`\alpha$ [*207·52 . *233·101]

$*233{\cdot}12.\ \vdash :.\,P \in \mathrm{Ser}\,.\sim\mathrm{E}\,!\,\mathrm{max}_P`(P\bar{R}Q)_{\mathrm{sc}}`\alpha\,.\,\supset :$
$y = (P\bar{R}Q)_{\mathrm{lmx}}`\alpha\,.\equiv.\,y \in C`P\,.\,\overrightarrow{P}`y = (P\bar{R}Q)_{\mathrm{sc}}`\alpha$

Dem.

$\vdash .\,*231{\cdot}13\,.\,*211{\cdot}41\,.\,\supset \vdash : \mathrm{Hp}\,.\,\supset\,.\,(P\bar{R}Q)_{\mathrm{sc}}`\alpha = P``(P\bar{R}Q)_{\mathrm{sc}}`\alpha$ (1)
$\vdash .\,(1)\,.\,*233{\cdot}11\,.\,\supset \vdash .\,\mathrm{Prop}$

$*233{\cdot}13.\ \vdash : P \in \mathrm{connex} \cap \mathrm{Ded}\,.\,\supset\,.$
$\mathrm{E}\,!\,(P\bar{R}Q)_{\mathrm{lmx}}`\alpha\,.\,(P\bar{R}Q)_{\mathrm{lmx}}`\alpha = \mathrm{limax}_P`(P\bar{R}Q)_{\mathrm{sc}}`\alpha$
[*233·102·103 . *214·11]

$*233{\cdot}14.\ \vdash : P \in \mathrm{Ser}\,.\,(P\bar{R}Q)_{\mathrm{os}}`\alpha \in 1\,.\,\supset\,.\,(P\bar{R}Q)_{\mathrm{lmx}}`\alpha = (\breve{P}\bar{R}Q)_{\mathrm{lmx}}`\alpha = \breve{\iota}`(P\bar{R}Q)_{\mathrm{os}}`\alpha$
[*231·193 . *233·102]

***233·141.** $\vdash :. P \in \text{Ser} . (P\overline{R}Q)_{\text{sc}}{}'\alpha \cup (\breve{P}\overline{R}Q)_{\text{sc}}{}'\alpha = C{}'P . (P\overline{R}Q)_{\text{os}}{}'\alpha = \Lambda . \supset :$

$$\text{E}\,!\,(P\overline{R}Q)_{\text{lmx}}{}'\alpha . \equiv . \text{E}\,!\,(\breve{P}\overline{R}Q)_{\text{lmx}}{}'\alpha$$

[*211·727 . *233·102 . *231·13]

***233·142.** $\vdash : P \in \text{Ser} . Q_* \upharpoonright \alpha \in \text{connex} .$

$R{}''(\alpha \cap C{}'Q) \subset C{}'P . (P\overline{R}Q)_{\text{os}}{}'Q_*{}''(\alpha \cap \mathrm{D}{}'R) \in 0 \cup 1 .$

$\text{E}\,!\,(P\overline{R}Q)_{\text{lmx}}{}'Q_*{}''(\alpha \cap \mathrm{D}{}'R) . (P\overline{R}Q)_{\text{lmx}}{}'Q_*{}''(\alpha \cap \mathrm{D}{}'R) \sim\in C{}'P_1 . \supset .$

$$(P\overline{R}Q)_{\text{lmx}}{}'\alpha = (\breve{P}\overline{R}Q)_{\text{lmx}}{}'\alpha = (P\overline{R}Q)_{\text{lmx}}{}'Q_*{}''(\alpha \cap \mathrm{D}{}'R)$$
$$= (\breve{P}\overline{R}Q)_{\text{lmx}}{}'Q_*{}''(\alpha \cap \mathrm{D}{}'R)$$

Dem.

$\vdash . \text{*}231{\cdot}252 . \supset \vdash : \text{Hp} . \supset . (P\overline{R}Q)_{\text{lmx}}{}'Q_*{}''(\alpha \cap \mathrm{D}{}'R) = (\breve{P}\overline{R}Q)_{\text{lmx}}{}'Q_*{}''(\alpha \cap \mathrm{D}{}'R)$ (1)

$\vdash . \text{*}232{\cdot}37 . \text{*}233{\cdot}14 . \supset \vdash : \text{Hp} . (P\overline{R}Q)_{\text{os}}{}'Q_*{}''(\alpha \cap \mathrm{D}{}'R) \in 1 . \supset .$

$$(P\overline{R}Q)_{\text{lmx}}{}'\alpha = (\breve{P}\overline{R}Q)_{\text{lmx}}{}'\alpha = (P\overline{R}Q)_{\text{lmx}}{}'Q_*{}''(\alpha \cap \mathrm{D}{}'R)$$
$$= (\breve{P}\overline{R}Q)_{\text{lmx}}{}'Q_*{}''(\alpha \cap \mathrm{D}{}'R) \quad (2)$$

$\vdash . (1) . \text{*}232{\cdot}34 . \supset \vdash : \text{Hp} . (P\overline{R}Q)_{\text{os}}{}'Q_*{}''(\alpha \cap \mathrm{D}{}'R) = \Lambda . \supset .$

$$(P\overline{R}Q)_{\text{lmx}}{}'\alpha = (P\overline{R}Q)_{\text{lmx}}{}'Q_*{}''(\alpha \cap \mathrm{D}{}'R) = (\breve{P}\overline{R}Q)_{\text{lmx}}{}'Q_*{}''(\alpha \cap \mathrm{D}{}'R)$$
$$= (\breve{P}\overline{R}Q)_{\text{lmx}}{}'\alpha \quad (3)$$

$\vdash . (2) . (3) . \supset \vdash . \text{Prop}$

***233·15.** $\vdash :. P \in \text{Ser} \cap \text{Ded} . (P\overline{R}Q)_{\text{sc}}{}'\alpha \cup (\breve{P}\overline{R}Q)_{\text{sc}}{}'\alpha = C{}'P . (P\overline{R}Q)_{\text{os}}{}'\alpha = \Lambda . \supset :$

$$(P\overline{R}Q)_{\text{lmx}}{}'\alpha = (\breve{P}\overline{R}Q)_{\text{lmx}}{}'\alpha . \vee . \{(P\overline{R}Q)_{\text{lmx}}{}'\alpha\} \, P_1 \, \{(\breve{P}\overline{R}Q)_{\text{lmx}}{}'\alpha\}$$

[*214·43 . *233·13 . *231·13]

***233·16.** $\vdash :. P \in \text{Ser} \cap \text{Ded} . P^2 = P . Q_* \in \text{connex} . R{}''C{}'Q \subset C{}'P . \supset :$

$$(P\overline{R}Q)_{\text{os}}{}'\alpha \in 0 \cup 1 . \supset_\alpha . (P\overline{R}Q)_{\text{lmx}}{}'\alpha = (\breve{P}\overline{R}Q)_{\text{lmx}}{}'\alpha$$

Dem.

$\vdash . \text{*}232{\cdot}22 . \supset \vdash : \text{Hp} . \supset . C{}'P = (P\overline{R}Q)_{\text{sc}}{}'\alpha \cup (\breve{P}\overline{R}Q)_{\text{sc}}{}'\alpha$ (1)

$\vdash . \text{*}201{\cdot}65 . \supset \vdash : \text{Hp} . \supset . P_1 = \dot{\Lambda}$ (2)

$\vdash . (1) . (2) . \text{*}233{\cdot}14{\cdot}15 . \supset \vdash . \text{Prop}$

***233·17.** $\vdash :. \alpha \cap C{}'Q \cap \mathrm{D}{}'R = \Lambda . \supset : y = (P\overline{R}Q)_{\text{lmx}}{}'\alpha . \equiv . y = B{}'\breve{P}$

Dem.

$\vdash . \text{*}232{\cdot}15 . \text{*}233{\cdot}101 . \supset$

$\vdash :. \text{Hp} . \supset : y = (P\overline{R}Q)_{\text{lmx}}{}'\alpha . \equiv . y = \text{limax}_P{}'C{}'P .$

[*206·2 . *93·117] $\equiv . y = B{}'\breve{P} :. \supset \vdash . \text{Prop}$

***233·171.** $\vdash : \alpha \cap C{}'Q \cap \mathrm{D}{}'R = \Lambda . \supset . \sim\{(P\overline{R}Q)_{\text{lmx}}{}'\alpha = (\breve{P}\overline{R}Q)_{\text{lmx}}{}'\alpha\}$

Dem.

$\vdash . \text{*}93{\cdot}102 . \supset \vdash . \sim(B{}'P = B{}'\breve{P})$ (1)

$\vdash . (1) . \text{*}233{\cdot}17 . \supset \vdash . \text{Prop}$

***233·172.** $\vdash : \alpha \cap C'Q \cap \text{Ꭰ}'R = \Lambda . E ! (P\overline{R}Q)_{\text{lmx}}`\alpha . E ! (\breve{P}\overline{R}Q)_{\text{lmx}}`\alpha . \supset .$
$(P\overline{R}Q)_{\text{os}}`\alpha \sim\epsilon 0 \cup 1$

Dem.

$\vdash . *233·171 . *232·15 . \supset$

$\vdash : \text{Hp} . \supset . (P\overline{R}Q)_{\text{lmx}}`\alpha, (\breve{P}\overline{R}Q)_{\text{lmx}}`\alpha \in (P\overline{R}Q)_{\text{os}}`\alpha . (P\overline{R}Q)_{\text{lmx}}`\alpha \neq (\breve{P}\overline{R}Q)_{\text{lmx}}`\alpha .$
$[*52·41] \supset . (P\overline{R}Q)_{\text{os}}`\alpha \sim\epsilon 0 \cup 1 : \supset \vdash . \text{Prop}$

***233·173.** $\vdash : (P\overline{R}Q)_{\text{os}}`\alpha \in 0 \cup 1 . E ! (P\overline{R}Q)_{\text{lmx}}`\alpha . E ! (\breve{P}\overline{R}Q)_{\text{lmx}}`\alpha . \supset .$
$\exists ! \alpha \cap C'Q \cap \text{Ꭰ}'R$ $[*233·172 . \text{Transp}]$

***233·174.** $\vdash : P \mathrel{\text{G}} J . (P\overline{R}Q)_{\text{os}}`\alpha \in 1 . \supset . \exists ! \alpha \cap C'Q \cap \text{Ꭰ}'R$

Dem.

$\vdash . *200·12 . \supset \vdash : \text{Hp} . \supset . \sim\{C'P \subset (P\overline{R}Q)_{\text{os}}`\alpha\} .$
$[*232·15] \quad \supset . \exists ! \alpha \cap C'Q \cap \text{Ꭰ}'R : \supset \vdash . \text{Prop}$

***233·2.** $\vdash : Q_* \upharpoonright \alpha \in \text{connex} . P \in \text{Ser} . R``(\alpha \cap C'Q) \subset C'P .$
$(P\overline{R}Q)_{\text{os}}`Q_*``(\alpha \cap \text{Ꭰ}'R) = \Lambda . E ! (P\overline{R}Q)_{\text{lmx}}`Q_*``(\alpha \cap \text{Ꭰ}'R) . \supset .$
$(P\overline{R}Q)_{\text{lmx}}`\alpha = (P\overline{R}Q)_{\text{lmx}}`Q_*``(\alpha \cap \text{Ꭰ}'R) . (\breve{P}\overline{R}Q)_{\text{lmx}}`\alpha = (\breve{P}\overline{R}Q)_{\text{lmx}}`Q_*``(\alpha \cap \text{Ꭰ}'R)$
$[*232·34 . *211·727 . *233·102]$

***233·21.** $\vdash : P_{\text{po}} \in \text{Ser} . \exists ! (P\overline{R}Q)_{\text{os}}`\alpha . (P\overline{R}Q)_{\text{os}}`Q_*``(\alpha \cap \text{Ꭰ}'R) \in 1 . \supset .$
$(P\overline{R}Q)_{\text{lmx}}`\alpha = (\breve{P}\overline{R}Q)_{\text{lmx}}`\alpha = (P\overline{R}Q)_{\text{lmx}}`Q_*``(\alpha \cap \text{Ꭰ}'R)$
$= (\breve{P}\overline{R}Q)_{\text{lmx}}`Q_*``(\alpha \cap \text{Ꭰ}'R) = \iota`(P\overline{R}Q)_{\text{os}}`Q_*``(\alpha \cap \text{Ꭰ}'R)$
$[*232·341 . *231·193]$

***233·22.** $\vdash :. P \in \text{Ser} . (P\overline{R}Q)_{\text{os}}`Q_*``(\alpha \cap \text{Ꭰ}'R) = \iota`x .$
$(P\overline{R}Q)_{\text{sc}}`\alpha \cup (\breve{P}\overline{R}Q)_{\text{sc}}`\alpha = C'P . \supset :$
$x = (P\overline{R}Q)_{\text{lmx}}`\alpha . \vee . (P\overline{R}Q)_{\text{lmx}}`\alpha P_1 x . (P\overline{R}Q)_{\text{lmx}}`\alpha = \max_P`(P\overline{R}Q)_{\text{sc}}`\alpha$

Dem.

$\vdash . *232·352 . \supset \vdash : \text{Hp} . x \in (P\overline{R}Q)_{\text{sc}}`\alpha . \supset . x = (P\overline{R}Q)_{\text{lmx}}`\alpha \quad (1)$

$\vdash . *232·356 . \supset \vdash : \text{Hp} . x \sim\epsilon (P\overline{R}Q)_{\text{sc}}`\alpha . \sim E ! \max_P`(P\overline{R}Q)_{\text{sc}}`\alpha . \supset .$
$x = (P\overline{R}Q)_{\text{lmx}}`\alpha \quad (2)$

$\vdash . *232·358 . *207·42 . \supset \vdash : \text{Hp} . x \sim\epsilon (P\overline{R}Q)_{\text{sc}}`\alpha . E ! \max_P`(P\overline{R}Q)_{\text{sc}}`\alpha . \supset .$
$\max_P`(P\overline{R}Q)`\alpha P_1 x . (P\overline{R}Q)_{\text{lmx}}`\alpha = \max_P`(P\overline{R}Q)_{\text{sc}}`\alpha \quad (3)$

$\vdash . (1) . (2) . (3) . \supset \vdash . \text{Prop}$

***233·23.** $\vdash : \text{Hp} *233·22 . \supset .$
$x = (P\overline{R}Q)_{\text{lmx}}`Q_*``(\alpha \cap \text{Ꭰ}'R) = (\breve{P}\overline{R}Q)_{\text{lmx}}`Q_*``(\alpha \cap \text{Ꭰ}'R)$ $[*231·193]$

***233·24.** $\vdash : \text{Hp} *233·22 . x \sim\epsilon \text{Ꭰ}'P_1 . \supset . x = (P\overline{R}Q)_{\text{lmx}}`\alpha$ $[*233·22]$

***233·241.** $\vdash : \text{Hp} *233·22 . x \sim\epsilon C'P_1 . \supset . x = (P\overline{R}Q)_{\text{lmx}}`\alpha = (\breve{P}\overline{R}Q)_{\text{lmx}}`\alpha$
$\left[*233·24 \dfrac{\breve{P}}{P} . *233·24\right]$

***233·25.** $\vdash :. P \epsilon \mathrm{Ser} \cap \mathrm{Ded} . P^2 = P . Q_* \epsilon \mathrm{connex} . R``C`Q \subset C`P . \supset :$
$(P\bar{R}Q)_{\mathrm{os}}`Q_*``(\alpha \cap \text{ɑ}`R) \epsilon 0 \cup 1 . \supset .$

$(P\bar{R}Q)_{\mathrm{lmx}}`\alpha = (P\bar{R}Q)_{\mathrm{lmx}}`Q_*``(\alpha \cap \text{ɑ}`R) = (\breve{P}\bar{R}Q)_{\mathrm{lmx}}`\alpha = (\breve{P}\bar{R}Q)_{\mathrm{lmx}}`Q_*``(\alpha \cap \text{ɑ}`R)$
[*232·39]

***233·4.** $\vdash : y \{R(PQ)\} a . \equiv . y \{(P\bar{R}Q)_{\mathrm{lmx}}\} \overrightarrow{Q}`a$ [(*233·02)]

***233·401.** $\vdash : y = R(PQ)`a . \equiv . y = (P\bar{R}Q)_{\mathrm{lmx}}`\overrightarrow{Q}`a$ [*233·4]

***233·402.** $\vdash : P \epsilon \mathrm{connex} . \supset . R(PQ) \epsilon 1 \rightarrow \mathrm{Cls}$ [*207·41]

***233·41.** $\vdash : y = R(PQ)`a . \equiv . y = (P\bar{R}Q)_{\mathrm{lmx}}`(\overrightarrow{Q}`a \cap \text{ɑ}`R)$

Dem.

$\vdash . *232·13 . \supset \vdash . (P\bar{R}Q)_{\mathrm{sc}}`\overrightarrow{Q}`a = (P\bar{R}Q)_{\mathrm{sc}}`(\overrightarrow{Q}`a \cap \text{ɑ}`R)$ (1)
$\vdash . (1) . *233·401·101 . \supset \vdash . \mathrm{Prop}$

***233·42.** $\vdash :. Q \epsilon \mathrm{trans} \cap \mathrm{connex} . E ! \max_Q`(\overrightarrow{Q}`a \cap \text{ɑ}`R) . \supset :$
$y = R(PQ)`a . \equiv . y = \mathrm{limax}_P`P_*``\overrightarrow{R}`\max_Q`(\overrightarrow{Q}`a \cap \text{ɑ}`R)$
[*232·24 . *233·401·101]

***233·421.** $\vdash : P \epsilon \mathrm{Rl}`J \cap \mathrm{trans} . Q \epsilon \mathrm{trans} \cap \mathrm{connex} . R`\max_Q`(\alpha \cap \text{ɑ}`R) \epsilon C`P . \supset .$
$R(PQ)`a = R`\max_Q`(\overrightarrow{Q}`a \cap \text{ɑ}`R)$

Dem.

$\vdash . *233·42 . \supset \vdash :. \mathrm{Hp} . \supset : y = R(PQ)`a . \equiv . y = \mathrm{limax}_P`\overrightarrow{P_*}`R`\max_Q`(\overrightarrow{Q}`a \cap \text{ɑ}`R).$
[*205·197] $\equiv . y = R`\max_Q`(\overrightarrow{Q}`a \cap \text{ɑ}`R) :. \supset \vdash . \mathrm{Prop}$

***233·422.** $\vdash :. \overrightarrow{Q}`a \cap \text{ɑ}`R = \Lambda . \supset : y = R(PQ)`a . \equiv . y = B`\breve{P}$ [*233·17]

***233·423.** $\vdash : \overrightarrow{Q}`a \cap \text{ɑ}`R = \Lambda . \supset . \sim \{R(PQ)`a = R(\breve{P}Q)`a\}$ [*233·171]

***233·424.** $\vdash : \overrightarrow{Q}`a \cap \text{ɑ}`R = \Lambda . E ! R(PQ)`a . E ! R(\breve{P}Q)`a . \supset . (P\bar{R}Q)_{\mathrm{os}}`\overrightarrow{Q}`a \sim \epsilon 0 \cup 1$
[*233·172]

***233·425.** $\vdash : (P\bar{R}Q)_{\mathrm{os}}`\overrightarrow{Q}`a \epsilon 0 \cup 1 . E ! R(PQ)`a . E ! R(\breve{P}Q)`a . \supset . \exists ! \overrightarrow{Q}`a \cap \text{ɑ}`R$
[*233·424 . Transp]

***233·426.** $\vdash : P \mathrel{\mathsf{G}} J . (P\bar{R}Q)_{\mathrm{os}}`\overrightarrow{Q}`a \epsilon 1 . \supset . \exists ! \overrightarrow{Q}`a \cap \text{ɑ}`R$ [*233·174]

***233·43.** $\vdash : P_{\mathrm{po}} \epsilon \mathrm{Ser} . (P\bar{R}Q)_{\mathrm{os}}`\overrightarrow{Q}`a \epsilon 1 . \supset .$
$R(PQ)`a = R(\breve{P}Q)`a = \breve{\iota}`(P\bar{R}Q)_{\mathrm{os}}`\overrightarrow{Q}`a$ [*231·193]

***233·431.** $\vdash : P \epsilon \mathrm{trans} \cap \mathrm{connex} . (P\bar{R}Q)_{\mathrm{os}}`\overrightarrow{Q}`a \sim \epsilon 0 \cup 1 .$
$E ! R(PQ)`a . E ! R(\breve{P}Q)`a . \supset . \{R(\breve{P}Q)`a\} P \{R(PQ)`a\}$
[*215·52 . *231·13·101]

***233·432.** $\vdash : P \epsilon \mathrm{trans} \cap \mathrm{connex} . (P\bar{R}Q)_{\mathrm{os}}`\overrightarrow{Q}`a = \Lambda .$
$E ! R(PQ)`a . E ! R(\breve{P}Q)`a . \supset . \{R(PQ)`a\} P_* \{R(\breve{P}Q)`a\}$ [*215·53]

***233·433.** $\vdash :. P \in \mathrm{Ser} . Q_* \upharpoonright \overrightarrow{Q}\text{‘}a \in \mathrm{connex} . R\text{“}\overrightarrow{Q}\text{‘}a \subset C\text{‘}P . (P\bar{R}Q)_{\mathrm{os}}\text{‘}\overrightarrow{Q}\text{‘}a = \Lambda .$
$\mathrm{E}!R(PQ)\text{‘}a . \mathrm{E}!R(\breve{P}Q)\text{‘}a . \supset : R(PQ)\text{‘}a = R(\breve{P}Q)\text{‘}a . \vee . \{R(PQ)\text{‘}a\} P_1 \{R(\breve{P}Q)\text{‘}a\}$
[*215·54 . *232·22]

***233·434.** $\vdash : P \in \mathrm{Ser} . Q_* \upharpoonright \overrightarrow{Q}\text{‘}a \in \mathrm{connex} . R\text{“}\overrightarrow{Q}\text{‘}a \subset C\text{‘}P . \mathrm{E}!R(PQ)\text{‘}a .$
$\mathrm{E}!R(\breve{P}Q)\text{‘}a . \supset . \{R(PQ)\text{‘}a\}(P_1 \cup \breve{P}_*)\{R(\breve{P}Q)\text{‘}a\}$ [*233·43·431·433]

***233·435.** $\vdash : P \in \mathrm{Ser} . R(PQ)\text{‘}a = R(\breve{P}Q)\text{‘}a . \supset . (P\bar{R}Q)_{\mathrm{os}}\text{‘}\overrightarrow{Q}\text{‘}a \in 0 \cup 1$
[*233·431 . Transp]

***233·44.** $\vdash :. P \in \mathrm{Ser} . Q_* \upharpoonright \overrightarrow{Q}\text{‘}a \in \mathrm{connex} . R\text{“}\overrightarrow{Q}\text{‘}a \subset C\text{‘}P . \mathrm{E}!R(PQ)\text{‘}a .$
$\mathrm{E}!R(\breve{P}Q)\text{‘}a . \sim\{R(PQ)\text{‘}a \in \mathrm{D}\text{‘}P_1 . R(\breve{P}Q)\text{‘}a \in \text{ᗡ}\text{‘}P_1\} . \supset :$
$R(PQ)\text{‘}a = R(\breve{P}Q)\text{‘}a . \equiv . (P\bar{R}Q)_{\mathrm{os}}\text{‘}\overrightarrow{Q}\text{‘}a \in 0 \cup 1$ [*233·426·43·433·435]

***233·45.** $\vdash :. P \in \mathrm{Ser} \cap \mathrm{Ded} . P^2 = P . Q_* \in \mathrm{connex} . R\text{“}C\text{‘}Q \subset C\text{‘}P . \supset :$
$R(PQ)\text{‘}a = R(\breve{P}Q)\text{‘}a . \equiv_a . (P\bar{R}Q)_{\mathrm{os}}\text{‘}\overrightarrow{Q}\text{‘}a \in 0 \cup 1$
[*233·13 . *201·65 . *233·44]

***233·5.** $\vdash : Q \in \mathrm{Ser} . a = \mathrm{lt}_Q\text{‘}(\alpha \cap \text{ᗡ}\text{‘}R) . \supset . \overrightarrow{Q}\text{‘}a = Q_*\text{“}(\alpha \cap \text{ᗡ}\text{‘}R)$ [*207·291]

***233·501.** $\vdash :. Q \in \mathrm{Ser} . a = \mathrm{lt}_Q\text{‘}(\alpha \cap \text{ᗡ}\text{‘}R) . \supset : \exists ! \overrightarrow{Q}\text{‘}a \cap \text{ᗡ}\text{‘}R . \equiv . \exists ! \alpha \cap C\text{‘}Q \cap \text{ᗡ}\text{‘}R$

Dem.

$\vdash . *233·5 . \supset \vdash :. \mathrm{Hp} . \supset : \exists ! \overrightarrow{Q}\text{‘}a \cap \text{ᗡ}\text{‘}R . \equiv . \exists ! Q_*\text{“}(\alpha \cap \text{ᗡ}\text{‘}R) \cap \text{ᗡ}\text{‘}R .$ (1)
[*37·29·265] $\supset . \exists ! \alpha \cap \text{ᗡ}\text{‘}R \cap C\text{‘}Q$ (2)
$\vdash . *90·33 . *22·43 . \supset \vdash : x \in \alpha \cap C\text{‘}Q \cap \text{ᗡ}\text{‘}R . \supset . x \in Q_*\text{“}(\alpha \cap \text{ᗡ}\text{‘}R) . x \in \text{ᗡ}\text{‘}R$ (3)
$\vdash . (3) . *10·28 . \supset \vdash : \exists ! \alpha \cap C\text{‘}Q \cap \text{ᗡ}\text{‘}R . \supset . \exists ! Q_*\text{“}(\alpha \cap \text{ᗡ}\text{‘}R) \cap \text{ᗡ}\text{‘}R$ (4)
$\vdash . (1) . (2) . (4) . \supset \vdash . \mathrm{Prop}$

***233·51.** $\vdash : \mathrm{Hp}\, *233·5 . P \in \mathrm{Ser} . R\text{“}(\alpha \cap C\text{‘}Q) \subset C\text{‘}P . (P\bar{R}Q)_{\mathrm{os}}\text{‘}\overrightarrow{Q}\text{‘}a = \Lambda .$
$\mathrm{E}!R(PQ)\text{‘}a . \supset . (P\bar{R}Q)_{\mathrm{lmx}}\text{‘}\alpha = R(PQ)\text{‘}a$ [*233·2·5]

***233·511.** $\vdash : \mathrm{Hp}\, *233·5 . P \in \mathrm{Ser} . \exists ! (P\bar{R}Q)_{\mathrm{os}}\text{‘}\alpha . (P\bar{R}Q)_{\mathrm{os}}\text{‘}\overrightarrow{Q}\text{‘}a \in 1 . \supset .$
$(P\bar{R}Q)_{\mathrm{lmx}}\text{‘}\alpha = (\breve{P}\bar{R}Q)_{\mathrm{lmx}}\text{‘}\alpha = R(PQ)\text{‘}a = R(\breve{P}Q)\text{‘}a = \breve{\iota}\text{‘}(P\bar{R}Q)_{\mathrm{os}}\text{‘}\overrightarrow{Q}\text{‘}a$
[*233·501·5·21]

***233·512.** $\vdash :. \mathrm{Hp}\, *233·5 . P \in \mathrm{Ser} . R\text{“}(\alpha \cap C\text{‘}Q) \subset C\text{‘}P . (P\bar{R}Q)_{\mathrm{os}}\text{‘}\overrightarrow{Q}\text{‘}a = \iota\text{‘}x . \supset :$
$x = R(PQ)\text{‘}a = R(\breve{P}Q)\text{‘}a : x = (P\bar{R}Q)_{\mathrm{lmx}}\text{‘}\alpha . \vee . (P\bar{R}Q)_{\mathrm{lmx}}\text{‘}\alpha\, P_1 x$
[*233·22·23 . *232·22]

***233·513.** $\vdash : \mathrm{Hp}\, *233·512 . x \sim\in \text{ᗡ}\text{‘}P_1 . \supset . x = (P\bar{R}Q)_{\mathrm{lmx}}\text{‘}\alpha$ [*233·512]

***233·514.** $\vdash : \mathrm{Hp}\, *233·512 . x \sim\in C\text{‘}P_1 . \supset . x = (P\bar{R}Q)_{\mathrm{lmx}}\text{‘}\alpha = (\breve{P}\bar{R}Q)_{\mathrm{lmx}}\text{‘}\alpha$
$\left[*233·513 \dfrac{\breve{P}}{P} . *233·513\right]$

$*233\cdot515$. $\vdash : P, Q \in \mathrm{Ser} \,.\, a = \mathrm{lt}_Q\text{‘}(\alpha \cap \mathrm{Cl}\text{‘}R) \,.\, R\text{“}(C\text{‘}Q \cap \alpha) \subset C\text{‘}P \,.$
$(P\overline{R}Q)_{\mathrm{os}}\text{‘}\overrightarrow{Q}\text{‘}a \in 0 \cup 1 \,.\, \mathrm{E}\,!\, R\,(PQ)\text{‘}a \,.\, R\,(PQ)\text{‘}a \sim\in C\text{‘}P_1 \,.\supset .$
$(P\overline{R}Q)_{\mathrm{lmx}}\text{‘}\alpha = (\breve{P}\overline{R}Q)_{\mathrm{lmx}}\text{‘}\alpha = R\,(PQ)\text{‘}a = R\,(\breve{P}Q)\text{‘}a$
[$*233\cdot142\cdot5$]

$*233\cdot516$. $\vdash : P, Q \in \mathrm{Ser} \,.\, \mathrm{E}\,!\, R\,(PQ)\text{‘}\mathrm{lt}_Q\text{‘}\alpha \,.\, R\,(PQ)\text{‘}\mathrm{lt}_Q\text{‘}\alpha \sim\in C\text{‘}P_1 \,.$
$R\text{“}(C\text{‘}Q \cap \alpha) \subset C\text{‘}P \,.\, (P\overline{R}Q)_{\mathrm{os}}\text{‘}\overrightarrow{Q}\text{‘}\mathrm{lt}_Q\text{‘}\alpha \in 0 \cup 1 \,.\supset .$
$(P\overline{R}Q)_{\mathrm{lmx}}\text{‘}\alpha = (\breve{P}\overline{R}Q)_{\mathrm{lmx}}\text{‘}\alpha = R\,(PQ)\text{‘}\mathrm{lt}_Q\text{‘}\alpha = R\,(\breve{P}Q)\text{‘}\mathrm{lt}_Q\text{‘}\alpha$
[$*233\cdot515$]

$*233\cdot52$. $\vdash :. \mathrm{Hp}\, *233\cdot5 \,.\, P \in \mathrm{Ser} \cap \mathrm{Ded} \,.\, P^2 = P \,.\, R\text{“}C\text{‘}Q \subset C\text{‘}P \,.\supset :$
$(P\overline{R}Q)_{\mathrm{os}}\text{‘}\overrightarrow{Q}\text{‘}a \in 0 \cup 1 \,.\supset .$
$(P\overline{R}Q)_{\mathrm{lmx}}\text{‘}\alpha = (\breve{P}\overline{R}Q)_{\mathrm{lmx}}\text{‘}\alpha = R\,(PQ)\text{‘}a = R\,(\breve{P}Q)\text{‘}a$ [$*233\cdot25$]

$*233\cdot53$. $\vdash : Q \in \mathrm{Ser} \,.\, P \in \mathrm{Ser} \cap \mathrm{Ded} \,.\, P^2 = P \,.\, R\text{“}C\text{‘}Q \subset C\text{‘}P \,.\, \alpha \subset \mathrm{Cl}\text{‘}R \,.\, \mathrm{E}\,!\, \mathrm{lt}_Q\text{‘}\alpha \,.$
$(P\overline{R}Q)_{\mathrm{os}}\text{‘}Q_*\text{“}\alpha \in 0 \cup 1 \,.\supset .$
$(P\overline{R}Q)_{\mathrm{lmx}}\text{‘}\alpha = (\breve{P}\overline{R}Q)_{\mathrm{lmx}}\text{‘}\alpha = R\,(PQ)\text{‘}\mathrm{lt}_Q\text{‘}\alpha = R\,(\breve{P}Q)\text{‘}\mathrm{lt}_Q\text{‘}\alpha$
[$*233\cdot52$]

$*234$. CONTINUITY OF FUNCTIONS

Summary of $*234$.

In the present number we are concerned with the definition and analysis of the continuity of functions. The following definition of continuity is given by Dini*:

"We call it [the function] *continuous* for $x=a$, or in the point a, in which it has the value $f(a)$, if, for every positive number σ, different from 0 but as small as we please, there exists a positive number ϵ, different from 0, such that, for all values of δ which are numerically less than ϵ, the difference $f(a+\delta)-f(a)$ is numerically less than σ. In other words, $f(x)$ is continuous in the point $x=a$, where it has the value $f(a)$, if the limit of its values to the right and left of a is the same and equal to $f(a)$...."

By the second form of the above definition, the function R of previous numbers is to be called continuous at the point a if

$$R(PQ)\text{‘}a = R(\breve{P}Q)\text{‘}a = R(P\breve{Q})\text{‘}a = R(\breve{P}\breve{Q})\text{‘}a = R\text{‘}a.$$

The first form of the definition can also be so stated as to be free from any reference to number, and derivable from the ideas dealt with in the previous numbers of the present section. For this purpose, instead of "a positive number σ," we take an interval in which $R\text{‘}a$ is contained, say $P(z-w)$. Similarly the "values of δ which are numerically less than ϵ" are replaced by arguments in a certain interval containing a.

By $*233{\cdot}423$, if the limits of the function as the argument approaches a are to be all equal, a must not be the maximum or minimum of $\text{Ɑ}\text{‘}R$. We therefore take the interval containing a to be an interval in which the end-points are not included, say $Q(y-y')$. Thus our definition becomes

(A) $R\text{‘}a \in P(z-w) \,.\, \supset_{z,w} .$

$$(\exists y, y') \,.\, y, y' \in \text{Ɑ}\text{‘}R \,.\, a \in Q(y-y') \,.\, R\text{‘‘}Q(y \vdash\!\dashv y') \subset P(z-w)$$

We require further, what is tacitly assumed in Dini's definition, that $R\text{‘}a$ is a member of $C\text{‘}P$ which has no immediate predecessor or successor, *i.e.*

$$R\text{‘}a \in C\text{‘}P - C\text{‘}P_1.$$

In order to deal more easily with the above definition, we analyse it into the product of four factors, which concern respectively P and Q, $\breve{P}$ and Q, P and $\breve{Q}$, $\breve{P}$ and $\breve{Q}$. In the first place, it is obvious that (A) is the product of

(B) $R\text{‘}a \in P(z-w) \,.\, \supset_{z,w} \,.\, (\exists y) \,.\, y \in \text{Ɑ}\text{‘}R \,.\, y \in \overrightarrow{\breve{Q}}\text{‘}a \,.\, R\text{‘‘}Q(y \vdash a) \subset P(z-w)$

* *Theorie der Functionen einer veränderlichen reellen Grösse*, Chap. IV. § 30, p. 50.

and a factor obtained by substituting $\breve{Q}$ for Q in (B). If $Q_* \epsilon \text{connex}$, and $P_{po} \epsilon \text{Ser}$, (B) is the product of

(C) $R'a \,\epsilon\, \overrightarrow{P_{po}}'w \,.\, \supset_w \,.\, (\exists y) \,.\, y \,\epsilon\, \text{G}'R \,.\, y \,\epsilon\, \overrightarrow{Q}'a \,.\, R''Q\,(y \vdash a) \subset \overrightarrow{P_*}'w$

and a factor obtained by writing $\breve{P}$ for P and z for w in (C); and in virtue of $R'a \sim\epsilon\, C'P_1$, (C) becomes

$$R'a \,\epsilon\, \overrightarrow{P_{po}}'w \,.\, \supset_w \,.\, (\exists y) \,.\, y \,\epsilon\, \text{G}'R \,.\, y \,\epsilon\, \overrightarrow{Q}'a \,.\, R''Q\,(y \vdash a) \subset \overrightarrow{P_{po}}'w,$$

i.e. if Q is transitive,

(D) $R'a \,\epsilon\, \overrightarrow{P_{po}}'w \,.\, \supset_w \,.\, R\,(Q_* \upharpoonright \overrightarrow{Q}'a)_{cn}\,(\overrightarrow{P_{po}}'w)$

Hence the function is continuous for the argument a if a satisfies (D) and the three other hypotheses resulting from replacing P by $\breve{P}$, or Q by $\breve{Q}$, or P and Q by $\breve{P}$ and $\breve{Q}$. If we substitute x for $R'a$, and Q for $Q_* \upharpoonright \overrightarrow{Q}'a$, (D) becomes

(E) $\overrightarrow{P_{po}}''\overleftarrow{P_{po}}'x \subset \overleftarrow{Q_{cn}}'R$

Hence continuity can be studied by studying the hypothesis (E), and replacing x by $R'a$ and Q by $Q_* \upharpoonright \overrightarrow{Q}'a$.

The hypothesis (E) is an interesting one on its own account. We put

$$\text{sc}\,(P, Q)'R = C'P \cap \hat{x}\,(\overrightarrow{P_{po}}''\overleftarrow{P_{po}}'x \subset \overleftarrow{Q_{cn}}'R) \quad \text{Df.}$$

Thus "$x \,\epsilon\, \text{sc}\,(P, Q)'R$" means that x is a member of the value-series such that, if y is any later member, the function ultimately becomes less than y. If we put further

$$\text{os}\,(P, Q)'R = \text{sc}\,(P, Q)'R \cap \text{sc}\,(\breve{P}, Q)'R \quad \text{Df,}$$

then, if x is a member of $\text{os}\,(P, Q)'R$, the function ultimately becomes less than any later member of $C'P$, and greater than any earlier member. Hence x is the limit of the function as the argument increases indefinitely. Hence, if we substitute $Q_* \upharpoonright \overrightarrow{Q}'a$ for Q, and if $x \,\epsilon\, \text{os}\,(P, Q_* \upharpoonright \overrightarrow{Q}'a)'R$, x is the limit of the function as the argument approaches a from below, *i.e.*

$$R\,(PQ)'a = R\,(\breve{P}Q)'a = x.$$

(This is proved in $*234{\cdot}462$.) Hence, putting $R'a$ in place of x, the function is continuous from below at the point a if

$$R'a \,\epsilon\, \text{os}\,(P, Q_* \upharpoonright \overrightarrow{Q}'a)'R,$$

and is continuous from above if

$$R'a \,\epsilon\, \text{os}\,(P, \breve{Q}_* \upharpoonright \overleftarrow{Q}'a)'R.$$

These results, and various others connected with them, are proved below. The equivalence of Dini's two definitions is proved in $*234{\cdot}63$. It will be observed that practically nothing in the theory of continuous functions requires the use of numbers.

We use the symbol "$\text{ct}(PQ)`R$" for the class of arguments $a$ for which the limit of the function for approaches to $a$ from below is $R`a$. Thus, in virtue of what was said above, we may put

$$\text{ct}(PQ)`R = \hat{a}\{R`a \in \text{os}(P, Q_* \upharpoonright \overrightarrow{Q_{\text{po}}}`a)`R - C`P_1\} \quad \text{Df.}$$

Then a function is continuous at the point a if a belongs to the two classes $\text{ct}(PQ)`R$ and $\text{ct}(P\breve{Q})`R$. Hence we put

$$\text{contin}(PQ)`R = \text{ct}(PQ)`R \cap \text{ct}(P\breve{Q})`R \quad \text{Df.}$$

The function R is continuous with respect to P and Q if it is continuous for all arguments in $C`Q$. Thus we put

$$P\,\overline{\text{contin}}\,Q = \hat{R}\{\exists ! C`Q \cap \text{Ꭰ}`R \,.\, C`Q \cap \text{Ꭰ}`R \subset \text{contin}(PQ)`R\} \quad \text{Df.}$$

Our propositions in this number begin with the properties of $\text{sc}(P, Q)`R$ and $\text{os}(P, Q)`R$. We have

***234·103.** $\vdash : P_{\text{po}} \in \text{Ser} \,.\, \exists ! \text{os}(P, Q)`R \,.\, \supset . \, P\overline{R}_{\text{os}}Q \in 0 \cup 1$

Thus the hypothesis $\exists ! \text{os}(P, Q)`R$ enables us to use propositions of previous numbers having the hypothesis $P\overline{R}_{\text{os}}Q \in 0 \cup 1$.

The identification of our definitions with the usual definitions of continuity of functions proceeds by means of the proposition

***234·12.** $\vdash :: Q_* \in \text{connex} \,.\, \supset :. \, x \in \text{os}(P, Q)`R \cap \text{D}`P \cap \text{Ꭰ}`P \,.\, \equiv :$
$x \in \text{D}`P \cap \text{Ꭰ}`P : x \in P(z - w) \,.\, \supset_{z, w} . \, RQ_{\text{cn}}\{P(z - w)\}$

We have a collection of propositions dealing with the relations of $\text{sc}(P, Q)`R$ to $P\overline{R}_{\text{sc}}Q$ and $\breve{P}\overline{R}_{\text{sc}}Q$. $\text{sc}(P, Q)`R$ is an upper section of P (*234·131); $\text{sc}(P, Q)`R$ is the complement of $P``(P\overline{R}_{\text{sc}}Q)$, *i.e.* of $P\overline{R}_{\text{sc}}Q$ without its maximum (if any). This is expressed in the following proposition:

***234·174.** $\vdash : P_{\text{po}} \in \text{Ser} \,.\, Q_* \in \text{connex} \,.\, R``C`Q \subset C`P \,.\, \supset .$
$C`P \cap p`\overrightarrow{P_{\text{po}}}``\text{sc}(P, Q)`R = P``(P\overline{R}_{\text{sc}}Q) = C`P - \text{sc}(P, Q)`R$

We thus arrive at

***234·182.** $\vdash : P \in \text{Ser} \,.\, Q_* \in \text{connex} \,.\, R``C`Q \subset C`P \,.\, \supset .$
$\overrightarrow{\text{limax}}_P`(P\overline{R}_{\text{sc}}Q) = \overrightarrow{\text{min}}_P`\text{sc}(P, Q)`R$

Thus $\text{os}(P, Q)`R$ is contained in $\overrightarrow{\text{max}}_P`(P\overline{R}_{\text{sc}}Q) \cup \overrightarrow{\text{min}}_P`(\breve{P}\overline{R}_{\text{sc}}Q)$ (*234·201), and therefore has not more than two terms (*234·202). If $P\overline{R}_{\text{os}}Q$ has one term, this is the only member of $\text{os}(P, Q)`R$ (*234·203). If $\text{os}(P, Q)`R$ has two terms, they have the relation $P_1$ (*234·242); hence if $P$ is a compact series, and $\text{os}(P, Q)`R$ is not null, its only member is both $\text{limax}_P`(P\overline{R}_{\text{sc}}Q)$ and $\text{limin}_P`(\breve{P}\overline{R}_{\text{sc}}Q)$ (*234·25), while conversely, if $\text{limax}_P`(P\overline{R}_{\text{sc}}Q)$ and $\text{limin}_P`(\breve{P}\overline{R}_{\text{sc}}Q)$ are equal, each is the only member of $\text{os}(P, Q)`R$ (*234·251).

We now apply the above results to the limits of a function as its argument approaches the limit of a class α. This is done, as before, by substituting $Q_* \upharpoonright \alpha$ for Q. We arrive at the proposition (*234·33) that if P has Dedekindian continuity, and $\mathrm{os}(P, Q_* \upharpoonright \alpha)\text{‘}R$ is not null, its only member is both $(P\bar{R}Q)_{\mathrm{lmx}}\text{‘}\alpha$ and $(\breve{P}\bar{R}Q)_{\mathrm{lmx}}\text{‘}\alpha$, *i.e.* is the limit of the function as the argument increases in α.

We then take for α the particular value $\overrightarrow{Q_{\mathrm{po}}}\text{‘}a$, so that we become concerned with what happens when the argument approaches a from below. For the comparison of our definition of continuity with such definitions as the one quoted from Dini above, we have

***234·41.** $\vdash :: Q \in \mathrm{trans} \,.\, Q_* \upharpoonright \overrightarrow{Q}\text{‘}a \in \mathrm{connex} \,.\supset :.$
$$x \in \mathrm{os}(P, Q_* \upharpoonright \overrightarrow{Q}\text{‘}a)\text{‘}R \cap \mathrm{D}\text{‘}P \cap \mathrm{Cl}\text{‘}P \,.\equiv:$$
$$x \in \mathrm{D}\text{‘}P \cap \mathrm{Cl}\text{‘}P : x \in P(z - w) \,.\supset_{z,w} .$$
$$(\exists y) \,.\, y \in \overrightarrow{Q}\text{‘}a \cap \mathrm{Cl}\text{‘}R \,.\, R\text{‘‘}Q(y \mapsto a) \subset P(z-w)$$

I.e. if x is neither the first nor the last member of the P-series, x belongs to $\mathrm{os}(P, Q_* \upharpoonright \overrightarrow{Q}\text{‘}a)\text{‘}R$ when, and only when, given any interval $P(z-w)$, however small, in which x is contained, there is an argument y earlier than a, such that the value of the function for all arguments earlier than a but not earlier than y lies in the interval $P(z-w)$.

We deduce from previous propositions that, with the usual hypothesis as to Q, if P is a Dedekindian series,
$$R(PQ)\text{‘}a = \mathrm{limin}_P\text{‘}\mathrm{sc}(P, Q_* \upharpoonright \overrightarrow{Q}\text{‘}a)\text{‘}R \quad (*234{\cdot}422),$$
and if P is a series and $\mathrm{os}(P, Q_* \upharpoonright \overrightarrow{Q}\text{‘}a)$ is a unit class, its only member is both $R(PQ)\text{‘}a$ and $R(\breve{P}Q)\text{‘}a$, *i.e.* is the limit of the function for approaches to a from below (*234·43). The following proposition sums up our results:

***234·45.** $\vdash :. P \in \mathrm{Ser} \,.\, Q \in \mathrm{trans} \,.\, Q_* \upharpoonright \overrightarrow{Q}\text{‘}a \in \mathrm{connex} \,.\, R\text{‘‘}\overrightarrow{Q}\text{‘}a \subset C\text{‘}P \,.\, P^2 = P \,.\supset:$
$$\exists !\, \mathrm{os}(P, Q_* \upharpoonright \overrightarrow{Q}\text{‘}a)\text{‘}R \,.\equiv.\, \mathrm{os}(P, Q_* \upharpoonright \overrightarrow{Q}\text{‘}a)\text{‘}R = \iota\text{‘}R(PQ)\text{‘}a \,.$$
$$\equiv.\, \mathrm{os}(P, Q_* \upharpoonright \overrightarrow{Q}\text{‘}a)\text{‘}R = \iota\text{‘}R(\breve{P}Q)\text{‘}a \,.$$
$$\equiv.\, R(PQ)\text{‘}a = R(\breve{P}Q)\text{‘}a$$

Thus $\exists !\, \mathrm{os}(P, Q_* \upharpoonright \overrightarrow{Q}\text{‘}a)\text{‘}R$ is, in a compact series, the necessary and sufficient condition for the existence of a definite limit of the function as the argument approaches a from below.

Without assuming $P^2 = P$, if x is a member of $\mathrm{os}(P, Q_* \upharpoonright \overrightarrow{Q}\text{‘}a)\text{‘}R$, and if x has no immediate predecessor or successor, so that in the neighbourhood of x the series is compact, we still have $x = R(PQ)\text{‘}a = R(\breve{P}Q)\text{‘}a$ (*234·462).

We next consider $\text{ct}\,(PQ)`R$. By the definition we have

***234·5.** $\vdash : a \,\epsilon\, \text{ct}\,(PQ)`R \,.\equiv.\, R`a \,\epsilon\, \text{os}\,(P, Q_* \upharpoonright \overrightarrow{Q_{po}}`a)`R - C`P_1$

Thus a is an argument for which the function has a single value which has no immediate predecessor or successor in P, and which, in virtue of *234·462, is the limit of the function as the argument approaches a from below (*234·52). The cases when $R`a = B`P$ or $R`a = B`\breve{P}$ require special attention; excluding these cases, we arrive at

***234·51.** $\vdash :: Q \,\epsilon\, \text{trans} \,.\, Q_* \upharpoonright \overrightarrow{Q}`a \,\epsilon\, \text{connex} \,.\, R`a \,\epsilon\, \text{D}`P \cap \text{Ꭰ}`P \,.\supset:.$
$a \,\epsilon\, \text{ct}\,(PQ)`R \,.\equiv: R`a \sim\epsilon\, C`P_1 : R`a \,\epsilon\, P\,(z - w) \,.\supset_{z,w}.$
$(\exists y) \,.\, y \,\epsilon\, \overrightarrow{Q}`a \cap \text{Ꭰ}`R \,.\, R``Q\,(y \mapsto a) \subset P\,(z - w)$

This proposition is analogous to *234·41.

We prove (*234·562) that if P, Q are series, and α is any class of arguments for which all the values belong to $C`P$, and if α has a limit at which the function is continuous from below, then the limit of the function, as the argument increases in α, is the value of the function at the limit of α.

We next consider $\text{contin}\,(PQ)`R$, which is defined as $\text{ct}\,(PQ)`R \cap \text{ct}\,(P\breve{Q})`R$. We show that if P is a series whose field contains $R``\overleftrightarrow{Q}`a$, and $Q$ is transitive, and $Q_* \upharpoonright \overleftrightarrow{Q}`a$ is connected, and $R`a$ is neither $B`P$ nor $B`\breve{P}$, then if $a$ belongs to the class $\text{contin}\,(PQ)`R$, $R`a$ is the limit of the function for the argument a for approaches either from below or from above (*234·62). If P is compact, the converse also holds (*234·63). Our definition of a point of continuity is thus identified with the second form of Dini's definition quoted above. It is identified with the first form by the following proposition: In the circumstances of *234·62, if $R`a \,\epsilon\, \text{D}`P \cap \text{Ꭰ}`P$, we have (*234·64)

$a \,\epsilon\, \text{contin}\,(PQ)`R \,.\equiv: R`a \,\epsilon\, C`P - C`P_1 : R`a \,\epsilon\, P\,(z - w) \,.\supset_{z,w}.$
$(\exists y, y') \,.\, y, y' \,\epsilon\, \text{Ꭰ}`R \,.\, a \,\epsilon\, Q\,(y - y') \,.\, R``Q\,(y \mapsto y') \subset P\,(z - w),$

i.e. a is a point of continuity when, and only when, the value $R`a$ for the argument $a$ is a member of the $P$-series having no immediate predecessor or successor, and if $R`a$ is contained in the interval $P\,(z - w)$, then, however small this interval may be, two arguments y, y' can be found such that a lies between them, and the values for all arguments from y to y' (both included) lie in the interval $P\,(z - w)$.

We end with a few propositions on continuous functions. The last of these (*234·73) states that, if P is a compact series and Q is transitive and connected, then R is continuous with respect to P and Q when, and only when, it has arguments in $C`Q$, and for all such arguments a we have

$$R\,(PQ)`a = R\,(\breve{P}Q)`a = R\,(P\breve{Q})`a = R\,(\breve{P}\breve{Q})`a = R`a,$$

i.e. the value for every argument is the limit for that argument for approaches either from above or from below.

***234·01.** $\mathrm{sc}(P,Q)`R = C`P \cap \hat{x}(\overrightarrow{P_{po}}``\overleftarrow{P_{po}}`x \subset \overleftarrow{Q_{cn}}`R)$ Df

***234·02.** $\mathrm{os}(P,Q)`R = \mathrm{sc}(P,Q)`R \cap \mathrm{sc}(\breve{P},Q)`R$ Df

***234·03.** $\mathrm{ct}(PQ)`R = \hat{a}\{R`a \in \mathrm{os}(P, Q_* \upharpoonright \overrightarrow{Q_{po}}`a)`R - C`P_1\}$ Df

***234·04.** $\mathrm{contin}(PQ)`R = \mathrm{ct}(PQ)`R \cap \mathrm{ct}(P\breve{Q})`R$ Df

***234·05.** $P \,\overline{\mathrm{contin}}\, Q = \hat{R}\{\exists ! C`Q \cap \text{Ɑ}`R \,.\, C`Q \cap \text{Ɑ}`R \subset \mathrm{contin}(PQ)`R\}$ Df

***234·1.** $\vdash :. x \in \mathrm{sc}(P,Q)`R \,.\equiv : x \in C`P : xP_{po}w \,.\supset_w .\, RQ_{cn}(\overrightarrow{P_{po}}`w) :$

$\equiv : x \in C`P : xP_{po}w \,.\supset_w .\, (\exists y) \,.\, y \in C`Q \cap \text{Ɑ}`R \,.\, R``\overleftarrow{Q_*}`y \subset \overrightarrow{P_{po}}`w$

[*230·11 . (*234·01)]

***234·101.** $\vdash : P_{po} \in \mathrm{Ser} \,.\, x \in \mathrm{sc}(P,Q)`R \,.\supset .\, P\overline{R}_{sc}Q \subset \overrightarrow{P_*}`x$

Dem.

$\vdash .\, *40·16 \,.\, (*234·01) \,.\supset$

$\vdash : \mathrm{Hp} \,.\supset .\, x \in C`P \,.\, p`P_*```\overleftarrow{Q_{cn}}`R \cap C`P \subset p`P_*```\overrightarrow{P_{po}}``\overleftarrow{P_{po}}`x \cap C`P$

[*91·574] $\subset p`\overrightarrow{P_{po}}``\overleftarrow{P_{po}}`x \cap C`P$

[*204·65.*91·602] $\subset \overrightarrow{P_*}`x$ (1)

$\vdash .\, (1) \,.\, *231·1 \,.\supset \vdash .$ Prop

***234·102.** $\vdash : P_{po} \in \mathrm{Ser} \,.\, x \in \mathrm{os}(P,Q)`R \,.\supset .\, P\overline{R}_{os}Q \subset \iota`x$

Dem.

$\vdash .\, *234·1·101 \,.\, (*234·02) \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, x \in C`P \,.\, P\overline{R}_{os}Q \subset \overrightarrow{P_*}`x \cap \overleftarrow{P_*}`x \,.$

[*200·39] $\supset .\, P\overline{R}_{os}Q \subset \iota`x : \supset \vdash .$ Prop

***234·103.** $\vdash : P_{po} \in \mathrm{Ser} \,.\, \exists ! \mathrm{os}(P,Q)`R \,.\supset .\, P\overline{R}_{os}Q \in 0 \cup 1$

Dem. $\vdash .\, *234·102 \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, (\exists x) \,.\, P\overline{R}_{os}Q \subset \iota`x \,.$

[*51·401] $\supset .\, P\overline{R}_{os}Q \in 0 \cup 1 : \supset \vdash .$ Prop

***234·104.** $\vdash : RQ_{cn}(\overrightarrow{P_*}`x) \,.\supset .\, x \in \mathrm{sc}(P,Q)`R$

Dem.

$\vdash .\, *91·52 \,.$ $\supset \vdash : xP_{po}z \,.\supset .\, \overrightarrow{P_*}`x \subset \overrightarrow{P_{po}}`z$ (1)

$\vdash .\, (1) \,.\, *230·211·151 \,.\supset \vdash :. \mathrm{Hp} \,.\supset : xP_{po}z \,.\supset_z .\, RQ_{cn}(\overrightarrow{P_{po}}`z) : x \in C`P :$

[*234·1] $\supset : x \in \mathrm{sc}(P,Q)`R :. \supset \vdash .$ Prop

***234·105.** $\vdash : P_{po} \in \mathrm{Ser} \,.\, x \in \mathrm{sc}(P,Q)`R \cap \mathrm{D}`P_1 \,.\supset .\, RQ_{cn}(\overrightarrow{P_*}`x)$

Dem.

$\vdash .\, *201·63 \,.\, *121·254 \,.\supset \vdash :: \mathrm{Hp} \,.\, xP_1z \,.\supset :. yP_{po}z \,.\supset : \sim(xP_{po}y) :$

[*202·103] $\supset : yP_{po}x \,.\vee .\, y = x$ (1)

$\vdash .\, (1) \,.\, *91·54 \,.$ $\supset \vdash :. \mathrm{Hp} \,.\, xP_1z \,.\supset : \overrightarrow{P_{po}}`z \subset \overrightarrow{P_*}`x :$

[*230·211] $\supset : RQ_{cn}(\overrightarrow{P_{po}}`z) \,.\supset .\, RQ_{cn}(\overrightarrow{P_*}`x)$ (2)

$\vdash .\, *234·1 \,.$ $\supset \vdash : \mathrm{Hp} \,.\supset .\, (\exists z) \,.\, xP_1z \,.\, RQ_{cn}(\overrightarrow{P_{po}}`z)$ (3)

$\vdash .\, (2) \,.\, (3) \,.\supset \vdash .$ Prop

When $x \sim\epsilon \text{D}`P_1$, the above proposition is not necessarily true: it may fail if $x = \min_P`\text{sc}(P, Q)`R$.

It is to be observed that $\text{sc}(P, Q)`R$ and $\text{os}(P, Q)`R$ are functions of P_{po}, so that they are unchanged when P_{po} is substituted for P. Hence the hypothesis $P_{\text{po}} \epsilon \text{Ser}$ is as effective, with regard to them, as the hypothesis $P \epsilon \text{Ser}$. This is stated in the following proposition.

***234·106.** $\vdash . \text{sc}(P, Q)`R = \text{sc}(P_{\text{po}}, Q)`R . \text{os}(P, Q)`R = \text{os}(P_{\text{po}}, Q)`R$ [*234·1]

***234·107.** $\vdash :. x \epsilon C`P - \text{D}`P_1 . \supset : x \epsilon \text{sc}(P, Q)`R . \equiv . \overrightarrow{P_*}``\overleftarrow{P}_{\text{po}}`x \subset \overleftarrow{Q}_{\text{cn}}`R$

Dem.

$\vdash . *121·254 . \supset \vdash :. \text{Hp} . \supset : x \sim\epsilon \text{D}`(P_{\text{po}})_1 :$

[*201·61] $\supset : x \sim\epsilon \text{D}`\{P_{\text{po}} \dot{-} P_{\text{po}}{}^2\} :$

[*10·51] $\supset : xP_{\text{po}}y . \supset . xP_{\text{po}}{}^2 y .$

[*91·574] $\supset . (\exists z) . xP_{\text{po}}z . \overrightarrow{P_*}`z \subset \overrightarrow{P}_{\text{po}}`y$ (1)

$\vdash . (1) . *230·211 . \supset$

$\vdash :. \text{Hp} : xP_{\text{po}}y . \supset_y . RQ_{\text{cn}}\overrightarrow{P_*}`y : \supset : xP_{\text{po}}y . \supset_y . RQ_{\text{cn}}\overrightarrow{P}_{\text{po}}`y$ (2)

$\vdash . *91·54 . *230·211 . \supset$

$\vdash :. xP_{\text{po}}y . \supset_y . RQ_{\text{cn}}\overrightarrow{P}_{\text{po}}`y : \supset : xP_{\text{po}}y . \supset_y . RQ_{\text{cn}}\overrightarrow{P_*}`y$ (3)

$\vdash . (2) . (3) . \supset \vdash :. \text{Hp} . \supset : \overrightarrow{P_*}``\overleftarrow{P}_{\text{po}}`x \subset \overleftarrow{Q}_{\text{cn}}`R . \equiv . \overrightarrow{P}_{\text{po}}``\overleftarrow{P}_{\text{po}}`x \subset \overleftarrow{Q}_{\text{cn}}`R$ (4)

$\vdash . (4) . *234·1 . \supset \vdash . \text{Prop}$

***234·11.** $\vdash :. x \epsilon \text{D}`P \cap \text{Ꮖ}`P : x \epsilon P(z - w) . \supset_{z, w} . RQ_{\text{cn}}\{P(z - w)\} : \equiv :$
$x \epsilon \text{D}`P \cap \text{Ꮖ}`P : x \epsilon P(z - w) . \supset_{z, w} .$
$(\exists y) . y \epsilon C`Q \cap \text{Ꮖ}`R . R``\overleftarrow{Q_*}`y \subset P(z - w)$ [*230·11]

***234·111.** $\vdash :. x \epsilon \text{D}`P \cap \text{Ꮖ}`P : x \epsilon P(z - w) . \supset_{z, w} . RQ_{\text{cn}}\{P(z - w)\} : \supset .$
$x \epsilon \text{os}(P, Q)`R$

Dem.

$\vdash . *230·211 . \supset$

$\vdash :: \text{Hp} . \supset :. x \epsilon \text{D}`P \cap \text{Ꮖ}`P :. xP_{\text{po}}w : (\exists z) . zP_{\text{po}}x : \supset_w . RQ_{\text{cn}}\overrightarrow{P}_{\text{po}}`w :.$

[*91·504] $\supset :. x \epsilon \text{D}`P : xP_{\text{po}}w . \supset_w . RQ_{\text{cn}}\overrightarrow{P}_{\text{po}}`w :.$

[*234·1] $\supset :. x \epsilon \text{sc}(P, Q)`R$ (1)

Similarly $\vdash : \text{Hp} . \supset . \text{sc}(\breve{P}, Q)`R$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***234·12.** $\vdash :: Q_* \epsilon \text{connex} . \supset :. x \epsilon \text{os}(P, Q)`R \cap \text{D}`P \cap \text{Ꮖ}`P . \equiv :$
$x \epsilon \text{D}`P \cap \text{Ꮖ}`P : x \epsilon P(z - w) . \supset_{z, w} . RQ_{\text{cn}}\{P(z - w)\}$

Dem.

$\vdash . *234·1 . \supset \vdash :. x \epsilon \text{os}(P, Q)`R \cap \text{D}`P \cap \text{Ꮖ}`P . \equiv :$
$x \epsilon \text{D}`P \cap \text{Ꮖ}`P : xP_{\text{po}}w . \supset_w . RQ_{\text{cn}}(\overrightarrow{P}_{\text{po}}`w) : zP_{\text{po}}x . \supset_z . RQ_{\text{cn}}(\overleftarrow{P}_{\text{po}}`z) :$

[*11·71] $\equiv : x \epsilon \text{D}`P \cap \text{Ꮖ}`P : zP_{\text{po}}x . xP_{\text{po}}w . \supset_{z, w} . RQ_{\text{cn}}(\overrightarrow{P}_{\text{po}}`w) . RQ_{\text{cn}}(\overleftarrow{P}_{\text{po}}`z)$ (1)

$\vdash . *230\cdot42 . \supset \vdash :. \text{Hp} . \supset : RQ_{\text{cn}} \overrightarrow{P}_{\text{po}}\text{‘}w . RQ_{\text{cn}} (\overleftarrow{P}_{\text{po}}\text{‘}z) . \equiv .$

$RQ_{\text{cn}} (\overleftarrow{P}_{\text{po}}\text{‘}z \cap \overrightarrow{P}_{\text{po}}\text{‘}w)$ (2)

$\vdash . (1) . (2) . *121\cdot1 . \supset \vdash . \text{Prop}$

$*234\cdot121$. $\vdash . \overrightarrow{B}\text{‘}\breve{P} \subset \text{sc}(P, Q)\text{‘}R$ [$*93\cdot104 . (*234\cdot01)$]

$*234\cdot122$. $\vdash :. P_{\text{po}} \epsilon \text{connex} . x = B\text{‘}P . \supset :$

$x \epsilon \text{os}(P, Q)\text{‘}R . \equiv . x \epsilon \text{sc}(P, Q)\text{‘}R . \equiv . \overrightarrow{P}_{\text{po}}\text{‘‘}\text{Ɑ}\text{‘}P \subset \overleftarrow{Q}_{\text{cn}}\text{‘}R$

[$*234\cdot121 . (*234\cdot02) . *234\cdot1 . *205\cdot253$]

$*234\cdot13$. $\vdash : x \epsilon \text{sc}(P, Q)\text{‘}R . \supset . \overleftarrow{P}_{*}\text{‘}x \subset \text{sc}(P, Q)\text{‘}R$

Dem.

$\vdash . *96\cdot3 . *91\cdot74 . *90\cdot13 . \supset \vdash : xP_{*}z . \supset . \overleftarrow{P}_{\text{po}}\text{‘}z \subset \overleftarrow{P}_{\text{po}}\text{‘}x . z \epsilon C\text{‘}P .$

[$*37\cdot2$] $\supset . \overrightarrow{P}_{\text{po}}\text{‘‘}\overleftarrow{P}_{\text{po}}\text{‘}z \subset \overrightarrow{P}_{\text{po}}\text{‘‘}\overleftarrow{P}_{\text{po}}\text{‘}x . z \epsilon C\text{‘}P$ (1)

$\vdash . (1) . (*234\cdot01) . \supset \vdash . \text{Prop}$

$*234\cdot131$. $\vdash . \text{sc}(P, Q)\text{‘}R = \breve{P}_{*}\text{‘‘}\text{sc}(P, Q)\text{‘}R . \text{sc}(P, Q)\text{‘}R \epsilon \text{sect}\text{‘}\breve{P}$

Dem.

$\vdash . *90\cdot21 . *234\cdot1 . \supset \vdash . \text{sc}(P, Q)\text{‘}R \subset \breve{P}_{*}\text{‘‘}\text{sc}(P, Q)\text{‘}R$ (1)

$\vdash . *234\cdot13 . \quad \supset \vdash . \breve{P}_{*}\text{‘‘}\text{sc}(P, Q)\text{‘}R \subset \text{sc}(P, Q)\text{‘}R$ (2)

$\vdash . (1) . (2) . *211\cdot13 . \supset \vdash . \text{Prop}$

$*234\cdot14$. $\vdash : Q_{*} \epsilon \text{connex} . x \epsilon \text{sc}(P, Q)\text{‘}R . \supset . x \epsilon C\text{‘}P . \overleftarrow{P}_{\text{po}}\text{‘}x \subset \breve{P}\overline{R}_{\text{sc}}Q$

Dem.

$\vdash . *234\cdot1 . \supset \vdash :. \text{Hp} . \supset : x \epsilon C\text{‘}P : xP_{\text{po}}z . \supset_z . RQ_{\text{cn}} (\overrightarrow{P}_{\text{po}}\text{‘}z) .$

[$*230\cdot211$] $\supset_z . RQ_{\text{cn}} (\overrightarrow{P}_{*}\text{‘}z) .$

[$*231\cdot141$] $\supset_z . z \epsilon \breve{P}\overline{R}_{\text{sc}}Q :. \supset \vdash . \text{Prop}$

$*234\cdot141$. $\vdash : Q_{*} \epsilon \text{connex} . \exists ! \text{sc}(P, Q)\text{‘}R . \supset . \exists ! \breve{P}\overline{R}_{\text{sc}}Q$ [$*234\cdot14$]

$*234\cdot142$. $\vdash : \exists ! \text{sc}(P, Q)\text{‘}R \cap \text{D}\text{‘}P . \supset . \exists ! C\text{‘}Q \cap \text{Ɑ}\text{‘}R$

Dem.

$\vdash . *234\cdot1 . \supset$

$\vdash :. x \epsilon \text{sc}(P, Q)\text{‘}R \cap \text{D}\text{‘}P . \supset : x \epsilon \text{D}\text{‘}P : (\exists w) . xP_{\text{po}}w . \supset . \exists ! C\text{‘}Q \cap \text{Ɑ}\text{‘}R :$

[$*91\cdot504$] $\supset : \exists ! C\text{‘}Q \cap \text{Ɑ}\text{‘}R :. \supset \vdash . \text{Prop}$

$*234\cdot15$. $\vdash : P_{*}, Q_{*} \epsilon \text{connex} . \exists ! \text{sc}(P, Q)\text{‘}R . \supset . P\overline{R}_{\text{sc}}Q \cup \breve{P}\overline{R}_{\text{sc}}Q = C\text{‘}P$

Dem.

$\vdash . *231\cdot202 . *234\cdot141 . \supset \vdash : \text{Hp} . \supset . C\text{‘}P - P\overline{R}_{\text{sc}}Q \subset \breve{P}\overline{R}_{\text{sc}}Q$ (1)

$\vdash . *231\cdot1 . \quad \supset \vdash . P\overline{R}_{\text{sc}}Q \cup \breve{P}\overline{R}_{\text{sc}}Q \subset C\text{‘}P$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***234·16.** $\vdash : P_{po} \epsilon \text{Ser} . Q_* \epsilon \text{connex} . \supset .$

$$P\overline{R}_{sc}Q \subset p`\overrightarrow{P_*}``\text{sc}(P, Q)`R . \breve{P}_{po}``\text{sc}(P, Q)`R \subset \breve{P}\overline{R}_{sc}Q \quad [*234·101·14]$$

***234·161.** $\vdash :. P_{po} \epsilon \text{Ser} . R``C`Q \subset C`P . P\overline{R}_{sc}Q \subset \overrightarrow{P_*}`x . \supset :$

$$P\overline{R}_{sc}Q = \overrightarrow{P_*}`x . \vee . RQ_{cn}(\overrightarrow{P_*}`x)$$

Dem.

$\vdash . *231·24 . \supset \vdash : \text{Hp} . \sim \{RQ_{cn}(\overrightarrow{P_*}`x)\} . \supset . \overrightarrow{P_*}`x \subset P\overline{R}_{sc}Q .$

$[\text{Hp}.*22·41] \quad \supset . \overrightarrow{P_*}`x = P\overline{R}_{sc}Q : \supset \vdash . \text{Prop}$

***234·162.** $\vdash : P_{po} \epsilon \text{Ser} . R``C`Q \subset C`P . \overrightarrow{P_*}`x = P\overline{R}_{sc}Q . x \epsilon C`P . \supset .$

$$x \epsilon \text{sc}(P, Q)`R$$

Dem.

$\vdash . *202·5 . \supset \vdash :. \text{Hp} . xP_{po}z . \supset : z \sim \epsilon P\overline{R}_{sc}Q :$

$[*231·12] \quad \supset : (\exists y) . y \epsilon C`Q \cap \text{G}`R . z \sim \epsilon P_*``R``\overleftarrow{Q_*}`y :$

$[*211·56] \quad \supset : (\exists y) . y \epsilon C`Q \cap \text{G}`R . P_*``R``\overleftarrow{Q_*}`y \subset \overrightarrow{P_{po}}`z :$

$[*90·33] \quad \supset : (\exists y) . y \epsilon C`Q \cap \text{G}`R . R``\overleftarrow{Q_*}`y \subset \overrightarrow{P_{po}}`z :$

$[*230·11] \quad \supset : RQ_{cn}(\overrightarrow{P_{po}}`z) \quad (1)$

$\vdash . (1) . *234·1 . \supset \vdash . \text{Prop}$

***234·17.** $\vdash :. P_{po} \epsilon \text{Ser} . R``C`Q \subset C`P . \supset :$

$$x \epsilon \text{sc}(P, Q)`R . \equiv . x \epsilon C`P . P\overline{R}_{sc}Q \subset \overrightarrow{P_*}`x$$

Dem.

$\vdash . *234·1·101 . \supset \vdash :. \text{Hp} . \supset : x \epsilon \text{sc}(P, Q)`R . \supset . x \epsilon C`P . P\overline{R}_{sc}Q \subset \overrightarrow{P_*}`x \quad (1)$

$\vdash . *234·161·162·104 . \supset \vdash :. \text{Hp} . \supset : x \epsilon C`P . P\overline{R}_{sc}Q \subset \overrightarrow{P_*}`x . \supset .$

$$x \epsilon \text{sc}(P, Q)`R \quad (2)$$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***234·171.** $\vdash : P_{po} \epsilon \text{Ser} . R``C`Q \subset C`P . x \epsilon C`P - \text{sc}(P, Q)`R . \supset .$

$$\overrightarrow{P_*}`x \subset P``(P\overline{R}_{sc}Q)$$

Dem.

$\vdash . *234·17 . \supset \vdash : \text{Hp} . \supset . \exists ! P\overline{R}_{sc}Q - \overrightarrow{P_*}`x \quad (1)$

$\vdash . (1) . *211·56 . *231·13 . \supset \vdash : \text{Hp} . \supset . \overrightarrow{P_*}`x \subset P_{po}``(P\overline{R}_{sc}Q) \quad (2)$

$\vdash . (2) . *231·134 . \supset \vdash . \text{Prop}$

***234·172.** $\vdash : P_{po} \epsilon \text{Ser} . \supset . C`P - \text{sc}(P, Q)`R = C`P \cap p`\overrightarrow{P_{po}}``\text{sc}(P, Q)`R$

Dem.

$\vdash . *200·5 . \supset \vdash : \text{Hp} . \supset . C`P \cap p`\overrightarrow{P_{po}}``\text{sc}(P, Q)`R \subset C`P - \text{sc}(P, Q)`R \quad (1)$

$\vdash . *234·131 . \supset$

$\vdash : x \epsilon \text{sc}(P, Q)`R . y \epsilon C`P - \text{sc}(P, Q)`R . \supset . \sim(xP_*y) . x, y \epsilon C`P .$

$[*202·103] \quad \supset . yP_{po}x \quad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***234·173.** $\vdash : P_{\mathrm{po}} \in \mathrm{Ser} . \exists ! \mathrm{sc}(P,Q)‘R . \supset . C‘P - \mathrm{sc}(P,Q)‘R = p‘\overrightarrow{P}_{\mathrm{po}}“\mathrm{sc}(P,Q)‘R$
[*234·172 . *40·61 . *37·15]

***234·174.** $\vdash : P_{\mathrm{po}} \in \mathrm{Ser} . Q_* \in \mathrm{connex} . R“C‘Q \subset C‘P . \supset .$

$$C‘P \cap p‘\overrightarrow{P}_{\mathrm{po}}“\mathrm{sc}(P,Q)‘R = P“(P\overline{R}_{\mathrm{sc}}Q) = C‘P - \mathrm{sc}(P,Q)‘R$$

Dem.

$\vdash . *234·171·172 . \supset \vdash : \mathrm{Hp} . \supset . P_*“\{C‘P \cap p‘\overrightarrow{P}_{\mathrm{po}}“\mathrm{sc}(P,Q)‘R\} \subset P“(P\overline{R}_{\mathrm{sc}}Q) .$

$[*90·21] \qquad \supset . C‘P \cap p‘\overrightarrow{P}_{\mathrm{po}}“\mathrm{sc}(P,Q)‘R \subset P“(P\overline{R}_{\mathrm{sc}}Q) \qquad (1)$

$\vdash . *234·16 . *37·2 . \supset \vdash : \mathrm{Hp} . \supset . P“(P\overline{R}_{\mathrm{sc}}Q) \subset P“p‘\overrightarrow{P}_*“\mathrm{sc}(P,Q)‘R$

$[*40·37 . *91·52] \qquad \subset p‘\overrightarrow{P}_{\mathrm{po}}“\mathrm{sc}(P,Q)‘R \qquad (2)$

$\vdash . *37·15 . \qquad \supset \vdash . P“(P\overline{R}_{\mathrm{sc}}Q) \subset \mathrm{D}‘P \qquad (3)$

$\vdash . (1) . (2) . (3) . *234·172 . \supset \vdash . \mathrm{Prop}$

***234·175.** $\vdash : \mathrm{Hp}\, *234·174 . \exists ! \mathrm{sc}(P,Q)‘R . \supset . p‘\overrightarrow{P}_{\mathrm{po}}“\mathrm{sc}(P,Q)‘R = P“(P\overline{R}_{\mathrm{sc}}Q)$
[*234·174 . *40·61 . *37·15]

***234·18.** $\vdash : P_{\mathrm{po}} \in \mathrm{Ser} . Q_* \in \mathrm{connex} . R“C‘Q \subset C‘P . \supset .$

$$C‘P = \mathrm{sc}(P,Q)‘R \cup P“(P\overline{R}_{\mathrm{sc}}Q) . \mathrm{sc}(P,Q)‘R \cap P“(P\overline{R}_{\mathrm{sc}}Q) = \Lambda .$$
$$\mathrm{sc}(P,Q)‘R = C‘P - P“(P\overline{R}_{\mathrm{sc}}Q)$$

Dem.

$\vdash . *234·174 . *24·411 . \supset \vdash : \mathrm{Hp} . \supset . C‘P = \mathrm{sc}(P,Q)‘R \cup P“(P\overline{R}_{\mathrm{sc}}Q) \qquad (1)$

$\vdash . *234·174 . \qquad \supset \vdash : \mathrm{Hp} . \supset . P“(P\overline{R}_{\mathrm{sc}}Q) \subset p‘\overrightarrow{P}_{\mathrm{po}}“\mathrm{sc}(P,Q)‘R .$

$[*200·5] \qquad \supset . \mathrm{sc}(P,Q)‘R \cap P“(P\overline{R}_{\mathrm{sc}}Q) = \Lambda \qquad (2)$

$\vdash . *24·492 . *234·174 . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{sc}(P,Q)‘R = C‘P - P“(P\overline{R}_{\mathrm{sc}}Q) \qquad (3)$

$\vdash . (1) . (2) . (3) . \supset \vdash . \mathrm{Prop}$

In virtue of this proposition, $P“(P\overline{R}_{\mathrm{sc}}Q)$ and $\mathrm{sc}(P,Q)‘R$ are complementary sections of P, *i.e.* they constitute a Dedekind cut in P.

***234·181.** $\vdash : P_{\mathrm{po}} \in \mathrm{Ser} . Q_* \in \mathrm{connex} . R“C‘Q \subset C‘P . \supset .$

$$P\overline{R}_{\mathrm{sc}}Q \cap \mathrm{sc}(P,Q)‘R = \overrightarrow{\max}_P‘(P\overline{R}_{\mathrm{sc}}Q) .$$
$$\mathrm{sc}(P,Q)‘R = (C‘P - P\overline{R}_{\mathrm{sc}}Q) \cup \overrightarrow{\max}_P‘(P\overline{R}_{\mathrm{sc}}Q)$$

Dem.

$\vdash . *234·18 . \supset \vdash : \mathrm{Hp} . \supset . P\overline{R}_{\mathrm{sc}}Q \cap \mathrm{sc}(P,Q)‘R = P\overline{R}_{\mathrm{sc}}Q - P“(P\overline{R}_{\mathrm{sc}}Q)$

$[*205·111] \qquad = \overrightarrow{\max}_P‘(P\overline{R}_{\mathrm{sc}}Q) \qquad (1)$

$\vdash . *24·412 . *231·13 . \supset$

$\vdash : \mathrm{Hp} . \supset . C‘P - P“(P\overline{R}_{\mathrm{sc}}Q) = \{C‘P - (P\overline{R}_{\mathrm{sc}}Q)\} \cup \{(P\overline{R}_{\mathrm{sc}}Q) - P“(P\overline{R}_{\mathrm{sc}}Q)\} .$

$[*234·18 . *205·111] \quad \supset . \mathrm{sc}(P,Q)‘R = (C‘P - P\overline{R}_{\mathrm{sc}}Q) \cup \overrightarrow{\max}_P‘(P\overline{R}_{\mathrm{sc}}Q) \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

$*234\cdot182$. $\vdash : P \in \text{Ser} \,.\, Q_* \in \text{connex} \,.\, R``C`Q \subset C`P \,.\, \supset .$

$$\overrightarrow{\text{limax}}_P`(P\overline{R}_{sc}Q) = \overrightarrow{\text{min}}_P`\text{sc}\,(P,Q)`R$$

Dem.

$\vdash . *207\cdot51 . \supset \vdash :. \text{Hp} . \supset : x = \text{limax}_P`(P\overline{R}_{sc}Q) . \equiv . x \in C`P . \overrightarrow{P}`x = P``(P\overline{R}_{sc}Q) .$

$[*234\cdot174]$ $\qquad \equiv . x \in C`P . \overrightarrow{P}`x = C`P \cap p`\overrightarrow{P}``\text{sc}\,(P,Q)`R \quad (1)$

$\vdash . *200\cdot52 . \supset$

$\vdash : \text{Hp} . x \in C`P . \overrightarrow{P}`x = C`P \cap p`\overrightarrow{P}``\text{sc}\,(P,Q)`R . \supset . C`P \neq C`P \cap p`\overrightarrow{P}``\text{sc}\,(P,Q)`R .$

$[*40\cdot2 . \text{Transp}]$ $\qquad \supset . \exists ! \,\text{sc}\,(P,Q)`R .$

$[*40\cdot62]$ $\qquad \supset . C`P \cap p`\overrightarrow{P}``\text{sc}\,(P,Q)`R = p`\overrightarrow{P}``\text{sc}\,(P,Q)`R .$

$[*13\cdot12]$ $\qquad \supset . \overrightarrow{P}`x = p`\overrightarrow{P}``\text{sc}\,(P,Q)`R \quad (2)$

$\vdash . *22\cdot621 . \supset$

$\vdash : \overrightarrow{P}`x = p`\overrightarrow{P}``\text{sc}\,(P,Q)`R . \supset . \overrightarrow{P}`x = C`P \cap p`\overrightarrow{P}``\text{sc}\,(P,Q)`R \quad (3)$

$\vdash . (2) . (3) . \supset \vdash :. \text{Hp} . x \in C`P . \supset :$

$$\overrightarrow{P}`x = C`P \cap p`\overrightarrow{P}``\text{sc}\,(P,Q)`R . \equiv . \overrightarrow{P}`x = p`\overrightarrow{P}``\text{sc}\,(P,Q)`R \quad (4)$$

$\vdash . (1) . (4) . \supset$

$\vdash :. \text{Hp} . \supset : x = \text{limax}_P`(P\overline{R}_{sc}Q) . \equiv . x \in C`P . \overrightarrow{P}`x = p`\overrightarrow{P}``\text{sc}\,(P,Q)`R .$

$[*205\cdot67]$ $\qquad \equiv . x = \text{min}_P`\text{sc}\,(P,Q)`R :. \supset \vdash . \text{Prop}$

$*234\cdot183$. $\vdash : \text{Hp}\, *234\cdot18 . \text{sc}\,(P,Q)`R = \Lambda . \supset . P\overline{R}_{sc}Q = C`P . \sim \text{E} ! \, B`\breve{P}$
$[*234\cdot181\cdot121]$

$*234\cdot2$. $\vdash : P_{po} \in \text{Ser} . R``C`Q \subset C`P . Q_* \in \text{connex} . \supset .$

$$\text{os}\,(P,Q)`R = \{\overrightarrow{\text{min}}_P`(\breve{P}\overline{R}_{sc}Q) - P\overline{R}_{sc}Q\} \cup \{\overrightarrow{\text{max}}_P`(P\overline{R}_{sc}Q) - \breve{P}\overline{R}_{sc}Q\} \cup \{\overrightarrow{\text{max}}_P`(P\overline{R}_{sc}Q) \cap \overrightarrow{\text{min}}_P`(\breve{P}\overline{R}_{sc}Q)\}$$

Dem.

$\vdash . *234\cdot181 . \supset \vdash : \text{Hp} . \supset . \text{os}\,(P,Q)`R = \{(C`P - P\overline{R}_{sc}Q) \cup \overrightarrow{\text{max}}_P`(P\overline{R}_{sc}Q)\} \cap$

$$\{(C`P - \breve{P}\overline{R}_{sc}Q) \cap \overrightarrow{\text{min}}_P`(\breve{P}\overline{R}_{sc}Q)\} \quad (1)$$

$\vdash . *231\cdot201 . \supset \vdash : \text{Hp} . \supset . (C`P - P\overline{R}_{sc}Q) \cap (C`P - \breve{P}\overline{R}_{sc}Q) = \Lambda \quad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

$*234\cdot201$. $\vdash : \text{Hp}\, *234\cdot2 . \supset . \text{os}\,(P,Q)`R \subset \overrightarrow{\text{max}}_P`(P\overline{R}_{sc}Q) \cup \overrightarrow{\text{min}}_P`(\breve{P}\overline{R}_{sc}Q)$
$[*234\cdot2]$

$*234\cdot202$. $\vdash : \text{Hp}\, *234\cdot2 . \supset . \text{os}\,(P,Q)`R \in 0 \cup 1 \cup 2$
$[*234\cdot201 . *205\cdot681 . *60\cdot391]$

$*234\cdot203$. $\vdash : \text{Hp}\, *234\cdot2 . P\overline{R}_{os}Q \in 1 . \supset .$

$$\text{os}\,(P,Q)`R \in 1 . \text{os}\,(P,Q)`R = \iota`\text{max}_P`(P\overline{R}_{sc}Q) = \iota`\text{min}_P`(\breve{P}\overline{R}_{sc}Q) = P\overline{R}_{os}Q$$
$[*231\cdot193\cdot103 . *205\cdot68 . *234\cdot2]$

$*234\cdot204$. $\vdash : P_{po} \in \text{Ser} . P\overline{R}_{os}Q \sim \in 0 \cup 1 . \supset . \text{os}\,(P,Q)`R = \Lambda \quad [*234\cdot103]$

***234·21.** $\vdash : \mathrm{Hp} * 234\cdot 2 \,.\, P\overline{R}_{\mathrm{os}}Q = \Lambda \,.\, \supset .$

$$\mathrm{os}\,(P, Q)`R = \overrightarrow{\max}_P`(P\overline{R}_{\mathrm{sc}}Q) \cup \overrightarrow{\min}_P`(\breve{P}\overline{R}_{\mathrm{sc}}Q)$$

Dem.

$\vdash . *205\cdot 11\cdot 111 . \supset$

$\vdash : \mathrm{Hp} . \supset . \overrightarrow{\max}_P`(P\overline{R}_{\mathrm{sc}}Q) \subset -(\breve{P}\overline{R}_{\mathrm{sc}}Q) . \overrightarrow{\min}_P`(\breve{P}\overline{R}_{\mathrm{sc}}Q) \subset -(P\overline{R}_{\mathrm{sc}}Q)$ (1)

$\vdash . (1) . *234\cdot 2 . \supset \vdash . \mathrm{Prop}$

***234·23.** $\vdash :. \mathrm{Hp} * 234\cdot 2 \,.\, P\overline{R}_{\mathrm{os}}Q \sim \epsilon\, 1 \,.\, \mathrm{os}\,(P, Q)`R \,\epsilon\, 1 \,.\, \supset :$

$$P\overline{R}_{\mathrm{os}}Q = \Lambda : \mathrm{os}\,(P, Q)`R = \iota`\max_P`(P\overline{R}_{\mathrm{sc}}Q) \,.\, \sim \mathrm{E}\,!\, \min_P`(\breve{P}\overline{R}_{\mathrm{sc}}Q) \,.\, \mathrm{v} \,.$$
$$\mathrm{os}\,(P, Q)`R = \iota`\min_P`(\breve{P}\overline{R}_{\mathrm{sc}}Q) \,.\, \sim \mathrm{E}\,!\, \max_P`(P\overline{R}_{\mathrm{sc}}Q)$$

Dem.

$\vdash . *234\cdot 103 . \supset \vdash : \mathrm{Hp} . \supset . P\overline{R}_{\mathrm{os}}Q = \Lambda$ (1)

[*234·21] $\supset . \mathrm{os}\,(P, Q)`R = \overrightarrow{\max}_P`(P\overline{R}_{\mathrm{sc}}Q) \cup \overrightarrow{\min}_P`(\breve{P}\overline{R}_{\mathrm{sc}}Q)$ (2)

$\vdash . *52\cdot 41 . \supset \vdash : P\overline{R}_{\mathrm{os}}Q = \Lambda . \mathrm{E}\,!\, \max_P`(P\overline{R}_{\mathrm{sc}}Q) . \mathrm{E}\,!\, \min_P`(\breve{P}\overline{R}_{\mathrm{sc}}Q) . \supset .$

$$\{\overrightarrow{\max}_P`(P\overline{R}_{\mathrm{sc}}Q) \cup \overrightarrow{\min}_P`(\breve{P}\overline{R}_{\mathrm{sc}}Q)\} \sim \epsilon\, 1 \quad (3)$$

$\vdash . (1) . (2) . (3) . \mathrm{Transp} . \supset$

$\vdash :. \mathrm{Hp} . \supset : \sim \mathrm{E}\,!\, \max_P`(P\overline{R}_{\mathrm{sc}}Q) . \mathrm{v} . \sim \mathrm{E}\,!\, \min_P`(\breve{P}\overline{R}_{\mathrm{sc}}Q)$ (4)

$\vdash . (2) . *205\cdot 681 . \supset \vdash :. \mathrm{Hp} . \supset : \mathrm{E}\,!\, \max_P`(P\overline{R}_{\mathrm{sc}}Q) . \mathrm{v} . \mathrm{E}\,!\, \min_P`(\breve{P}\overline{R}_{\mathrm{sc}}Q)$ (5)

$\vdash . (1) . (2) . (4) . (5) . \supset \vdash . \mathrm{Prop}$

***234·24.** $\vdash :. P \,\epsilon\, \mathrm{Ser} \,.\, Q_* \,\epsilon\, \mathrm{connex} \,.\, R``C`Q \subset C`P \,.\, \supset :$

$$\mathrm{os}\,(P, Q)`R \,\epsilon\, 1 \,.\, \supset .\, \mathrm{os}\,(P, Q)`R = \iota`\mathrm{limax}_P`(P\overline{R}_{\mathrm{sc}}Q) = \iota`\mathrm{limin}_P`(\breve{P}\overline{R}_{\mathrm{sc}}Q)$$

Dem.

$\vdash . *234\cdot 203 . *207\cdot 42 . \supset$

$\vdash : \mathrm{Hp} . P\overline{R}_{\mathrm{os}}Q \,\epsilon\, 1 . \supset . \mathrm{os}\,(P, Q)`R = \iota`\mathrm{limax}_P`(P\overline{R}_{\mathrm{sc}}Q) = \iota`\mathrm{limin}_P`(\breve{P}\overline{R}_{\mathrm{sc}}Q)$ (1)

$\vdash . *234\cdot 23 . *211\cdot 728 . *207\cdot 42 . \supset$

$\vdash : \mathrm{Hp} . P\overline{R}_{\mathrm{os}}Q \sim \epsilon\, 1 . \supset . \mathrm{os}\,(P, Q)`R = \iota`\mathrm{limax}_P`(P\overline{R}_{\mathrm{sc}}Q) = \iota`\mathrm{limin}_P`(\breve{P}\overline{R}_{\mathrm{sc}}Q)$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***234·241.** $\vdash : \mathrm{Hp} * 234\cdot 2 \,.\, \mathrm{os}\,(P, Q)`R \,\epsilon\, 2 \,.\, \supset .\, P\overline{R}_{\mathrm{os}}Q = \Lambda$

Dem.

$\vdash . *234\cdot 103 . \quad \supset \vdash : \mathrm{Hp} . \supset . P\overline{R}_{\mathrm{os}}Q \,\epsilon\, 0 \cup 1$ (1)

$\vdash . *234\cdot 203 . \mathrm{Transp} . \supset \vdash : \mathrm{Hp} . \supset . P\overline{R}_{\mathrm{os}}Q \sim \epsilon\, 1$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***234·242.** $\vdash : \mathrm{Hp} * 234\cdot 2 \,.\, \mathrm{os}\,(P, Q)`R \,\epsilon\, 2 \,.\, \supset .$

$$\mathrm{os}\,(P, Q)`R = \iota`\max_P`(P\overline{R}_{\mathrm{sc}}Q) \cup \iota`\min_P`(\breve{P}\overline{R}_{\mathrm{sc}}Q) \,.\, \max_P`(P\overline{R}_{\mathrm{sc}}Q)\, P_1 \min_P`(\breve{P}\overline{R}_{\mathrm{sc}}Q)$$

Dem.

$\vdash . *234\cdot 201 . *205\cdot 3 . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{E}\,!\, \max_P`(P\overline{R}_{\mathrm{sc}}Q) . \mathrm{E}\,!\, \min_P`(\breve{P}\overline{R}_{\mathrm{sc}}Q) .$

$$\max_P`(P\overline{R}_{\mathrm{sc}}Q) \neq \min_P`(\breve{P}\overline{R}_{\mathrm{sc}}Q) \quad (1)$$

$\vdash . *234{\cdot}241{\cdot}15 . \supset \vdash : \mathrm{Hp} . \supset . \breve{P}\overline{R}_{sc}Q = C`P - P\overline{R}_{sc}Q .$

[$*211{\cdot}8$.(1)] $\supset . \max_P`(P\overline{R}_{sc}Q) = \max(P_{po})`(P\overline{R}_{sc}Q) .$

$\min_P`(\breve{P}\overline{R}_{sc}Q) = \mathrm{seq}(P_{po})`(P\overline{R}_{sc}Q) .$

[$*206{\cdot}5 . *201{\cdot}63$] $\supset . \{\max_P`(P\overline{R}_{sc}Q)\}\,(P_{po})_1\,\{\min_P`(\breve{P}\overline{R}_{sc}Q)\} .$

[$*121{\cdot}254$] $\supset . \{\max_P`(P\overline{R}_{sc}Q)\}\,P_1\,\{\min_P`(\breve{P}\overline{R}_{sc}Q)\}$ (2)

$\vdash . (1) . (2) . *234{\cdot}201 . \supset \vdash .$ Prop

$*234{\cdot}243$. $\vdash : \mathrm{Hp}\,*234{\cdot}24 . \exists ! \,\mathrm{os}(P,Q)`R . \supset .$

$E ! \,\mathrm{limax}_P`(P\overline{R}_{sc}Q) . E ! \,\mathrm{limin}_P`(\breve{P}\overline{R}_{sc}Q)$

Dem.

$\vdash . *234{\cdot}202 . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{os}(P,Q)`R \in 1 \cup 2$ (1)

$\vdash . (1) . *234{\cdot}24{\cdot}242 . \supset \vdash .$ Prop

$*234{\cdot}244$. $\vdash : \mathrm{Hp}\,*234{\cdot}2 . P^2 = P . \supset . \mathrm{os}(P,Q)`R \in 0 \cup 1$

Dem.

$\vdash . *234{\cdot}242{\cdot}202 . \supset \vdash : \mathrm{Hp}\,*234{\cdot}2 . \mathrm{os}(P,Q)`R \sim\in 0 \cup 1 . \supset . \dot{\exists} ! \,P_1$ (1)

$\vdash . (1) .$ Transp $. *201{\cdot}65 . \supset \vdash .$ Prop

$*234{\cdot}25$. $\vdash : \mathrm{Hp}\,*234{\cdot}2 . P^2 = P . \exists ! \,\mathrm{os}(P,Q)`R . \supset .$

$\mathrm{os}(P,Q)`R = \iota`\mathrm{limax}_P`(P\overline{R}_{sc}Q) = \iota`\mathrm{limin}_P`(\breve{P}\overline{R}_{sc}Q)$

[$*234{\cdot}244{\cdot}24$]

$*234{\cdot}251$. $\vdash : \mathrm{Hp}\,*234{\cdot}24 . \mathrm{limax}_P`(P\overline{R}_{sc}Q) = \mathrm{limin}_P`(\breve{P}\overline{R}_{sc}Q) . \supset .$

$\mathrm{os}(P,Q)`R = \iota`\mathrm{limax}_P`(P\overline{R}_{sc}Q) = \iota`\min_P`\mathrm{sc}(P,Q)`R = \iota`\max_P`\mathrm{sc}(\breve{P},Q)`R$

Dem.

$\vdash . *234{\cdot}18 . *207{\cdot}51 . \supset$

$\vdash : \mathrm{Hp} . \supset . \mathrm{sc}(P,Q)`R = C`P - \overrightarrow{P}`\mathrm{limax}_P`(P\overline{R}_{sc}Q) .$

$\mathrm{sc}(\breve{P},Q)`R = C`P - \overleftarrow{P}`\mathrm{limin}_P`(\breve{P}\overline{R}_{sc}Q) .$

[Hp.$*202{\cdot}101$] $\supset . \mathrm{os}(P,Q)`R = C`P \cap \iota`\mathrm{limax}_P`(P\overline{R}_{sc}Q) .$

[$*51{\cdot}31$] $= \iota`\mathrm{limax}_P`(P\overline{R}_{sc}Q)$ (1)

[$*234{\cdot}182$] $= \iota`\min_P`\mathrm{sc}(P,Q)`R$ (2)

$\left[(2)\dfrac{\breve{P}}{P}\right]$ $= \iota`\max_P`\mathrm{sc}(\breve{P},Q)`R$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash .$ Prop

$*234{\cdot}26$. $\vdash :. \mathrm{Hp}\,*234{\cdot}2 . P^2 = P . \supset :$

$\exists ! \,\mathrm{os}(P,Q)`R . \equiv . \mathrm{os}(P,Q)`R = \iota`\mathrm{limax}_P`(P\overline{R}_{sc}Q) .$

$\equiv . \mathrm{os}(P,Q)`R = \iota`\mathrm{limin}_P`(\breve{P}\overline{R}_{sc}Q) .$

$\equiv . \mathrm{os}(P,Q)`R = \iota`\min_P`\mathrm{sc}(PQ)`R .$

$\equiv . \mathrm{os}(P,Q)`R = \iota`\max_P`\mathrm{sc}(\breve{P}Q)`R .$

$\equiv . \mathrm{limax}_P`(P\overline{R}_{sc}Q) = \mathrm{limin}_P`(\breve{P}\overline{R}_{sc}Q)$

[$*234{\cdot}25{\cdot}251{\cdot}182 . *51{\cdot}161$]

$*234{\cdot}27.\quad \vdash : \mathrm{Hp}\, *234{\cdot}24 \,.\, x \in \mathrm{os}\,(P, Q)`R - \text{Ɑ}`P_1 \,.\, \supset .\, x = \mathrm{limax}_P`(P\overline{R}_{\mathrm{sc}}Q)$

Dem.

$\vdash .\, *234{\cdot}24 \;.\, \supset \vdash : \mathrm{Hp} \,.\, \mathrm{os}\,(P, Q)`R \in 1 \,.\, \supset .\, x = \mathrm{limax}_P`(P\overline{R}_{\mathrm{sc}}Q) \qquad (1)$

$\vdash .\, *234{\cdot}242 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \mathrm{os}\,(P, Q)`R \in 2 \,.\, \supset .\, x = \mathrm{limax}_P`(P\overline{R}_{\mathrm{sc}}Q) \qquad (2)$

$\vdash .\, *234{\cdot}202 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset .\, \mathrm{os}\,(P, Q)`R \in 1 \cup 2 \qquad (3)$

$\vdash .\, (1) \,.\, (2) \,.\, (3) \,.\, \supset \vdash .\, \mathrm{Prop}$

$*234{\cdot}271.\quad \vdash : \mathrm{Hp}\, *234{\cdot}24 \,.\, x \in \mathrm{os}\,(P, Q)`R - \mathrm{D}`P_1 \,.\, \supset .\, x = \mathrm{limin}_P`(\breve{P}\overline{R}_{\mathrm{sc}}Q)$

$\left[*234{\cdot}27 \dfrac{\breve{P}}{P}\right]$

$*234{\cdot}272.\quad \vdash : \mathrm{Hp}\, *234{\cdot}24 \,.\, x \in \mathrm{os}\,(P, Q)`R - C`P_1 \,.\, \supset .$

$\qquad x = \mathrm{limax}_P`(P\overline{R}_{\mathrm{sc}}Q) = \mathrm{limin}_P`(\breve{P}\overline{R}_{\mathrm{sc}}Q) \quad [*234{\cdot}27{\cdot}271]$

The remaining propositions of the present number are for the most part immediate consequences of those already proved. In order to obtain, from propositions already proved, propositions concerning the limit of a function as the argument approaches the limit of some class of arguments α, we only have to substitute $Q_* \upharpoonright \alpha$ for Q. In order to obtain the limit of a function as the argument approaches a given term a, we take $Q_* \upharpoonright \overrightarrow{Q}`a$ in place of Q.

$*234{\cdot}3.\quad \vdash :.\, x \in \mathrm{sc}\,(P, Q_* \upharpoonright \alpha)`R \,.\, \equiv :$

$\qquad x \in C`P : xP_{\mathrm{po}}w \,.\, \supset_w .\, (\exists y) \,.\, y \in \alpha \cap C`Q \cap \text{Ɑ}`R \,.\, R``(\alpha \cap \overrightarrow{Q_*}`y) \subset \overrightarrow{P_{\mathrm{po}}}`w$

$[*234{\cdot}1]$

$*234{\cdot}301.\quad \vdash :: Q_* \upharpoonright \alpha \in \mathrm{connex} \,.\, \supset :.\, x \in \mathrm{os}\,(P, Q_* \upharpoonright \alpha)`R \cap \mathrm{D}`P \cap \text{Ɑ}`P \,.\, \equiv :$

$\qquad x \in \mathrm{D}`P \cap \text{Ɑ}`P : x \in P\,(z - w) \,.\, \supset_{z,w} .$

$\qquad (\exists y) \,.\, y \in \alpha \cap C`Q \cap \text{Ɑ}`R \,.\, R``(\alpha \cap \overleftarrow{Q_*}`y) \subset P\,(z - w)$

$[*234{\cdot}12]$

$*234{\cdot}31.\quad \vdash : P_{\mathrm{po}} \in \mathrm{Ser} \,.\, Q_* \upharpoonright \alpha \in \mathrm{connex} \,.\, R``(\alpha \cap C`Q) \subset C`P \,.\, \supset .$

$\qquad C`P - \mathrm{sc}\,(P, Q_* \upharpoonright \alpha)`R = C`P \cap p`\overrightarrow{P_{\mathrm{po}}}``\mathrm{sc}\,(P, Q_* \upharpoonright \alpha)`R = P``(P\overline{R}Q)_{\mathrm{sc}}`\alpha$

$[*234{\cdot}174]$

$*234{\cdot}311.\quad \vdash : \mathrm{Hp}\, *234{\cdot}31 \,.\, \supset .\, C`P = \mathrm{sc}\,(P, Q_* \upharpoonright \alpha)`R \cup P``(P\overline{R}Q)_{\mathrm{sc}}`\alpha \,.$

$\qquad \mathrm{sc}\,(P, Q_* \upharpoonright \alpha)`R \cap P``(P\overline{R}Q)_{\mathrm{sc}}`\alpha = \Lambda \,.$

$\qquad \mathrm{sc}\,(P, Q_* \upharpoonright \alpha)`R = C`P - P``(P\overline{R}Q)_{\mathrm{sc}}`\alpha$

$[*234{\cdot}18]$

$*234{\cdot}312.\quad \vdash :.\, P \in \mathrm{Ser} \,.\, Q_* \upharpoonright \alpha \in \mathrm{connex} \,.\, R``(\alpha \cap C`Q) \subset C`P \,.\, \supset :$

$\qquad \mathrm{E}\,!\,(P\overline{R}Q)_{\mathrm{lmx}}`\alpha \,.\, \equiv .\, \mathrm{E}\,!\,\mathrm{min}_P`\mathrm{sc}\,(P, Q_* \upharpoonright \alpha)`R \,.$

$\qquad \equiv .\, (P\overline{R}Q)_{\mathrm{lmx}}`\alpha = \mathrm{min}_P`\mathrm{sc}\,(P, Q_* \upharpoonright \alpha)`R$

$[*234{\cdot}182]$

$*234{\cdot}32.\quad \vdash :.\, P_{\mathrm{po}} \in \mathrm{Ser} \,.\, Q_* \upharpoonright \alpha \in \mathrm{connex} \,.\, R``(\alpha \cap C`Q) \subset C`P \,.\, \supset : (P\overline{R}Q)_{\mathrm{os}}`\alpha \in 1 \,.\, \supset .$

$\qquad \mathrm{os}\,(P, Q_* \upharpoonright \alpha)`R = (P\overline{R}Q)_{\mathrm{os}}`\alpha = \iota`\mathrm{max}_P`(P\overline{R}Q)_{\mathrm{sc}}`\alpha = \iota`\mathrm{min}_P`(\breve{P}\overline{R}Q)_{\mathrm{sc}}`\alpha$

$[*234{\cdot}203]$

$*234{\cdot}321$. $\vdash :: \mathrm{Hp}\,*234{\cdot}32\,.\,\mathrm{os}\,(P, Q_*\upharpoonright\alpha)\text{‘}R \,\epsilon\, 1\,.\supset :.\,(P\overline{R}Q)_{\mathrm{os}}\text{‘}\alpha \sim\epsilon\, 1\,.\supset :$

$(P\overline{R}Q)_{\mathrm{os}}\text{‘}\alpha = \Lambda : \mathrm{os}\,(P, Q_*\upharpoonright\alpha)\text{‘}R = \iota\text{‘}\mathrm{max}_P\text{‘}(P\overline{R}Q)_{\mathrm{sc}}\text{‘}\alpha\,.\sim\mathrm{E}\,!\,\mathrm{min}_P\text{‘}(\breve{P}\overline{R}Q)_{\mathrm{sc}}\text{‘}\alpha\,.$

$\mathsf{v}\,.\,\mathrm{os}\,(P, Q_*\upharpoonright\alpha)\text{‘}R = \iota\text{‘}\mathrm{min}_P\text{‘}(\breve{P}\overline{R}Q)_{\mathrm{sc}}\text{‘}\alpha\,.\sim\mathrm{E}\,!\,\mathrm{max}_P\text{‘}(P\overline{R}Q)_{\mathrm{sc}}\text{‘}\alpha$

[$*234{\cdot}23$]

$*234{\cdot}322$. $\vdash : \mathrm{Hp}\,*234{\cdot}312\,.\,\mathrm{os}\,(P, Q_*\upharpoonright\alpha)\text{‘}R \,\epsilon\, 1\,.\supset .$

$\mathrm{os}\,(P, Q_*\upharpoonright\alpha)\text{‘}R = \iota\text{‘}(P\overline{R}Q)_{\mathrm{lmx}}\text{‘}\alpha = \iota\text{‘}(\breve{P}\overline{R}Q)_{\mathrm{lmx}}\text{‘}\alpha$ [$*234{\cdot}24$]

$*234{\cdot}329$. $\vdash : \mathrm{Hp}\,*234{\cdot}32\,.\,\mathrm{os}\,(P, Q_*\upharpoonright\alpha)\text{‘}R \,\epsilon\, 2\,.\supset .$

$\mathrm{os}\,(P, Q_*\upharpoonright\alpha)\text{‘}R = \iota\text{‘}\mathrm{max}_P\text{‘}(P\overline{R}Q)_{\mathrm{sc}}\text{‘}\alpha \cup \iota\text{‘}\mathrm{min}_P\text{‘}(\breve{P}\overline{R}Q)_{\mathrm{sc}}\text{‘}\alpha\,.$

$\{\mathrm{max}_P\text{‘}(P\overline{R}Q)_{\mathrm{sc}}\text{‘}\alpha\}\,P_1\,\{\mathrm{min}_P\text{‘}(\breve{P}\overline{R}Q)_{\mathrm{sc}}\text{‘}\alpha\}$

[$*234{\cdot}242$]

$*234{\cdot}33$. $\vdash : \mathrm{Hp}\,*234{\cdot}32\,.\,P^2 = P\,.\,\exists\,!\,\mathrm{os}\,(P, Q_*\upharpoonright\alpha)\text{‘}R\,.\supset .$

$\mathrm{os}\,(P, Q_*\upharpoonright\alpha)\text{‘}R = \iota\text{‘}(P\overline{R}Q)_{\mathrm{lmx}}\text{‘}\alpha = \iota\text{‘}(\breve{P}\overline{R}Q)_{\mathrm{lmx}}\text{‘}\alpha$ [$*234{\cdot}25$

$*234{\cdot}331$. $\vdash : \mathrm{Hp}\,*234{\cdot}312\,.\,(P\overline{R}Q)_{\mathrm{lmx}}\text{‘}\alpha = (\breve{P}\overline{R}Q)_{\mathrm{lmx}}\text{‘}\alpha\,.\supset .$

$\mathrm{os}\,(P, Q_*\upharpoonright\alpha)\text{‘}R = \iota\text{‘}(P\overline{R}Q)_{\mathrm{lmx}}\text{‘}\alpha = \iota\text{‘}(\breve{P}\overline{R}Q)_{\mathrm{lmx}}\text{‘}\alpha$

$= \iota\text{‘}\mathrm{min}_P\text{‘}\mathrm{sc}\,(P, Q_*\upharpoonright\alpha)\text{‘}R = \iota\text{‘}\mathrm{max}_P\text{‘}\mathrm{sc}\,(\breve{P}, Q_*\upharpoonright\alpha)\text{‘}R$

[$*234{\cdot}251$]

$*234{\cdot}34$. $\vdash :.\,\mathrm{Hp}\,*234{\cdot}32\,.\,P^2 = P\,.\supset :$

$\exists\,!\,\mathrm{os}\,(P, Q_*\upharpoonright\alpha)\text{‘}R\,.\equiv .\,\mathrm{os}\,(P, Q_*\upharpoonright\alpha)\text{‘}R = \iota\text{‘}(P\overline{R}Q)_{\mathrm{lmx}}\text{‘}\alpha\,.$

$\equiv .\,\mathrm{os}\,(P, Q_*\upharpoonright\alpha)\text{‘}R = \iota\text{‘}(\breve{P}\overline{R}Q)_{\mathrm{lmx}}\text{‘}\alpha\,.$

$\equiv .\,(P\overline{R}Q)_{\mathrm{lmx}}\text{‘}\alpha = (\breve{P}\overline{R}Q)_{\mathrm{lmx}}\text{‘}\alpha$

[$*234{\cdot}26$]

$*234{\cdot}35$. $\vdash : \mathrm{Hp}\,*234{\cdot}312\,.\,x \,\epsilon\, \mathrm{os}\,(P, Q_*\upharpoonright\alpha)\text{‘}R - \text{Ɑ}\text{‘}P_1\,.\supset .\,x = (P\overline{R}Q)_{\mathrm{lmx}}\text{‘}\alpha$

[$*234{\cdot}27$]

$*234{\cdot}351$. $\vdash : \mathrm{Hp}\,*234{\cdot}312\,.\,x \,\epsilon\, \mathrm{os}\,(P, Q_*\upharpoonright\alpha)\text{‘}R - \mathrm{D}\text{‘}P_1\,.\supset .\,x = (\breve{P}\overline{R}Q)_{\mathrm{lmx}}\text{‘}\alpha$

$\left[*234{\cdot}35\,\dfrac{\breve{P}}{P}\right]$

$*234{\cdot}352$. $\vdash : \mathrm{Hp}\,*234{\cdot}312\,.\,x \,\epsilon\, \mathrm{os}\,(P, Q_*\upharpoonright\alpha)\text{‘}R - C\text{‘}P_1\,.\supset .$

$x = (P\overline{R}Q)_{\mathrm{lmx}}\text{‘}\alpha = (\breve{P}\overline{R}Q)_{\mathrm{lmx}}\text{‘}\alpha$ [$*234{\cdot}35{\cdot}351$]

$*234{\cdot}4$. $\vdash :.\,x \,\epsilon\, \mathrm{sc}\,(P, Q_*\upharpoonright\overrightarrow{Q_{\mathrm{po}}}\text{‘}a)\text{‘}R\,.\equiv :$

$x \,\epsilon\, C\text{‘}P : xP_{\mathrm{po}}w\,.\supset_w .\,(\exists y)\,.\,y \,\epsilon\, \overrightarrow{Q_{\mathrm{po}}}\text{‘}a \cap \text{Ɑ}\text{‘}R\,.\,R\text{“}Q\,(y \vdash a) \subset \overrightarrow{P_{\mathrm{po}}}\text{‘}w$

[$*234{\cdot}3\,.\,(*121{\cdot}012)$]

$*234{\cdot}41$. $\vdash :: Q \,\epsilon\, \mathrm{trans}\,.\,Q_*\upharpoonright\overrightarrow{Q}\text{‘}a \,\epsilon\, \mathrm{connex}\,.\supset :.$

$x \,\epsilon\, \mathrm{os}\,(P, Q_*\upharpoonright\overrightarrow{Q}\text{‘}a)\text{‘}R \cap \mathrm{D}\text{‘}P \cap \text{Ɑ}\text{‘}P\,.\equiv : x \,\epsilon\, \mathrm{D}\text{‘}P \cap \text{Ɑ}\text{‘}P :$

$x \,\epsilon\, P\,(z - w)\,.\supset_{z,w} .\,(\exists y)\,.\,y \,\epsilon\, \overrightarrow{Q}\text{‘}a \cap \text{Ɑ}\text{‘}R\,.\,R\text{“}Q\,(y \vdash a) \subset P\,(z - w)$

[$*234{\cdot}301\,.\,(*121{\cdot}012)\,.\,*201{\cdot}18$]

***234·42.** $\vdash :. P \in \mathrm{Ser} . Q \in \mathrm{trans} . Q_* \upharpoonright \overrightarrow{Q}‘a \in \mathrm{connex} . R‘‘\overrightarrow{Q}‘a \subset C‘P . \supset :$

$$\overrightarrow{R(PQ)}‘a = \mathrm{min}_P‘\mathrm{sc}(P, Q_* \upharpoonright \overrightarrow{Q}‘a)‘R \quad [*234·182]$$

***234·421.** $\vdash :. P_{\mathrm{po}} \in \mathrm{Ser} . Q \in \mathrm{trans} . Q_* \upharpoonright \overrightarrow{Q}‘a \in \mathrm{connex} . R‘‘\overrightarrow{Q}‘a \subset C‘P . \supset :$

$$\mathrm{sc}(P, Q_* \upharpoonright \overrightarrow{Q}‘a)‘R = \Lambda . \supset . \overrightarrow{R(PQ)}‘a = \overrightarrow{B}‘\breve{P} \quad [*234·183]$$

***234·422.** $\vdash : \mathrm{Hp}\, *234·42 . P \in \mathrm{Ded} . \supset . R(PQ)‘a = \mathrm{limin}_P‘\mathrm{sc}(P, Q_* \upharpoonright \overrightarrow{Q}‘a)‘R$
[*233·13 . *234·42]

***234·43.** $\vdash : \mathrm{Hp}\, *234·42 . \mathrm{os}(P, Q_* \upharpoonright \overrightarrow{Q}‘a)‘R \in 1 . \supset .$

$$\mathrm{os}(P, Q_* \upharpoonright \overrightarrow{Q}‘a) = \iota‘R(PQ)‘a = \iota‘R(\breve{P}Q)‘a \quad [*234·322]$$

***234·439.** $\vdash : \mathrm{Hp}\, *234·421 . \mathrm{os}(P, Q_* \upharpoonright \overrightarrow{Q}‘a)‘R \in 2 . \supset .$

$$\mathrm{os}(P, Q_* \upharpoonright \overrightarrow{Q}‘a)‘R = \iota‘R(PQ)‘a \cup \iota‘R(\breve{P}Q)‘a .$$
$$\{R(PQ)‘a\}\, P_1 \,\{R(\breve{P}Q)‘a\} \quad [*234·329]$$

***234·44.** $\vdash : \mathrm{Hp}\, *234·421 . P^2 = P . \exists ! \,\mathrm{os}(P, Q_* \upharpoonright \overrightarrow{Q}‘a)‘R . \supset .$

$$\mathrm{os}(P, Q_* \upharpoonright \overrightarrow{Q}‘a)‘R = \iota‘R(PQ)‘a = \iota‘R(\breve{P}Q)‘a \quad [*234·33]$$

***234·441.** $\vdash : \mathrm{Hp}\, *234·42 . R(PQ)‘a = R(\breve{P}Q)‘a . \supset .$

$$\mathrm{os}(P, Q_* \upharpoonright \overrightarrow{Q}‘a) = \iota‘R(PQ)‘a = \iota‘R(\breve{P}Q)‘a \quad [*234·331]$$

***234·45.** $\vdash :. P \in \mathrm{Ser} . Q \in \mathrm{trans} . Q_* \upharpoonright \overrightarrow{Q}‘a \in \mathrm{connex} . R‘‘\overrightarrow{Q}‘a \subset C‘P . P^2 = P . \supset :$

$$\begin{aligned} \exists ! \,\mathrm{os}(P, Q_* \upharpoonright \overrightarrow{Q}‘a)‘R . \equiv . &\mathrm{os}(P, Q_* \upharpoonright \overrightarrow{Q}‘a)‘R = \iota‘R(PQ)‘a . \\ \equiv . &\mathrm{os}(P, Q_* \upharpoonright \overrightarrow{Q}‘a)‘R = \iota‘R(\breve{P}Q)‘a . \\ \equiv . &R(PQ)‘a = R(\breve{P}Q)‘a \qquad [*234·34] \end{aligned}$$

***234·46.** $\vdash : \mathrm{Hp}\, *234·42 . x \in \mathrm{os}(P, Q_* \upharpoonright \overrightarrow{Q}‘a)‘R - \text{Ꭰ}‘P_1 . \supset . x = R(PQ)‘a$
[*234·35]

***234·461.** $\vdash : \mathrm{Hp}\, *234·42 . x \in \mathrm{os}(P, Q_* \upharpoonright \overrightarrow{Q}‘a)‘R - \mathrm{D}‘P_1 . \supset . x = R(\breve{P}Q)‘a$
$\left[*234·46 \dfrac{\breve{P}}{P}\right]$

***234·462.** $\vdash : \mathrm{Hp}\, *234·42 . x \in \mathrm{os}(P, Q_* \upharpoonright \overrightarrow{Q}‘a)‘R - C‘P_1 . \supset .$

$$x = R(PQ)‘a = R(\breve{P}Q)‘a \quad [*234·46·461]$$

***234·5.** $\vdash : a \in \mathrm{ct}(PQ)‘R . \equiv . R‘a \in \mathrm{os}(P, Q_* \upharpoonright \overrightarrow{Q}_{\mathrm{po}}‘a)‘R - C‘P_1 \quad [(*234·03)]$

***234·51.** $\vdash :: Q \in \text{trans} . Q_* \upharpoonright \overrightarrow{Q}\text{‘}a \in \text{connex} . R\text{‘}a \in \text{D‘}P \cap \text{Ɑ‘}P . \supset :.$
$a \in \text{ct}(PQ)\text{‘}R . \equiv : R\text{‘}a \sim\in C\text{‘}P_1 : R\text{‘}a \in P(z-w) . \supset_{z,w} .$
$(\exists y) . y \in \overrightarrow{Q}\text{‘}a \cap \text{Ɑ‘}R . R\text{“}Q(y \mapsto a) \subset P(z-w)$

Dem.

$\vdash . *234{\cdot}5{\cdot}4 . *53{\cdot}31 . \supset$

$\vdash :: \text{Hp} . \supset :. a \in \text{ct}(PQ)\text{‘}R . \equiv : R\text{‘}a \in \text{D‘}P \cap \text{Ɑ‘}P - C\text{‘}P_1 : R\text{‘}a \in P(z-w) . \supset_{z,w} .$
$(\exists y) . y \in \overrightarrow{Q}\text{‘}a \cap \text{Ɑ‘}R . R\text{“}Q(y \vdash a) \subset P(z-w) . R\text{“}\iota\text{‘}a \subset P(z-w)$ (1)

$\vdash . (1) . *121{\cdot}242 . \supset \vdash . \text{Prop}$

***234·52.** $\vdash :. P \in \text{Ser} . Q \in \text{trans} . Q_* \upharpoonright \overrightarrow{Q}\text{‘}a \in \text{connex} . R\text{“}\overrightarrow{Q}\text{‘}a \subset C\text{‘}P . \supset :$
$a \in \text{ct}(PQ)\text{‘}R . \supset . R(PQ)\text{‘}a = R(\breve{P}Q)\text{‘}a = R\text{‘}a$ [*234·462·5]

***234·521.** $\vdash : \text{Hp} *234{\cdot}52 . a \in \text{ct}(PQ)\text{‘}R . \supset . \text{os}(P, Q_* \upharpoonright \overrightarrow{Q}\text{‘}a)\text{‘}R = \iota\text{‘}R\text{‘}a$
[*234·441·52]

***234·522.** $\vdash :. \text{Hp} *234{\cdot}52 . P^2 = P . \supset :$
$a \in \text{ct}(PQ)\text{‘}R . \equiv . R(PQ)\text{‘}a = R(\breve{P}Q)\text{‘}a = R\text{‘}a$

Dem.

$\vdash . *234{\cdot}45 . \supset$

$\vdash :. \text{Hp} . \supset : R(PQ)\text{‘}a = R(\breve{P}Q)\text{‘}a = R\text{‘}a . \supset . \text{os}(P, Q_* \upharpoonright \overrightarrow{Q}\text{‘}a)\text{‘}R = \iota\text{‘}R\text{‘}a .$
[*234·5.*201·65] $\supset . a \in \text{ct}(PQ)\text{‘}R$ (1)

$\vdash . (1) . *234{\cdot}52 . \supset \vdash . \text{Prop}$

***234·53.** $\vdash :: P_{\text{po}} \in \text{connex} . Q \in \text{trans} . R\text{‘}a = B\text{‘}P . \supset :.$
$a \in \text{ct}(PQ)\text{‘}R . \equiv : B\text{‘}P \sim\in \text{D‘}P_1 : w \in \text{Ɑ‘}P . \supset_w .$
$(\exists y) . y \in \overrightarrow{Q}\text{‘}a \cap \text{Ɑ‘}R . R\text{“}Q(y \mapsto a) \subset \overrightarrow{P}_{\text{po}}\text{‘}w$

Dem.

$\vdash . *234{\cdot}122 . *53{\cdot}31 . *234{\cdot}5 . \supset$

$\vdash :: \text{Hp} . \supset :. a \in \text{ct}(PQ)\text{‘}R . \equiv : B\text{‘}P \sim\in \text{D‘}P_1 : (B\text{‘}P) P_{\text{po}} w . \supset_w .$
$(\exists y) . y \in \overrightarrow{Q}\text{‘}a \cap \text{Ɑ‘}R . R\text{“}(\overleftarrow{Q_*}\text{‘}y \cap \overrightarrow{Q}\text{‘}a) \subset \overrightarrow{P}_{\text{po}}\text{‘}w . R\text{“}\iota\text{‘}a \subset \overrightarrow{P}_{\text{po}}\text{‘}w :$
[*202·522.*205·253.*201·18] $\equiv : B\text{‘}P \sim\in \text{D‘}P_1 :$
$w \in \text{Ɑ‘}P . \supset_w . (\exists y) . y \in \overrightarrow{Q}\text{‘}a \cap \text{Ɑ‘}R . R\text{“}Q(y \mapsto a) \subset \overrightarrow{P}_{\text{po}}\text{‘}w :: \supset \vdash . \text{Prop}$

***234·54.** $\vdash : a \in \text{ct}(PQ)\text{‘}R . \supset . a \in \text{Ɑ‘}R \cap \breve{Q}_{\text{po}}\text{“Ɑ‘}R . R\text{‘}a \in C\text{‘}P$

Dem.

$\vdash . *234{\cdot}5{\cdot}1 . (*234{\cdot}02) . \supset \vdash : \text{Hp} . \supset . R\text{‘}a \in C\text{‘}P$ (1)

$\vdash . (1) . *234{\cdot}5 . (*234{\cdot}02) . \supset$

$\vdash :. \text{Hp} . \supset : \exists ! \text{sc}(P, Q_* \upharpoonright \overrightarrow{Q}_{\text{po}}\text{‘}a)\text{‘}R \cap \text{D‘}P . \text{v} . \exists ! \text{sc}(\breve{P}, Q_* \upharpoonright \overrightarrow{Q}_{\text{po}}\text{‘}a)\text{‘}R \cap \text{Ɑ‘}P :$
[*234·142] $\supset : \exists ! \overrightarrow{Q}_{\text{po}}\text{‘}a \cap \text{Ɑ‘}R :$
[*37·46] $\supset : a \in \breve{Q}_{\text{po}}\text{“Ɑ‘}R$ (2)

$\vdash . (1) . *14{\cdot}21 . *33{\cdot}43 . \supset \vdash : \text{Hp} . \supset . a \in \text{Ɑ‘}R$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash . \text{Prop}$

$*234\cdot55.\ \vdash . \sim\{\min(Q_{po})\text{‘}\mathrm{G}\text{‘}R \in \mathrm{ct}\,(PQ)\text{‘}R\}\quad [*234\cdot54 . \mathrm{Transp}]$

$*234\cdot56.\ \vdash : \mathrm{Hp}\,*234\cdot52 . a \in \mathrm{ct}\,(PQ)\text{‘}R . \supset .$

$(P\bar{R}Q)_{\mathrm{os}}\text{‘}\overrightarrow{Q}\text{‘}a \in 0 \cup 1 . \mathrm{E}\,!\,R\,(PQ)\text{‘}a . R\,(PQ)\text{‘}a \sim\in C\text{‘}P_1 . R\text{‘}a = R\,(PQ)\text{‘}a$

Dem.

$\vdash . *234\cdot5 . \supset \vdash : \mathrm{Hp} . \supset . \exists\,!\,\mathrm{os}\,(P, Q_* \upharpoonright \overrightarrow{Q}\text{‘}a)\text{‘}R . R\text{‘}a \sim\in C\text{‘}P_1 .$

$[*234\cdot103]\quad \supset . (P\bar{R}Q)_{\mathrm{os}}\text{‘}\overrightarrow{Q}\text{‘}a \in 0 \cup 1 . R\text{‘}a \sim\in C\text{‘}P_1 \quad (1)$

$\vdash . *234\cdot52 . \supset \vdash : \mathrm{Hp} . \supset . R\text{‘}a = R\,(PQ)\text{‘}a . \mathrm{E}\,!\,R\,(PQ)\text{‘}a \quad (2)$

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

$*234\cdot561.\ \vdash : P, Q \in \mathrm{Ser} . a \in \mathrm{ct}\,(PQ)\text{‘}R . a = \mathrm{lt}_Q\text{‘}(\alpha \cap \mathrm{G}\text{‘}R) . R\text{‘‘}\overrightarrow{Q}\text{‘}a \subset C\text{‘}P . \supset .$

$(P\bar{R}Q)_{\mathrm{lmx}}\text{‘}\alpha = R\text{‘}a = (\breve{P}\bar{R}\dot{Q})_{\mathrm{lmx}}\text{‘}\alpha \quad [*233\cdot515 . *234\cdot56]$

$*234\cdot562.\ \vdash : P, Q \in \mathrm{Ser} . \mathrm{lt}_Q\text{‘}(\alpha \cap \mathrm{G}\text{‘}R) \in \mathrm{ct}\,(PQ)\text{‘}R . R\text{‘‘}(\alpha \cap C\text{‘}Q) \subset C\text{‘}P . \supset .$

$(P\bar{R}Q)_{\mathrm{lmx}}\text{‘}\alpha = (\breve{P}\bar{R}Q)_{\mathrm{lmx}}\text{‘}\alpha = R\text{‘}\mathrm{lt}_Q\text{‘}\alpha \quad [*233\cdot516 . *234\cdot56]$

That is, if α is any class of arguments having a limit at which the function is continuous, then the limit of the function, as the argument approaches the limit of the set of arguments, is the value of the function for that limit.

$*234\cdot6.\ \vdash : a \in \mathrm{contin}\,(PQ)\text{‘}R . \equiv . a \in \mathrm{ct}\,(PQ)\text{‘}R \cap \mathrm{ct}\,(P\breve{Q})\text{‘}R \quad [(*234\cdot04)]$

$*234\cdot61.\ \vdash :: P_{\mathrm{po}} \in \mathrm{Ser} . Q \in \mathrm{trans} . Q_* \upharpoonright \overleftrightarrow{Q}\text{‘}a \in \mathrm{connex} . R\text{‘}a \in \mathrm{D}\text{‘}P \cap \mathrm{G}\text{‘}P . \supset :.$

$a \in \mathrm{contin}\,(PQ)\text{‘}R . \equiv : R\text{‘}a \sim\in C\text{‘}P_1 : R\text{‘}a \in P\,(z - w) . \supset_{z, w} .$

$(\exists y, y') . a \in Q\,(y - y') . y, y' \in \mathrm{G}\text{‘}R . R\text{‘‘}Q\,(y \mapsto\!\dashv y') \subset P\,(z - w)$

Dem.

$\vdash . *234\cdot51 . \supset \vdash :: \mathrm{Hp} . \supset :. a \in \mathrm{contin}\,(PQ)\text{‘}R . \equiv :$

$R\text{‘}a \in \mathrm{D}\text{‘}P \cap \mathrm{G}\text{‘}P - C\text{‘}P_1 : R\text{‘}a \in P\,(z - w) . \supset_{z, w} .$

$(\exists y, y') . y \in \overrightarrow{Q}\text{‘}a \cap \mathrm{G}\text{‘}R . y' \in \overleftarrow{Q}\text{‘}a \cap \mathrm{G}\text{‘}R .$

$R\text{‘‘}Q\,(y \mapsto\!\dashv a) \cup R\text{‘‘}Q\,(a \mapsto\!\dashv y') \subset P\,(z - w) \quad (1)$

$\vdash . (1) . *201\cdot19 . *202\cdot17 . \supset \vdash . \mathrm{Prop}$

$*234\cdot62.\ \vdash :. \mathrm{Hp}\,*234\cdot61 . P \in \mathrm{trans} . R\text{‘‘}\overleftrightarrow{Q}\text{‘}a \subset C\text{‘}P . \supset : a \in \mathrm{contin}\,(PQ)\text{‘}R . \supset .$

$R\,(PQ)\text{‘}a = R\,(\breve{P}Q)\text{‘}a = R\,(P\breve{Q})\text{‘}a = R(\breve{P}\breve{Q})\text{‘}a = R\text{‘}a$

$[*234\cdot52\cdot6]$

$*234\cdot63.\ \vdash :. \mathrm{Hp}\,*234\cdot62 . P^2 = P . \supset : a \in \mathrm{contin}\,(PQ)\text{‘}R . \equiv .$

$R\,(PQ)\text{‘}a = R\,(\breve{P}Q)\text{‘}a = R\,(P\breve{Q})\text{‘}a = R\,(\breve{P}\breve{Q})\text{‘}a = R\text{‘}a$

$[*234\cdot522\cdot6]$

$*234\cdot64.\ \vdash :: \mathrm{Hp}\,*234\cdot62 . R\text{‘}a \in \mathrm{D}\text{‘}P \cap \mathrm{G}\text{‘}P . \supset :. a \in \mathrm{contin}\,(PQ)\text{‘}R . \equiv :$

$R\text{‘}a \in C\text{‘}P - C\text{‘}P_1 : R\text{‘}a \in P\,(z - w) . \supset_{z, w} .$

$(\exists y, y') . y, y' \in \mathrm{G}\text{‘}R . a \in Q\,(y - y') . R\text{‘‘}Q\,(y \mapsto\!\dashv y') \subset P\,(z - w)$

$[*234\cdot51\cdot6]$

***234·7.** $\vdash : R \in P\,\overline{\text{contin}}\,Q \,.\equiv .\, \exists !\, C\text{‘}Q \cap \text{Ɑ}\text{‘}R \,.\, C\text{‘}Q \cap \text{Ɑ}\text{‘}R \subset \text{contin}\,(PQ)\text{‘}R$
[(*234·04)]

***234·71.** $\vdash : R \in P\,\overline{\text{contin}}\,Q \,.\supset .\, R \upharpoonright C\text{‘}Q \in 1 \to \text{Cls} \,.\, R\text{‘‘}C\text{‘}Q \subset C\text{‘}P$

Dem.

$\vdash .\, *234\cdot 7\cdot 6\cdot 5 \,.\supset$

$\vdash :.\, \text{Hp} \,.\supset : a \in C\text{‘}Q \cap \text{Ɑ}\text{‘}R \,.\supset .\, R\text{‘}a \in \text{os}\,(P, Q_* \upharpoonright \overrightarrow{Q_{\text{po}}}\text{‘}a)\text{‘}R \,.$

$[*234\cdot 1] \qquad \supset .\, R\text{‘}a \in C\text{‘}P \,. \qquad (1)$

$[*14\cdot 21] \qquad \supset .\, E\,!\, R\text{‘}a \qquad (2)$

$\vdash .\, (2) \,.\, *71\cdot 572 \,. \quad \supset \vdash : \text{Hp} \,.\supset .\, R \upharpoonright C\text{‘}Q \in 1 \to \text{Cls} \qquad (3)$

$\vdash .\, (1) \,.\, (2) \,.\, *37\cdot 61 \,.\supset \vdash : \text{Hp} \,.\supset .\, R\text{‘‘}(C\text{‘}Q \cap \text{Ɑ}\text{‘}R) \subset C\text{‘}P \qquad (4)$

$\vdash .\, (3) \,.\, (4) \,.\, *37\cdot 26 \,.\supset \vdash .\, \text{Prop}$

***234·72.** $\vdash :.\, P \in \text{Ser} \,.\, Q \in \text{trans} \cap \text{connex} \,.\, R \in P\,\overline{\text{contin}}\,Q \,.\supset :$

$a \in C\text{‘}Q \cap \text{Ɑ}\text{‘}R \,.\supset_a .\, R\,(PQ)\text{‘}a = R\,(\breve{P}Q)\text{‘}a = R\,(P\breve{Q})\text{‘}a = R\,(\breve{P}\breve{Q})\text{‘}a = R\text{‘}a$
[*234·62·7]

***234·73.** $\vdash ::\, P \in \text{Ser} \,.\, P^2 = P \,.\, Q \in \text{trans} \cap \text{connex} \,.\supset :.$

$R \in P\,\overline{\text{contin}}\,Q \,.\equiv : \exists !\, C\text{‘}Q \cap \text{Ɑ}\text{‘}R : a \in C\text{‘}Q \cap \text{Ɑ}\text{‘}R \,.\supset_a .$

$R\,(PQ)\text{‘}a = R\,(\breve{P}Q)\text{‘}a = R\,(P\breve{Q})\text{‘}a = R\,(\breve{P}\breve{Q})\text{‘}a = R\text{‘}a$

Dem.

$\vdash .\, *234\cdot 7\cdot 71 \,.\supset \vdash ::\, \text{Hp} \,.\supset :.\, R \in P\,\overline{\text{contin}}\,Q \,.\equiv : \exists !\, C\text{‘}Q \cap \text{Ɑ}\text{‘}R \,.\, R\text{‘‘}C\text{‘}Q \subset C\text{‘}P :$

$a \in C\text{‘}Q \cap \text{Ɑ}\text{‘}R \,.\supset_a .\, a \in \text{contin}\,(PQ)\text{‘}R :$

$[*234\cdot 63] \equiv : \exists !\, C\text{‘}Q \cap \text{Ɑ}\text{‘}R \,.\, R\text{‘‘}C\text{‘}Q \subset C\text{‘}P : a \in C\text{‘}Q \cap \text{Ɑ}\text{‘}R \,.\supset_a .$

$R\,(PQ)\text{‘}a = R\,(\breve{P}Q)\text{‘}a = R\,(P\breve{Q})\text{‘}a = R\,(\breve{P}\breve{Q})\text{‘}a = R\text{‘}a \qquad (1)$

$\vdash .\, *233\cdot 401\cdot 101 \,.\supset$

$\vdash :.\, a \in C\text{‘}Q \cap \text{Ɑ}\text{‘}R \,.\supset_a .\, R\,(PQ)\text{‘}a = R\text{‘}a : \supset : a \in C\text{‘}Q \cap \text{Ɑ}\text{‘}R \,.\supset_a .\, R\text{‘}a \in C\text{‘}P :$

$[*37\cdot 61\cdot 26] \qquad \supset : R\text{‘‘}C\text{‘}Q \subset C\text{‘}P \qquad (2)$

$\vdash .\, (1) \,.\, (2) \,.\supset \vdash .\, \text{Prop}$

PRINCIPIA MATHEMATICA

BY

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AND

BERTRAND RUSSELL

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PREFACE TO VOLUME III

THE present volume continues the theory of series begun in Volume II, and then proceeds to the theory of measurement. Geometry we have found it necessary to reserve for a separate final volume.

In the theory of well-ordered series and compact series, we have followed Cantor closely, except in dealing with Zermelo's theorem (*257—8), and in cases where Cantor's work tacitly assumes the multiplicative axiom. Thus what novelty there is, is in the main negative. In particular, the multiplicative axiom is required in all known proofs of the fundamental proposition that the limit of a progression of ordinals of the second class (*i.e.* applicable to series whose fields have $\aleph_0$ terms) is an ordinal of the second class (cf. *265). In consequence of this fact, a very large part of the recognized theory of transfinite ordinals must be considered doubtful.

Part VI, on the theory of ratio and measurement, on the other hand, is new, though it is a development of the method initiated in Euclid Book V and continued by Burali-Forti*. Among other points in our treatment of quantity to which we wish to draw attention we may mention the following. (1) We regard our quantities as in a generalized sense "vectors," and therefore we regard ratios as holding between *relations*. (2) The hypothesis that the vectors concerned in any context form a *group*, which has generally been made prominent in such investigations, sinks with us into a very subordinate position, being sometimes not verified at all, and at other times a consequence of other more fruitful hypotheses. (3) We have developed a theory of ratios and real numbers which is prior to our theory of measurement, and yet is not purely arithmetical, *i.e.* does not treat ratios as mere couples of integers, but as relations between actual quantities such as two distances or two periods of time. (4) In our theory of "vector families," which are families of the kind to which some form of measurement is

* Cf. Peano's *Formulaire*, I. (1895), pp. 28—57.

applicable, we have been able to develop a very large part of their properties before introducing numbers; thus the theory of measurement results from the combination of two other theories, one a pure arithmetic of ratios and real numbers without reference to vectors, the other a pure theory of vectors without reference to ratios or real numbers. (5) With a view to geometrical applications, we have devoted a special Section to cyclic families, such as the angles about a given point in a given plane.

The theory of measurement developed in Part VI will be required in the next volume for the introduction of coordinates in Geometry.

We have to thank various friends for their kindness in bringing to our notice mistakes and misprints noted in the Errata, both in this and in previous volumes.

A. N. W.
B. R.

15 *February* 1913

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SECTION D.

WELL-ORDERED SERIES.

Summary of Section D.

A "well-ordered" series is one which is such that every existent class contained in it has a first term, or, what comes to the same thing, one which is such that every class which has successors has a sequent. We will call a relation in general well-ordered if every existent class contained in its field has one or more minima. Then a well-ordered series is a series which is a well-ordered relation.

Well-ordered series have many important properties not possessed by series in general. A well-ordered series is Dedekindian, except for the fact that it may have no last term; *i.e.* every section having a last term is Dedekindian. A well-ordered series which is not null has a first term, and every term of the series (except the last, if there is one) has an immediate successor. A very important property of well-ordered series is that they obey an extended form of mathematical induction, which we shall call "transfinite induction," namely the following: If σ is a class such that the sequent (if any) of any class contained in σ and in the series is a member of σ, then the whole series is contained in σ. (It will be observed that Λ is contained in σ, and therefore, by $*206{\cdot}14$, $B{`}P$ is a member of σ.) This differs from ordinary mathematical induction by the fact that, instead of dealing with the successors of single terms, it deals with the successors of classes. A closely analogous property, which holds for all well-ordered relations, whether serial or not, is the following. If σ is a class such that, whenever $\overrightarrow{P}{`}x \subset \sigma$, where $x$ is any member of $C{`}P$, x itself belongs to σ, then $C{`}P \subset \sigma$. If P is well-ordered, this property holds for all σ's; and conversely, if this property holds for all σ's, P is well-ordered. Hence this property is equivalent to well-orderedness.

If P is a well-ordered series, $\min_P$ selects one term out of each member of $\mathrm{Cl\,ex}{`}C{`}P$. Hence $C{`}P$, which is $\min_P{``}\mathrm{Cl\,ex}{`}C{`}P$, is a member of the multiplicative class of $\mathrm{Cl\,ex}{`}C{`}P$; hence the multiplicative class of $\mathrm{Cl\,ex}{`}C{`}P$ exists, and therefore the multiplicative class of any class contained in $\mathrm{Cl\,ex}{`}C{`}P$ exists (by $*88{\cdot}22$). It follows that if $s{`}\kappa$ can be well-ordered, and $\Lambda \sim\in \kappa$, the multiplicative class of κ exists; and that, if every class can be

well-ordered, the multiplicative axiom holds. The converse of this latter proposition also holds, as has been proved by Zermelo (cf. $*258$).

Another important set of properties of well-ordered series results from $*208{\cdot}41$ ff. Two ordinally similar well-ordered series can only be correlated in one way; and no proper section of a well-ordered series is ordinally similar to the whole series. (A "proper" section is a section not the whole.)

From the uniqueness of the correlator of two similar well-ordered series, it follows that all the uses of the multiplicative axiom in $*164$ can be avoided if the fields of the relations concerned consist of well-ordered series. *I.e.* taking $*164{\cdot}45$, which is the fundamental proposition in this subject, we have, without assuming the multiplicative axiom,

$$P, Q \in \text{Rel}^2\,\text{excl} \,.\supset :\, \exists\,!\, P \,\overline{\text{smor}}\, Q \cap \text{Rl}`\text{smor} \,.\equiv .\, P \,\text{smor smor}\, Q,$$

whenever $C`P$ and $C`Q$ consist of well-ordered series. Hence, under this hypothesis, the multiplicative axiom disappears from the hypotheses of all the consequences of $*164{\cdot}45$.

Ordinal numbers ($*251$) are defined as the relation-numbers of well-ordered series. (This definition is in accordance with usage: otherwise, there would be no special reason against defining "ordinal numbers" as the relation-numbers of series in general. The relation-numbers of series will be called *serial numbers.*) Sums of an ordinal number of ordinal numbers are ordinal numbers, but products of an ordinal number of ordinal numbers are not in general ordinal numbers. The product of an ordinal number of serial numbers is a serial number, and the product of an ordinal number (not zero) of ordinal numbers other than zero is not zero, *i.e.* a product of ordinal numbers, in which the number of factors is an ordinal number, does not vanish unless one of the factors vanishes. (For relations in general, the corresponding proposition requires the multiplicative axiom.) If ν is an ordinal number, and μ is any serial number, $\mu \exp_r \nu$ (*i.e.* μ^ν as it would naturally be called) is a serial number; but if $\mu > 1$, $\mu \exp_r \nu$ is not an *ordinal* number unless ν is finite.

The theory of sections and segments ($*252$, $*253$) is much simplified for well-ordered series, owing to the fact that every proper section has a sequent. Proper sections are identical with proper segments, and both are identical with $\overrightarrow{P}``C`P$. The series of sections, $\varsigma`P_*$, is $\overrightarrow{P};P \nrightarrow C`P$. The series of segments, $\varsigma`P$, is $\overrightarrow{P};P$ or $\overrightarrow{P};P \nrightarrow C`P$ according as there is or is not a last term of $C`P$. The series of sectional relations, P_ς, is $P \upharpoonright ; \overrightarrow{P} ; P \upharpoonright \text{Ɑ}`P \nrightarrow P$; its domain is $P \upharpoonright `` \overrightarrow{P}``C`P$, and its field is $P \upharpoonright `` \overrightarrow{P}``C`P \cup \iota`P$. If $x \in C`P$, $P \upharpoonright \overrightarrow{P}`x$ is never similar to P.

The theory of greater and less among well-ordered series and ordinal numbers is dealt with in *254 and *255. Cantor has proved, by means of segments, that of any two different ordinal numbers one must be the greater. This is proved by showing that of any two well-ordered series which are not similar, one must be similar to a segment of the other. We define an ordinal number α as *less than* another β if series P and Q can be found such that P is an α and Q is a β and P is similar to some relation contained in Q, but not to Q. It can be proved that all the ordinals less than $\mathrm{Nr}`Q$ belong, one each, to the proper segments of Q. Hence to say that the ordinal number of P is less than that of Q is equivalent to saying that there is a proper segment of Q to which P is similar.

When two series have the same ordinal, they also have the same cardinal, in virtue of *151·18, but the converse does not hold. When the cardinal number of one series is greater than that of the other, so is the ordinal number. When two classes can be well-ordered, any well-ordering will make the one class similar to a part of the other, or the other similar to a part of the one, in virtue of the properties of segments of well-ordered series. Hence of two different cardinals each of which is applicable to classes which can be well-ordered, one must be the greater—a property which cannot be proved concerning cardinals in general.

In *256 we deal with the series of ordinals in order of magnitude. We show that this is a well-ordered series. and that the series of all ordinals of a given type has an ordinal number which is greater than any of the ordinals of the given type. This constitutes the solution of Burali-Forti's paradox concerning the greatest ordinal: there is no greatest ordinal in any one type, and all the ordinals of a given type are surpassed by ordinals of higher types.

*257, *258 and *259 deal with "transfinite induction" and its applications, of which the most important is Zermelo's theorem, namely,

***258·34.** $\vdash :. \mu \sim \epsilon 1 . \supset : S \epsilon \epsilon_\Delta`\mathrm{Cl\,ex}`\mu . \equiv .$
$$(\exists P) . P \epsilon \Omega . C`P = \mu . S = \mathrm{min}_P \upharpoonright \mathrm{Cl\,ex}`\mu$$

where Ω is the class of well-ordered series. This proposition leads to the following:

***258·36.** $\vdash : \mu \epsilon C``\Omega \cup 1 . \equiv . \exists ! \epsilon_\Delta`\mathrm{Cl\,ex}`\mu$

I.e. a class can be well-ordered or is a unit class when, and only when, a selection can be made from its existent sub-classes. Hence we arrive at

***258·37.** $\vdash : \mathrm{Mult\,ax} . \equiv . C``\Omega \cup 1 = \mathrm{Cls}$

I.e. the multiplicative axiom is equivalent to the assumption that every class can be well-ordered or consists of a single member.

The proof of Zermelo's theorem uses an extension to transfinite induction of the ideas of *90 and *91, which is explained in *257.

$*250$. ELEMENTARY PROPERTIES OF WELL-ORDERED SERIES.

Summary of $*250$.

A relation is called "well-ordered" when every existent sub-class of its field has one or more minima. A well-ordered series is defined as a well-ordered relation which is a series. We shall denote the class of well-ordered relations by "Bord," which is an abbreviation for "bene ordinata" or "bien ordonnée." The class of well-ordered series will be denoted by Ω. Thus our definitions are

$$\mathrm{Bord} = \hat{P}(\mathrm{Cl}\,\mathrm{ex}`C`P \subset \text{Ɑ}`\mathrm{min}_P) \quad \mathrm{Df},$$

$$\Omega = \mathrm{Ser} \cap \mathrm{Bord} \quad \mathrm{Df}.$$

Well-ordered relations other than series will be seldom referred to after the present number.

By applying the definition of "Bord" to unit classes, it appears that a well-ordered relation must be contained in diversity ($*250{\cdot}104$). A well-ordered relation is one whose existent upper sections all have minima ($*250{\cdot}102$). Hence by $*211{\cdot}17$,

$*250{\cdot}103$. $\vdash : P \in \mathrm{Bord} . \equiv . P_{\mathrm{po}} \in \mathrm{Bord}$

Hence by $*250{\cdot}104$,

$*250{\cdot}105$. $\vdash : P \in \mathrm{Bord} . \supset . P_{\mathrm{po}} \mathrel{\dot{\subset}} J$

By considering couples, it can be shown ($*250{\cdot}111$) that a well-ordered relation in which no class has more than one minimum is connected; hence by $*204{\cdot}16$ and $*250{\cdot}105$, it is a series. Thus we have

$*250{\cdot}125$. $\vdash : P \in \Omega . \equiv . \mathrm{E}\,!!\, \mathrm{min}_P``\mathrm{Cl}\,\mathrm{ex}`C`P$,

I.e. a well-ordered series is a relation such that every existent sub-class of the field has a unique minimum. This might have been taken as the definition of Ω.

By the definition of Ω we have

$$*250{\cdot}121.\ \vdash :. P \in \Omega . \equiv : P \in \mathrm{Ser} : \alpha \subset C`P . \exists ! \alpha . \supset_\alpha . \mathrm{E}!\,\mathrm{min}_P`\alpha :$$
$$\equiv : P \in \mathrm{Ser} : \exists ! \alpha \cap C`P . \supset_\alpha . \mathrm{E}!\,\mathrm{min}_P`\alpha$$

Applying this to $C`P$ we have

$*250{\cdot}13$. $\vdash : P \in \Omega - \iota`\dot{\Lambda} . \supset . \mathrm{E}!\,B`P$

We have also

$*250{\cdot}141$. $\vdash : P \epsilon \Omega . \supset . P \upharpoonleft \alpha \epsilon \Omega$

$*250{\cdot}17$. $\vdash :. P, Q \epsilon \Omega - \iota`\dot{\Lambda} . \supset : P \text{ smor } Q . \equiv . P \upharpoonleft \text{Ǝ}`P \text{ smor } Q \upharpoonleft \text{Ǝ}`Q$

This proposition justifies the subtraction of $\dot{1}$ from the beginning, and is useful in the theory of segments of well-ordered series.

We have next ($*250{\cdot}2$—$\cdot243$) an important set of propositions on P_1 when $P \epsilon \Omega$. The most useful of these is

$*250{\cdot}21$. $\vdash : P \epsilon \Omega . \supset . \text{D}`P = \text{D}`P_1$

I.e. in a well-ordered series every term except the last (if any) has an immediate successor. (It is not in general the case that every term except the first has an immediate predecessor.) Another useful proposition is

$*250{\cdot}242$. $\vdash : P \epsilon \Omega . \supset . P = P_1 \cup P_1 | P$

The next set of propositions ($*250{\cdot}3$—$\cdot362$) is concerned with "transfinite induction." We have

$*250{\cdot}33$. $\vdash . \Omega = \text{connex} \cap \hat{P} \{\alpha \subset C`P \cap \sigma . \supset_\alpha . \overrightarrow{\text{seq}_P}`\alpha \subset \sigma : \supset_\sigma . C`P \subset \sigma\}$

I.e. a well-ordered series is a connected relation P such that the whole field of P is contained in every class σ which is such that the sequent (if any) of every sub-class of $C`P \cap \sigma$ is a member of σ.

$*250{\cdot}35$. $\vdash . \text{Bord} = \hat{P} \{x \epsilon C`P . \overrightarrow{P}`x \subset \sigma . \supset_x . x \epsilon \sigma : \supset_\sigma . C`P \subset \sigma\}$

I.e. a well-ordered relation is a relation P whose field is contained in every class σ which contains every member of $C`P$ whose predecessors are all contained in $\sigma$. We may say that a property is "transfinitely hereditary" in $P$ if it belongs to the sequents of all classes composed of members of $C`P$ which possess the property. In virtue of $*250{\cdot}33$, if P is well-ordered, every transfinitely hereditary property belongs to every member of $C`P$, and conversely.

Our next set of propositions ($*250{\cdot}4$—$\cdot44$) is concerned with $\dot{\Lambda}$ and couples. We prove that $\dot{\Lambda} \epsilon \Omega$ ($*250{\cdot}4$) and that $x \neq y . \supset . x \downarrow y \epsilon \Omega$ ($*250{\cdot}41$).

$*250{\cdot}5$—$\cdot54$ are concerned with selections. We have

$*250{\cdot}5$. $\vdash : P \epsilon \Omega . \supset .$

$$\text{min}_P \upharpoonright \text{Cl ex}`C`P \epsilon \epsilon_\Delta`\text{Cl ex}`C`P . \iota`C`P = \text{Prod}`\text{Cl ex}`C`P$$

whence

$*250{\cdot}51$. $\vdash : \alpha \epsilon C``\Omega . \supset . \exists ! \epsilon_\Delta`\text{Cl ex}`\alpha$

Observe that $C``\Omega$ is the class of those classes that can be well-ordered. From $*250{\cdot}51$ we deduce

$*250{\cdot}54$. $\vdash : C``\Omega \cup 1 = \text{Cls} . \supset . \text{Mult ax}$

The converse, which is Zermelo's theorem, is proved in $*258$.

*250·6—·67 are concerned with consequences of *208. We show that two well-ordered series cannot have more than one correlator (*250·6); that if P is a well-ordered series, and β is contained in a proper section of P, $P \upharpoonright \beta$ is not similar to P (*250·65); and that if P is any well-ordered relation, and α is any class such that there are terms in $C'P$ which are later than any member of $\alpha \cap C'P$, then P is not similar to $P \upharpoonright \alpha$ (*250·67).

***250·01.** $\text{Bord} = \hat{P}\,(\text{Cl ex}'C'P \subset \text{Q}'\text{min}_P)$ Df

***250·02.** $\Omega = \text{Ser} \cap \text{Bord}$ Df

***250·1.** $\vdash : P \in \text{Bord} . \equiv . \text{Cl ex}'C'P \subset \text{Q}'\text{min}_P$ [(*250·01)]

***250·101.** $\vdash :. P \in \text{Bord} . \equiv : \exists ! \alpha \cap C'P . \supset_\alpha . \exists ! \overrightarrow{\text{min}}_P{}'\alpha$ [*250·1 . *205·15]

***250·102.** $\vdash : P \in \text{Bord} . \equiv . \text{sect}'\breve{P} - \iota'\Lambda \subset \text{Q}'\text{min}_P$

Dem.

$\vdash . *250·1 . \supset \vdash : P \in \text{Bord} . \supset . \text{sect}'\breve{P} - \iota'\Lambda \subset \text{Q}'\text{min}_P$ (1)

$\vdash . *205·19 . \supset \vdash . \overrightarrow{\text{min}}\,(P_{\text{po}})'\alpha = \overrightarrow{\text{min}}\,(P_{\text{po}})'\breve{P}_*{}''\alpha$

[*205·68] $= \overrightarrow{\text{min}}_P{}'\breve{P}_*{}''\alpha$ (2)

$\vdash . *90·331 . *211·13 . \supset \vdash : \exists ! \alpha \cap C'P . \supset . \breve{P}_*{}''\alpha \in \text{sect}'\breve{P} - \iota'\Lambda$ (3)

$\vdash . (3) . \supset \vdash :. \text{sect}'\breve{P} - \iota'\Lambda \subset \text{Q}'\text{min}_P . \supset : \exists ! \alpha \cap C'P . \supset_\alpha . \exists ! \overrightarrow{\text{min}}_P{}'(\breve{P}_*{}''\alpha) .$

[(2)] $\supset_\alpha . \exists ! \overrightarrow{\text{min}}\,(P_{\text{po}})'\alpha .$

[*205·26] $\supset_\alpha . \exists ! \overrightarrow{\text{min}}_P{}'\alpha :$

[*250·101] $\supset : P \in \text{Bord}$ (4)

$\vdash . (1) . (4) . \supset \vdash . \text{Prop}$

***250·103.** $\vdash : P \in \text{Bord} . \equiv . P_{\text{po}} \in \text{Bord}$ [*250·102 . *211·17]

***250·104.** $\vdash . \text{Bord} \subset \text{Rl}'J$

Dem.

$\vdash . *250·1 . \supset \vdash : P \in \text{Bord} . x \in C'P . \supset . x \in \overrightarrow{\text{min}}_P{}'\iota'x .$

[*205·194] $\supset . \sim(xPx) : \supset \vdash . \text{Prop}$

***250·105.** $\vdash : P \in \text{Bord} . \supset . P_{\text{po}} \mathrel{\text{G}} J$ [*250·103·104]

***250·11.** $\vdash :: P \in \text{connex} . \supset :. P \in \text{Bord} . \equiv : \exists ! \alpha \cap C'P . \supset_\alpha . \text{E} ! \text{min}_P{}'\alpha :$

$\equiv : \alpha \subset C'P . \exists ! \alpha . \supset_\alpha . \text{E} ! \text{min}_P{}'\alpha$

[*250·1·101 . *205·32]

***250·111.** $\vdash :. P \in \text{Bord} . \supset : P \in \text{connex} . \equiv . \text{min}_P \in 1 \rightarrow \text{Cls}$

Dem.

$\vdash . *250·1 . *71·1 . \supset$

$\vdash :: P \in \text{Bord} . \text{min}_P \in 1 \rightarrow \text{Cls} . \supset :. x, y \in C'P . \supset : (\iota'x \cup \iota'y) - \breve{P}''(\iota'x \cup \iota'y) \in 1 :$

[*54·4] $\supset : \iota'x \cup \iota'y - \breve{P}''(\iota'x \cup \iota'y) = \iota'x . \mathbf{v} .$

$\iota'x \cup \iota'y - \breve{P}''(\iota'x \cup \iota'y) = \iota'y$ (1)

$\vdash . (1) . \supset \vdash :. P \in \text{Bord} . \min_P \in 1 \rightarrow \text{Cls} . x, y \in C'P . x \neq y . \supset :$

$\quad y \in \breve{P}``(\iota'x \cup \iota'y) . \vee . x \in \breve{P}``(\iota'x \cup \iota'y) :$

$[*250{\cdot}104] \quad \supset : xPy . \vee . yPx \quad (2)$

$\vdash . (2) . *202{\cdot}103 . \supset \vdash : P \in \text{Bord} . \min_P \in 1 \rightarrow \text{Cls} . \supset . P \in \text{connex} \quad (3)$

$\vdash . (3) . *205{\cdot}31 . \supset \vdash . \text{Prop}$

$*250{\cdot}112$. $\vdash : P \in \text{connex} \cap \text{Bord} . \equiv . E !! \min_P``\text{Cl ex}'C'P$

Dem.

$\vdash . *250{\cdot}1{\cdot}111 . \supset$

$\vdash : P \in \text{connex} \cap \text{Bord} . \equiv . \min_P \in 1 \rightarrow \text{Cls} . \text{Cl ex}'C'P \subset \text{Q}'\min_P .$

$[*71{\cdot}16] \quad \equiv . E !! \min_P``\text{Q}'\min_P . \text{Cl ex}'C'P \subset \text{Q}'\min_P .$

$[*205{\cdot}15{\cdot}16] \quad \equiv . E !! \min_P``\text{Cl ex}'C'P : \supset \vdash . \text{Prop}$

$*250{\cdot}113$. $\vdash . \text{connex} \cap \text{Bord} = \Omega$

Dem.

$\vdash . *204{\cdot}1 . (*250{\cdot}02) . \supset \vdash . \Omega \subset \text{connex} \cap \text{Bord} \quad (1)$

$\vdash . *250{\cdot}105 . \supset \vdash : P \in \text{connex} \cap \text{Bord} . \supset . P \in \text{connex} . P_{\text{po}} \in J .$

$[*204{\cdot}16] \quad \supset . P \in \text{Ser} \quad (2)$

$\vdash . (2) . (*250{\cdot}02) . \supset \vdash : P \in \text{connex} \cap \text{Bord} . \supset . P \in \Omega \quad (3)$

$\vdash . (1) . (3) . \supset \vdash . \text{Prop}$

$*250{\cdot}12$. $\vdash : P \in \Omega . \equiv . P \in \text{Ser} \cap \text{Bord} \quad [(*250{\cdot}02)]$

$*250{\cdot}121$. $\vdash :. P \in \Omega . \equiv : P \in \text{Ser} : \alpha \subset C'P . \exists ! \alpha . \supset_\alpha . E ! \min_P{}'\alpha :$

$\quad \equiv : P \in \text{Ser} : \exists ! \alpha \cap C'P . \supset_\alpha . E ! \min_P{}'\alpha \quad [*250{\cdot}12{\cdot}11]$

$*250{\cdot}122$. $\vdash :. P \in \Omega . \equiv : P \in \text{Ser} : \exists ! C'P \cap p'\overleftarrow{P}``(\alpha \cap C'P) . \supset_\alpha . E ! \text{seq}_P{}'\alpha$

Dem.

$\vdash . *206{\cdot}13 . *250{\cdot}121 . \supset$

$\vdash :. P \in \Omega . \supset : P \in \text{Ser} : \exists ! C'P \cap p'\overleftarrow{P}``(\alpha \cap C'P) . \supset_\alpha . E ! \text{seq}_P{}'\alpha \quad (1)$

$\vdash . *204{\cdot}62 . \supset$

$\vdash : P \in \text{Ser} . \exists ! \alpha \cap C'P . \supset . \exists ! C'P \cap p'\overleftarrow{P}``p'\overrightarrow{P}``(\alpha \cap C'P) .$

$[*40{\cdot}62] \quad \supset . \exists ! C'P \cap p'\overleftarrow{P}``\{C'P \cap p'\overrightarrow{P}``(\alpha \cap C'P)\} \quad (2)$

$\vdash . (2) . *10{\cdot}1 . \supset$

$\vdash :. P \in \text{Ser} : \exists ! C'P \cap p'\overleftarrow{P}``(\alpha \cap C'P) . \supset_\alpha . E ! \text{seq}_P{}'\alpha : \supset :$

$\quad \exists ! \alpha \cap C'P . \supset_\alpha . E ! \text{seq}_P{}'\{C'P \cap p'\overrightarrow{P}``(\alpha \cap C'P)\} .$

$[*206{\cdot}131{\cdot}54] \quad \supset_\alpha . E ! \min_P{}'\alpha :$

$[*250{\cdot}121] \supset : P \in \Omega \quad (3)$

$\vdash . (1) . (3) . \supset \vdash . \text{Prop}$

∗250·123. $\vdash :. P \epsilon \Omega - \iota`\dot{\Lambda} . \equiv : P \epsilon \mathrm{Ser} : \exists ! p`\overleftarrow{P}``(\alpha \cap C`P) . \supset_\alpha . E ! \mathrm{seq}_P`\alpha$

Dem.

$\vdash . ∗250·122 . \supset$

$\vdash :. P \epsilon \mathrm{Ser} : \exists ! p`\overleftarrow{P}``(\alpha \cap C`P) . \supset_\alpha . E ! \mathrm{seq}_P`\alpha : \supset . P \epsilon \Omega$ (1)

$\vdash . ∗40·6 . ∗24·52 . \supset$

$\vdash :. \exists ! p`\overleftarrow{P}``(\alpha \cap C`P) . \supset_\alpha . E ! \mathrm{seq}_P`\alpha : \supset . E ! \mathrm{seq}_P`\Lambda .$

[∗206·18] $\supset . \dot{\exists} ! P$ (2)

$\vdash . ∗250·122 . ∗40·62 . \supset$

$\vdash :. P \epsilon \Omega . \supset : P \epsilon \mathrm{Ser} : \exists ! \alpha \cap C`P . \exists ! p`\overleftarrow{P}``(\alpha \cap C`P) . \supset_\alpha . E ! \mathrm{seq}_P`\alpha$ (3)

$\vdash . ∗206·14 . \supset \vdash : \alpha \cap C`P = \Lambda . \supset . \overrightarrow{\mathrm{seq}_P}`\alpha = \overrightarrow{B}`P$

[∗205·12] $= \overrightarrow{\min_P}`C`P$ (4)

$\vdash . ∗33·24 . ∗250·121 . \supset \vdash : P \epsilon \Omega - \iota`\dot{\Lambda} . \supset . E ! \min_P`C`P$ (5)

$\vdash . (4) . (5) . \supset \vdash : P \epsilon \Omega - \iota`\dot{\Lambda} . \alpha \cap C`P = \Lambda . \supset . E ! \mathrm{seq}_P`\alpha$ (6)

$\vdash . (3) . (6) . \supset$

$\vdash :. P \epsilon \Omega - \iota`\dot{\Lambda} . \supset : P \epsilon \mathrm{Ser} : \exists ! p`\overleftarrow{P}``(\alpha \cap C`P) . \supset_\alpha . E ! \mathrm{seq}_P`\alpha$ (7)

$\vdash . (1) . (2) . (7) . \supset \vdash .$ Prop

∗250·124. $\vdash : P \epsilon \Omega . \equiv . P \epsilon \mathrm{Ser} . \mathrm{sect}`P - \iota`C`P \subset \mathrm{Cl}`\mathrm{seq}_P$

Dem.

$\vdash . ∗250·122 . ∗211·703 . \supset \vdash : P \epsilon \Omega . \supset . P \epsilon \mathrm{Ser} . \mathrm{sect}`P - \iota`C`P \subset \mathrm{Cl}`\mathrm{seq}_P$ (1)

$\vdash . ∗211·7 . \quad \supset \vdash :. P \epsilon \mathrm{Ser} . \mathrm{sect}`P - \iota`C`P \subset \mathrm{Cl}`\mathrm{seq}_P . \supset :$

$\beta \epsilon \mathrm{sect}`\breve{P} - \iota`\Lambda . \supset_\beta . E ! \mathrm{seq}_P`(C`P - \beta) .$

[∗211·723] $\supset_\beta . E ! \min_P`\beta :$

[∗250·102·12] $\supset : P \epsilon \Omega$ (2)

$\vdash . (1) . (2) . \supset \vdash .$ Prop

∗250·125. $\vdash : P \epsilon \Omega . \equiv . E !! \min_P``\mathrm{Cl\,ex}`C`P$ [∗250·112·113]

The above proposition might be demonstrated, independently of ∗250·112·113, as follows:

(*a*) If $E !! \min_P``\mathrm{Cl\,ex}`C`P$, it follows that $x \epsilon C`P . \supset . E ! \min_P`\iota`x$, whence $x \epsilon C`P . \supset . \sim(xPx)$, whence $P \mathrel{\mathsf{G}} J$.

(*b*) If $E !! \min_P``\mathrm{Cl\,ex}`C`P$, it follows that

$$x, y \epsilon C`P . x \neq y . \supset . E ! \min_P`(\iota`x \cup \iota`y).$$

whence it follows that

$$xPy . \sim(yPx) . \vee . yPx . \sim(xPy).$$

Hence $$P \epsilon \mathrm{connex} . P^2 \mathrel{\mathsf{G}} J.$$

(*c*) If $E !! \min_P``\mathrm{Cl\,ex}`C`P$ it follows that

$$xPy . yPz . \supset . E ! \min_P`(\iota`x \cup \iota`y \cup \iota`z),$$

whence $xPy \,.\, yPz \,.\, \supset \,.\, \sim(zPx)$,

and by $P^2 \mathrel{\mathsf{G}} J$ (which has just been proved)

$$xPy \,.\, yPz \,.\, \supset \,.\, x \neq z.$$

Hence, since, by (b), $P \in \text{connex}$, we must have

$$xPy \,.\, yPz \,.\, \supset \,.\, xPz, \;\; i.e. \;\; P \in \text{trans}.$$

Hence $\quad E \,!!\, \min_P``\text{Cl ex}`C`P \,.\, \supset \,.\, P \in \text{Ser}.$

Hence the above proposition is obvious.

***250·126.** $\vdash : P \in \Omega \,.\, E\,!\, \max_P`\alpha \,.\, \sim E\,!\, \text{seq}_P`\alpha \,.\, \supset \,.\, B`\breve{P} \in \alpha \,.\, B`\breve{P} = \max_P`\alpha$

Dem.

$\vdash \,.\, *250·123 \,.\, \text{Transp} \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, \sim \exists\,!\, p`\overleftarrow{P}``(\alpha \cap C`P) \,.$

$[*205·65] \qquad \supset \,.\, \sim \exists\,!\, \overleftarrow{P}`\max_P`\alpha \,.$

$[*33·4] \qquad \supset \,.\, \max_P`\alpha \sim\in D`P \,.$

$[*93·103] \qquad \supset \,.\, \max_P`\alpha \in \overrightarrow{B}`\breve{P} \,.$

$[*202·52] \qquad \supset \,.\, \max_P`\alpha = B`\breve{P} : \supset \vdash \,.\, \text{Prop}$

***250·13.** $\vdash : P \in \Omega - \iota`\dot{\Lambda} \,.\, \supset \,.\, E\,!\, B`P$

Dem.

$\vdash \,.\, *33·24 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, \exists\,!\, C`P \,.$

$[*250·121] \qquad \supset \,.\, E\,!\, \min_P`C`P \,.$

$[*205·12] \qquad \supset \,.\, E\,!\, B`P : \supset \vdash \,.\, \text{Prop}$

***250·131.** $\vdash :. P \in \Omega \,.\, \supset : \dot{\exists}\,!\, P \,.\, \equiv \,.\, E\,!\, B`P$

Dem.

$\vdash \,.\, *93·102 \,.\, *33·24 \,.\, \supset \vdash : E\,!\, B`P \,.\, \supset \,.\, \dot{\exists}\,!\, P \qquad (1)$

$\vdash \,.\, (1) \,.\, *250·13 \,.\, \supset \vdash \,.\, \text{Prop}$

***250·14.** $\vdash : P \in \text{Bord} \,.\, \supset \,.\, \text{Rl}`P \subset \text{Bord}$

Dem.

$\vdash \,.\, *250·1 \,.\, *205·26 \,.\, \supset$

$\vdash : P \in \text{Bord} \,.\, Q \mathrel{\mathsf{G}} P \,.\, \supset \,.\, \text{Cl ex}`C`P \subset \text{Cl}`\min_P \,.\, \min_P \upharpoonright \text{Cl ex}`C`Q \mathrel{\mathsf{G}} \min_Q \,. \qquad (1)$

$[*60·42 . *35·64] \supset \,.\, \text{Cl ex}`C`Q \subset \text{Cl ex}`C`P \,.\, \text{Cl}`\min_P \cap \text{Cl ex}`C`Q \subset \text{Cl}`\min_Q \qquad (2)$

$\vdash \,.\, (1) \,.\, (2) \,.\, *22·44·621 \,.\, \supset \vdash : P \in \text{Bord} \,.\, Q \mathrel{\mathsf{G}} P \,.\, \supset \,.\, \text{Cl ex}`C`Q \subset \text{Cl}`\min_Q \,.$

$[*250·1] \qquad \supset \,.\, Q \in \text{Bord} : \supset \vdash \,.\, \text{Prop}$

***250·141.** $\vdash : P \in \Omega \,.\, \supset \,.\, P \upharpoonright\!\!\upharpoonright \alpha \in \Omega \quad [*250·14 \,.\, *204·4]$

***250·142.** $\vdash : P \in \text{Bord} \,.\, \supset \,.\, \text{Rl}`P \cap \text{connex} \subset \Omega$

Dem.

$\vdash \,.\, *250·14 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, \text{Rl}`P \cap \text{connex} \subset \text{Bord} \cap \text{connex}$

$[*250·113] \qquad \subset \Omega : \supset \vdash \,.\, \text{Prop}$

***250·15.** $\vdash : P \epsilon \Omega \,.\, \mathrm{E} ! B`\breve{P} \,.\, \supset \,.\, P \epsilon \mathrm{Ded}$

Dem.

$$\vdash \,.\, *250{\cdot}101 \,.\, \supset \vdash :.\, \mathrm{Hp} \,.\, \supset : \exists ! \alpha \cap C`P \,.\, \supset_\alpha \,.\, \exists ! \overrightarrow{\mathrm{min}_P}`\alpha \quad (1)$$

$$\vdash \,.\, *206{\cdot}14 \,.\, \supset \vdash :.\, \mathrm{Hp} \,.\, \supset : \alpha \cap C`P = \Lambda \,.\, \supset_\alpha \,.\, \exists ! \overrightarrow{\mathrm{prec}_P}`\alpha \quad (2)$$

$$\vdash \,.\, (1) \,.\, (2) \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, (\alpha) \,.\, \exists ! (\overrightarrow{\mathrm{min}_P}`\alpha \cup \overrightarrow{\mathrm{prec}_P}`\alpha) \,.$$

$$[*214{\cdot}1] \qquad \supset \,.\, \breve{P} \epsilon \mathrm{Ded} \,.$$

$$[*214{\cdot}14] \qquad \supset \,.\, P \epsilon \mathrm{Ded} : \supset \vdash \,.\, \mathrm{Prop}$$

***250·151.** $\vdash : P \epsilon \Omega \,.\, x \epsilon \mathrm{G}`P \,.\, \supset \,.\, P \upharpoonright \overrightarrow{P_*}`x \epsilon \mathrm{Ded}$

Dem.

$$\vdash \,.\, *250{\cdot}141 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, P \upharpoonright \overrightarrow{P_*}`x \epsilon \Omega \quad (1)$$

$$\vdash \,.\, *205{\cdot}41 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, \overrightarrow{B}`\mathrm{Cnv}`(P \upharpoonright \overrightarrow{P_*}`x) = \overrightarrow{\mathrm{max}_P}`\overrightarrow{P_*}`x$$

$$[*205{\cdot}197] \qquad = \iota`x \,.$$

$$[*53{\cdot}3] \qquad \supset \,.\, \mathrm{E} ! B`\mathrm{Cnv}`(P \upharpoonright \overrightarrow{P_*}`x) \quad (2)$$

$$\vdash \,.\, (1) \,.\, (2) \,.\, *250{\cdot}15 \,.\, \supset \vdash \,.\, \mathrm{Prop}$$

***250·152.** $\vdash \,.\, \Omega \subset \mathrm{semi\,Ded} \quad [*214{\cdot}7 \,.\, *250{\cdot}124]$

***250·16.** $\vdash : P \epsilon \Omega \,.\, \exists ! \alpha \cap C`P \,.\, \supset \,.\, \overrightarrow{P}`\mathrm{min}_P`\alpha = p`\overrightarrow{P}``(\alpha \cap C`P)$
$[*205{\cdot}65 \,.\, *250{\cdot}121]$

***250·17.** $\vdash :.\, P, Q \epsilon \Omega - \iota`\dot{\Lambda} \,.\, \supset : P \,\mathrm{smor}\, Q \,.\, \equiv \,.\, P \upharpoonright \mathrm{G}`P \,\mathrm{smor}\, Q \upharpoonright \mathrm{G}`Q$
$[*204{\cdot}47 \,.\, *250{\cdot}13]$

This proposition is useful in connection with the series of segmental relations in a well-ordered series, for the series of proper segmental relations in a well-ordered series is (as will be proved later)

$$P \upharpoonright ; \overrightarrow{P} ; P \upharpoonright \mathrm{G}`P,$$

and this is ordinally similar to $P \upharpoonright \mathrm{G}`P$. Hence, by the above proposition, two well-ordered series which are not null are ordinally similar when, and only when, the series of their segmental relations are ordinally similar.

***250·2.** $\vdash : P \epsilon \mathrm{Bord} \,.\, \supset \,.\, \mathrm{D}`P = \mathrm{D}`(P \dot{-} P^2)$

Dem.

$$\vdash \,.\, *33{\cdot}4 \,.\, \supset \vdash : x \epsilon \mathrm{D}`P \,.\, \equiv \,.\, \exists ! \overleftarrow{P}`x \quad (1)$$

$$\vdash \,.\, *250{\cdot}1 \,.\, *205{\cdot}16 \,.\, \supset \vdash :.\, P \epsilon \mathrm{Bord} \,.\, \supset : \exists ! \overleftarrow{P}`x \,.\, \equiv \,.\, \exists ! \overrightarrow{\mathrm{min}_P}`\overleftarrow{P}`x \,.$$

$$[*205{\cdot}251] \qquad \equiv \,.\, x \epsilon \mathrm{D}`(P \dot{-} P^2) \quad (2)$$

$$\vdash \,.\, (1) \,.\, (2) \,.\, \supset \vdash \,.\, \mathrm{Prop}$$

***250·21.** $\vdash : P \in \Omega . \supset . \mathrm{D}'P = \mathrm{D}'P_1$ [*201·63 . *250·2]

In virtue of this proposition, every term of a well-ordered series (except the last, if any) has an immediate successor.

***250·22.** $\vdash : P \in \mathrm{Ser} \cap \mathrm{Ded} . \mathrm{D}'P = \mathrm{D}'P_1 . \supset . P \in \Omega - \iota'\dot{\Lambda}$

Dem.

$\vdash . *214\cdot101 . \supset \vdash : \mathrm{Hp} . \sim \mathrm{E}!\, \mathrm{max}_P`\alpha . \supset . \mathrm{E}!\, \mathrm{seq}_P`\alpha$ (1)

$\vdash . *206\cdot45 . \supset \vdash : \mathrm{Hp} . \mathrm{max}_P`\alpha \in \mathrm{D}'P . \supset . \mathrm{E}!\, \overrightarrow{\mathrm{seq}_P}`\mathrm{max}_P`\alpha .$

[*206·46] $\supset . \mathrm{E}!\, \breve{\mathrm{seq}}_P`\alpha$ (2)

$\vdash . (1) . (2) . \supset \vdash :. \mathrm{Hp} . \supset : \sim(\mathrm{max}_P`\alpha = B`\breve{P}) . \supset_\alpha . \mathrm{E}!\, \mathrm{seq}_P`\alpha :$

[*93·118] $\supset : \sim(B`\breve{P} \in \alpha) . \supset_\alpha . \mathrm{E}!\, \mathrm{seq}_P`\alpha :$

[*202·511.*214·5] $\supset : \exists!\, p`\overleftarrow{P}``(\alpha \cap C`P) . \supset_\alpha . \mathrm{E}!\, \mathrm{seq}_P`\alpha :$

[*250·123] $\supset : P \in \Omega - \iota'\dot{\Lambda} :. \supset \vdash . \mathrm{Prop}$

***250·23.** $\vdash : P \in \Omega . \mathrm{E}!\, B`\breve{P} . \equiv . P \in \mathrm{Ser} \cap \mathrm{Ded} . \mathrm{D}'P = \mathrm{D}'P_1$

Dem.

$\vdash . *250\cdot22 . *214\cdot5 . \supset \vdash : P \in \mathrm{Ser} \cap \mathrm{Ded} . \mathrm{D}'P = \mathrm{D}'P_1 . \supset . P \in \Omega . \mathrm{E}!\, B`\breve{P}$ (1)

$\vdash . *250\cdot15\cdot21 . \supset \vdash : P \in \Omega . \mathrm{E}!\, B`\breve{P} . \supset . P \in \mathrm{Ser} \cap \mathrm{Ded} . \mathrm{D}'P = \mathrm{D}'P_1$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***250·24.** $\vdash : P \in \Omega . \supset . P^2 \mid \breve{P}_1 = P \upharpoonright \mathrm{D}'P$

Dem.

$\vdash . *201\cdot1 . *13\cdot12 . \supset \vdash :. \mathrm{Hp} . xP^2z . \supset : yPx . \supset . yP^2z : y = x . \supset . yP^2z :$

[Transp] $\supset : \sim(yP^2z) . \supset . \sim(yPx) . y \neq x :$

[*201·63.*202·103] $\supset : yP_1z . \supset . xPy$ (1)

$\vdash . (1) . *201\cdot63 . \supset \vdash : \mathrm{Hp} . xP^2z . z\breve{P}_1y . \supset . xPy . x, y \in \mathrm{D}'P$ (2)

$\vdash . *250\cdot21 . \supset \vdash : \mathrm{Hp} . x, y \in \mathrm{D}'P . xPy . \supset . (\exists z) . yP_1z .$

[*201·63] $\supset . (\exists z) . yPz . z\breve{P}_1y .$

[*34·1] $\supset . x(P^2 \mid \breve{P}_1)y$ (3)

$\vdash . (2) . (3) . \supset \vdash . \mathrm{Prop}$

***250·241.** $\vdash : P \in \Omega . \supset . \breve{P}_1 \mid P^2 = (\mathrm{G}'P_1) \upharpoonleft P$ [Proof as in *250·24]

***250·242.** $\vdash : P \in \Omega . \supset . P = P_1 \cup P_1 \mid P$

Dem.

$\vdash . *201\cdot63 . \supset \vdash :: \mathrm{Hp} . \supset :. xPy . \equiv : xP_1y . \vee . xP^2y :$

[*250·21] $\equiv : xP_1y . \vee . (\exists z) . xP_1z . xP^2y :$

[*250·241] $\equiv : xP_1y . \vee . (\exists z) . xP_1z . zPy :: \supset \vdash . \mathrm{Prop}$

***250·243.** $\vdash : P \in \Omega . \supset . P \upharpoonright \text{Ɑ}`P_1 = (\text{Ɑ}`P_1) \upharpoonleft (P_1 \cup P \mid P_1)$

[Proof as in *250·242]

The following propositions deal with the extended form of mathematical induction which is characteristic of well-ordered series.

***250·3.** $\vdash :. P \in \text{Bord} : \alpha \subset C`P \cap \sigma . \supset_\alpha . \overrightarrow{\text{seq}}_P`\alpha \subset \sigma : \supset . C`P \subset \sigma$

Dem.

$\vdash . *250·101 . \supset \vdash : P \in \text{Bord} . \exists ! C`P - \sigma . \supset . \exists ! \overrightarrow{\text{min}}_P`(C`P - \sigma) .$

$[*205·14] \qquad \supset . (\exists x) . x \in C`P - \sigma . \overrightarrow{P}`x \subset \sigma .$

$[*206·4.*250·104] \supset . (\exists x) . x \in C`P - \sigma . \overrightarrow{P}`x \subset \sigma . x \,\text{seq}_P (\overrightarrow{P}`x) .$

$[*13·195] \qquad \supset . (\exists x, \alpha) . \alpha = \overrightarrow{P}`x . \alpha \subset C`P \cap \sigma . x \in \overrightarrow{\text{seq}}_P`\alpha - \sigma .$

$[*10·24] \qquad \supset . (\exists \alpha) . \alpha \subset C`P \cap \sigma . \exists ! \overrightarrow{\text{seq}}_P`\alpha - \sigma \qquad (1)$

$\vdash . (1) . \text{Transp} . \supset \vdash . \text{Prop}$

***250·301.** $\vdash : P \in \text{connex} . \sim \exists ! \overrightarrow{\text{min}}_P`\tau . \sigma = C`P - \breve{P}``\tau . \alpha \subset \sigma . \supset . \overrightarrow{\text{seq}}_P`\alpha \subset \sigma$

Dem.

$\vdash . *205·122 . *202·501 . \supset \vdash : \text{Hp} . \supset . \sigma \subset p`\overrightarrow{P}``\tau .$

$[*40·67] \qquad \supset . \tau \subset p`\overleftarrow{P}``\sigma \qquad (1)$

$\vdash . *206·134 . \supset \vdash : \text{Hp} . x \,\text{seq}_P\, \alpha . \supset . \overrightarrow{P}`x \subset - p`\overleftarrow{P}``\alpha$

$[*40·16] \qquad \subset - p`\overleftarrow{P}``\sigma$

$[(1)] \qquad \subset - \tau .$

$[*37·462] \qquad \supset . x \sim\in \breve{P}``\tau .$

$[*206·18.\text{Hp}] \qquad \supset . x \in \sigma : \supset \vdash . \text{Prop}$

***250·31.** $\vdash :: P \in \text{connex} :. \alpha \subset C`P \cap \sigma . \supset_\alpha . \overrightarrow{\text{seq}}_P`\alpha \subset \sigma : \supset_\sigma . C`P \subset \sigma :. \supset . P \in \Omega$

Dem.

$\vdash . *250·301 . \supset$

$\vdash :. P \in \text{connex} . \exists ! C`P \cap \tau . \sim \exists ! \overrightarrow{\text{min}}_P`\tau . \sigma = C`P - \breve{P}``\tau . \supset :$

$\alpha \subset \sigma . \supset_\alpha . \text{seq}_P`\alpha \subset \sigma : \exists ! C`P - \sigma \qquad (1)$

$\vdash . (1) . *10·28 . \supset$

$\vdash :. P \in \text{connex} : (\exists \tau) . \exists ! C`P \cap \tau . \sim \exists ! \overrightarrow{\text{min}}_P`\tau : \supset :$

$(\exists \sigma) : \alpha \subset \sigma . \supset_\alpha . \overrightarrow{\text{seq}}_P`\alpha \subset \sigma : \exists ! C`P - \sigma \qquad (2)$

$\vdash . (2) . \text{Transp} . \supset$

$\vdash :: P \in \text{connex} :. \alpha \subset \sigma . \supset_\alpha . \overrightarrow{\text{seq}}_P`\alpha \subset \sigma : \supset_\sigma . C`P \subset \sigma :. \supset :$

$\exists ! C`P \cap \tau . \supset_\tau . \exists ! \overrightarrow{\text{min}}_P`\tau :$

$[*250·101] \quad \supset : P \in \text{Bord} \qquad (3)$

$\vdash . (3) . *250·113 . \supset \vdash . \text{Prop}$

***250·32.** $\vdash :: . P \epsilon \text{connex} . \supset :: P \epsilon \text{Bord} . \equiv :.$

$$\alpha \subset C'P \cap \sigma . \supset_\alpha . \overrightarrow{\text{seq}}_P{}'\alpha \subset \sigma : \supset_\sigma . C'P \subset \sigma \quad [*250·3·31]$$

***250·33.** $\vdash . \Omega = \text{connex} \cap \hat{P} \{\alpha \subset C'P \cap \sigma . \supset_\alpha . \overrightarrow{\text{seq}}_P{}'\alpha \subset \sigma : \supset_\sigma . C'P \subset \sigma\}$
[*250·32·113]

***250·34.** $\vdash :. P \epsilon \text{Bord} : x \epsilon C'P . \overrightarrow{P}'x \subset \sigma . \supset_x . x \epsilon \sigma : \supset . C'P \subset \sigma$

Dem.

$\vdash . *250·11 . \supset \vdash : P \epsilon \text{Bord} . \exists ! C'P - \sigma . \supset . \exists ! \overrightarrow{\min}_P{}'(C'P - \sigma) .$

[*205·14] $\supset . (\exists x) . x \epsilon C'P - \sigma . \overrightarrow{P}'x \subset \sigma$ (1)

$\vdash . (1) . \text{Transp} . \supset \vdash . \text{Prop}$

***250·341.** $\vdash :: x \epsilon C'P . \overrightarrow{P}'x \subset \sigma . \supset_x . x \epsilon \sigma : \supset_\sigma . C'P \subset \sigma :. \supset . P \epsilon \text{Bord}$

Dem.

$\vdash . *205·122 . *37·462 . \supset$

$\vdash : \exists ! C'P \cap \tau . \sim \exists ! \overrightarrow{\min}_P{}'\tau . \sigma = C'P - \breve{P}``\tau . x \epsilon C'P . \overrightarrow{P}'x \subset \sigma . \supset .$

$x \sim \epsilon \breve{P}``\tau . \exists ! C'P - \sigma .$

[Hp] $\supset . x \epsilon \sigma . \exists ! C'P - \sigma$ (1)

$\vdash . (1) . *10·28 . \supset \vdash :. (\exists \tau) . \exists ! C'P \cap \tau . \sim \exists ! \overrightarrow{\min}_P{}'\tau . \supset :$

$(\exists \sigma) : x \epsilon C'P . \overrightarrow{P}'x \subset \sigma . \supset_x . x \epsilon \sigma : \exists ! C'P - \sigma$ (2)

$\vdash . (2) . \text{Transp} . \supset \vdash :. \text{Hp} . \supset : \exists ! C'P \cap \tau . \supset_\tau . \exists ! \overrightarrow{\min}_P{}'\tau :$

[*250·101] $\supset : P \epsilon \text{Bord} :. \supset \vdash . \text{Prop}$

***250·35.** $\vdash . \text{Bord} = \hat{P} \{x \epsilon C'P . \overrightarrow{P}'x \subset \sigma . \supset_x . x \epsilon \sigma : \supset_\sigma . C'P \subset \sigma\}$
[*250·34·341]

***250·36.** $\vdash :. P \epsilon \Omega : \lambda \subset \sigma . \exists ! \lambda \cap C'P . \supset_\lambda . \overrightarrow{\text{seq}}_P{}'\lambda \subset \sigma : \supset . \breve{P}``\sigma \subset \sigma$

Dem.

$\vdash . *250·121 . \supset \vdash : P \epsilon \Omega . \exists ! \breve{P}``\sigma - \sigma . \supset . E ! \min_P{}'(\breve{P}``\sigma - \sigma)$ (1)

$\vdash . *205·14 . *37·46 . \supset$

$\vdash : x = \min_P{}'(\breve{P}``\sigma - \sigma) . \supset . \exists ! \sigma \cap \overrightarrow{P}'x . \overrightarrow{P}'x \cap (\breve{P}``\sigma - \sigma) = \Lambda .$

[*24·311] $\supset . \exists ! \sigma \cap \overrightarrow{P}'x . \overrightarrow{P}'x - \sigma \subset - \breve{P}``\sigma$ (2)

$\vdash . (2) . *202·501 . \supset$

$\vdash : P \epsilon \text{Ser} . x = \min_P{}'(\breve{P}``\sigma - \sigma) . \supset . \exists ! \sigma \cap \overrightarrow{P}'x . \overrightarrow{P}'x - \sigma \subset p'\overrightarrow{P}``(\sigma \cap C'P) .$

[*40·16] $\supset . \exists ! \sigma \cap \overrightarrow{P}'x . \overrightarrow{P}'x - \sigma \subset p'\overrightarrow{P}``(\sigma \cap \overrightarrow{P}'x) .$

[*40·61] $\supset . \overrightarrow{P}'x - \sigma \subset P``(\sigma \cap \overrightarrow{P}'x)$ (3)

$\vdash . (3) . \supset \vdash : \text{Hp} (3) . \supset . \overrightarrow{P}'x \subset (\sigma \cap \overrightarrow{P}'x) \cup P``(\sigma \cap \overrightarrow{P}'x) .$

[*206·171] $\supset . x = \mathrm{seq}_P`(\sigma \cap \overrightarrow{P}`x) .$

[(2)] $\supset . \exists ! \sigma \cap \overrightarrow{P}`x . \sigma \cap \overrightarrow{P}`x \subset \sigma . \sim \{\mathrm{seq}_P`(\sigma \cap \overrightarrow{P}`x) \subset \sigma\} .$

[*10·24] $\supset . (\exists \lambda) . \lambda \subset \sigma . \exists ! \lambda \cap C`P . \sim (\overrightarrow{\mathrm{seq}}_P`\lambda \subset \sigma)$ (4)

$\vdash . (4) . \mathrm{Transp} . \supset \vdash : \mathrm{Hp} . \supset . \sim E ! \mathrm{min}_P`(\breve{P}``\sigma - \sigma) .$

[(1).Transp] $\supset . \breve{P}``\sigma - \sigma = \Lambda : \supset \vdash . \mathrm{Prop}$

***250·361.** $\vdash :. P \in \Omega . \breve{P}_1``\sigma \subset \sigma : \lambda \subset \sigma . \exists ! (\lambda \cap C`P) . \supset_\lambda . \overrightarrow{\mathrm{limax}}_P`\lambda \subset \sigma : \supset .$
$\breve{P}``\sigma \subset \sigma$

Dem.

$\vdash . *206·46·43 . \supset \vdash : \mathrm{Hp} . \lambda \subset \sigma . E ! \mathrm{max}_P`\lambda . \supset . \overrightarrow{\mathrm{seq}}_P`\lambda = \overleftarrow{P}_1`\mathrm{max}_P`\lambda .$

[Hp] $\supset . \overrightarrow{\mathrm{seq}}_P`\lambda \subset \sigma$ (1)

$\vdash . *207·4 . \supset \vdash : \mathrm{Hp} . \lambda \subset \sigma . \exists ! (\lambda \cap C`P) . \sim E ! \mathrm{max}_P`\lambda . \supset . \overrightarrow{\mathrm{seq}}_P`\lambda = \overrightarrow{\mathrm{limax}}_P`\lambda .$

[Hp] $\supset . \overrightarrow{\mathrm{seq}}_P`\lambda \subset \sigma$ (2)

$\vdash . (1) . (2) . \supset \vdash :. \mathrm{Hp} . \supset : \lambda \subset \sigma . \exists ! (\lambda \cap C`P) . \supset_\lambda . \overrightarrow{\mathrm{seq}}_P`\lambda \subset \sigma :$

[*250·36] $\supset : \breve{P}``\sigma \subset \sigma :. \supset \vdash . \mathrm{Prop}$

***250·362.** $\vdash :. \breve{P} \in \Omega . P_1``\sigma \subset \sigma : \lambda \subset \sigma . \exists ! \lambda \cap C`P . \supset_\lambda . \overrightarrow{\mathrm{limin}}_P`\lambda \subset \sigma : \supset .$
$P``\sigma \subset \sigma$

$\left[*250·361 \dfrac{\breve{P}}{P} . *121·26 \right]$

***250·4.** $\vdash . \dot{\Lambda} \in \Omega$

Dem.

$\vdash . *60·33 . \supset \vdash . \mathrm{Cl}\,\mathrm{ex}`C`\dot{\Lambda} \subset \text{Ɑ}`\mathrm{min}(\dot{\Lambda})$ (1)

$\vdash . (1) . *250·1 . \supset \vdash . \dot{\Lambda} \in \mathrm{Bord}$ (2)

$\vdash . (2) . *204·24 . \supset \vdash . \mathrm{Prop}$

***250·41.** $\vdash : x \neq y . \supset . x \downarrow y \in \Omega$

Dem.

$\vdash . *60·39 . \supset \vdash . \mathrm{Cl}\,\mathrm{ex}`C`(x \downarrow y) = \iota`\iota`x \cup \iota`\iota`y \cup \iota`(\iota`x \cup \iota`y)$ (1)

$\vdash . *205·18 . \supset \vdash : \mathrm{Hp} . P = x \downarrow y . \supset . \mathrm{min}_P`\iota`x = x . \mathrm{min}_P`\iota`y = y$ (2)

$\vdash . *205·181 . \supset \vdash : \mathrm{Hp}(2) . \supset . \mathrm{min}_P`(\iota`x \cup \iota`y) = x$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash : \mathrm{Hp}(2) . \supset . \mathrm{Cl}\,\mathrm{ex}`C`(x \downarrow y) \subset \text{Ɑ}`\mathrm{min}_P .$

[*250·1] $\supset . x \downarrow y \in \mathrm{Bord}$ (4)

$\vdash . (4) . *204·25 . \supset \vdash . \mathrm{Prop}$

***250·42.** $\vdash : P \epsilon \Omega - \iota`\dot{\Lambda} . \supset . \mathrm{E} ! 2_P . 2_P = \breve{P}_1`B`P . \overrightarrow{P}`2_P = \iota`B`P . P \upharpoonright \overrightarrow{P}`2_P = \dot{\Lambda}$

Dem.

$\vdash . *121·13 . \supset \vdash : x = 2_P . \equiv . x = \breve{P}_1`B`P$ (1)

$\vdash . *250·13 . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{E} ! B`P .$

[*250·21.*204·7] $\supset . \mathrm{E} ! \breve{P}_1`B`P$ (2)

$\vdash . (1) . (2) . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{E} ! 2_P . 2_P = \breve{P}_1`B`P$ (3)

[*204·71] $\supset . \overrightarrow{P}`2_P = \iota`B`P$ (4)

[*200·35] $\supset . P \upharpoonright \overrightarrow{P}`2_P = \dot{\Lambda}$ (5)

$\vdash . (3) . (4) . (5) . \supset \vdash .$ Prop

***250·43.** $\vdash . 0_r = \Omega \cap \breve{C}``0$

Dem.

$\vdash . *56·104 . \supset \vdash : P \epsilon 0_r . \equiv . P = \dot{\Lambda} .$

[*250·4.*33·241] $\equiv . P \epsilon \Omega . C`P = \Lambda .$

[*71·37.*54·1] $\equiv . P \epsilon \Omega \cap \breve{C}``0 : \supset \vdash .$ Prop

***250·44.** $\vdash . 2_r = \Omega \cap \breve{C}``2$

Dem.

$\vdash . *56·11 . \supset \vdash :. P \epsilon 2_r . \equiv : (\exists x, y) . x \neq y . P = x \downarrow y :$

[*250·41] $\equiv : P \epsilon \Omega : (\exists x, y) . x \neq y . P = x \downarrow y :$

[*56·11·38] $\equiv : P \epsilon \Omega \cap \breve{C}``2 . P \dot{\cap} \breve{P} = \dot{\Lambda} :$

[*204·14] $\equiv : P \epsilon \Omega \cap \breve{C}``2 :. \supset \vdash .$ Prop

***250·5.** $\vdash : P \epsilon \Omega . \supset . \mathrm{min}_P \upharpoonright \mathrm{Cl\,ex}`C`P \epsilon \epsilon_\Delta`\mathrm{Cl\,ex}`C`P .$
$\iota`C`P = \mathrm{Prod}`\mathrm{Cl\,ex}`C`P$ [*205·33 . *250·1 . *115·17]

This proposition is of great importance, since it gives the existence-theorem for selections from any class of existent classes whose sum can be well-ordered (cf. *250·53, below). Observe that "$\alpha \epsilon C``\Omega$" means "$\alpha$ is a class which can be well-ordered."

***250·51.** $\vdash : \alpha \epsilon C``\Omega . \supset . \exists ! \epsilon_\Delta`\mathrm{Cl\,ex}`\alpha$ [*250·5]

***250·52.** $\vdash : \alpha \epsilon C``\Omega . \beta \subset \alpha . \supset . \exists ! \epsilon_\Delta`\mathrm{Cl\,ex}`\beta$ [*83·22·2 . *250·51]

***250·53.** $\vdash : s`\kappa \epsilon C``\Omega . \Lambda \sim \epsilon \kappa \,\dot{.}\, \supset . \exists ! \epsilon_\Delta`\kappa$

Dem.

$\vdash . *60·23·57 . \supset \vdash : \mathrm{Hp} . \supset . \kappa \subset \mathrm{Cl\,ex}`s`\kappa .$

[*88·22.*250·51] $\supset \exists ! \epsilon_\Delta`\kappa : \supset \vdash .$ Prop

***250·54.** $\vdash : C``\Omega \cup 1 = \mathrm{Cls} . \supset . \mathrm{Mult\,ax}$

Dem.

$\vdash . *250·53 . *83·4 . \supset \vdash :. \mathrm{Hp} . \supset : \Lambda \sim \epsilon \kappa . \supset_\kappa . \exists ! \epsilon_\Delta`\kappa :$

[*88·37] $\supset : \mathrm{Mult\,ax} :. \supset \vdash .$ Prop

The above proposition states that if every class which is not a unit class is the field of some well-ordered series, then the multiplicative axiom holds. The converse of this proposition has been proved by Zermelo (cf. ∗258·47).

∗250·6. $\vdash : P, Q \epsilon \Omega . P \,\mathrm{smor}\, Q . \supset \therefore P \,\overline{\mathrm{smor}}\, Q \epsilon 1$ [∗208·41 . ∗250·12·1]

This proposition is very useful, since it enables us, when two similar series of similar well-ordered series are given, to pick out the correlators of all the pairs without assuming the multiplicative axiom. *I.e.* given $P, Q \epsilon \mathrm{Rel}^2 \mathrm{excl} . S \epsilon P \,\overline{\mathrm{smor}}\, Q . S \subseteq \mathrm{smor}$, if $N \epsilon C\text{'}Q$, the correlator of $\breve{S}\text{'}N$ and N will be $\iota\text{'}(\breve{S}\text{'}N) \,\overline{\mathrm{smor}}\, N$ if $\breve{S}\text{'}N, N \epsilon \Omega$. This enables us to dispense with the multiplicative axiom in the hypotheses of ∗164·44 and its consequences, whenever the relations concerned have fields whose members are well-ordered series.

∗250·61. $\vdash : P \epsilon \Omega . \supset . P \,\overline{\mathrm{smor}}\, P = \iota\text{'}(I \upharpoonright C\text{'}P)$ [∗208·42]

∗250·62. $\vdash : P \epsilon \mathrm{Bord} . S \epsilon \mathrm{cror}\text{'}P . \supset . \sim (\exists x) . (S\text{'}x) P x$ [∗208·43]

∗250·63. $\vdash : P \epsilon \Omega \cap \mathrm{Cnv}\text{''}\Omega . \supset . \mathrm{Rl}\text{'}P \cap \mathrm{Nr}\text{'}P = \iota\text{'}P$ [∗208·45]

This proposition will be useful in showing that a *finite* series is not similar to any proper part of itself, and is a series which is well-ordered and has a converse which is also well-ordered.

∗250·64. $\vdash : P \epsilon \mathrm{Bord} . S \epsilon \mathrm{cror}\text{'}P . \supset . C\text{'}P \cap p\text{'}\overleftarrow{P}\text{''}\mathrm{D}\text{'}S = \Lambda$ [∗208·46]

In virtue of this proposition, a part of a well-ordered series can only be similar to the whole if the part extends to the end of the series. Thus *e.g.* no proper section of a well-ordered series can be similar to the whole.

∗250·65. $\vdash : P \epsilon \Omega . \alpha \epsilon \mathrm{sect}\text{'}P - \iota\text{'}C\text{'}P . \beta \subset \alpha . \supset . \sim \{P \,\mathrm{smor}\, P \upharpoonright \beta\}$

Dem.

$\vdash . ∗40·16 . \supset \vdash : \mathrm{Hp} . \supset . p\text{'}\overleftarrow{P}\text{''}C\text{'}(P \upharpoonright \alpha) \subset p\text{'}\overleftarrow{P}\text{''}C\text{'}(P \upharpoonright \beta)$ (1)

$\vdash . ∗211·133 . \supset \vdash : \mathrm{Hp} . \alpha \sim \epsilon 1 . \supset . \alpha = C\text{'}(P \upharpoonright \alpha) .$

[∗211·703] $\supset . \exists ! p\text{'}\overleftarrow{P}\text{''}C\text{'}(P \upharpoonright \alpha) .$

[(1)] $\supset . \exists ! p\text{'}\overleftarrow{P}\text{''}C\text{'}(P \upharpoonright \beta)$ (2)

$\vdash . (2) . ∗40·6·62 . \supset \vdash : \mathrm{Hp} . \alpha \sim \epsilon 1 . \dot{\exists} ! P . \supset . \exists ! C\text{'}P \cap p\text{'}\overleftarrow{P}\text{''}C\text{'}(P \upharpoonright \beta) .$

[∗208·47] $\supset . \sim \{P \,\mathrm{smor}\, (P \upharpoonright \beta)\}$ (3)

$\vdash . ∗211·1 . ∗24·13 . \supset \vdash : P = \dot{\Lambda} . \supset . \mathrm{sect}\text{'}P - \iota\text{'}C\text{'}P = \Lambda$ (4)

$\vdash . (4) . \mathrm{Transp} . \supset \vdash : \mathrm{Hp} . \supset . \dot{\exists} ! P$ (5)

$\vdash . ∗200·35 . ∗250·104 . \supset \vdash : \mathrm{Hp} . \dot{\exists} ! P . \alpha \epsilon 1 . \supset . \sim \{P \,\mathrm{smor}\, (P \upharpoonright \beta)\}$ (6)

$\vdash . (3) . (5) . (6) . \supset \vdash . \mathrm{Prop}$

∗250·651. $\vdash : P \epsilon \Omega . \supset . \mathrm{Nr}\text{'}P \cap P \upharpoonright\text{''}(\mathrm{sect}\text{'}P - \iota\text{'}C\text{'}P) = \Lambda$ [∗250·65]

***250·652.** $\vdash : P \epsilon \text{Bord} . Q \subseteq P . \exists ! C'P \cap p'\overleftarrow{P}``C'Q . \supset . \sim (P \text{ smor } Q)$
[*208·47]

***250·653.** $\vdash : P \epsilon \text{Bord} . \exists ! C'P \cap p'\overleftarrow{P}``(\alpha \cap C'P) . \supset . \sim (P \text{ smor } P \upharpoonright \alpha)$

Dem.

$\vdash . *37·41 . \supset \vdash . C'(P \upharpoonright \alpha) \subset \alpha \cap C'P .$

$[*40·16] \quad \supset \vdash . p'\overleftarrow{P}``(\alpha \cap C'P) \subset p'\overleftarrow{P}``C'(P \upharpoonright \alpha) \quad (1)$

$\vdash . (1) . \quad \supset \vdash : \text{Hp} . \supset . \exists ! C'P \cap p'\overleftarrow{P}``C'(P \upharpoonright \alpha) .$

$[*250·652] \quad \supset . \sim \{P \text{ smor} (P \upharpoonright \alpha)\} : \supset \vdash . \text{Prop}$

***250·66.** $\vdash : P \epsilon \Omega . \alpha \epsilon \text{sect}'P . P \text{ smor} (P \upharpoonright \alpha) . \supset . \alpha = C'P \quad [*250·65 . \text{Transp}]$

***250·67.** $\vdash : P \epsilon \Omega . x \epsilon C'P . \supset . \sim \{P \text{ smor} (P \upharpoonright \overrightarrow{P}'x)\}$

Dem.

$\vdash . *211·302 . \supset \vdash : \text{Hp} . \supset . \overrightarrow{P}'x \epsilon \text{sect}'P \quad (1)$

$\vdash . *200·52 . \supset \vdash : \text{Hp} . \supset . \overrightarrow{P}'x \neq C'P \quad (2)$

$\vdash . (1) . (2) . *250·65 . \supset \vdash . \text{Prop}$

***250·7.** $\vdash :. P \epsilon \Omega . \equiv : x \epsilon C'P . \supset_x . P \upharpoonright \overrightarrow{P_*}'x \epsilon \Omega : P \epsilon \text{Ser}$

Dem.

$\vdash . *250·141 . \supset \vdash :. P \epsilon \Omega . \supset : x \epsilon C'P . \supset_x . P \upharpoonright \overrightarrow{P_*}'x \epsilon \Omega \quad (1)$

$\vdash . *250·121 . \supset$

$\vdash :. x \epsilon C'P . \supset_x . P \upharpoonright \overrightarrow{P_*}'x \epsilon \Omega : \equiv : x \epsilon C'P . \exists ! \alpha \cap C'(P \upharpoonright \overrightarrow{P_*}'x) . \supset_{x,\alpha} .$
$\text{E} ! \min (P \upharpoonright \overrightarrow{P_*}'x)'\alpha :$

$[*202·55] \supset : x \epsilon \text{G}'P \cap \alpha . \supset_{x,\alpha} . \text{E} ! \min (P \upharpoonright \overrightarrow{P_*}'x)'\alpha :$

$[*205·27] \quad \supset_{x,\alpha} . \text{E} ! \min_P{}'\alpha :$

$[*10·23] \quad \supset : \exists ! \text{G}'P \cap \alpha . \supset_\alpha . \text{E} ! \min_P{}'\alpha \quad (2)$

$\vdash . *205·18 . *202·52 . \supset \vdash : P \epsilon \text{Ser} . \alpha = \overrightarrow{B}'P . \supset . \text{E} ! \min_P{}'\alpha \quad (3)$

$\vdash . (2) . (3) . \quad \supset \vdash :. x \epsilon C'P . \supset_x . P \upharpoonright \overrightarrow{P_*}'x \epsilon \Omega : P \epsilon \text{Ser} : \supset :$
$\exists ! \alpha \cap C'P . \supset_\alpha . \text{E} ! \min_P{}'\alpha :$

$[*250·121] \quad \supset : P \epsilon \Omega \quad (4)$

$\vdash . (1) . (4) . \supset \vdash . \text{Prop}$

This proposition is used in proving that the series of ordinals in order of magnitude is well-ordered (*256·3). We prove first that if $P \epsilon \Omega$, the ordinals up to and including $\text{Nr}'P$ are well-ordered; thence, by the above proposition, it follows that the whole series of ordinals is well-ordered.

$*251$. ORDINAL NUMBERS.

Summary of $*251$.

The name "ordinal numbers" is commonly confined to the relation-numbers of well-ordered series, and will be so confined in what follows. The relation-numbers of series in general are commonly called "order-types*." Thus α is an order-type if $\alpha \in \mathrm{Nr}``\mathrm{Ser}$, and $\alpha$ is an ordinal number if $\alpha \in \mathrm{Nr}``\Omega$. In the present number we shall be concerned with a few of the simpler properties of ordinal numbers and of the sums, products, and powers of well-ordered series.

We put $$\mathrm{NO} = \mathrm{Nr}``\Omega \quad \mathrm{Df},$$
where "NO" stands for "ordinal number."

We prove in this number that any relation similar to a well-ordered relation is well-ordered ($*251 \cdot 11$), and therefore any relation similar to a well-ordered series is a well-ordered series ($*251 \cdot 111$). We prove

$*251 \cdot 132 \cdot 142$. $\vdash : \alpha \in \mathrm{NO} . \equiv . \alpha \dot{+} \dot{1} \in \mathrm{NO} . \equiv . \dot{1} \dot{+} \alpha \in \mathrm{NO}$

$*251 \cdot 15 \cdot 16$. $\vdash . 0_r, 2_r \in \mathrm{NO}$

$*251 \cdot 24$. $\vdash : \alpha, \beta \in \mathrm{NO} . \supset . \alpha \dot{+} \beta \in \mathrm{NO}$

We prove that if P is a well-ordered series of mutually exclusive well-ordered series, $\Sigma`P$ is a well-ordered series ($*251 \cdot 21$); that if $P$ is a well-ordered series of series, $\Pi`P$ is a series ($*251 \cdot 3$); that if P is a series and Q is a well-ordered series, P^Q and $P \exp Q$ are series ($*251 \cdot 42$); that if P, Q are well-ordered series, so is $P \times Q$ ($*251 \cdot 55$), and therefore the product of two ordinal numbers is an ordinal number ($*251 \cdot 56$).

In virtue of the uniqueness of the correlator of two well-ordered series, we have

$*251 \cdot 61$. $\vdash :. P, Q \in \mathrm{Rel}^2 \mathrm{excl} . C`P \subset \Omega . \supset :$
$$\exists ! (P \, \overline{\mathrm{smor}} \, Q) \cap \mathrm{Rl}`\mathrm{smor} . \equiv . P \, \mathrm{smor} \, \mathrm{smor} \, Q$$

whence, without assuming the multiplicative axiom,

* We shall also speak of them as "serial numbers."

***251·621.** $\vdash : C`P \subset \Omega \,.\, \exists ! (P \,\overline{\text{smor}}\, Q) \cap \text{Rl}`\text{smor} \,.\, \supset .$
$$\Sigma \text{Nr}`P = \Sigma \text{Nr}`Q \,.\, \Pi \text{Nr}`P = \Pi \text{Nr}`Q$$

***251·65.** $\vdash : \alpha \in \text{NO} - \iota`\Lambda \,.\, \beta \in \text{NR} \,.\, P \in \beta \,.\, C`P \subset \alpha \,.\, \supset .$
$$\Sigma \text{Nr}`P = \beta \dot{\times} \alpha \,.\, \Pi \text{Nr}`P = \alpha \exp_r \beta$$

Finally, we have propositions (*251·7·71) showing that the existence of an existent Ω in any type is equivalent to the existence of 2_r in that type, and therefore holds for every type of homogeneous relations, except (possibly, so far as our primitive propositions can show) in the type of relations of individuals to individuals.

***251·01.** $\text{NO} = \text{Nr}``\Omega$ Df

***251·1.** $\vdash : \alpha \in \text{NO} \,.\equiv .\, (\exists P) \,.\, P \in \Omega \,.\, \alpha = \text{Nr}`P$ [(*251·01)]

***251·11.** $\vdash : P \in \text{Bord} \,.\, P \text{ smor } Q \,.\, \supset .\, Q \in \text{Bord}$

Dem.

$\vdash . *205·8 . *250·1 . *37·431 . \supset$

$\vdash :. P \in \text{Bord} \,.\, S \in P \,\overline{\text{smor}}\, Q \,.\, \supset : \alpha \subset C`P \,.\, \exists ! \alpha \,.\, \supset_\alpha .\, \exists ! \overrightarrow{\text{min}}_Q`\breve{S}``\alpha :$

[*37·63·431] $\supset : \beta \in \breve{S}```\text{Cl ex}`C`P \,.\, \exists ! \beta \,.\, \supset_\beta .\, \exists ! \overrightarrow{\text{min}}_Q`\beta :$

[*71·491] $\supset : \beta \in \text{Cl ex}`\breve{S}``C`P \,.\, \supset_\beta .\, \exists ! \overrightarrow{\text{min}}_Q`\beta :$

[*151·11·131.*37·25] $\supset : \beta \in \text{Cl ex}`C`Q \,.\, \supset_\beta .\, \exists ! \overrightarrow{\text{min}}_Q`\beta :$

[*250·1] $\supset : Q \in \text{Bord} :. \supset \vdash . \text{Prop}$

***251·111.** $\vdash : P \in \Omega \,.\, P \text{ smor } Q \,.\, \supset .\, Q \in \Omega$ [*251·11 . *204·21]

***251·12.** $\vdash : P \in \text{Bord} \,.\, \supset .\, \text{Nr}`P \subset \text{Bord}$ [*251·11]

***251·121.** $\vdash : P \in \Omega \,.\, \supset .\, \text{Nr}`P \subset \Omega$ [*251·111]

***251·122.** $\vdash : \alpha \in \text{NO} \,.\, \supset .\, \alpha \subset \Omega$ [*251·121·1]

***251·13.** $\vdash : P \in \text{Bord} \,.\, z \sim\in C`P \,.\equiv .\, P \nrightarrow z \in \text{Bord}$

Dem.

$\vdash . *205·83 . *250·1 . \supset \vdash : \text{Hp} \,.\, \exists ! C`P \cap \alpha \,.\, \supset .\, \exists ! \overrightarrow{\text{min}}\,(P \nrightarrow z)`\alpha$ (1)

$\vdash . *205·831 . \quad \supset \vdash : \text{Hp} \,.\, C`(P \nrightarrow z) \cap \alpha = \iota`z \,.\, \supset .\, \exists ! \overrightarrow{\text{min}}\,(P \nrightarrow z)`\alpha$ (2)

$\vdash . *161·14 . \quad \supset \vdash :. \text{Hp} \,.\, \exists ! C`(P \nrightarrow z) \cap \alpha \,.\, \supset :$
$$\exists ! C`P \cap \alpha \,.\, \vee .\, C`P \cap \alpha = \Lambda \,.\, \exists ! \iota`z \cap \alpha :$$

[*161·14] $\supset : \exists ! C`P \cap \alpha \,.\, \vee .\, C`(P \nrightarrow z) \cap \alpha = \iota`z$ (3)

$\vdash . (1) . (2) . (3) . \supset$

$\vdash :. \text{Hp} \,.\, \supset : \exists ! C`(P \nrightarrow z) \cap \alpha \,.\, \supset_\alpha .\, \exists ! \overrightarrow{\text{min}}\,(P \nrightarrow z)`\alpha$ (4)

$\vdash . (4) . *250·101 . \quad \supset \vdash : P \in \text{Bord} \,.\, z \sim\in C`P \,.\, \supset .\, P \nrightarrow z \in \text{Bord}$ (5)

$\vdash . *250·14·104 . *200·41 . \supset \vdash : P \nrightarrow z \in \text{Bord} \,.\, \supset .\, P \in \text{Bord} \,.\, z \sim\in C`P$ (6)

$\vdash . (5) . (6) . \supset \vdash . \text{Prop}$

***251·131.** $\vdash : P \in \Omega \,.\, z \sim \in C`P \,.\, \equiv \,.\, P \dotplus\!\!\rightarrow z \in \Omega$ [*204·51 . *251·13]

***251·132.** $\vdash : \alpha \in \text{NO} \,.\, \equiv \,.\, \alpha \dotplus \dot{1} \in \text{NO}$

Dem.

$\vdash . *251{\cdot}111 . *181{\cdot}12 . \supset \vdash : P \in \Omega . \equiv . \downarrow \Lambda_x \mathbin{;} \iota \mathbin{;} P \in \Omega .$

$[*181{\cdot}11.(*181{\cdot}01).*251{\cdot}131] \quad \equiv . P \dotplus\!\!\rightarrow x \in \Omega .$

$[*181{\cdot}3.*251{\cdot}1] \quad \equiv . \text{Nr}`P \dotplus \dot{1} \in \text{NO} \quad (1)$

$\vdash . (1) . *251{\cdot}1 . \supset \vdash . \text{Prop}$

***251·14.** $\vdash : P \in \text{Bord} \,.\, z \sim \in C`P \,.\, \equiv \,.\, z \leftarrow\!\!\dotplus P \in \text{Bord}$

Dem.

$\vdash . *205{\cdot}832 . *161{\cdot}12 . \supset$

$\vdash :. \text{Hp} . \supset : z \sim \in \alpha . \supset . \overrightarrow{\text{min}} (z \leftarrow\!\!\dotplus P)`\alpha = \overrightarrow{\text{min}}_P`\alpha :$

$[*250{\cdot}101] \supset : \exists ! (\alpha \cap C`P) . z \sim \in \alpha . \supset . \exists ! \overrightarrow{\text{min}} (z \leftarrow\!\!\dotplus P)`\alpha \quad (1)$

$\vdash . *205{\cdot}833 . *161{\cdot}12 . \supset$

$\vdash : \text{Hp} . z \in \alpha . \dot{\exists} ! P . \supset . \exists ! \overrightarrow{\text{min}} (z \leftarrow\!\!\dotplus P)`\alpha \quad (2)$

$\vdash . (1) . (2) . \supset$

$\vdash :. \text{Hp} . \dot{\exists} ! P . \supset : \exists ! \alpha \cap C`(z \leftarrow\!\!\dotplus P) . \supset_\alpha . \exists ! \overrightarrow{\text{min}} (z \leftarrow\!\!\dotplus P)`\alpha :$

$[*250{\cdot}101] \quad \supset : z \leftarrow\!\!\dotplus P \in \text{Bord} \quad (3)$

$\vdash . *161{\cdot}201 . *250{\cdot}4 . \quad \supset \vdash : P = \dot{\Lambda} . \supset . z \leftarrow\!\!\dotplus P \in \text{Bord} \quad (4)$

$\vdash . (3) . (4) . \quad \supset \vdash : P \in \text{Bord} . z \sim \in C`P . \supset . z \leftarrow\!\!\dotplus P \in \text{Bord} \quad (5)$

$\vdash . *250{\cdot}14{\cdot}104 . *200{\cdot}41 . \supset \vdash : z \leftarrow\!\!\dotplus P \in \text{Bord} . \supset . P \in \text{Bord} . z \sim \in C`P \quad (6)$

$\vdash . (5) . (6) . \supset \vdash . \text{Prop}$

***251·141.** $\vdash : P \in \Omega \,.\, z \sim \in C`P \,.\, \equiv \,.\, z \leftarrow\!\!\dotplus P \in \Omega$ [*204·51 . *251·14]

***251·142.** $\vdash : \alpha \in \text{NO} \,.\, \equiv \,.\, \dot{1} \dotplus \alpha \in \text{NO}$ [Proof as in *251·132]

***251·15.** $\vdash . 0_r \in \text{NO}$ [*250·4 . *153·11]

***251·16.** $\vdash . 2_r \in \text{NO}$ [*250·41 . *153·211]

***251·17.** $\vdash : x \neq y . x \neq z . y \neq z . \supset . x \downarrow y \dotplus\!\!\rightarrow z \in \Omega$ [*251·131 . *250·41]

***251·171.** $\vdash . 2_r \dotplus \dot{1} \in \text{NO}$ [*251·16·132]

***251·2.** $\vdash : P \in \text{Rel}^2\text{excl} \cap \text{Bord} \,.\, C`P \subset \text{Bord} \,.\, \supset \,.\, \Sigma`P \in \text{Bord}$

Dem.

$\vdash . *162{\cdot}23 . \quad \supset \vdash : \exists ! \alpha \cap C`\Sigma`P . \supset . \exists ! \alpha \cap F``C`P .$

$[*37{\cdot}264] \quad \supset . \exists ! C`P \cap \breve{F}``\alpha \quad (1)$

$\vdash . *37{\cdot}46 . *33{\cdot}5 . \supset \vdash : Q \in \breve{F}``\alpha . \supset . \exists ! \alpha \cap C`Q \quad (2)$

$\vdash . (1) . (2) . *250{\cdot}101 . \supset$

$\vdash :. \text{Hp} . \supset : \exists ! \alpha \cap C`\Sigma`P . \supset . (\exists Q) . Q \, \text{min}_P \breve{F}``\alpha . \exists ! \overrightarrow{\text{min}}_Q`\alpha .$

$[*205{\cdot}85] \quad \supset . \exists ! \overrightarrow{\text{min}} (\Sigma`P)`\alpha \quad (3)$

$\vdash . (3) . *250{\cdot}101 . \supset \vdash . \text{Prop}$

***251·21.** $\vdash : P \epsilon \mathrm{Rel}^2\,\mathrm{excl} \cap \Omega \,.\, C\text{‘}P \subset \Omega \,.\, \supset \,.\, \Sigma\text{‘}P \epsilon \Omega$ [*204·52 . *251·2]

***251·211.** $\vdash : \mathrm{Nr}\text{‘}P \epsilon \mathrm{NO} \,.\, \mathrm{Nr}\text{“}C\text{‘}P \subset \mathrm{NO} \,.\, \supset \,.\, \Sigma\,\mathrm{Nr}\text{‘}P \epsilon \mathrm{NO}$

Dem.

$\vdash . *182{\cdot}16{\cdot}162 . \quad \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, \mathrm{Nr}\text{‘}\overset{\frown}{\underset{\cdot,}{\downarrow}}\,;P \epsilon \mathrm{NO} \,.\, \overset{\frown}{\underset{\cdot,}{\downarrow}}\,;P \epsilon \mathrm{Rel}^2\,\mathrm{excl}$ (1)

$\vdash . *182{\cdot}05{\cdot}11 . *151{\cdot}65 . \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, \mathrm{Nr}\text{“}C\text{‘}\overset{\frown}{\underset{\cdot,}{\downarrow}}\,;P \subset \mathrm{NO}$ (2)

$\vdash . (1) . (2) . *251{\cdot}122 . \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, \overset{\frown}{\underset{\cdot,}{\downarrow}}\,;P \epsilon \mathrm{Rel}^2\,\mathrm{excl} \cap \Omega \,.\, C\text{‘}\overset{\frown}{\underset{\cdot,}{\downarrow}}\,;P \subset \Omega \,.$

[*251·21] $\quad \supset \,.\, \Sigma\text{‘}\overset{\frown}{\underset{\cdot,}{\downarrow}}\,;P \epsilon \Omega \,.$

[*251·1.(*183·01)] $\quad \supset \,.\, \Sigma\,\mathrm{Nr}\text{‘}P \epsilon \mathrm{NO} : \supset \vdash . \mathrm{Prop}$

***251·22.** $\vdash : P, Q \epsilon \mathrm{Bord} \,.\, C\text{‘}P \cap C\text{‘}Q = \Lambda \,.\, \supset \,.\, P \ddagger Q \epsilon \mathrm{Bord}$

Dem.

$\vdash . *162{\cdot}3 . *163{\cdot}42 . \supset \vdash : \mathrm{Hp} \,.\, \sim(P = \dot{\Lambda} \,.\, Q = \dot{\Lambda}) \,.\, \supset \,.$

$\quad P \downarrow Q \epsilon \mathrm{Bord} \,.\, C\text{‘}(P \downarrow Q) \subset \mathrm{Bord} \,.\, P \downarrow Q \epsilon \mathrm{Rel}^2\,\mathrm{excl} \,.$

$\quad \Sigma\text{‘}(P \downarrow Q) = P \ddagger Q \,.$

[*251·2] $\quad \supset \,.\, P \ddagger Q \epsilon \mathrm{Bord}$ (1)

$\vdash . *160{\cdot}21 . *250{\cdot}4 . \supset \vdash : P = \dot{\Lambda} \,.\, Q = \dot{\Lambda} \,.\, \supset \,.\, P \ddagger Q \epsilon \mathrm{Bord}$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***251·23.** $\vdash : P, Q \epsilon \Omega \,.\, C\text{‘}P \cap C\text{‘}Q = \Lambda \,.\, \supset \,.\, P \ddagger Q \epsilon \Omega$ [*204·5 . *251·22]

***251·24.** $\vdash : \alpha, \beta \epsilon \mathrm{NO} \,.\, \supset \,.\, \alpha \dot{+} \beta \epsilon \mathrm{NO}$

Dem.

$\vdash . *251{\cdot}111 . *180{\cdot}12{\cdot}11 . \supset$

$\vdash : P, Q \epsilon \Omega \,.\, \supset \,.\, \downarrow(\Lambda \cap C\text{‘}Q)\,;\iota\,;P \epsilon \Omega \,.\, (\Lambda \cap C\text{‘}P)\downarrow\,;\iota\,;Q \epsilon \Omega \,.$

$\quad C\text{‘}\downarrow(\Lambda \cap C\text{‘}Q)\,;\iota\,;P \cap C\text{‘}(\Lambda \cap C\text{‘}P)\downarrow\,;\iota\,;Q = \Lambda \,.$

[*251·23.(*181·01)] $\supset \,.\, P + Q \epsilon \Omega \,.$

[*180·3.*251·1] $\quad \supset \,.\, \mathrm{Nr}\text{‘}P \dot{+} \mathrm{Nr}\text{‘}Q \epsilon \mathrm{NO}$ (1)

$\vdash . (1) . *251{\cdot}1 . \supset \vdash . \mathrm{Prop}$

***251·25.** $\vdash : P \ddagger Q \epsilon \Omega \,.\, \equiv \,.\, P, Q \epsilon \Omega \,.\, C\text{‘}P \cap C\text{‘}Q = \Lambda$

Dem.

$\vdash . *204{\cdot}5 . \quad \supset \vdash : P \ddagger Q \epsilon \Omega \,.\, \supset \,.\, P, Q \epsilon \mathrm{Ser} \,.\, C\text{‘}P \cap C\text{‘}Q = \Lambda$ (1)

$\vdash . (1) . *205{\cdot}84 . \quad \supset \vdash :. P \ddagger Q \epsilon \Omega \,.\, \supset : \exists\,!\, C\text{‘}P \cap \alpha \,.\, \supset_\alpha \,.\, \exists\,!\, \overrightarrow{\min}_P\text{‘}\alpha :$

[*250·11] $\quad \supset : P \epsilon \mathrm{Bord}$ (2)

$\vdash . (1) . *205{\cdot}841 . \supset \vdash :. P \ddagger Q \epsilon \Omega \,.\, \supset :$

$\quad \exists\,!\, \alpha - C\text{‘}P \cap C\text{‘}(P \ddagger Q) \,.\, \supset_\alpha \,.\, \exists\,!\, \overrightarrow{\min}_Q\text{‘}(\alpha - C\text{‘}P) :$

[*160·14.(1)] $\supset : \exists ! \alpha \cap C'Q . \supset_\alpha . \exists ! \overrightarrow{\min}_Q{}'(\alpha - C'P)$.

[*205·15.(1)] $\supset_\alpha . \exists ! \overrightarrow{\min}_Q{}'\alpha$:

[*250·101] $\supset : Q \epsilon \text{Bord}$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash : P \dotplus Q \epsilon \Omega . \supset . P, Q \epsilon \Omega . C'P \cap C'Q = \Lambda$ (4)

$\vdash . (4) . *251·23 . \supset \vdash$. Prop

***251·26.** $\vdash : \alpha, \beta \epsilon \text{NO} - \iota'\Lambda . \equiv . \alpha \dotplus \beta \epsilon \text{NO} - \iota'\Lambda$ [*251·25]

***251·3.** $\vdash : P \epsilon \Omega . C'P \subset \text{Ser} . \supset . \Pi'P \epsilon \text{Ser}$ [*204·57 . *250·1]

***251·31.** $\vdash : \text{E} !! B``C'P . \supset . B \upharpoonright C'P \epsilon F_\Delta{}'C'P$

Dem.

$\vdash . *71·571 . \supset \vdash : \text{Hp} . \supset . B \upharpoonright C'P \epsilon 1 \rightarrow \text{Cls} . \text{Q}'(B \upharpoonright C'P) = C'P$ (1)

$\vdash . *93·103 . \supset \vdash . B \mathrel{\text{G}} F$ (2)

$\vdash . (1) . (2) . *80·14 . \supset \vdash$. Prop

***251·32.** $\vdash : \text{E} !! B``C'P . \dot{\exists} ! P . \supset . B \upharpoonright C'P = B'\Pi'P$

Dem.

$\vdash . *172·162 . \supset \vdash : \text{Hp} . \supset . \overrightarrow{B}'\Pi'P = B_\Delta{}'C'P$

[*82·21] $= \iota'(B \upharpoonright C'P) : \supset \vdash$. Prop

***251·33.** $\vdash : C'P \subset \Omega - \iota'\dot{\Lambda} . \dot{\exists} ! P . \supset . \dot{\exists} ! \Pi'P . B \upharpoonright C'P = B'\Pi'P$ [*250·13 . *251·32]

***251·34.** $\vdash : P \epsilon \text{Rel}^2 \text{excl} . C'P \subset \Omega - \iota'\dot{\Lambda} . \supset . \exists ! \epsilon_\Delta{}'C``C'P$

Dem.

$\vdash . *251·33 . *173·16 . \supset \vdash : \text{Hp} . \dot{\exists} ! P . \supset . \dot{\exists} ! \text{Prod}'P$.

[*173·161] $\supset . \exists ! \text{Prod}'C``C'P$

[*115·1] $\supset . \exists ! \epsilon_\Delta{}'C``C'P$ (1)

$\vdash . *83·15 . \supset \vdash : P = \dot{\Lambda} . \supset . \exists ! \epsilon_\Delta{}'C``C'P$ (2)

$\vdash . (1) . (2) . \supset \vdash$. Prop

***251·35.** $\vdash :: P \epsilon \Omega . \supset :.$

$$\alpha P_{\text{cl}} \beta . \equiv : \alpha, \beta \epsilon \text{Cl}'C'P : (\exists z) . z \epsilon \alpha - \beta . \alpha \cap \overrightarrow{P}'z = \beta \cap \overrightarrow{P}'z$$

Dem.

$\vdash . *170·2 . \supset$

$\vdash :. \alpha, \beta \epsilon \text{Cl}'C'P : (\exists z) . z \epsilon \alpha - \beta . \alpha \cap \overrightarrow{P}'z = \beta \cap \overrightarrow{P}'z : \supset . \alpha P_{\text{cl}} \beta$ (1)

$\vdash . *170·231 . *250·121 . \supset$

$\vdash :: \text{Hp} . \supset :. \alpha P_{\text{cl}} \beta . \supset : \alpha, \beta \epsilon \text{Cl}'C'P : (\exists z) . z \epsilon \alpha - \beta . \alpha \cap \overrightarrow{P}'z = \beta \cap \overrightarrow{P}'z$ (2)

$\vdash . (1) . (2) . \supset \vdash$. Prop

$*251\cdot351.\ \vdash :: \breve{P} \epsilon \Omega . \supset :. \alpha P_{\mathrm{lc}} \beta . \equiv :$

$\alpha, \beta \epsilon \mathrm{Cl}`C`P : (\exists z) . z \epsilon \beta - \alpha . \alpha \cap \overleftarrow{P}`z = \beta \cap \overleftarrow{P}`z$ [$*251\cdot35 . *170\cdot101$]

$*251\cdot36.\ \vdash : P \epsilon \Omega . \supset . P_{\mathrm{cl}} \epsilon \mathrm{Ser}$

Dem.

$\vdash . *170\cdot17 . \supset \vdash . P_{\mathrm{cl}} \in J$ (1)

$\vdash . *251\cdot35 . \supset \vdash :: \mathrm{Hp} . \supset :. \alpha P_{\mathrm{cl}} \beta . \beta P_{\mathrm{cl}} \gamma . \supset :$

$(\exists z, w) . z \epsilon \alpha - \beta . w \epsilon \beta - \gamma . \alpha \cap \overrightarrow{P}`z = \beta \cap \overrightarrow{P}`z . \beta \cap \overrightarrow{P}`w = \gamma \cap \overrightarrow{P}`w$ (2)

$\vdash . *201\cdot14 . \supset$

$\vdash :. \mathrm{Hp} . z \epsilon \alpha - \beta . w \epsilon \beta - \gamma . \alpha \cap \overrightarrow{P}`z = \beta \cap \overrightarrow{P}`z . \beta \cap \overrightarrow{P}`w = \gamma \cap \overrightarrow{P}`w . \supset :$

$zPw . \supset . z \epsilon \alpha - \gamma . \alpha \cap \overrightarrow{P}`z = \gamma \cap \overrightarrow{P}`z$ (3)

$\vdash . *201\cdot14 . \supset \vdash :. \mathrm{Hp}\,(3) . \supset : wPz . \supset . w \epsilon \alpha - \gamma . \alpha \cap \overrightarrow{P}`w = \gamma \cap \overrightarrow{P}`w$ (4)

$\vdash . (2) . (3) . (4) . *202\cdot104 . *251\cdot35 . \supset \vdash :. \mathrm{Hp} . \supset : \alpha P_{\mathrm{cl}} \beta . \beta P_{\mathrm{cl}} \gamma . \supset . \alpha P_{\mathrm{cl}} \gamma$ (5)

$\vdash . *250\cdot121 . \supset$

$\vdash : \mathrm{Hp} . \alpha, \beta \epsilon \mathrm{Cl}`C`P . \alpha \neq \beta . \supset . (\exists z) . z = \min_P`\{(\alpha - \beta) \cup (\beta - \alpha)\} .$

[$*205\cdot14$] $\supset . (\exists z) . z \epsilon \{(\alpha - \beta) \cup (\beta - \alpha)\} . \alpha \cap \overrightarrow{P}`z = \beta \cap \overrightarrow{P}`z .$

[$*251\cdot35$] $\supset . \alpha (P_{\mathrm{cl}} \cup \breve{P}_{\mathrm{cl}}) \beta$ (6)

$\vdash . (1) . (5) . (6) . \supset \vdash .$ Prop

$*251\cdot361.\ \vdash : \breve{P} \epsilon \Omega . \supset . P_{\mathrm{lc}} \epsilon \mathrm{Ser}$ [$*251\cdot36 . *170\cdot101$]

$*251\cdot37.\ \vdash : P \epsilon \Omega . \supset . P_{\mathrm{cl}} = P_{\mathrm{df}}$ [$*251\cdot35 . *171\cdot2$]

$*251\cdot371.\ \vdash : \breve{P} \epsilon \Omega . \supset . P_{\mathrm{lc}} = P_{\mathrm{fd}}$ [$*251\cdot37 . *170\cdot101 . *171\cdot101$]

$*251\cdot4.\ \vdash : P \epsilon \mathrm{Rel}^3 \mathrm{arithm} \cap \mathrm{Bord} . C`P \subset \mathrm{Bord} . C`\Sigma`P \subset \mathrm{Bord} . \supset .$

$\Sigma`\Sigma`P \epsilon \mathrm{Bord}$

Dem.

$\vdash . *251\cdot2 . \supset : \mathrm{Hp} . \supset . \Sigma`P \epsilon \mathrm{Rel}^2 \mathrm{excl} \cap \mathrm{Bord} . C`\Sigma`P \subset \mathrm{Bord} .$

[$*251\cdot2$] $\supset . \Sigma`\Sigma`P \epsilon \mathrm{Bord} : \supset \vdash .$ Prop

$*251\cdot41.\ \vdash : P \epsilon \mathrm{Rel}^3 \mathrm{arithm} \cap \Omega . C`P \subset \Omega . C`\Sigma`P \subset \Omega . \supset . \Sigma`\Sigma`P \epsilon \Omega$

[$*204\cdot54 . *251\cdot4$]

$*251\cdot42.\ \vdash : P \epsilon \mathrm{Ser} . Q \epsilon \Omega . \supset . P^Q, (P \exp Q) \epsilon \mathrm{Ser}$ [$*204\cdot59 . *250\cdot1$]

$*251\cdot43.\ \vdash : \alpha \epsilon \mathrm{NR} . \alpha \subset \mathrm{Ser} . \beta \epsilon \mathrm{NO} . \supset . (\alpha \exp_r \beta) \epsilon \mathrm{NR} . (\alpha \exp_r \beta) \subset \mathrm{Ser}$

[$*186\cdot13 . *251\cdot42$]

$*251\cdot44.\ \vdash : \alpha \epsilon \mathrm{NO} - \iota`0_r . \beta \epsilon \mathrm{NO} - \iota`0_r . \supset . \alpha \exp_r \beta \neq 0_r$

Dem.

$\vdash . *165\cdot27 . \supset$

$\vdash : \mathrm{Hp} . P \epsilon \alpha . Q \epsilon \beta . \supset . P \downarrow ; Q \epsilon \Omega - \iota`\dot{\Lambda} . C`P \downarrow ; Q \subset \Omega - \iota`\dot{\Lambda} .$

[$*251\cdot33 . *176\cdot1$] $\supset . \dot{\exists} ! (P \exp Q)$ (1)

$\vdash . (1) . *186\cdot13 . \supset \vdash .$ Prop

$*251{\cdot}5.\quad \vdash : \dot{\exists} \,!\, P \,.\, Q \in \mathrm{Bord} \,.\, \supset \,.\, P \underset{\cdot\cdot}{\downarrow} {}^{;}Q \in \mathrm{Bord} \quad [*165{\cdot}25 \,.\, *251{\cdot}11]$

$*251{\cdot}51.\quad \vdash : \dot{\exists} \,!\, P \,.\, Q \in \Omega \,.\, \supset \,.\, P \underset{\cdot\cdot}{\downarrow} {}^{;}Q \in \Omega \quad [*165{\cdot}25 \,.\, *204{\cdot}21 \,.\, *251{\cdot}5]$

$*251{\cdot}52.\quad \vdash : P \in \mathrm{Bord} \,.\, \supset \,.\, C\text{‘}P \underset{\cdot\cdot}{\downarrow} {}^{;}Q \subset \mathrm{Bord} \quad [*165{\cdot}26 \,.\, *251{\cdot}12]$

$*251{\cdot}53.\quad \vdash : P \in \Omega \,.\, \supset \,.\, C\text{‘}P \underset{\cdot\cdot}{\downarrow} {}^{;}Q \in \Omega \quad [*165{\cdot}26 \,.\, *204{\cdot}22 \,.\, *251{\cdot}52]$

$*251{\cdot}54.\quad \vdash : P, Q \in \mathrm{Bord} \,.\, \supset \,.\, P \times Q \in \mathrm{Bord}$

Dem.

$\vdash \,.\, *165{\cdot}21 \,.\, *251{\cdot}5{\cdot}52 \,.\, \supset$

$\vdash : \mathrm{Hp} \,.\, \dot{\exists} \,!\, Q \,.\, \supset \,.\, Q \underset{\cdot\cdot}{\downarrow} {}^{;}P \in \mathrm{Rel}^2\,\mathrm{excl} \cap \mathrm{Bord} \,.\, C\text{‘}Q \underset{\cdot\cdot}{\downarrow} {}^{;}P \subset \mathrm{Bord} \,.$

$[*251{\cdot}2 \,.\, *166{\cdot}1] \supset \,.\, P \times Q \in \mathrm{Bord} \quad (1)$

$\vdash \,.\, *166{\cdot}13 \,.\, *250{\cdot}4 \,.\, \supset \vdash : Q = \dot{\Lambda} \,.\, \supset \,.\, P \times Q \in \mathrm{Bord} \quad (2)$

$\vdash \,.\, (1) \,.\, (2) \,.\, \supset \vdash \,.\, \mathrm{Prop}$

$*251{\cdot}55.\quad \vdash : P, Q \in \Omega \,.\, \supset \,.\, P \times Q \in \Omega \quad [*251{\cdot}54 \,.\, *204{\cdot}55]$

$*251{\cdot}56.\quad \vdash : \alpha, \beta \in \mathrm{NO} \,.\, \supset \,.\, \alpha \dot{\times} \beta \in \mathrm{NO} \quad [*184{\cdot}13 \,.\, *251{\cdot}55{\cdot}1]$

$*251{\cdot}6.\quad \vdash : P, Q \in \mathrm{Rel}^2\,\mathrm{excl} \,.\, C\text{‘}P \subset \Omega \,.\, S \in P \,\overline{\mathrm{smor}}\, Q \cap \mathrm{Rl}\text{‘}\mathrm{smor} \,.$

$\breve{\mu} = \hat{\lambda} \{(\exists N) \,.\, N \in C\text{‘}Q \,.\, \lambda = (\breve{S}\text{‘}N) \,\overline{\mathrm{smor}}\, N\} \,.\, \supset \,.$

$\iota \upharpoonright \breve{\mu} \in \epsilon_\Delta\text{‘}\breve{\mu} \,.\, \dot{s}\text{‘}\iota\text{“}\breve{\mu} \in P \,\overline{\mathrm{smor}}\,\overline{\mathrm{smor}}\, Q$

Dem.

$\vdash \,.\, *250{\cdot}6 \,.\, *251{\cdot}111 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, \breve{\mu} \subset 1 \,.$

$[*83{\cdot}43] \qquad \supset \,.\, \iota \upharpoonright \breve{\mu} \in \epsilon_\Delta\text{‘}\breve{\mu} \,. \quad (1)$

$[*164{\cdot}43] \qquad \supset \,.\, \dot{s}\text{‘}\iota\text{“}\breve{\mu} \in P \,\overline{\mathrm{smor}}\,\overline{\mathrm{smor}}\, Q \quad (2)$

$\vdash \,.\, (1) \,.\, (2) \,.\, \supset \vdash \,.\, \mathrm{Prop}$

$*251{\cdot}61.\quad \vdash :. P, Q \in \mathrm{Rel}^2\,\mathrm{excl} \,.\, C\text{‘}P \subset \Omega \,.\, \supset :$

$\exists \,!\, (P \,\overline{\mathrm{smor}}\, Q) \cap \mathrm{Rl}\text{‘}\mathrm{smor} \,.\, \equiv \,.\, P \,\mathrm{smor}\,\mathrm{smor}\, Q$

Dem.

$\vdash \,.\, *251{\cdot}6 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \exists \,!\, (P \,\overline{\mathrm{smor}}\, Q) \cap \mathrm{Rl}\text{‘}\mathrm{smor} \,.\, \supset \,.\, P \,\mathrm{smor}\,\mathrm{smor}\, Q \quad (1)$

$\vdash \,.\, (1) \,.\, *164{\cdot}17 \,.\, \supset \vdash \,.\, \mathrm{Prop}$

$*251{\cdot}62.\quad \vdash : \mathrm{Hp}\, *251{\cdot}61 \,.\, \exists \,!\, P \,\overline{\mathrm{smor}}\, Q \cap \mathrm{Rl}\text{‘}\mathrm{smor} \,.\, \supset \,.$

$\Sigma\text{‘}P \,\mathrm{smor}\, \Sigma\text{‘}Q \,.\, \Pi\text{‘}P \,\mathrm{smor}\, \Pi\text{‘}Q \,.$

$\Sigma\,\mathrm{Nr}\text{‘}P = \Sigma\,\mathrm{Nr}\text{‘}Q \,.\, \Pi\,\mathrm{Nr}\text{‘}P = \Pi\,\mathrm{Nr}\text{‘}Q$

Dem.

$\vdash \,.\, *164{\cdot}151 \,.\, *251{\cdot}61 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, \Sigma\text{‘}P \,\mathrm{smor}\, \Sigma\text{‘}Q \quad (1)$

$\vdash \,.\, *172{\cdot}44 \,.\, *251{\cdot}61 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, \Pi\text{‘}P \,\mathrm{smor}\, \Pi\text{‘}Q \quad (2)$

$\vdash \,.\, (1) \,.\, *183{\cdot}13 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, \Sigma\,\mathrm{Nr}\text{‘}P = \Sigma\,\mathrm{Nr}\text{‘}Q \quad (3)$

$\vdash \,.\, (2) \,.\, *185{\cdot}1 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, \Pi\,\mathrm{Nr}\text{‘}P = \Pi\,\mathrm{Nr}\text{‘}Q \quad (4)$

$\vdash \,.\, (1) \,.\, (2) \,.\, (3) \,.\, (4) \,.\, \supset \vdash \,.\, \mathrm{Prop}$

In the above proposition, the hypothesis "$P, Q \in \text{Rel}^2\,\text{excl}$" is unnecessary for $\Sigma \text{Nr}\text{‘}P = \Sigma\text{Nr}\text{‘}Q$ and $\Pi\text{Nr}\text{‘}P = \Pi\text{Nr}\text{‘}Q$, as appears from ∗183·14 and ∗185·12. Thus we have

∗251·621. $\vdash : C\text{‘}P \subset \Omega \,.\, \exists\,! \,(P \,\overline{\text{smor}}\, Q) \cap \text{Rl}\text{‘smor} \,.\, \supset .$
$$\Sigma\text{Nr}\text{‘}P = \Sigma\text{Nr}\text{‘}Q \,.\, \Pi\text{Nr}\text{‘}P = \Pi\text{Nr}\text{‘}Q$$

Dem.

$\vdash . ∗151·65 . ∗182·05·162 . \supset \vdash . \underset{\cdot\cdot}{\hat{\downarrow}} \upharpoonright C\text{‘}P \in (\underset{\cdot\cdot}{\hat{\downarrow}} \,;P)\,\overline{\text{smor}}\, P \cap \text{Rl}\text{‘smor}$ (1)

$\vdash . (1) . ∗151·162 . \qquad \supset \vdash : \text{Hp} \,.\, \supset .\, \exists\,!\,\{(\underset{\cdot\cdot}{\hat{\downarrow}}\,;P)\,\overline{\text{smor}}\,(\underset{\cdot\cdot}{\hat{\downarrow}}\,;Q)\} \cap \text{Rl}\text{‘smor}$ (2)

$\vdash . (1) . ∗251·111 . ∗182·16 . \supset \vdash : \text{Hp} \,.\, \supset .\, C\text{‘}\underset{\cdot\cdot}{\hat{\downarrow}}\,;P \subset \Omega \,.\, \underset{\cdot\cdot}{\hat{\downarrow}}\,;P, \underset{\cdot\cdot}{\hat{\downarrow}}\,;Q \in \text{Rel}^2\,\text{excl}$ (3)

$\vdash . (2) . (3) . ∗251·62 . ∗183·14 . ∗185·12 . \supset \vdash . \text{Prop}$

∗251·63. $\vdash : \alpha \in \text{NO} - \iota\text{‘}\Lambda \,.\, \beta \in \text{NR} \,.\, P \in \text{Rel}^2\,\text{excl} \,.\, P \in \beta \,.\, C\text{‘}P \subset \alpha \,.\, \supset .$
$$\Sigma\text{‘}P \in \beta \dot{\times} \alpha \,.\, \Sigma\text{Nr}\text{‘}P = \beta \dot{\times} \alpha$$

Dem.

$\vdash . ∗164·47 . ∗165·27·21 . \supset$

$\vdash : \text{Hp} \,.\, Q \in \alpha \,.\, \alpha \neq 0_r \,.\, \supset .\, Q\underset{\cdot\cdot}{\downarrow}\,;P \in \beta \,.\, C\text{‘}Q\underset{\cdot\cdot}{\downarrow}\,;P \subset \alpha \,.\, P, Q\underset{\cdot\cdot}{\downarrow}\,;P \in \text{Rel}^2\,\text{excl} .$

[∗164·47] $\supset .\, \exists\,!\,(Q\underset{\cdot\cdot}{\downarrow}\,;P)\,\overline{\text{smor}}\, P \cap \text{Rl}\text{‘smor} \,.\, P, Q\underset{\cdot\cdot}{\downarrow}\,;P \in \text{Rel}^2\,\text{excl} .$

[∗251·61] $\supset .\, (Q\underset{\cdot\cdot}{\downarrow}\,;P)\,\text{smor smor}\, P .$

[∗164·151.∗166·1] $\supset .\, (P \times Q)\,\text{smor}\,\Sigma\text{‘}P .$

[∗184·13] $\supset .\, \Sigma\text{‘}P \in \beta \dot{\times} \alpha$ (1)

$\vdash . (1) . \supset \vdash : \text{Hp} \,.\, \alpha \neq 0_r \,.\, \supset .\, \Sigma\text{‘}P \in \beta \dot{\times} \alpha$ (2)

$\vdash . ∗162·42 . \text{Transp} . \supset \vdash : \text{Hp} \,.\, \alpha = 0_r \,.\, \supset .\, \Sigma\text{‘}P = \dot{\Lambda} .$

[∗184·16] $\supset .\, \Sigma\text{‘}P \in \beta \dot{\times} \alpha$ (3)

$\vdash . (2) . (3) . \supset \vdash : \text{Hp} \,.\, \supset .\, \Sigma\text{‘}P \in \beta \dot{\times} \alpha$ (4)

[∗183·13] $\supset .\, \Sigma\text{Nr}\text{‘}P = \beta \dot{\times} \alpha$ (5)

$\vdash . (4) . (5) . \supset \vdash . \text{Prop}$

∗251·64. $\vdash : \text{Hp}\,∗251·63 \,.\, \supset .\, \Pi\text{‘}P \in (\alpha\,\text{exp}_r\,\beta) \,.\, \Pi\text{Nr}\text{‘}P = \alpha\,\text{exp}_r\,\beta$
[Proof as in ∗251·63]

∗251·65. $\vdash : \alpha \in \text{NO} - \iota\text{‘}\Lambda \,.\, \beta \in \text{NR} \,.\, P \in \beta \,.\, C\text{‘}P \subset \alpha \,.\, \supset .$
$$\Sigma\text{Nr}\text{‘}P = \beta \dot{\times} \alpha \,.\, \Pi\text{Nr}\text{‘}P = \alpha\,\text{exp}_r\,\beta$$

Dem.

$\vdash . ∗182·16 . ∗183·231 . \supset$

$\vdash : \text{Hp} \,.\, Q \in \alpha \,.\, \supset .\, \underset{\cdot\cdot}{\hat{\downarrow}}\,;P \in \text{Rel}^2\,\text{excl} \,.\, \underset{\cdot\cdot}{\hat{\downarrow}}\,;P \in \text{Nr}\text{‘}P \,.\, C\text{‘}\underset{\cdot\cdot}{\hat{\downarrow}}\,;P \subset \text{Nr}\text{‘}Q .$ (1)

$[*251\cdot63]\quad \supset . \Sigma \mathrm{Nr}`\overset{\frown}{\downarrow};P = \mathrm{Nr}`P \dot{\times} \mathrm{Nr}`Q .$

$[*183\cdot14]\quad \supset . \Sigma \mathrm{Nr}`P = \mathrm{Nr}`P \dot{\times} \mathrm{Nr}`Q$

$[*152\cdot45]\quad = \beta \dot{\times} \alpha \quad (2)$

$\vdash . (2) . *10\cdot23 . \supset \vdash : \mathrm{Hp} . \supset . \Sigma \mathrm{Nr}`P = \beta \dot{\times} \alpha \quad (3)$

$\vdash . (1) . *251\cdot64 . \supset \vdash : \mathrm{Hp} . Q \in \alpha . \supset . \Pi \mathrm{Nr}`\overset{\frown}{\downarrow};P = (\mathrm{Nr}`Q) \exp_r (\mathrm{Nr}`P) .$

$[*185\cdot1\cdot12]\quad \supset . \Pi \mathrm{Nr}`P = (\mathrm{Nr}`Q) \exp_r (\mathrm{Nr}`P)$

$[*152\cdot45]\quad = \alpha \exp_r \beta \quad (4)$

$\vdash . (4) . *10\cdot23 . \supset \vdash : \mathrm{Hp} . \supset . \Pi \mathrm{Nr}`P = \alpha \exp_r \beta \quad (5)$

$\vdash . (3) . (5) . \supset \vdash . \mathrm{Prop}$

In virtue of the above proposition, the usual relations of addition to multiplication, and of multiplication to exponentiation, when the summands or the factors are all equal, can be established without the multiplicative axiom, provided the summands, or the factors, are *ordinal* numbers.

$*251\cdot7$. $\vdash : \exists ! \Omega - \iota`\dot{\Lambda} \cap t_{00}`\alpha . \equiv . \exists ! 2_r \cap t_{00}`\alpha . \equiv . \exists ! 2 \cap t`\alpha . \equiv . \exists ! 2_\alpha$

Dem.

$\vdash . *64\cdot55 . \supset \vdash : \exists ! \Omega - \iota`\dot{\Lambda} \cap t_{00}`\alpha . \equiv . (\exists P) . P \in \Omega - \iota`\dot{\Lambda} . C`P \subset t_0`\alpha \quad (1)$

$\vdash . *200\cdot12 . \supset \vdash : P \in \Omega - \iota`\dot{\Lambda} . \supset . (\exists x, y) . x, y \in C`P . x \neq y .$

$[*153\cdot201 . *55\cdot3]\quad \supset . \exists ! 2_r \cap \mathrm{Rl}`P \quad (2)$

$\vdash . (1) . (2) . \supset \vdash : \exists ! \Omega - \iota`\dot{\Lambda} \cap t_{00}`\alpha . \supset . (\exists P) . C`P \subset t_0`\alpha . \exists ! 2_r \cap \mathrm{Rl}`P .$

$[*33\cdot265]\quad \supset . (\exists Q) . Q \in 2_r . C`Q \subset t_0`\alpha .$

$[*64\cdot55]\quad \supset . \exists ! 2_r \cap t_{00}`\alpha \quad (3)$

$\vdash . *251\cdot16\cdot122 . \supset \vdash : \exists ! 2_r \cap t_{00}`\alpha . \supset . \exists ! \Omega - \iota`\dot{\Lambda} \cap t_{00}`\alpha \quad (4)$

$\vdash . (3) . (4) . \quad \supset \vdash : \exists ! \Omega - \iota`\dot{\Lambda} \cap t_{00}`\alpha . \equiv . \exists ! 2_r \cap t_{00}`\alpha \quad (5)$

$\vdash . *64\cdot55 . \quad \supset \vdash : \exists ! 2_r \cap t_{00}`\alpha . \equiv . (\exists x, y) . x \neq y . x, y \in t_0`\alpha .$

$[*63\cdot62]\quad \equiv . (\exists x, y) . x \neq y . \iota`x \cup \iota`y \in t`\alpha .$

$[*54\cdot26]\quad \equiv . \exists ! 2 \cap t`\alpha \quad (6)$

$\vdash . (5) . (6) . (*65\cdot01) . \supset \vdash . \mathrm{Prop}$

$*251\cdot71$. $\vdash . \exists ! \Omega - \iota`\dot{\Lambda} \cap t_{00}`\mathrm{Cls} . \exists ! \Omega - \iota`\dot{\Lambda} \cap t_{00}`\mathrm{Rel}$

$[*251\cdot7 . *101\cdot42\cdot43]$

$*252$. SEGMENTS OF WELL-ORDERED SERIES.

Summary of $*252$.

The properties of sections and segments are greatly simplified in the case of series which are well-ordered, owing to the fact that every proper section has a sequent, whence it follows that the class of proper sections is $\overrightarrow{P}``C`P$; and this is also the class of proper segments. Hence also the series of proper sections or of proper segments is the series $\overrightarrow{P};P$ ($*252\cdot37$). The series of all sections is $\overrightarrow{P};P \rightarrow\!\!\!\!+ C`P$ ($*252\cdot38$); hence ($*252\cdot381$)

$$\mathrm{Nr}`\varsigma`P_* = \mathrm{Nr}`P \dot{+} \dot{1}.$$

The most useful propositions in this number are (apart from the above)

$*252\cdot12$. $\vdash : P \in \Omega . \supset .$

$$\mathrm{sect}`P - \iota`C`P = \mathrm{D}`P_\epsilon - \iota`C`P = \overrightarrow{P}``C`P . \mathrm{sect}`P = \overrightarrow{P}``C`P \cup \iota`C`P$$

$*252\cdot17$. $\vdash : P \in \Omega - \iota`\dot{\Lambda} . \supset . \mathrm{sect}`P - \iota`\Lambda = \overrightarrow{P}``\text{Ɑ}`P \cup \iota`C`P$

$*252\cdot171$. $\vdash : P \in \Omega . \supset . \mathrm{sect}`P - \iota`\Lambda - \iota`C`P = \overrightarrow{P}``\text{Ɑ}`P$

$*252\cdot372$. $\vdash :. P \in \Omega . \supset : \varsigma`P \in \Omega : \mathrm{E} ! B`\breve{P} . \supset . \mathrm{Nr}`\varsigma`P = \mathrm{Nr}`P :$

$$\sim \mathrm{E} ! B`\breve{P} . \supset . \mathrm{Nr}`\varsigma`P = \mathrm{Nr}`P \dot{+} \dot{1}$$

$*252\cdot4$. $\vdash : P \in \Omega . \lambda \subset \mathrm{sect}`P . \exists ! \lambda . \supset . p`\lambda \in \lambda$

$*252\cdot1$. $\vdash : P \in \Omega . \alpha \in \mathrm{sect}`P - \iota`C`P . \supset . \mathrm{E} ! \mathrm{seq}_P`\alpha$ [$*250\cdot124$]

$*252\cdot11$. $\vdash : P \in \Omega . \supset . \mathrm{sect}`P - \iota`C`P = \mathrm{sect}`P \cap \text{Ɑ}`\mathrm{seq}_P$

Dem.

$\vdash . *206\cdot18\cdot2 . \supset \vdash . C`P \sim\in \text{Ɑ}`\mathrm{seq}_P$ (1)

$\vdash . (1) . *252\cdot1 . \supset \vdash .$ Prop

***252·12.** $\vdash : P \in \Omega . \supset .$

$$\mathrm{sect}\text{‘}P - \iota\text{‘}C\text{‘}P = \mathrm{D}\text{‘}P_\epsilon - \iota\text{‘}C\text{‘}P = \overrightarrow{P}\text{“}C\text{‘}P . \mathrm{sect}\text{‘}P = \overrightarrow{P}\text{“}C\text{‘}P \cup \iota\text{‘}C\text{‘}P$$

Dem.

$$\vdash . *211\cdot24 . *252\cdot11 . \supset \vdash : \mathrm{Hp} . \alpha \in \mathrm{sect}\text{‘}P - \iota\text{‘}C\text{‘}P . \supset . \alpha \in \mathrm{D}\text{‘}P_\epsilon \quad (1)$$

$$\vdash . *211\cdot15 . \supset \vdash : \mathrm{Hp} . \alpha \in \mathrm{D}\text{‘}P_\epsilon - \iota\text{‘}C\text{‘}P . \supset . \alpha \in \mathrm{sect}\text{‘}P - \iota\text{‘}C\text{‘}P \quad (2)$$

$$\vdash . (1) . (2) . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{sect}\text{‘}P - \iota\text{‘}C\text{‘}P = \mathrm{D}\text{‘}P_\epsilon - \iota\text{‘}C\text{‘}P \quad (3)$$

$$\vdash . *211\cdot302 . *252\cdot11 . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{sect}\text{‘}P - \iota\text{‘}C\text{‘}P = \overrightarrow{P}\text{“}C\text{‘}P \quad (4)$$

$$\vdash . (3) . (4) . *211\cdot26 . \supset \vdash . \mathrm{Prop}$$

In dealing with sections and segments of well-ordered series, it is necessary to distinguish series with a last term from such as have no last term. If a series has no last term, $C\text{‘}P = P\text{“}C\text{‘}P$, so that $C\text{‘}P \in \mathrm{D}\text{‘}P_\epsilon$. But if a series has a last term, $C\text{‘}P \sim\in \mathrm{D}\text{‘}P_\epsilon$; in this case, $\mathrm{D}\text{‘}P_\epsilon = \overrightarrow{P}\text{“}C\text{‘}P$. Thus $\mathrm{D}\text{‘}P_\epsilon$ is either $\overrightarrow{P}\text{“}C\text{‘}P$ or $\mathrm{sect}\text{‘}P$, according as there is or is not a last term. In either case,

$$\mathrm{sect}\text{‘}P = \overrightarrow{P}\text{“}C\text{‘}P \cup \iota\text{‘}C\text{‘}P,$$

as has been already proved in *252·12.

***252·13.** $\vdash : P \in \Omega . \mathrm{E} ! B\text{‘}\breve{P} . \supset . \mathrm{sect}\text{‘}P - \iota\text{‘}C\text{‘}P = \mathrm{D}\text{‘}P_\epsilon = \overrightarrow{P}\text{“}C\text{‘}P .$

$$\mathrm{sect}\text{‘}P = \mathrm{D}\text{‘}P_\epsilon \cup \iota\text{‘}C\text{‘}P = \overrightarrow{P}\text{“}C\text{‘}P \cup \iota\text{‘}C\text{‘}P$$

Dem.

$$\vdash . *250\cdot21 . *211\cdot36 . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{sect}\text{‘}P - \mathrm{D}\text{‘}P_\epsilon = \iota\text{‘}C\text{‘}P .$$

$$[*24\cdot492 . *211\cdot15] \qquad \supset . \mathrm{sect}\text{‘}P - \iota\text{‘}C\text{‘}P = \mathrm{D}\text{‘}P_\epsilon \quad (1)$$

$$[*252\cdot12] \qquad = \overrightarrow{P}\text{“}C\text{‘}P \quad (2)$$

$$\vdash . (1) . (2) . *211\cdot26 . \supset \vdash . \mathrm{Prop}$$

***252·14.** $\vdash : P \in \Omega . \sim\mathrm{E} ! B\text{‘}\breve{P} . \supset . \mathrm{sect}\text{‘}P = \mathrm{D}\text{‘}P_\epsilon = \overrightarrow{P}\text{“}C\text{‘}P \cup \iota\text{‘}C\text{‘}P$
[*250·21 . *211·361 . *252·12]

***252·15.** $\vdash : P \in \Omega . \supset . \mathrm{D}\text{‘}P_\epsilon = \overrightarrow{P}\text{“}\mathrm{D}\text{‘}P \cup \iota\text{‘}\mathrm{D}\text{‘}P$

Dem.

$$\vdash . *252\cdot13 . \supset \vdash : \mathrm{Hp} . \mathrm{E} ! B\text{‘}\breve{P} . \supset . \mathrm{D}\text{‘}P_\epsilon = \overrightarrow{P}\text{“}\mathrm{D}\text{‘}P \cup \iota\text{‘}\overrightarrow{P}\text{‘}B\text{‘}\breve{P}$$

$$[*202\cdot524] \qquad = \overrightarrow{P}\text{“}\mathrm{D}\text{‘}P \cup \iota\text{‘}\mathrm{D}\text{‘}P \quad (1)$$

$$\vdash . *252\cdot14 . \supset \vdash : \mathrm{Hp} . \sim\mathrm{E} ! B\text{‘}\breve{P} . \supset . \mathrm{D}\text{‘}P_\epsilon = \overrightarrow{P}\text{“}\mathrm{D}\text{‘}P \cup \iota\text{‘}\mathrm{D}\text{‘}P \quad (2)$$

$$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$$

***252·16.** $\vdash : P \in \Omega - 2_r . \supset . \mathrm{D}\text{‘}P_\epsilon = \mathrm{sect}\text{‘}(P \upharpoonright \mathrm{D}\text{‘}P)$

Dem.

$$\vdash . *204\cdot271 . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{D}\text{‘}P \sim\in 1 .$$

$$[*202\cdot55] \qquad \supset . C\text{‘}(P \upharpoonright \mathrm{D}\text{‘}P) = \mathrm{D}\text{‘}P .$$

$$[*250\cdot141 . *252\cdot12] \qquad \supset . \mathrm{sect}\text{‘}(P \upharpoonright \mathrm{D}\text{‘}P) = \overrightarrow{P \upharpoonright \mathrm{D}\text{‘}P}\text{“}\mathrm{D}\text{‘}P \cup \iota\text{‘}\mathrm{D}\text{‘}P$$

$$[*37\cdot42\cdot421] \qquad = \overrightarrow{P}\text{“}\mathrm{D}\text{‘}P \cup \iota\text{‘}\mathrm{D}\text{‘}P$$

$$[*252\cdot15] \qquad = \mathrm{D}\text{‘}P_\epsilon : \supset \vdash . \mathrm{Prop}$$

***252·17.** $\vdash : P \epsilon \Omega - \iota`\dot{\Lambda} . \supset . \text{sect}`P - \iota`\Lambda = \overrightarrow{P}``\text{G}`P \cup \iota`C`P$

Dem.

$\vdash . *252\cdot12 . \supset \vdash : \text{Hp} . \supset . \text{sect}`P - \iota`\Lambda = (\overrightarrow{P}``C`P - \iota`\Lambda) \cup \iota`C`P$

$[*33\cdot41] \qquad = \overrightarrow{P}``\text{G}`P \cup \iota`C`P : \supset \vdash . \text{Prop}$

***252·171.** $\vdash : P \epsilon \Omega . \supset . \text{sect}`P - \iota`\Lambda - \iota`C`P = \overrightarrow{P}``\text{G}`P$

Dem.

$\vdash . *252\cdot12 . \supset \vdash : \text{Hp} . \supset . (\text{sect}`P - \iota`C`P) - \iota`\Lambda = \overrightarrow{P}``C`P - \iota`\Lambda$

$[*33\cdot41] \qquad = \overrightarrow{P}``\text{G}`P : \supset \vdash . \text{Prop}$

***252·3.** $\vdash : P \epsilon \Omega . \supset . \text{D}`\varsigma`P_* = \overrightarrow{P}``C`P \quad [*212\cdot171 . *252\cdot12]$

***252·31.** $\vdash : P \epsilon \Omega . \dot{\exists} ! P . \supset . C`\varsigma`P_* = \overrightarrow{P}``C`P \cup \iota`C`P$
$[*212\cdot172 . *252\cdot12]$

***252·311.** $\vdash : P \epsilon \Omega . \dot{\exists} ! P . \supset . \text{G}`\varsigma`P_* = \overrightarrow{P}``\text{G}`P \cup \iota`C`P$
$[*212\cdot171 . *252\cdot17]$

***252·32.** $\vdash : P \epsilon \Omega . \supset . \text{D}`\varsigma`P = \overrightarrow{P}``\text{D}`P \quad [*212\cdot132 . *252\cdot15]$

***252·33.** $\vdash : P \epsilon \Omega - \iota`\dot{\Lambda} . \supset . C`\varsigma`P = \overrightarrow{P}``\text{D}`P \cup \iota`\text{D}`P$
$[*212\cdot133 . *252\cdot15]$

***252·34.** $\vdash : P \epsilon \Omega . \text{E} ! B`\breve{P} . \supset . C`\varsigma`P = \overrightarrow{P}``C`P$

Dem.

$\vdash . *202\cdot524 . \supset \vdash : \text{Hp} . \supset . \overrightarrow{P}`B`\breve{P} = \text{D}`P .$

$[*252\cdot33] \qquad \supset . C`\varsigma`P = \overrightarrow{P}``C`P : \supset \vdash . \text{Prop}$

***252·35.** $\vdash : P \epsilon \Omega - \iota`\dot{\Lambda} . \sim \text{E} ! B`\breve{P} . \supset . C`\varsigma`P = \overrightarrow{P}``C`P \cup \iota`C`P$
$[*212\cdot133 . *252\cdot14]$

***252·36.** $\vdash : P \epsilon \Omega . \text{E} ! B`\breve{P} . \supset . \varsigma`P = \overrightarrow{P} ; P$

Dem.

$\vdash . *212\cdot25 . *252\cdot34 . \supset \vdash : \text{Hp} . \supset . \overrightarrow{P} ; P = (\varsigma`P) \upharpoonright C``\varsigma`P$

$[*36\cdot33] \qquad = \varsigma`P : \supset \vdash . \text{Prop}$

***252·37.** $\vdash : P \epsilon \Omega . \supset . (\varsigma`P) \upharpoonright (-\iota`C`P) = \overrightarrow{P} ; P$

Dem.

$\vdash . *36\cdot3 . \qquad \supset \vdash . (\varsigma`P) \upharpoonright (-\iota`C`P) = (\varsigma`P) \upharpoonright (C`\varsigma`P - \iota`C`P)$

$[*212\cdot133\cdot134] \qquad = (\varsigma`P) \upharpoonright (\text{D}`P_\epsilon - \iota`C`P) \quad (1)$

$\vdash . (1) . *252\cdot12 . \supset \vdash : \text{Hp} . \supset . (\varsigma`P) \upharpoonright (-\iota`C`P) = (\varsigma`P) \upharpoonright (\overrightarrow{P}``C`P)$

$[*212\cdot25] \qquad = \overrightarrow{P} ; P : \supset \vdash . \text{Prop}$

***252·371.** $\vdash : P \epsilon \Omega . \sim \mathrm{E} ! B`\breve{P} . \supset . \varsigma`P = \overrightarrow{P};P \dotplus\!\!\!\rightarrow C`P$

Dem.

$\vdash . *212\cdot25 . *252\cdot32 . \quad \supset \vdash : \mathrm{Hp} . \supset . \overrightarrow{P};P = (\varsigma`P) \upharpoonright (\mathrm{D}`\varsigma`P)$ (1)

$\vdash . *212\cdot133 . \quad \supset \vdash : \mathrm{Hp} . \dot{\exists} ! P . \supset . C`P = B`\mathrm{Cnv}`\varsigma`P$ (2)

$\vdash . *252\cdot32 . \quad \supset \vdash : \mathrm{Hp} . \supset . \mathrm{D}`\varsigma`P = \overrightarrow{P}``C`P .$

$[*200\cdot12 . *204\cdot34] \quad \supset . \mathrm{D}`\varsigma`P \sim \epsilon\, 1$ (3)

$\vdash . (1) . (2) . (3) . *204\cdot461 . \supset \vdash : \mathrm{Hp} . \dot{\exists} ! P . \supset . \overrightarrow{P};P \dotplus\!\!\!\rightarrow C`P = \varsigma`P$ (4)

$\vdash . *212\cdot134 . *161\cdot2 . \quad \supset \vdash : \mathrm{Hp} . P = \dot{\Lambda} . \supset . \varsigma`P = \dot{\Lambda} . \overrightarrow{P};P \dotplus\!\!\!\rightarrow C`P = \dot{\Lambda}$ (5)

$\vdash . (4) . (5) . \supset \vdash . \mathrm{Prop}$

***252·372.** $\vdash :. P \epsilon \Omega . \supset : \varsigma`P \epsilon \Omega : \mathrm{E} ! B`\breve{P} . \supset . \mathrm{Nr}`\varsigma`P = \mathrm{Nr}`P :$
$\sim \mathrm{E} ! B`\breve{P} . \supset . \mathrm{Nr}`\varsigma`P = \mathrm{Nr}`P \dotplus \dot{1}$

Dem.

$\vdash . *252\cdot36 . *204\cdot35 . \supset \vdash : \mathrm{Hp} . \mathrm{E} ! B`\breve{P} . \supset . \varsigma`P \,\mathrm{smor}\, P .$

$[*251\cdot111 . *152\cdot321] \quad \supset . \varsigma`P \epsilon \Omega . \mathrm{Nr}`\varsigma`P = \mathrm{Nr}`P$ (1)

$\vdash . *252\cdot371 . *204\cdot35 . *200\cdot52 . \supset$

$\vdash : \mathrm{Hp} . \sim \mathrm{E} ! B`\breve{P} . \supset . \mathrm{Nr}`\varsigma`P = \mathrm{Nr}`P \dotplus \dot{1} .$ (2)

$[*251\cdot132] \quad \supset . \varsigma`P \epsilon \Omega$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash . \mathrm{Prop}$

***252·38.** $\vdash : P \epsilon \Omega . \supset . \varsigma`P_* = \overrightarrow{P};P \dotplus\!\!\!\rightarrow C`P$

Dem.

$\vdash . *252\cdot12 . *212\cdot24 . \supset$

$\vdash :: \mathrm{Hp} . \supset :. \alpha (\varsigma`P_*) \beta . \equiv : \alpha, \beta \epsilon \overrightarrow{P}``C`P \cup \iota`C`P . \alpha \subset \beta . \alpha \neq \beta :$

$[*37\cdot6 . *200\cdot52]$

$\equiv : (\exists x, y) . x, y \epsilon C`P . \alpha = \overrightarrow{P}`x . \beta = \overrightarrow{P}`y . \overrightarrow{P}`x \subset \overrightarrow{P}`y . \overrightarrow{P}`x \neq \overrightarrow{P}`y . \mathbf{v} .$
$(\exists x) . x \epsilon C`P . \alpha = \overrightarrow{P}`x . \beta = C`P :$

$[*204\cdot33\cdot34] \equiv : (\exists x, y) . xPy . \alpha = \overrightarrow{P}`x . \beta = \overrightarrow{P}`y . \mathbf{v} .$
$(\exists x) . x \epsilon C`P . \alpha = \overrightarrow{P}`x . \beta = C`P :$

$[*150\cdot5\cdot22] \quad \equiv : \alpha (\overrightarrow{P};P) \beta . \mathbf{v} . \alpha \epsilon C`\overrightarrow{P};P . \beta = C`P :$

$[*161\cdot11] \quad \equiv : \alpha (\overrightarrow{P};P \dotplus\!\!\!\rightarrow C`P) \beta :: \supset \vdash . \mathrm{Prop}$

***252·381.** $\vdash : P \epsilon \Omega . \supset . \varsigma`P_* \epsilon \Omega . \mathrm{Nr}`\varsigma`P_* = \mathrm{Nr}`P \dotplus \dot{1}$
$[*252\cdot38 . *200\cdot52 . *204\cdot35 . *251\cdot131]$

***252·4.** $\vdash : P \epsilon \Omega . \lambda \subset \text{sect}`P . \exists ! \lambda . \supset . p`\lambda \epsilon \lambda$

Dem.

$\vdash . *211·44·1 . \supset \vdash : \text{Hp} . P = \dot{\Lambda} . \supset . \lambda = \iota`\dot{\Lambda} .$

$[*53·01] \qquad \supset . p`\lambda \epsilon \lambda \qquad (1)$

$\vdash . *212·172 . \supset \vdash : \text{Hp} . \dot{\exists} ! P . \supset . \lambda \subset C`\varsigma`P_* . \exists ! \lambda .$

$[*252·381 . *250·121] \qquad \supset . E ! \min (\varsigma`P_*)`\lambda .$

$[*210·222 . *211·67·66] \qquad \supset . p`\lambda \epsilon \lambda \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***252·41.** $\vdash : \breve{P} \epsilon \Omega . \lambda \subset \text{sect}`P . \exists ! \lambda . \supset . s`\lambda \epsilon \lambda$ [Proof as in *252·4]

***252·42.** $\vdash :. P \epsilon \Omega . (\text{Cnv}`\varsigma`P_*)_1``\sigma \subset \sigma :$

$\lambda \subset \sigma . \exists ! \lambda \cap C`\varsigma`P_* . \supset_\lambda . s`(\lambda \cap C`\varsigma`P_*) \epsilon \sigma : \supset .$

$(\text{Cnv}`\varsigma`P_*)``\sigma \subset \sigma$

[*250·361 . *252·381 . *212·322]

***252·43.** $\vdash :. \breve{P} \epsilon \Omega . (\varsigma`P_*)_1``\sigma \subset \sigma :$

$\lambda \subset \sigma . \exists ! \lambda \cap C`\varsigma`P_* . \supset_\lambda . p`(\lambda \cap C`\varsigma`P_*) \epsilon \sigma : \supset . (\varsigma`P_*)``\sigma \subset \sigma$

Dem.

$\vdash . *212·181 . \qquad \supset \vdash . (\text{Cnv}`\varsigma`P_*) \text{ smor } (\varsigma`\breve{P}_*) \qquad (1)$

$\vdash . (1) . *252·381 . \supset \vdash : \text{Hp} . \supset . \text{Cnv}`\varsigma`P_* \epsilon \Omega \qquad (2)$

$\vdash . (2) . *212·34 . *250·362 . \supset \vdash . \text{Prop}$

$*253$. SECTIONAL RELATIONS OF WELL-ORDERED SERIES.

Summary of $*253$.

In the present number we shall consider the properties of the relation P_s (defined in $*213$) when $P \in \Omega$. The relation P_s has great importance in this case, owing to the fact (to be proved later) that $\mathrm{Nr}``\mathrm{D}`P_s$ is the class of all ordinals less than $\mathrm{Nr}`P$, and that, if P, Q are any two well-ordered series, either P is similar to a member of $C`Q_s$, or $Q$ is similar to a member of $C`P_s$, whence it follows that of any two unequal ordinals one must be the greater.

The present number consists merely of the more elementary properties of P_s when $P \in \Omega$. The interesting properties connected with greater and less will be treated in the following number.

The most useful propositions of the present number are the following:

$*253\cdot13$. $\vdash : P \in \Omega . \supset . \mathrm{D}`P_s = P \upharpoonright ``\overrightarrow{P}``\text{Ɑ}`P = P \upharpoonright ``\overrightarrow{P}``C`P$

$*253\cdot18$. $\vdash : P \in \Omega . \supset . C`P_s \subset P \upharpoonright ``\overrightarrow{P}``\text{Ɑ}`P \cup \iota`P . C`P_s \subset \Omega$

Instead of $C`P_s \subset P \upharpoonright ``\overrightarrow{P}``\text{Ɑ}`P \cup \iota`P$ we shall have equality, unless $P = \dot{\Lambda}$ ($*253\cdot15$).

$*253\cdot2$. $\vdash : P \in \Omega - 2_r . \supset . \mathrm{Nr}`P_s = \mathrm{Nr}`(P \upharpoonright \text{Ɑ}`P) \dot{+} \dot{1}$

The case when $P \in 2_r$ has to be excluded, because then $P \upharpoonright \text{Ɑ}`P = \dot{\Lambda}$.

$*253\cdot21$. $\vdash : P \in \Omega . \supset . \dot{1} \dot{+} \mathrm{Nr}`P_s = \mathrm{Nr}`P \dot{+} \dot{1}$

This proposition involves $\mathrm{Nr}`P_s = \mathrm{Nr}`P$ when P is finite, but when P is infinite it involves $\mathrm{Nr}`P_s = \mathrm{Nr}`P \dot{+} \dot{1}$ (cf. $*261\cdot38$).

$*253\cdot22$. $\vdash : P \in \Omega . \supset . P_s \upharpoonright \mathrm{D}`P_s \;\mathrm{smor}\; P \upharpoonright \text{Ɑ}`P$

$*253\cdot24$. $\vdash : P \in \Omega . \supset . P_s \in \Omega$

$*253\cdot4$. $\vdash : P \in \Omega - \iota`\dot{\Lambda} . \supset .$

$$C`P_s = \hat{Q}\{(\exists R) . P = Q \mathbin{\Uparrow} R . \mathrm{v} . (\exists x) . P = Q \mathbin{+\!\!\!\to} x\}$$

$*253\cdot421$. $\vdash : P \in \Omega . Q \in \mathrm{D}`P_s . \supset . \sim(Q \;\mathrm{smor}\; P)$

$*253\cdot44$. $\vdash : \alpha, \beta \in \mathrm{NO} - \iota`\Lambda . \beta \neq 0_r . \supset . \alpha \dot{+} \beta \neq \alpha$

This proposition marks a difference between ordinals and cardinals. An ordinal is always increased by the addition of anything at the end, whereas this is (often if not always) not the case with a cardinal if it is reflexive and greater than the addendum. The above proposition ceases to be true if we add β at the beginning instead of the end: $\beta \dotplus \alpha = \alpha$ will be true if α is infinite and $\omega \dot{\times} \beta$ is not greater than α. (For the definition of ω, cf. $*263$.)

$*253{\cdot}45$. $\vdash : \alpha \in \mathrm{NO} - \iota`\Lambda - \iota`0_r . \supset . \alpha \dotplus \dot{1} \neq \alpha$

Similar remarks apply to this proposition as to $*253{\cdot}44$.

$*253{\cdot}46$. $\vdash : P \in \Omega . Q, R \in C`P_s . Q \,\mathrm{smor}\, R . \supset . Q = R$

I.e. no two different sections of a well-ordered series are similar.

It follows from $*253{\cdot}46$ that the series of the ordinals of proper sections of a well-ordered series P is similar to the series of proper sections, and therefore, by $*253{\cdot}22$, to the series P with its first term omitted ($*253{\cdot}463$).

We have next a set of propositions ($*253{\cdot}5$—$\cdot574$) on the circumstances under which $\mathrm{Nr}`P_s = \mathrm{Nr}`P$ and those under which $\mathrm{Nr}`P_s = \mathrm{Nr}`P \dotplus \dot{1}$. As a matter of fact, the former holds when P is finite, the latter when P is infinite. But the distinction of finite and infinite will not be introduced till the next section. In the present number, we prove that (assuming $P \in \Omega$) $\mathrm{Nr}`P_s = \mathrm{Nr}`P$ if $C`P_1 = C`P . \mathrm{E} ! B`\breve{P}$, and if not, then $\mathrm{Nr}`P_s = \mathrm{Nr}`P \dotplus \dot{1}$ ($*253{\cdot}56$). This is proved by using $P_1$ as a correlator. ($P_1$ as a correlator moves every term one place down, except the first, which disappears.) For, if $P \in \Omega$, we have $P_1 ; P = P \upharpoonright \mathrm{D}`P$ ($*253{\cdot}5$); hence we prove $P \upharpoonright C`P_1 \,\mathrm{smor}\, P \upharpoonright \mathrm{D}`P$ ($*253{\cdot}502$), and hence, if $C`P_1 = C`P$, we obtain $P \upharpoonright C`P \,\mathrm{smor}\, P \upharpoonright \mathrm{D}`P$ ($*253{\cdot}503$). Hence by $*253{\cdot}2$ (with special consideration of the case when $P \in 2_r$) we have the two propositions

$*253{\cdot}51$. $\vdash : P \in \Omega . C`P_1 = C`P . \mathrm{E} ! B`\breve{P} . \supset . \mathrm{Nr}`P_s = \mathrm{Nr}`P$

$*253{\cdot}511$. $\vdash : P \in \Omega . C`P_1 = C`P . \sim \mathrm{E} ! B`\breve{P} . \supset .$

$$\mathrm{Nr}`P_s = \mathrm{Nr}`P \dotplus \dot{1} . \mathrm{Nr}`P \upharpoonright C`P = \mathrm{Nr}`P$$

But if there is a term, say x, belonging to $C`P - C`P_1$, use P_1 as a correlator for the predecessors of x; we thus find that, in this case, $P \,\mathrm{smor}\, P \upharpoonright C`P$. Hence, by $*253{\cdot}2$, $\mathrm{Nr}`P_s = \mathrm{Nr}`P \dotplus \dot{1}$.

The hypothesis $C`P_1 = C`P . \mathrm{E} ! B`\breve{P}$ means that there is a last term, and every other term has an immediate successor. This, as we shall prove later, and as is indeed obvious, is equivalent to the assumption that P is finite but not null.

From the above propositions it results immediately that

$*253{\cdot}573$. $\vdash :. P \in \Omega . \supset : C`P_1 = C`P . \mathrm{E} ! B`\breve{P} . \equiv . \dot{1} \dotplus \mathrm{Nr}`P \neq \mathrm{Nr}`P$

Hence it will follow that finite ordinals other than 0_r are those which are increased by the addition of $\dot{1}$ at the beginning. We have also

***253·574.** $\vdash :. P \in \Omega - \iota`\dot{\Lambda} . \supset : \text{Ɑ}`P_1 = \text{Ɑ}`P . E ! B`\breve{P} . \equiv . \dot{1} \dot{+} \mathrm{Nr}`P = \mathrm{Nr}`P \dot{+} \dot{1}$

Whence it will follow that finite ordinals are those for which the addition of $\dot{1}$ is commutative.

***253·1.** $\vdash :. P \in \Omega . \supset : QP_sR . \equiv .$

$$(\exists \alpha, \beta) . \alpha, \beta \in \overrightarrow{P}``\text{Ɑ}`P \cup \iota`C`P . \exists ! \beta - \alpha . Q = P \upharpoonright \alpha . R = P \upharpoonright \beta$$

Dem.

$\vdash . *213·1 . *252·17 . \supset \vdash :. \mathrm{Hp} . \dot{\exists} ! P . \supset : QP_sR . \equiv .$

$$(\exists \alpha, \beta) . \alpha, \beta \in \overrightarrow{P}``\text{Ɑ}`P \cup \iota`C`P . \exists ! \beta - \alpha . Q = P \upharpoonright \alpha . R = P \upharpoonright \beta \quad (1)$$

$\vdash . *33·241 . \supset \vdash :. P = \dot{\Lambda} . \supset : \overrightarrow{P}``\text{Ɑ}`P \cup \iota`C`P = \iota`\Lambda :$

$[*24·53] \qquad \supset : \sim (\exists \alpha, \beta) . \alpha, \beta \in \overrightarrow{P}``\text{Ɑ}`P \cup \iota`C`P . \exists ! \beta - \alpha :$

$[*213·3] \supset : QP_sR . \equiv .$

$$(\exists \alpha, \beta) . \alpha, \beta \in \overrightarrow{P}``\text{Ɑ}`P \cup \iota`C`P . \exists ! \beta - \alpha . Q = P \upharpoonright \alpha . R = P \upharpoonright \beta \quad (2)$$

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***253·11.** $\vdash :: P \in \Omega . \supset :. QP_sR . \equiv :$

$$(\exists x, y) . x \in \text{Ɑ}`P . xPy . Q = P \upharpoonright \overrightarrow{P}`x . R = P \upharpoonright \overrightarrow{P}`y . \vee .$$
$$(\exists x) . x \in \text{Ɑ}`P . Q = P \upharpoonright \overrightarrow{P}`x . R = P$$

Dem.

$\vdash . *33·152 . \qquad \supset \vdash : \alpha = C`P . \beta \in \overrightarrow{P}``\text{Ɑ}`P \cup \iota`C`P . \supset . \sim \exists ! \beta - \alpha \quad (1)$

$\vdash . *200·52 . (1) . \qquad \supset \vdash : \mathrm{Hp} . \alpha \in \overrightarrow{P}``\text{Ɑ}`P . \beta = C`P . \supset . \exists ! \beta - \alpha \quad (2)$

$\vdash . (1) . (2) . *253·1 . \supset \vdash :: \mathrm{Hp} . \supset :. QP_sR . \equiv :$

$$(\exists \alpha, \beta) . \alpha, \beta \in \overrightarrow{P}``\text{Ɑ}`P . \exists ! \beta - \alpha . Q = P \upharpoonright \alpha . R = P \upharpoonright \beta . \vee .$$
$$(\exists \alpha, \beta) . \alpha \in \overrightarrow{P}``\text{Ɑ}`P . \beta = C`P . Q = P \upharpoonright \alpha . R = P \upharpoonright \beta :$$

$[*37·6 . *36·33]$

$$\equiv : (\exists x, y) . x, y \in \text{Ɑ}`P . \exists ! \overrightarrow{P}`y - \overrightarrow{P}`x . Q = P \upharpoonright \overrightarrow{P}`x . R = P \upharpoonright \overrightarrow{P}`y . \vee .$$
$$(\exists x) . x \in \text{Ɑ}`P . Q = P \upharpoonright \overrightarrow{P}`x . R = P :$$

$[*211·61 . *210·1]$

$$\equiv : (\exists x, y) . x, y \in \text{Ɑ}`P . \overrightarrow{P}`x \subset \overrightarrow{P}`y . \overrightarrow{P}`x \neq \overrightarrow{P}`y . Q = P \upharpoonright \overrightarrow{P}`x . R = P \upharpoonright \overrightarrow{P}`y . \vee .$$
$$(\exists x) . x \in \text{Ɑ}`P . Q = P \upharpoonright \overrightarrow{P}`x . R = P :$$

$[*204·33·34] \equiv : (\exists x, y) . x, y \in \text{Ɑ}`P . xPy . Q = P \upharpoonright \overrightarrow{P}`x . R = P \upharpoonright \overrightarrow{P}`y . \vee .$

$$(\exists x) . x \in \text{Ɑ}`P . Q = P \upharpoonright \overrightarrow{P}`x . R = P \quad (3)$$

$\vdash . (3) . *33·14 . \supset \vdash . \mathrm{Prop}$

***253·12.** $\vdash : P \in \Omega . P \sim\in 2_r . \supset . P_s = (P \upharpoonright ; \overrightarrow{P} ; P \upharpoonright \text{Ɑ}`P) \dashv\!\!\!\!\rightarrow P$

Dem.

$\vdash . *204·272 . \supset \vdash : \mathrm{Hp} . \supset . \text{Ɑ}`P \sim\in 1 .$

$[*202·55 . *213·151] \qquad \supset . P \upharpoonright `` \overrightarrow{P}``\text{Ɑ}`P = C`P \upharpoonright ; \overrightarrow{P} ; P \upharpoonright \text{Ɑ}`P \quad (1)$

$\vdash . (1) . *253\cdot11 . \supset \vdash :: \text{Hp} . \supset :. QP_s R . \equiv :$

$Q(P \upharpoonright ; \overrightarrow{P} ; P \upharpoonright \text{Ɑ}`P) R . \vee . Q \in C`(P \upharpoonright ; \overrightarrow{P} ; P \upharpoonright \text{Ɑ}`P) . R = P :$

$[*161\cdot11] \quad \equiv : Q \{(P \upharpoonright ; \overrightarrow{P} ; P \upharpoonright \text{Ɑ}`P) \rightarrow\!\!\!+ P\} R :: \supset \vdash . \text{Prop}$

$*253\cdot121$. $\vdash : P \in \Omega . \supset . P \sim\in C`P \upharpoonright ; \overrightarrow{P} ; P \upharpoonright \text{Ɑ}`P$

Dem.

$\vdash . *200\cdot52 . \supset \vdash : \text{Hp} . \supset . C`P \sim\in \overrightarrow{P}``\text{Ɑ}`P .$

$[*36\cdot25] \qquad \supset . P \sim\in C`P \upharpoonright ; \overrightarrow{P} ; P \upharpoonright \text{Ɑ}`P : \supset \vdash . \text{Prop}$

$*253\cdot13$. $\vdash : P \in \Omega . \supset . \text{D}`P_s = P \upharpoonright``\overrightarrow{P}``\text{Ɑ}`P = P \upharpoonright``\overrightarrow{P}``C`P$

Dem.

$\vdash . *213\cdot141 . *252\cdot171 . \supset \vdash : \text{Hp} . \supset . \text{D}`P_s = P \upharpoonright``\overrightarrow{P}``\text{Ɑ}`P \qquad (1)$

$\vdash . *37\cdot22 . *250\cdot13 . \supset$

$\vdash : \text{Hp} . \exists ! P . \supset . P \upharpoonright``\overrightarrow{P}``C`P = P \upharpoonright``\overrightarrow{P}``\text{Ɑ}`P \cup \iota`P \upharpoonright \overrightarrow{P}`B`P$

$[*33\cdot41.\text{Transp}] \qquad = P \upharpoonright``\overrightarrow{P}``\text{Ɑ}`P \cup \iota`\dot{\Lambda} \qquad (2)$

$\vdash . *250\cdot42 . \supset \vdash : \text{Hp} . \dot{\exists} ! P . \supset . \dot{\Lambda} \in P \upharpoonright``\overrightarrow{P}``\text{Ɑ}`P \qquad (3)$

$\vdash . (2) . (3) . \supset \vdash : \text{Hp} . \dot{\exists} ! P . \supset . P \upharpoonright``\overrightarrow{P}``C`P = P \upharpoonright``\overrightarrow{P}``\text{Ɑ}`P \qquad (4)$

$\vdash . *33\cdot241 . \supset \vdash : P = \dot{\Lambda} . \supset . P \upharpoonright``\overrightarrow{P}``C`P = \Lambda . P \upharpoonright``\overrightarrow{P}``\text{Ɑ}`P = \Lambda \qquad (5)$

$\vdash . (4) . (5) . \supset \vdash : \text{Hp} . \supset . P \upharpoonright``\overrightarrow{P}``C`P = P \upharpoonright``\overrightarrow{P}``\text{Ɑ}`P \qquad (6)$

$\vdash . (1) . (6) . \supset \vdash . \text{Prop}$

$*253\cdot14$. $\vdash : P \in \Omega . \supset .$

$\text{Ɑ}`P_s = (P \upharpoonright``\overrightarrow{P}``\text{Ɑ}`P \cup \iota`P) - \iota`\dot{\Lambda} = (P \upharpoonright``\overrightarrow{P}``C`P \cup \iota`P) - \iota`\dot{\Lambda}$

Dem.

$\vdash . *213\cdot162 . \supset \vdash : \text{Hp} . \supset . \text{Ɑ}`P_s = P \upharpoonright``\text{sect}`P - \iota`\dot{\Lambda}$

$[*252\cdot12.*36\cdot33] \qquad = (P \upharpoonright``\overrightarrow{P}``C`P \cup \iota`P) - \iota`\dot{\Lambda} \qquad (1)$

$[*253\cdot13] \qquad = (P \upharpoonright``\overrightarrow{P}``\text{Ɑ}`P \cup \iota`P) - \iota`\dot{\Lambda} \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

$*253\cdot15$. $\vdash : P \in \Omega - \iota`\dot{\Lambda} . \supset . C`P_s = P \upharpoonright``\overrightarrow{P}``\text{Ɑ}`P \cup \iota`P = P \upharpoonright``\overrightarrow{P}``C`P \cup \iota`P$
$[*253\cdot13\cdot14]$

$*253\cdot16$. $\vdash : P \in \Omega - \iota`\dot{\Lambda} . \supset . B`P_s = \dot{\Lambda} . B`\breve{P}_s = P \quad [*213\cdot155\cdot158 . *250\cdot13]$

$*253\cdot17$. $\vdash : P \in \Omega . \supset . P_s \upharpoonright \text{D}`P_s = P \upharpoonright ; \overrightarrow{P} ; P \upharpoonright \text{Ɑ}`P$

Dem.

$\vdash . *253\cdot11 . \supset$

$\vdash :: \text{Hp} . \supset :. QP_s R . \equiv : Q(P \upharpoonright ; \overrightarrow{P} ; P \upharpoonright \text{Ɑ}`P) R . \vee . Q \in P \upharpoonright``\overrightarrow{P}``\text{Ɑ}`P . R = P :.$

$[*253\cdot121] \supset :. Q(P_s \upharpoonright \text{D}`P_s) R . \equiv . Q(P \upharpoonright ; \overrightarrow{P} ; P \upharpoonright \text{Ɑ}`P) R :: \supset \vdash . \text{Prop}$

***253·18.** $\vdash : P \epsilon \Omega . \supset . C'P_s \subset P \upharpoonright {}^{\prime\prime}\overrightarrow{P}{}^{\prime\prime}\text{Ɑ}'P \cup \iota'P . C'P_s \subset \Omega$

Dem.

$\vdash . *253{\cdot}11 . \supset$

$\vdash :: \text{Hp} . \supset :. Q \epsilon C'P_s . \supset : (\exists x) . x \epsilon \text{Ɑ}'P . Q = P \upharpoonright \overrightarrow{P}'x . \text{v} . Q = P :$

$[*37{\cdot}6] \qquad \supset : Q \epsilon P \upharpoonright {}^{\prime\prime}\overrightarrow{P}{}^{\prime\prime}\text{Ɑ}'P \cup \iota'P \qquad (1)$

$\vdash . (1) . *250{\cdot}141 . \supset \vdash : \text{Hp} . \supset . C'P_s \subset \Omega \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***253·181.** $\vdash : P \epsilon \Omega . \supset . C'P_s \subset \text{D}'P_s \cup \iota'P \quad [*253{\cdot}18{\cdot}13]$

***253·2.** $\vdash : P \epsilon \Omega - 2_r . \supset . \text{Nr}'P_s = \text{Nr}'(P \upharpoonright \text{Ɑ}'P) \dotplus \dot{1}$

Dem.

$\vdash . *253{\cdot}12{\cdot}121 . \supset \vdash : \text{Hp} . \supset . \text{Nr}'P_s = \text{Nr}'P \upharpoonright ; \overrightarrow{P} ; P \upharpoonright \text{Ɑ}'P \dotplus \dot{1}$

$[*213{\cdot}151 . *252{\cdot}171] \qquad = \text{Nr}'\overrightarrow{P} ; P \upharpoonright \text{Ɑ}'P \dotplus \dot{1}$

$[*204{\cdot}34] \qquad = \text{Nr}'(P \upharpoonright \text{Ɑ}'P) \dotplus \dot{1} : \supset \vdash . \text{Prop}$

***253·21.** $\vdash : P \epsilon \Omega . \supset . \dot{1} \dotplus \text{Nr}'P_s = \text{Nr}'P \dotplus \dot{1}$

Dem.

$\vdash . *253{\cdot}2 . \supset \vdash : \text{Hp} . P \sim \epsilon 2_r . \supset . \dot{1} \dotplus \text{Nr}'P_s = \dot{1} \dotplus \text{Nr}'(P \upharpoonright \text{Ɑ}'P) \dotplus \dot{1}$

$[*204{\cdot}46{\cdot}272] \qquad = \text{Nr}'P \dotplus \dot{1} \qquad (1)$

$\vdash . *213{\cdot}32 . \supset \vdash : P \epsilon 2_r . \supset . \dot{1} \dotplus \text{Nr}'P_s = \dot{1} \dotplus 2_r$

$[*161{\cdot}211] \qquad = 2_r \dotplus \dot{1}$

$[\text{Hp}] \qquad = \text{Nr}'P \dotplus \dot{1} \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

It would be an error to infer from the above proposition that $\text{Nr}'P_s = \text{Nr}'P$, since addition of ordinals is not in general commutative. When $P \epsilon \Omega$, $\text{Nr}'P_s = \text{Nr}'P$ holds when $C'P$ is finite, but not otherwise. When $C'P$ is not finite, $\dot{1} \dotplus \text{Nr}'P_s = \text{Nr}'P_s$, so that $\text{Nr}'P_s = \text{Nr}'P \dotplus \dot{1}$; but $\text{Nr}'P \neq \text{Nr}'P \dotplus \dot{1}$.

***253·22.** $\vdash : P \epsilon \Omega . \supset . P_s \upharpoonright \text{D}'P_s \,\text{smor}\, P \upharpoonright \text{Ɑ}'P$

$[*253{\cdot}17 . *213{\cdot}151 . *252{\cdot}171 . *204{\cdot}34]$

***253·23.** $\vdash :. P \epsilon \Omega . \supset : \text{Nr}'P = \text{Nr}'Q . \equiv . \text{Nr}'P_s = \text{Nr}'Q_s :$

$P \,\text{smor}\, Q . \equiv . P_s \,\text{smor}\, Q_s$

Dem.

$\vdash . *181{\cdot}33 . \supset \vdash : \text{Nr}'P = \text{Nr}'Q . \equiv . \text{Nr}'P \dotplus \dot{1} = \text{Nr}'Q \dotplus \dot{1} \qquad (1)$

$\vdash . (1) . *253{\cdot}21 . \supset$

$\vdash :. \text{Hp} . \supset : \text{Nr}'P = \text{Nr}'Q . \equiv . \dot{1} \dotplus \text{Nr}'P_s = \dot{1} \dotplus \text{Nr}'Q_s .$

$[*181{\cdot}33] \qquad \equiv . \text{Nr}'P_s = \text{Nr}'Q_s :. \supset \vdash . \text{Prop}$

***253·24.** $\vdash : P \in \Omega . \supset . P_s \in \Omega$

Dem.

$\vdash . *253 \cdot 2 . *250 \cdot 141 . *251 \cdot 132 . \supset \vdash : \text{Hp} . P \sim \in 2_r . \supset . \text{Nr}'P_s \in \text{NO}$ (1)

$\vdash . *213 \cdot 32 . *251 \cdot 16 . \supset \vdash : P \in 2_r . \supset . \text{Nr}'P_s \in \text{NO}$ (2)

$\vdash . (1) . (2) . \quad \supset \vdash : \text{Hp} . \supset . \text{Nr}'P_s \in \text{NO} .$

$[*251 \cdot 122] \quad \supset . P_s \in \Omega : \supset \vdash . \text{Prop}$

***253·25.** $\vdash :. P, Q \in \Omega - \iota'\dot{\Lambda} . \supset : P_s \upharpoonleft \mathrm{D}'P_s \text{ smor } Q_s \upharpoonleft \mathrm{D}'Q_s . \equiv . P \text{ smor } Q$
$[*253 \cdot 22 . *250 \cdot 17]$

***253·3.** $\vdash : P \in \Omega . \supset . \overrightarrow{P_s}'P = P \upharpoonleft \text{“}\overrightarrow{P}\text{“}\mathrm{G}'P = P \upharpoonleft \text{“}\overrightarrow{P}\text{“}C'P = \mathrm{D}'P_s$
$[*213 \cdot 243 . *253 \cdot 13]$

***253·31.** $\vdash :. P \in \Omega . \supset : QP_sR . \equiv . R \in P \upharpoonleft \text{“}\overrightarrow{P}\text{“}C'P \cup \iota'P . Q \in R \upharpoonleft \text{“}\overrightarrow{R}\text{“}C'R$

Dem.

$\vdash . *213 \cdot 245 . *253 \cdot 13 . \supset$

$\vdash :. \text{Hp} . \supset : QP_sR . \equiv . R \in C'P_s . Q \in R \upharpoonleft \text{“}\overrightarrow{R}\text{“}C'R .$

$[*33 \cdot 24 . *213 \cdot 3] \quad \equiv . R \in C'P_s . \dot{\exists} ! P . Q \in R \upharpoonleft \text{“}\overrightarrow{R}\text{“}C'R .$

$[*253 \cdot 15] \quad \equiv . R \in P \upharpoonleft \text{“}\overrightarrow{P}\text{“}C'P \cup \iota'P . \dot{\exists} ! P . Q \in R \upharpoonleft \text{“}\overrightarrow{R}\text{“}C'R$ (1)

$\vdash . *37 \cdot 29 . *33 \cdot 24 . \supset \vdash : Q \in R \upharpoonleft \text{“}\overrightarrow{R}\text{“}C'R . \supset . \dot{\exists} ! R :$ (2)

$[*13 \cdot 12] \quad \supset \vdash : Q \in R \upharpoonleft \text{“}\overrightarrow{R}\text{“}C'R . R = P . \supset . \dot{\exists} ! P$ (3)

$\vdash . (2) \dfrac{R, P}{Q, R} . \quad \supset \vdash : R \in P \upharpoonleft \text{“}\overrightarrow{P}\text{“}C'P . \supset . \dot{\exists} ! P$ (4)

$\vdash . (3) . (4) . \quad \supset \vdash : R \in P \upharpoonleft \text{“}\overrightarrow{P}\text{“}C'P \cup \iota'P . Q \in R \upharpoonleft \text{“}\overrightarrow{R}\text{“}C'R . \supset . \dot{\exists} ! P$ (5)

$\vdash . (1) . (5) . \supset \vdash . \text{Prop}$

***253·32.** $\vdash : P \in \Omega . R \in C'P_s . \supset . \overrightarrow{P_s}'R = R \upharpoonleft \text{“}\overrightarrow{R}\text{“}C'R = \mathrm{D}'R_s$
$[*213 \cdot 246 . *253 \cdot 13]$

***253·33.** $\vdash :. P \in \Omega . \supset : Q (P_s \upharpoonleft \mathrm{D}'P_s) R . \equiv . R \in P \upharpoonleft \text{“}\overrightarrow{P}\text{“}C'P . Q \in R \upharpoonleft \text{“}\overrightarrow{R}\text{“}C'R$
$[*213 \cdot 247 . *253 \cdot 13]$

If α is any ordinal number, and $P \in \alpha$, the ordinal numbers of the sectional relations of P are all those ordinals which can be made equal to α by being added to, *i.e.* all ordinals β such that, for a suitable γ, $\alpha = \beta \dotplus \gamma$. (Here γ must be an ordinal or $\dot{1}$.) Further, in virtue of $*250 \cdot 67$, no member of $\mathrm{D}'P_s$ is similar to P; hence, if α is an ordinal, and $\alpha = \beta \dotplus \gamma$, where $\gamma \neq 0_r$, it follows that $\alpha \neq \beta$. (Observe that $\alpha \neq \gamma$ does not follow from $\beta \neq 0_r . \alpha = \beta \dotplus \gamma$.) These and kindred propositions, which are important in the theory of ordinals, are now to be proved.

***253·4.** $\vdash : P \in \Omega - \iota'\dot{\Lambda} . \supset . C'P_s = \hat{Q}\{(\exists R) . P = Q \mathbin{\updownarrow} R . \vee . (\exists x) . P = Q \rightarrow x\}$
$[*213 \cdot 41 . *250 \cdot 13]$

***253·401.** $\vdash : P \in \Omega . \supset .$

$$P \upharpoonright \text{“}\overrightarrow{P}\text{“}C\text{‘}P \cup \iota\text{‘}P = \hat{Q}\{(\exists R) . P = Q \mathbin{+\!\!\!\uparrow} R . \vee . (\exists x) . P = Q \mathbin{+\!\!\!\rightarrow} x\}$$

Dem.

$\vdash . *253\cdot4\cdot15 . \supset \vdash : \text{Hp} . \dot{\exists} ! P . \supset .$

$P \upharpoonright \text{“}\overrightarrow{P}\text{“}C\text{‘}P \cup \iota\text{‘}P = \hat{Q}\{(\exists R) . P = Q \mathbin{+\!\!\!\uparrow} R . \vee . (\exists x) . P = Q \mathbin{+\!\!\!\rightarrow} x\}$ (1)

$\vdash . *37\cdot29 . \quad \supset \vdash : P = \dot{\Lambda} . \supset . P \upharpoonright \text{“}\overrightarrow{P}\text{“}C\text{‘}P \cup \iota\text{‘}P = \iota\text{‘}\dot{\Lambda}$ (2)

$\vdash . *160\cdot14 . *33\cdot241 . \supset \vdash :. P = \dot{\Lambda} . \supset : P = Q \mathbin{+\!\!\!\uparrow} R . \equiv . Q = \dot{\Lambda} . R = \dot{\Lambda} :$

$[*10\cdot281] \quad \supset : (\exists R) . P = Q \mathbin{+\!\!\!\uparrow} R . \equiv . Q = \dot{\Lambda}$ (3)

$\vdash . *161\cdot13 . *33\cdot241 . \supset \vdash :. P = \dot{\Lambda} . \supset : P = Q \mathbin{+\!\!\!\rightarrow} x . \equiv . Q = \dot{\Lambda} :$

$[*10\cdot24\cdot23] \quad \supset : (\exists x) . P = Q \mathbin{+\!\!\!\rightarrow} x . \equiv . Q = \dot{\Lambda}$ (4)

$\vdash . (3) . (4) . \supset \vdash :: P = \dot{\Lambda} . \supset :. (\exists R) . P = Q \mathbin{+\!\!\!\uparrow} R . \vee . (\exists x) . P = Q \mathbin{+\!\!\!\rightarrow} x : \equiv . Q = \dot{\Lambda} .$

$[(2)] \quad \equiv . Q \in P \upharpoonright \text{“}\overrightarrow{P}\text{“}C\text{‘}P \cup \iota\text{‘}P$ (5)

$\vdash . (1) . (5) . \supset \vdash . \text{Prop}$

***253·402.** $\vdash : P \in \Omega - \iota\text{‘}\dot{\Lambda} . \supset .$

$$\text{D‘}P_s = \hat{Q}\{(\exists R) . R \neq \dot{\Lambda} . P = Q \mathbin{+\!\!\!\uparrow} R . \vee . (\exists x) . P = Q \mathbin{+\!\!\!\rightarrow} x\}$$

Dem.

$\vdash . *253\cdot16\cdot4 . \supset$

$\vdash :: \text{Hp} . \supset :. Q \in \text{D‘}P_s . \equiv : Q \neq P : (\exists R) . P = Q \mathbin{+\!\!\!\uparrow} R . \vee . (\exists x) . P = Q \mathbin{+\!\!\!\rightarrow} x$ (1)

$\vdash . *161\cdot14 . *200\cdot41 . \supset \vdash : \text{Hp} . P = Q \mathbin{+\!\!\!\rightarrow} x . \supset . x \in C\text{‘}P . x \sim\in C\text{‘}Q .$

$[*13\cdot14] \quad \supset . Q \neq P$ (2)

$\vdash . *160\cdot21 . \supset \vdash : Q \neq P . P = Q \mathbin{+\!\!\!\uparrow} R . \supset . \dot{\exists} ! R$ (3)

$\vdash . *160\cdot14 . *200\cdot4 . \supset$

$\vdash : \text{Hp} . P = Q \mathbin{+\!\!\!\uparrow} R . \dot{\exists} ! R . \supset . \exists ! C\text{‘}P \cap C\text{‘}R . \sim\exists ! C\text{‘}Q \cap C\text{‘}R .$

$[*13\cdot14] \quad \supset . P \neq Q$ (4)

$\vdash . (3) . (4) . \supset$

$\vdash :: \text{Hp} . \supset :. Q \neq P : (\exists R) . P = Q \mathbin{+\!\!\!\uparrow} R : \equiv . (\exists R) . R \neq \dot{\Lambda} . P = Q \mathbin{+\!\!\!\uparrow} R$ (5)

$\vdash . (1) . (2) . (5) . \supset \vdash :: \text{Hp} . \supset :. Q \in \text{D‘}P_s . \equiv :$

$(\exists R) . R \neq \dot{\Lambda} . P = Q \mathbin{+\!\!\!\uparrow} R . \vee . (\exists x) . P = Q \mathbin{+\!\!\!\rightarrow} x :: \supset \vdash . \text{Prop}$

***253·41.** $\vdash :. P \in \Omega . Q \in C\text{‘}P_s . \supset :$

$$(\exists \alpha) . \alpha \in \text{NO} . \text{Nr‘}P = \text{Nr‘}Q \dotplus \alpha . \vee . \text{Nr‘}P = \text{Nr‘}Q \dotplus \dot{1}$$

Dem.

$\vdash . *213\cdot3 . \supset \vdash :. \text{Hp} . \supset : P \neq \dot{\Lambda} :$

$[*253\cdot4] \quad \supset : (\exists R) . P = Q \mathbin{+\!\!\!\uparrow} R . \vee . (\exists x) . P = Q \mathbin{+\!\!\!\rightarrow} x :$

$[*211\cdot283 . *200\cdot41]$

$\supset : (\exists R) . P = Q \mathbin{+\!\!\!\uparrow} R . C\text{‘}Q \cap C\text{‘}R = \Lambda . \vee . (\exists x) . P = Q \mathbin{+\!\!\!\rightarrow} x . x \sim\in C\text{‘}Q :$

$[*180\cdot32 . *181\cdot32] \supset : (\exists R) . \text{Nr‘}P = \text{Nr‘}Q \dotplus \text{Nr‘}R . \vee . \text{Nr‘}P = \text{Nr‘}Q \dotplus \dot{1} :$

$[*251\cdot26] \supset : (\exists \alpha) . \alpha \in \text{NO} . \text{Nr‘}P = \text{Nr‘}Q \dotplus \alpha . \vee . \text{Nr‘}P = \text{Nr‘}Q \dotplus \dot{1} :. \supset \vdash . \text{Prop}$

$*253\cdot42.\ \vdash : P \in \Omega . \supset . \mathrm{Nr}`P \cap \mathrm{D}`P_s = \Lambda$ [$*250\cdot651 . *213\cdot141$]

$*253\cdot421.\ \vdash : P \in \Omega . Q \in \mathrm{D}`P_s . \supset . \sim (Q \,\mathrm{smor}\, P)$ [$*253\cdot42$]

$*253\cdot43.\ \vdash :. P \in \Omega . x, y \in C`P . \supset : P \upharpoonright \overrightarrow{P}`x \,\mathrm{smor}\, P \upharpoonright \overrightarrow{P}`y . \equiv . x = y$

Dem.

$\vdash . *253\cdot11 . \supset \vdash : \mathrm{Hp} . xPy . \supset . (P \upharpoonright \overrightarrow{P}`x)\, P_s\, (P \upharpoonright \overrightarrow{P}`y) .$

[$*213\cdot245$] $\supset . P \upharpoonright \overrightarrow{P}`x \in \mathrm{D}`(P \upharpoonright \overrightarrow{P}`y)_s .$

[$*253\cdot421$] $\supset . \sim \{(P \upharpoonright \overrightarrow{P}`x) \,\mathrm{smor}\, (P \upharpoonright \overrightarrow{P}`y)\}$ (1)

Similarly $\vdash : \mathrm{Hp} . yPx . \supset . \sim \{(P \upharpoonright \overrightarrow{P}`x) \,\mathrm{smor}\, (P \upharpoonright \overrightarrow{P}`y)\}$ (2)

$\vdash . (1) . (2) . \supset \vdash :. \mathrm{Hp} . \supset : (P \upharpoonright \overrightarrow{P}`x) \,\mathrm{smor}\, (P \upharpoonright \overrightarrow{P}`y) . \supset . \sim (xPy) . \sim (yPx) .$

[$*202\cdot103$] $\supset . x = y$ (3)

$\vdash . (3) . *151\cdot13 . \supset \vdash . \mathrm{Prop}$

$*253\cdot431.\ \vdash : P \dotplus Q \in \Omega . \dot{\exists} ! Q . \supset . \mathrm{Nr}`P \neq \mathrm{Nr}`(P \dotplus Q)$

Dem.

$\vdash . *253\cdot402 . \supset \vdash : \mathrm{Hp} . \supset . P \in \mathrm{D}`(P \dotplus Q)_s$ (1)

$\vdash . (1) . *253\cdot421 . \supset \vdash . \mathrm{Prop}$

$*253\cdot432.\ \vdash : P \rightarrow\!\!\!\!+ x \in \Omega . \dot{\exists} ! P . \supset . \mathrm{Nr}`P \neq \mathrm{Nr}`(P \rightarrow\!\!\!\!+ x)$ [$*253\cdot402\cdot421$]

$*253\cdot44.\ \vdash : \alpha, \beta \in \mathrm{NO} - \iota`\Lambda . \beta \neq 0_r . \supset . \alpha \dotplus \beta \neq \alpha$

Dem.

$\vdash . *251\cdot1 . *155\cdot34 . \supset$

$\vdash : \mathrm{Hp} . \supset . (\exists P, Q) . P, Q \in \Omega . \alpha = \mathrm{N_0r}`P . \beta = \mathrm{N_0r}`Q . \dot{\exists} ! Q .$

[$*180\cdot3$]

$\supset . (\exists P, Q) . P, Q \in \Omega . \alpha = \mathrm{N_0r}`P . \beta = \mathrm{N_0r}`Q . \dot{\exists} ! Q . \alpha \dotplus \beta = \mathrm{Nr}`(P + Q)$ (1)

$\vdash . *180\cdot12 . *253\cdot431 . (*180\cdot01) . \supset$

$\vdash : P, Q \in \Omega . \dot{\exists} ! Q . \supset . \mathrm{Nr}`(P + Q) \neq \mathrm{Nr}`P .$

[$*155\cdot16$] $\supset . \mathrm{Nr}`(P + Q) \neq \mathrm{N_0r}`P$ (2)

$\vdash . (1) . (2) . \supset$

$\vdash : \mathrm{Hp} . \supset . (\exists P, Q) . P, Q \in \Omega . \alpha = \mathrm{N_0r}`P . \beta = \mathrm{N_0r}`Q . \alpha \dotplus \beta \neq \mathrm{N_0r}`P .$

[$*13\cdot195$] $\supset . \alpha \dotplus \beta \neq \alpha : \supset \vdash . \mathrm{Prop}$

$*253\cdot45.\ \vdash : \alpha \in \mathrm{NO} - \iota`\Lambda - \iota`0_r . \supset . \alpha \dotplus \dot{1} \neq \alpha$

[Proof as in $*253\cdot44$, using $*253\cdot432$ instead of $*253\cdot431$]

$*253\cdot46.\ \vdash : P \in \Omega . Q, R \in C`P_s . Q \,\mathrm{smor}\, R . \supset . Q = R$

Dem.

$\vdash . *253\cdot421\cdot16 . \supset \vdash : \mathrm{Hp} . Q = P . \supset . R = Q$ (1)

$\vdash . *253\cdot16 . \supset \vdash : \mathrm{Hp} . Q \neq P . R \neq P . \supset . Q, R \in \mathrm{D}`P_s .$

[$*253\cdot13$] $\supset . (\exists x, y) . x, y \in C`P . Q = P \upharpoonright \overrightarrow{P}`x . R = P \upharpoonright \overrightarrow{P}`y .$

[$*253\cdot43 . \mathrm{Hp}$] $\supset . Q = R$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***253·461.** $\vdash : P \in \Omega . \supset . \mathrm{Nr} \upharpoonright C\text{‘}P_s \in 1 \rightarrow 1$

Dem.

$\vdash . *253{\cdot}46 . \supset \vdash : \mathrm{Hp} . Q, R \in C\text{‘}P_s . \mathrm{Nr}\text{‘}Q = \mathrm{Nr}\text{‘}R . \supset . Q = R : \supset \vdash . \mathrm{Prop}$

***253·462.** $\vdash : P \in \Omega . \supset .$

$$\mathrm{Nr}\ (P \sqsubset)\ \overrightarrow{P} \upharpoonright \text{ᗡ}\text{‘}P \in 1 \rightarrow 1 . \mathrm{Nr} ; P \sqsubset ; \overrightarrow{P} ; P \sqsubset \text{ᗡ}\text{‘}P \ \mathrm{smor}\ P \sqsubset \text{ᗡ}\text{‘}P$$

[*253·43]

***253·463.** $\vdash : P \in \Omega . \supset .$

$$\mathrm{Nr} ; (P_s \sqsubset \mathrm{D}\text{‘}P_s)\ \mathrm{smor}\ P_s \sqsubset \mathrm{D}\text{‘}P_s . \mathrm{Nr} ; (P_s \sqsubset \mathrm{D}\text{‘}P\ )\ \mathrm{smor}\ P \sqsubset \text{ᗡ}\text{‘}P$$

[*253·462·17·22]

***253·47.** $\vdash : P \in \Omega - \iota\text{‘}\dot{\Lambda} . \supset .$

$$\mathrm{Nr}\text{‘‘}C\text{‘}P_s = \hat{\alpha}\{(\exists \beta) . \alpha \dot{+} \beta = \mathrm{Nr}\text{‘}P . \mathrm{v} . \alpha \dot{+} \dot{1} = \mathrm{Nr}\text{‘}P\} \quad [*253{\cdot}4]$$

***253·471.** $\vdash : P \in \Omega . \supset .$

$$\mathrm{Nr}\text{‘‘}(\mathrm{D}\text{‘}P_s \cup \iota\text{‘}P) = \hat{\alpha}\{(\exists \beta) . \alpha \dot{+} \beta = \mathrm{Nr}\text{‘}P . \mathrm{v} . \alpha \dot{+} \dot{1} = \mathrm{Nr}\text{‘}P\}$$

[*253·401·13]

The following propositions are concerned in proving that $\mathrm{Nr}\text{‘}P_s$ is either $\mathrm{Nr}\text{‘}P$ or $\mathrm{Nr}\text{‘}P \dot{+} \dot{1}$. This is proved by using P_1 as a correlator. The methods employed anticipate the discussion of finite and infinite series; in fact, when P is finite, $\mathrm{Nr}\text{‘}P_s = \mathrm{Nr}\text{‘}P$, and when P is infinite, $\mathrm{Nr}\text{‘}P_s = \mathrm{Nr}\text{‘}P \dot{+} \dot{1}$. But it is important at this stage to know that $\mathrm{Nr}\text{‘}P_s$ is either equal to or greater than $\mathrm{Nr}\text{‘}P$, and the propositions are therefore inserted here.

***253·5.** $\vdash : P \in \Omega . \supset . P_1 ; P = P \sqsubset \mathrm{D}\text{‘}P$

Dem.

$\vdash . *201{\cdot}63 . *25{\cdot}411 . \supset \vdash :: \mathrm{Hp} . \supset :. P = P_1 \cup P^2 :.$

$[*150{\cdot}11] \supset :. x (P_1 ; P) w . \equiv : (\exists y, z) : x P_1 y : y P_1 z . \mathrm{v} . y P^2 z : w P_1 z :$

$[*204{\cdot}7] \qquad \equiv : (\exists z) . x P_1 w . w P_1 z . \mathrm{v} . (\exists y, z) . x P_1 y . y P^2 z . w P_1 z :$

$[*250{\cdot}21{\cdot}24] \qquad \equiv : x P_1 w . w \in \mathrm{D}\text{‘}P . \mathrm{v} . (\exists y) . x P_1 y . y, w \in \mathrm{D}\text{‘}P . y P w :$

$[*33{\cdot}14 . *34{\cdot}1] \qquad \equiv : x (P_1 \cup P_1 | P) w . w \in \mathrm{D}\text{‘}P :$

$[*33{\cdot}14 . *250{\cdot}242] \equiv : x . w \in \mathrm{D}\text{‘}P . x P w :: \supset \vdash . \mathrm{Prop}$

***253·501.** $\vdash : P \in \Omega . \supset . \breve{P}_1 ; P = P \sqsubset \text{ᗡ}\text{‘}P_1$

Dem.

$\vdash . *250{\cdot}242 . \supset \vdash : \mathrm{Hp} . \supset . \breve{P}_1 \ P = \breve{P}_1 \ P_1 \cup \breve{P}_1 \ P_1 \ P$

$[*71{\cdot}191 . *204{\cdot}7] \qquad = I \upharpoonright \text{ᗡ}\text{‘}P_1 \cup (\text{ᗡ}\text{‘}P_1) \upharpoonright P .$

$[*150{\cdot}1 . *50{\cdot}65] \qquad \supset . \breve{P}_1 ; P = (\text{ᗡ}\text{‘}P_1) \upharpoonright P_1 \cup (\text{ᗡ}\text{‘}P_1) \upharpoonright P \ P_1$

$[*250{\cdot}243] \qquad = P \sqsubset \text{ᗡ}\text{‘}P_1 : \supset \vdash . \mathrm{Prop}$

***253·502.** $\vdash : P \in \Omega . \supset . P \upharpoonright \text{C}'P_1 \text{ smor } P \upharpoonright \text{D}'P$

Dem.

$\vdash . *253{\cdot}5 . *150{\cdot}36 . \supset \vdash : \text{Hp} . \supset . P \upharpoonright \text{D}'P = P_1 ; (P \upharpoonright \text{C}'P_1)$ (1)

$\vdash . *151{\cdot}21 . *204{\cdot}7 . \supset \vdash : \text{Hp} . \supset . P_1 ; (P \upharpoonright \text{C}'P_1) \text{ smor } P \upharpoonright \text{C}'P_1$ (2)

$\vdash . (1) . (2) . \supset \vdash .$ Prop

***253·503.** $\vdash : P \in \Omega . \text{C}'P_1 = \text{C}'P . \supset . P \upharpoonright \text{C}'P \text{ smor } P \upharpoonright \text{D}'P$ [*253·502]

This proposition shows that if P is a well-ordered series in which every term except the first has an immediate predecessor, the series obtained by omitting the last term (if any) is similar to that obtained by omitting the first term. The converse also holds, as will be shown later. The hypothesis $P \in \Omega . \text{C}'P_1 = \text{C}'P$ is equivalent to the hypothesis that P is finite or a progression. (Here a *progression* is not what was defined as "Prog" in *121, but what Cantor calls ω; *i.e.* if $R \in$ Prog, R_{po} is a progression in our present sense.)

***253·51.** $\vdash : P \in \Omega . \text{C}'P_1 = \text{C}'P . \text{E} ! B'\breve{P} . \supset . \text{Nr}'P_s = \text{Nr}'P$

Dem.

$\vdash . *253{\cdot}2 . \supset \vdash : \text{Hp} . P \sim\in 2_r . \supset . \text{Nr}'P_s = \text{Nr}'(P \upharpoonright \text{C}'P) \dot{+} \dot{1}$

[*253·503] $= \text{Nr}'(P \upharpoonright \text{D}'P) \dot{+} \dot{1}$

[*204·461·272] $= \text{Nr}'P$ (1)

$\vdash . *213{\cdot}32 . \supset \vdash : P \in 2_r . \supset . \text{Nr}'P_s = \text{Nr}'P$ (2)

$\vdash . (1) . (2) . \supset \vdash .$ Prop

***253·511.** $\vdash : P \in \Omega . \text{C}'P_1 = \text{C}'P . \sim \text{E} ! B'\breve{P} . \supset .$
$\text{Nr}'P_s = \text{Nr}'P \dot{+} \dot{1} . \text{Nr}'P \upharpoonright \text{C}'P = \text{Nr}'P$

Dem.

$\vdash . *93{\cdot}103 . *202{\cdot}52 . \supset \vdash : \text{Hp} . \supset . P \upharpoonright \text{D}'P = P .$

[*253·503] $\supset . \text{Nr}'P \upharpoonright \text{C}'P = \text{Nr}'P .$ (1)

[*253·2] $\supset . \text{Nr}'P_s = \text{Nr}'P \dot{+} \dot{1}$ (2)

$\vdash . (1) . (2) . \supset \vdash .$ Prop

***253·52.** $\vdash : P \in \Omega . x = \min_P{}'(\text{C}'P - \text{C}'P_1) . \supset .$
$\text{C}'P \cap \overrightarrow{P}'x \subset \text{C}'P_1 . P_1{}''\overrightarrow{P}'x = \overrightarrow{P}'x . \breve{P}_1{}''\overrightarrow{P}'x = \overrightarrow{P}'x - \iota'B'P$

Dem.

$\vdash . *205{\cdot}14 . \supset \vdash : \text{Hp} . \supset . \text{C}'P \cap \overrightarrow{P}'x \subset \text{C}'P_1$ (1)

$\vdash . *250{\cdot}242 . \supset \vdash : \text{Hp} . \supset . \overrightarrow{P}'x = \overrightarrow{P}_1'x \cup P_1{}''\overrightarrow{P}'x$

[*33·41.Hp] $= P_1{}''\overrightarrow{P}'x .$ (2)

[*72·501.*204·7] $\supset . \breve{P}_1{}''\overrightarrow{P}'x = \overrightarrow{P}'x \cap \text{C}'P_1$ (3)

$\vdash . (1) . \supset \vdash : \text{Hp} . \supset . \text{C}'P \cap \overrightarrow{P}'x = \text{C}'P \cap \overrightarrow{P}'x \cap \text{C}'P_1$

[*121·305] $= \text{C}'P_1 \cap \overrightarrow{P}'x$ (4)

$\vdash . (3) . (4) . \supset \vdash : \text{Hp} . \supset . \breve{P}_1{}''\overrightarrow{P}'x = \overrightarrow{P}'x \cap \text{C}'P$

[*33·15.*202·52] $= \overrightarrow{P}'x - \iota'B'P$ (5)

$\vdash . (1) . (2) . (5) . \supset \vdash .$ Prop

***253·521.** $\vdash : P \in \Omega \,.\, x \in \text{Ɑ}`P - \text{Ɑ}`P_1 \,.\, \supset \,.\, \overrightarrow{P}`x, \text{Ɑ}`P \sim\in 1$

Dem.

$\vdash \,.\, *201{\cdot}66 \,.\quad \supset \vdash : P \in \Omega \,.\, \overrightarrow{P}`x \in 1 \,.\, \supset \,.\, x \in \text{Ɑ}`P_1 \quad (1)$

$\vdash \,.\, (1) \,.\, \text{Transp} \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, \overrightarrow{P}`x \sim\in 1 \quad (2)$

$\vdash \,.\, *201{\cdot}662 \,.\quad \supset \vdash : \text{Hp} \,.\, \supset \,.\, \text{Ɑ}`P \sim\in 1 \quad (3)$

$\vdash \,.\, (2) \,.\, (3) \,.\, \supset \vdash \,.\, \text{Prop}$

***253·522.** $\vdash : P \in \Omega \,.\, x = \min_P`(\text{Ɑ}`P - \text{Ɑ}`P_1) \,.\, S = P_1 \upharpoonright \overrightarrow{P}`x \cup I \upharpoonright \overleftarrow{P_*}`x \,.\, \supset \,.$
$S \mathbin{;} (P \mathbin{\text{£}} \text{Ɑ}`P) = P$

Dem.

$\vdash \,.\, *34{\cdot}25{\cdot}26 \,.\, *50{\cdot}5{\cdot}51 \,.\, \supset$

$\vdash : \text{Hp} \,.\, \supset \,.\, S \mathbin{;} (P \mathbin{\text{£}} \text{Ɑ}`P) = (P_1 \upharpoonright \overrightarrow{P}`x) \mathbin{;} P \mathbin{\text{£}} \text{Ɑ}`P \cup (I \upharpoonright \overleftarrow{P_*}`x) \mathbin{;} P \cup$
$(P_1 \upharpoonright \overrightarrow{P}`x) \,|\, P \,|\, I \upharpoonright \overleftarrow{P_*}`x \cup I \upharpoonright \overleftarrow{P_*}`x \,|\, P \,|\, \overrightarrow{P}`x \uparrow \breve{P}_1$

$[*50{\cdot}6{\cdot}61 . *150{\cdot}36 . *35{\cdot}452] = (P_1 \upharpoonright \overrightarrow{P}`x) \mathbin{;} P \cup P \mathbin{\text{£}} \overleftarrow{P_*}`x \cup P_1 \upharpoonright \overrightarrow{P}`x \,|\, P \upharpoonright \overleftarrow{P_*}`x \cup$
$\overleftarrow{P_*}`x \uparrow P \upharpoonright \overrightarrow{P}`x \,|\, \breve{P}_1$

$[*74{\cdot}141 . *253{\cdot}52 . *200{\cdot}381] = (P_1 \upharpoonright \overrightarrow{P}`x) \mathbin{;} P \cup P \mathbin{\text{£}} \overleftarrow{P_*}`x \cup \overrightarrow{P}`x \uparrow P_1 \,|\, P \upharpoonright \overleftarrow{P_*}`x$

$[*250{\cdot}242 . \text{Hp}] \quad = (P_1 \upharpoonright \overrightarrow{P}`x) \mathbin{;} P \cup P \mathbin{\text{£}} \overleftarrow{P_*}`x \cup \overrightarrow{P}`x \uparrow P \upharpoonright \overleftarrow{P_*}`x$

$[*150{\cdot}36] \quad = (P_1 \mathbin{;} P) \mathbin{\text{£}} P_1``\overrightarrow{P}`x \cup P \mathbin{\text{£}} \overleftarrow{P_*}`x \cup \overrightarrow{P}`x \uparrow P \upharpoonright \overleftarrow{P_*}`x$

$[*253{\cdot}5{\cdot}52] \quad = P \mathbin{\text{£}} \overrightarrow{P}`x \cup P \mathbin{\text{£}} \overleftarrow{P_*}`x \cup \overrightarrow{P}`x \uparrow P \upharpoonright \overleftarrow{P_*}`x$

$[*35{\cdot}413 . *200{\cdot}381] \quad = P \mathbin{\text{£}} (\overrightarrow{P}`x \cup \overleftarrow{P_*}`x)$

$[*202{\cdot}101] \quad = P : \supset \vdash \,.\, \text{Prop}$

***253·53.** $\vdash : P \in \Omega \,.\, x = \min_P`(\text{Ɑ}`P - \text{Ɑ}`P_1) \,.\, \supset \,.$
$P_1 \upharpoonright \overrightarrow{P}`x \cup I \upharpoonright \overleftarrow{P_*}`x \in \{P \,\overline{\text{smor}}\, (P \mathbin{\text{£}} \text{Ɑ}`P)\}$

Dem.

$\vdash \,.\, *204{\cdot}7 \,.\, *200{\cdot}381 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, P_1 \upharpoonright \overrightarrow{P}`x \cup I \upharpoonright \overleftarrow{P_*}`x \in 1 \to 1 \quad (1)$

$\vdash \,.\, *253{\cdot}52 \,.\, *50{\cdot}5{\cdot}52 \,.\, \supset$

$\vdash : \text{Hp} \,.\, \supset \,.\, \text{Ɑ}`(P_1 \upharpoonright \overrightarrow{P}`x \cup I \upharpoonright \overleftarrow{P_*}`x) = (\overrightarrow{P}`x - \iota`B`P) \cup \overleftarrow{P_*}`x$

$[*202{\cdot}101] \quad = C`P - \iota`B`P$

$[*93{\cdot}103] \quad = \text{Ɑ}`P$

$[*202{\cdot}55 . *253{\cdot}521] \quad = C`(P \mathbin{\text{£}} \text{Ɑ}`P) \quad (2)$

$\vdash \,.\, *253{\cdot}522 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, (P_1 \upharpoonright \overrightarrow{P}`x \cup I \upharpoonright \overleftarrow{P_*}`x) \mathbin{;} (P \mathbin{\text{£}} \text{Ɑ}`P) = P \quad (3)$

$\vdash \,.\, (1) \,.\, (2) \,.\, (3) \,.\, *151{\cdot}11 \,.\, \supset \vdash \,.\, \text{Prop}$

***253·54.** $\vdash : P \in \Omega \,.\, \exists ! \,\text{Ɑ}`P - \text{Ɑ}`P_1 \,.\, \supset \,.\, P \,\text{smor}\, P \mathbin{\text{£}} \text{Ɑ}`P$

Dem.

$\vdash \,.\, *250{\cdot}121 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, E ! \min_P`(\text{Ɑ}`P - \text{Ɑ}`P_1) \quad (1)$

$\vdash \,.\, (1) \,.\, *253{\cdot}53 \,.\, \supset \vdash \,.\, \text{Prop}$

***253·55.** $\vdash : P \in \Omega \,.\, \exists ! \, C\text{‘}P - C\text{‘}P_1 \,.\, \supset .\, \mathrm{Nr}\text{‘}P_s = \mathrm{Nr}\text{‘}P \dotplus \dot{1}$

Dem.

$\vdash .\, *253{\cdot}521 \,.\, *204{\cdot}272 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset .\, P \sim\in 2_r$ (1)

$\vdash .\, (1) \,.\, *253{\cdot}54{\cdot}2 \,.\, \supset \vdash .\, \mathrm{Prop}$

***253·56.** $\vdash :.\, P \in \Omega \,.\, \supset : C\text{‘}P_1 = C\text{‘}P \,.\, E ! \, B\text{‘}\breve{P} \,.\, \supset .\, \mathrm{Nr}\text{‘}P_s = \mathrm{Nr}\text{‘}P :$
$\sim (C\text{‘}P_1 = C\text{‘}P \,.\, E ! \, B\text{‘}\breve{P}) \,.\, \supset .\, \mathrm{Nr}\text{‘}P_s = \mathrm{Nr}\text{‘}P \dotplus \dot{1}$

[*253·51·511·55]

***253·57.** $\vdash : P \in \Omega \,.\, C\text{‘}P_1 = C\text{‘}P \,.\, E ! \, B\text{‘}\breve{P} \,.\, \supset .$
$\dot{1} \dotplus \mathrm{Nr}\text{‘}P = \mathrm{Nr}\text{‘}P \dotplus \dot{1} \,.\, \dot{1} \dotplus \mathrm{Nr}\text{‘}P \neq \mathrm{Nr}\text{‘}P$

Dem.

$\vdash .\, *253{\cdot}51 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset .\, \mathrm{Nr}\text{‘}P_s = \mathrm{Nr}\text{‘}P \,.$

[*253·21] $\supset .\, \dot{1} \dotplus \mathrm{Nr}\text{‘}P = \mathrm{Nr}\text{‘}P \dotplus \dot{1} \,.$ (1)

[*253·45] $\supset .\, \dot{1} \dotplus \mathrm{Nr}\text{‘}P \neq \mathrm{Nr}\text{‘}P$ (2)

$\vdash .\, (1) \,.\, (2) \,.\, \supset \vdash .\, \mathrm{Prop}$

***253·571.** $\vdash : P \in \Omega \,.\, \sim (C\text{‘}P_1 = C\text{‘}P \,.\, E ! \, B\text{‘}\breve{P}) \,.\, \supset .\, \dot{1} \dotplus \mathrm{Nr}\text{‘}P = \mathrm{Nr}\text{‘}P$

Dem.

$\vdash .\, *253{\cdot}56 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset .\, \mathrm{Nr}\text{‘}P_s = \mathrm{Nr}\text{‘}P \dotplus \dot{1} \,.$

[*253·21] $\supset .\, \dot{1} \dotplus \mathrm{Nr}\text{‘}P \dotplus \dot{1} = \mathrm{Nr}\text{‘}P \dotplus \dot{1} \,.$

[*181·33] $\supset .\, \dot{1} \dotplus \mathrm{Nr}\text{‘}P = \mathrm{Nr}\text{‘}P : \supset \vdash .\, \mathrm{Prop}$

***253·572.** $\vdash : P \in \Omega - \iota\text{‘}\dot{\Lambda} \,.\, \sim (C\text{‘}P_1 = C\text{‘}P \,.\, E ! \, B\text{‘}\breve{P}) \,.\, \supset .\, \dot{1} \dotplus \mathrm{Nr}\text{‘}P \neq \mathrm{Nr}\text{‘}P \dotplus \dot{1}$

[*253·571·45]

***253·573.** $\vdash :.\, P \in \Omega \,.\, \supset : C\text{‘}P_1 = C\text{‘}P \,.\, E ! \, B\text{‘}\breve{P} \,.\, \equiv .\, \dot{1} \dotplus \mathrm{Nr}\text{‘}P \neq \mathrm{Nr}\text{‘}P$

[*253·57·571]

***253·574.** $\vdash :.\, P \in \Omega - \iota\text{‘}\dot{\Lambda} \,.\, \supset : C\text{‘}P_1 = C\text{‘}P \,.\, E ! \, B\text{‘}\breve{P} \,.\, \equiv .\, \dot{1} \dotplus \mathrm{Nr}\text{‘}P = \mathrm{Nr}\text{‘}P \dotplus \dot{1}$

[*253·57·572]

$*254$. GREATER AND LESS AMONG WELL-ORDERED SERIES.

Summary of $*254$.

In the present number we have to prove that of any two well-ordered series one must be similar to a sectional relation of the other. From this it will follow that of any two unequal ordinals one must be the greater. The propositions of the present number are due to Cantor*.

Our procedure is as follows. We define a relation "$RP_{\mathrm{sm}}Q$," meaning "R is a proper section of P, and is similar to Q," *i.e.*

$$RP_{\mathrm{sm}}Q \,.\equiv.\, R \in \mathrm{D}`P_\varsigma \,.\, R \,\mathrm{smor}\, Q.$$

In virtue of $*253{\cdot}46$, if $P, Q \in \Omega$, $P_{\mathrm{sm}} \in 1 \rightarrow \mathrm{Cls}$ ($*254{\cdot}22$) and $P_{\mathrm{sm}} \upharpoonright \mathrm{D}`Q_\varsigma \in 1 \rightarrow 1$ ($*254{\cdot}222$). Thus if $S$ is any proper section of $Q$ which is similar to some proper section of $P_\varsigma$, the proper section of $P$ to which it is similar is $P_{\mathrm{sm}}`S$. It is easy to prove that $P_{\mathrm{sm}} ; Q_\varsigma \mathbin{\text{⁅}} \mathrm{D}`Q_\varsigma$ is a section of $P$; and if $\mathrm{D}`P_\varsigma \subset \text{ᗡ}`Q_{\mathrm{sm}}$, *i.e.* if every proper section of P is similar to some proper section of Q, we shall have ($*254{\cdot}261$)

$$P_\varsigma \mathbin{\text{⁅}} \mathrm{D}`P_\varsigma = P_{\mathrm{sm}} ; Q_\varsigma \mathbin{\text{⁅}} \mathrm{D}`Q_\varsigma.$$

Hence it follows ($*254{\cdot}27$) that if, further, $\mathrm{D}`Q_\varsigma \subset \text{ᗡ}`P_{\mathrm{sm}}$, we shall have

$$P_\varsigma \mathbin{\text{⁅}} \mathrm{D}`P_\varsigma \,\mathrm{smor}\, Q_\varsigma \mathbin{\text{⁅}} \mathrm{D}`Q_\varsigma,$$

i.e. by $*253{\cdot}25$, $\qquad P \,\mathrm{smor}\, Q \quad (*254{\cdot}31)$.

Thus (A) if every proper section of P is similar to some proper section of Q, and vice versa, then P is similar to Q.

Consider next the case in which every proper section of P is similar to a proper section of Q (*i.e.* $\mathrm{D}`P_\varsigma \subset \text{ᗡ}`Q_{\mathrm{sm}}$), but not vice versa, so that $\exists ! \mathrm{D}`Q_\varsigma - \text{ᗡ}`P_{\mathrm{sm}}$. It is easy to prove that, under this hypothesis, if $S \in \mathrm{D}`Q_\varsigma - \text{ᗡ}`P_{\mathrm{sm}}$, then $\mathrm{D}`P_\varsigma \subset \text{ᗡ}`S_{\mathrm{sm}}$ ($*254{\cdot}32$). But if S is the minimum (in the order Q_ς) of the class $\mathrm{D}`Q_\varsigma - \text{ᗡ}`P_{\mathrm{sm}}$, then $\mathrm{D}`S_\varsigma \subset \text{ᗡ}`P_{\mathrm{sm}}$. Hence, by (A),

$$S \,\mathrm{smor}\, P \quad (*254{\cdot}321).$$

Thus (B) if every proper section of P is similar to a proper section of Q, but not vice versa, then P is similar to a proper section of Q ($*254{\cdot}33$).

* *Math. Annalen*, Vol. 49.

From (B), by transposition, we find that if every proper section of P is similar to a proper section of Q, but P itself is not similar to any proper section of Q, then every proper section of Q is similar to a proper section of P, whence, by (A), P is similar to Q (*254·34). Hence, if there are proper sections of P which are not similar to any proper section of Q, the smallest of such sections (say P') must be similar to Q, since it is not itself similar to any proper section of Q, but all its proper sections are similar to proper sections of Q. Hence (C) if there are proper sections of P which are not similar to any proper section of Q, then there is a proper section of P which is similar to Q, *i.e.*

$$\vdash : P, Q \epsilon \Omega \,.\, \exists ! \, \mathrm{D}`P_\varsigma - \mathrm{G}`Q_{\mathrm{sm}} \,.\, \supset \,.\, Q \epsilon \mathrm{G}`P_{\mathrm{sm}} \quad (*254·35).$$

Thus either (1) $\exists ! \, \mathrm{D}`P_\varsigma - \mathrm{G}`Q_{\mathrm{sm}}$, in which case $Q \epsilon \mathrm{G}`P_{\mathrm{sm}}$, or (2) $\exists ! \, \mathrm{D}`Q_\varsigma - \mathrm{G}`P_{\mathrm{sm}}$, in which case $P \epsilon \mathrm{G}`Q_{\mathrm{sm}}$, or (3) $\mathrm{D}`P_\varsigma \subset \mathrm{G}`Q_{\mathrm{sm}}$ and $\mathrm{D}`Q_\varsigma \subset \mathrm{G}`P_{\mathrm{sm}}$, in which case, by (A), $P \,\mathrm{smor}\, Q$. Thus (D) if P and Q are any two well-ordered series, either they are similar or one is similar to a proper section of the other (*254·37).

We now proceed to define one well-ordered series P as *less than* another well-ordered series Q if P is similar to a part of Q, but not to Q, *i.e.* we put

$$\mathrm{less} = \hat{P}\hat{Q} \{P, Q \epsilon \Omega \,.\, \exists ! \, \mathrm{Rl}`Q \cap \mathrm{Nr}`P \,.\, \sim (P \,\mathrm{smor}\, Q)\} \quad \mathrm{Df.}$$

(Observe that we have $\mathrm{Rl}`Q$ in this definition, not $\mathrm{D}`Q_\varsigma$.)

It follows from (D) that, P and Q being well-ordered series, if P and Q are not similar, one must be less than the other (*254·4). It follows also from *250·65 that if P is similar to a proper section of Q Q cannot be less than P (*254·181). Hence P is less than Q when, and only when, P is similar to a proper section of Q, *i.e.*

$$P \,\mathrm{less}\, Q \,.\, \equiv \,.\, P, Q \epsilon \Omega \,.\, P \epsilon \mathrm{G}`Q_{\mathrm{sm}} \quad (*254·41).$$

Hence if each of two well-ordered series is similar to a part of the other, the two series are similar (*254·45); and in any other case, one of them is similar to a proper section of the other.

From the above results we easily obtain the following propositions, which are useful in the ordinal theory of finite and infinite.

***254·51.** $\vdash : P \,\mathrm{less}\, Q \,.\, \equiv \,.\, P, Q \epsilon \Omega \,.\, \mathrm{Rl}`P \cap \mathrm{Nr}`Q = \Lambda$

I.e. one well-ordered series is less than another when, and only when, no part of it is similar to the other.

***254·52.** $\vdash : P \epsilon \Omega \,.\, \alpha \subset C`P \,.\, \exists ! \, C`P \cap p`\overleftarrow{P}``\alpha \,.\, \supset \,.\, P \upharpoonright \alpha \,\mathrm{less}\, P$

I.e. any part of a well-ordered series which stops short of the end is less than the whole series

$*254{\cdot}55.\quad \vdash :. \, Q \,\text{less}\, P \,.\equiv: P, Q \in \Omega :(\exists R) \,.\, R \,\text{smor}\, Q \,.\, R \Subset P \,.\, \exists ! \, C\text{‘}P \cap p\text{‘}\overleftarrow{P}\text{“}C\text{‘}R$

I.e. one well-ordered series is less than another when, and only when, it is similar to a part of the other which stops short of the end.

$*254{\cdot}01.\quad \text{less} = \hat{P}\hat{Q}\{P, Q \in \Omega \,.\, \exists ! \, \text{Rl}\text{‘}Q \cap \text{Nr}\text{‘}P \,.\, \sim (P \,\text{smor}\, Q)\} \quad \text{Df}$

$*254{\cdot}02.\quad P_{\text{sm}} = (\text{D}\text{‘}P_s) \upharpoonright \text{smor} \quad \text{Df}$

$*254{\cdot}1.\quad \vdash : P \,\text{less}\, Q \,.\equiv.\, P, Q \in \Omega \,.\, \exists ! \, \text{Rl}\text{‘}Q \cap \text{Nr}\text{‘}P \,.\, \sim (P \,\text{smor}\, Q) \quad [(*254{\cdot}01)]$

$*254{\cdot}101.\quad \vdash : P, Q \in \Omega \,.\, P \Subset Q \,.\, \sim (P \,\text{smor}\, Q) \,.\supset.\, P \,\text{less}\, Q \quad [*254{\cdot}1]$

$*254{\cdot}11.\quad \vdash : R P_{\text{sm}} Q \,.\equiv.\, R \in \text{D}\text{‘}P_s \,.\, R \,\text{smor}\, Q \quad [(*254{\cdot}02)]$

$*254{\cdot}111.\quad \vdash \,.\, \overrightarrow{P_{\text{sm}}}\text{‘}Q = \text{D}\text{‘}P_s \cap \text{Nr}\text{‘}Q \quad [*254{\cdot}11]$

$*254{\cdot}12.\quad \vdash : Q \in \text{G}\text{‘}P_{\text{sm}} \,.\equiv.\, \exists ! \, \text{D}\text{‘}P_s \cap \text{Nr}\text{‘}Q \quad [*254{\cdot}111]$

$*254{\cdot}121.\quad \vdash \,.\, \text{D}\text{‘}P_s \subset \text{G}\text{‘}P_{\text{sm}} \quad [*254{\cdot}12 \,.\, *152{\cdot}3]$

$*254{\cdot}13.\quad \vdash :. \, P \,\text{smor}\, P' \,.\, Q \,\text{smor}\, Q' \,.\supset: P \,\text{less}\, Q \,.\equiv.\, P' \,\text{less}\, Q'$
$[*151{\cdot}15 \,.\, *152{\cdot}321 \,.\, *254{\cdot}1]$

$*254{\cdot}14.\quad \vdash : S \in \text{D}\text{‘}Q_s \,.\, T \in P \,\overline{\text{smor}}\, Q \,.\supset.\, T ; S \in \text{D}\text{‘}P_s \cap \text{Nr}\text{‘}S$

Dem.

$\vdash \,.\, *213{\cdot}141 \,.\supset \vdash : \text{Hp} \,.\supset.\, (\exists \beta) \,.\, \beta \in \text{sect}\text{‘}Q - \iota\text{‘}\Lambda - \iota\text{‘}C\text{‘}Q \,.\, S = Q \upharpoonleft \beta \quad (1)$

$\vdash \,.\, *150{\cdot}37 \,.\quad \supset \vdash : \text{Hp} \,.\, S = Q \upharpoonleft \beta \,.\supset.\, T ; S = (T ; Q) \upharpoonleft T\text{“}\beta$

$[*151{\cdot}11] \qquad = P \upharpoonleft T\text{“}\beta \quad (2)$

$\vdash \,.\, *212{\cdot}7 \,.\quad \supset \vdash : \text{Hp} \,.\, \beta \in \text{sect}\text{‘}Q \,.\supset.\, T\text{“}\beta \in \text{sect}\text{‘}P \quad (3)$

$\vdash \,.\, *37{\cdot}43 \,.\quad \supset \vdash : \text{Hp} \,.\, \beta \in \text{sect}\text{‘}Q - \iota\text{‘}\Lambda \,.\supset.\, \exists ! \, T\text{“}\beta \quad (4)$

$\vdash \,.\, *150{\cdot}22 \,.\quad \supset \vdash : \text{Hp} \,.\, T\text{“}\beta = C\text{‘}P \,.\supset.\, T\text{“}\beta = T\text{“}C\text{‘}Q :$

$[*72{\cdot}481] \quad \supset \vdash : \text{Hp} \,.\, T\text{“}\beta = C\text{‘}P \,.\, \beta \in \text{sect}\text{‘}Q \,.\supset.\, \beta = C\text{‘}Q :$

$[\text{Transp}] \quad \supset \vdash : \text{Hp} \,.\, \beta \in \text{sect}\text{‘}Q - \iota\text{‘}C\text{‘}Q \,.\supset.\, T\text{“}\beta \neq C\text{‘}P \quad (5)$

$\vdash \,.\, (3) \,.\, (4) \,.\, (5) \,.\supset$

$\vdash : \text{Hp} \,.\, \beta \in \text{sect}\text{‘}Q - \iota\text{‘}\Lambda - \iota\text{‘}C\text{‘}Q \,.\supset.\, T\text{“}\beta \in \text{sect}\text{‘}P - \iota\text{‘}\Lambda - \iota\text{‘}C\text{‘}P \quad (6)$

$\vdash \,.\, (1) \,.\, (2) \,.\, (6) \,.\supset \vdash : \text{Hp} \,.\supset.\, (\exists \alpha) \,.\, \alpha \in \text{sect}\text{‘}P - \iota\text{‘}\Lambda - \iota\text{‘}C\text{‘}P \,.\, T ; S = P \upharpoonleft \alpha \,.$

$[*213{\cdot}141] \qquad \supset.\, T ; S \in \text{D}\text{‘}P_s \quad (7)$

$\vdash \,.\, *151{\cdot}21 \,.\quad \supset \vdash : \text{Hp} \,.\supset.\, (T ; S) \,\text{smor}\, S \quad (8)$

$\vdash \,.\, (7) \,.\, (8) \,.\supset \vdash \,.\, \text{Prop}$

$*254{\cdot}141.\quad \vdash : P \,\text{smor}\, Q \,.\supset.\, \text{D}\text{‘}Q_s \subset \text{G}\text{‘}P_{\text{sm}} \,.\, \text{D}\text{‘}P_s \subset \text{G}\text{‘}Q_{\text{sm}}$

Dem.

$\vdash \,.\, *254{\cdot}12{\cdot}14 \,.\supset \vdash :. \, \text{Hp} \,.\supset: S \in \text{D}\text{‘}Q_s \,.\supset.\, S \in \text{G}\text{‘}P_{\text{sm}} \quad (1)$

$\vdash \,.\, (1) \,.\, *151{\cdot}14 \,.\supset \vdash \,.\, \text{Prop}$

$*254{\cdot}142.\quad \vdash : R \in C\text{‘}P_s \,.\supset.\, R_{\text{sm}} \Subset P_{\text{sm}}$

Dem.

$\vdash \,.\, *213{\cdot}241 \,.\supset \vdash : \text{Hp} \,.\supset.\, \text{D}\text{‘}R_s \subset \text{D}\text{‘}P_s \quad (1)$

$\vdash \,.\, (1) \,.\, *254{\cdot}11 \,.\supset \vdash \,.\, \text{Prop}$

***254·143.** $\vdash : Q \in \mathrm{Cl}`P_{sm} . \supset . C`Q_s \subset \mathrm{Cl}`P_{sm}$

Dem.

$\vdash . *254\cdot12 . \supset \vdash : \mathrm{Hp} . \supset . (\exists R) . R \in \mathrm{D}`P_s . R \,\mathrm{smor}\, Q .$

[*254·141] $\supset . (\exists R) . R \in \mathrm{D}`P_s . \mathrm{D}`Q_s \subset \mathrm{Cl}`R_{sm} .$

[*254·142] $\supset . \mathrm{D}`Q_s \subset \mathrm{Cl}`P_{sm} .$

[*213·16.Hp] $\supset . Q \upharpoonright``(\mathrm{sect}`Q - \iota`\Lambda) \subset \mathrm{Cl}`P_{sm} .$

[*213·1] $\supset . C`Q_s \subset \mathrm{Cl}`P_{sm} : \supset \vdash . \mathrm{Prop}$

***254·144.** $\vdash : P = \dot{\Lambda} . \supset . P_{sm} = \dot{\Lambda}$ [*213·3 . *254·11]

***254·15.** $\vdash :. Q_{po} \Subset J . \exists ! \overrightarrow{B}`P . P_{po} \Subset J . \supset : Q \in \mathrm{Cl}`P_{sm} . \equiv . C`Q_s \subset \mathrm{Cl}`P_{sm}$

Dem.

$\vdash . *254\cdot143 . \supset \vdash : Q \in \mathrm{Cl}`P_{sm} . \supset . C`Q_s \subset \mathrm{Cl}`P_{sm}$ (1)

$\vdash . *213\cdot142 . *211\cdot26 . \supset \vdash :. \mathrm{Hp} . \dot{\exists} ! Q . \supset : Q \in C`Q_s :$

[*22·441] $\supset : C`Q_s \subset \mathrm{Cl}`P_{sm} . \supset . Q \in \mathrm{Cl}`P_{sm}$ (2)

$\vdash . *211\cdot18 . \supset \vdash : \mathrm{Hp} . \supset . \exists ! \mathrm{sect}`P \cap 1 .$

[*200·35] $\supset . \dot{\Lambda} \in P \upharpoonright``(\mathrm{sect}`P - \iota`\Lambda) .$

[*213·16] $\supset . \dot{\Lambda} \in \mathrm{D}`P_s .$

[*254·121] $\supset . \dot{\Lambda} \in \mathrm{Cl}`P_{sm}$ (3)

$\vdash . (2) . (3) . \supset \vdash :. \mathrm{Hp} . \supset : C`Q_s \subset \mathrm{Cl}`P_{sm} . \supset . Q \in \mathrm{Cl}`P_{sm}$ (4)

$\vdash . (1) . (4) . \supset \vdash . \mathrm{Prop}$

***254·16.** $\vdash :. Q \,\mathrm{smor}\, Q' . \supset : \overrightarrow{P_{sm}}`Q = \overrightarrow{P_{sm}}`Q' : Q \in \mathrm{Cl}`P_{sm} . \equiv . Q' \in \mathrm{Cl}`P_{sm}$

Dem.

$\vdash . *254\cdot111 . *152\cdot321 . \supset \vdash :. \mathrm{Hp} . \supset : \overrightarrow{P_{sm}}`Q = \overrightarrow{P_{sm}}`Q' :$ (1)

[*13·12] $\supset : \exists ! \overrightarrow{P_{sm}}`Q . \equiv . \exists ! \overrightarrow{P_{sm}}`Q' :$

[*33·41] $\supset : Q \in \mathrm{Cl}`P_{sm} . \equiv . Q' \in \mathrm{Cl}`P_{sm}$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***254·161.** $\vdash : P \,\mathrm{smor}\, P' . \supset . \mathrm{Cl}`P_{sm} = \mathrm{Cl}`P'_{sm}$

Dem.

$\vdash . *254\cdot14 . \supset \vdash : T \in P \,\overline{\mathrm{smor}}\, P' . S \in \mathrm{D}`P'_s \cap \mathrm{Nr}`Q . \supset . T ; S \in \mathrm{D}`P_s \cap \mathrm{Nr}`Q :$

[*254·12] $\supset \vdash : T \in P \,\overline{\mathrm{smor}}\, P' . Q \in \mathrm{Cl}`P'_{sm} . \supset . Q \in \mathrm{Cl}`P_{sm} :$

[*151·12] $\supset \vdash : P \,\mathrm{smor}\, P' . \supset . \mathrm{Cl}`P'_{sm} \subset \mathrm{Cl}`P_{sm}$ (1)

$\vdash . (1) . *151\cdot14 . \supset \vdash : P \,\mathrm{smor}\, P' . \supset . \mathrm{Cl}`P_{sm} \subset \mathrm{Cl}`P'_{sm}$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***254·162.** $\vdash :. P \,\mathrm{smor}\, P' . Q \,\mathrm{smor}\, Q' . \supset : Q \in \mathrm{Cl}`P_{sm} . \equiv . Q' \in \mathrm{Cl}`P'_{sm}$ [*254·16·161]

***254·163.** $\vdash : R \in \mathrm{Cl}`Q_{sm} . \supset . \mathrm{Cl}`R_{sm} \subset \mathrm{Cl}`Q_{sm}$

Dem.

$\vdash . *254\cdot12 . \supset \vdash : \mathrm{Hp} . \supset . (\exists S) . R \,\mathrm{smor}\, S . S \in \mathrm{D}`Q_s .$

[*254·161·142] $\supset . (\exists S) . \mathrm{Cl}`R_{sm} = \mathrm{Cl}`S_{sm} . \mathrm{Cl}`S_{sm} \subset \mathrm{Cl}`Q_{sm} .$

[*13·195] $\supset . \mathrm{Cl}`R_{sm} \subset \mathrm{Cl}`Q_{sm} : \supset \vdash . \mathrm{Prop}$

***254·164.** $\vdash : \mathrm{D}`P_s \subset \mathrm{Cl}`Q_{sm} . \supset . \mathrm{D}`P_s = P_{sm}``(\mathrm{D}`Q_s \cap \mathrm{Cl}`P_{sm}) = P_{sm}``\mathrm{D}`Q_s$

Dem.

$\vdash . *254·11 . \supset \vdash : \mathrm{Hp} . R \epsilon \mathrm{D}`P_s . \supset . (\exists S) . S \epsilon \mathrm{D}`Q_s . R \text{ smor } S .$

[*254·11] $\supset . (\exists S) . S \epsilon \mathrm{D}`Q_s . R P_{sm} S .$

[*37·1] $\supset . R \epsilon P_{sm}``\mathrm{D}`Q_s$ (1)

$\vdash . *254·11 . \supset \vdash . P_{sm}``\mathrm{D}`Q_s \subset \mathrm{D}`P_s$ (2)

$\vdash . (1) . (2) . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{D}`P_s = P_{sm}``\mathrm{D}`Q_s$

[*37·26] $= P_{sm}``(\mathrm{D}`Q_s \cap \mathrm{Cl}`P_{sm}) : \supset \vdash . \mathrm{Prop}$

***254·17.** $\vdash : P \epsilon \Omega . Q \epsilon \mathrm{D}`P_s . R \in Q . \supset . \sim (R \text{ smor } P)$

Dem.

$\vdash . *204·21 . \supset \vdash : P \epsilon \Omega . R \in P . R \text{ smor } P . \supset . R \epsilon \mathrm{Ser} .$

[*204·41] $\supset . R = P \upharpoonright C`R$ (1)

$\vdash . *250·65 . \mathrm{Transp} . \supset$

$\vdash : P \epsilon \Omega . R \text{ smor } P . R = P \upharpoonright C`R . \supset . \sim (\exists \alpha) . \alpha \epsilon \mathrm{sect}`P - \iota`C`P . C`R \subset \alpha .$

[*211·133·44] $\supset . \sim (\exists Q) . Q \epsilon P \upharpoonright ``(\mathrm{sect}`P - \iota`C`P) . R \in Q .$

[*213·141] $\supset . \sim (\exists Q) . Q \epsilon \mathrm{D}`P_s . R \in Q$ (2)

$\vdash . (1) . (2) . \supset \vdash : P \epsilon \Omega . R \text{ smor } P . R \in P . \supset . \sim (\exists Q) . Q \epsilon \mathrm{D}`P_s . R \in Q$ (3)

$\vdash . (3) . \mathrm{Transp} . \supset \vdash . \mathrm{Prop}$

***254·18.** $\vdash : Q \epsilon \mathrm{D}`P_s . \supset . \sim (P \text{ less } Q)$ [*254·17·1]

***254·181.** $\vdash : Q \epsilon \mathrm{Cl}`P_{sm} . \supset . \sim (P \text{ less } Q)$

Dem.

$\vdash . *254·18·12 . \supset \vdash : \mathrm{Hp} . \supset . (\exists R) . R \text{ smor } Q . \sim (P \text{ less } R) .$

[*254·13] $\supset . \sim (P \text{ less } Q) : \supset \vdash . \mathrm{Prop}$

***254·182.** $\vdash : P \epsilon \Omega . Q \epsilon \mathrm{D}`P_s . \supset . Q \text{ less } P$ [*254·101 . *253·421·18]

***254·2.** $\vdash : P \epsilon \Omega . Q \epsilon \mathrm{Cl}`P_{sm} . \supset . Q \text{ less } P$

Dem.

$\vdash . *254·11 . \supset \vdash : \mathrm{Hp} . \supset . (\exists R) . R \epsilon \mathrm{D}`P_s . R \text{ smor } Q .$

[*254·182] $\supset . (\exists R) . R \text{ less } P . R \text{ smor } Q .$

[*254·13] $\supset . Q \text{ less } P : \supset \vdash . \mathrm{Prop}$

***254·21.** $\vdash : P \epsilon \Omega . Q \epsilon \mathrm{Cl}`P_{sm} . R \in Q . R \epsilon \Omega . \supset . R \text{ less } P$

Dem.

$\vdash . *254·12 . \supset \vdash : \mathrm{Hp} . \supset . (\exists S, T) . S \epsilon \mathrm{D}`P_s . T \epsilon S \text{ smor } Q .$

[*151·21 . *150·31] $\supset . (\exists S, T) . S \epsilon \mathrm{D}`P_s . T \epsilon S \text{ smor } Q . T ; R \text{ smor } R . T ; R \in S .$

[*254·17] $\supset . (\exists T) . T ; R \text{ smor } R . T ; R \in P . \sim (T ; R \text{ smor } P) .$

[*151·17] $\supset . (\exists T) . T ; R \text{ smor } R . T ; R \in P . \sim (R \text{ smor } P) .$

[*254·1] $\supset . R \text{ less } P : \supset \vdash . \mathrm{Prop}$

***254·22.** $\vdash : P \epsilon \Omega . \supset . P_{\mathrm{sm}} \epsilon 1 \rightarrow \mathrm{Cls}$

Dem.

$\vdash . *254·11 . \supset \vdash :. RP_{\mathrm{sm}}Q . SP_{\mathrm{sm}}Q . \supset : R, S \epsilon \mathrm{D}‘P_s . R \,\mathrm{smor}\, S :$

$[*253·46] \quad \supset : P \epsilon \Omega . \supset . R = S \quad (1)$

$\vdash . (1) . \mathrm{Comm} . \supset \vdash . \mathrm{Prop}$

***254·221.** $\vdash : P \epsilon \Omega . \supset . \text{ꓷ}‘P_{\mathrm{sm}} \subset \Omega$

Dem.

$\vdash . *254·12 . *253·13 . \supset$

$\vdash : \mathrm{Hp} . Q \epsilon \text{ꓷ}‘P_{\mathrm{sm}} . \supset . (\exists R, \alpha) . R = P \upharpoonleft \alpha . R \,\mathrm{smor}\, Q .$

$[*250·141 . *251·111] \supset . Q \epsilon \Omega : \supset \vdash . \mathrm{Prop}$

***254·222.** $\vdash : P, Q \epsilon \Omega . \supset . P_{\mathrm{sm}} \upharpoonright \mathrm{D}‘Q_s \epsilon 1 \rightarrow 1$

Dem.

$\vdash . *254·11 . \supset \vdash :. R (P_{\mathrm{sm}} \upharpoonright \mathrm{D}‘Q_s) S . R (P_{\mathrm{sm}} \upharpoonright \mathrm{D}‘Q_s) S' . \supset :$

$S, S' \epsilon \mathrm{D}‘Q_s . R \,\mathrm{smor}\, S . R \,\mathrm{smor}\, S' :$

$[*253·46] \quad \supset : Q \epsilon \Omega . \supset . S = S' \quad (1)$

$\vdash . (1) . \mathrm{Comm} . \supset \vdash : \mathrm{Hp} . \supset . P_{\mathrm{sm}} \upharpoonright \mathrm{D}‘Q_s \epsilon \mathrm{Cls} \rightarrow 1 \quad (2)$

$\vdash . (2) . *254·22 . \supset \vdash . \mathrm{Prop}$

***254·223.** $\vdash . \mathrm{Cnv}‘(P_{\mathrm{sm}} \upharpoonright \mathrm{D}‘Q_s) = Q_{\mathrm{sm}} \upharpoonright \mathrm{D}‘P_s$

Dem.

$\vdash . *254·11 . \supset \vdash : R (P_{\mathrm{sm}} \upharpoonright \mathrm{D}‘Q_s) S . \equiv . R \epsilon \mathrm{D}‘P_s . S \epsilon \mathrm{D}‘Q_s . R \,\mathrm{smor}\, S .$

$[*151·14] \quad \equiv . S \epsilon \mathrm{D}‘Q_s . R \epsilon \mathrm{D}‘P_s . S \,\mathrm{smor}\, R .$

$[*254·11] \quad \equiv . S (Q_{\mathrm{sm}} \upharpoonright \mathrm{D}‘P_s) R : \supset \vdash . \mathrm{Prop}$

***254·224.** $\vdash : Q \epsilon \Omega . \mathrm{E}\,!\, P_{\mathrm{sm}}‘S . S \epsilon \mathrm{D}‘Q_s . \supset . S = Q_{\mathrm{sm}}‘P_{\mathrm{sm}}‘S$

Dem.

$\vdash . *254·223 . \supset \vdash :. \mathrm{Hp} . \supset : SQ_{\mathrm{sm}} (P_{\mathrm{sm}}‘S) . \equiv . (P_{\mathrm{sm}}‘S) P_{\mathrm{sm}} S \quad (1)$

$\vdash . (1) . *30·32 . *254·22 . \supset \vdash . \mathrm{Prop}$

***254·23.** $\vdash : P \epsilon \Omega . Q \epsilon \text{ꓷ}‘P_{\mathrm{sm}} . \supset . P_{\mathrm{sm}}‘Q = \check{\iota}‘(\mathrm{D}‘P_s \cap \mathrm{Nr}‘Q) \quad [*254·22·111]$

***254·24.** $\vdash : P, Q \epsilon \Omega . R \epsilon \mathrm{D}‘P_s \cap \text{ꓷ}‘Q_{\mathrm{sm}} . S \epsilon \mathrm{Rl}‘R \cap \mathrm{D}‘P_s . \supset . S \epsilon \text{ꓷ}‘Q_{\mathrm{sm}}$

Dem.

$\vdash . *213·24 . \supset \vdash : \mathrm{Hp} . \supset . S \epsilon \mathrm{D}‘R_s .$

$[*254·143 . \mathrm{Hp}] \quad \supset . S \epsilon \text{ꓷ}‘Q_{\mathrm{sm}} : \supset \vdash . \mathrm{Prop}$

***254·241.** $\vdash :. P \in \Omega \,.\, Q, R \in C\text{‘}P_s \,.\, \supset : R \in \text{Ɑ}\text{‘}Q_{sm} \,.\, \equiv \,.\, R \in \text{D}\text{‘}Q_s$

Dem.

$\vdash \,.\, *254\cdot121 \,.\, \supset \vdash : R \in \text{D}\text{‘}Q_s \,.\, \supset \,.\, R \in \text{Ɑ}\text{‘}Q_{sm}$ (1)

$\vdash \,.\, *254\cdot142 \,.\, \supset \vdash : \text{Hp} \,.\, Q \in C\text{‘}R_s \,.\, \supset \,.\, Q_{sm} \subset R_{sm}$ (2)

$\vdash \,.\, *253\cdot42 \,.\, \supset \vdash : R \in \Omega \,.\, \supset \,.\, R \sim\in \text{Ɑ}\text{‘}R_{sm}$ (3)

$\vdash \,.\, (2) \,.\, (3) \,.\, \supset \vdash : \text{Hp} \,.\, Q \in C\text{‘}R_s \,.\, \supset \,.\, R \sim\in \text{Ɑ}\text{‘}Q_{sm}$ (4)

$\vdash \,.\, (4) \,.\, \text{Transp} \,.\, (3) \,.\, \supset \vdash : \text{Hp} \,.\, R \in \text{Ɑ}\text{‘}Q_{sm} \,.\, \supset \,.\, Q \sim\in C\text{‘}R_s \,.\, Q \neq R \,.$

$[*213\cdot245]$ $\supset \,.\, \sim(QP_sR) \,.\, Q \neq R \,.$

$[*213\cdot153.\text{Hp}]$ $\supset \,.\, RP_sQ \,.$

$[*213\cdot245]$ $\supset \,.\, R \in \text{D}\text{‘}Q_s$ (5)

$\vdash \,.\, (1) \,.\, (5) \,.\, \supset \vdash \,.\, \text{Prop}$

***254·242.** $\vdash : Q \in \Omega \,.\, T \in P \,\overline{\text{smor}}\, Q \,.\, S \in \text{D}\text{‘}Q_s \,.\, \supset \,.\, T ; S = P_{sm}\text{‘}S$

Dem.

$\vdash \,.\, *254\cdot14 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, T ; S \in \text{D}\text{‘}P_s \cap \text{Nr}\text{‘}S \,.$

$[*254\cdot11]$ $\supset \,.\, (T ; S)\, P_{sm} S \,.$

$[*254\cdot22 . *251\cdot111]$ $\supset \,.\, T ; S = P_{sm}\text{‘}S : \supset \vdash \,.\, \text{Prop}$

***254·243.** $\vdash : Q \in \Omega \,.\, S \in \text{D}\text{‘}Q_s \,.\, T \in P \,\text{smor}\, S \,.\, S' Q_s S \,.\, \supset \,.\, T ; S' = P_{sm}\text{‘}S'$

Dem.

$\vdash \,.\, *213\cdot245 \,.\, *253\cdot18 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, S \in \Omega \,.\, S' \in \text{D}\text{‘}S_s \,.$

$[*254\cdot242]$ $\supset \,.\, T ; S' = P_{sm}\text{‘}S' : \supset \vdash \,.\, \text{Prop}$

***254·244.** $\vdash : P, Q \in \Omega \,.\, S \in \text{D}\text{‘}Q_s \cap \text{Ɑ}\text{‘}P_{sm} \,.\, T \in (P_{sm}\text{‘}S) \,\overline{\text{smor}}\, S \,.\, S' Q_s S \,.\, \supset \,.$
$T ; S = P_{sm}\text{‘}S \,.\, T ; S' = P_{sm}\text{‘}S' \,.\, (T ; S')\, P_s\, (T ; S)$

Dem.

$\vdash \,.\, *254\cdot243 \,.\, \supset \vdash : \text{Hp} \,.\, R = P_{sm}\text{‘}S \,.\, \supset \,.\, T ; S' = R_{sm}\text{‘}S'$ (1)

$\vdash \,.\, *254\cdot11 \,.\, \supset \vdash : \text{Hp}(1) \,.\, \supset \,.\, R \in \text{D}\text{‘}P_s \,.$ (2)

$[*254\cdot142]$ $\supset \,.\, R_{sm} \subset P_{sm}$ (3)

$\vdash \,.\, (1) \,.\, (3) \,.\, *254\cdot22 \,.\, \supset \vdash : \text{Hp}(1) \,.\, \supset \,.\, T ; S' = P_{sm}\text{‘}S'$ (4)

$\vdash \,.\, *151\cdot11 \,.\, \supset \vdash : \text{Hp}(1) \,.\, \supset \,.\, R = T ; S \,.$ (5)

$[(2)]$ $\supset \,.\, T ; S \in \text{D}\text{‘}P_s$ (6)

$\vdash \,.\, (1) \,.\, (5) \,.\, *254\cdot11 \,.\, \supset \vdash : \text{Hp}(1) \,.\, \supset \,.\, T ; S' \in \text{D}\text{‘}(T ; S)$ (7)

$\vdash \,.\, (6) \,.\, (7) \,.\, *213\cdot244 \,.\, \supset \vdash : \text{Hp}(1) \,.\, \supset \,.\, (T ; S')\, P_s\, (T ; S)$ (8)

$\vdash \,.\, (5) \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, T ; S = P_{sm}\text{‘}S$ (9)

$\vdash \,.\, (9) \,.\, (4) \,.\, (8) \,.\, \supset \vdash \,.\, \text{Prop}$

***254·245.** $\vdash : P, Q \in \Omega \,.\, S \in \text{D}\text{‘}Q_s \cap \text{Ɑ}\text{‘}P_{sm} \,.\, S' Q_s S \,.\, \supset \,.\, (P_{sm}\text{‘}S')\, P_s\, (P_{sm}\text{‘}S)$

Dem.

$\vdash \,.\, *254\cdot22\cdot11 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, (P_{sm}\text{‘}S) \,\text{smor}\, S$ (1)

$\vdash \,.\, (1) \,.\, *254\cdot244 \,.\, \supset \vdash \,.\, \text{Prop}$

***254·25.** $\vdash :. P, Q \epsilon \Omega . S, S' \epsilon \mathrm{D}`Q_s \cap \mathrm{G}`P_{sm} . \supset : S' Q_s S . \equiv . (P_{sm}`S') P_s (P_{sm}`S)$

Dem.

$\vdash . *254·245 . \supset \vdash :. \mathrm{Hp} . \supset : S' Q_s S . \supset . (P_{sm}`S') P_s (P_{sm}`S)$ (1)

$\vdash . (1) \dfrac{P_{sm}`S, P_{sm}`S', P, Q}{S, \quad S', \quad Q, P} . \supset$

$\vdash :. \mathrm{Hp} . \supset : (P_{sm}`S') P_s (P_{sm}`S) . \supset . (Q_{sm}`P_{sm}`S') Q_s (Q_{sm}`P_{sm}`S) .$

[*254·224] $\supset . S' Q_s S$ (2)

$\vdash . (1) . (2) . \supset \vdash .$ Prop

***254·26.** $\vdash : P, Q \epsilon \Omega . \supset . Q_s \upharpoonright (\mathrm{D}`Q_s \cap \mathrm{G}`P_{sm}) = Q_{sm} ; (P_s \upharpoonright \mathrm{D}`P_s)$

Dem.

$\vdash . *254·25 . \supset \vdash :: \mathrm{Hp} . \supset :. S' \{Q_s \upharpoonright (\mathrm{D}`Q_s \cap \mathrm{G}`P_{sm})\} S . \equiv :$

$S, S' \epsilon \mathrm{D}`Q_s \cap \mathrm{G}`P_{sm} . (P_{sm}`S') P_s (P_{sm}`S) :$

[*254·22] $\equiv : S, S' \epsilon \mathrm{D}`Q_s : (\exists R, R') . R P_{sm} S . R' P_{sm} S' . R' P_s R :$

[*254·223] $\equiv : (\exists R, R') . S Q_{sm} R . S' Q_{sm} R' . R, R' \epsilon \mathrm{D}`P_s . R' P_s R :$

[*150·11] $\equiv : S' \{Q_{sm} ; (P_s \upharpoonright \mathrm{D}`P_s)\} S :: \supset \vdash .$ Prop

***254·261.** $\vdash : P, Q \epsilon \Omega . \mathrm{D}`Q_s \subset \mathrm{G}`P_{sm} . \supset . Q_s \upharpoonright \mathrm{D}`Q_s = Q_{sm} ; (P_s \upharpoonright \mathrm{D}`P_s)$
[*254·26]

***254·27.** $\vdash : P, Q \epsilon \Omega . \mathrm{D}`P_s \subset \mathrm{G}`Q_{sm} . \mathrm{D}`Q_s \subset \mathrm{G}`P_{sm} . \supset .$

$Q_{sm} \upharpoonright C`(P_s \upharpoonright \mathrm{D}`P_s) \epsilon (Q_s \upharpoonright \mathrm{D}`Q_s) \overline{\mathrm{smor}} (P_s \upharpoonright \mathrm{D}`P_s)$

Dem.

$\vdash . *254·222 . \supset \vdash : \mathrm{Hp} . \supset . Q_{sm} \upharpoonright C`(P_s \upharpoonright \mathrm{D}`P_s) \epsilon 1 \rightarrow 1$ (1)

$\vdash . *37·41 . \quad \supset \vdash : \mathrm{Hp} . \supset . C`(P_s \upharpoonright \mathrm{D}`P_s) \subset \mathrm{G}`Q_{sm}$ (2)

$\vdash . (1) . (2) . *254·261 . *151·22 . \supset \vdash .$ Prop

In virtue of the above proposition, we have, when its hypothesis is realized,

$$(Q_s \upharpoonright \mathrm{D}`Q_s) \,\mathrm{smor}\, (P_s \upharpoonright \mathrm{D}`P_s),$$

whence, by *253·25, $\quad Q \,\mathrm{smor}\, P$.

This proposition is the converse of *254·141.

In the above proposition we take $Q_{sm} \upharpoonright C`(P_s \upharpoonright \mathrm{D}`P_s)$ as the correlator, rather than $Q_{sm} \upharpoonright \mathrm{D}`P_s$, so as not to have to make an exception for the case when $P \epsilon 2_r$. For if $P \epsilon 2_r$, $\mathrm{D}`P_s \epsilon 1$, but $P_s \upharpoonright \mathrm{D}`P_s = \dot{\Lambda}$. Thus $Q_{sm} \upharpoonright \mathrm{D}`P_s$ is not a correlator in this case.

The following propositions, down to the end of the present number, are important, and give the foundations of the theory of inequality between well-ordered series and between ordinals.

***254·31.** $\vdash : P, Q \in \Omega \,.\, \mathrm{D}'P_s \subset \mathrm{C}'Q_{sm} \,.\, \mathrm{D}'Q_s \subset \mathrm{C}'P_{sm} \,.\, \supset \,.\, P \operatorname{smor} Q$

Dem.

$\vdash . *254·27 . \supset \vdash :. \mathrm{Hp} . \supset : (P_s \upharpoonright \mathrm{D}'P_s) \operatorname{smor} (Q_s \upharpoonright \mathrm{D}'Q_s) :$

$[*253·25] \quad \supset : \dot{\exists} ! P . \dot{\exists} ! Q . \supset . P \operatorname{smor} Q \quad (1)$

$\vdash . *254·144 . \supset \vdash : \mathrm{Hp} . P = \dot{\Lambda} . \supset . \mathrm{D}'Q_s = \Lambda .$

$[*213·302] \quad \supset . Q = \dot{\Lambda} .$

$[*153·101] \quad \supset . P \operatorname{smor} Q \quad (2)$

Similarly $\vdash : \mathrm{Hp} . Q = \dot{\Lambda} . \supset . P \operatorname{smor} Q \quad (3)$

$\vdash . (1) . (2) . (3) . \supset \vdash . \mathrm{Prop}$

***254·311.** $\vdash :. P, Q \in \Omega . \supset : \mathrm{D}'P_s \subset \mathrm{C}'Q_{sm} . \mathrm{D}'Q_s \subset \mathrm{C}'P_{sm} . \equiv . P \operatorname{smor} Q$

[*254·31·141]

***254·32.** $\vdash : P, Q \in \Omega . \mathrm{D}'P_s \subset \mathrm{C}'Q_{sm} . S \in \mathrm{D}'Q_s - \mathrm{C}'P_{sm} . \supset . \mathrm{D}'P_s \subset \mathrm{C}'S_{sm}$

Dem.

$\vdash . *254·24 . \quad \supset \vdash : \mathrm{Hp} . R, S' \in \mathrm{D}'Q_s . S' \Subset R . R \in \mathrm{C}'P_{sm} . \supset . S' \in \mathrm{C}'P_{sm} \quad (1)$

$\vdash . (1) . \mathrm{Transp} . \supset \vdash : \mathrm{Hp} . R \in \mathrm{D}'Q_s \cap \mathrm{C}'P_{sm} . \supset . \sim (S \Subset R) .$

$[*213·21] \quad \supset . R Q_s S .$

$[*254·22·11 . *213·245] \quad \supset . (P_{sm}{}'R) \operatorname{smor} R . R \in \mathrm{D}'S_s .$

$[*254·12] \quad \supset . (P_{sm}{}'R) \in \mathrm{C}'S_{sm} \quad (2)$

$\vdash . (2) . *37·61 . \supset \vdash : \mathrm{Hp} . \supset . P_{sm}{}''(\mathrm{D}'Q_s \cap \mathrm{C}'P_{sm}) \subset \mathrm{C}'S_{sm} .$

$[*254·164] \quad \supset . \mathrm{D}'P_s \subset \mathrm{C}'S_{sm} : \supset \vdash . \mathrm{Prop}$

***254·321.** $\vdash : P, Q \in \Omega . \mathrm{D}'P_s \subset \mathrm{C}'Q_{sm} . S = \min(Q_s)'(\mathrm{D}'Q_s - \mathrm{C}'P_{sm}) . \supset . S \operatorname{smor} P$

Dem.

$\vdash . *205·14 . \supset \vdash : \mathrm{Hp} . \supset . \overrightarrow{Q_s}'S \subset \mathrm{C}'P_{sm} .$

$[*213·246] \quad \supset . \mathrm{D}'S_s \subset \mathrm{C}'P_{sm} \quad (1)$

$\vdash . *254·32 . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{D}'P_s \subset \mathrm{C}'S_{sm} \quad (2)$

$\vdash . (1) . (2) . *254·31 . \supset \vdash . \mathrm{Prop}$

***254·33.** $\vdash : P, Q \in \Omega . \mathrm{D}'P_s \subset \mathrm{C}'Q_{sm} . \exists ! \mathrm{D}'Q_s - \mathrm{C}'P_{sm} . \supset . P \in \mathrm{C}'Q_{sm}$

Dem.

$\vdash . *253·24 . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{E} ! \min(Q_s)'(\mathrm{D}'Q_s - \mathrm{C}'P_{sm}) .$

$[*254·321] \quad \supset . (\exists S) . S \in \mathrm{D}'Q_s . S \operatorname{smor} P .$

$[*254·11] \quad \supset . P \in \mathrm{C}'Q_{sm} : \supset \vdash . \mathrm{Prop}$

***254·34.** $\vdash : P, Q \in \Omega . P \sim\in \mathrm{C}'Q_{sm} . \mathrm{D}'P_s \subset \mathrm{C}'Q_{sm} . \supset . P \operatorname{smor} Q$

Dem.

$\vdash . *254·33 . \mathrm{Transp} . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{D}'Q_s \subset \mathrm{C}'P_{sm} . \mathrm{D}'P_s \subset \mathrm{C}'Q_{sm} .$

$[*254·31] \quad \supset . P \operatorname{smor} Q : \supset \vdash . \mathrm{Prop}$

***254·35.** $\vdash : P, Q \in \Omega \,.\, \exists ! \, \mathrm{D}'Q_s - \mathrm{Cl}'P_{sm} \,.\supset .\, P \in \mathrm{Cl}'Q_{sm}$

Dem.

$\vdash .\, *253\cdot24 \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, \mathrm{E}! \min (Q_s)'(\mathrm{D}'Q_s - \mathrm{Cl}'P_{sm}) \,.$

$[*205\cdot14] \qquad \supset .\, (\exists S) \,.\, S \in \mathrm{D}'Q_s - \mathrm{Cl}'P_{sm} \,.\, \overrightarrow{Q_s}'S \subset \mathrm{Cl}'P_{sm} \,.$

$[*213\cdot246] \qquad \supset .\, (\exists S) \,.\, S \in \mathrm{D}'Q_s - \mathrm{Cl}'P_{sm} \,.\, \mathrm{D}'S_s \subset \mathrm{Cl}'P_{sm} \,.$

$[*254\cdot34] \qquad \supset .\, (\exists S) \,.\, S \in \mathrm{D}'Q_s \,.\, S \,\mathrm{smor}\, P \,.$

$[*254\cdot11] \qquad \supset .\, P \in \mathrm{Cl}'Q_{sm} : \supset \vdash .\, \mathrm{Prop}$

***254·36.** $\vdash : P, Q \in \Omega \,.\, \exists ! \, \mathrm{D}'Q_s - \mathrm{Cl}'P_{sm} \,.\supset .\, C'P_s \subset \mathrm{Cl}'Q_{sm}$ $\quad [*254\cdot35\cdot143]$

***254·37.** $\vdash :.\, P, Q \in \Omega \,.\supset : P \,\mathrm{smor}\, Q \,.\vee .\, P \in \mathrm{Cl}'Q_{sm} \,.\vee .\, Q \in \mathrm{Cl}'P_{sm}$

Dem.

$\vdash .\, *254\cdot31 \,.\supset \vdash : \mathrm{Hp} \,.\, \mathrm{D}'P_s \subset \mathrm{Cl}'Q_{sm} \,.\, \mathrm{D}'Q_s \subset \mathrm{Cl}'P_{sm} \,.\supset .\, P \,\mathrm{smor}\, Q \quad (1)$

$\vdash .\, *254\cdot35 \,.\supset \vdash : \mathrm{Hp} \,.\, \exists ! \, \mathrm{D}'Q_s - \mathrm{Cl}'P_{sm} \,.\supset .\, P \in \mathrm{Cl}'Q_{sm} \quad (2)$

$\vdash .\, *254\cdot35 \,.\supset \vdash : \mathrm{Hp} \,.\, \exists ! \, \mathrm{D}'P_s - \mathrm{Cl}'Q_{sm} \,.\supset .\, Q \in \mathrm{Cl}'P_{sm} \quad (3)$

$\vdash .\, (1) \,.\, (2) \,.\, (3) \,.\supset \vdash .\, \mathrm{Prop}$

This proposition is the most important on the relations of two well-ordered series to each other's segments. It shows that of every two well-ordered series which are not similar, one must be similar to a segment of the other.

***254·4.** $\vdash :.\, P, Q \in \Omega \,.\supset : P \,\mathrm{less}\, Q \,.\vee .\, P \,\mathrm{smor}\, Q \,.\vee .\, Q \,\mathrm{less}\, P$

Dem.

$\vdash .\, *254\cdot2 \,.\supset \vdash : \mathrm{Hp} \,.\, P \in \mathrm{Cl}'Q_{sm} \,.\supset .\, P \,\mathrm{less}\, Q \quad (1)$

$\vdash .\, *254\cdot2 \,.\supset \vdash : \mathrm{Hp} \,.\, Q \in \mathrm{Cl}'P_{sm} \,.\supset .\, Q \,\mathrm{less}\, P \quad (2)$

$\vdash .\, *254\cdot37 \,.\supset \vdash : \mathrm{Hp} \,.\, P \sim\in \mathrm{Cl}'Q_{sm} \,.\, Q \sim\in \mathrm{Cl}'P_{sm} \,.\supset .\, P \,\mathrm{smor}\, Q \quad (3)$

$\vdash .\, (1) \,.\, (2) \,.\, (3) \,.\supset \vdash .\, \mathrm{Prop}$

***254·401.** $\vdash :.\, P, Q \in \Omega \,.\supset : \overrightarrow{\mathrm{less}}'P = \overrightarrow{\mathrm{less}}'Q \,.\equiv .\, P \,\mathrm{smor}\, Q$

Dem.

$\vdash .\, *254\cdot1 \,.\supset \vdash : \mathrm{Hp} \,.\, \overrightarrow{\mathrm{less}}'P = \overrightarrow{\mathrm{less}}'Q \,.\supset .\, \sim(P \,\mathrm{less}\, Q) \,.\, \sim(Q \,\mathrm{less}\, P) \,.$

$[*254\cdot4] \qquad \supset .\, P \,\mathrm{smor}\, Q \quad (1)$

$\vdash .\, *254\cdot13 \,.\supset \vdash : \mathrm{Hp} \,.\, P \,\mathrm{smor}\, Q \,.\supset .\, \overrightarrow{\mathrm{less}}'P = \overrightarrow{\mathrm{less}}'Q \quad (2)$

$\vdash .\, (1) \,.\, (2) \,.\supset \vdash .\, \mathrm{Prop}$

***254·41.** $\vdash : P \,\mathrm{less}\, Q \,.\equiv .\, P, Q \in \Omega \,.\, P \in \mathrm{Cl}'Q_{sm} \,.\equiv .\, Q \in \Omega \,.\, P \in \mathrm{Cl}'Q_{sm}$

Dem.

$\vdash .\, *254\cdot2 \,. \qquad \supset \vdash : Q \in \Omega \,.\, P \in \mathrm{Cl}'Q_{sm} \,.\supset .\, P \,\mathrm{less}\, Q \quad (1)$

$\vdash .\, *254\cdot181 \,. \qquad \supset \vdash : Q \in \mathrm{Cl}'P_{sm} \,.\supset .\, \sim(P \,\mathrm{less}\, Q) \quad (2)$

$\vdash .\, *253\cdot421 \,. \qquad \supset \vdash : Q \in \Omega \,.\, R \in \mathrm{D}'Q_s \,.\, P \,\mathrm{smor}\, R \,.\supset .\, \sim(P \,\mathrm{smor}\, Q) :$

$[*254\cdot11] \qquad \supset \vdash : Q \in \Omega \,.\, P \in \mathrm{Cl}'P_{sm} \,.\supset .\, \sim(P \,\mathrm{smor}\, Q) \quad (3)$

$\vdash .\, (2) \,.\, (3) \,.\, *254\cdot4 \,.\supset \vdash : Q \in \Omega \,.\, P \in \mathrm{Cl}'P_{sm} \,.\supset .\, P \,\mathrm{less}\, Q \quad (4)$

$\vdash .\, (1) \,.\, (4) \,. \qquad \supset \vdash : P \,\mathrm{less}\, Q \,.\equiv .\, Q \in \Omega \,.\, P \in \mathrm{Cl}'Q_{sm} \,.$

$[*254\cdot1] \qquad \equiv .\, P, Q \in \Omega \,.\, P \in \mathrm{Cl}'Q_{sm} : \supset \vdash .\, \mathrm{Prop}$

$*254\cdot42.\quad \vdash . \text{less} \Subset J . \text{less}^2 \Subset \text{less}$

Dem.

$$\vdash . *254\cdot1 . \quad \supset \vdash : P \,\text{less}\, Q . \supset . \sim (P \,\text{smor}\, Q) .$$

$$[*151\cdot13] \qquad \supset . P \neq Q \qquad (1)$$

$$\vdash . *254\cdot163 . \supset \vdash : R \in \text{Ɑ}`Q_{\text{sm}} . S \in \text{Ɑ}`R_{\text{sm}} . \supset . S \in \text{Ɑ}`Q_{\text{sm}} :$$

$$[*254\cdot41] \quad \supset \vdash : R \,\text{less}\, Q . S \,\text{less}\, R . \supset . S \,\text{less}\, Q \qquad (2)$$

$$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$$

The relation "less" fails to generate a series, because it is not connected, two similar well-ordered series being neither greater nor less than each other. On the other hand, the relation Nr⁏less is serial, since two similar well-ordered series both contribute the same term to the field of Nr⁏less, and therefore connection does not fail. The relation Nr⁏less will be dealt with in the next number.

$*254\cdot43.\quad \vdash : Q \in \Omega - \iota`\dot{\Lambda} . \supset . \dot{\Lambda} \,\text{less}\, Q \quad [*254\cdot1 . *250\cdot4 . *152\cdot11]$

$*254\cdot431.\quad \vdash . \text{Ɑ}`\text{less} = \Omega - \iota`\dot{\Lambda} . C`\text{less} \subset \Omega$

Dem.

$$\vdash . *254\cdot43 . \qquad \supset \vdash : Q \in \Omega - \iota`\dot{\Lambda} . \supset . \dot{\Lambda} \,\text{less}\, Q \qquad (1)$$

$$\vdash . *254\cdot1 . *25\cdot13 . \quad \supset \vdash : Q = \dot{\Lambda} . \supset . Q \sim\in \text{Ɑ}`\text{less} \qquad (2)$$

$$\vdash . *254\cdot1 . \qquad \supset \vdash . C`\text{less} \subset \Omega \qquad (3)$$

$$\vdash . (3) . (2) . \text{Transp} . \supset \vdash . \text{Ɑ}`\text{less} \subset \Omega - \iota`\dot{\Lambda} \qquad (4)$$

$$\vdash . (1) . (4) . \qquad \supset \vdash . \text{Ɑ}`\text{less} = \Omega - \iota`\dot{\Lambda} \qquad (5)$$

$$\vdash . (3) . (5) . \supset \vdash . \text{Prop}$$

In order to obtain $C`\text{less} = \Omega$, we need, as appears from (1) in the above proof, $\exists ! \Omega - \iota`\dot{\Lambda}$. In virtue of $*251\cdot7$, this requires $\exists ! 2$. By $*101\cdot42\cdot43$, this holds if "less" has its field defined as belonging to a class-type or a relation-type. If, however, "less" has its field defined as composed of individuals, the primitive propositions assumed in the present work do not enable us to prove $\exists ! 2$, nor therefore to prove $\dot{\exists} ! \text{less}$.

It should be observed that "less," like "sm" and "smor," is significant when it is not homogeneous; but "$C`$less" is only significant for homogeneous typical determinations of "less," because only homogeneous relations have fields.

$*254\cdot432.\quad \vdash : \exists ! 2_\alpha . \equiv . \dot{\exists} ! \text{less} \mathbin{\dot{\cap}} t_{00}`\alpha \uparrow t_{00}`\alpha . \equiv . \exists ! \Omega - \iota`\dot{\Lambda} \cap t_{00}`\alpha$

Dem.

$$\vdash . *251\cdot7 . \supset \vdash : \exists ! 2_\alpha . \equiv . \exists ! \Omega - \iota`\dot{\Lambda} \cap t_{00}`\alpha . \qquad (1)$$

$$[*254\cdot43] \qquad \equiv . (\exists Q) . Q \in \Omega - \iota`\dot{\Lambda} \cap t_{00}`\alpha . \dot{\Lambda} \,\text{less}\, Q .$$

$$[*55\cdot37] \qquad \supset . (\exists Q) . \dot{\Lambda} \,\text{less}\, Q . \Lambda \downarrow Q \Subset t_{00}`\alpha \uparrow t_{00}`\alpha .$$

$$[*55\cdot3] \qquad \supset . \dot{\exists} ! \text{less} \mathbin{\dot{\cap}} t_{00}`\alpha \uparrow t_{00}`\alpha \qquad (2)$$

$\vdash . *35\cdot103 . \supset \vdash : \dot{\exists} ! \text{less} \mathbin{\dot{\cap}} t_{00}\text{‘}\alpha \uparrow t_{00}\text{‘}\alpha . \supset . (\exists P, Q) . P \text{ less } Q . P, Q \in t_{00}\text{‘}\alpha .$

$[*254\cdot431] \qquad \supset . \exists ! \Omega - \iota\text{‘}\dot{\Lambda} \cap t_{00}\text{‘}\alpha .$

$[(1)] \qquad \supset . \exists ! 2_\alpha \qquad (3)$

$\vdash . (1) . (2) . (3) . \supset \vdash . \text{Prop}$

$*254\cdot433$. $\vdash . \dot{\exists} ! \text{less} \mathbin{\dot{\cap}} t_{00}\text{‘Cls} \uparrow t_{00}\text{‘Cls} . \dot{\exists} ! \text{less} \mathbin{\dot{\cap}} t_{00}\text{‘Rel} \uparrow t_{00}\text{‘Rel}$

$[*254\cdot432 . *101\cdot42\cdot43]$

$*254\cdot434$. $\vdash : \dot{\exists} ! \text{less} . \equiv . C\text{‘less} = \Omega . \equiv . B\text{‘less} = \dot{\Lambda}$

Dem.

$\vdash . *250\cdot4 . *33\cdot24 . \supset \vdash : C\text{‘less} = \Omega . \supset . \dot{\exists} ! \text{less} \qquad (1)$

$\vdash . *93\cdot102 . *33\cdot24 . \supset \vdash : B\text{‘less} = \dot{\Lambda} . \supset . \dot{\exists} ! \text{less} \qquad (2)$

$\vdash . *254\cdot43 . \qquad \supset \vdash : Q \in \Omega - \iota\text{‘}\dot{\Lambda} . \supset . \dot{\Lambda} \text{ less } Q \qquad (3)$

$\vdash . (3) . \qquad \supset \vdash : \exists ! \Omega - \iota\text{‘}\dot{\Lambda} . \supset . \dot{\Lambda} \in \text{D‘less} .$

$[*254\cdot431] \qquad \supset . \dot{\Lambda} = B\text{‘less} \qquad (4)$

$\vdash . (4) . *254\cdot431 . \qquad \supset \vdash : \exists ! \Omega - \iota\text{‘}\dot{\Lambda} . \supset . C\text{‘less} = \Omega \qquad (5)$

$\vdash . (1) . (2) . (4) . (5) . \supset \vdash . \text{Prop}$

$*254\cdot44$. $\vdash : P \in C\text{‘less} . \supset . C\text{‘less} = \overrightarrow{\text{less}}\text{‘}P \cup \text{Nr‘}P \cup \overleftarrow{\text{less}}\text{‘}P$

Dem.

$\vdash . *254\cdot13 . \qquad \supset \vdash : \text{Hp} . \supset . \text{Nr‘}P \subset C\text{‘less} \qquad (1)$

$\vdash . (1) . *33\cdot152 . \supset \vdash : \text{Hp} . \supset . \overrightarrow{\text{less}}\text{‘}P \cup \text{Nr‘}P \cup \overleftarrow{\text{less}}\text{‘}P \subset C\text{‘less} \qquad (2)$

$\vdash . *254\cdot1 . \qquad \supset \vdash . C\text{‘less} \subset \Omega .$

$[*254\cdot4] \supset \vdash :. P \in C\text{‘less} . \supset : Q \in C\text{‘less} . \supset . Q \in \overrightarrow{\text{less}}\text{‘}P \cup \text{Nr‘}P \cup \overleftarrow{\text{less}}\text{‘}P \qquad (3)$

$\vdash . (2) . (3) . \supset \vdash . \text{Prop}$

$*254\cdot45$. $\vdash : P, Q \in \Omega . \exists ! \text{Rl‘}P \cap \text{Nr‘}Q . \exists ! \text{Rl‘}Q \cap \text{Nr‘}P . \supset . P \text{ smor } Q$

Dem.

$\vdash . *254\cdot42 . \supset \vdash : P \text{ less } Q . \supset . \sim(Q \text{ less } P) \qquad (1)$

$\vdash . *254\cdot1 . \quad \supset \vdash : P, Q \in \Omega . \exists ! \text{Rl‘}Q \cap \text{Nr‘}P . \sim(P \text{ smor } Q) . \supset . P \text{ less } Q .$

$[(1)] \qquad \supset . \sim(Q \text{ less } P) .$

$[*254\cdot1 . \text{Transp}] \qquad \supset . \sim \exists ! \text{Rl‘}P \cap \text{Nr‘}Q \qquad (2)$

$\vdash . (2) . \text{Transp} . \supset \vdash . \text{Prop}$

This proposition is the analogue, for ordinals, of the Schröder-Bernstein theorem.

***254·46.** $\vdash : P \text{ less } Q . \equiv . P, Q \in \Omega . \exists ! \text{Rl}`Q \cap \text{Nr}`P . \sim \exists ! \text{Rl}`P \cap \text{Nr}`Q$

Dem.

$\vdash . *152 \cdot 11 . *61 \cdot 34 . \supset$

$\vdash : P, Q \in \Omega . \exists ! \text{Rl}`Q \cap \text{Nr}`P . \sim \exists ! \text{Rl}`P \cap \text{Nr}`Q . \supset .$

$P, Q \in \Omega . \exists ! \text{Rl}`Q \cap \text{Nr}`P . \sim (P \text{ smor } Q) .$

[*254·1] $\supset . P \text{ less } Q$ (1)

$\vdash . *254 \cdot 1 \cdot 45 . \text{Transp} . \supset$

$\vdash : P \text{ less } Q . \supset . P, Q \in \Omega . \exists ! \text{Rl}`Q \cap \text{Nr}`P . \sim \exists ! \text{Rl}`P \cap \text{Nr}`Q$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***254·47.** $\vdash : P \in \Omega . \supset . P_s = \text{less} \upharpoonright C`P_s$

Dem.

$\vdash . *213 \cdot 245 . \supset \vdash :. \text{Hp} . \supset : R P_s Q . \equiv . R \in \text{D}`Q_s . Q \in C`P_s .$

[*254·121] $\supset . R \in \text{Cl}`Q_{sm} .$

[*254·41] $\supset . R \text{ less } Q$ (1)

$\vdash . *254 \cdot 181 . \text{Transp} . \supset \vdash : \text{Hp} . Q, R \in C`P_s . R \text{ less } Q . \supset . Q \sim \in \text{Cl}`R_{sm} .$

[*254·121] $\supset . Q \sim \in \text{D}`R_s$ (2)

$\vdash . (2) . *213 \cdot 25 . *254 \cdot 42 . \supset \vdash : \text{Hp} . Q, R \in C`P_s . R \text{ less } Q . \supset . R \in \text{D}`Q_s .$

[*213·245] $\supset . R P_s Q$ (3)

$\vdash . (1) . (3) . \supset \vdash . \text{Prop}$

***254·5.** $\vdash :. P, Q \in \Omega . \supset :$

$\text{Rl}`P \cap \text{Nr}`Q = \Lambda . \equiv . \exists ! \text{Rl}`Q \cap \text{Nr}`P . \sim (P \text{ smor } Q) . \equiv . P \text{ less } Q$

Dem.

$\vdash . *254 \cdot 46 . \supset \vdash : \text{Hp} . \text{Rl}`P \cap \text{Nr}`Q = \Lambda . \supset . \sim (Q \text{ less } P)$ (1)

$\vdash . *61 \cdot 34 . *152 \cdot 11 . \supset \vdash : P \text{ smor } Q . \supset . P \in \text{Rl}`P \cap \text{Nr}`Q$ (2)

$\vdash . (2) . \text{Transp} . \supset \vdash : \text{Rl}`P \cap \text{Nr}`Q = \Lambda . \supset . \sim (P \text{ smor } Q)$ (3)

$\vdash . (1) . (3) . *254 \cdot 4 . \supset \vdash : \text{Hp} . \text{Rl}`P \cap \text{Nr}`Q = \Lambda . \supset . P \text{ less } Q$ (4)

$\vdash . *254 \cdot 46 . \supset \vdash : P \text{ less } Q . \supset . \text{Rl}`P \cap \text{Nr}`Q = \Lambda$ (5)

$\vdash . (4) . (5) . \supset \vdash :. \text{Hp} . \supset : \text{Rl}`P \cap \text{Nr}`Q = \Lambda . \equiv . P \text{ less } Q .$

[*254·1] $\equiv . \exists ! \text{Rl}`Q \cap \text{Nr}`P . \sim (P \text{ smor } Q) :. \supset \vdash . \text{Prop}$

***254·51.** $\vdash : P \text{ less } Q . \equiv . P, Q \in \Omega . \text{Rl}`P \cap \text{Nr}`Q = \Lambda$ [*254·5·1]

***254·52.** $\vdash : P \in \Omega . \alpha \subset C`P . \exists ! C`P \cap p`\overleftarrow{P}``\alpha . \supset . P \upharpoonright \alpha \text{ less } P$

Dem.

$\vdash . *250 \cdot 141 . \supset \vdash : \text{Hp} . \supset . P \upharpoonright \alpha \in \Omega$ (1)

$\vdash . *250 \cdot 653 . \supset \vdash : \text{Hp} . \supset . \sim (P \upharpoonright \alpha \text{ smor } P)$ (2)

$\vdash . (1) . (2) . *254 \cdot 101 . \supset \vdash . \text{Prop}$

***254·53.** $\vdash : P, Q \in \Omega \,.\, Q \subsetneq P \,.\, \exists ! C'P \cap p'\overleftarrow{P}``C'Q \,.\, \supset \,.\, Q \text{ less } P$

Dem.

$\vdash \,.\, *250·652 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, \sim (Q \text{ smor } P)$ (1)

$\vdash \,.\, (1) \,.\, *254·101 \,.\, \supset \vdash \,.\, \text{Prop}$

***254·54.** $\vdash : P, Q \in \Omega \,.\, R \text{ smor } Q \,.\, R \subsetneq P \,.\, \exists ! C'P \cap p'\overleftarrow{P}``C'R \,.\, \supset \,.\, Q \text{ less } P$
[*254·53·13]

***254·55.** $\vdash :. Q \text{ less } P \,.\, \equiv : P, Q \in \Omega : (\exists R) \,.\, R \text{ smor } Q \,.\, R \subsetneq P \,.\, \exists ! C'P \cap p'\overleftarrow{P}``C'R$

Dem.

$\vdash \,.\, *254·41 \,.\, \supset \vdash :. Q \text{ less } P \,.\, \supset : P, Q \in \Omega : (\exists R) \,.\, R \text{ smor } Q \,.\, R \in \text{D}'R_s :$

$[*213·18] \supset : P, Q \in \Omega : (\exists R) \,.\, R \text{ smor } Q \,.\, R \subsetneq P \,.\, \exists ! C'P \cap p'\overleftarrow{P}``C'R$ (1)

$\vdash \,.\, (1) \,.\, *254·54 \,.\, \supset \vdash \,.\, \text{Prop}$

*255. GREATER AND LESS AMONG ORDINAL NUMBERS.

Summary of *255.

If P and Q are well-ordered series, we say that $\text{Nr}'P$ is less than $\text{Nr}'Q$ if P is less than Q. Thus if μ and ν are ordinal numbers, we say that μ is less than ν if there are well-ordered series P, Q, such that $\mu = \text{Nr}'P$ and $\nu = \text{Nr}'Q$ and P is less than Q. In order to exclude the case where, in the type concerned, we have $\text{Nr}'P = \Lambda$ or $\text{Nr}'Q = \Lambda$, we assume $\mu = \text{N}_0\text{r}'P$ and $\nu = \text{N}_0\text{r}'Q$. Thus we put

$$\mu \lessdot \nu \,.\equiv.\, (\exists P, Q) \,.\, \mu = \text{N}_0\text{r}'P \,.\, \nu = \text{N}_0\text{r}'Q \,.\, P \,\text{less}\, Q,$$

i.e. we put $$\lessdot = \text{N}_0\text{r}{;}\text{less} \quad \text{Df.}$$

In order to be able to speak of $\text{Nr}'P$ (where the type of "Nr" is left ambiguous) as greater or less than $\text{Nr}'Q$, we put

$$\mu \lessdot \text{Nr}'P \,.=.\, \mu \lessdot \text{N}_0\text{r}'P \quad \text{Df,}$$

$$\text{Nr}'P \lessdot \mu \,.=.\, \text{N}_0\text{r}'P \lessdot \mu \quad \text{Df.}$$

The treatment of types proceeds, mutatis mutandis, as in *117, to which, together with the prefatory statement in Vol. II, the reader is referred for explanations.

In virtue of *254·46 and *117·1, there is a close analogy between cardinal and ordinal inequality. That is to say, most of the properties of cardinal inequality have exact analogues for ordinal inequality, and these analogues have analogous proofs. (In the present number, when a proposition is analogous to the proposition with the same decimal part in *117, and has an analogous proof, we shall omit the proof.) But ordinal inequality has a good many properties which have no analogues for cardinal inequality. The chief of these, upon which most of the rest depend, is

***255·112.** $\vdash :. \mu, \nu \in \text{N}_0\text{O} \,.\supset:\, \mu \lessdot \nu \,.\vee.\, \mu = \text{smor}``\nu \,.\vee.\, \nu \lessdot \mu$

where "N_0O" stands for "homogeneous ordinals," *i.e.* $\text{NO} \cap \text{N}_0\text{R}$. We have also, what is often important,

***255·17.** $\vdash : \text{Nr}'P \gtrdot \text{Nr}'Q \,.\equiv.\, Q \,\text{less}\, P \,.\equiv.\, P, Q \in \Omega \,.\, Q \in \text{Cl}'P_{\text{sm}} \,.$
$$\equiv.\, P, Q \in \Omega \,.\, \exists \,!\, \text{D}'P_{\text{s}} \cap \text{Nr}'Q$$

so that

***255·171.** $\vdash :. P \in \Omega . \supset : \mu \lessdot \mathrm{Nr}'P . \equiv . \mu \in \mathrm{Nr}``\mathrm{D}'P_\varsigma - \iota'\Lambda$

and more generally,

***255·172.** $\vdash :. P \in \Omega . \supset :$

$$\mu \lessdot \mathrm{Nr}'P . \equiv . (\exists \alpha) . \alpha \subset C'P . \exists ! C'P \cap p'\overleftarrow{P}``\alpha . \mu = \mathrm{Nr}'P \upharpoonright \alpha . \exists ! \mu$$

As in cardinals, μ is greater than ν if (and only if) μ is the sum of ν and an ordinal other than zero, including $\dot{1}$ except when $\nu = 0_r$ (*255·33). But it is necessary to the truth of this proposition that the addendum should come after ν, not before it; *i.e.* $\nu \dotplus \varpi \gtrdot \nu$ unless $\varpi = 0_r$ (*255·32·321), but $\varpi \dotplus \nu$ is often equal to ν.

If α, β, γ are ordinals, and $\alpha \gtrdot \beta$, we shall have

$$\gamma \dotplus \alpha \gtrdot \gamma \dotplus \beta \quad (*255·561),$$

$$\alpha \dot{\times} \beta \gtrdot \beta \text{ if } \alpha \neq 0_r . \beta \neq 0_r \quad (*255·571),$$

$$\alpha \dot{\times} \gamma \gtrdot \beta \dot{\times} \gamma \text{ if } \gamma \neq 0_r \quad (*255·58),$$

$$\gamma \dot{\times} \beta \gtrdot \gamma \text{ if } \gamma \text{ is of the form } \delta \dotplus \dot{1} \quad (*255·573),$$

$$\gamma \dot{\times} \alpha \gtrdot \gamma \dot{\times} \beta \text{ if } \gamma \text{ is of the form } \delta \dotplus \dot{1} \quad (*255·582).$$

From the above propositions it follows that if α, β, γ are ordinals,

$$\gamma \dotplus \alpha = \gamma \dotplus \beta . \supset . \alpha = \beta$$

(*255·565, where β may be substituted for smor$``\beta$ whenever significance permits; cf. note to *120·51), which gives the uniqueness of subtraction from the end (subtraction from the beginning is not unique);

$$\alpha \dot{\times} \gamma = \beta \dot{\times} \gamma . \supset . \alpha = \beta \text{ unless } \gamma = 0_r \quad (*255·59).$$

which gives the uniqueness of division by an end-factor;

$$\gamma \dot{\times} \alpha = \gamma \dot{\times} \beta . \supset . \alpha = \beta \text{ if } \gamma = \delta \dotplus \dot{1} \quad (*255·591),$$

which gives the uniqueness of division by a beginning-factor of the form $\delta \dotplus \dot{1}$.

We do not have generally

$$\alpha, \beta, \gamma \in \mathrm{N_0O} . \alpha \lessdot \beta . \supset . \alpha \exp_r \gamma \lessdot \beta \exp_r \gamma,$$

because $\alpha \exp_r \gamma$ and $\beta \exp_r \gamma$ are in general not *ordinal* numbers, since series having these numbers are in general not well-ordered. Thus the theory of ordinal inequality has only a restricted application to exponentiation. This subject cannot be adequately dealt with until we have considered finite and infinite series.

If α is an ordinal, $C``\alpha$ is the corresponding cardinal, *i.e.* the cardinal number of terms in a series whose ordinal number is α. Thus the cardinal numbers of classes which can be well-ordered are $C```\mathrm{NO}$, *i.e.*

***255·7.** $\vdash . \mathrm{Nc}``C``\Omega = C```\mathrm{NO}$

It is evident that

***255·71.** $\vdash : P \,\text{less}\, Q \,.\supset .\, \text{Nc}`C`P \leqslant \text{Nc}`C`Q$

whence, by *254·4,

***255·73.** $\vdash :. P, Q \,\epsilon\, \Omega \,.\supset :$

$$\text{Nc}`C`P < \text{Nc}`C`Q \,.\vee .\, \text{Nc}`C`P = \text{Nc}`C`Q \,.\vee .\, \text{Nc}`C`P > \text{Nc}`C`Q$$

whence also

***255·74.** $\vdash :. \alpha, \beta \,\epsilon\, C``\text{NO} - \iota`\Lambda \,.\supset : \alpha \leqslant \beta \,.\vee .\, \alpha > \beta$

Thus if two classes can both be well-ordered, they either have the same cardinal, or the cardinal of one is less than that of the other

We have

***255·75.** $\vdash : P\ Q \,\epsilon\, \Omega \,.\, \text{Nc}`C`P < \text{Nc}`C`Q \,.\supset .\, P \,\text{less}\, Q$

or, what comes to the same thing,

***255·76.** $\vdash : \alpha, \beta \,\epsilon\, \text{NO} \,.\, C``\alpha < C``\beta \,.\supset .\, \alpha \lessdot \beta$

The converse of this proposition only holds for finite ordinals. If α is an infinite ordinal, $\alpha \dot{+} \dot{1}$ always exists and is greater than α, but $C``\alpha = C``(\alpha \dot{+} \dot{1})$. (The existence of $\alpha \dot{+} \dot{1}$ is deduced from that of α by taking a member of α, and removing its first term to the end. The result is a series whose number is $\alpha \dot{+} \dot{1}$, in virtue of *253·503·54.)

***255·01.** $\lessdot = \text{N}_0\text{r}`\text{less}$ Df

***255·02.** $\gtrdot = \text{Cnv}`\lessdot$ Df

***255·03.** $\text{N}_0\text{O} = \text{NO} \cap \text{N}_0\text{R}$ Df

Thus "N_0O" means "homogeneous ordinals." In virtue of *155·34·22, this is the same as "ordinals other than Λ." It is not, however, strictly correct to put $\text{N}_0\text{O} = \text{NO} - \iota`\Lambda$, because if the "NO" on the right is derived from an ascending Nr, it will not contain all the ordinals in the type to which it takes us, but only those which are not too big to be derived from the lower type from which "Nr" starts. Thus in this case N_0O will be a larger class than $\text{NO} - \iota`\Lambda$. If, however, the "Nr" from which the "NO" on the right is derived is homogeneous or descending, we shall have

$$\text{N}_0\text{O} = \text{NO} - \iota`\Lambda.$$

***255·04.** $\leqslant = \lessdot \cup \text{smor}_\epsilon \upharpoonright \text{N}_0\text{O}$ Df

This definition leads to the usual meaning of "less than or equal to." We want the relation "less than or equal to" to hold only between numbers of the sort in question (cardinal or ordinal), and we want "equal to" to hold between two numbers which are merely different typical determinations of a given number, provided neither of these typical determinations is Λ. That is, if μ is an ordinal which is not Λ, $\text{smor}``\mu$ is to be reckoned equal to $\mu$ in every type in which it is not $\Lambda$. Thus if $\nu = \text{smor}``\mu$, *i.e.* if $\nu = \text{smor}_\epsilon`\mu$, we

shall reckon ν equal to μ if both are ordinals and neither is Λ, *i.e.* in virtue of $*155{\cdot}34{\cdot}22$, if $\mu, \nu \in \mathrm{N_0O}$. This leads to the above definition.

$*255{\cdot}05.\quad \geqslant\, = \mathrm{Cnv}`\leqslant \qquad$ Df

$*255{\cdot}06.\quad \mu < \mathrm{Nr}`P \,.=.\, \mu < \mathrm{N_0r}`P \quad$ Df

On this definition, compare the remarks on $*117{\cdot}02$.

$*255{\cdot}07.\quad \mathrm{Nr}`P < \mu \,.=.\, \mathrm{N_0r}`P < \mu \quad$ Df

The following propositions (down to $*255{\cdot}108$) merely re-state the above definitions.

$*255{\cdot}1.\quad \vdash : \mu < \nu \,.\equiv.\, (\exists P, Q) \,.\, \mu = \mathrm{N_0r}`P \,.\, \nu = \mathrm{N_0r}`Q \,.\, P \text{ less } Q$

$*255{\cdot}101.\quad \vdash : \mu < \mathrm{Nr}`Q \,.\equiv.\, \mu < \mathrm{N_0r}`Q$

$*255{\cdot}102.\quad \vdash : \mathrm{Nr}`P < \nu \,.\equiv.\, \mathrm{N_0r}`P < \nu$

$*255{\cdot}103.\quad \vdash : \mu > \nu \,.\equiv.\, \nu < \mu$

$*255{\cdot}104.\quad \vdash :.\, \mu \leqslant \nu \,.\equiv : \mu < \nu \,.\vee.\, \mu, \nu \in \mathrm{N_0O} \,.\, \mu = \mathrm{smor}``\nu$

$*255{\cdot}105.\quad \vdash :.\, \mu \geqslant \nu \,.\equiv : \nu \leqslant \mu : \equiv : \nu < \mu \,.\vee.\, \mu, \nu \in \mathrm{N_0O} \,.\, \mu = \mathrm{smor}``\nu$
[$*255{\cdot}104 \,.\, (*255{\cdot}05) \,.\, *155{\cdot}44$]

$*255{\cdot}106.\quad \vdash : \mathrm{Nr}`P < \mathrm{Nr}`Q \,.\equiv.\, \mathrm{N_0r}`P < \mathrm{N_0r}`Q \quad$ [$*255{\cdot}101{\cdot}102$]

$*255{\cdot}107.\quad \vdash : \mathrm{Nr}`P \leqslant \mathrm{Nr}`Q \,.\equiv.\, \mathrm{N_0r}`P \leqslant \mathrm{N_0r}`Q$

$*255{\cdot}108.\quad \vdash :.\, \mathrm{Nr}`P \leqslant \mathrm{Nr}`Q \,.\equiv : \mathrm{N_0r}`P < \mathrm{N_0r}`Q \,.\vee.\, \mathrm{Nr}`P = \mathrm{Nr}`Q \,.\, P \in \Omega$
[$*255{\cdot}107{\cdot}104 \,.\, *155{\cdot}16 \,.\, *152{\cdot}53$]

$*255{\cdot}11.\quad \vdash : \mu < \nu \,.\equiv.\, (\exists P, Q) \,.\, P, Q \in \Omega \,.\, \mu = \mathrm{N_0r}`P \,.\, \nu = \mathrm{N_0r}`Q \,.$
$\exists ! \,\mathrm{Rl}`Q \cap \mathrm{Nr}`P \,.\, \sim \exists ! \,\mathrm{Rl}`P \cap \mathrm{Nr}`Q \quad$ [$*255{\cdot}1 \,.\, *254{\cdot}46$]

$*255{\cdot}111.\quad \vdash : \mu > \nu \,.\equiv.\, (\exists P, Q) \,.\, P, Q \in \Omega \,.\, \mu = \mathrm{N_0r}`P \,.\, \nu = \mathrm{N_0r}`Q \,.$
$\exists ! \,\mathrm{Rl}`P \cap \mathrm{Nr}`Q \,.\, \sim \exists ! \,\mathrm{Rl}`Q \cap \mathrm{Nr}`P \quad$ [$*255{\cdot}11{\cdot}103$]

This proposition is exactly analogous to $*117{\cdot}1$, except for the addition $P, Q \in \Omega$. Hence except where this addition is relevant, the analogues of the propositions of $*117$ follow by analogous proofs. Such analogues will be given without proof in what follows, and will have the same decimal part as the corresponding propositions in $*117$. Where proofs are given, there are no analogues in $*117$, or else the method of proof is not analogous.

$*255{\cdot}112.\quad \vdash :.\, \mu, \nu \in \mathrm{N_0O} \,.\supset : \mu < \nu \,.\vee.\, \mu = \mathrm{smor}``\nu \,.\vee.\, \nu < \mu$

Dem.

$\vdash \,.\, *255{\cdot}1 \,.\, *254{\cdot}4 \,.\supset \vdash :.\, \mathrm{Hp} \,.\supset :$

$\mu < \nu \,.\vee.\, \nu < \mu \,.\vee.\, (\exists P, Q) \,.\, P, Q \in \Omega \,.\, \mu = \mathrm{N_0r}`P \,.\, \nu = \mathrm{N_0r}`Q \,.\, P \text{ smor } Q :$

[$*155{\cdot}4 . *152{\cdot}321$]

$\supset : \mu < \nu \,.\vee.\, \nu < \mu \,.\vee.\, (\exists P, Q) \,.\, \mu = \mathrm{N_0r}`P \,.\, \mathrm{Nr}`P = \mathrm{Nr}`Q \,.\, \mathrm{Nr}`Q = \mathrm{smor}``\nu :$

[$*155{\cdot}16$]

$\supset : \mu < \nu \,.\vee.\, \nu < \mu \,.\vee.\, (\exists P, Q) \,.\, \mu = \mathrm{N_0r}`P \,.\, \mathrm{N_0r}`P = \mathrm{Nr}`Q \,.\, \mathrm{Nr}`Q = \mathrm{smor}``\nu :$

[$*13{\cdot}17$] $\supset : \mu < \nu \,.\vee.\, \nu < \mu \,.\vee.\, \mu = \mathrm{smor}``\nu :.\supset \vdash \,.\,$ Prop

***255·113.** $\vdash :. P, Q \epsilon \Omega . \supset : \mathrm{Nr}`P \mathrel{<\!\!\cdot} \mathrm{Nr}`Q . \vee . \mathrm{Nr}`P = \mathrm{Nr}`Q . \vee . \mathrm{Nr}`Q \mathrel{<\!\!\cdot} \mathrm{Nr}`P$

Dem.

$\vdash . *255·112·106 . \supset \vdash :. \mathrm{Hp} . \supset :$

$\mathrm{Nr}`P \mathrel{<\!\!\cdot} \mathrm{Nr}`Q . \vee . \mathrm{N_0r}`P = \mathrm{smor}``\mathrm{N_0r}`Q . \vee . \mathrm{Nr}`Q \mathrel{<\!\!\cdot} \mathrm{Nr}`P :$

$[*155·4·16] \supset : \mathrm{Nr}`P \mathrel{<\!\!\cdot} \mathrm{Nr}`Q . \vee . \mathrm{Nr}`P = \mathrm{Nr}`Q . \vee . \mathrm{Nr}`Q \mathrel{<\!\!\cdot} \mathrm{Nr}`P :. \supset \vdash . \mathrm{Prop}$

***255·114.** $\vdash :. \mu, \nu \epsilon \mathrm{N_0O} . \supset : \mu \leqslant \nu . \vee . \nu \mathrel{<\!\!\cdot} \mu : \mu \geqslant \nu . \vee . \nu \mathrel{>\!\!\cdot} \mu$ [*255·112·104·105·103]

***255·115.** $\vdash :. P, Q \epsilon \Omega . \supset : \mathrm{Nr}`P \leqslant \mathrm{Nr}`Q . \vee . \mathrm{Nr}`Q \mathrel{<\!\!\cdot} \mathrm{Nr}`P :$
$\mathrm{Nr}`P \geqslant \mathrm{Nr}`Q . \vee . \mathrm{Nr}`Q \mathrel{>\!\!\cdot} \mathrm{Nr}`P$ [*255·113·108]

***255·12.** $\vdash :. \mu \mathrel{>\!\!\cdot} \nu . \equiv : \mu, \nu \epsilon \mathrm{N_0O} :$
$P \epsilon \mu . Q \epsilon \nu . \supset_{P,Q} . \exists ! \mathrm{Rl}`P \cap \mathrm{Nr}`Q . \sim \exists ! \mathrm{Rl}`Q \cap \mathrm{Nr}`P$

***255·121.** $\vdash :. \mu \mathrel{>\!\!\cdot} \nu . \equiv : \mu, \nu \epsilon \mathrm{N_0O} :$
$P \epsilon \mu . \supset_P . (\exists Q) . Q \epsilon \nu . \exists ! \mathrm{Rl}`P \cap \mathrm{Nr}`Q . \sim \exists ! \mathrm{Rl}`Q \cap \mathrm{Nr}`P$

***255·13.** $\vdash : \mathrm{Nr}`P \mathrel{>\!\!\cdot} \mathrm{Nr}`Q . \equiv . P, Q \epsilon \Omega . \exists ! \mathrm{Rl}`P \cap \mathrm{Nr}`Q . \sim \exists ! \mathrm{Rl}`Q \cap \mathrm{Nr}`P$

***255·131.** $\vdash : \mathrm{Nr}`P \mathrel{>\!\!\cdot} \mathrm{Nr}`Q . \equiv . \mathrm{Nr}`P \geqslant \mathrm{Nr}`Q . \mathrm{Nr}`P \neq \mathrm{Nr}`Q$ [*255·13 . *254·45]

***255·14.** $\vdash : \mu \mathrel{>\!\!\cdot} \nu . \equiv . (\exists P, Q) . P, Q \epsilon \Omega . \mu = \mathrm{N_0r}`P . \nu = \mathrm{N_0r}`Q . \mathrm{Nr}`P \mathrel{>\!\!\cdot} \mathrm{Nr}`Q$

***255·141.** $\vdash : \mu \mathrel{>\!\!\cdot} \nu . \equiv . \mu \geqslant \nu . \mu \neq \mathrm{smor}``\nu$ [*255·131·14]

***255·15.** $\vdash : \mu \mathrel{>\!\!\cdot} \nu . \equiv . \mu, \nu \epsilon \mathrm{N_0O} . \exists ! s`\mathrm{Rl}``\mu \cap \mathrm{smor}``\nu . \sim \exists ! s`\mathrm{Rl}``\nu \cap \mathrm{smor}``\mu$

***255·16.** $\vdash :. \mu, \nu \epsilon \mathrm{N_0O} . \supset :$
$\mu \mathrel{>\!\!\cdot} \nu . \equiv . \mathrm{smor}``\mu \mathrel{>\!\!\cdot} \nu . \equiv . \mu \mathrel{>\!\!\cdot} \mathrm{smor}``\nu . \equiv . \mathrm{smor}``\mu \mathrel{>\!\!\cdot} \mathrm{smor}``\nu$

***255·17.** $\vdash : \mathrm{Nr}`P \mathrel{>\!\!\cdot} \mathrm{Nr}`Q . \equiv . Q \,\mathrm{less}\, P . \equiv . P, Q \epsilon \Omega . Q \epsilon \mathrm{C\!I}`P_{\mathrm{sm}} .$
$\equiv . P, Q \epsilon \Omega . \exists ! \mathrm{D}`P_s \cap \mathrm{Nr}`Q$

Dem.

$\vdash . *255·13 . *254·46 . \supset \vdash : \mathrm{Nr}`P \mathrel{>\!\!\cdot} \mathrm{Nr}`Q . \equiv . Q \,\mathrm{less}\, P .$ (1)

$[*254·41] \quad \equiv . P, Q \epsilon \Omega . Q \epsilon \mathrm{C\!I}`P_{\mathrm{sm}} .$ (2)

$[*254·12] \quad \equiv . P, Q \epsilon \Omega . \exists ! \mathrm{D}`P_s \cap \mathrm{Nr}`Q$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash . \mathrm{Prop}$

***255·171.** $\vdash :. P \epsilon \Omega . \supset : \mu \mathrel{<\!\!\cdot} \mathrm{Nr}`P . \equiv . \mu \epsilon \mathrm{Nr}``\mathrm{D}`P_s - \iota`\Lambda$

Dem.

$\vdash . *255·14 . \supset \vdash :. \mathrm{Hp} . \supset : \mu \mathrel{<\!\!\cdot} \mathrm{Nr}`P . \equiv . (\exists Q) . \mu = \mathrm{N_0r}`Q . \mathrm{Nr}`Q \mathrel{<\!\!\cdot} \mathrm{Nr}`P .$

$[*255·17] \quad \equiv . (\exists Q) . \mu = \mathrm{N_0r}`Q . Q \epsilon \Omega . \exists ! \mathrm{D}`P_s \cap \mathrm{Nr}`Q .$

$[*152·1] \quad \equiv . (\exists Q, R) . \mu = \mathrm{N_0r}`Q . Q \epsilon \Omega . Q \,\mathrm{smor}\, R . R \epsilon \mathrm{D}`P_s .$

$[*152·35 . *155·16] \equiv . (\exists R) . \mu = \mathrm{Nr}`R . R \epsilon \Omega . R \epsilon \mathrm{D}`P_s . \exists ! \mu .$

$[*253·18 . *37·6] \quad \equiv . \mu \epsilon \mathrm{Nr}``\mathrm{D}`P_s - \iota`\Lambda :. \supset \vdash . \mathrm{Prop}$

***255·172.** $\vdash :. P \epsilon \Omega . \supset :$

$$\mu < \text{Nr}`P . \equiv . (\exists \alpha) . \alpha \subset C`P . \exists ! C`P \cap p`\overleftarrow{P}``\alpha . \mu = \text{Nr}`P \upharpoonright \alpha . \exists ! \mu$$

Dem.

$\vdash . *211{\cdot}703 . *213{\cdot}141 . \supset$

$\vdash : Q \epsilon \text{D}`P_s . \supset . (\exists \alpha) . \alpha \subset C`P . \exists ! C`P \cap p`\overleftarrow{P}``\alpha . Q = P \upharpoonright \alpha \quad (1)$

$\vdash . (1) . *255{\cdot}171 . \supset \vdash : \text{Hp} . \mu < \text{Nr}`P . \supset .$

$(\exists \alpha) . \alpha \subset C`P . \exists ! C`P \cap p`\overleftarrow{P}``\alpha . \mu = \text{Nr}`P \upharpoonright \alpha . \exists ! \mu \quad (2)$

$\vdash . *250{\cdot}653 . *254{\cdot}47 . \supset$

$\vdash : \text{Hp} . \alpha \subset C`P . \exists ! C`P \cap p`\overleftarrow{P}``\alpha . \supset . P \upharpoonright \alpha \text{ less } P .$

$[*255{\cdot}17] \quad \supset . \text{Nr}`P \upharpoonright \alpha < \text{Nr}`P \quad (3)$

$\vdash . (2) . (3) . \supset \vdash . \text{Prop}$

***255·173.** $\vdash :. P \epsilon \Omega . \supset :$

$$\text{Nr}`Q < \text{Nr}`P . \equiv . (\exists \alpha) . \alpha \subset C`P . \exists ! C`P \cap p`\overleftarrow{P}``\alpha . Q \text{ smor } (P \upharpoonright \alpha)$$

Dem.

$\vdash . *255{\cdot}172{\cdot}102 . *155{\cdot}22 . \supset$

$\vdash :. \text{Hp} . \supset : \text{Nr}`Q < \text{Nr}`P . \equiv . (\exists \alpha) . \alpha \subset C`P . \exists ! C`P \cap p`\overleftarrow{P}``\alpha . \text{N}_0\text{r}`Q = \text{Nr}`P \upharpoonright \alpha .$

$[*152{\cdot}35 . *155{\cdot}22] \equiv . (\exists \alpha) . \alpha \subset C`P . \exists ! C`P \cap p`\overleftarrow{P}``\alpha . Q \text{ smor } (P \upharpoonright \alpha) :. \supset \vdash . \text{Prop}$

***255·174.** $\vdash : \text{Nr}`Q < \text{Nr}`P . \equiv . P \epsilon \Omega . \text{Nr}`Q \epsilon \text{Nr}``\text{D}`P_s$

Dem.

$\vdash . *255{\cdot}171{\cdot}102{\cdot}13 . \supset$

$\vdash :. \text{Nr}`Q < \text{Nr}`P . \equiv : P \epsilon \Omega . \text{N}_0\text{r}`Q \epsilon \text{Nr}``\text{D}`P_s - \iota`\Lambda :$

$[*37{\cdot}6 . *155{\cdot}22] \quad \equiv : P \epsilon \Omega : (\exists R) . R \epsilon \text{D}`P_s . \text{N}_0\text{r}`Q = \text{Nr}`R :$

$[*155{\cdot}16] \quad \equiv : P \epsilon \Omega : (\exists R) . R \epsilon \text{D}`P_s . \text{Nr}`Q = \text{Nr}`R :$

$[*37{\cdot}6] \quad \equiv : P \epsilon \Omega . \text{Nr}`Q \epsilon \text{Nr}``\text{D}`P_s :. \supset \vdash . \text{Prop}$

***255·175.** $\vdash : \text{Nr}`Q \leqslant \text{Nr}`P . \equiv . P \epsilon \Omega . \text{Nr}`Q \epsilon \text{Nr}``(\text{D}`P_s \cup \iota`P)$ [*255·174·108]

***255·176.** $\vdash :. \dot{\exists} ! P . \supset : \text{Nr}`Q \leqslant \text{Nr}`P . \equiv . P \epsilon \Omega . \text{Nr}`Q \epsilon \text{Nr}`C`P_s$

[*213·158 . *255·175]

***255·21.** $\vdash : \text{Nr}`P < \text{Nr}`Q . \equiv . P, Q \epsilon \Omega . \text{Rl}`P \cap \text{Nr}`Q = \Lambda$ [*254·51 . *255·17]

This proposition has no analogue in cardinals, because it depends upon *254·4. In cardinals, if $\text{Cl}`\alpha \cap \text{Nc}`\beta = \Lambda$, it does not follow that $\dot{\exists} ! \text{Cl}`\beta \cap \text{Nc}`\alpha$, so that $\text{Nc}`\alpha$ may be neither less than, nor equal to, nor greater than $\text{Nc}`\beta$.

***255·211.** $\vdash :. P, Q \epsilon \Omega . \supset : \exists ! \text{Rl}`P \cap \text{Nr}`Q . \exists ! \text{Rl}`Q \cap \text{Nr}`P . \equiv . \text{Nr}`P = \text{Nr}`Q$

[*254·45]

This proposition is the ordinal analogue of the Schröder-Bernstein theorem. If P and Q are series which may be not well-ordered, the proposition fails. Thus *e.g.* the series of rationals is like the series of proper fractions, which is

a part of the series of rationals >0 and $\leqslant 1$, and this latter series is part of the series of rationals, but is not similar to the series of rationals, since it has a last term, which the series of rationals has not.

***255·22.** $\vdash : P, Q \in \Omega \,.\, \exists \,!\, \mathrm{Rl}`P \cap \mathrm{Nr}`Q \,.\equiv.\, \mathrm{Nr}`P \geqslant \mathrm{Nr}`Q$

***255·221.** $\vdash :. \mathrm{Nr}`P \geqslant \mathrm{Nr}`Q \,.\equiv:\, P, Q \in \Omega : (\exists R) \,.\, R \subset P \,.\, R \,\mathrm{smor}\, Q$

***255·222.** $\vdash : Q \subset P \,.\, P, Q \in \Omega \,.\supset.\, \mathrm{Nr}`P \geqslant \mathrm{Nr}`Q$

***255·23.** $\vdash : \mathrm{Nr}`P \geqslant \mathrm{Nr}`Q \,.\, \mathrm{Nr}`Q \geqslant \mathrm{Nr}`P \,.\equiv.\, P, Q \in \Omega \,.\, \mathrm{Nr}`P = \mathrm{Nr}`Q$

***255·24.** $\vdash : \mu \geqslant \nu \,.\equiv.\, (\exists P, Q) \,.\, \mu = \mathrm{N_0r}`P \,.\, \nu = \mathrm{N_0r}`Q \,.\, \mathrm{Nr}`P \geqslant \mathrm{Nr}`Q$

***255·241.** $\vdash : \mu \geqslant \nu \,.\equiv.\, (\exists P, Q) \,.\, \mu = \mathrm{N_0r}`P \,.\, \nu = \mathrm{N_0r}`Q \,.\, P, Q \in \Omega \,.\, \exists \,!\, \mathrm{Rl}`P \cap \mathrm{Nr}`Q$

***255·242.** $\vdash :. \mu, \nu \in \mathrm{NO} \,.\supset: \mu \geqslant \nu \,.\equiv.\, (\exists P, Q) \,.\, P \in \mu \,.\, Q \in \nu \,.\, \exists \,!\, \mathrm{Rl}`P \cap \mathrm{Nr}`Q$

***255·243.** $\vdash :. \mu \geqslant \nu \,.\equiv:$

$(\exists P, Q) : P, Q \in \Omega \,.\, \mu = \mathrm{N_0r}`P \,.\, \nu = \mathrm{N_0r}`Q : (\exists R) \,.\, R \subset P \,.\, R \,\mathrm{smor}\, Q$

***255·244.** $\vdash :. \mu, \nu \in \mathrm{N_0O} \,.\supset:$

$\mu \geqslant \nu \,.\equiv.\, \mathrm{smor}``\mu \geqslant \nu \,.\equiv.\, \mu \geqslant \mathrm{smor}``\nu \,.\equiv.\, \mathrm{smor}``\mu \geqslant \mathrm{smor}``\nu$

***255·25.** $\vdash : \mu \geqslant \nu \,.\, \nu \geqslant \mu \,.\equiv.\, \mu, \nu \in \mathrm{N_0O} \,.\, \mathrm{smor}``\mu = \mathrm{smor}``\nu$

***255·27.** $\vdash : \mathrm{Nr}`P < \mathrm{Nr}`Q \,.\equiv.\, \mathrm{Nr}`P \leqslant \mathrm{Nr}`Q \,.\, \mathrm{Nr}`P \neq \mathrm{Nr}`Q$

***255·28.** $\vdash : \mathrm{Nr}`P > \mathrm{Nr}`Q \,.\equiv.\, \mathrm{Nr}`P \geqslant \mathrm{Nr}`Q \,.\, \sim (\mathrm{Nr}`Q \geqslant \mathrm{Nr}`P) \,.$

$\equiv.\, P, Q \in \Omega \,.\, \sim (\mathrm{Nr}`Q \geqslant \mathrm{Nr}`P)$ [*255·13·22·21]

***255·281.** $\vdash : \mu > \nu \,.\equiv.\, \mu \geqslant \nu \,.\, \sim (\nu \geqslant \mu) \,.\equiv.\, \mu, \nu \in \mathrm{N_0O} \,.\, \sim (\nu \geqslant \mu)$ [*255·114]

***255·29.** $\vdash : \mathrm{Nr}`P < \mathrm{Nr}`Q \,.\equiv.\, \mathrm{Nr}`P \leqslant \mathrm{Nr}`Q \,.\, \sim (\mathrm{Nr}`Q \leqslant \mathrm{Nr}`P) \,.$

$\equiv.\, P, Q \in \Omega \,.\, \sim (\mathrm{Nr}`Q \leqslant \mathrm{Nr}`P)$ [*255·115]

***255·291.** $\vdash : \mu < \nu \,.\equiv.\, \mu \leqslant \nu \,.\, \sim (\nu \leqslant \mu) \,.\equiv.\, \mu, \nu \in \mathrm{N_0O} \,.\, \sim (\nu \leqslant \mu)$ [*255·114]

In the following proposition, we employ an abbreviation which is justified by its convenience, namely we put

$$(\exists \varpi) \,.\, \varpi \in \mathrm{NO} \cup \iota`\dot{1} \,.\, \mathrm{Nr}`P = \mathrm{Nr}`Q \dotplus \varpi$$

instead of

$$(\exists \varpi) \,.\, \varpi \in \mathrm{NO} \,.\, \mathrm{Nr}`P = \mathrm{Nr}`Q \dotplus \varpi \,.\vee.\, \mathrm{Nr}`P = \mathrm{Nr}`Q \dotplus \dot{1}.$$

In virtue of *51·239, these two expressions would be equivalent if $\dot{1}$ had any independent meaning; but as $\dot{1}$ is only significant as an addendum, *51·239 cannot be applied. We will, however, adopt the following definitions:

***255·298.** $(\exists \varpi) \,.\, \varpi \in \kappa \cup \iota`\dot{1} \,.\, f(\mu \dotplus \varpi) \,.=:\, (\exists \varpi) \,.\, \varpi \in \kappa \,.\, f(\mu \dotplus \varpi) \,.\vee.\, f(\mu \dotplus \dot{1})$ Df

***255·299.** $\varpi \in \kappa \cup \iota`\dot{1} \,.\supset_\varpi.\, f(\mu \dotplus \varpi) \,.=:\, \varpi \in \kappa \,.\supset_\varpi.\, f(\mu \dotplus \varpi) : f(\mu \dotplus \dot{1})$ Df

These definitions enable us to state many propositions, in which $\dot{1}$ occurs, as though $\dot{1}$ were an ordinal number.

***255·3.** $\vdash :. \mathrm{Nr}'P \geqslant \mathrm{Nr}'Q . \equiv : P, Q \in \Omega : (\exists \varpi) . \varpi \in \mathrm{NO} \cup \iota'\dot{1} . \mathrm{Nr}'P = \mathrm{Nr}'Q \dot{+} \varpi$

Dem.

$\vdash . *255·175 . *253·471 . \supset$

$\vdash :. \mathrm{Nr}'P \geqslant \mathrm{Nr}'Q . \equiv : P \in \Omega : (\exists \varpi) . \mathrm{Nr}'Q \dot{+} \varpi = \mathrm{Nr}'P . \mathbf{v} . \mathrm{Nr}'Q \dot{+} \dot{1} = \mathrm{Nr}'P :$

[*251·132·26] $\equiv : P \in \Omega : (\exists \varpi) . \mathrm{Nr}'Q, \varpi \in \mathrm{NO} . \mathrm{Nr}'Q \dot{+} \varpi = \mathrm{Nr}'P . \mathbf{v} .$
$\mathrm{Nr}'Q \in \mathrm{NO} . \mathrm{Nr}'Q \dot{+} \dot{1} = \mathrm{Nr}'P :$

[*251·1·111] $\equiv : P, Q \in \Omega : (\exists \varpi) . \varpi \in \mathrm{NO} . \mathrm{Nr}'Q \dot{+} \varpi = \mathrm{Nr}'P . \mathbf{v} .$
$\mathrm{Nr}'Q \dot{+} \dot{1} = \mathrm{Nr}'P :$

[(*255·298)] $\equiv : P, Q \in \Omega : (\exists \varpi) . \varpi \in \mathrm{NO} \cup \iota'\dot{1} . \mathrm{Nr}'P = \mathrm{Nr}'Q \dot{+} \varpi :. \supset \vdash . \mathrm{Prop}$

***255·31** $\vdash :. \mu \geqslant \nu . \equiv : \mu, \nu \in \mathrm{N_0O} : (\exists \varpi) . \varpi \in \mathrm{NO} \cup \iota'\dot{1} . \mu = \nu \dot{+} \varpi$
[*255·3·14]

***255·32.** $\vdash :. \nu, \varpi \in \mathrm{N_0O} . \supset : \nu \dot{+} \varpi > \nu . \equiv . \varpi \neq 0_r$

Dem.

$\vdash . *253·44 . \qquad \supset \vdash : \mathrm{Hp} . \varpi \neq 0_r . \supset . \nu \dot{+} \dot{1} \neq \nu \qquad (1)$

$\vdash . *255·31 . \qquad \supset \vdash : \mathrm{Hp} . \supset . \nu \dot{+} \varpi \geqslant \nu \qquad (2)$

$\vdash . (1) . (2) . *255·141 . \supset \vdash : \mathrm{Hp} . \varpi \neq 0_r . \supset . \nu \dot{+} \varpi > \nu \qquad (3)$

$\vdash . *255·141 . \qquad \supset \vdash : \mathrm{Hp} . \nu \dot{+} \varpi > \nu . \supset . \nu \dot{+} \varpi \neq \mathrm{smor}''\nu .$

[*180·6] $\qquad \supset . \varpi \neq 0_r \qquad (4)$

$\vdash . (3) . (4) . \supset \vdash . \mathrm{Prop}$

***255·321.** $\vdash :. \nu \in \mathrm{N_0O} . \supset : \nu \neq 0_r . \equiv . \nu \dot{+} \dot{1} > \nu$

Dem.

$\vdash . *253·45 . \qquad \supset \vdash : \mathrm{Hp} . \nu \neq 0_r . \supset . \nu \dot{+} \dot{1} \neq \nu \qquad (1)$

$\vdash . *255·31 . \qquad \supset \vdash : \mathrm{Hp} . \supset . \nu \dot{+} \dot{1} \geqslant \nu \qquad (2)$

$\vdash . (1) . (2) . *255·141 . \supset \vdash : \mathrm{Hp} . \nu \neq 0_r . \supset . \nu \dot{+} \dot{1} > \nu \qquad (3)$

$\vdash . *255·141 . \qquad \supset \vdash : \mathrm{Hp} . \nu \dot{+} \dot{1} > \nu . \supset . \nu \dot{+} \dot{1} \neq \mathrm{smor}''\nu .$

[*161·2] $\qquad \supset . \nu \neq 0_r \qquad (4)$

$\vdash . (3) . (4) . \supset \vdash . \mathrm{Prop}$

***255·33.** $\vdash :. \mu > \nu . \equiv :$
$\mu, \nu \in \mathrm{N_0O} : (\exists \varpi) . \varpi \in \mathrm{NO} - \iota'0_r . \mu = \nu \dot{+} \varpi . \mathbf{v} . \nu \neq 0_r . \mu = \nu \dot{+} \dot{1}$

Dem.

$\vdash . *255·31 . \supset$

$\vdash :. \mu > \nu . \equiv : \mu, \nu \in \mathrm{N_0O} : (\exists \varpi) . \varpi \in \mathrm{NO} . \mu = \nu \dot{+} \varpi . \mu > \nu . \mathbf{v} . \mu = \nu \dot{+} \dot{1} . \mu > \nu :$

[*255·32·321]

$\equiv : \mu, \nu \in \mathrm{N_0O} : (\exists \varpi) . \varpi \in \mathrm{NO} - \iota'0_r . \mu = \nu \dot{+} \varpi . \mathbf{v} . \nu \neq 0_r . \mu = \nu \dot{+} \dot{1} :. \supset \vdash . \mathrm{Prop}$

***255·4.** $\vdash : \mu \geqslant \nu \,.\, \nu \geqslant \varpi \,.\, \supset .\, \mu \geqslant \varpi$

***255·41.** $\vdash : \mu \leqslant \nu \,.\, \nu \leqslant \varpi \,.\, \supset .\, \mu \leqslant \varpi$

***255·42.** $\vdash .\, \sim (\mu > \mu) \,.\, \sim (\mu < \mu)$

***255·43.** $\vdash : \mu \geqslant \nu \,.\, \sim (\mu \geqslant \varpi) \,.\, \supset .\, \sim (\nu \geqslant \varpi)$

***255·431.** $\vdash : \mu \geqslant \nu \,.\, \varpi \in \mathrm{N_0O} \,.\, \sim (\mu \geqslant \varpi) \,.\, \supset .\, \varpi > \nu$ [*255·43·114]

***255·44.** $\vdash : \nu \geqslant \varpi \,.\, \sim (\mu \geqslant \varpi) \,.\, \supset .\, \sim (\mu \geqslant \nu)$

***255·441.** $\vdash : \nu \geqslant \varpi \,.\, \mu \in \mathrm{N_0O} \,.\, \sim (\mu \geqslant \varpi) \,.\, \supset .\, \nu > \mu$ [*255·44·114]

***255·45.** $\vdash : \mu \geqslant \nu \,.\, \nu > \varpi \,.\, \supset .\, \mu > \varpi$

***255·46.** $\vdash : \mu > \nu \,.\, \nu \geqslant \varpi \,.\, \supset .\, \mu > \varpi$

***255·47.** $\vdash : \mu > \nu \,.\, \nu > \varpi \,.\, \supset .\, \mu > \varpi$

***255·471.** $\vdash : \mu < \nu \,.\, \nu < \varpi \,.\, \supset .\, \mu < \varpi$

***255·482.** $\vdash : \mu \geqslant \nu \,.\, \equiv .\, \mu, \nu \in \mathrm{N_0O} \,.\, \sim (\nu > \mu)$

***255·483.** $\vdash : \mu \leqslant \nu \,.\, \equiv .\, \mu, \nu \in \mathrm{N_0O} \,.\, \sim (\nu < \mu)$

***255·5.** $\vdash : \mu \in \mathrm{N_0O} \,.\, \equiv .\, \mu \geqslant 0_r$

Dem.

$\vdash .\, *255{\cdot}31 \,.\, \supset \vdash :.\, \mu \geqslant 0_r \,.\, \equiv : \mu \in \mathrm{N_0O} : (\exists \varpi) \,.\, \varpi \in \mathrm{NO} \cup \iota\text{‘}\dot{1} \,.\, \mu = 0_r \dot{+} \varpi :$

[*180·61] $\equiv : \mu \in \mathrm{N_0O} :.\, \supset \vdash .\, \mathrm{Prop}$

***255·51.** $\vdash : \mu \in \mathrm{N_0O} - \iota\text{‘}0_r \,.\, \equiv .\, \mu > 0_r$ [*255·141·5 . *153·15]

***255·52.** $\vdash : P \in \Omega - \iota\text{‘}\dot{\Lambda} \,.\, \equiv .\, \mathrm{Nr}\text{‘}P \geqslant 2_r$

Dem.

$\vdash .\, *250{\cdot}13 \,.\, \supset \vdash : P \in \Omega - \iota\text{‘}\dot{\Lambda} \,.\, \supset .\, \mathrm{E}\,!\, B\text{‘}P \,.$

[*93·101] $\supset .\, (\exists y) \,.\, (B\text{‘}P)\, Py \,.\, B\text{‘}P \neq y \,.$

[*56·11.*55·3] $\supset .\, (\exists y) \,.\, (B\text{‘}P) \downarrow y \in 2_r \cap \mathrm{Rl}\text{‘}P \,.$

[*13·195] $\supset .\, \exists\,!\, 2_r \cap \mathrm{Rl}\text{‘}P \,.$

[*255·22] $\supset .\, \mathrm{Nr}\text{‘}P \geqslant 2_r$ (1)

$\vdash .\, *255{\cdot}22 \,.\, \supset \vdash : \mathrm{Nr}\text{‘}P \geqslant 2_r \,.\, \supset .\, P \in \Omega \,.\, \exists\,!\, 2_r \cap \mathrm{Rl}\text{‘}P \,.$

[*61·361] $\supset .\, P \in \Omega - \iota\text{‘}\dot{\Lambda}$ (2)

$\vdash .\, (1) \,.\, (2) \,.\, \supset \vdash .\, \mathrm{Prop}$

***255·53.** $\vdash : \mu \in \mathrm{N_0O} - \iota\text{‘}0_r \,.\, \equiv .\, \mu \geqslant 2_r$ [*255·52]

***255·54.** $\vdash :.\, 2_r \geqslant \mu \,.\, \equiv : \mu = 0_r \,.\, \vee \,.\, \mu = 2_r$

Dem.

$\vdash .\, *255{\cdot}53 \,.\, \mathrm{Transp} \,.\, *255{\cdot}281 \,.\, \supset \vdash : 2_r > \mu \,.\, \equiv .\, \mu = 0$ (1)

$\vdash .\, (1) \,.\, *255{\cdot}105 \,.\, \supset \vdash .\, \mathrm{Prop}$

***255·55.** $\vdash : \mu > 2_r . \equiv . \mu \in N_0O - \iota'0_r - \iota'2_r$

Dem.

$\vdash . *255·54 . \text{Transp} . *255·281 . \supset$

$\vdash : \mu > 2_r . \equiv . \mu \in N_0O . \mu \neq 0_r . \mu \neq 2_r : \supset \vdash . \text{Prop}$

***255·56.** $\vdash : R \in \Omega . \text{Nr}'P > \text{Nr}'Q . \supset . \text{Nr}'R \dot{+} \text{Nr}'P > \text{Nr}'R \dot{+} \text{Nr}'Q$

Dem.

$\vdash . *255·3 . \supset \vdash :. \text{Hp} . \supset : P, Q, R \in \Omega : (\exists \varpi) . \varpi \in \text{NO} \cup \iota'\dot{1} . \text{Nr}'P = \text{Nr}'Q \dot{+} \varpi :$

[*180·56]

$\supset : P, Q, R \in \Omega : (\exists \varpi) . \varpi \in \text{NO} \cup \iota'\dot{1} . \text{Nr}'R \dot{+} \text{Nr}'P = (\text{Nr}'R \dot{+} \text{Nr}'Q) \dot{+} \varpi :$

[*255·31.*251·26] $\supset : \text{Nr}'R \dot{+} \text{Nr}'P > \text{Nr}'R \dot{+} \text{Nr}'Q :. \supset \vdash . \text{Prop}$

***255·561.** $\vdash : \gamma \in N_0O . \alpha > \beta . \supset . \gamma \dot{+} \alpha > \gamma \dot{+} \beta$ [*255·56]

***255·562.** $\vdash : R \in \Omega . \text{Nr}'P \geqslant \text{Nr}'Q . \supset . \text{Nr}'R \dot{+} \text{Nr}'P \geqslant \text{Nr}'R \dot{+} \text{Nr}'Q$

Dem.

$\vdash . *180·3 . \supset \vdash : \text{Nr}'P = \text{Nr}'Q . \supset . \text{Nr}'R \dot{+} \text{Nr}'P = \text{Nr}'R \dot{+} \text{Nr}'Q$ (1)

$\vdash . (1) . *255·108·56 . \supset$

$\vdash :. \text{Hp} . \supset : \text{Nr}'R \dot{+} \text{Nr}'P > \text{Nr}'R \dot{+} \text{Nr}'Q . \vee . \text{Nr}'R \dot{+} \text{Nr}'P = \text{Nr}'R \dot{+} \text{Nr}'Q :$

[*255·108] $\supset : \text{Nr}'R \dot{+} \text{Nr}'P \geqslant \text{Nr}'R \dot{+} \text{Nr}'Q :. \supset \vdash . \text{Prop}$

***255·563.** $\vdash : \gamma \in N_0O . \alpha \geqslant \beta . \supset . \gamma \dot{+} \alpha \geqslant \gamma \dot{+} \beta$ [*255·562]

***255·564.** $\vdash : P, Q, R \in \Omega . \text{Nr}'R \dot{+} \text{Nr}'P = \text{Nr}'R \dot{+} \text{Nr}'Q . \supset . \text{Nr}'P = \text{Nr}'Q$

Dem.

$\vdash . *255·42 . \supset \vdash : \text{Hp} . \supset . \sim(\text{Nr}'R \dot{+} \text{Nr}'P > \text{Nr}'R \dot{+} \text{Nr}'Q) .$

[*255·56.Transp] $\supset . \sim(\text{Nr}'P > \text{Nr}'Q)$ (1)

Similarly $\vdash : \text{Hp} . \supset . \sim(\text{Nr}'Q > \text{Nr}'P)$ (2)

$\vdash . (1) . (2) . *255·113 . \supset \vdash . \text{Prop}$

This proposition establishes the uniqueness of subtraction from the end. Owing to the fact that ordinal addition is not commutative, we have to distinguish "subtraction from the end" from "subtraction from the beginning." They may be called *terminal* and *initial* subtraction respectively. Thus by the above proposition, *terminal* subtraction among ordinals is unique. This does not hold in general for *initial* subtraction among ordinals.

***255·565.** $\vdash : \alpha, \beta, \gamma \in N_0O . \gamma \dot{+} \alpha = \gamma \dot{+} \beta . \supset . \alpha = \text{smor}``\beta$ [*255·564]

The above proposition is still true if we put $\alpha = \beta$ instead of $\alpha = \text{smor}``\beta$ in the conclusion, but in that case it is only significant when α and β are of the same type, whereas in the above form it is free from this limitation.

***255·57.** $\vdash : P, Q \in \Omega - \iota‘\Lambda . \supset . Q \,\mathrm{less}\, (P \times Q) . \mathrm{Nr}‘Q < \mathrm{Nr}‘P \dot{\times} \mathrm{Nr}‘Q$

Dem.

$\vdash . *250\cdot13 . \quad \supset \vdash : \mathrm{Hp} . \supset . \mathrm{E} ! B‘P .$ (1)

$[*165\cdot251] \quad \supset . Q \,\mathrm{smor}\, Q \underset{;}{\downarrow} (B‘P)$ (2)

$\vdash (1) . *166\cdot1 . \supset \vdash : \mathrm{Hp} . \supset . Q \underset{;}{\downarrow} (B‘P) \mathrel{\dot\subset} P \times Q$ (3)

$\vdash . (1) . *93\cdot101 . \supset \vdash : \mathrm{Hp} . \supset . (\exists x) . (B‘P) P x$ (4)

$\vdash . *166\cdot113 . \supset \vdash : (B‘P) P x . R \in C‘Q \underset{;}{\downarrow} (B‘P) . y \in C‘Q . \supset . R (P \times Q) (y \downarrow x)$ (5)

$\vdash . (5) . (4) . *33\cdot24 . *166\cdot12 . *113\cdot106 . \supset$

$\vdash :. \mathrm{Hp} . \supset : (\exists x, y) : R \in C‘Q \underset{;}{\downarrow} (B‘P) . \supset_R . R (P \times Q) (y \downarrow x) : y \downarrow x \in C‘(P \times Q)$ (6)

$\vdash . (2) . (3) . (6) . \supset \vdash : \mathrm{Hp} . \supset .$

$Q \underset{;}{\downarrow} (B‘P) \,\mathrm{smor}\, Q . Q \underset{;}{\downarrow} (B‘P) \mathrel{\dot\subset} P \times Q . \exists ! C‘(P \times Q) \cap p‘\overleftarrow{P \times Q}“C‘Q \underset{;}{\downarrow} (B‘P) .$

$[*254\cdot54] \supset . Q \,\mathrm{less}\, (P \times Q)$ (7)

$\vdash . (7) . *255\cdot17 . \supset \vdash . \mathrm{Prop}$

***255·571.** $\vdash : \alpha, \beta \in \mathrm{N_0O} - \iota‘0_r . \supset . \beta < \alpha \dot{\times} \beta \quad [*255\cdot57]$

***255·572.** $\vdash : P, Q \in \Omega - \iota‘\Lambda . \mathrm{E} ! B‘\breve{P} . \supset . P \,\mathrm{less}\, (P \times Q) . \mathrm{Nr}‘P < \mathrm{Nr}‘P \dot{\times} \mathrm{Nr}‘Q$

Dem.

$\vdash . *250\cdot13 . \quad \supset \vdash : \mathrm{Hp} . \supset . \mathrm{E} ! B‘Q .$ (1)

$[*166\cdot111] \quad \supset . (B‘Q) \downarrow ; P \mathrel{\dot\subset} P \times Q$ (2)

$\vdash . *151\cdot64 . (1) . \supset \vdash : \mathrm{Hp} . \supset . (B‘Q) \downarrow ; P \,\mathrm{smor}\, P$ (3)

$\vdash . *202\cdot511 . \supset \vdash :. \mathrm{Hp} . \supset : B‘\breve{P} \in p‘\overleftarrow{P}“\mathrm{D}‘P :$

$[*166\cdot111] \quad \supset : x \in \mathrm{D}‘P . y \in \text{Ɑ}‘Q . \supset . \{(B‘Q) \downarrow x\} (P \times Q) \{y \downarrow (B‘\breve{P})\}$ (4)

$\vdash . *202\cdot511 . \supset \vdash :. \mathrm{Hp} . \supset : B‘Q \in p‘\overrightarrow{Q}“\text{Ɑ}‘Q :$

$[*166\cdot111] \quad \supset : x = B‘\breve{P} . y \in \text{Ɑ}‘Q . \supset . \{(B‘Q) \downarrow x\} (P \times Q) \{y \downarrow (B‘\breve{P})\}$ (5)

$\vdash . (4) . (5) . \supset \vdash :. \mathrm{Hp} . \supset : x \in C‘P . y \in \text{Ɑ}‘Q . \supset . \{(B‘Q) \downarrow x\} (P \times Q) \{y \downarrow (B‘\breve{P})\} :$

$[*150\cdot22] \quad \supset : M \in C‘(B‘Q) \downarrow ; P . y \in \text{Ɑ}‘Q . \supset . M (P \times Q) \{y \downarrow (B‘\breve{P})\} :$

$[\mathrm{Hp} . *33\cdot24 . *166\cdot111]$

$\supset : (\exists N) : N \in C‘(P \times Q) : M \in C‘(B‘Q) \downarrow ; P . \supset_M . M (P \times Q) N$ (6)

$\vdash . (2) . (3) . (6) . *254\cdot54 . \supset \vdash : \mathrm{Hp} . \supset . P \,\mathrm{less}\, (P \times Q)$ (7)

$\vdash . (7) . *255\cdot17 . \supset \vdash . \mathrm{Prop}$

***255·573.** $\vdash :. \alpha, \beta \in \mathrm{N_0O} - \iota‘0_r : (\exists \gamma) . \gamma \in \mathrm{NO} - \iota‘0_r \cup \iota‘\dot{1} . \alpha = \gamma \dot{+} \dot{1} : \supset . \alpha < \alpha \dot{\times} \beta$

Dem.

$\vdash . *204\cdot483 . \supset \vdash : \mathrm{Hp} . \supset . (\exists P, Q) . \alpha = \mathrm{N_0r}‘P . \beta = \mathrm{N_0r}‘Q . \exists ! B‘\breve{P}$ (1)

$\vdash . (1) . *255\cdot572 . \supset \vdash . \mathrm{Prop}$

***255·58.** $\vdash : \gamma \in \mathrm{N_0O} - \iota`0_r \,.\, \alpha > \beta \,.\supset .\, \alpha \dot{\times} \gamma > \beta \dot{\times} \gamma$

Dem.

$\vdash .\, *255{\cdot}31 \,.\supset$

$\vdash :.\, \mathrm{Hp} \,.\supset : (\exists \varpi) \,.\, \varpi \in \mathrm{NO} - \iota`0_r \,.\, \alpha = \beta \dot{+} \varpi \,.\vee .\, \beta \neq 0_r \,.\, \alpha = \beta \dot{+} \dot{1}$ (1)

$\vdash .\, *184{\cdot}35 \,. \quad \supset \vdash : \alpha = \beta \dot{+} \varpi \,.\supset .\, \alpha \dot{\times} \gamma = (\beta \dot{\times} \gamma) \dot{+} (\varpi \dot{\times} \gamma)$ (2)

$\vdash .\, *184{\cdot}16 \,. \quad \supset \vdash : \mathrm{Hp} \,.\, \varpi \neq 0_r \,.\supset .\, \varpi \dot{\times} \gamma \neq 0_r$ (3)

$\vdash .\, (2) \,.\, (3) \,.\, *255{\cdot}32 \,.\supset \vdash : \mathrm{Hp} \,.\, \varpi \in \mathrm{NO} - \iota`0_r \,.\, \alpha = \beta \dot{+} \varpi \,.\supset .\, \alpha \dot{\times} \gamma > \beta \dot{\times} \gamma$ (4)

$\vdash .\, *184{\cdot}41 \,. \quad \supset \vdash : \mathrm{Hp} \,.\, \alpha = \beta \dot{+} \dot{1} \,.\supset .\, \alpha \dot{\times} \gamma = (\beta \dot{\times} \gamma) \dot{+} \gamma \,.$

[*255·32] $\quad \supset .\, \alpha \dot{\times} \gamma > \beta \dot{\times} \gamma$ (5)

$\vdash .\, (1) \,.\, (4) \,.\, (5) \,.\supset \vdash .\, \mathrm{Prop}$

***255·581.** $\vdash : P \in \Omega \,.\, \mathrm{E} \,!\, B`\breve{P} \,.\, Q \,\mathrm{less}\, R \,.\supset .$

$P \times Q \,\mathrm{less}\, P \times R \,.\, \mathrm{Nr}`P \dot{\times} \mathrm{Nr}`Q < \mathrm{Nr}`P \dot{\times} \mathrm{Nr}`R$

Dem.

$\vdash .\, *254{\cdot}55 \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, (\exists S) \,.\, S \,\mathrm{smor}\, Q \,.\, S \subset R \,.\, \exists \,!\, C`R \cap p`\overleftarrow{R}``C`S$ (1)

$\vdash .\, *166{\cdot}11 \,.\supset \vdash : S \subset R \,.\supset .\, P \times S \subset P \times R$ (2)

$\vdash .\, *166{\cdot}23 \,.\supset \vdash : S \,\mathrm{smor}\, Q \,.\supset .\, P \times S \,\mathrm{smor}\, P \times Q$ (3)

$\vdash .\, *202{\cdot}524 \,.\, *40{\cdot}53 \,.\supset \vdash :.\, \mathrm{Hp} \,.\, z \in C`P \,.\, w \in C`S \,.\, y \in C`R \cap p`\overleftarrow{R}``C`S \,.\supset :$

$zP(B`\breve{P}) \,.\vee .\, z = B`\breve{P} : wRy :$

[*166·113] $\quad \supset : (w \downarrow z)(P \times R)\{y \downarrow (B`\breve{P})\}$ (4)

$\vdash .\, (4) \,.\, *166{\cdot}111 \,.\supset \vdash :.\, \mathrm{Hp} \,.\, y \in C`R \cap p`\overleftarrow{R}``C`S \,.\supset :$

$M \in C`(P \times S) \,.\supset_M .\, M(P \times R)\{y \downarrow (B`\breve{P})\}$ (5)

$\vdash .\, (5) \,.\, *10{\cdot}28 \,.\supset \vdash :.\, \mathrm{Hp} \,.\, \exists \,!\, C`R \cap p`\overleftarrow{R}``C`S \,.\supset :$

$(\exists N) : N \in C`(P \times R) : M \in C`(P \times S) \,.\supset_M .\, M(P \times R)N$ (6)

$\vdash .\, (2) \,.\, (3) \,.\, (6) \,.\supset \vdash :.\, \mathrm{Hp} \,.\, S \,\mathrm{smor}\, Q \,.\, S \subset R \,.\, \exists \,!\, C`R \cap p`\overleftarrow{R}``C`S \,.\supset :$

$(P \times S) \,\mathrm{smor}\, (P \times Q) \,.\, P \times S \subset P \times R \,.\, \exists \,!\, C`(P \times R) \cap p`\overleftarrow{P \times R}``C`(P \times S) :$

[*254·54] $\quad \supset .\, P \times Q \,\mathrm{less}\, P \times R$ (7)

$\vdash .\, (1) \,.\, (7) \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, P \times Q \,\mathrm{less}\, P \times R$ (8)

$\vdash .\, (8) \,.\, *255{\cdot}17 \,.\supset \vdash .\, \mathrm{Prop}$

***255·582.** $\vdash :.\, \alpha \in \mathrm{N_0O} : (\exists \delta) \,.\, \delta \in \mathrm{NO} - \iota`0_r \cup \iota`\dot{1} \,.\, \alpha = \delta \dot{+} \dot{1} : \beta < \gamma : \supset .$

$\alpha \dot{\times} \beta < \alpha \dot{\times} \gamma$ [*255·581 . *204·483]

***255·59.** $\vdash : \alpha, \beta, \gamma \in \mathrm{N_0O} \,.\, \gamma \neq 0_r \,.\, \alpha \dot{\times} \gamma = \beta \dot{\times} \gamma \,.\supset .\, \alpha = \mathrm{smor}``\beta$

Dem.

$\vdash .\, *255{\cdot}58 \,.\, \mathrm{Transp} \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, \sim(\alpha > \beta) \,.\, \sim(\alpha < \beta) \,.$

[*255·112] $\quad \supset .\, \alpha = \mathrm{smor}``\beta : \supset \vdash .\, \mathrm{Prop}$

This proposition establishes the uniqueness of *terminal* division, *i.e.* division by an end-factor. *Initial* division (*i.e.* division by a beginning-factor) is only unique if the divisor is of the form $\delta \dot{+} \dot{1}$.

***255·591.** $\vdash :. \alpha, \beta, \gamma \in \mathrm{N_0O} : (\exists \delta) . \delta \in \mathrm{NO} - \iota`0_r \cup \iota`\dot{1} . \alpha = \delta \dot{+} \dot{1} :$
$\alpha \dot{\times} \beta = \alpha \dot{\times} \gamma : \supset . \beta = \mathrm{smor}``\gamma$ [*255·582·112]

***255·6.** $\vdash : \mathrm{Nr}`P > \mathrm{Nr}`Q . \supset . \dot{1} \dot{+} \mathrm{Nr}`P > \dot{1} \dot{+} \mathrm{Nr}`Q$

Dem.

$\vdash . *255·33 . \supset \vdash :. \mathrm{Hp} . \supset : (\exists \varpi) . \varpi \in \mathrm{NO} - \iota`0_r . \mathrm{Nr}`P = \mathrm{Nr}`Q \dot{+} \varpi . \mathbf{v} .$
$\mathrm{Nr}`P \neq 0_r . \mathrm{Nr}`P = \mathrm{Nr}`Q \dot{+} \dot{1} :$

[*181·55] $\supset : (\exists \varpi) . \varpi \in \mathrm{NO} - \iota`0_r . \dot{1} \dot{+} \mathrm{Nr}`P = (\dot{1} \dot{+} \mathrm{Nr}`Q) \dot{+} \varpi . \mathbf{v} .$
$\mathrm{Nr}`P \neq 0_r . \dot{1} \dot{+} \mathrm{Nr}`P = (\dot{1} \dot{+} \mathrm{Nr}`Q) \dot{+} \dot{1} :$

[*255·33] $\supset : \dot{1} \dot{+} \mathrm{Nr}`P > \dot{1} \dot{+} \mathrm{Nr}`Q :. \supset \vdash . \mathrm{Prop}$

***255·601.** $\vdash : \mathrm{Nr}`P > \mathrm{Nr}`Q . \equiv . \dot{1} \dot{+} \mathrm{Nr}`P > \dot{1} \dot{+} \mathrm{Nr}`Q$

Dem.

$\vdash . *255·6 \dfrac{Q, P}{P, Q} . *255·103 . \supset$

$\vdash : \mathrm{Nr}`P < \mathrm{Nr}`Q . \supset . \dot{1} \dot{+} \mathrm{Nr}`P < \dot{1} \dot{+} \mathrm{Nr}`Q$ (1)

$\vdash . (1) . *255·108 . \supset \vdash : \mathrm{Nr}`P \leqslant \mathrm{Nr}`Q . \supset . \dot{1} \dot{+} \mathrm{Nr}`P \leqslant \dot{1} \dot{+} \mathrm{Nr}`Q$ (2)

$\vdash . (2) . \mathrm{Transp} . *251·142 . \supset$

$\vdash : \dot{1} \dot{+} \mathrm{Nr}`P, \dot{1} \dot{+} \mathrm{Nr}`Q \in \mathrm{NO} . \sim (\dot{1} \dot{+} \mathrm{Nr}`P \leqslant \dot{1} \dot{+} \mathrm{Nr}`Q) . \supset .$
$\mathrm{Nr}`P, \mathrm{Nr}`Q \in \mathrm{NO} . \sim (\mathrm{Nr}`P \leqslant \mathrm{Nr}`Q)$ (3)

$\vdash . (3) . *255·281 . \supset \vdash : \dot{1} \dot{+} \mathrm{Nr}`P > \dot{1} \dot{+} \mathrm{Nr}`Q . \supset . \mathrm{Nr}`P > \mathrm{Nr}`Q$ (4)

$\vdash . (4) . *255·6 . \supset \vdash . \mathrm{Prop}$

***255·61.** $\vdash : Q, R \in \Omega . \mathrm{Nr}`P = \mathrm{Nr}`Q \dot{+} \mathrm{Nr}`R . \mathrm{C}`R_1 = \mathrm{C}`R . \mathrm{E} ! B`\breve{R} . \supset .$
$\mathrm{Nr}`P \dot{+} \dot{1} > \mathrm{Nr}`Q \dot{+} \dot{1}$

Dem.

$\vdash . *253·57 . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{Nr}`P \dot{+} \dot{1} = \mathrm{Nr}`Q \dot{+} \dot{1} \dot{+} \mathrm{Nr}`R .$

[*255·32] $\supset . \mathrm{Nr}`P \dot{+} \dot{1} > \mathrm{Nr}`Q \dot{+} \dot{1} : \supset \vdash . \mathrm{Prop}$

***255·62.** $\vdash : Q, R \in \Omega . \mathrm{Nr}`P = \mathrm{Nr}`Q \dot{+} \mathrm{Nr}`R . \mathrm{Nr}`R \neq 0_r .$
$\sim (\mathrm{C}`R_1 = \mathrm{C}`R . \mathrm{E} ! B`\breve{R}) . \supset .$
$\mathrm{Nr}`P > \mathrm{Nr}`Q \dot{+} \dot{1} . \mathrm{Nr}`P \dot{+} \dot{1} > \mathrm{Nr}`Q \dot{+} \dot{1}$

Dem.

$\vdash . *253·571 . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{Nr}`P = \mathrm{Nr}`Q \dot{+} \dot{1} \dot{+} \mathrm{Nr}`R .$

[*255·32] $\supset . \mathrm{Nr}`P > \mathrm{Nr}`Q \dot{+} \dot{1} .$ (1)

[*255·321] $\supset . \mathrm{Nr}`P \dot{+} \dot{1} > \mathrm{Nr}`Q \dot{+} \dot{1}$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***255·63.** $\vdash : \mathrm{Nr}'P > \mathrm{Nr}'Q \,.\supset .\, \mathrm{Nr}'P \dot{+} \dot{1} > \mathrm{Nr}'Q \dot{+} \dot{1}$

Dem.

$\vdash . *255{\cdot}33 \,.\supset \vdash :. \mathrm{Hp} \,.\supset : (\exists R) \,.\, \mathrm{Nr}'R \neq 0_r \,.\, \mathrm{Nr}'P = \mathrm{Nr}'Q \dot{+} \mathrm{Nr}'R \,.\vee.$
$\mathrm{Nr}'Q \neq 0_r \,.\, \mathrm{Nr}'P = \mathrm{Nr}'Q \dot{+} \dot{1} :$

[*255·62·321] $\supset : \mathrm{Nr}'P \dot{+} \dot{1} > \mathrm{Nr}'Q \dot{+} \dot{1} :. \supset \vdash .$ Prop

***255·64.** $\vdash : \mathrm{Nr}'P > \mathrm{Nr}'Q \,.\equiv .\, \mathrm{Nr}'P \dot{+} \dot{1} > \mathrm{Nr}'Q \dot{+} \dot{1}$

Dem.

$\vdash . *255{\cdot}63{\cdot}103 \,.\quad \supset \vdash : \mathrm{Nr}'P < \mathrm{Nr}'Q \,.\supset .\, \mathrm{Nr}'P \dot{+} \dot{1} < \mathrm{Nr}'Q \dot{+} \dot{1}$ (1)

$\vdash . *181{\cdot}31 \,.\quad \supset \vdash : \mathrm{Nr}'P = \mathrm{Nr}'Q \,.\supset .\, \mathrm{Nr}'P \dot{+} \dot{1} = \mathrm{Nr}'Q \dot{+} \dot{1}$ (2)

$\vdash . (1) . (2) . *255{\cdot}113 \,.\supset \vdash : P, Q \in \Omega \,.\, \sim(\mathrm{Nr}'P > \mathrm{Nr}'Q) \,.\supset .$
$\mathrm{Nr}'P \dot{+} \dot{1} \leqslant \mathrm{Nr}'Q \dot{+} \dot{1} \,.$

[*255·483] $\supset .\, \sim(\mathrm{Nr}'P \dot{+} \dot{1} > \mathrm{Nr}'Q \dot{+} \dot{1})$ (3)

$\vdash . *251{\cdot}132 \,.\quad \supset \vdash : \sim(P, Q \in \Omega) \,.\supset .\, \sim(\mathrm{Nr}'P \dot{+} \dot{1}, \mathrm{Nr}'Q \dot{+} \dot{1} \in \mathrm{NR}) \,.$

[*255·12] $\supset .\, \sim(\mathrm{Nr}'P \dot{+} \dot{1} > \mathrm{Nr}'Q \dot{+} \dot{1})$ (4)

$\vdash . (3) . (4) \,.\quad \supset \vdash : \sim(\mathrm{Nr}'P > \mathrm{Nr}'Q) \,.\supset .\, \sim(\mathrm{Nr}'P \dot{+} \dot{1} > \mathrm{Nr}'Q \dot{+} \dot{1})$ (5)

$\vdash . (5) . *255{\cdot}63 \,.\supset \vdash .$ Prop

***255·65.** $\vdash :. \mu \in \mathrm{N_0O} - \iota'0_r \,.\supset : \nu > \mu \,.\equiv .\, \nu \geqslant \mu \dot{+} \dot{1}$

Dem.

$\vdash . *255{\cdot}33 \,.\supset \vdash :. \nu > \mu \,.\supset : (\exists \varpi) \,.\, \varpi \in \mathrm{NO} - \iota'0_r \,.\, \nu = \mu \dot{+} \varpi \,.\vee .\, \nu = \mu \dot{+} \dot{1}$ (1)

$\vdash . *255{\cdot}53{\cdot}31 \,.\supset$

$\vdash :. \mathrm{Hp} \,.\, \varpi \in \mathrm{NO} - \iota'0_r \,.\, \nu = \mu \dot{+} \varpi \,.\supset : (\exists \rho) \,.\, \rho \in \mathrm{NO} \cup \iota'\dot{1} \,.\, \nu = \mu \dot{+} \dot{2} \dot{+} \rho :$

[*181·56] $\supset : (\exists \rho) \,.\, \rho \in \mathrm{NO} \cup \iota'\dot{1} \,.\, \nu = \mu \dot{+} \dot{1} \dot{+} \dot{1} \dot{+} \rho :$

[(*255·298)] $\supset : \nu = \mu \dot{+} \dot{1} \dot{+} \dot{1} \,.\vee .\, \nu = \mu \dot{+} \dot{1} \dot{+} \dot{1} \dot{+} \dot{1} \,.\vee .$
$(\exists \rho) \,.\, \rho \in \mathrm{NO} - \iota'0_r \,.\, \nu = \mu \dot{+} \dot{1} \dot{+} \dot{1} \dot{+} \rho :$

[*255·33] $\supset : \nu > \mu \dot{+} \dot{1}$ (2)

$\vdash . (1) . (2) \,.\quad \supset \vdash : \nu > \mu \,.\supset .\, \nu \geqslant \mu \dot{+} \dot{1}$ (3)

$\vdash . *255{\cdot}45{\cdot}321 \,.\supset \vdash : \mathrm{Hp} \,.\, \nu \geqslant \mu \dot{+} \dot{1} \,.\supset .\, \nu > \mu$ (4)

$\vdash . (3) . (4) \,.\supset \vdash .$ Prop

The following propositions are concerned with the relations of ordinals to the corresponding cardinals, *i.e.* to the cardinals of the fields of well-ordered series having the given ordinals. If P is a well-ordered series whose ordinal is α, $C``\alpha = \mathrm{Nc}'C'P$, so that $C``\alpha$ is a cardinal whose members can be well-ordered. Such cardinals have the property that of any two which are not equal, one must be the greater.

If the cardinal number of one series is greater than that of another, so is the ordinal number; but the converse does not hold except for finite numbers.

***255·7.** $\vdash . \mathrm{Nc}``C``\Omega = C```\mathrm{NO}$ [*152·7 . (*251·01)]

***255·701.** $\vdash . \mathrm{Nc}``C``\Omega - \iota`\Lambda = C```(\mathrm{NO} - \iota`\Lambda) = C```\mathrm{NO} - \iota`\Lambda$ [*255·7 . *37·45]

***255·71.** $\vdash : P \text{ less } Q . \supset . \mathrm{Nc}`C`P \leqslant \mathrm{Nc}`C`Q$

Dem.

$\vdash . *254{\cdot}1 . \supset \vdash : \mathrm{Hp} . \supset . \exists ! \mathrm{Rl}`Q \cap \mathrm{Nr}`P .$

[*154·1] $\supset . \exists ! \mathrm{Cl}`C`Q \cap \mathrm{Nc}`C`P .$

[*117·22] $\supset . \mathrm{Nc}`C`P \leqslant \mathrm{Nc}`C`Q : \supset \vdash . \mathrm{Prop}$

***255·711.** $\vdash : \mathrm{Nr}`P \leqslant \mathrm{Nr}`Q . \supset . \mathrm{Nc}`C`P \leqslant \mathrm{Nc}`C`Q$

[Proof as in *255·71, using *255·22]

***255·72.** $\vdash : \alpha \leqslant \beta . \supset . C``\alpha \leqslant C``\beta$

Dem.

$\vdash . *255{\cdot}24 . \supset \vdash : \mathrm{Hp} . \supset . (\exists P, Q) . \alpha = \mathrm{N_0r}`P . \beta = \mathrm{N_0r}`Q . \mathrm{Nr}`P \leqslant \mathrm{Nr}`Q .$

[*255·711] $\supset . (\exists P, Q) . \alpha = \mathrm{N_0r}`P . \beta = \mathrm{N_0r}`Q . \mathrm{Nc}`C`P \leqslant \mathrm{Nc}`C`Q .$

[*152·7] $\supset . C``\alpha \leqslant C``\beta : \supset \vdash . \mathrm{Prop}$

***255·73.** $\vdash :. P, Q \in \Omega . \supset :$

$\mathrm{Nc}`C`P < \mathrm{Nc}`C`Q . \mathrm{v} . \mathrm{Nc}`C`P = \mathrm{Nc}`C`Q . \mathrm{v} . \mathrm{Nc}`C`P > \mathrm{Nc}`C`Q$

Dem.

$\vdash . *255{\cdot}711 . \supset \vdash : \mathrm{Hp} . \mathrm{Nr}`P \leqslant \mathrm{Nr}`Q . \supset . \mathrm{Nc}`C`P \leqslant \mathrm{Nc}`C`Q$ (1)

$\vdash . *255{\cdot}71 . \supset \vdash : \mathrm{Hp} . \mathrm{Nr}`Q \lessdot \mathrm{Nr}`P . \supset . \mathrm{Nc}`C`Q < \mathrm{Nc}`C`P$ (2)

$\vdash . (1) . (2) . *255{\cdot}115 . \supset \vdash . \mathrm{Prop}$

***255·74.** $\vdash :. \alpha, \beta \in C```\mathrm{NO} - \iota`\Lambda . \supset : \alpha \leqslant \beta . \mathrm{v} . \alpha > \beta$

Dem.

$\vdash . *255{\cdot}701 . \supset \vdash : \mathrm{Hp} . \supset . \alpha, \beta \in C```(\mathrm{NO} - \iota`\Lambda) .$

[*155·34] $\supset . (\exists P, Q) . P, Q \in \Omega . \alpha = C``\mathrm{N_0r}`P . \beta = C``\mathrm{N_0r}`Q .$

[*152·7] $\supset . (\exists P, Q) . P, Q \in \Omega . \alpha = \mathrm{N_0c}`C`P . \beta = \mathrm{N_0c}`C`Q$ (1)

$\vdash . *255{\cdot}73 . *117{\cdot}106{\cdot}107{\cdot}108 . \supset$

$\vdash :. P, Q \in \Omega . \supset : \mathrm{N_0c}`C`P \leqslant \mathrm{N_0c}`C`Q . \mathrm{v} . \mathrm{N_0c}`C`P > \mathrm{N_0c}`C`Q$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***255·75.** $\vdash : P, Q \in \Omega . \mathrm{Nc}`C`P < \mathrm{Nc}`C`Q . \supset . P \text{ less } Q$

Dem.

$\vdash . *117{\cdot}291 . \supset \vdash : \mathrm{Hp} . \supset . \sim (\mathrm{Nc}`C`Q \leqslant \mathrm{Nc}`C`P) .$

[*255·711.Transp] $\supset . \sim (\mathrm{Nr}`Q \leqslant \mathrm{Nr}`P) .$

[*255·29] $\supset . \mathrm{Nr}`P \lessdot \mathrm{Nr}`Q .$

[*255·17] $\supset . P \text{ less } Q : \supset \vdash . \mathrm{Prop}$

***255·76.** $\vdash : \alpha, \beta \in \mathrm{NO} . C``\alpha < C``\beta . \supset . \alpha \lessdot \beta$ [*255·75 . *152·7]

*256. THE SERIES OF ORDINALS.

Summary of *256.

In the present number, we have to consider the series of ordinals in order of magnitude. Propositions on this subject deserve close attention, because it is in this connection that Burali-Forti's paradox* arises. This paradox, as we shall show in the present number, is avoided by the doctrine of types. But before discussing the paradox, it will be well to explain various propositions which raise no difficulty.

For convenience of notation, we shall, in the present number, employ the letter M for the relation "$<$" (This letter is chosen as the initial of "minor.") Thus "$\alpha M \beta$" means that α and β are ordinals of which α is less than β. $\overrightarrow{M}{}^{\text{‘}}\beta$ will be the class of ordinals less than β, $\breve{M}_1{}^{\text{‘}}\beta$ will be $\beta \dot{+} \dot{1}$, and $M_1{}^{\text{‘}}\beta$, when it exists, will be such that either $M_1{}^{\text{‘}}\beta \dot{+} \dot{1} = \beta$, or $\beta = 2_r \,.\, M_1{}^{\text{‘}}\beta = 0_r$. Thus $\text{Ɑ}{}^{\text{‘}}M_1$ is the class of ordinals having immediate predecessors, and $\overrightarrow{B}{}^{\text{‘}}M_1$ is the class of ordinals not having immediate predecessors.

We have (*256·12)

$$\vdash :.\ \alpha M \beta \,.\equiv :\ \alpha, \beta \in \mathrm{N_0 O} : (\exists \gamma) \,.\, \gamma \in \mathrm{NO} - \iota{}^{\text{‘}}0_r \cup \iota{}^{\text{‘}}\dot{1} \,.\, \beta = \alpha \dot{+} \gamma,$$

that is, one ordinal is less than another when something not zero can be added to the first to make it equal to the second;

***256·11.** $\vdash : P \in \Omega \,.\supset .\, \overrightarrow{M}{}^{\text{‘}}\mathrm{Nr}{}^{\text{‘}}P = \mathrm{Nr}{}^{\text{‘‘}}\mathrm{D}{}^{\text{‘}}P_s$

I.e. the numbers less than that of P are the numbers of the proper segments of P. Also, if $P \in \Omega$,

$$M \upharpoonright \overrightarrow{M}{}^{\text{‘}}\mathrm{Nr}{}^{\text{‘}}P = \mathrm{N_0 r} ; (P_s \upharpoonright \mathrm{D}{}^{\text{‘}}P_s) \,.\, \mathrm{N_0 r} \upharpoonright \mathrm{D}{}^{\text{‘}}P_s \in 1 \to 1 \quad (*256{\cdot}2{\cdot}201),$$

so that (*256·202) the series of ordinals less than that of P is similar to the series of the proper segments of P, *i.e.* to $P \upharpoonright \text{Ɑ}{}^{\text{‘}}P$ (in virtue of *253·22). It follows (*256·22) that every section of M is well-ordered, and therefore that M is well-ordered (*256·3), *i.e.* that the ordinals in order of magnitude form a well-ordered series.

* "Una questione sui numeri transfiniti," *Rendiconti del circolo matematico di Palermo*, Vol. XI. (1897).

For the purposes of the present number, it is convenient to include 1_s (cf. *153) in the series of ordinals; we therefore get

$$N = M \cup 0_r \downarrow 1_s \cup (\iota`1_s) \uparrow \text{C}`M \quad \text{Dft [*256]}.$$

The effect of this definition is merely to insert 1_s in the series M between 0_r and 2_r. We then have (*256·42)

$$\text{Nr}`N = \dot{1} \dotplus \text{Nr}`M.$$

Now if $P \epsilon \Omega$, $P \upharpoonright \text{C}`P$ (as we have just seen) is similar to a proper segment of M, so that if we omit to mention types we obtain

$$\vdash : P \epsilon \Omega . \supset . \text{Nr}`P \upharpoonright \text{C}`P < \text{Nr}`M.$$

Hence $\text{Nr}`P$, which is $\dot{1} \dotplus \text{Nr}`P \upharpoonright \text{C}`P$, is less than $\dot{1} \dotplus \text{Nr}`M$ (by *255·63), *i.e.* is less than N. Hence

$$\vdash : P \epsilon \Omega . \supset . \text{Nr}`P < \text{Nr}`N.$$

Nevertheless $N \epsilon \Omega$, so that it might seem as if $\text{Nr}`N$ must be less than itself, which is impossible by *255·42. Hence we are led to Burali-Forti's paradox concerning the ordinal number of all ordinals.

Burali-Forti's own statement of his paradox, which is somewhat different from the above, may be summarized as follows. Assuming

$$\alpha, \beta \epsilon \text{N}_0\text{O} . \supset : \alpha < \beta . \text{v} . \alpha = \beta . \text{v} . \alpha > \beta \qquad \text{(A)},$$

we shall have $\qquad \alpha \epsilon \text{N}_0\text{O} . \supset . \alpha < \alpha \dotplus \dot{1}.$

But we also have $\qquad \alpha \epsilon \text{N}_0\text{O} . \supset . \alpha \leqslant \text{Nr}`N.$

Hence $\qquad \text{Nr}`N < \text{Nr}`N \dotplus \dot{1} . \text{Nr}`N \dotplus \dot{1} \leqslant \text{Nr}`N,$

which is impossible. The conclusion drawn by Burali-Forti is that the above proposition (A) is false. This, however, cannot be maintained in view of Cantor's proof, reproduced above (*255·112, depending on *254·4). The solution of the paradox must therefore be sought elsewhere.

With regard to Burali-Forti's statement of the paradox, it is to be observed that "$\alpha < \alpha \dotplus \dot{1}$" only holds if $\exists ! \alpha \dotplus \dot{1}$, *i.e.* if $(\exists P) . P \epsilon \alpha . C`P \neq \text{V}$. This will always hold if $\alpha$ exists and is infinite, because then, if $P \epsilon \alpha$, $P \upharpoonright \text{C}`P \rightarrow B`P \epsilon \alpha \dotplus \dot{1}$. But if α is finite, this method fails, since

$$P \upharpoonright \text{C}`P \rightarrow B`P \epsilon \alpha.$$

Thus if the total number of entities in the universe (of any one type) is finite, "$\alpha < \alpha \dotplus \dot{1}$" fails when $C``\alpha = \iota`\text{V}$, which is just the crucial case for Burali-Forti's proof. Hence as it stands, his proof is only applicable if we assume the axiom of infinity; it might, therefore, be regarded as a reductio ad absurdum of the axiom of infinity, *i.e.* as showing that the total number of entities of any one type is finite.

In order to make it plain that the paradox does not depend upon the axiom of infinity, we have above stated it in a form independent of this

axiom. The paradox, stated simply, is as follows: The ordinal number of the series of ordinals from 0_r (including 1_s) to any ordinal α is $\alpha \dot{+} \dot{1}$; hence $\alpha \dot{+} \dot{1}$ exists, and is therefore $> \alpha$. But the ordinal α is similar to the segment of the series of ordinals consisting of the predecessors of α, and is therefore less than the ordinal number of all ordinals. Hence the ordinal number of all ordinals is greater than every ordinal, and therefore than itself, which is absurd; moreover, though the greatest of all ordinals, it can be increased by the addition of $\dot{1}$, which is again absurd.

In order to dispel the above paradox, it is only necessary to make the types explicit. In the proposition

$$P \epsilon \Omega . \supset . P \text{ less } N \qquad \text{(B)},$$

upon which the paradox depends, the relation "less" is not homogeneous. N is of the same type as M, which is defined as $\text{Nr}\text{'less}$, where $C\text{'less} = \Omega$. Thus $\text{Nr}\text{'}P \epsilon C\text{'}N$. Thus N, as it occurs in (B), should really be $N \upharpoonright t\text{'}\text{N}_0\text{r}\text{'}P$, *i.e.* $N \upharpoonright t\text{'}t\text{'}P$, *i.e.* $N(P, P)$, according to the definition $*65{\cdot}12$. We have therefore

***256·53.** $\vdash : P \epsilon \Omega . \supset . P \text{ less } N \upharpoonright t\text{'}\text{N}_0\text{r}\text{'}P$

but this does not allow the inference

$$N \upharpoonright t\text{'}\text{N}_0\text{r}\text{'}P \text{ less } N \upharpoonright t\text{'}\text{N}_0\text{r}\text{'}P,$$

which is what would be required in order to elicit a paradox. The correct inference is, substituting for $N \upharpoonright t\text{'}\text{N}_0\text{r}\text{'}P$ the equivalent form $N(P, P)$,

$$N(P, P) \text{ less } N\{N(P, P), N(P, P)\}.$$ or, more generally,

***256·56.** $\vdash . (N \upharpoonright \lambda) \text{ less } \{N \upharpoonright (t\text{'}t_{00}\text{'}\lambda)\}$

Thus in higher types there are greater ordinals than any to be found in lower types. This fact is what gave rise to the paradox, as the corresponding fact in cardinals gave rise to the paradox of the greatest cardinal.

***256·01.** $M = <$ Dft [*256]

***256·02.** $N = M \cup 0_r \downarrow 1_s \cup (\iota\text{'}1_s) \uparrow \text{C}\text{'}M$ Dft [*256]

***256·1.** $\vdash . M \epsilon \text{Ser} . C\text{'}M \subset \text{N}_0\text{O}$

Dem.

$\vdash . *255{\cdot}42 . \qquad \supset \vdash . M \in J \qquad (1)$

$\vdash . *255{\cdot}471 . \qquad \supset \vdash . M \epsilon \text{trans} \qquad (2)$

$\vdash . *255{\cdot}12 . \qquad \supset \vdash . C\text{'}M \subset \text{N}_0\text{O} \qquad (3)$

$\vdash . (3) . *255{\cdot}112 . *155{\cdot}43 . \supset \vdash . M \epsilon \text{connex} \qquad (4)$

$\vdash . (1) . (2) . (3) . (4) . \supset \vdash . \text{Prop}$

The above proposition assumes that M is homogeneous, since otherwise "$C\text{'}M$" is not significant. But M is significant even when it is not homogeneous. Thus the conditions of significance in the above proposition impose a limitation upon M which is not always imposed upon M.

***256·101.** $\vdash : \dot{\exists} ! M . \supset . C\text{‘}M = \mathrm{N_0O} . 0_r = B\text{‘}M : \mathrm{N_0O} - \iota\text{‘}0_r = \text{ᗡ}\text{‘}M$

Dem.

$\vdash . *200{\cdot}12 . *256{\cdot}1 . \supset \vdash . C\text{‘}M \sim\epsilon\, 1$ (1)

$\vdash . (1) . *51{\cdot}4 . \quad \supset \vdash : \dot{\exists} ! M . \supset . \exists ! C\text{‘}M - \iota\text{‘}0_r .$

$[*256{\cdot}1] \quad \supset . \exists ! \mathrm{N_0O} - \iota\text{‘}0_r$ (2)

$\vdash . *255{\cdot}51 . \quad \supset \vdash : \mu \,\epsilon\, \mathrm{N_0O} - \iota\text{‘}0_r . \equiv . 0_r M \mu$ (3)

$\vdash . (3) . \quad \supset \vdash . \mathrm{N_0O} - \iota\text{‘}0_r \subset \text{ᗡ}\text{‘}M . 0_r \sim\epsilon\, \text{ᗡ}\text{‘}M$ (4)

$\vdash . (2) . (3) . \quad \supset \vdash : \dot{\exists} ! M . \supset . 0_r \,\epsilon\, \mathrm{D}\text{‘}M$ (5)

$\vdash . (4) . *256{\cdot}1 . \quad \supset \vdash . \text{ᗡ}\text{‘}M \subset \mathrm{N_0O} - \iota\text{‘}0_r$ (6)

$\vdash . (4) . (5) . (6) . \supset \vdash .$ Prop

The hypothesis $\dot{\exists} ! M$ will fail in the lowest type for which M is significant, if the universe contains only one individual. Under any other circumstances, $\dot{\exists} ! M$ must hold.

***256·102.** $\vdash : \exists ! \mathrm{N_0O} - \iota\text{‘}0_r . \supset . \dot{\exists} ! M$

Dem.

$\vdash . *256{\cdot}101 . \supset \vdash : \mathrm{Hp} . \supset . \exists ! \text{ᗡ}\text{‘}M$ (1)

$\vdash . (1) . *33{\cdot}24 . \supset \vdash .$ Prop

***256·11.** $\vdash : P \,\epsilon\, \Omega . \supset . \overrightarrow{M}\text{‘}\mathrm{Nr}\text{‘}P = \mathrm{Nr}\text{“}\mathrm{D}\text{‘}P_s$ [*225·174]

***256·12.** $\vdash :. \alpha M \beta . \equiv : \alpha, \beta \,\epsilon\, \mathrm{N_0O} :$

$(\exists \gamma) . \gamma \,\epsilon\, \mathrm{NO} - \iota\text{‘}0_r . \beta = \alpha \dot{+} \gamma . \vee . \alpha \neq 0_r . \beta = \alpha \dot{+} \dot{1}$ [*255·33]

***256·2.** $\vdash : P \,\epsilon\, \Omega . \supset .$

$M \mathbin{⊏} (\overrightarrow{M_*}\text{‘}\mathrm{Nr}\text{‘}P) = \mathrm{N_0r};P_s . M \mathbin{⊏} (\overrightarrow{M}\text{‘}\mathrm{Nr}\text{‘}P) = \mathrm{N_0r};(P_s \mathbin{⊏} \mathrm{D}\text{‘}P_s)$

Dem.

$\vdash . *256{\cdot}101 . \supset \vdash : \mathrm{Hp} . P \,\epsilon\, 0_r . \supset . M \mathbin{⊏} \overrightarrow{M_*}\text{‘}\mathrm{Nr}\text{‘}P = \dot{\Lambda} . M \mathbin{⊏} (\overrightarrow{M}\text{‘}\mathrm{Nr}\text{‘}P) = \dot{\Lambda}$ (1)

$\vdash . *213{\cdot}3 . \quad \supset \vdash : \mathrm{Hp} . P \,\epsilon\, 0_r . \supset . \mathrm{N_0r};P_s = \dot{\Lambda} . \mathrm{N_0r};(P_s \mathbin{⊏} \mathrm{D}\text{‘}P_s) = \dot{\Lambda}$ (2)

$\vdash . *256{\cdot}11 . *213{\cdot}158 . \supset \vdash : \mathrm{Hp} . P \sim\epsilon\, 0_r . \supset . \overrightarrow{M_*}\text{‘}\mathrm{Nr}\text{‘}P = \mathrm{Nr}\text{“}C\text{‘}P_s$ (3)

$\vdash . (3) . *255{\cdot}17 . \supset \vdash :. \mathrm{Hp} . P \sim\epsilon\, 0_r . \supset : \alpha \{M \mathbin{⊏} (\overrightarrow{M_*}\text{‘}\mathrm{Nr}\text{‘}P)\} \beta . \equiv .$

$(\exists Q, R) . \alpha = \mathrm{N_0r}\text{‘}Q . \beta = \mathrm{N_0r}\text{‘}R . Q, R \,\epsilon\, C\text{‘}P_s . Q \text{ less } R .$

$[*254{\cdot}47] \quad \equiv . (\exists Q, R) . \alpha = \mathrm{N_0r}\text{‘}Q . \beta = \mathrm{N_0r}\text{‘}R . Q P_s R .$

$[*150{\cdot}4] \quad \equiv . \alpha (\mathrm{N_0r};P_s) \beta$ (4)

Similarly $\vdash :. \mathrm{Hp} . P \sim\epsilon\, 0_r . \supset : \alpha \{M \mathbin{⊏} (\overrightarrow{M}\text{‘}\mathrm{Nr}\text{‘}P)\} \beta . \equiv . \alpha \{\mathrm{N_0r};(P_s \mathbin{⊏} \mathrm{D}\text{‘}P_s)\} \beta$ (5)

$\vdash . (1) . (2) . (4) . (5) . \supset \vdash .$ Prop

***256·201.** $\vdash : P \,\epsilon\, \Omega . \supset . \mathrm{N_0r} \upharpoonright \mathrm{D}\text{‘}P_s \,\epsilon\, \{M \mathbin{⊏} (\overrightarrow{M}\text{‘}\mathrm{Nr}\text{‘}P)\} \,\overline{\mathrm{smor}}\, (P_s \mathbin{⊏} \mathrm{D}\text{‘}P_s) .$

$\mathrm{N_0r} \upharpoonright C\text{‘}P_s \,\epsilon\, \{M \mathbin{⊏} (\overrightarrow{M_*}\text{‘}\mathrm{Nr}\text{‘}P)\} \,\overline{\mathrm{smor}}\, P_s$ [*253·461 . *256·2]

***256·202.** $\vdash : P \epsilon \Omega . \supset . \mathrm{Nr}'\{M \upharpoonright (\overrightarrow{M}'\mathrm{Nr}'P)\} = \mathrm{Nr}'(P_s \upharpoonright \mathrm{D}'P_s) = \mathrm{Nr}'(P \upharpoonright \mathrm{C\!\!\!I}'P)$
[*256·201 . *253·22]

***256·203.** $\vdash : P \epsilon \Omega . \supset . \mathrm{Nr}'\{M \upharpoonright (\overrightarrow{M_*}'\mathrm{Nr}'P)\} = \mathrm{Nr}'P_s$ [*256·201]

***256·204.** $\vdash : \alpha \epsilon \mathrm{N_0O} - \iota'2_r . \supset . \dot{1} \dot{+} \mathrm{Nr}'(M \upharpoonright \overrightarrow{M}'\alpha) = \alpha$

Dem.

$\vdash . *255·101 . *256·202 . \supset$

$\vdash :. P \epsilon \Omega . \alpha = \mathrm{N_0r}'P . \supset : \mathrm{Nr}'\{M \upharpoonright \overrightarrow{M}'\alpha\} = \mathrm{Nr}'(P \upharpoonright \mathrm{C\!\!\!I}'P) :$

[*204·46·272] $\supset : P \sim \epsilon\, 2_r . \supset . \dot{1} \dot{+} \mathrm{Nr}'(M \upharpoonright \overrightarrow{M}'\alpha) = \mathrm{Nr}'P :. \supset \vdash . \mathrm{Prop}$

***256·21.** $\vdash : \mu \epsilon \mathrm{NO} . P \epsilon \mu . \supset . \overrightarrow{M}'\mu = \mathrm{Nr}``\mathrm{D}'P_s$ [*256·11]

***256·211.** $\vdash : \mu \epsilon \mathrm{NO} - \iota'0_r . P \epsilon \mu . \supset . \overrightarrow{M_*}'\mu = \mathrm{Nr}``C'P_s$ [*213·158 . *256·21]

***256·22.** $\vdash : \mu \epsilon \mathrm{NO} . \supset . M \upharpoonright \overrightarrow{M_*}'\mu \epsilon \Omega$

Dem.

$\vdash . *256·203 . \supset \vdash : \mathrm{Hp} . P \epsilon \mu . \supset . \mathrm{Nr}'(M \upharpoonright \overrightarrow{M_*}'\mu) = \mathrm{Nr}'P_s .$

[*253·24] $\supset . M \upharpoonright \overrightarrow{M_*}'\mu \epsilon \Omega$ (1)

$\vdash . (1) . \supset \vdash : \mu \neq \Lambda . \supset . M \upharpoonright \overrightarrow{M_*}'\mu \epsilon \Omega$ (2)

$\vdash . (2) . *250·4 . \supset \vdash . \mathrm{Prop}$

***256·221.** $\vdash : \mu \epsilon \mathrm{NO} . \supset . M \upharpoonright \overrightarrow{M}'\mu \epsilon \Omega$ [*256·202]

***256·3.** $\vdash . M \epsilon \Omega$ [*256·22·1 . *250·7]

***256·31.** $\vdash : \exists ! M . \supset . 2_r = 2_M = \breve{M}_1'0_r$

Dem.

$\vdash . *255·51·53 . \supset \vdash : \mathrm{Hp} . \supset . \overleftarrow{M}'0_r = \iota'2_r \cup \overleftarrow{M}'2_r .$

[*205·196.*256·1] $\supset . 2_r = \min_M{}'\overleftarrow{M}'0_r$

[*206·42.*201·63] $= \breve{M}_1'0_r$

[*250·42.*256·101] $= 2_M : \supset \vdash . \mathrm{Prop}$

We shall have, for every finite ν, $\nu_r = \nu_M$, where ν_r will be defined as the ordinal corresponding to ν, *i.e.* as

$$\Omega \cap \breve{C}``\nu.$$

(This is a single ordinal when ν is finite; otherwise, it is the sum of a class of ordinals.) This subject will be considered in the next section.

***256·32.** $\vdash :. \alpha M_1 \beta . \equiv : \alpha, \beta \in \mathrm{N_0O} : \alpha \neq 0_r . \beta = \alpha \dot{+} \dot{1} . \mathrm{v} . \alpha = 0_r . \beta = 2_r$

Dem.

$\vdash . *255{\cdot}65 . \supset \vdash : \alpha \in \mathrm{N_0O} - \iota`0_r . \supset . \overleftarrow{M}`\alpha = \iota`(\alpha \dot{+} \dot{1}) \cup \overleftarrow{M}`(\alpha \dot{+} \dot{1}) .$

$[*205{\cdot}196] \qquad \supset . \alpha \dot{+} \dot{1} = \min_M`\overleftarrow{M}`\alpha .$

$[*206{\cdot}42 . *201{\cdot}63] \qquad \supset . \alpha \dot{+} \dot{1} = \breve{M}_1`\alpha \qquad (1)$

$\vdash . (1) . *256{\cdot}31 . \supset \vdash . \mathrm{Prop}$

***256·4.** $\vdash . 1_s \sim \in \mathrm{NO}$

Dem

$\vdash . *153{\cdot}36 . \qquad \supset \vdash : R \in 1_s . \supset . C`R \in 1 .$

$[*200{\cdot}12 . *250{\cdot}12] \qquad \supset . R \sim \in \Omega \qquad (1)$

$\vdash . (1) . *251{\cdot}122 . \supset \vdash : \alpha \in \mathrm{NO} . \supset . \alpha \cap 1_s = \Lambda \qquad (2)$

$\vdash . (2) . *153{\cdot}34 . \supset \vdash . \mathrm{Prop}$

***256·41.** $\vdash . N = M \cup 0_r \downarrow 1_s \cup (\iota`1_s) \uparrow \text{Ɑ}`M \quad [(*256{\cdot}02)]$

***256·411.** $\vdash :. \alpha N \beta . \equiv : \alpha = 0_r . \beta \in \iota`1_s \cup \text{Ɑ}`M . \mathrm{v} .$

$\alpha = 1_s . \beta \in \text{Ɑ}`M . \mathrm{v} . \alpha, \beta \in \text{Ɑ}`M . \alpha M \beta \quad [*256{\cdot}41]$

***256·412.** $\vdash : M = \dot{\Lambda} . \supset . N = 0_r \downarrow 1_s . N \in 2_r \quad [*256{\cdot}41]$

***256·413.** $\vdash : M = 0_r \downarrow 2_r . \supset . N = 0_r \downarrow 1_s \cup 0_r \downarrow 2_r \cup 1_r \downarrow 2_r . N \in \dot{1} \dot{+} 2_r$

$[*256{\cdot}41 . *161{\cdot}211]$

***256·414.** $\vdash : \text{Ɑ}`M \sim \in 1 . \supset . N = 0_r \downarrow 1_s \dotplus M \upharpoonright \text{Ɑ}`M$

Dem.

$\vdash . *204{\cdot}46 . *256{\cdot}101 . \supset$

$\vdash : \mathrm{Hp} . \dot{\exists} ! M . \supset . N = 0_r \leftarrow\!\!\!+ M \upharpoonright \text{Ɑ}`M \cup 0_r \downarrow 1_s \cup (\iota`1_s) \uparrow C`(M \upharpoonright \text{Ɑ}`M)$

$[*161{\cdot}101] \qquad = 0_r \downarrow 1_s \cup (\iota`0_r \cup \iota`1_s) \uparrow C`(M \upharpoonright \text{Ɑ}`M) \cup M \upharpoonright \text{Ɑ}`M$

$[*160{\cdot}1] \qquad = 0_r \downarrow 1_s \dotplus M \upharpoonright \text{Ɑ}`M \qquad (1)$

$\vdash . (1) . *256{\cdot}412 . \supset \vdash . \mathrm{Prop}$

***256·42.** $\vdash : \dot{\exists} ! M . \supset . \mathrm{Nr}`N = 1 \dot{+} \mathrm{Nr}`M$

Dem.

$\vdash . *256{\cdot}414 . \supset \vdash : \mathrm{Hp} . \text{Ɑ}`M \sim \in 1 . \supset . \mathrm{Nr}`N = 2_r \dot{+} \mathrm{Nr}`(M \upharpoonright \text{Ɑ}`M)$

$[*181{\cdot}57] \qquad = \dot{1} \dot{+} \dot{1} \dot{+} \mathrm{Nr}`(M \upharpoonright \text{Ɑ}`M)$

$[*204{\cdot}46] \qquad = \dot{1} \dot{+} \mathrm{Nr}`M \qquad (1)$

$\vdash . (1) . *256{\cdot}413 . \supset \vdash . \mathrm{Prop}$

***256·43.** $\vdash : N \in \Omega - \iota`\dot{\Lambda} \quad [*256{\cdot}412{\cdot}42]$

***256·44.** $\vdash :. P \in \Omega . \supset : P \upharpoonright \text{C}'P \text{ less } M . \equiv . P \text{ less } N . \dot{\exists} ! M$

Dem.

$\vdash . *255·17·601 . \supset$

$\vdash :. \text{Hp} . \supset : P \upharpoonright \text{C}'P \text{ less } M . \equiv . \dot{1} \dot{+} \text{Nr}'P \upharpoonright \text{C}'P < \dot{1} \dot{+} \text{Nr}'M$ (1)

$\vdash . *256·412·42 . \supset \vdash : P = \dot{\Lambda} . \supset . P \text{ less } N$ (2)

$\vdash . *255·51 . \supset \vdash :. P = \dot{\Lambda} . \supset : P \upharpoonright \text{C}'P \text{ less } M . \equiv . \dot{\exists} ! M$ (3)

$\vdash . (2) . (3) . \supset \vdash :. P = \dot{\Lambda} . \supset : P \upharpoonright \text{C}'P \text{ less } M . \equiv . P \text{ less } N . \dot{\exists} ! M$ (4)

$\vdash . *200·35 . *255·51 . \supset \vdash :. \text{C}'P \in 1 . \supset : P \upharpoonright \text{C}'P \text{ less } M . \equiv . \dot{\exists} ! M$ (5)

$\vdash . *256·42 . \supset \vdash :. \text{Hp} . \text{C}'P \in 1 . \dot{\exists} ! M . \supset . P \text{ less } N$ (6)

$\vdash . (5) . (6) . \supset \vdash :. \text{Hp} . \text{C}'P \in 1 . \supset : P \upharpoonright \text{C}'P \text{ less } M . \equiv . \dot{\exists} ! M . P \text{ less } N$ (7)

$\vdash . *204·46 . \supset \vdash :. \text{Hp} . \dot{\exists} ! P . \text{C}'P \sim \in 1 . \supset : \dot{1} \dot{+} \text{Nr}'P \upharpoonright \text{C}'P = \text{Nr}'P :$

[(1)] $\supset : P \upharpoonright \text{C}'P \text{ less } M . \equiv . \text{Nr}'P < \dot{1} \dot{+} \text{Nr}'M .$

[*256·101·42] $\equiv . \text{Nr}'P < \text{Nr}'N . \dot{\exists} ! M$ (8)

$\vdash . (4) . (7) . (8) . \supset \vdash . \text{Prop}$

We now make use of the above propositions to show that every well-ordered relation P of the type we start from is less than N, where N is to hold between ordinals of the type to which $\text{N}_0\text{r}'P$ belongs. This proposition embodies what Burali-Forti's paradox becomes when account is taken of types.

***256·5.** $\vdash : \dot{\exists} ! M . P \in \Omega . \supset . \text{N}_0\text{r};(P_s \upharpoonright \text{D}'P_s) \in \text{D}'(M \upharpoonright t'\text{N}_0\text{r}'P)_s$

Dem.

$\vdash . *256·2 . *253·13 . \supset \vdash : \text{Hp} . \supset . \text{N}_0\text{r};(P_s \upharpoonright \text{D}'P_s) \in \text{D}'M_s$ (1)

$\vdash . (1) . *150·22 . \supset \vdash : \text{Hp} . \supset . \text{N}_0\text{r}``\text{D}'P_s \subset t_0`C`M_s .$

[*213·141] $\supset . \text{N}_0\text{r}'P \in t_0`C`M_s .$

[*63·53] $\supset . t_0`C`M_s = t'\text{N}_0\text{r}'P$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***256·51.** $\vdash : P \in \Omega . \supset . \text{N}_0\text{r};(P_s \upharpoonright \text{D}'P_s) \text{ smor } P \upharpoonright \text{C}'P$ [*253·463]

***256·52.** $\vdash : \dot{\exists} ! M . P \in \Omega . \supset . P \upharpoonright \text{C}'P \text{ less } M \upharpoonright t'\text{N}_0\text{r}'P$ [*256·5·51 . *254·182]

***256·53.** $\vdash : P \in \Omega . \supset . P \text{ less } N \upharpoonright t'\text{N}_0\text{r}'P$

Dem.

$\vdash . *256·44·52 . \supset \vdash : \text{Hp} . \dot{\exists} ! M . \supset . P \text{ less } N \upharpoonright t'\text{N}_0\text{r}'P$ (1)

$\vdash . *256·102 . \supset \vdash : \text{Hp} . M = \dot{\Lambda} . \supset . P = \dot{\Lambda} .$

[*256·43] $\supset . P \text{ less } N$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***256·54.** $\vdash : P \in \Omega . \supset . \text{Nr}(P)`(N \upharpoonright t'\text{N}_0\text{r}'P) = \Lambda$

Dem.

$\vdash . *256·53 . \supset \vdash :. \text{Hp} . \supset : Q \in t'P . \supset_Q . \sim \{Q \text{ smor } N \upharpoonright t'\text{N}_0\text{r}'P\} :$

[*152·11] $\supset : t'P \cap \text{Nr}'(N \upharpoonright t'\text{N}_0\text{r}'P) = \Lambda :$

[(*65·04)] $\supset : \text{Nr}(P)`(N \upharpoonright t'\text{N}_0\text{r}'P) = \Lambda :. \supset \vdash . \text{Prop}$

***256·55.** $\vdash : P \epsilon \Omega . \supset .$

$$\mathrm{Nr}\,(P)\text{‘}(N \upharpoonright t\text{‘}\mathrm{N_0r}\text{‘}P) = \mathrm{Nr}\,(P)\text{‘}(N \upharpoonright t\text{‘}t\text{‘}P) = \mathrm{Nr}\,(P)\text{‘}\{N\,(P, P)\} = \Lambda$$

Dem.

$\vdash . *155{\cdot}12 . \supset \vdash . P \epsilon \mathrm{N_0r}\text{‘}P .$

$[*63{\cdot}105] \quad \supset \vdash . P \epsilon t_0\text{‘}\mathrm{N_0r}\text{‘}P .$

$[*63{\cdot}53] \quad \supset \vdash . t\text{‘}t\text{‘}P = t\text{‘}\mathrm{N_0r}\text{‘}P \quad (1)$

$\vdash . (1) . \quad \supset \vdash . \mathrm{Nr}\,(P)\text{‘}(N \upharpoonright t\text{‘}\mathrm{N_0r}\text{‘}P) = \mathrm{Nr}\,(P)\text{‘}(N \upharpoonright t\text{‘}t\text{‘}P) \quad (2)$

$[(*65{\cdot}12)] \quad = \mathrm{Nr}\,(P)\text{‘}\{N\,(P, P)\} \quad (3)$

$\vdash . (2) . (3) . *256{\cdot}54 . \supset \vdash .$ Prop

***256·56.** $\vdash . (N \upharpoonright \lambda)$ less $\{N \upharpoonright (t\text{‘}t_{00}\text{‘}\lambda)\}$

Dem.

$\vdash . *256{\cdot}43{\cdot}53 . \supset \vdash . (N \upharpoonright \lambda)$ less $\{N \upharpoonright (t\text{‘}\mathrm{N_0r}\text{‘}N \upharpoonright \lambda)\} \quad (1)$

$\vdash . *155{\cdot}12 . \quad \supset \vdash . N \upharpoonright \lambda \epsilon \mathrm{N_0r}\text{‘}N \upharpoonright \lambda .$

$[*63{\cdot}105] \quad \supset \vdash . N \upharpoonright \lambda \epsilon t_0\text{‘}\mathrm{N_0r}\text{‘}N \upharpoonright \lambda .$

$[*63{\cdot}53] \quad \supset \vdash . t\text{‘}t\text{‘}N \upharpoonright \lambda = t\text{‘}\mathrm{N_0r}\text{‘}N \upharpoonright \lambda \quad (2)$

$\vdash . *64{\cdot}16 . \quad \supset \vdash . N \upharpoonright \lambda \epsilon t\text{‘}(t_0\text{‘}\lambda \uparrow t_0\text{‘}\lambda) .$

$[(*64{\cdot}01)] \quad \supset \vdash . N \upharpoonright \lambda \epsilon t_{00}\text{‘}\lambda \quad (3)$

$\vdash . (2) . (3) . \quad \supset \vdash . t\text{‘}t_{00}\text{‘}\lambda = t\text{‘}\mathrm{N_0r}\text{‘}N \upharpoonright \lambda \quad (4)$

$\vdash . (1) . (4) . \supset \vdash .$ Prop

When types are neglected, the above proposition appears as

$$N \text{ less } N,$$

which is impossible, and embodies Burali-Forti's paradox. In the form proved above, however, the paradox has disappeared, and we have instead the proposition that in higher types longer series are possible than in lower ones.

*257. THE TRANSFINITE ANCESTRAL RELATION.

Summary of *257.

In this number, we are concerned with an extension of the notions of R_* and R_{po}. This extension requires two relations, R and Q. It is most easily explained by first defining the "transfinite posterity" of a term x with respect to R and Q; this class is an extension of $\overleftarrow{R_*}\text{‘}x$. This class is generated as follows. Let us suppose, to aid the imagination, that Q is more or less serial in character, and that R is a many-one relation contained in Q. Then the transfinite posterity of x with respect to R and Q is generated as follows: Starting from x, we travel down the posterity of x with respect to R (*i.e.* $\overleftarrow{R_*}\text{‘}x$) as long as we can; if the whole class $\overleftarrow{R_*}\text{‘}x$ has a limit with respect to Q, we begin again with this limit, which is to be included in the transfinite posterity of x with respect to R and Q; if the limit is y, we travel down $\overleftarrow{R_*}\text{‘}y$, and include the limit of this class with respect to Q, and so on, as long as we still have either terms belonging to $\text{D}\text{‘}R$ or classes belonging to $\text{Ɑ}\text{‘}\text{lt}_Q$. The whole of the terms so obtainable constitute the transfinite posterity of x with respect to R and Q. which we will denote* by $(R*Q)\text{‘}x$.

In order to obtain a symbolic definition of this class, let us call a class σ "transfinitely hereditary" when not only $\breve{R}\text{“}\sigma \subset \sigma$, as in the ordinary hereditary class, but also if we take any existent sub-class μ of $\sigma \cap C\text{‘}Q$, if μ has a limit with respect to Q, that limit is to be a member of σ. Thus σ is to be such that the R-successor of any member of σ belongs to σ and the Q-limit of any existent sub-class of $\sigma \cap C\text{‘}Q$ belongs to σ (so long as these exist). That is, $\breve{R}\text{“}\sigma \subset \sigma$ and $\mu \subset \sigma \,.\, \exists ! \mu \cap C\text{‘}Q \,.\, \supset_\mu .\, \overrightarrow{\text{lt}_Q}\text{‘}\mu \subset \sigma$. Using the notion of the derivative of a class with respect to Q, introduced in *216, the condition $\mu \subset \sigma \,.\, \exists ! \mu \cap C\text{‘}Q \,.\, \supset_\mu .\, \overrightarrow{\text{lt}_Q}\text{‘}\mu \subset \sigma$ reduces to $\delta_Q\text{‘}\sigma \subset \sigma$, in virtue of *216·1. Hence σ is transfinitely hereditary with respect to R and Q if

$$\breve{R}\text{“}\sigma \cup \delta_Q\text{‘}\sigma \subset \sigma.$$

* This meaning for $R*Q$ has no connection with the meaning temporarily assigned to this symbol in *95.

We may now define the transfinite posterity of x with respect to R and Q as all members of $C'Q$ which belong to every transfinitely hereditary class to which x belongs, *i.e.* we put

$$(R*Q)'x = C'Q \cap \hat{y}\{x \in \sigma \,.\, \breve{R}``\sigma \cup \delta_Q`\sigma \subset \sigma \,.\, \supset_\sigma \,.\, y \in \sigma\} \quad \text{Df.}$$

Then the analogue of R_* is $\hat{x}\hat{y}\{y \in (R*Q)'x\}$. This relation, however, is less important than the analogue of R_{po} limited to the posterity of x. This analogue, assuming Q to be transitive, will be $Q \upharpoonleft (R*Q)'x$. For this we introduce the two notations Q_{Rx} and $Q(R, x)$, the latter being more convenient when either R or x is replaced by a more complicated expression. Thus we put

$$Q_{Rx} = Q(R, x) = Q \upharpoonleft (R*Q)'x \quad \text{Df.}$$

If Q is a well-ordered series and $R = Q_1$, Q_{Rx} is merely the series Q beginning with x, and $(R*Q)'x = \overleftarrow{Q_*}'x = \overleftarrow{Q}'x \cup \iota'x$ if $x \in C'Q$. Thus in this case, if $x = B'Q$, $Q_{Rx} = Q$. But the importance of Q_{Rx} is in cases where Q is not completely serial, but becomes so when limited to $(R*Q)'x$. In these cases, Q will, in applications, almost always be logical inclusion combined with diversity, or the converse of this; *i.e.* it will be either

$$\hat{\alpha}\hat{\beta}(\alpha \subset \beta \,.\, \alpha \neq \beta)$$

or

$$\hat{M}\hat{N}(M \subseteq N \,.\, M \neq N),$$

or the converse of one of these. In the case of $\hat{\alpha}\hat{\beta}(\alpha \subset \beta \,.\, \alpha \neq \beta)$, we have

$$\text{lt}_Q = s \upharpoonright (-\text{Cl}'\text{max}_Q) \,.\, \text{tl}_Q = p \upharpoonright (-\text{Cl}'\text{min}_Q),$$

as will be proved in *258.

In the present number, we are concerned in proving that, under certain circumstances, $Q_{Rx} \in \Omega$. The proof proceeds on the lines of Zermelo's second proof* of his theorem that if a selection exists from all the existent sub-classes of a given class, then the given class can be well-ordered.

Before proceeding to treat of this subject, however, it is necessary to prove some elementary properties of $(R*Q)'x$. These are given in the propositions preceding *257·2.

We have

***257·11.** $\vdash : x \in \sigma \,.\, \breve{R}``\sigma \cup \delta_Q`\sigma \subset \sigma \,.\, \supset \,.\, (R*Q)'x \subset \sigma$

Thus in order to prove that $(R*Q)'x$ is contained in a class σ, we have to prove (1) that x belongs to σ, (2) that the R-successors of members of σ are members of σ, *i.e.* that σ is hereditary with respect to R, (3) that the derivative of σ with respect to Q is contained in σ, *i.e.* that if μ is any existent sub-class of $\sigma \cap C'Q$ which has a Q-limit, this limit is a member of σ.

* "Neuer Beweis für die Möglichkeit einer Wohlordnung," *Math. Annalen*, LXV. p. 107 (1907). His first proof, which was somewhat more complicated, was published in *Math. Annalen*, LIX. p. 514 (1904).

***257·111.** $\vdash . (R*Q)`x \subset C`Q$

***257·12.** $\vdash : x \in C`Q . \equiv . x \in (R*Q)`x$

***257·123.** $\vdash : R \mathrel{\mathsf{G}} Q . \supset . \breve{R}``(R*Q)`x \subset (R*Q)`x$

I.e. if $R \mathrel{\mathsf{G}} Q, (R*Q)`x$ is hereditary with respect to $R$. The hypothesis $R \mathrel{\mathsf{G}} Q$ is required for most of the properties of $(R*Q)`x$.

***257·125.** $\vdash : R \mathrel{\mathsf{G}} Q . x \in C`Q . \supset . \overleftarrow{R_*}`x \subset (R*Q)`x$

Thus if $x \in C`Q$, the $R$-posterity of $x$ is contained in $(R*Q)`x$.

***257·13.** $\vdash : \mu \subset (R*Q)`x . \exists ! \mu . \supset . \overrightarrow{\mathrm{lt}_Q}`\mu \subset (R*Q)`x$

***257·14.** $\vdash : R \mathrel{\mathsf{G}} Q . \supset . (R*Q)`x \subset \overleftarrow{Q_*}`x$

Thus $(R*Q)`x$ is wholly contained in the Q-posterity of x.

The following propositions (*257·2—·36) are concerned in proving $Q_{Rx} \in \Omega$, with a suitable hypothesis. This hypothesis is

$$Q \in \mathrm{Rl}`J \cap \mathrm{trans} . R \in \mathrm{Rl}`Q \cap \mathrm{Cls} \to 1 . \mathrm{lt}_Q \upharpoonright \mathrm{Cl\,ex}`(R*Q)`x \in 1 \to \mathrm{Cls}.$$

We assume, to begin with, only part of this hypothesis, namely,

$$Q \in \mathrm{Rl}`J \cap \mathrm{trans} . R \in \mathrm{Rl}`Q \cap \mathrm{Cls} \to 1.$$

Thus to prove $Q_{Rx} \in \mathrm{Ser}$, we only have to prove $Q_{Rx} \in \mathrm{connex}$, *i.e.*

$$y \in (R*Q)`x . \supset . (R*Q)`x \subset \overleftrightarrow{Q}`y,$$

or, what comes to the same thing,

$$(R*Q)`x \subset p`\overleftrightarrow{Q}``(R*Q)`x.$$

Let us put $$\sigma_1 = (R*Q)`x \cap p`\overleftrightarrow{Q}``(R*Q)`x.$$

Then any member of σ_1 may be called a "connected term," because it is connected by Q or $\breve{Q}$ with every other term of $(R*Q)`x$. (A connected relation is then a relation whose field consists entirely of connected terms.) We wish to prove that $\sigma_1$ is a transfinitely hereditary class, and therefore equal to $(R*Q)`x$. We do this, not directly, but by combining σ_1 with another class σ_2 defined as follows. Consider those members z of $(R*Q)`x$ which are such that their successors in $Q_{Rx}$ consist of $\breve{R}`z$ and its successors in Q_{Rx}, *i.e.* put

$$\tau = (R*Q)`x \cap \hat{z}\{\overleftarrow{Q_{Rx}}`z = \overleftarrow{(Q_{Rx})_*}`\breve{R}`z\}.$$

It will be observed that, even when Q is transitive, Q_* and $(Q_{Rx})_*$ are still useful. In this case, $(Q_{Rx})_* = Q_{Rx} \cup I \upharpoonright C`Q_{Rx}$, so that $\overleftarrow{(Q_{Rx})_*}`\breve{R}`z$ consists of $\breve{R}`z$ and its successors in Q_{Rx}. We then consider the class σ_2 consisting of those terms y whose predecessors are all members of τ, *i.e.* we put

$$\sigma_2 = (R*Q)`x \cap \hat{y}\{zQy . z \in (R*Q)`x . \supset_z . \overleftarrow{Q_{Rx}}`z = \overleftarrow{(Q_{Rx})_*}`\breve{R}`z\}.$$

Finally we put $\sigma = \sigma_1 \cap \sigma_2$, *i.e.*

$$\sigma = (R*Q)`x \cap p`\overleftrightarrow{Q}``(R*Q)`x \cap \hat{y}\{zQy . z \in (R*Q)`x . \supset_z . \overleftarrow{Q_{Rx}}`z = \overleftarrow{(Q_{Rx})_*}`\breve{R}`z\}.$$

The reason for this process is that it is easier to prove that σ is a transfinitely hereditary class than it is to prove this directly for σ_1; and the result follows immediately for σ_1 when it has been proved for σ.

We have then to prove $\breve{R}\text{‘‘}\sigma \subset \sigma \,.\, \delta_Q\text{‘}\sigma \subset \sigma$.

The first step is to prove

$$y \in \sigma \,.\, \supset \,.\, \overleftarrow{Q_{Rx}}\text{‘}y = \overleftarrow{Q_{Rx}}\text{‘}\breve{R}\text{‘}y \cup \iota\text{‘}\breve{R}\text{‘}y.$$

This is proved by transfinite induction, by showing that

$$\overrightarrow{Q_*}\text{‘}y \cup \overleftarrow{Q_*}\text{‘}\breve{R}\text{‘}y$$

is a transfinitely hereditary class, whence the result, because, by hypothesis,

$$(R*Q)\text{‘}x = \overrightarrow{(Q_{Rx})_*}\text{‘}y \cup \overleftarrow{Q_{Rx}}\text{‘}y.$$

The proof that $\overrightarrow{Q_*}\text{‘}y \cup \overleftarrow{Q_*}\text{‘}\breve{R}\text{‘}y$ is a transfinitely hereditary class is as follows.

If $z \in \overleftarrow{Q_*}\text{‘}\breve{R}\text{‘}y$, $\breve{R}\text{‘}z \in \overleftarrow{Q_*}\text{‘}\breve{R}\text{‘}y$. If $z = y$, $\breve{R}\text{‘}z = \breve{R}\text{‘}y$.

If $z \in \overrightarrow{Q_{Rx}}\text{‘}y$, then since by the hypothesis $\overleftarrow{Q_{Rx}}\text{‘}z = \overleftarrow{(Q_{Rx})_*}\text{‘}\breve{R}\text{‘}z$, we have

$$y \in \overleftarrow{(Q_{Rx})_*}\text{‘}\breve{R}\text{‘}z, \text{ i.e. } \breve{R}\text{‘}z \in \overrightarrow{Q_*}\text{‘}y.$$

Hence $z \in (R*Q)\text{‘}x \cap (\overrightarrow{Q_*}\text{‘}y \cup \overleftarrow{Q_*}\text{‘}\breve{R}\text{‘}y) \,.\, \supset \,.\, \breve{R}\text{‘}z \in \overrightarrow{Q_*}\text{‘}y \cup \overleftarrow{Q_*}\text{‘}\breve{R}\text{‘}y$.

We have next to prove

$$\mu \subset (R*Q)\text{‘}x \cap (\overrightarrow{Q_*}\text{‘}y \cup \overleftarrow{Q_*}\text{‘}\breve{R}\text{‘}y) \,.\, \exists ! \mu \,.\, \supset \,.\, \overrightarrow{\mathrm{lt}_Q}\text{‘}\mu \subset \overrightarrow{Q_*}\text{‘}y \cup \overleftarrow{Q_*}\text{‘}\breve{R}\text{‘}y.$$

If $\exists ! \mu \cap \overleftarrow{Q_*}\text{‘}\breve{R}\text{‘}y$, then $\overrightarrow{\mathrm{lt}_Q}\text{‘}\mu \subset \overleftarrow{Q_*}\text{‘}\breve{R}\text{‘}y$.

If $\mu \subset \overrightarrow{Q_*}\text{‘}y \,.\, y \in \mu$, then $y \in \overrightarrow{\max_Q}\text{‘}\mu$, and $\overrightarrow{\mathrm{lt}_Q}\text{‘}\mu = \Lambda$.

If $\mu \subset \overrightarrow{Q}\text{‘}y$, we have $y \in p\text{‘}\overleftarrow{Q}\text{‘‘}\mu$, whence $w \,\mathrm{lt}_Q \mu \,.\, \supset \,.\, \sim(yQw)$, whence, since y, by hypothesis, is a connected term, wQ_*y.

Hence in any case $\overrightarrow{\mathrm{lt}_Q}\text{‘}\mu \subset \overrightarrow{Q_*}\text{‘}y \cup \overleftarrow{Q_*}\text{‘}\breve{R}\text{‘}y$. Hence $\overrightarrow{Q_*}\text{‘}y \cup \overleftarrow{Q_*}\text{‘}\breve{R}\text{‘}y$ is hereditary, and therefore contains $(R*Q)\text{‘}x$; and hence

$$\overleftarrow{Q_{Rx}}\text{‘}y = \overleftarrow{(Q_{Rx})_*}\text{‘}\breve{R}\text{‘}y \,.\, \overrightarrow{(Q_{Rx})_*}\text{‘}y = \overrightarrow{Q_{Rx}}\text{‘}\breve{R}\text{‘}y.$$

This shows that $\breve{R}\text{‘}y$ is a member of σ_2. For by hypothesis this holds of all predecessors of y, and we have now shown (1) that it also holds of y, (2) that y is the only predecessor of $\breve{R}\text{‘}y$ which does not precede y. This is the first step towards proving that σ is transfinitely hereditary.

It follows immediately, from what has now been proved, that if $y \in \sigma$, $\breve{R}\text{‘}y$ (if it exists) is a connected term. For by hypothesis

$$(R*Q)\text{‘}x \subset \overrightarrow{Q_*}\text{‘}y \cup \overleftarrow{Q}\text{‘}y,$$

whence, by what we have just proved,

$$(R*Q)\text{‘}x \subset \overrightarrow{Q}\text{‘}\breve{R}\text{‘}y \cup \overleftarrow{Q_*}\text{‘}\breve{R}\text{‘}y,$$

whence $\breve{R}`y$ is a connected term. Hence $\breve{R}`y \in \sigma$. Hence $\breve{R}``\sigma \subset \sigma$.

It remains to prove $\qquad \delta_Q`\sigma \subset \sigma.$

Just as $\breve{R}``\sigma \subset \sigma$ was proved by proving $\overleftarrow{Q}`y = \overleftarrow{Q_*}`\breve{R}`y$, so $\delta_Q`\sigma \subset \sigma$ is proved by proving

$$p`\overleftarrow{Q}``\mu \subset \breve{Q}_*``\overrightarrow{\mathrm{lt}_Q}`\mu,$$

provided $\qquad \mu \subset \sigma \,.\, \exists\,!\,\mu \,.\, \sim\exists\,!\,\overrightarrow{\mathrm{max}_Q}`\mu\,;$

and this is proved by showing that $Q``\mu \cup \breve{Q}_*``\overrightarrow{\mathrm{lt}_Q}`\mu$ is a transfinitely hereditary class.

To show that $Q``\mu \cup \breve{Q}_*``\overrightarrow{\mathrm{lt}_Q}`\mu$ is a transfinitely hereditary class if

$$\mu \subset \sigma \,.\, \exists\,!\,\mu \,.\, \sim\exists\,!\,\overrightarrow{\mathrm{max}_Q}`\mu,$$

we observe that by hypothesis

$$z \in Q_{Rx}``\mu \,.\, \supset .\, \overleftarrow{Q_{Rx}}`z = \overleftarrow{(Q_{Rx})_*}`\breve{R}`z \,.\, \supset .\, \exists\,!\,\mu \cap \overleftarrow{(Q_{Rx})_*}`\breve{R}`z.$$

Hence $\breve{R}`z \in (Q_{Rx})_*``\mu$; and hence, since by hypothesis $\mu \subset Q``\mu$,

$$\breve{R}`z \in Q_{Rx}``\mu.$$

Hence $\qquad \breve{R}``\{(Q*R)`x \cap Q``\mu\} \subset (Q*R)`x \cap Q``\mu.$

Also obviously $\qquad \breve{R}``\breve{Q}_*``\overrightarrow{\mathrm{lt}_Q}`\mu \subset \breve{Q}_*``\overrightarrow{\mathrm{lt}_Q}`\mu.$

Hence putting $\qquad \rho = (Q*R)`x \cap (Q``\mu \cup \breve{Q}_*``\overrightarrow{\mathrm{lt}_Q}`\mu),$

we have $\qquad \breve{R}``\rho \subset \rho.$

We have now to prove $\qquad \delta_Q`\rho \subset \rho,$

i.e. $\qquad \alpha \subset \rho \,.\, \exists\,!\,\alpha \,.\, \sim\exists\,!\,\overrightarrow{\mathrm{max}_Q}`\alpha \,.\, \supset .\, \overrightarrow{\mathrm{lt}_Q}`\alpha \subset \rho.$

If $\alpha \subset Q``\mu$, it is obvious (since $\mu$ is composed entirely of connected terms) that $\overrightarrow{\mathrm{seq}_Q}`\alpha \subset Q``\mu \cup \overrightarrow{\mathrm{lt}_Q}`\mu$.

On the other hand, if $\exists\,!\,\alpha \cap \breve{Q}_*``\overrightarrow{\mathrm{lt}_Q}`\mu$, then $\alpha \cap Q``\mu$, if it exists, does not affect the value of the limit of α, which is the limit of $\alpha \cap \breve{Q}_*``\overrightarrow{\mathrm{lt}_Q}`\mu$, which is obviously contained in $\breve{Q}_*``\overrightarrow{\mathrm{lt}_Q}`\mu$. Hence $\delta_Q`\mu \subset \mu$. Hence μ is transfinitely hereditary, and we have

$$\mu \subset \sigma \,.\, \exists\,!\,\mu \,.\, \sim\exists\,!\,\overrightarrow{\mathrm{max}_Q}`\mu \,.\, \supset .\, (R*Q)`x \subset Q``\mu \cup \breve{Q}_*``\overrightarrow{\mathrm{lt}_Q}`\mu.$$

At this point it is necessary to assume

$$\mathrm{lt}_Q \upharpoonright \mathrm{Cl\,ex}`(R*Q)`x \in 1 \rightarrow \mathrm{Cls}.$$

This being assumed, we have, by what has just been proved,

$$\mu \subset \sigma \,.\, \exists\,!\,\mu \,.\, \exists\,!\,\overrightarrow{\mathrm{lt}_Q}`\mu \,.\, \supset .\, (R*Q)`x \subset Q``\mu \cup \overleftarrow{Q_*}`\mathrm{lt}_Q`\mu \,.$$
$$\supset .\, (R*Q)`x \subset \overrightarrow{Q}`\mathrm{lt}_Q`\mu \cup \overleftarrow{Q_*}`\mathrm{lt}_Q`\mu.$$

Hence $\mathrm{lt}_Q‘\mu$ is a connected term. Hence

$$\delta_Q‘\sigma \subset p‘\overleftrightarrow{Q}“(R*Q)‘x.$$

We only require further

$$\mu \subset \sigma \,.\, \exists\,!\,\mu \,.\, \exists\,!\,\overrightarrow{\mathrm{lt}_Q}‘\mu \,.\supset: zQ\,\mathrm{lt}_Q‘\mu \,.\, z \in (R*Q)‘x \,.\supset_z.\, \overleftarrow{Q_{Rx}}‘z = \overleftarrow{(Q_{Rx})_*}‘\breve{R}‘z.$$

Now by what we have just proved, $zQ\,\mathrm{lt}_Q‘\mu \,.\equiv.\, z \in Q“\mu$; and by the definition of σ, since $\mu \subset \sigma$, we have

$$z \in Q“\mu \,.\supset.\, \overleftarrow{Q_{Rx}}‘z = \overleftarrow{(Q_{Rx})_*}‘\breve{R}‘z.$$

Hence we arrive at $\delta_Q‘\sigma \subset \sigma$. Since we have already proved $\breve{R}“\sigma \subset \sigma$, it follows that σ is hereditary, and $(R*Q)‘x \subset \sigma$, *i.e.*

$$y \in (R*Q)‘x \,:\supset_y: y \in p‘\overleftrightarrow{Q}“(R*Q)‘x : zQ_{Rx}y \,.\supset_z.\, \overleftarrow{Q_{Rx}}‘z = \overleftarrow{(Q_{Rx})_*}‘\breve{R}‘z,$$

i.e. $\qquad Q_{Rx} \in \mathrm{connex} : z \in \mathrm{D}‘Q_{Rx} \,.\supset_z.\, \overleftarrow{Q_{Rx}}‘z = \overleftarrow{(Q_{Rx})_*}‘\breve{R}‘z.$

Hence $Q_{Rx} \in \mathrm{Ser}$. Hence also the immediate successor of every term z in $\mathrm{D}‘Q_{Rx}$ is $\breve{R}‘z$, so that

$$\mathrm{D}‘Q_{Rx} \subset \mathrm{D}‘R \,.\, (Q_{Rx})_1 = R \upharpoonleft (R*Q)‘x.$$

To show that $Q_{Rx} \in \Omega$, we observe that every class contained in $\mathrm{D}‘Q_{Rx}$ has a sequent, namely

$$\mathrm{seq}\,(Q_{Rx})‘\Lambda = x,$$

$$\alpha \subset \mathrm{D}‘Q_{Rx} \,.\, \exists\,!\,\overrightarrow{\max_Q}‘\alpha \,.\supset.\, \mathrm{seq}\,(Q_{Rx})‘\alpha = \breve{R}‘\max_Q‘\alpha,$$

$$\alpha \subset \mathrm{D}‘Q_{Rx} \,.\, \exists\,!\,\alpha \,.\, \sim\exists\,!\,\overrightarrow{\max_Q}‘\alpha \,.\supset.\, \mathrm{seq}\,(Q_{Rx})‘\alpha = \mathrm{lt}_Q‘\alpha,$$

whence $\qquad \alpha \subset \mathrm{D}‘Q_{Rx} \,.\supset_\alpha.\, \mathrm{E}\,!\,\mathrm{seq}\,(Q_{Rx})‘\alpha,$

which shows that $Q_{Rx} \in \Omega$.

The first derivative of Q_{Rx} is $\delta_Q‘(Q*R)‘x$, and its last term, if any, is

$$\breve{\iota}‘\{(Q*R)‘x - \mathrm{D}‘R\}, \text{ i.e. } \mathrm{lt}_Q‘\{(Q*R)‘x \cap \mathrm{D}‘R\}.$$

The hypothesis required for $Q_{Rx} \in \Omega$ is the same as for $Q_{Rx} \in \mathrm{Ser}$, namely,

$$Q \in \mathrm{Rl}‘J \cap \mathrm{trans} \,.\, R \in \mathrm{Rl}‘Q \cap \mathrm{Cls} \to 1 \,.\, \mathrm{lt}_Q \upharpoonright \mathrm{Cl\,ex}‘(R*Q)‘x \in 1 \to \mathrm{Cls}.$$

In order that Q_{Rx} may not be null, we require further $x \in \mathrm{D}‘R$.

The next set of propositions (*257·5—·56) are designed to prove that, subject to the above hypothesis together with $x \in \mathrm{D}‘R$, Q_{Rx} is the only value of P fulfilling the following conditions:

(1) P is transitive.

(2) $C‘P$ is contained in $(R*Q)‘x$.

(3) If z is any member of $\mathrm{D}‘P$, $\breve{R}‘z$ is its immediate successor.

(4) If α is any existent class contained in $C‘P$ and having no maximum, $\mathrm{lt}_Q‘\alpha$ is its P-limit.

This proposition is essential for what may be called "transfinite inductive definitions," *i.e.* definitions of a series by defining the successor of every term, and the successor of every class having no maximum.

The following illustration may make this clear. Suppose R is a many-one relation of classes to individuals; suppose we start with some class α, and proceed to $\alpha \cup \iota'\breve{R}'\alpha$, $\alpha \cup \iota'\breve{R}'\alpha \cup \iota'\breve{R}'(\alpha \cup \iota'\breve{R}'\alpha)$, and so on. At the end of this series we put its sum, *i.e.* its limit with respect to the relation $(\subset \dot{\cap} J)$; let the sum be β. We then proceed with $\beta \cup \iota'\breve{R}'\beta$, and so on, as long as possible. The series ends with a sum which is not a member of $\mathrm{D}'R$, if there is such a sum. It is evident that the series is uniquely determined by the above method of generation; the above-mentioned propositions give symbolic expression to the process expressed in words by "and so on, as long as possible."

***257·01.** $(R*Q)'x = C'Q \cap \hat{y}\{x \in \sigma \,.\, \breve{R}``\sigma \cup \delta_Q'\sigma \subset \sigma \,.\supset_\sigma.\, y \in \sigma\}$ Df

***257·02.** $Q_{Rx} = Q(R, x) = Q \upharpoonright (R*Q)'x$ Df

***257·1.** $\vdash :. \, y \in (R*Q)'x \,.\equiv:\, y \in C'Q : x \in \sigma \,.\, \breve{R}``\sigma \cup \delta_Q'\sigma \subset \sigma \,.\supset_\sigma.\, y \in \sigma$
[(*257·01)]

***257·101.** $\vdash :: y \in (R*Q)'x \,.\equiv:.\, y \in C'Q :.$
$x \in \sigma \,.\, \breve{R}``\sigma \subset \sigma : \mu \subset \sigma \,.\, \exists!\, \mu \cap C'Q \,.\supset_\mu.\, \overrightarrow{\mathrm{lt}}_Q'\mu \subset \sigma : \supset_\sigma . \, y \in \sigma$
[*257·1 . *216·1]

***257·102.** $\vdash :: y \in (R*Q)'x \,.\equiv:.\, y \in C'Q :.$
$x \in \sigma . \breve{R}``\sigma \subset \sigma : \mu \subset \sigma \,.\, \exists!\, \mu \cap C'Q \,.\, \sim\exists!\, \overrightarrow{\mathrm{max}}_Q'\mu \,.\supset_\mu.\, \overrightarrow{\mathrm{seq}}_Q'\mu \subset \sigma : \supset_\sigma . \, y \in \sigma$
[*257·101 . *207·1]

***257·11.** $\vdash : x \in \sigma \,.\, \breve{R}``\sigma \cup \delta_Q'\sigma \subset \sigma \,.\supset.\, (R*Q)'x \subset \sigma$ [*257·1]

Almost all proofs of propositions concerning $(R*Q)'x$ use this proposition.

***257·111.** $\vdash . (R*Q)'x \subset C'Q$ [*257·1]

***257·12.** $\vdash : x \in C'Q \,.\equiv.\, x \in (R*Q)'x$ [*257·1]

***257·121.** $\vdash : R \Subset Q \,.\, y \in (R*Q)'x \,.\supset.\, \overleftarrow{R}'y \subset (R*Q)'x$

Dem.

$\vdash . *257{\cdot}1 . \supset \vdash :. \mathrm{Hp} \,.\, yRz \,.\supset: x \in \sigma \,.\, \breve{R}``\sigma \cup \delta_Q'\sigma \subset \sigma \,.\supset_\sigma.\, y \in \sigma : yRz \,.\, z \in C'Q :$

[*37·1] $\supset : z \in C'Q : x \in \sigma \,.\, \breve{R}``\sigma \subset \sigma \,.\, \delta_Q'\sigma \subset \sigma \,.\supset_\sigma.\, z \in \sigma :$

[*257·1] $\supset : z \in (R*Q)'x :. \supset \vdash . \mathrm{Prop}$

***257·122.** $\vdash : R \mathrel{G} Q \,.\, \mu \subset (R*Q)`x \,.\supset .\, \breve{R}``\mu \subset (R*Q)`x$ [*257·121]

***257·123.** $\vdash : R \mathrel{G} Q \,.\supset .\, \breve{R}``(R*Q)`x \subset (R*Q)`x$ [*257·122]

***257·124.** $\vdash : R \mathrel{G} Q \,.\supset .\, \breve{R}_*``(R*Q)`x \subset (R*Q)`x$ [*257·123]

***257·125.** $\vdash : R \mathrel{G} Q \,.\, x \in C`Q \,.\supset .\, \overleftarrow{R_*}`x \subset (R*Q)`x$ [*257·12·124]

***257·126.** $\vdash : R \mathrel{G} Q \,.\, x \in D`R \,.\, \sim(xRx) \,.\supset .\, (R*Q)`x \sim\in 0 \cup 1$ [*257·125]

***257·13.** $\vdash : \mu \subset (R*Q)`x \,.\, \exists ! \mu \,.\supset .\, \overrightarrow{\mathrm{lt}_Q}`\mu \subset (R*Q)`x$

Dem.

$\vdash .\, *257·101 .\, *10·1 .\, *22·1 .\supset \vdash :: \mu \subset (R*Q)`x \,.\supset :.$

$x \in \sigma \,.\, \breve{R}``\sigma \subset \sigma : \nu \subset \sigma \,.\, \exists ! \nu \cap C`Q \,.\supset_\nu .\, \overrightarrow{\mathrm{lt}_Q}`\nu \subset \sigma : \supset .\, \mu \subset \sigma$ (1)

$\vdash .\, (1) .\, \mathrm{Fact} .\supset \vdash :: \mathrm{Hp} .\supset :.$

$x \in \sigma \,.\, \breve{R}``\sigma \subset \sigma : \nu \subset \sigma \,.\, \exists ! \nu \cap C`Q \,.\supset_\nu .\, \overrightarrow{\mathrm{lt}_Q}`\nu \subset \sigma : \supset .\, \mu \subset \sigma \,.\, \exists ! \mu$ (2)

$\vdash .\, *10·1 .\, *257·111 .\supset$

$\vdash :.\, \nu \subset \sigma \,.\, \exists ! \nu \cap C`Q \,.\supset_\nu .\, \overrightarrow{\mathrm{lt}_Q}`\nu \subset \sigma : \supset : \mathrm{Hp} \,.\, \mu \subset \sigma \,.\, y \,\mathrm{lt}_Q \mu \,.\supset .\, y \in \sigma$ (3)

$\vdash .\, (2) .\, (3) .\supset \vdash :: \mathrm{Hp} \,.\, y \,\mathrm{lt}_Q \mu \,.\supset :.$

$x \in \sigma \,.\, \breve{R}``\sigma \subset \sigma : \nu \subset \sigma \,.\, \exists ! \nu \cap C`Q \,.\supset_\nu .\, \overrightarrow{\mathrm{lt}_Q}`\nu \subset \sigma : \supset .\, y \in \sigma$ (4)

$\vdash .\, (4) .\, *10·11·21 .\, *257·101 .\supset \vdash : \mathrm{Hp} \,.\, y \,\mathrm{lt}_Q \mu \,.\supset .\, y \in (R*Q)`\mu : \supset \vdash .\, \mathrm{Prop}$

***257·131.** $\vdash .\, \delta_Q`(R*Q)`x \subset (R*Q)`x$ [*257·13 . *216·1]

***257·132.** $\vdash : \kappa \subset \mathrm{Cl\,ex}`(R*Q)`x \,.\supset .\, \mathrm{lt}_Q``\kappa \subset (R*Q)`x$ [*257·13]

***257·14.** $\vdash : R \mathrel{G} Q \,.\supset .\, (R*Q)`x \subset \overleftarrow{Q_*}`x$

Dem.

$\vdash .\, *90·163 .\supset \vdash : \mathrm{Hp} .\supset .\, \breve{R}``\overleftarrow{Q_*}`x \subset \overleftarrow{Q_*}`x$ (1)

$\vdash .\, *206·15 .\supset \vdash : \mu \subset \overleftarrow{Q_*}`x \,.\, z \,\mathrm{lt}_Q \mu \,.\, \exists ! \mu \,.\supset .\, z \in p`\overleftarrow{Q}``\mu \,.\, \exists ! \mu \,.\, \mu \subset \overleftarrow{Q_*}`x \,.$

[*40·61.*90·163] $\supset .\, z \in \breve{Q}``\mu \,.\, \breve{Q}``\mu \subset \overleftarrow{Q_*}`x \,.$

[*22·46] $\supset .\, z \in \overleftarrow{Q_*}`x$ (2)

$\vdash .\, (1) .\, (2) .\, *257·11 .\supset \vdash : \mathrm{Hp} \,.\, x \in C`Q \,.\supset .\, (R*Q)`x \subset \overleftarrow{Q_*}`x$ (3)

$\vdash .\, *37·261·29 .\, *60·33 .\, (*216·01) .\supset$

$\vdash : \mathrm{Hp} .\supset .\, \breve{R}``(-C`Q) = \Lambda \,.\, \delta_Q`(-C`Q) = \Lambda$ (4)

$\vdash .\, (4) .\, *257·11 .\supset \vdash : \mathrm{Hp} \,.\, x \sim\in C`Q \,.\supset .\, (R*Q)`x \subset -C`Q \,.$

[*257·111] $\supset .\, (R*Q)`x = \Lambda$ (5)

$\vdash .\, (3) .\, (5) .\supset \vdash .\, \mathrm{Prop}$

***257·141.** $\vdash : R \mathrel{G} Q \,.\supset .\, \breve{R}``C`Q \cup \delta_Q`C`Q \subset C`Q$ [*216·111 . *37·201·16]

***257·142.** $\vdash : R \Subset Q \,.\, x \in C\text{‘}Q \,.\, \supset \,.\, (R_* Q)\text{‘}x = \hat{y}\,\{x \in \sigma \,.\, \breve{R}\text{“}\sigma \cup \delta_Q\text{‘}\sigma \subset \sigma \,.\, \supset_\sigma .\, y \in \sigma\}$

Dem.

$\vdash \,.\, *257\cdot141 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, \hat{y}\,\{x \in \sigma \,.\, \breve{R}\text{“}\sigma \cup \delta_Q\text{‘}\sigma \subset \sigma \,.\, \supset_\sigma .\, y \in \sigma\} \subset C\text{‘}Q \quad (1)$

$\vdash \,.\, (1) \,.\, *257\cdot1 \,.\, \supset \vdash \,.\, \text{Prop}$

***257·15.** $\vdash : y \in (R_* Q)\text{‘}x \,.\, z \in (R_* Q)\text{‘}y \,.\, \supset \,.\, z \in (R_* Q)\text{‘}x$

Dem.

$\vdash \,.\, *257\cdot1 \,.\, \supset \vdash :.\, \breve{R}\text{“}\sigma \cup \delta_Q\text{‘}\sigma \subset \sigma \,.\, \supset : x \in \sigma \,.\, \supset \,.\, y \in \sigma : y \in \sigma \,.\, \supset \,.\, z \in \sigma :$

[Syll] $\supset : x \in \sigma \,.\, \supset \,.\, z \in \sigma \quad (1)$

$\vdash \,.\, (1) \,.\, *257\cdot1 \,.\, \supset \vdash \,.\, \text{Prop}$

***257·16.** $\vdash : x \in C\text{‘}Q - \text{D‘}R \,.\, \supset \,.\, (R_* Q)\text{‘}x = \iota\text{‘}x$

Dem.

$\vdash \,.\, *257\cdot12 \,.\quad \supset \vdash : \text{Hp} \,.\, \supset \,.\, x \in (R_* Q)\text{‘}x \quad (1)$

$\vdash \,.\, *37\cdot261\cdot29 \,.\quad \supset \vdash : \text{Hp} \,.\, \supset \,.\, \breve{R}\text{“}\iota\text{‘}x = \Lambda \quad (2)$

$\vdash \,.\, *205\cdot18 \,.\quad \supset \vdash : \text{Hp} \,.\, \sim \exists \,!\, \overrightarrow{\text{max}}_Q\text{‘}\iota\text{‘}x \,.\, \supset \,.\, xQx \,.$

[*206·42] $\supset \,.\, \overrightarrow{\text{seq}}_Q\text{‘}\iota\text{‘}x = \Lambda \quad (3)$

$\vdash \,.\, (3) \,.\, *216\cdot101 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, \delta_Q\text{‘}\iota\text{‘}x = \Lambda \quad (4)$

$\vdash \,.\, (2) \,.\, (4) \,.\quad \supset \vdash : \text{Hp} \,.\, \supset \,.\, \breve{R}\text{“}\iota\text{‘}x \cup \delta_Q\text{‘}\iota\text{‘}x \subset \iota\text{‘}x \,.$

[*257·11] $\supset \,.\, (R_* Q)\text{‘}x \subset \iota\text{‘}x \quad (5)$

$\vdash \,.\, (1) \,.\, (5) \,.\, \supset \vdash \,.\, \text{Prop}$

We now begin the proof (completed in *257·34) that under certain circumstances $Q_{Rx} \in \Omega$. We first prove that the class σ introduced in *257·2 is transfinitely hereditary, and this requires as a preliminary the proof that if $y \in \sigma$, the class $\overrightarrow{(Q_{Rx})_*}\text{‘}y \cup \overleftarrow{(Q_{Rx})_*}\text{‘}\breve{R}\text{‘}y$ is transfinitely hereditary. This preliminary is provided by *257·2·21. The hypothesis of *257·2 is not all used in *257·2, but is introduced because it is required in the set of propositions of which this is the first.

***257·2.** $\vdash :.\, Q \in \text{Rl‘}J \cap \text{trans} \,.\, R \in \text{Rl‘}Q \cap \text{Cls} \rightarrow 1 \,.$

$\sigma = (R_* Q)\text{‘}x \cap p\text{‘}\overleftrightarrow{Q}\text{“}(R_* Q)\text{‘}x \cap \hat{y}\,\{zQ_{Rx}y \,.\, \supset_z .\, \overleftarrow{Q}_{Rx}\text{‘}z = \overleftarrow{(Q_{Rx})_*}\text{‘}\breve{R}\text{‘}z\} \,.\, \supset :$

$y \in \sigma \,.\, z \in \overrightarrow{(Q_{Rx})_*}\text{‘}y \cup \overleftarrow{(Q_{Rx})_*}\text{‘}\breve{R}\text{‘}y \,.\, z \in \text{D‘}R \,.\, \supset \,.\, \breve{R}\text{‘}z \in \overrightarrow{(Q_{Rx})_*}\text{‘}y \cup \overleftarrow{(Q_{Rx})_*}\text{‘}\breve{R}\text{‘}y$

Dem.

$\vdash \,.\, *90\cdot163 \,.\, *37\cdot62 \,.\, *257\cdot123 \,.\, \supset$

$\vdash :.\, R \Subset Q \,.\, \text{E} \,!\, \breve{R}\text{‘}z \,.\, \supset : z \in \overleftarrow{(Q_{Rx})_*}\text{‘}\breve{R}\text{‘}y \,.\, \supset \,.\, \breve{R}\text{‘}z \in \overleftarrow{(Q_{Rx})_*}\text{‘}\breve{R}\text{‘}y \quad (1)$

$\vdash \,.\, *30\cdot37 \,.\, \supset \vdash : \text{E} \,!\, \breve{R}\text{‘}z \,.\, z = y \,.\, \supset \,.\, \breve{R}\text{‘}z = \breve{R}\text{‘}y \quad (2)$

$\vdash \,.\, *201\cdot18 \,.\, *91\cdot52 \,.\, *32\cdot182 \,.\, \supset$

$\vdash : \text{Hp} \,.\, y \in \sigma \,.\, z \in \overrightarrow{Q}_{Rx}\text{‘}y \,.\, \supset \,.\, \overleftarrow{Q}_{Rx}\text{‘}z = \overleftarrow{(Q_{Rx})_*}\text{‘}\breve{R}\text{‘}z \,.\, y \in \overleftarrow{Q}_{Rx}\text{‘}z$

[*13·13] $\supset \,.\, y \in \overleftarrow{(Q_{Rx})_*}\text{‘}\breve{R}\text{‘}z \,.$

[*32·182] $\supset \,.\, \breve{R}\text{‘}z \in \overrightarrow{(Q_{Rx})_*}\text{‘}y \quad (3)$

$\vdash \,.\, (1) \,.\, (2) \,.\, (3) \,.\, *71\cdot161 \,.\, \supset \vdash \,.\, \text{Prop}$

***257·21.** $\vdash : \text{Hp} *257\cdot 2 . y \in \sigma . \mu \subset \overrightarrow{(Q_{Rx})_*}`y \cup \overleftarrow{(Q_{Rx})_*}`\breve{R}`y . \exists ! \mu . \supset . \overrightarrow{\text{lt}_Q}`\mu \subset \overrightarrow{Q_*}`y \cup \overleftarrow{Q_*}`\breve{R}`y$

Dem.

$\vdash . *201\cdot 14\cdot 15 . *206\cdot 134 . \supset$

$\vdash : \text{Hp} . \exists ! \mu \cap \overleftarrow{Q_*}`\breve{R}`y . \supset . \overrightarrow{\text{lt}_Q}`\mu \subset \overleftarrow{Q_*}`\breve{R}`y$ (1)

$\vdash . *205\cdot 38 . \supset \vdash : \text{Hp} . \mu \subset \overrightarrow{Q_*}`y . y \in \mu . \supset . y \in \overrightarrow{\max_Q}`\mu .$

[*207·11] $\supset . \overrightarrow{\text{lt}_Q}`\mu = \Lambda$ (2)

$\vdash . *40\cdot 55 . *206\cdot 143 . \supset$

$\vdash : \mu \subset \overrightarrow{Q}`y . w \,\text{lt}_Q \mu . \supset . y \in p`\overleftarrow{Q}``\mu . w \sim\in \breve{Q}``p`\overleftarrow{Q}``\mu .$

[*37·1] $\supset . \sim (yQw)$ (3)

$\vdash . *257\cdot 13 . \supset \vdash :. \text{Hp}(3) . \text{Hp} . \supset : yQw . \lor . wQ_* y :$

[(3)] $\supset : wQ_* y$ (4)

$\vdash . (1) . (2) . (4) . \supset \vdash . \text{Prop}$

***257·211.** $\vdash : \text{Hp} *257\cdot 2 . y \in \sigma . \supset . (R * Q)`x \subset \overrightarrow{(Q_{Rx})_*}`y \cup \overleftarrow{(Q_{Rx})_*}`\breve{R}`y$

Dem.

$\vdash . *257\cdot 14 . \supset \vdash : \text{Hp} . \supset . x \in \overrightarrow{(Q_{Rx})_*}`y$ (1)

$\vdash . (1) . *257\cdot 2\cdot 21\cdot 11 . \supset \vdash . \text{Prop}$

***257·22.** $\vdash : \text{Hp} *257\cdot 2 . y \in \sigma . \supset . \overleftarrow{Q_{Rx}}`y = \overleftarrow{(Q_{Rx})_*}`\breve{R}`y . \overrightarrow{(Q_{Rx})_*}`y = \overrightarrow{Q_{Rx}}`\breve{R}`y$

Dem.

$\vdash . *257\cdot 211 . \supset \vdash : \text{Hp} . \supset . \overleftarrow{(Q_{Rx})_*}`\breve{R}`y = (R * Q)`x - \overrightarrow{(Q_{Rx})_*}`y$

[Hp] $= \overleftarrow{Q_{Rx}}`y$ (1)

Similarly $\vdash : \text{Hp} . \supset . \overrightarrow{(Q_{Rx})_*}`y = \overrightarrow{Q_{Rx}}`\breve{R}`y$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

It is to be understood that $\overleftarrow{(Q_{Rx})_*}`\breve{R}`y = \Lambda$ if $\sim E ! \breve{R}`y$.

***257·23.** $\vdash : \text{Hp} *257\cdot 2 . \supset . \breve{R}``\sigma \subset \sigma$

Dem.

$\vdash . *257\cdot 22 . \quad \supset \vdash :. \text{Hp} . y \in \sigma \cap \text{D}`R . \supset : zQ\breve{R}`y . \supset_z . \overleftarrow{Q_{Rx}}`z = \overleftarrow{(Q_{Rx})_*}`\breve{R}`z$ (1)

$\vdash . *257\cdot 22\cdot 211 . \supset \vdash : \text{Hp} . y \in \sigma \cap \text{D}`R . \supset . (R * Q)`x = \overrightarrow{Q_{Rx}}`\breve{R}`y \cup \overleftarrow{(Q_{Rx})_*}`\breve{R}`y$ (2)

$\vdash . (1) . (2) . \quad \supset \vdash : \text{Hp} . y \in \sigma \cap \text{D}`R . \supset . \breve{R}`y \in \sigma : \supset \vdash . \text{Prop}$

The above proposition gives the first stage in the proof that σ is transfinitely hereditary. The second stage, similarly, requires as a preliminary the proof that if μ is an existent sub-class of σ having no maximum, then

$$Q_{Rx}``\mu \cup (\breve{Q}_{Rx})_*``\overrightarrow{\text{lt}_Q}`\mu$$

is a transfinitely hereditary class. This proof is provided by *257·24·241·242.

***257·24.** $\vdash : \mathrm{Hp} \, {*}257{\cdot}2 \,.\, \mu \subset \sigma \,.\, \exists ! \mu \,.\, \sim \exists ! \overrightarrow{\max}_Q{}^{\backprime}\mu \,.\, \supset \,.\, \breve{R}``Q_{Rx}``\mu \subset Q_{Rx}``\mu$

Dem.

$\vdash . {*}91{\cdot}52 . {*}201{\cdot}18 . \supset \vdash : \mathrm{Hp} . z \in Q_{Rx}``\mu . \supset . \overleftarrow{Q}_{Rx}{}^{\backprime}z = \overleftarrow{(Q_{Rx})_*}{}^{\backprime}\breve{R}{}^{\backprime}z .$

$[{*}37{\cdot}46 . {*}13{\cdot}12] \quad \supset . \exists ! \overleftarrow{(Q_{Rx})_*}{}^{\backprime}\breve{R}{}^{\backprime}z \cap \mu .$

$[{*}37{\cdot}46] \quad \supset . \breve{R}{}^{\backprime}z \in (Q_{Rx})_*``\mu \qquad (1)$

$\vdash . {*}205{\cdot}123 . \quad \supset \vdash : \mathrm{Hp} . \supset . \mu \subset Q_{Rx}``\mu \qquad (2)$

$\vdash . (1) . (2) . \quad \supset \vdash : \mathrm{Hp} . z \in Q_{Rx}``\mu . \supset . \breve{R}{}^{\backprime}z \in Q_{Rx}``\mu : \supset \vdash . \mathrm{Prop}$

***257·241.** $\vdash : \mathrm{Hp} \, {*}257{\cdot}24 . \supset . \breve{R}``\{Q_{Rx}``\mu \cup (\breve{Q}_{Rx})_*``\overrightarrow{\mathrm{lt}}_Q{}^{\backprime}\mu\} \subset Q_{Rx}``\mu \subset (\breve{Q}_{Rx})_*``\overrightarrow{\mathrm{lt}}_Q{}^{\backprime}\mu$

Dem.

$\vdash . {*}90{\cdot}164 . \supset \vdash : R \in Q . \supset . \breve{R}``(\breve{Q}_{Rx})_*``\overrightarrow{\mathrm{lt}}_Q{}^{\backprime}\mu \subset (\breve{Q}_{Rx})_*``\overrightarrow{\mathrm{lt}}_Q{}^{\backprime}\mu \qquad (1)$

$\vdash . (1) . {*}257{\cdot}24 . \supset \vdash . \mathrm{Prop}$

***257·242.** $\vdash : \mathrm{Hp} \, {*}257{\cdot}24 . \rho = Q_{Rx}``\mu \cup (\breve{Q}_{Rx})_*``\overrightarrow{\mathrm{lt}}_Q{}^{\backprime}\mu .$

$\alpha \subset \rho . \exists ! \alpha . \sim \exists ! \overrightarrow{\max}_Q{}^{\backprime}\alpha . \supset . \overrightarrow{\mathrm{lt}}_Q{}^{\backprime}\alpha \subset \rho$

Dem.

$\vdash . {*}206{\cdot}15 . \quad \supset \vdash : \mathrm{Hp} . \exists ! \mu \cap p{}^{\backprime}\overleftarrow{Q}``\alpha . w \, \mathrm{lt}_Q \alpha . \supset . \exists ! \mu - \overrightarrow{Q}{}^{\backprime}w \qquad (1)$

$\vdash . {*}201{\cdot}521 . \supset \vdash : \mathrm{Hp} . \mu \subset \sigma . \supset . \mu - \overrightarrow{Q}{}^{\backprime}w \subset \overleftarrow{Q_*}{}^{\backprime}w \qquad (2)$

$\vdash . (1) . (2) . \quad \supset \vdash : \mathrm{Hp}(1) . \supset . \exists ! \mu \cap \overleftarrow{Q_*}{}^{\backprime}w \qquad (3)$

$\vdash . {*}205{\cdot}123 . \supset \vdash : \mathrm{Hp} . \supset . \mu \subset Q``\mu \qquad (4)$

$\vdash . (3) . (4) . \quad \supset \vdash : \mathrm{Hp}(1) . \supset . w \in Q_{Rx}``\mu \qquad (5)$

$\vdash . {*}206{\cdot}24 . \quad \supset \vdash : \mathrm{Hp} . \mu \subset Q``\alpha . \alpha \subset Q``\mu . \supset . \overrightarrow{\mathrm{lt}}_Q{}^{\backprime}\alpha = \overrightarrow{\mathrm{lt}}_Q{}^{\backprime}\mu \qquad (6)$

$\vdash . {*}206{\cdot}15 . \quad \supset \vdash : \mathrm{Hp} . \exists ! \alpha \cap (\breve{Q}_{Rx})_*``\overrightarrow{\mathrm{lt}}_Q{}^{\backprime}\mu . \supset . \overrightarrow{\mathrm{lt}}_Q{}^{\backprime}\alpha \subset (\breve{Q}_{Rx})_*``\overrightarrow{\mathrm{lt}}_Q{}^{\backprime}\mu \qquad (7)$

$\vdash . (5) . (6) . (7) . \supset \vdash . \mathrm{Prop}$

***257·243.** $\vdash : \mathrm{Hp} \, {*}257{\cdot}24 . \supset . (R{*}Q){}^{\backprime}x = Q_{Rx}``\mu \cup p{}^{\backprime}\overleftarrow{Q}_{Rx}``\mu \quad [{*}40{\cdot}53 . {*}205{\cdot}123]$

***257·25.** $\vdash : \mathrm{Hp} \, {*}257{\cdot}24 . \supset . (R{*}Q){}^{\backprime}x = Q_{Rx}``\mu \cup (\breve{Q}_{Rx})_*``\overrightarrow{\mathrm{lt}}_Q{}^{\backprime}\mu$

Dem.

$\vdash . {*}257{\cdot}242 . \supset \vdash : \mathrm{Hp} . \supset . \delta_Q{}^{\backprime}\{Q_{Rx}``\mu \cup (\breve{Q}_{Rx})_*``\overrightarrow{\mathrm{lt}}_Q{}^{\backprime}\mu\} \subset Q_{Rx}``\mu \cup (\breve{Q}_{Rx})_*``\overrightarrow{\mathrm{lt}}_Q{}^{\backprime}\mu \qquad (1)$

$\vdash . (1) . {*}257{\cdot}241 . \supset \vdash . \mathrm{Prop}$

***257·251.** $\vdash : \mathrm{Hp} \, {*}257{\cdot}24 . \supset . (\breve{Q}_{Rx})_*``\overrightarrow{\mathrm{lt}}_Q{}^{\backprime}\mu = p{}^{\backprime}\overleftarrow{Q}_{Rx}``\mu$

Dem.

$\vdash . {*}257{\cdot}25{\cdot}243 . \supset \vdash : \mathrm{Hp} . \supset . Q_{Rx}``\mu \cup (\breve{Q}_{Rx})_*``\overrightarrow{\mathrm{lt}}_Q{}^{\backprime}\mu = Q_{Rx}``\mu \cup p{}^{\backprime}\overleftarrow{Q}_{Rx}``\mu .$

$[{*}200{\cdot}53 . {*}24{\cdot}481] \quad \supset . (\breve{Q}_{Rx})_*``\overrightarrow{\mathrm{lt}}_Q{}^{\backprime}\mu = p{}^{\backprime}\overleftarrow{Q}_{Rx}``\mu : \supset \vdash . \mathrm{Prop}$

***257·252.** $\vdash : \mathrm{Hp}\, *257\cdot24 \,.\, \exists !\, p\text{‘}\overleftarrow{Q}_{Rx}\text{“}\mu \,.\supset .\, Q_{Rx}\text{“}\mu = p\text{‘}\overrightarrow{Q}_{Rx}\text{“}\overrightarrow{\mathrm{lt}}_Q\text{‘}\mu \,.\, \exists !\, \overrightarrow{\mathrm{lt}}_Q\text{‘}\mu$

Dem.

$\vdash . *257\cdot251 . *37\cdot29 . \supset \vdash : \mathrm{Hp} . \supset . \exists !\, \overrightarrow{\mathrm{lt}}_Q\text{‘}\mu$ (1)

[*200·53.*40·62] $\supset . p\text{‘}\overrightarrow{Q}_{Rx}\text{“}\overrightarrow{\mathrm{lt}}_Q\text{‘}\mu \subset (R*Q)\text{‘}x - (\breve{Q}_{Rx})_*\text{“}\overrightarrow{\mathrm{lt}}_Q\text{‘}\mu$

[*257·251] $\subset (R*Q)\text{‘}x - p\text{‘}\overleftarrow{Q}_{Rx}\text{“}\mu$

[Hp.*10·57.*257·243] $\subset Q_{Rx}\text{“}\mu$ (2)

$\vdash . *201\cdot51 . *40\cdot67 . \supset \vdash : \mathrm{Hp} . \supset . Q_{Rx}\text{“}\mu \subset p\text{‘}\overrightarrow{Q}_{Rx}\text{“}\overrightarrow{\mathrm{lt}}_Q\text{‘}\mu$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash . \mathrm{Prop}$

In order to complete the proof that σ is a hereditary class, we have to introduce the additional hypothesis

$$\mathrm{lt}_Q \upharpoonright \mathrm{Cl\,ex}\text{‘}(R*Q)\text{‘}x \in 1 \rightarrow \mathrm{Cls}.$$

With the help of this hypothesis, the last stage of the proof is provided by the following proposition.

***257·26.** $\vdash : \mathrm{Hp}\, *257\cdot2 \,.\, \mathrm{lt}_Q \upharpoonright \mathrm{Cl\,ex}\text{‘}(R*Q)\text{‘}x \in 1 \rightarrow \mathrm{Cls} \,.\supset .\, \delta_Q\text{‘}\sigma \subset \sigma$

Dem.

$\vdash . *257\cdot251\cdot252 . \supset \vdash :. \mathrm{Hp} . \mu \subset \sigma . \exists !\, \mu . \exists !\, \overrightarrow{\mathrm{lt}}_Q\text{‘}\mu . \supset :$

$(R*Q)\text{‘}x = \overrightarrow{Q}_{Rx}\text{‘}\mathrm{lt}_Q\text{‘}\mu \cup \overleftarrow{(Q_{Rx})_*}\text{‘}\mathrm{lt}_Q\text{‘}\mu \,.\, \overrightarrow{Q}_{Rx}\text{‘}\mathrm{lt}_Q\text{‘}\mu = Q_{Rx}\text{“}\mu :$

$[\mathrm{Hp}] \supset : \mathrm{lt}_Q\text{‘}\mu \in p\text{‘}\overleftrightarrow{Q}\text{“}(R*Q)\text{‘}x : yQ_{Rx}\mathrm{lt}_Q\text{‘}\mu . \supset_y . \overleftarrow{Q}_{Rx}\text{‘}y = \overleftarrow{(Q_{Rx})_*}\text{‘}\breve{R}\text{‘}y :$

$[\mathrm{Hp}] \supset : \mathrm{lt}_Q\text{‘}\mu \in \sigma :. \supset \vdash . \mathrm{Prop}$

***257·261.** $\vdash : \mathrm{Hp}\, *257\cdot26 \,.\supset .\, (R*Q)\text{‘}x = \sigma$ [*257·11·23·26]

***257·27.** $\vdash : Q \in \mathrm{Rl}\text{‘}J \cap \mathrm{trans} \,.\, R \in \mathrm{Rl}\text{‘}Q \cap \mathrm{Cls} \rightarrow 1 \,.$

$\mathrm{lt}_Q \upharpoonright \mathrm{Cl\,ex}\text{‘}(R*Q)\text{‘}x \in 1 \rightarrow \mathrm{Cls} \,.\supset .$

$Q_{Rx} \in \mathrm{Ser} \,.\, Q_{Rx} = (R \,|\, Q_*) \upharpoonleft (R*Q)\text{‘}x$

Dem.

$\vdash . *257\cdot261 . \supset$

$\vdash : \mathrm{Hp} . \supset . (R*Q)\text{‘}x \subset p\text{‘}\overleftrightarrow{Q}\text{“}(R*Q)\text{‘}x \cap \hat{y}\,\{zQ_{Rx}y . \supset_z . \overleftarrow{Q}_{Rx}\text{‘}z = \overleftarrow{(Q_{Rx})_*}\text{‘}\breve{R}\text{‘}z\}$ (1)

$\vdash . (1) . \supset \vdash :: \mathrm{Hp} . \supset :. Q_{Rx} \in \mathrm{connex} :. z \in \mathrm{D}\text{‘}Q_{Rx} . \supset_z : zQ_{Rx}w . \equiv_w . zR \,|\, (Q_{Rx})_* w :.$

[*5·32.*4·71.*257·121]

$\supset :. Q_{Rx} \in \mathrm{connex} :. zQ_{Rx}w . \equiv_{z,w} . z \in \mathrm{D}\text{‘}Q_{Rx} . zR \,|\, Q_* w . w \in C\text{‘}Q_{Rx} :.$

[*36·13.*257·121] $\supset :. Q_{Rx} \in \mathrm{connex} \,.\, Q_{Rx} = (R \,|\, Q_*) \upharpoonleft (R*Q)\text{‘}x :: \supset \vdash . \mathrm{Prop}$

We have thus proved that Q_{Rx} is a series. No additional hypothesis is required to prove that it is well-ordered, as we shall now show.

***257·28.** $\vdash : \mathrm{Hp}\, *257\cdot27 \,.\, \mu \subset (R*Q)\text{‘}x \,.\, \exists !\, \mu \,.\, \overrightarrow{\mathrm{max}}_Q\text{‘}\mu = \Lambda \,.\, \exists !\, p\text{‘}\overleftarrow{Q}_{Rx}\text{“}\mu \,.\supset .$

$p\text{‘}\overleftarrow{Q}_{Rx}\text{“}\mu = (\breve{Q}_{Rx})_*\text{“}\overleftarrow{\mathrm{lt}}_Q\text{‘}\mu \,.\, Q_{Rx}\text{“}\mu = p\text{‘}\overrightarrow{Q}_{Rx}\text{“}\overrightarrow{\mathrm{lt}}_Q\text{‘}\mu$ [*257·251·27]

***257·281.** $\vdash : \text{Hp} *257\cdot28 . \text{E} ! \text{lt}_Q{}^{\backprime}\mu . \supset .$
$$p^{\backprime}\overleftarrow{Q}_{Rx}{}^{\backprime\backprime}\mu = \overleftarrow{(Q_{Rx})}_{*}{}^{\backprime}\text{lt}_Q{}^{\backprime}\mu . Q_{Rx}{}^{\backprime\backprime}\mu = \overrightarrow{Q}_{Rx}{}^{\backprime}\text{lt}_Q{}^{\backprime}\mu \quad [*257\cdot28]$$

***257·29.** $\vdash : \text{Hp} *257\cdot27 . x \in \text{D}^{\backprime}R . \supset . C^{\backprime}Q_{Rx} = (R*Q)^{\backprime}x . B^{\backprime}Q_{Rx} = x$

Dem.

$\vdash . *257\cdot27\cdot126 . *202\cdot55 . \supset \vdash : \text{Hp} . \supset . C^{\backprime}Q_{Rx} = (R*Q)^{\backprime}x \quad (1)$

$\vdash . *257\cdot14 . \quad \supset \vdash : \text{Hp} . \supset . (R*Q)^{\backprime}x - \iota^{\backprime}x \subset \overleftarrow{Q}_{Rx}{}^{\backprime}x \quad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***257·291.** $\vdash : \text{Hp} *257\cdot27 . x \sim\in \text{D}^{\backprime}R . \supset . Q_{Rx} = \dot{\Lambda} \quad [*257\cdot16 . *200\cdot35]$

***257·3.** $\vdash : \text{Hp} *257\cdot27 . \supset . \text{D}^{\backprime}Q_{Rx} = \text{D}^{\backprime}R \cap (R*Q)^{\backprime}x$

Dem.

$\vdash . *257\cdot27 . \supset \vdash :. \text{Hp} . y \in (R*Q)^{\backprime}x . \supset : \exists ! \overleftarrow{Q}^{\backprime}y . \equiv . \exists ! \overleftarrow{Q}_{*}{}^{\backprime}\breve{R}^{\backprime}y .$

$[*257\cdot141] \quad \equiv . \text{E} ! \breve{R}^{\backprime}y :. \supset \vdash . \text{Prop}$

***257·31.** $\vdash : \text{Hp} *257\cdot27 . \mu \subset (R*Q)^{\backprime}x . \exists ! \mu . \sim\exists ! \overrightarrow{\max}_Q{}^{\backprime}\mu . \exists ! p^{\backprime}\overleftarrow{Q}_{Rx}{}^{\backprime\backprime}\mu . \supset .$
$$\text{seq}\,(Q_{Rx})^{\backprime}\mu = \text{lt}_Q{}^{\backprime}\mu \quad [*257\cdot28]$$

***257·32.** $\vdash : \text{Hp} *257\cdot27 . \mu \subset (R*Q)^{\backprime}x . \exists ! \overrightarrow{\max}_Q{}^{\backprime}\mu . \exists ! p^{\backprime}\overleftarrow{Q}_{Rx}{}^{\backprime\backprime}\mu . \supset .$
$$\text{seq}\,(Q_{Rx})^{\backprime}\mu = \breve{R}^{\backprime}\max\,(Q_{Rx})^{\backprime}\mu$$

Dem.

$\vdash . *257\cdot3 . \supset \vdash : \text{Hp} . \supset . \mu \subset \text{D}^{\backprime}R .$

$[*257\cdot27.\text{Transp}] \quad \supset . \overrightarrow{Q}_{*}{}^{\backprime}\max\,(Q_{Rx})^{\backprime}\mu = \overrightarrow{Q}^{\backprime}\breve{R}^{\backprime}\max\,(Q_{Rx})^{\backprime}\mu : \supset \vdash . \text{Prop}$

***257·33.** $\vdash : \text{Hp} *257\cdot27 . \mu \subset (R*Q)^{\backprime}x . \exists ! \mu . \exists ! p^{\backprime}\overleftarrow{Q}_{Rx}{}^{\backprime\backprime}\mu . \supset . \text{E} ! \text{seq}\,(Q_{Rx})^{\backprime}\mu$
$[*257\cdot31\cdot32]$

The above proposition together with *257·27 shows that Q_{Rx} is well-ordered, in virtue of *250·123.

***257·34.** $\vdash : \text{Hp} *257\cdot27 . \supset . Q_{Rx} \in \Omega$

Dem.

$\vdash . *257\cdot291 . \quad \supset \vdash : \text{Hp} . x \sim\in \text{D}^{\backprime}R . \supset . Q_{Rx} \in \Omega \quad (1)$

$\vdash . *257\cdot29 . *206\cdot14 . \supset \vdash : \text{Hp} . x \in \text{D}^{\backprime}R . \supset . \text{seq}_P{}^{\backprime}\Lambda = x \quad (2)$

$\vdash . (2) . *257\cdot33 . \supset$

$\vdash :. \text{Hp} . x \in \text{D}^{\backprime}R . \supset : \mu \subset (R*Q)^{\backprime}x . \exists ! p^{\backprime}\overleftarrow{Q}_{Rx}{}^{\backprime\backprime}\mu . \supset_\mu . \text{E} ! \text{seq}\,(Q_{Rx})^{\backprime}\mu :$

$[*257\cdot29.*206\cdot131] \supset : \exists ! p^{\backprime}\overleftarrow{Q}_{Rx}{}^{\backprime\backprime}(\mu \cap C^{\backprime}Q_{Rx}) . \supset_\mu . \text{E} ! \text{seq}\,(Q_{Rx})^{\backprime}\mu :$

$[*250\cdot123.*257\cdot27] \supset : Q_{Rx} \in \Omega \quad (3)$

$\vdash . (1) . (3) . \supset \vdash . \text{Prop}$

***257·35.** $\vdash : \mathrm{Hp} *257{\cdot}27 . \supset . R \upharpoonright (R*Q)`x = (Q_{Rx})_1 . R \upharpoonright (R*Q)`x \in 1 \to 1$

Dem.

$\vdash . *257{\cdot}32 . \supset \vdash :. \mathrm{Hp} . \supset : y \in \mathrm{D}`Q_{Rx} . \supset . \mathrm{seq}\,(Q_{Rx})`\iota`y = \breve{R}`y$ (1)

$\vdash . (1) . *206{\cdot}43 . *204{\cdot}7 . \supset \vdash . \mathrm{Prop}$

***257·36.** $\vdash : \mathrm{Hp} *257{\cdot}27 . x \in \mathrm{D}`R . \supset .$

$$C`Q_{Rx} = (R*Q)`x . \mathrm{G}`Q_{Rx} = (R*Q)`x - \iota`x .$$

$$B`Q_{Rx} = x . \overrightarrow{B}`\breve{Q}_{Rx} = (R*Q)`x - \mathrm{D}`R \quad [*257{\cdot}29{\cdot}3]$$

The following propositions are concerned in showing that a relation P which satisfies the hypothesis of *257·5 is identical with Q_{Rx}, thus showing that this hypothesis is sufficient to determine P.

***257·5.** $\vdash : \mathrm{Hp} *257{\cdot}27 . P \in \mathrm{trans} . C`P \subset (R*Q)`x . P \dot{-} P^2 = R \upharpoonright (R*Q)`x$

$\mathrm{lt}_P \upharpoonright \mathrm{Cl\,ex}`(R*Q)`x = \mathrm{lt}_Q \upharpoonright \mathrm{Cl\,ex}`(R*Q)`x . \supset . P \mathrel{\mathsf{G}} J . C`P = (R*Q)`x$

The above hypothesis is not all necessary for the present proposition, but it is necessary for the series of propositions of which this is the first.

Dem.

$\vdash . *37{\cdot}41 . \supset \vdash :. \mathrm{Hp} . \supset : \mathrm{D}`(P \dot{-} P^2) = R``(R*Q)`x \cap (R*Q)`x$

$[*257{\cdot}36] \qquad = (R*Q)`x \cap \mathrm{D}`R$ (1)

$\vdash . *32{\cdot}14 . \supset \vdash : \mathrm{Hp} . \supset . \overrightarrow{\mathrm{lt}_P}`\{(R*Q)`x \cap \mathrm{D}`R\} = \overrightarrow{\mathrm{lt}_Q}`\{(R*Q)`x \cap \mathrm{D}`R\}$

$[*257{\cdot}36] \qquad = (R*Q)`x - \mathrm{D}`R$ (2)

$\vdash . (1) . (2) . \supset \vdash : \mathrm{Hp} . \supset . (R*Q)`x \subset C`P .$

$[\mathrm{Hp}] \qquad \supset . (R*Q)`x = C`P$ (3)

$\vdash . (3) . \qquad \supset \vdash : \mathrm{Hp} . \supset : x \in \mathrm{D}`P . \supset . x P \dot{-} P^2 (\breve{R}`x) .$

$[*34{\cdot}5 . \mathrm{Transp}] \qquad \supset . \sim (xPx)$ (4)

$\vdash . (3) . (4) . \supset \vdash . \mathrm{Prop}$

***257·51.** $\vdash : \mathrm{Hp} *257{\cdot}5 . \supset . C`P = \overleftarrow{P_*}`x$

Dem.

$\vdash . *257{\cdot}123 . *90{\cdot}16 . \supset \vdash : \mathrm{Hp} . \supset . \breve{R}``\overleftarrow{P_*}`x \subset \overleftarrow{P_*}`x$ (1)

$\vdash . *90{\cdot}13 . \qquad \supset \vdash : \mathrm{Hp} . \supset . \mathrm{lt}_Q``\mathrm{Cl\,ex}`\overleftarrow{P_*}`x = \mathrm{lt}_P``\mathrm{Cl\,ex}`\overleftarrow{P_*}`x .$

$[*90{\cdot}163 . *40{\cdot}61] \qquad \supset . \mathrm{lt}_Q``\mathrm{Cl\,ex}`\overleftarrow{P_*}`x \subset \overleftarrow{P_*}`x$ (2)

$\vdash . (1) . (2) . \qquad \supset \vdash : \mathrm{Hp} . \supset . (R*Q)`x \subset \overleftarrow{P_*}`x$ (3)

$\vdash . (3) . *257{\cdot}5 . \supset \vdash . \mathrm{Prop}$

In order to prove $P = Q_{Rx}$ we first prove $P \in \Omega$. The proof proceeds as for Q_{Rx}, but in some points it is easier. It is merely outlined below, as it closely resembles the proof for Q_{Rx}.

***257·52.** $\vdash : \mathrm{Hp} *257{\cdot}5 .$

$$\sigma = C`P \cap p`\overleftrightarrow{P}``C`P \cap \hat{y}(zPy . \supset_z . \overleftarrow{P}`z = \overleftarrow{P_*}`\breve{R}`z) . \supset . \breve{R}``\sigma \subset \sigma$$

Dem.

$\vdash . *34\cdot5 . \text{Transp} . *201\cdot18 . \supset \vdash :. P_1 = R \upharpoonright (R*Q)`x . y \in p`\overleftrightarrow{P}``C`P . \supset :$

$zP(\breve{R}`y) . \supset . \sim(yPz) : zP_*y . \supset . zP(\breve{R}`y) :$

[Hp] $\supset : zP(\breve{R}`y) . \equiv . zP_*y$ (1)

As in $*257\cdot2\cdot21$, using $\text{lt}_P \upharpoonright \text{Cl ex}`(R*Q)`x = \text{lt}_Q \upharpoonright \text{Cl ex}`(R*Q)`x$. we prove

$\vdash : \text{Hp} . y \in \sigma \cap \text{D}`R . \rho = \overrightarrow{P_*}`y \cup \overleftarrow{P_*}`\breve{R}`y . \supset . \breve{R}``\rho \subset \rho . \delta_Q`\rho \subset \rho .$

$\supset . (R*Q)`x = \overrightarrow{P_*}`y \cup \overleftarrow{P_*}`\breve{R}`y$ (2)

$\vdash . (1) . (2) . \supset \vdash : \text{Hp} . y \in \sigma \cap \text{D}`R . \supset . \overleftarrow{P}`y = \overleftarrow{P_*}`\breve{R}`y$ (3)

$\vdash . (1) . (3) . \supset \vdash : \text{Hp} . y \in \sigma \cap \text{D}`R . \supset . \breve{R}`y \in \sigma :. \supset \vdash . \text{Prop}$

$*257\cdot521$. $\vdash : \text{Hp } *257\cdot52 . \mu \subset \sigma . \exists ! \mu . \sim\exists ! \overrightarrow{\text{max}_P}`\mu . \supset .$

$(R*Q)`x = P``\mu \cup \breve{P}_*``\overrightarrow{\text{lt}_P}`\mu$

[Proof as in $*257\cdot25$, by similar stages]

$*257\cdot53$. $\vdash :. \text{Hp } *257\cdot5 . \supset : P \in \text{Ser} : z \in \text{D}`P . \supset_z . \overleftarrow{P}`z = \overleftarrow{P_*}`\breve{R}`z$

[Proof as in $*257\cdot27$]

$*257\cdot54$. $\vdash : \text{Hp } *257\cdot5 . \supset . P \in \Omega$ [Proof as in $*257\cdot34$]

$*257\cdot55$. $\vdash : \text{Hp } *257\cdot5 . \sigma = \hat{y}(\overrightarrow{P}`y = \overrightarrow{Q_{Rx}}`y) . \supset . \breve{R}``\sigma \subset \sigma$

Dem.

$\vdash . *257\cdot53 . \supset \vdash : \text{Hp} . y \in C`P . \supset . \overrightarrow{P}`\breve{R}`y = C`P - \overleftarrow{P_*}`\breve{R}`y$

[$*257\cdot53$] $= C`P - \overleftarrow{P}`y$

[$*257\cdot53$] $= \overrightarrow{P}`y \cup \iota`y$ (1)

$\vdash . (1) . \supset \vdash : \text{Hp} . y \in \sigma . \supset . \overrightarrow{P}`\breve{R}`y = \overrightarrow{Q_{Rx}}`y \cup \iota`y$

[$*257\cdot22$] $= \overrightarrow{Q_{Rx}}`\breve{R}`y : \supset \vdash . \text{Prop}$

$*257\cdot551$. $\vdash : \text{Hp } *257\cdot55 . \supset . \delta_Q`\sigma \subset \sigma$

Dem.

$\vdash . *257\cdot53 . \supset$

$\vdash : \text{Hp} . \mu \subset \sigma . \exists ! \mu . z = \text{lt}_Q`\mu . \supset . \overrightarrow{P}`z = \{(R*Q)`x \cap \mu\} \cup P``\mu$

[Hp] $= \{(R*Q)`x \cap \mu\} \cup Q_{Rx}``\mu$

[$*257\cdot27$] $= \overrightarrow{Q_{Rx}}`z : \supset \vdash . \text{Prop}$

$*257\cdot56$. $\vdash : \text{Hp } *257\cdot5 . \supset . P = Q_{Rx}$

Dem.

$\vdash . *257\cdot51\cdot54 . \supset \vdash : \text{Hp} . \supset . \overrightarrow{P}`x = \Lambda .$

[$*257\cdot36$] $\supset . \overrightarrow{P}`x = \overrightarrow{Q_{Rx}}`x$ (1)

$\vdash . (1) . *257\cdot55\cdot551 . \supset \vdash :. \text{Hp} . \supset : y \in C`P . \supset_y . \overrightarrow{P}`y = \overrightarrow{Q_{Rx}}`y :. \supset \vdash . \text{Prop}$

This proves that the conditions in the hypothesis of $*257\cdot5$ are sufficient to determine P.

Summary of *258.

In this number, we shall first show the applicability of the propositions of *257 to the case where the Q of that number is replaced by logical inclusion combined with diversity, *i.e.* by any one of the four relations:

$$\hat{\alpha}\hat{\beta}(\alpha \subset \beta \,.\, \alpha \neq \beta), \quad \hat{\alpha}\hat{\beta}(\beta \subset \alpha \,.\, \alpha \neq \beta),$$

$$\hat{M}\hat{N}(M \Subset N \,.\, M \neq N), \quad \hat{M}\hat{N}(N \Subset M \,.\, M \neq N).$$

If we put $\qquad Q = \hat{\alpha}\hat{\beta}(\alpha \subset \beta \,.\, \alpha \neq \beta),$

and if κ is any class of classes, then $s`\kappa$ is the maximum of $\kappa$ with respect to $Q$ if $s`\kappa \in \kappa$, and the sequent of κ with respect to Q if $s`\kappa \sim\in \kappa$ (*258·1·11); similarly $p`\kappa$ is the minimum of κ if $p`\kappa \in \kappa$ and the precedent of $\kappa$ if $p`\kappa \sim\in \kappa$ (*258·101·111). Hence every class of classes has a unique maximum or a unique sequent with respect to Q, and every class of classes has a unique minimum or a unique precedent (*258·12); we have, moreover,

$$\text{lt}_Q = s \upharpoonright (-\text{Ꭰ}`\text{max}_Q) \,.\, \text{tl}_Q = p \upharpoonright (-\text{Ꭰ}`\text{min}_Q) \quad (*258{\cdot}13{\cdot}131).$$

Hence $\text{lt}_Q, \text{tl}_Q \in 1 \to \text{Cls}$ (*258·14), and Q and $\breve{Q}$ therefore satisfy the most exacting part of the hypothesis of *257·27. Also Q and $\breve{Q}$ are Dedekindian relations (*258·14). (They are not series, because they are not connected.)

An exactly similar argument applies to $\hat{M}\hat{N}(M \Subset N \,.\, M \neq N)$. Hence if Q is any one of the above four relations, and if R is a many-one contained in Q, it follows from *257·34 that Q with its field limited to the transfinite posterity of any term is a well-ordered series. If we take $Q = \hat{\alpha}\hat{\beta}(\alpha \subset \beta \,.\, \alpha \neq \beta)$, and take any initial term α, our series proceeds to continually larger classes, proceeding to the limit by taking the logical sum, *i.e.* if κ is any existent sub-class of the posterity of α, $s`\kappa = \text{limax}_Q`\kappa = \text{limax}\,(Q_{R\alpha})`\kappa$ (*258·21·22), where $Q_{R\alpha}$ has the meaning defined in *257. This process stops with $s`\{\text{D}`R \cap (R_*Q)`x\}$ if $\text{D}`R \cap (R_*Q)`x$ has no maximum; otherwise, it stops with the R-successor of this maximum, which is $\text{max}_Q`\{C`R \cap (R_*Q)`x\}$. If, on the other hand, we take $Q$ to be the converse of the above, we proceed to continually smaller classes, and the limit of any set of classes $\kappa$ having no last term is $p`\kappa$. In this case, if, starting from α, every existent sub-class of α belongs to $\text{D}`R$, the process of diminution cannot stop short of Λ. This is

the process applied in Zermelo's theorem. We have the e a class μ, assumed to be not a unit class, and a selective relation S for existent sub-classes of μ, *i.e.* a relation S for which $S \epsilon \epsilon_\Delta$'Cl ex'$\mu$. Then our relation R is the relation of α to $\alpha - \iota$'S'α, *i.e.* the relation of an existent sub-class of μ to the class resulting from taking away its S-representative. Thus $Q_{R\mu}$ is a well-ordered series, which starts from μ and ends with Λ. Omitting the final Λ, S selects a representative from every member of the field of $Q_{R\mu}$, and the series of these representatives. *i.e.* $S;Q_{R\mu}$, is similar to $Q_{R\mu}$ with the final Λ omitted. Moreover every member of μ occurs among these representatives, for, if x be any member of μ, let κ be the class of those members of C'$Q_{R\mu}$ of which x is a member. (There are such classes, because $\mu \epsilon C$'Q_{Rx} and $x \epsilon \mu$.) Then $x \epsilon p$'κ, and by what was said earlier, p'κ is a member of C'$Q_{R\mu}$. Hence, by the definition of κ, p'$\kappa \epsilon \kappa$, and therefore p'$\kappa = \max_Q$'κ. But no class smaller than p'κ can belong to κ, and therefore p'$\kappa - \iota$'S'p'κ is not a member of κ, and therefore x is not a member of p'$\kappa - \iota$'S'p'κ. Hence $x = S$'p'κ, and therefore x occurs among the representatives of members of C'$Q_{R\mu}$, which was to be proved. (The above is an abbreviated rendering of the symbolic proof given below in ∗258·301.) Hence the field of $S;Q_{R\mu}$ is μ, and therefore there is a well-ordered series having μ for its field, provided ϵ_Δ'Cl ex'μ is not null (∗258·32). This is Zermelo's theorem.

The converse of Zermelo's theorem has been already proved (∗250·51). Hence the assumption that a selection can be made from all the existent sub-classes of μ is equivalent to the assumption that μ can be well-ordered or is a unit class, *i.e.*

∗258·36. $\vdash : \mu \epsilon C``\Omega \cup 1 . \equiv . \exists ! \epsilon_\Delta$'Cl ex'$\mu$

Hence also, by ∗88·33, the multiplicative axiom is equivalent to the assumption that all classes except unit classes can be well-ordered, *i.e.*

∗258·37. $\vdash :$ Mult ax $. \equiv . C``\Omega \cup 1 =$ Cls

Hence also, in virtue of ∗255·73, the multiplicative axiom implies that of any two unequal existent cardinals one must be the greater, *i.e.*

∗258·39. $\vdash ::$ Mult ax $. \supset :. \mu, \nu \epsilon N_0C . \supset : \mu \leqslant \nu . \vee . \mu > \nu$

∗258·1. $\vdash :. Q = \hat{\alpha}\hat{\beta}(\alpha \subset \beta . \alpha \neq \beta) . \supset : s$'$\kappa \epsilon \kappa . \supset . s$'$\kappa = \max_Q$'$\kappa$

Dem.

$\vdash .$ ∗205·101 $. \supset \vdash ::$ Hp $. \supset :. \gamma \max_Q \kappa . \equiv : \gamma \epsilon \kappa : \alpha \epsilon \kappa . \supset_\alpha . \sim(\gamma \subset \alpha . \gamma \neq \alpha) :$

[Transp] $\equiv : \gamma \epsilon \kappa : \alpha \epsilon \kappa . \alpha \neq \gamma . \supset_\alpha . \sim(\gamma \subset \alpha)$ (1)

$\vdash . (1) .$ ∗10·1 $. \supset \vdash ::$ Hp $. s$'$\kappa \epsilon \kappa . \supset :.$

$\gamma \max_Q \kappa . \equiv : \gamma \epsilon \kappa : \alpha \epsilon \kappa . \alpha \neq \gamma . \supset_\alpha . \sim(\gamma \subset \alpha) : s$'$\kappa \neq \gamma . \supset . \sim(\gamma \subset s$'$\kappa) :$

[∗40·13] $\equiv : \gamma \epsilon \kappa : \alpha \epsilon \kappa . \alpha \neq \gamma . \supset_\alpha . \sim(\gamma \subset \alpha) : s$'$\kappa = \gamma :$

[Transp.∗40·13] $\equiv : \gamma \epsilon \kappa . s$'$\kappa = \gamma :$

[Hp] $\equiv : s$'$\kappa = \gamma :: \supset \vdash .$ Prop

***258·101.** $\vdash : \mathrm{Hp}\,*258\cdot1 \,.\, p`\kappa \in \kappa \,.\supset .\, p`\kappa = \min_Q`\kappa$ [Proof as in *258·1]

***258·11.** $\vdash : \mathrm{Hp}\,*258\cdot1 \,.\, s`\kappa \sim\in \kappa \,.\supset .\, \mathrm{seq}_Q`\kappa = s`\kappa$

Dem.

$\vdash .\, *40\cdot53 \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, p`\overleftarrow{Q}``\kappa = \hat{\gamma}(\alpha \in \kappa \,.\supset_\alpha .\, \alpha \subset \gamma \,.\, \alpha \neq \gamma)$

$[\mathrm{Hp}.*40\cdot151.*10\cdot29] \qquad = \hat{\gamma}(s`\kappa \subset \gamma)$ (1)

$\vdash .\, *40\cdot1 \,.\, *22\cdot42\cdot46 \,.\supset \vdash .\, s`\kappa = p`\hat{\gamma}(s`\kappa \subset \gamma)$ (2)

$\vdash .\, (2) \,.\, *258\cdot101 \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, s`\kappa = \min_Q`\hat{\gamma}(s`\kappa \subset \gamma)$

$[(1)] \qquad = \mathrm{seq}_Q`\kappa : \supset \vdash .\, \mathrm{Prop}$

***258·111.** $\vdash : \mathrm{Hp}\,*258\cdot1 \,.\, p`\kappa \sim\in \kappa \,.\supset .\, \mathrm{prec}_Q`\kappa = p`\kappa$ [Proof as in *258·11]

***258·12.** $\vdash :.\, \mathrm{Hp}\,*258\cdot1 \,.\supset : E!\max_Q`\kappa \,.\vee .\, E!\mathrm{seq}_Q`\kappa :$
$E!\min_Q`\kappa \,.\vee .\, E!\mathrm{prec}_Q`\kappa$ [*258·1·101·11·111]

***258·13.** $\vdash : \mathrm{Hp}\,*258\cdot1 \,.\supset .\, \mathrm{lt}_Q = s \upharpoonright (-\mathrm{G}`\max_Q)$

Dem.

$\vdash .\, *258\cdot1 \,.\, \mathrm{Transp} \,.\supset \vdash : \mathrm{Hp} \,.\sim \exists ! \overrightarrow{\max_Q}`\kappa \,.\supset .\, s`\kappa \sim\in \kappa \,.$

$[*258\cdot11] \qquad \supset .\, \mathrm{lt}_Q`\kappa = s`\kappa : \supset \vdash .\, \mathrm{Prop}$

***258·131.** $\vdash : \mathrm{Hp}\,*258\cdot1 \,.\supset .\, \mathrm{tl}_Q = p \upharpoonright (-\mathrm{G}`\min_Q)$ [Proof as in *258·13]

***258·14.** $\vdash : \mathrm{Hp}\,*258\cdot1 \,.\supset .\, Q, \breve{Q} \in \mathrm{Ded} \,.\, \mathrm{lt}_Q, \mathrm{tl}_Q \in 1 \to \mathrm{Cls}$ [*258·12·13·131]

***258·2.** $\vdash : \mathrm{Hp}\,*258\cdot1 \,.\, R \in \mathrm{Rl}`Q \cap \mathrm{Cls} \to 1 \,.\supset .\, Q_{R\alpha} \in \Omega$

Dem.

$\vdash .\, *258\cdot14 \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, \mathrm{Hp}\,*257\cdot27$ (1)

$\vdash .\, (1) \,.\, *257\cdot34 \,.\supset \vdash .\, \mathrm{Prop}$

***258·201.** $\vdash .\, Q = \hat{\alpha}\hat{\beta}(\beta \subset \alpha \,.\, \alpha \neq \beta) \,.\, R \in \mathrm{Rl}`Q \cap \mathrm{Cls} \to 1 \,.\supset .\, Q_{R\alpha} \in \Omega$
[Proof as in *258·2]

***258·202.** $\vdash : Q = \hat{M}\hat{N}(M \mathrel{G} N \,.\, M \neq N) \,.\, R \in \mathrm{Rl}`Q \cap \mathrm{Cls} \to 1 \,.\supset .\, Q_{RX} \in \Omega$

***258·203.** $\vdash : Q = \hat{M}\hat{N}(N \mathrel{G} M \,.\, M \neq N) \,.\, R \in \mathrm{Rl}`Q \cap \mathrm{Cls} \to 1 \,.\supset .\, Q_{RX} \in \Omega$

***258·21.** $\vdash : \mathrm{Hp}\,*258\cdot2 \,.\, \kappa \subset (R*Q)`\alpha \,.\supset .\, s`\kappa = \mathrm{limax}_Q`\kappa$

Dem.

$\vdash .\, *258\cdot13 \,.\supset \vdash : \mathrm{Hp} \,.\sim \exists ! \overrightarrow{\max_Q}`\kappa \,.\supset .\, s`\kappa = \mathrm{lt}_Q`\kappa$ (1)

$\vdash .\, *258\cdot2 \,.\; \supset \vdash :.\, \mathrm{Hp} \,.\, \exists ! \overrightarrow{\max_Q}`\kappa \,.\supset : (\exists\gamma) : \gamma \in \kappa : \alpha \in \kappa \,.\supset_\alpha .\, \alpha \subset \gamma :$

$[*40\cdot151] \qquad \supset : s`\kappa \in \kappa :$

$[*258\cdot1] \qquad \supset : s`\kappa = \max_Q`\kappa$ (2)

$\vdash .\, (1) \,.\, (2) \,.\supset \vdash .\, \mathrm{Prop}$

***258·211.** $\vdash : \mathrm{Hp}\,*258\cdot201 \,.\, \kappa \subset (R*Q)`\alpha \,.\supset .\, p`\kappa = \mathrm{limax}_Q`\kappa$

$*258\cdot22$. $\vdash : \text{Hp} *258\cdot2 . \alpha \in \text{D}`R . \kappa \subset (R*Q)`\alpha . \exists ! \kappa . \supset . s`\kappa = \text{limax}\,(Q_{R\alpha})`\kappa$

Dem.

$\vdash . *258\cdot21 . \supset \vdash : \text{Hp} . s`\kappa \sim\in \kappa . \supset . s`\kappa = \text{lt}_Q`\kappa .$

$[*257\cdot13] \quad \supset . s`\kappa \in (R*Q)`\alpha .$

$[*210\cdot233] \quad \supset . s`\kappa = \text{limax}\,(Q_{R\alpha})`\kappa : \supset \vdash . \text{Prop}$

$*258\cdot221$. $\vdash : \text{Hp} *258\cdot201 . \alpha \in \text{D}`R . \kappa \subset (R*Q)`\alpha . \supset . p`\kappa = \text{limax}\,(Q_{R\alpha})`\kappa$

$*258\cdot23$. $\vdash : \text{Hp} *258\cdot2 . \alpha \in \text{D}`R . \supset . Q_{R\alpha} \in \text{Ded} . s`(R*Q)`\alpha = B`\breve{Q}_{R\alpha}$
$[*258\cdot2\cdot22 . *250\cdot23 . *205\cdot121]$

$*258\cdot231$. $\vdash : \text{Hp} *258\cdot201 . \alpha \in \text{D}`R . \supset . Q_{R\alpha} \in \text{Ded} . p`(R*Q)`\alpha = B`\breve{Q}_{R\alpha}$

$*258\cdot24$. $\vdash : \text{Hp} *258\cdot2 . \supset .$

$$(R*Q)`\alpha = \hat{\beta}\,(\alpha \in \sigma . \breve{R}``\sigma \subset \sigma . s``\text{Cl ex}`\sigma \subset \sigma . \supset_\sigma . \beta \in \sigma)$$

Dem.

$\vdash . *258\cdot1\cdot13 . *257\cdot1 . \supset$

$\vdash : \text{Hp} . \supset . (R*Q)`\alpha \subset \hat{\beta}\,(\alpha \in \sigma . \breve{R}``\sigma \subset \sigma . s``\text{Cl ex}`\sigma \subset \sigma . \supset_\sigma . \beta \in \sigma) \quad (1)$

$\vdash . *257\cdot123 . \supset \vdash : \text{Hp} . \supset . \breve{R}``(R*Q)`\alpha \subset (R*Q)`\alpha \quad (2)$

$\vdash . *258\cdot22 . \supset \vdash : \text{Hp} . \mu \subset (R*Q)`\alpha . \exists ! \mu . \supset . s`\mu \in (R*Q)`\alpha \quad (3)$

$\vdash . *257\cdot12 . \supset \vdash : \text{Hp} . \supset . \alpha \in (R*Q)`\alpha \quad (4)$

$\vdash . (2) . (3) . (4) . \supset$

$\vdash :. \text{Hp} : \alpha \in \sigma . \breve{R}``\sigma \subset \sigma . s``\text{Cl ex}`\sigma \subset \sigma . \supset_\sigma . \beta \in \sigma : \supset . \beta \in (R*Q)`x \quad (5)$

$\vdash . (1) . (5) . \supset \vdash . \text{Prop}$

$*258\cdot241$. $\vdash : \text{Hp} *258\cdot201 . \supset .$

$$(R*Q)`\alpha = \hat{\beta}\,(\alpha \in \sigma . \breve{R}``\sigma \subset \sigma . p``\text{Cl ex}`\sigma \subset \sigma . \supset_\sigma . \beta \in \sigma)$$

$*258\cdot242$. $\vdash : \text{Hp} *258\cdot202 . \supset .$

$$(R*Q)`X = \hat{Y}\,(X \in \sigma . \breve{R}``\sigma \subset \sigma . \dot{s}``\text{Cl ex}`\sigma \subset \sigma . \supset_\sigma . Y \in \sigma)$$

$*258\cdot243$. $\vdash : \text{Hp} *258\cdot203 . \supset .$

$$(R*Q)`X = \hat{Y}\,(X \in \sigma . \breve{R}``\sigma \subset \sigma . \dot{p}``\text{Cl ex}`\sigma \subset \sigma . \supset_\sigma . Y \in \sigma)$$

$*258\cdot3$. $\vdash : Q = \hat{\alpha}\hat{\beta}\,(\beta \subset \alpha . \alpha \neq \beta) . S \in \epsilon_\Delta`\text{Cl ex}`\mu .$
$R = \hat{\alpha}\hat{\beta}\,(\alpha \in \text{Cl ex}`\mu . \beta = \alpha - \iota`S`\alpha) . \supset . Q_{R\mu} \in \Omega . S ; Q_{R\mu} \,\text{smor}\, Q_{R\mu} \upharpoonright (-\iota`\Lambda)$

Dem.

$\vdash . *80\cdot14 . \supset \vdash : \text{Hp} . \supset . R \in Q . R \in \text{Cls} \to 1 . \text{D}`R = \text{Cl ex}`\mu . C`R = \text{Cl}`\mu \quad (1)$

$\vdash . (1) . *258\cdot201 . \supset \vdash : \text{Hp} . \supset . Q_{R\mu} \in \Omega \quad (2)$

$\vdash . *257\cdot35 . \quad \supset \vdash : \text{Hp} . \supset . R \upharpoonright C`Q_{R\mu} \in 1 \to 1 .$

$[(1).\text{Hp}] \quad \supset . S \upharpoonright C`Q_{R\mu} \in 1 \to 1 \quad (3)$

$\vdash . *257\cdot14 . \quad \supset \vdash : \text{Hp} . \supset . C`Q_{R\mu} \subset \text{Cl}`\mu \quad (4)$

$\vdash . *80\cdot14 . \quad \supset \vdash : \text{Hp} . \supset . \text{Q}`S = \text{Cl ex}`\mu \quad (5)$

$\vdash . (3) . (4) . (5) . \quad \supset \vdash : \text{Hp} . \supset . S ; Q_{R\mu} \,\text{smor}\, Q_{R\mu} \upharpoonright (-\iota`\Lambda) \quad (6)$

$\vdash . (2) . (6) . \supset \vdash . \text{Prop}$

***258·301.** $\vdash : \text{Hp} *258\cdot3 \,.\, x \in \mu \,.\, \kappa = C`Q_{R_\mu} \cap \overleftarrow{\epsilon}`x \,.\, \supset .\, x = S`p`\kappa$

Dem.

$\vdash . *257\cdot36 . \quad \supset \vdash : \text{Hp} . \supset . \mu \in C`Q_{R_\mu} .$

$[\text{Hp}] \quad \supset . \exists ! \kappa \qquad (1)$

$\vdash . (1) . *258\cdot241 . \supset \vdash : \text{Hp} . \supset . p`\kappa \in (R * Q)`\mu .$

$[*257\cdot36] \quad \supset . p`\kappa \in C`Q_{R_\mu} \qquad (2)$

$\vdash . *40\cdot1 . \quad \supset \vdash : \text{Hp} . \supset . x \in p`\kappa \qquad (3)$

$\vdash . (2) . (3) . \quad \supset \vdash : \text{Hp} . \supset . p`\kappa \in \kappa .$

$[*258\cdot101] \quad \supset . p`\kappa = \max_Q`\kappa \qquad (4)$

$\vdash . (4) . \quad \supset \vdash : \text{Hp} . \supset . (p`\kappa - \iota`S`p`\kappa) \sim \in \kappa .$

$[*257\cdot121.\text{Hp}] \quad \supset . x \sim \in (p`\kappa - \iota`S`p`\kappa) \qquad (5)$

$\vdash . (3) . (5) . \quad \supset \vdash : \text{Hp} . \supset . x \in \iota`S`p`\kappa : \supset \vdash . \text{Prop}$

***258·31.** $\vdash : \text{Hp} *258\cdot3 \,.\, \mu \sim \in 1 \,.\, \supset .\, C`S;Q_{R_\mu} = \mu$

Dem.

$\vdash . *80\cdot14 . \supset \vdash : \text{Hp} . \supset . \text{G}`S = \text{Cl ex}`\mu .$

$[*150\cdot36.*257\cdot14] \quad \supset . S;Q_{R_\mu} = S;Q_{R_\mu} \upharpoonright (-\iota`\Lambda) . C`Q_{R_\mu} \upharpoonright (-\iota`\Lambda) \subset \text{G}`S .$

$[*150\cdot22] \quad \supset . C`S;Q_{R_\mu} = S``C`Q_{R_\mu} \upharpoonright (-\iota`\Lambda) .$

$[*202\cdot54.*257\cdot125] \quad \supset . C`S;Q_{R_\mu} = S``(C`Q_{R_\mu} - \iota`\Lambda) \qquad (1)$

$\vdash . *83\cdot21 . \supset \vdash : \text{Hp} . \supset . S``C`Q_{R_\mu} \subset \mu \qquad (2)$

$\vdash . *258\cdot241\cdot301 . \supset \vdash : \text{Hp} . x \in \mu . \supset . x \in S``\{(R*Q)`\mu - \iota`\Lambda\} .$

$[*257\cdot36] \quad \supset . x \in S``(C`Q_{R_\mu} - \iota`\Lambda) \qquad (3)$

$\vdash . (2) . (3) . \supset \vdash : \text{Hp} . \supset . S``(C`Q_{R_\mu} - \iota`\Lambda) = \mu \qquad (4)$

$\vdash . (1) . (4) . \supset \vdash . \text{Prop}$

***258·32.** $\vdash : \mu \sim \in 1 \,.\, \exists ! \epsilon_\Delta`\text{Cl ex}`\mu \,.\, \supset .\, \mu \in C``\Omega \quad [*258\cdot3\cdot31]$

This is Zermelo's theorem.

***258·321.** $\vdash : \text{Hp} *258\cdot3 \,.\, \beta Q_{R_\mu} \alpha \,.\, \supset .\, S`\beta \sim \in \alpha$

Dem.

$\vdash . *250\cdot242 . \supset \vdash :. \text{Hp} . \supset : \alpha = (\breve{Q}_{R_\mu})_1`\beta . \vee . (\breve{Q}_{R_\mu})_1`\beta Q_{R_\mu} \alpha :$

$[*257\cdot35.\text{Hp}] \quad \supset : \alpha \subset \beta - \iota`S`\beta :. \supset \vdash . \text{Prop}$

***258·33.** $\vdash : \text{Hp} *258\cdot3 \,.\, \mu \sim \in 1 \,.\, P = S;Q_{R_\mu} \,.\, \supset .\, S = \min_P \upharpoonright \text{Cl ex}`\mu$

Dem.

$\vdash . *80\cdot14 . \quad \supset \vdash : \text{Hp} . \alpha \subset \mu . \exists ! \alpha . \supset . S`\alpha \in \alpha \qquad (1)$

$\vdash . *258\cdot321 . \quad \supset \vdash : \text{Hp}(1) . x \in \alpha . \supset . \sim (\exists \beta) . \beta Q_{R_\mu} \alpha . x = S`\beta .$

$[*150\cdot4.\text{Hp}] \quad \supset . \sim (x P S`\alpha) \qquad (2)$

$\vdash . (1) . (2) . *205\cdot1 . \supset \vdash : \text{Hp}(1) . \supset . S`\alpha \min_P \alpha .$

$[*258\cdot3] \quad \supset . S`\alpha = \min_P`\alpha : \supset \vdash . \text{Prop}$

***258·34.** $\vdash :. \mu \sim \epsilon 1 . \supset :$

$$S \epsilon \epsilon_\Delta \text{`Cl ex`}\mu . \equiv . (\exists P) . P \epsilon \Omega . C\text{`}P = \mu . S = \min_P \upharpoonright \text{Cl ex`}\mu$$

[*250·5 . *258·33]

***258·35.** $\vdash : \mu \epsilon C\text{``}\Omega . \equiv . \mu \sim \epsilon 1 . \exists ! \epsilon_\Delta \text{`Cl ex`}\mu$ [*200·12 . *250·51 . *258·32]

***258·36.** $\vdash : \mu \epsilon C\text{``}\Omega \cup 1 . \equiv . \exists ! \epsilon_\Delta \text{`Cl ex`}\mu$ [*258·35 . *60·37 . *83·901]

***258·37.** $\vdash : \text{Mult ax} . \equiv . C\text{``}\Omega \cup 1 = \text{Cls}$ [*258·36 . *88·33]

***258·38.** $\vdash :. \text{Mult ax} . \supset : \text{Nc`}\alpha < \text{Nc`}\beta . \vee . \text{Nc`}\alpha = \text{Nc`}\beta . \vee . \text{Nc`}\alpha > \text{Nc`}\beta$

[*255·73 . *258·37 . *117·54·55]

***258·39.** $\vdash :: \text{Mult ax} . \supset :. \mu, \nu \epsilon \text{N}_0\text{C} . \supset : \mu \leqslant \nu . \vee . \mu > \nu$ [*258·38]

*259. INDUCTIVELY DEFINED CORRELATIONS.

Summary of *259.

In the theory of well-ordered relations, we often have occasion to define a relation (which is generally of the nature of a correlation) by the following process: Given a relation S, let $W\text{‘}S$ be a relation (generally a couple) which is a function of S. Let us put

$$\breve{A}_W\text{‘}S = S \cup W\text{‘}S.$$

Then, starting from $\dot\Lambda$, we form the series

$$\dot\Lambda,\ \breve{A}_W\text{‘}\dot\Lambda,\ \breve{A}_W\text{‘}\breve{A}_W\text{‘}\dot\Lambda,\ \text{etc.},$$

each of which contains all its predecessors. We proceed to the limit by taking the sum of all these relations, *i.e.* $\dot{s}\text{‘}\overleftarrow{(A_W)}_*\text{‘}\dot\Lambda$; we then proceed to $\breve{A}_W\text{‘}\dot{s}\text{‘}\overleftarrow{(A_W)}_*\text{‘}\dot\Lambda$, and so on, as long as possible. The sum of all the relations so obtained is a function of W, and is often important.

As an example, we may consider the correlation of two well-ordered series P, Q, which is dealt with in *259·2—·25 below. In this case, we put

$$W = \hat{X}\hat{T}\{X = \mathrm{seq}_P\text{‘}\mathrm{D}\text{‘}T \downarrow \mathrm{seq}_Q\text{‘}\text{Ɑ}\text{‘}T\}.$$

Hence $$W\text{‘}\dot\Lambda = \breve{A}_W\text{‘}\dot\Lambda = B\text{‘}P \downarrow B\text{‘}Q = 1_P \downarrow 1_Q,$$

$$\breve{A}_W\text{‘}\breve{A}_W\text{‘}\dot\Lambda = 1_P \downarrow 1_Q \cup 2_P \downarrow 2_Q,$$

and so on.

Proceeding in this fashion, we can continue until one at least of the two series P, Q is exhausted. We thus obtain a new proof that, of any two well-ordered series, one must be similar to a section of the other.

For convenience of notation, let us put temporarily

$$A = \hat{S}\hat{T}(S \Subset T \,.\, S \neq T) \quad \text{Dft.}$$

We then have $A \in \mathrm{Rl}\text{‘}J \cap \mathrm{trans} \,.\, A_W \in \mathrm{Rl}\text{‘}A \cap \mathrm{Cls} \to 1$, which is part of the hypothesis of *257·27 and following propositions. The rest of this hypothesis follows by analogy from *258·14 We now put

$$W_A = \dot{s}\text{‘}(A_W * A)\text{‘}\dot\Lambda \quad \text{Df.}$$

Then W_A correlates the whole of P with part or the whole of Q, or vice versa. This is proved in *259·25, below.

For other values of W, we get other results, often of a useful kind; for example we shall have occasion to use the methods of this number in $*273$, which deals with series similar to the series of rationals.

The present number gives, first, some elementary properties of $(A_W * A)`\dot{\Lambda}$ and $W_A$ for a general relation $W$, concerning which we only assume that $W`S$ is never contained in S, *i.e.* $W \dot{\cap} (\subseteq) = \dot{\Lambda}$ (except in $*259{\cdot}121{\cdot}13$, where we also assume $W \in 1 \to \mathrm{Cls}$). We then proceed to deal specially with the case where

$$W = \hat{X}\hat{T}\{X = \mathrm{seq}_P`\mathrm{D}`T \downarrow \mathrm{seq}_Q`\mathrm{G}`T\}$$

as explained above.

$*259{\cdot}01.\quad A = \hat{S}\hat{T}(S \subseteq T \,.\, S \neq T) \quad$ Dft [$*259$]

$*259{\cdot}02.\quad A_W = \hat{S}\hat{T}(T = S \cup W`S) \quad$ Dft [$*259$]

$*259{\cdot}03.\quad W_A = \dot{s}`(A_W * A)`\dot{\Lambda} \quad$ Df

In the following propositions, which result from those of $*258$, it is essential to have $A_W \subseteq A$. For this we require that $W`S$, when it exists, shall not be contained in S. It will be observed that, according to the above definition,

$$A_* = \hat{S}\hat{T}(S \subseteq T).$$

Hence instead of using "$\subseteq$" as a relation, which is notationally awkward, we shall use A_*. Thus the condition we wish to impose upon W is that we are never to have $(W`S)A_*S$. This is insured by

$$W \dot{\cap} A_* = \dot{\Lambda},$$

which accordingly appears as hypothesis in the following propositions.

$*259{\cdot}1.\quad \vdash : A \in \mathrm{Rl}`J \cap \mathrm{trans} \,.\, \mathrm{lt}_A \in 1 \to \mathrm{Cls} :$
$$W \dot{\cap} A_* = \dot{\Lambda} \,.\supset .\, A_W \in \mathrm{Rl}`A \cap \mathrm{Cls} \to 1 \,.\, A\,(A_W, \dot{\Lambda}) \in \Omega$$

Dem.

As in $*258{\cdot}14$, $\quad \vdash .\, \mathrm{lt}_A \in 1 \to \mathrm{Cls} \quad (1)$

$\vdash .\, *201{\cdot}18 \,. \quad \supset \vdash :.\, \mathrm{Hp} \,.\supset : MWS \,.\supset .\, \sim(M \subseteq S) \quad (2)$

$\vdash .\, (2) \,.\, (*259{\cdot}02) \,.\supset \vdash :.\, \mathrm{Hp} \,.\supset : SA_WT \,.\supset .\, S \subseteq T \,.\, S \neq T \,.$

$[(*259{\cdot}01)] \quad \supset .\, SAT \quad (3)$

$\vdash .\, (1) \,.\, (3) \,.\, *258{\cdot}202 \,.\supset \vdash .\, \mathrm{Prop}$

In the following proposition, the notation $A\,(A_W, \dot{\Lambda})$ is that defined in $*257{\cdot}02$, adopted because A_W cannot conveniently be used as a suffix.

$*259{\cdot}11.\quad \vdash : \mathrm{E}\,!\, W`\dot{\Lambda} \,.\, W \dot{\cap} A_* = \dot{\Lambda} \,.\supset .$
$$W_A = B`\mathrm{Cnv}`A\,(A_W, \dot{\Lambda}) \,.\, \dot{s}``\mathrm{Cl}`(A_W * A)`\dot{\Lambda} \subset (A_W * A)`\dot{\Lambda}$$

Dem.

$\vdash .\, *258{\cdot}242 \,.\, *259{\cdot}1 \,.\supset \vdash : \mathrm{Hp} \,.\, \lambda \subset (A_W * A)`\dot{\Lambda} \,.\supset .\, \dot{s}`\lambda \in (A_W * A)`\dot{\Lambda} \quad (1)$

$\vdash .\, (1) \,. \quad \supset \vdash : \mathrm{Hp} \,.\supset .\, W_A \in (A_W * A)`\dot{\Lambda} \quad (2)$

$\vdash .\, *41{\cdot}13 \,. \quad \supset \vdash : \mathrm{Hp} \,.\, T \in (A_W * A)`\dot{\Lambda} - \iota`W_A \,.\supset .\, TAW_A \quad (3)$

$\vdash .\, (1) \,.\, (2) \,.\, (3) \,.\supset \vdash .\, \mathrm{Prop}$

***259·111.** $\vdash :. W \dot{\frown} A_* = \dot{\Lambda} . S, T \epsilon (A_W * A)`\dot{\Lambda} . \supset : S \subsetneq T . \vee . T \subsetneq S$
[*259·1 . *257·36]

***259·12.** $\vdash : S \epsilon \mathrm{D}`A_W . \equiv . \mathrm{E}!\, W`S$ [(*259·02)]

***259·121.** $\vdash : W \epsilon 1 \to \mathrm{Cls} . \supset . \mathrm{D}`A_W = \mathrm{C}`W$ [*259·12]

***259·122.** $\vdash : W \dot{\frown} A_* = \dot{\Lambda} . x W_A y . \lambda = (A_W * A)`\dot{\Lambda} \frown \hat{T}\{\sim(xTy)\} . \supset . x(W`\dot{s}`\lambda)y$

Dem.

$\vdash . *259·11 . \supset \vdash : \mathrm{Hp} . \supset . \dot{s}`\lambda \epsilon (A_W * A)`\dot{\Lambda}$ (1)

[Hp] $\supset . \dot{s}`\lambda \epsilon \lambda$ (2)

$\vdash . (1) . (2) . *257·3 . \supset \vdash : \mathrm{Hp} . \supset . \dot{s}`\lambda \epsilon \mathrm{D}`A_W .$

[*259·12] $\supset . \mathrm{E}!\, W`\dot{s}`\lambda$ (3)

$\vdash . (3) . \supset \vdash : \mathrm{Hp} . \supset . (\dot{s}`\lambda)\, A\, (A_W`\dot{s}`\lambda) .$

[*257·121] $\supset . A_W`\dot{s}`\lambda \epsilon (A_W * A)`\dot{\Lambda} - \lambda .$

[Hp] $\supset . x (A_W`\dot{s}`\lambda) y$ (4)

$\vdash . (2) . (4) . \supset \vdash : \mathrm{Hp} . \supset . \sim \{x (\dot{s}`\lambda) y\} . x (A_W`\dot{s}`\lambda) y .$

[(*259·02)] $\supset . x\, (W`\dot{s}`\lambda)\, y : \supset \vdash . \mathrm{Prop}$

***259·13.** $\vdash : W \dot{\frown} A_* = \dot{\Lambda} . W \epsilon 1 \to \mathrm{Cls} . \supset . W_A = \dot{s}`W``(A_W * A)`\dot{\Lambda}$

Dem.

$\vdash . *259·122 . \supset \vdash : \mathrm{Hp} . \supset . W_A \mathrel{\dot{\subset}} \dot{s}`W``(A_W * A)`\dot{\Lambda}$ (1)

$\vdash . *257·123 . \supset \vdash : \mathrm{Hp} . \supset . \dot{s}`W``(A_W * A)`\dot{\Lambda} \mathrel{\dot{\subset}} W_A$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***259·14.** $\vdash :. W \dot{\frown} A_* = \dot{\Lambda} : S \epsilon (A_W * A)`\dot{\Lambda} \frown 1 \to \mathrm{Cls} \frown \mathrm{C}`W . \supset_S .$
$W`S \epsilon 1 \to \mathrm{Cls} . \mathrm{C}`S \frown \mathrm{C}`W`S = \Lambda : \supset . W_A \epsilon 1 \to \mathrm{Cls}$

Dem.

$\vdash . *71·24 . (*259·02) . \supset \vdash :. \mathrm{Hp} . \supset :$

$S \epsilon (A_W * A)`\dot{\Lambda} \frown 1 \to \mathrm{Cls} . \supset . \breve{A}_W`S \epsilon (A_W * A)`\dot{\Lambda} \frown 1 \to \mathrm{Cls}$ (1)

$\vdash . *259·111 . \supset \vdash :. \mathrm{Hp} . S, T \epsilon (A_W * A)`\dot{\Lambda} . \supset : S \subsetneq T . \vee . T \subsetneq S$ (2)

$\vdash . (2) . \supset \vdash : \mathrm{Hp} . \lambda \subset (A_W * A)`\dot{\Lambda} . x(\dot{s}`\lambda)z . y(\dot{s}`\lambda)z . \supset . (\exists T) . T \epsilon \lambda . xTz . yTz$ (3)

$\vdash . (3) . \supset \vdash : \mathrm{Hp} . \lambda \subset (A_W * A)`\dot{\Lambda} \frown 1 \to \mathrm{Cls} . x (\dot{s}`\lambda) z . y (\dot{s}`\lambda) z . \supset . x = y$ (4)

$\vdash . (4) . \supset \vdash : \mathrm{Hp} . \lambda \subset (A_W * A)`\dot{\Lambda} \frown 1 \to \mathrm{Cls} . \supset . \dot{s}`\lambda \epsilon 1 \to \mathrm{Cls}$ (5)

$\vdash . (1) . (5) . *258·242 . \supset \vdash : \mathrm{Hp} . \supset . (A_W * A)`\dot{\Lambda} \subset 1 \to \mathrm{Cls} .$

[*259·11] $\supset . W_A \epsilon 1 \to \mathrm{Cls} : \supset \vdash . \mathrm{Prop}$

***259·141.** $\vdash :. W \dot{\frown} A_* = \dot{\Lambda} : S \epsilon (A_W * A)`\dot{\Lambda} \frown \mathrm{Cls} \to 1 \frown \mathrm{C}`W . \supset_S .$
$W`S \epsilon \mathrm{Cls} \to 1 . \mathrm{D}`S \frown \mathrm{D}`W`S = \Lambda : \supset . W_A \epsilon \mathrm{Cls} \to 1$

[Proof as in *259·14]

***259·15.** $\vdash :. W \mathbin{\dot\cap} A_* = \dot\Lambda : S \epsilon (A_W * A)`\dot\Lambda \cap 1 \to 1 \cap \text{Cl}`W . \supset_S .$
$W`S \epsilon 1 \to 1 . \text{D}`S \cap \text{D}`W`S = \Lambda . \text{Cl}`S \cap \text{Cl}`W`S = \Lambda : \supset . W_A \epsilon 1 \to 1$
[*259·14·141]

The following proposition is a lemma for *273·23.

***259·16.** $\vdash :. W \mathbin{\dot\cap} A_* = \dot\Lambda : T \epsilon (A_W * A)`\dot\Lambda \cap \text{Cl}`W . P \upharpoonleft \text{D}`T = T;Q . \supset_T .$
$P \upharpoonleft (\breve{A}_W`T) = (\breve{A}_W`T);Q : \supset :$
$P \upharpoonleft \text{D}`W_A = W_A;Q : T \epsilon (A_W * A)`\dot\Lambda . \supset_T . P \upharpoonleft \text{D}`T = T;Q$

Dem.

$\vdash . *259·111 . \supset \vdash :. \text{Hp} . \lambda \subset (A_W * A)`\dot\Lambda . \supset :$
$x (P \upharpoonleft \text{D}`\dot s`\lambda) y . \equiv . (\exists T) . T \epsilon \lambda . x (P \upharpoonleft \text{D}`T) y \qquad (1)$

$\vdash . (1) . \supset \vdash :. \text{Hp} . \lambda \subset (A_W * A)`\dot\Lambda : T \epsilon \lambda . \supset_T . P \upharpoonleft \text{D}`T = T;Q : \supset :$
$x (P \upharpoonleft \text{D}`\dot s`\lambda) y . \equiv . (\exists T) . T \epsilon \lambda . x (T;Q) y .$

[*259·111] $\equiv . (\exists S, T) . S, T \epsilon \lambda . x (S \mid Q \mid \breve{T}) y .$

[*150·1] $\equiv . x \{(\dot s`\lambda);Q\} y \qquad (2)$

$\vdash . (2) . *258·242 . \supset \vdash : \text{Hp} . T \epsilon (A_W * A)`\dot\Lambda . \supset . P \upharpoonleft \text{D}`T = T;Q \qquad (3)$

$\vdash . (3) . *259·11 . \supset \vdash . \text{Prop}$

The two following propositions are lemmas for *273·22·212.

***259·17.** $\vdash :. W \mathbin{\dot\cap} A_* = \dot\Lambda : S \epsilon (A_W * A)`\dot\Lambda \cap \text{Cl}`W . \supset_S .$
$\text{Cl}`S \cap \text{Cl}`W`S = \Lambda : \supset . \text{Cl} \upharpoonright (A_W * A)`\dot\Lambda \epsilon 1 \to 1$

Dem.

$\vdash . *250·242 . *257·35 . *259·1 . \supset$

$\vdash :. \text{Hp} . S, T \epsilon (A_W * A)`\dot\Lambda . S \neq T . \supset : A_W`S \mathrel{\dot\subset} T . \text{v} . \breve{A}_W`T \mathrel{\dot\subset} S :$

[(*259·02)] $\supset : \text{Cl}`W`S \subset \text{Cl}`T . \text{v} . \text{Cl}`W`T \subset \text{Cl}`S :$

[Hp] $\supset : \text{Cl}`S \neq \text{Cl}`T :. \supset \vdash . \text{Prop}$

***259·171.** $\vdash :. W \mathbin{\dot\cap} A_* = \dot\Lambda : S \epsilon (A_W * A)`\dot\Lambda \cap \text{Cl}`W . \supset_S .$
$\text{D}`S \cap \text{D}`W`S = \Lambda : \supset . \text{D} \upharpoonright (A_W * A)`\dot\Lambda \epsilon 1 \to 1$

[Proof as in *259·17]

***259·2.** $\vdash : W = \hat X \hat T \{X = \text{seq}_P`\text{D}`T \downarrow \text{seq}_Q`\text{Cl}`T\} . \supset . W_A \epsilon 1 \to 1 . W \mathbin{\dot\cap} A_* = \dot\Lambda$

Dem.

$\vdash . *72·182 . \supset \vdash :. \text{Hp} . \supset : T \epsilon \text{Cl}`W . \supset . W`T \epsilon 1 \to 1 \qquad (1)$

$\vdash . *206·2 . \supset \vdash :. \text{Hp} . \supset : T \epsilon \text{Cl}`W . \supset . \text{D}`T \cap \text{D}`W`T = \Lambda . \text{Cl}`T \cap \text{Cl}`W`T = \Lambda \qquad (2)$

$\vdash . (2) . *55·134 . \supset \vdash : \text{Hp} . T \epsilon \text{Cl}`W . \supset . \sim (W`T \mathrel{\dot\subset} T) \qquad (3)$

$\vdash . (1) . (2) . (3) . *259·15 . \supset \vdash . \text{Prop}$

$*259\cdot21$. $\vdash : \text{Hp} *259\cdot2 . Q^2 \in J . \supset . W_A{}^{\text{;}}Q \in P . \text{D}`W_A \subset C`P . \text{ɑ}`W_A \subset C`Q$

Dem.

$\vdash . *206\cdot133 . \supset \vdash : \text{Hp} . T \in \text{ɑ}`W . \supset . (W`T){}^{\text{;}}Q = \dot\Lambda$ (1)

$\vdash . *206\cdot21 . \supset \vdash : \text{Hp}(1) . \supset . \text{seq}_Q`\text{ɑ}`T \sim\in Q``\text{ɑ}`T .$

$[*37\cdot461] \quad \supset . (W`T) \mid Q \mid \breve{T} = \dot\Lambda$ (2)

$\vdash . *206\cdot18 . \supset \vdash : \text{Hp}(1) . \supset . \text{D}`A_W \subset C`P$ (3)

$\vdash . (3) . *41\cdot43 . *258\cdot242 . \supset \vdash : \text{Hp} . \supset . \text{D}`W_A \subset C`P$ (4)

Similarly $\quad \vdash : \text{Hp} . \supset . \text{ɑ}`W_A \subset C`Q$ (5)

$\vdash . (4) . *206\cdot132 . \supset \vdash : \text{Hp}(1) . T \in (A_W * A)`\dot\Lambda . \supset . \text{seq}_P`\text{D}`T \in p`\overleftarrow{P}``\text{D}`T .$

$[*40\cdot16] \quad \supset . \text{seq}_P`\text{D}`T \in p`\overleftarrow{P}``T``\overrightarrow{Q}`\text{seq}_Q`\text{ɑ}`T .$

$[*40\cdot67] \quad \supset . (T``\overrightarrow{Q}`\text{seq}_Q`\text{ɑ}`T) \uparrow \iota`\text{seq}_P`\text{D}`T \in P$ (6)

$\vdash . (1) . (2) . (6) . \supset \vdash : \text{Hp}(1) . T \in (A_W * A)`\dot\Lambda . T{}^{\text{;}}Q \in P . \supset . (\breve{A}_W`T){}^{\text{;}}Q \in P$ (7)

$\vdash . *259\cdot111 . \supset \vdash :: \lambda \subset (A_W * A)`\dot\Lambda . x \{(\dot s`\lambda){}^{\text{;}}Q\} y . \supset :.$

$(\exists T) . T \in \lambda . x (T{}^{\text{;}}Q) y :.$

$[*11\cdot62 . *10\cdot23] \quad \supset :. T \in \lambda . \supset_T . T{}^{\text{;}}Q \in P : \supset . xPy$ (8)

$\vdash . (8) . \text{Comm} . \supset \vdash :. \lambda \subset (A_W * A)`\dot\Lambda : T \in \lambda . \supset_T . T{}^{\text{;}}Q \in P : \supset . (\dot s`\lambda){}^{\text{;}}Q \in P$ (9)

$\vdash . (7) . (9) . *258\cdot242 . \supset \vdash :. \text{Hp} . \supset : T \in (A_W * A)`\dot\Lambda . \supset . T{}^{\text{;}}Q \in P :$

$[*259\cdot11] \quad \supset : W_A{}^{\text{;}}Q \in P$ (10)

$\vdash . (10) . (4) . (5) . \supset \vdash . \text{Prop}$

$*259\cdot211$. $\vdash : \text{Hp} *259\cdot2 . P^2 \in J . \supset . \breve{W}_A{}^{\text{;}}P \in Q$ [Proof as in $*259\cdot21$]

$*259\cdot22$. $\vdash : \text{Hp} *259\cdot2 . P \in \text{connex} . \supset . \text{D}``(A_W * A)`\dot\Lambda \subset \text{sect}`P$

Dem.

$\vdash . *211\cdot22 . \supset \vdash : \text{Hp} . T \in \text{ɑ}`W . \text{D}`T \in \text{sect}`P . \supset . \text{D}`\breve{A}_W`T \in \text{sect}`P$ (1)

$\vdash . *211\cdot63 . \supset \vdash : \text{D}``\lambda \subset \text{sect}`P . \supset . \text{D}`\dot s`\lambda \in \text{sect}`P$ (2)

$\vdash . (1) . (2) . *258\cdot242 . \supset \vdash . \text{Prop}$

$*259\cdot221$. $\vdash : \text{Hp} *259\cdot2 . Q \in \text{connex} . \supset . \text{ɑ}``(A_W * A)`\dot\Lambda \subset \text{sect}`Q$

$*259\cdot222$. $\vdash : \text{Hp} *259\cdot2 . P \in \text{Ser} . \text{E} ! B`P . Q^2 \in J . T \in (A_W * A)`\dot\Lambda . \supset .$

$T{}^{\text{;}}Q \in C`P_s$ [$*259\cdot21\cdot22 . *213\cdot161$]

$*259\cdot223$. $\vdash : \text{Hp} *259\cdot2 . Q \in \text{Ser} . \text{E} ! B`Q . P^2 \in J . T \in (A_W * A)`\dot\Lambda . \supset .$

$\breve{T}{}^{\text{;}}P \in C`Q_s$

$*259\cdot23$. $\vdash : \text{Hp} *259\cdot2 . P, Q \in \text{Ser} \cap \text{ɑ}`B . T \in (A_W * A)`\dot\Lambda . \supset .$

$(\exists M, N) . M \in C`P_s . N \in C`Q_s . T \in M \,\text{smor}\, N$ [$*259\cdot2\cdot21\cdot222\cdot223$]

***259·24.** $\vdash :. \text{Hp} *259\cdot2 . P, Q \in \Omega . \supset : D`W_A = C`P . \vee . \text{D}`W_A = C`Q$

Dem.

$\vdash . *206\cdot18 . \supset \vdash : \text{Hp} . P = \dot{\Lambda} . \supset . W_A = \dot{\Lambda}$ (1)

$\vdash . *206\cdot18 . \supset \vdash : \text{Hp} . Q = \dot{\Lambda} . \supset . W_A = \dot{\Lambda}$ (2)

$\vdash . (1) . (2) . \supset \vdash :. \text{Hp} : P = \dot{\Lambda} . \vee . Q = \dot{\Lambda} : \supset : D`W_A = C`P . \vee . \text{D}`W_A = C`Q$ (3)

$\vdash . *259\cdot11 . *257\cdot36 . \supset \vdash : \text{Hp} . \dot{\exists} ! P . \dot{\exists} ! Q . \supset . W_A \sim \in D`A_W .$

$[*259\cdot12] \quad \supset . \sim (E ! \text{seq}_P`D`W_A . E ! \text{seq}_Q`\text{D}`W_A)$ (4)

$\vdash . (4) . *252\cdot1 . *259\cdot22\cdot221 . \supset$

$\vdash :. \text{Hp} . \dot{\exists} ! P . \dot{\exists} ! Q . \supset : D`W_A = C`P . \vee . \text{D}`W_A = C`Q$ (5)

$\vdash . (3) . (5) . \supset \vdash . \text{Prop}$

***259·25.** $\vdash :. \text{Hp} *259\cdot24 . \supset : (\exists\beta) . \beta \in \text{sect}`Q . W_A \in P \,\overline{\text{smor}}\, (Q \upharpoonright \beta) . \vee .$
$(\exists\alpha) . \alpha \in \text{sect}`P . W_A \in (P \upharpoonright \alpha) \,\overline{\text{smor}}\, Q$ [*259·23·24]

The above affords a new proof of *254·37, which asserts that if P and Q are well-ordered series, one must be similar to a section of the other. In virtue of *259·25 (which has been proved without using the propositions of *254), W_A is the correlator which correlates the whole of one series with part or the whole of the other.

It will be observed that the relations $(A_W * A)`\dot{\Lambda}$ are the class of correlators of sections of $P$ with sections of $Q$, provided $P, Q \in \Omega - \iota`\dot{\Lambda}$; *i.e.*

$\vdash : \text{Hp} *259\cdot2 . P, Q \in \Omega - \iota`\dot{\Lambda} . \supset .$

$$(A_W * A)`\dot{\Lambda} = \hat{T}\{(\exists M, N) . M \in C`P_\varsigma . N \in C`Q_\varsigma . T \in M \,\overline{\text{smor}}\, N\}.$$

SECTION E.

FINITE AND INFINITE SERIES AND ORDINALS.

Summary of Section E.

In the present section we shall be concerned first with the distinction of finite and infinite as applied to series and ordinals. We shall then establish the distinguishing properties of finite ordinals, and shall deal with the smallest of infinite ordinals, namely ω, the ordinal number of a *progression*. Finally we shall briefly consider certain special ordinals, and the series of cardinals applicable to well-ordered infinite series, namely the series of "Alephs," as they are called after Cantor's usage.

In dealing with the finite and the infinite as applied to series, we have constant need of the relation $(P_1)_{\text{po}}$, where P is the generating relation of the series. We have

$$x (P_1)_{\text{po}}\, y \,.\equiv.\, P(x \mapsto y) \,\epsilon\, \text{Cls induct} - \iota\text{`}\Lambda,$$

i.e. "$x (P_1)_{\text{po}}\, y$" holds when, and only when, there is a finite number of intermediaries between x and y. When P is finite, we have

$$P = (P_1)_{\text{po}},$$

but we may have this when P is not finite. The infinite series for which this holds are progressions and their converses (which we will call regressions), and series consisting of a regression followed by a progression, of which an instance is afforded by the negative and positive finite integers in order of magnitude.

*260. ON FINITE INTERVALS IN A SERIES.

Summary of *260.

In the present number we are concerned with the relation which holds between x and y when the interval $P(x \vdash y)$ is an inductive class other than Λ, or when the interval $P(x \vdash\!\dashv y)$ is an inductive class of at least two terms. This relation holds if x and y have any relation of the class $\text{fin}`P$ (defined in *121). We will call this relation P_{fn}. Thus we put

$$P_{\text{fn}} = \dot{s}`\text{fin}`P \quad \text{Df.}$$

Then $xP_{\text{fn}}y$ holds when $xP_\nu y$, where ν is an inductive cardinal other than 0 (*260·1). This relation will take us from x to any later term which can be reached without passing to the limit. But if in the interval $P(x \dashv y)$ there is any term which has no immediate predecessor, *i.e.* any member of $C`P - \text{Cl}`P_1$, then we shall not have $xP_{\text{fn}}y$. Thus P_{fn} confines us to terms which are at a finite distance from our starting-point. We shall find that if $P \in \Omega$, a necessary condition for the finitude of P is $P = P_{\text{fn}}$. This is not a *sufficient* condition, since it does not exclude progressions, but these are the only infinite series it admits, and these are excluded by the assumption $\text{E}\,!\,B`\breve{P}$.

Although P_{fn} is not in general serial when P or P_{po} is serial, it becomes serial when confined to the posterity or the ancestry or the family of any term with respect to itself (*260·32·4). When a series P is well-ordered, the whole series can be divided into constituent series, each of which is the family of any one of its members with respect to P_{fn} (except when P has a last term which has no immediate predecessor, in which case this last term must be omitted). (Cf. *264.) Each of these series (except the last, possibly) is a progression, and the last is either finite or a progression. Hence every infinite well-ordered series consists of a series of progressions followed by a finite tail (which may be null); hence the cardinal of the field of an infinite well-ordered series is a multiple of $\aleph_0$. These results will be proved later; for the present we are concerned with the proof that the family of any term with respect to P_{fn} is a series of which the generating relation is P_{fn} with its field confined to that family.

In the present number we are chiefly concerned with the relations of P_{fn} to P_1. We have

***260·27.** $\vdash : P_{\text{po}} \in \text{Ser} \,.\supset .\, P_{\text{fn}} = (P_1)_{\text{po}}$

This proposition will be used very frequently throughout this section.

Without any hypothesis we have

***260·12.** $\vdash .\, P_{\text{fn}} \subset P_{\text{po}}$

We have also

***260·15.** $\vdash .\, P_{\text{fn}} = (P_{\text{po}})_{\text{fn}}$

Hence whatever properties of P_{fn} result from the hypothesis that P is a series will result from the weaker hypothesis that P_{po} is a series.

If P_{po} is a series, P_{fn} is contained in diversity and is transitive (*260·202), but not in general connected.

In comparing P_{fn} and $(P_1)_{\text{po}}$, we constantly need the proposition

***260·22.** $\vdash : P_{\text{po}} \in \text{Ser} \,.\supset .\, (P_1)_1 = P_1 \,.\, P_1 \in 1 \to 1 \,.\, (P_1)_{\text{po}} \subset J$

From *260·3 to the end of the number, we are concerned with the result of limiting the field of P_{fn} to the ancestry, posterity or family of some member of its field. We have

***260·33.** $\vdash : P_{\text{po}} \in \text{Ser} \,.\, x \in \text{D}`P_1 \,.\, P_1 = R \,.\supset .$
$$P_{\text{fn}} \upharpoonleft (\iota`x \cup \overleftarrow{P_{\text{fn}}}`x) = (\overleftarrow{R_*}`x) \rceil R_{\text{po}} = \{(\overleftarrow{R_*}`x) \rceil R\}_{\text{po}} = \{R \upharpoonright (\overleftarrow{R_{\text{po}}}`x)\}_{\text{po}}$$

***260·34.** $\vdash : \text{Hp} \,\text{*}260·33 \,.\supset .\, \{P_{\text{fn}} \upharpoonleft (\iota`x \cup \overleftarrow{P_{\text{fn}}}`x)\}_1 = (\overleftarrow{R_*}`x) \rceil R = R \upharpoonright \overleftarrow{R_{\text{po}}}`x$

***260·01.** $P_{\text{fn}} = \dot{s}`\text{fin}`P \quad \text{Df}$

***260·1.** $\vdash : xP_{\text{fn}}y \,.\equiv .\, (\exists\nu) \,.\, \nu \in \text{NC induct} - \iota`0 \,.\, xP_\nu y$
[*121·121 . (*260·01)]

***260·11.** $\vdash : xP_{\text{fn}}y \,.\equiv .\, P(x \mapsto y) \in \text{Cls induct} - 0 - 1$

Dem.

$\vdash .\, \text{*}260·1 \,.\, \text{*}121·11 \,.\supset$
$\vdash : xP_{\text{fn}}y \,.\equiv .\, (\exists\nu) \,.\, \nu \in \text{NC induct} - \iota`0 \,.\, P(x \mapsto y) \in \nu +_c 1 \,.$
$[\text{*}120·472] \equiv .\, (\exists\mu) \,.\, \mu \in \text{NC induct} - \iota`0 - \iota`1 \,.\, P(x \mapsto y) \in \mu \,.$
$[\text{*}120·2] \quad \equiv .\, P(x \mapsto y) \in \text{Cls induct} - 0 - 1 :\supset \vdash .\, \text{Prop}$

***260·12.** $\vdash .\, P_{\text{fn}} \subset P_{\text{po}}$

Dem.

$\vdash .\, \text{*}121·321 \,.\, \text{*}117·511 \,.\supset \vdash : \nu \in \text{NC induct} - \iota`0 \,.\supset .\, P_\nu \subset P_{\text{po}} \quad (1)$
$\vdash .\, (1) \,.\, \text{*}260·1 \,.\supset \vdash .\, \text{Prop}$

$*260\cdot13$. $\vdash : xP_{\text{fn}}\,y \,.\supset .\, P(x \vdash y), P(x \dashv y) \,\epsilon\, \text{Cls induct} - \iota'\Lambda$

Dem.

$\vdash . *260\cdot12 . *121\cdot21\cdot22 . \supset \vdash : \text{Hp} . \supset . P(x \vdash y), P(x \dashv y) \,\epsilon\, -\iota'\Lambda$ (1)

$\vdash . *91\cdot54 . (*121\cdot011\cdot012\cdot013) . \supset$

$\vdash . P(x \vdash y) \subset P(x \vdash\!\dashv y) . P(x \dashv y) \subset P(x \vdash\!\dashv y) .$

$[*120\cdot481 . *260\cdot11] \supset \vdash : \text{Hp} . \supset . P(x \vdash y), P(x \dashv y) \,\epsilon\, \text{Cls induct}$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

$*260\cdot131$. $\vdash :. P_{\text{po}} \Subset J . \supset : xP_{\text{fn}}\,y \,.\equiv .\, P(x \vdash y) \,\epsilon\, \text{Cls induct} - \iota'\Lambda \,.$
$\equiv . P(x \dashv y) \,\epsilon\, \text{Cls induct} - \iota'\Lambda$

Dem.

$\vdash . *121\cdot22 . \supset \vdash : P(x \vdash y) \,\epsilon\, \text{Cls induct} - \iota'\Lambda . \supset . xP_{\text{po}}\,y .$ (1)

$[*121\cdot242 . *91\cdot54] \quad \supset . P(x \vdash\!\dashv y) = P(x \vdash y) \cup \iota'y$

$[*120\cdot251] \quad \supset . P(x \vdash\!\dashv y) \,\epsilon\, \text{Cls induct}$ (2)

$\vdash . (1) . *121\cdot242 . \quad \supset \vdash : \text{Hp} . \text{Hp}(1) . \supset . x, y \,\epsilon\, P(x \vdash\!\dashv y) . x \neq y .$

$[*52\cdot41] \quad \supset . P(x \vdash\!\dashv y) \sim\epsilon\, 0 \cup 1$ (3)

$\vdash . (2) . (3) . *260\cdot11 . \supset \vdash : \text{Hp} . \text{Hp}(1) . \supset . xP_{\text{fn}}\,y$ (4)

Similarly $\quad \vdash : \text{Hp} . P(x \dashv y) \,\epsilon\, \text{Cls induct} . \supset . xP_{\text{fn}}\,y$ (5)

$\vdash . (4) . (5) . *260\cdot13 . \supset \vdash . \text{Prop}$

$*260\cdot14$. $\vdash : P \,\epsilon\, (\text{Cls} \to 1) \cup (1 \to \text{Cls}) . P_{\text{po}} \Subset J . \supset . P_{\text{fn}} = P_{\text{po}}$

Dem.

$\vdash . *121\cdot52 . \supset \vdash : \text{Hp} . \supset . \dot{s}'\text{finid}'P = P_* .$

$[(*260\cdot01)] \quad \supset . P_{\text{fn}} = P_* \mathbin{\dot{-}} P_0$

$[*121\cdot302] \quad = P_* \mathbin{\dot{-}} I \upharpoonright C'P$

$[*91\cdot541] \quad = P_{\text{po}} : \supset \vdash . \text{Prop}$

$*260\cdot15$. $\vdash . P_{\text{fn}} = (P_{\text{po}})_{\text{fn}} \quad [*260\cdot1 . *121\cdot254]$

$*260\cdot16$. $\vdash . (\breve{P})_{\text{fn}} = \breve{P}_{\text{fn}} \quad [*260\cdot1 . *121\cdot26]$

$*260\cdot17$. $\vdash : P_{\text{po}} \,\epsilon\, \text{Ser} . xP_{\text{po}}\,y . \supset . P(x \vdash\!\dashv y) = C'\{P_{\text{po}} \sqsubset P(x \vdash\!\dashv y)\} .$
$x = B'\{P_{\text{po}} \sqsubset P(x \vdash\!\dashv y)\} . y = B'\text{Cnv}'\{P_{\text{po}} \sqsubset P(x \vdash\!\dashv y)\}$

Dem.

$\vdash . *121\cdot242 . \supset \vdash : \text{Hp} . \supset . x, y \,\epsilon\, P(x \vdash\!\dashv y) . x \neq y .$ (1)

$[*52\cdot41] \quad \supset . P(x \vdash\!\dashv y) \sim\epsilon\, 1 .$

$[*202\cdot55] \quad \supset . C'\{P_{\text{po}} \sqsubset P(x \vdash\!\dashv y)\} = P(x \vdash\!\dashv y)$ (2)

$\vdash . *91\cdot542 . \quad \supset \vdash :. \text{Hp} . \supset : z \,\epsilon\, P(x \vdash\!\dashv y) . z \neq x . \supset . x \{P_{\text{po}} \sqsubset P(x \vdash\!\dashv y)\} z :$

$[(1) . *205\cdot35] \quad \supset : x = \min\{P_{\text{po}} \sqsubset P(x \vdash\!\dashv y)\}'P(x \vdash\!\dashv y) :$

$[(2) . *205\cdot12] \quad \supset : x = B'\{P_{\text{po}} \sqsubset P(x \vdash\!\dashv y)\}$ (3)

Similarly $\quad \vdash : \text{Hp} . \supset . y = B'\text{Cnv}'\{P_{\text{po}} \sqsubset P(x \vdash\!\dashv y)\}$ (4)

$\vdash . (2) . (3) . (4) . \supset \vdash . \text{Prop}$

The following propositions are concerned in proving that if $P_{\text{po}} \epsilon \text{Ser}$, $P_{\text{fn}} = (P_1)_{\text{po}}$ and $P_\nu = (P_1)_\nu$. Note that "$x (P_1)_{\text{po}} y$" means that we can get from x to y by a finite number of steps from one term to the next, so that the series contains no limit-points between x and y. The relation "$x (P_1)_\nu y$" means that $\nu -_c 1$ intermediate terms

$$z_1, z_2, z_3, \ldots z_{\nu -_c 1}$$

can be found, each of which has the relation P_1 to its neighbour, and such that $x P_1 z_1$ and $z_{\nu -_c 1} P_1 y$. Thus we have to prove that, provided P_{po} is a series, this occurs when, and only when, the number of terms in the interval $P(x \mapsto y)$ is $\nu +_c 1$.

***260·2.** $\vdash : P_{\text{po}} \epsilon \text{connex} . x P_* y . y P_* z . \supset . P(x \mapsto z) = P(x \mapsto y) \cup P(y \mapsto z)$

Dem.

$\vdash . *201{\cdot}14{\cdot}15 . \supset \vdash : \text{Hp} . \supset . P(x \mapsto y) \subset P(x \mapsto z) . P(y \mapsto z) \subset P(x \mapsto z)$ (1)

$\vdash . *202{\cdot}13{\cdot}103 . \supset \vdash :. \text{Hp} . x P_* w . \supset : w P_* y . \vee . y P_* w$ (2)

$\vdash . (2) . *121{\cdot}103 . \supset$

$\vdash :. \text{Hp} . w \epsilon P(x \mapsto z) . \supset : x P_* w . w P_* y . \vee . y P_* w . w P_* z :$

[*121·103] $\supset : w \epsilon P(x \mapsto y) \cup P(y \mapsto z)$ (3)

$\vdash . (1) . (3) . \supset \vdash . \text{Prop}$

***260·201.** $\vdash : P_{\text{po}} \epsilon \text{connex} . \supset . P_{\text{fn}} \epsilon \text{trans}$

Dem.

$\vdash . *260{\cdot}12 . \supset \vdash : x P_{\text{fn}} y . y P_{\text{fn}} z . \supset . x P_* y . y P_* z$ (1)

$\vdash . (1) . *260{\cdot}2 . \supset$

$\vdash : \text{Hp} . x P_{\text{fn}} y . y P_{\text{fn}} z . \supset . P(x \mapsto z) = P(x \mapsto y) \cup P(y \mapsto z) .$ (2)

[*260·11.*120·71] $\supset . P(x \mapsto z) \epsilon \text{Cls induct}$ (3)

$\vdash . *60{\cdot}32{\cdot}371 . \supset \vdash : \alpha \epsilon 0 \cup 1 . \beta \subset \alpha . \supset . \beta \epsilon 0 \cup 1 :$

[Transp] $\supset \vdash : \beta \sim\epsilon 0 \cup 1 . \beta \subset \alpha . \supset . \alpha \sim\epsilon 0 \cup 1$ (4)

$\vdash . (2) . *260{\cdot}11 . \supset$

$\vdash : \text{Hp} . x P_{\text{fn}} y . y P_{\text{fn}} z . \supset . P(x \mapsto y) \sim\epsilon 0 \cup 1 . P(x \mapsto y) \subset P(x \mapsto z) .$

[(4)] $\supset . P(x \mapsto z) \sim\epsilon 0 \cup 1$ (5)

$\vdash . (3) . (5) . *260{\cdot}11 . \supset \vdash : \text{Hp} . x P_{\text{fn}} y . y P_{\text{fn}} z . \supset . x P_{\text{fn}} z : \supset \vdash . \text{Prop}$

***260·202.** $\vdash : P_{\text{po}} \epsilon \text{Ser} . \supset . P_{\text{fn}} \epsilon \text{Rl}`J \cap \text{trans}$

Dem.

$\vdash . *260{\cdot}12 . \supset \vdash : P_{\text{po}} \mathrel{\text{G}} J . \supset . P_{\text{fn}} \mathrel{\text{G}} J$ (1)

$\vdash . (1) . *260{\cdot}201 . \supset \vdash . \text{Prop}$

We shall not have in general $P_{\text{po}} \epsilon \text{Ser} . \supset . P_{\text{fn}} \epsilon \text{Ser}$, because P_{fn} is in general not connected. P_{fn} only relates two terms which are at a finite distance from each other, and hence divides P_{po} into a number of mutually exclusive parts. We shall only have $P_{\text{fn}} \epsilon \text{Ser}$ when every interval in the series is finite.

∗260·21. $\vdash : P_{\text{po}} \in \text{Ser} \,.\, xP_*y \,.\, yP_1z \,.\, \supset \,.\, P(x \vdash\!\dashv z) = P(x \vdash\!\dashv y) \cup \iota`z$

Dem.

$\vdash \,.\, *121\cdot304 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, P(y \vdash\!\dashv z) = \iota`y \cup \iota`z$ (1)

$\vdash \,.\, *121\cdot242 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, y \in P(x \vdash\!\dashv y)$ (2)

$\vdash \,.\, *260\cdot2 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, P(x \vdash\!\dashv z) = P(x \vdash\!\dashv y) \cup P(y \vdash\!\dashv z)$

[(1).(2)] $= P(x \vdash\!\dashv y) \cup \iota`z : \supset \vdash \,.\, \text{Prop}$

∗260·22. $\vdash : P_{\text{po}} \in \text{Ser} \,.\, \supset \,.\, (P_1)_1 = P_1 \,.\, P_1 \in 1 \to 1 \,.\, (P_1)_{\text{po}} \subseteq J$

Dem.

$\vdash \,.\, *121\cdot254 \,.\, \supset \vdash \,.\, P_1 = (P_{\text{po}})_1$ (1)

$\vdash \,.\, (1) \,.\, *204\cdot7 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, P_1 \in 1 \to 1$ (2)

$\vdash \,.\, *121\cdot305 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, P_1 \subseteq P \,.$

[∗91·59] $\supset \,.\, (P_1)_{\text{po}} \subseteq P_{\text{po}} \,.$

[∗204·1] $\supset \,.\, (P_1)_{\text{po}} \subseteq J$ (3)

$\vdash \,.\, (1) \,.\, (2) \,.\, (3) \,.\, *121\cdot31 \,.\, \supset \vdash \,.\, \text{Prop}$

∗260·23. $\vdash : P_{\text{po}} \in \text{Ser} \,.\, \nu \in \text{NC induct} \,.\, \supset \,.\, (P_1)_\nu \in 1 \to 1$ [∗121·342 . ∗260·22]

∗260·24. $\vdash : P_{\text{po}} \in \text{Ser} \,.\, \nu \in \text{NC induct} \,.\, x(P_1)_\nu y \,.\, x(P_1)_{\nu +_c 1} z \,.\, \supset \,.\, yP_1z$

Dem.

$\vdash \,.\, *121\cdot35 \,.\, *260\cdot22 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, x\{(P_1)_\nu \,|\, P_1\}z \,.$

[∗34·1] $\supset \,.\, (\exists w) \,.\, x(P_1)_\nu w \,.\, wP_1z \,.$

[∗260·23.Hp] $\supset \,.\, yP_1z : \supset \vdash \,.\, \text{Prop}$

∗260·25. $\vdash : P_{\text{po}} \in \text{Ser} \,.\, R = P_1 \,.\, xR_*y \,.\, \supset \,.\, P(x \vdash\!\dashv y) = R(x \vdash\!\dashv y)$

Dem.

$\vdash \,.\, *260\cdot24 \,.\, \supset \vdash : \text{Hp} \,.\, \nu \in \text{NC induct} \,.\, xR_\nu y \,.\, xR_{\nu +_c 1}z \,.\, P$ [illegible] $= R(x \vdash\!\dashv y) \,.\, \supset \,.$
$yRz \,.\, P(x \vdash\!\dashv y) = R(x \vdash\!\dashv y) \,.$

[∗260·21] $\supset \,.\, P(x \vdash\!\dashv z) = R(x \vdash\!\dashv y) \cup \iota`z$

[∗260·22.∗121·371·304] $= R(x \vdash\!\dashv z)$ (1)

$\vdash \,.\, (1) \,.\, \supset \vdash :. \text{Hp} \,.\, \nu \in \text{NC induct} : xR_\nu y \,.\, \supset_y \,.\, P(x \vdash\!\dashv y) = R(x \vdash\!\dashv y) : \supset :$
$xR_{\nu +_c 1}z \,.\, \supset_z \,.\, P(x \vdash\!\dashv z) = R(x \vdash\!\dashv z)$ (2)

$\vdash \,.\, *121\cdot301\cdot22\cdot242 \,.\, \supset \vdash : \text{Hp} \,.\, xR_0y \,.\, \supset \,.\, P(x \vdash\!\dashv y) = \iota`x = R(x \vdash\!\dashv y)$ (3)

$\vdash \,.\, (2) \,.\, (3) \,.\, \text{Induct} \,.\, \supset$

$\vdash :. \text{Hp} \,.\, \supset : \nu \in \text{NC induct} \,.\, xR_\nu y \,.\, \supset \,.\, P(x \vdash\!\dashv y) = R(x \vdash\!\dashv y) :$

[∗121·12] $\supset : S \in \text{finid}`R \,.\, xSy \,.\, \supset \,.\, P(x \vdash\!\dashv y) = R(x \vdash\!\dashv y) :$

[∗121·52.∗260·22] $\supset : xR_*y \,.\, \supset \,.\, P(x \vdash\!\dashv y) = R(x \vdash\!\dashv y) :. \supset \vdash \,.\, \text{Prop}$

In the above proposition, "Induct" refers to ∗120·13. The "$\phi\xi$" of ∗120·13 is replaced by

$$xR_\xi y \,.\, \supset_y \,.\, P(x \vdash\!\dashv y) = R(x \vdash\!\dashv y).$$

Thus (2), in the above proof, is (when ν is replaced by ξ)

$$\xi \epsilon \text{NC induct} . \phi\xi . \supset . \phi(\xi +_c 1),$$

and (3) is $\phi 0$.

Hence, by $*120{\cdot}13$, we have

$$\alpha \epsilon \text{NC induct} . \supset . \phi\alpha,$$

i.e. $\nu \epsilon \text{NC induct} . \supset : xR_\nu y . \supset_y . R(x \mapsto y),$

which is the inference drawn in the above proof.

Wherever "Induct" is given as a reference, it indicates a process such as the above, making use of $*120{\cdot}13$ or $*120{\cdot}11$.

$*260{\cdot}251$. $\vdash : P_{po} \epsilon \text{Ser} . \supset . (P_1)_{po} \subseteq P_{fn}$

Dem.

$\vdash . *260{\cdot}25 . \supset \vdash : \text{Hp} . R = P_1 . xR_{po}y . \supset . P(x \mapsto y) = R(x \mapsto y) .$ (1)

$[*121{\cdot}45 . *260{\cdot}22]$ $\supset . P(x \mapsto y) \epsilon \text{Cls induct}$ (2)

$\vdash . *121{\cdot}242 . (1) . *260{\cdot}22 . \supset \vdash : \text{Hp}(1) . \supset . x, y \epsilon P(x \mapsto y) . x \neq y .$

$[*52{\cdot}41]$ $\supset . P(x \mapsto y) \sim \epsilon 0 \cup 1$ (3)

$\vdash . (2) . (3) . \supset \vdash : \text{Hp} . x(P_1)_{po}y . \supset . P(x \mapsto y) \epsilon \text{Cls induct} - 0 - 1 .$

$[*260{\cdot}11]$ $\supset . xP_{fn}y : \supset \vdash . \text{Prop}$

$*260{\cdot}26$. $\vdash :. P_{po} \epsilon \text{Ser} . R = P_1 . xR_*y . \supset : xP_\nu y . \equiv . xR_\nu y$

Dem.

$\vdash . *260{\cdot}25 . \supset \vdash :. \text{Hp} . \supset : P(x \mapsto y) = R(x \mapsto y) :$

$[*121{\cdot}11]$ $\supset : xP_\nu y . \equiv . xR_\nu y :. \supset \vdash . \text{Prop}$

$*260{\cdot}261$. $\vdash : P_{po} \epsilon \text{Ser} . \nu \epsilon \text{NC induct} - \iota`0 . xP_\nu y . xP_{\nu +_c 1}z . \supset . yP_1 z$

Dem.

$\vdash . *121{\cdot}11 . \supset \vdash : \text{Hp} . \supset . \text{Nc}`P(x \mapsto y) = \nu +_c 1 . \text{Nc}`P(x \mapsto z) = \nu +_c 2 .$ (1)

$[*120{\cdot}32]$ $\supset . y \neq z$ (2)

$\vdash . (1) . *120{\cdot}428 . \supset \vdash : \text{Hp} . \supset . \text{Nc}`P(x \mapsto z) > \text{Nc}`P(x \mapsto y) .$

$[*117{\cdot}222 . \text{Transp}]$ $\supset . \sim \{P(x \mapsto z) \subset P(x \mapsto y)\} .$

$[*121{\cdot}103 . *201{\cdot}14{\cdot}15]$ $\supset . \sim (zP_* y) .$ (3)

$[*202{\cdot}103]$ $\supset . yP_{po}z .$

$[*202{\cdot}171]$ $\supset . P(x \mapsto z) = P(x \mapsto y) \cup P(y \dashv z) .$

$[*120{\cdot}41 . (1) . (3)]$ $\supset . P(y \dashv z) \epsilon 1 .$

$[*121{\cdot}242 . (2)]$ $\supset . P(y \mapsto z) \epsilon 2 .$

$[*121{\cdot}11]$ $\supset . yP_1 z : \supset \vdash . \text{Prop}$

$*260{\cdot}27$. $\vdash : P_{po} \epsilon \text{Ser} . \supset . P_{fn} = (P_1)_{po}$

Dem.

$\vdash . *260{\cdot}261 . \supset \vdash : \text{Hp} . \nu \epsilon \text{NC induct} - \iota`0 . xP_\nu y . xP_{\nu +_c 1}z . x(P_1)_{po}y . \supset .$

$yP_1 z . x(P_1)_{po}y .$

$[*91{\cdot}511]$ $\supset . x(P_1)_{po}z$ (1)

$\vdash . (1) . \supset \vdash :. \mathrm{Hp} . \nu \in \mathrm{NC\ induct} - \iota\text{'}0 : xP_\nu y . \supset_y . x (P_1)_{\mathrm{po}} y : \supset :$

$xP_{\nu +_c 1} z . \supset_z . x (P_1)_{\mathrm{po}} z \quad (2)$

$\vdash . *91{\cdot}502 . \supset \vdash : xP_1 y . \supset . x (P_1)_{\mathrm{po}} y \quad (3)$

$\vdash . (2) . (3) . *120{\cdot}47 . \supset \vdash :. \mathrm{Hp} . \supset : \nu \in \mathrm{NC\ induct} - \iota\text{'}0 . \supset_\nu . P_\nu \subseteq (P_1)_{\mathrm{po}} :$

$[*260{\cdot}1] \quad \supset : P_{\mathrm{fn}} \subseteq (P_1)_{\mathrm{po}} \quad (4)$

$\vdash . (4) . *260{\cdot}251 . \supset \vdash . \mathrm{Prop}$

***260·28.** $\vdash : P_{\mathrm{po}} \in \mathrm{Ser} . \nu \in \mathrm{NC\ induct} - \iota\text{'}0 . \supset . P_\nu = (P_1)_\nu = (P_{\mathrm{fn}})_\nu$

Dem.

$\vdash . *260{\cdot}26 . \quad \supset \vdash :. \mathrm{Hp} . \supset : x (P_1)_{\mathrm{po}} y . xP_\nu y . \equiv . x (P_1)_{\mathrm{po}} y . x (P_1)_\nu y \quad (1)$

$\vdash . *260{\cdot}1 . \quad \supset \vdash : \mathrm{Hp} . xP_\nu y . \supset . xP_{\mathrm{fn}} y .$

$[*260{\cdot}27] \quad \supset . x (P_1)_{\mathrm{po}} y \quad (2)$

$\vdash . *121{\cdot}321 . \quad \supset \vdash : \mathrm{Hp} . x (P_1)_\nu y . \supset . x (P_1)_{\mathrm{po}} y \quad (3)$

$\vdash . (1) . (2) . (3) . \supset \vdash :. \mathrm{Hp} . \supset : xP_\nu y . \equiv . x (P_1)_\nu y \quad (4)$

$\vdash . *121{\cdot}254 . \quad \supset \vdash . (P_1)_\nu = \{(P_1)_{\mathrm{po}}\}_\nu .$

$[*260{\cdot}27] \quad \supset \vdash : \mathrm{Hp} . \supset . (P_1)_\nu = (P_{\mathrm{fn}})_\nu \quad (5)$

$\vdash . (4) . (5) . \supset \vdash . \mathrm{Prop}$

The above proposition does not hold in general when $\nu = 0$, for if P is a compact series, $P_1 = \dot{\Lambda}$, so that $(P_1)_0 = \dot{\Lambda}$, but $P_0 = I \upharpoonright C\text{'}P$.

***260·29.** $\vdash : P_{\mathrm{po}} \in \mathrm{Ser} . xP_{\mathrm{fn}} y . \supset . P (x \mapsto\!\dashv y) = P_1 (x \mapsto\!\dashv y) = P_{\mathrm{fn}} (x \mapsto\!\dashv y)$

Dem.

$\vdash . *260{\cdot}27{\cdot}25 . \supset \vdash : \mathrm{Hp} . \supset . P (x \mapsto\!\dashv y) = P_1 (x \mapsto\!\dashv y)$

$[*121{\cdot}253 . *260{\cdot}27] \quad = P_{\mathrm{fn}} (x \mapsto\!\dashv y) : \supset \vdash . \mathrm{Prop}$

The following propositions are mainly concerned with the result of confining the field of P_{fn} to the posterity of a single term.

***260·3.** $\vdash : P_{\mathrm{po}} \in \mathrm{Ser} . \supset . \mathrm{D}\text{'}P_{\mathrm{fn}} = \mathrm{D}\text{'}P_1 . \mathrm{G}\text{'}P_{\mathrm{fn}} = \mathrm{G}\text{'}P_1 . C\text{'}P_{\mathrm{fn}} = C\text{'}P_1$

$[*260{\cdot}27 . *91{\cdot}504]$

***260·31.** $\vdash : P_{\mathrm{po}} \in \mathrm{Ser} . x \in \mathrm{D}\text{'}P_1 . \supset .$

$C\text{'}\{P_{\mathrm{fn}} \sqsubset (\iota\text{'}x \cup \overleftarrow{P_{\mathrm{fn}}}\text{'}x)\} = \overleftarrow{(P_1)_*}\text{'}x = \iota\text{'}x \cup \overleftarrow{P_{\mathrm{fn}}}\text{'}x$

Dem.

$\vdash . *260{\cdot}27 . \supset \vdash : \mathrm{Hp} . \supset . \iota\text{'}x \cup \overleftarrow{P_{\mathrm{fn}}}\text{'}x = \iota\text{'}x \cup \overleftarrow{(P_1)_{\mathrm{po}}}\text{'}x$

$[*96{\cdot}14] \quad = \overleftarrow{(P_1)_*}\text{'}x \quad (1)$

$\vdash . *260{\cdot}3 . \supset \vdash : \mathrm{Hp} . \supset . \exists ! \overleftarrow{P_{\mathrm{fn}}}\text{'}x .$

$[*36{\cdot}13] \quad \supset . (\exists y) . x \{P_{\mathrm{fn}} \sqsubset (\iota\text{'}x \cup \overleftarrow{P_{\mathrm{fn}}}\text{'}x)\} y \quad (2)$

$\vdash . *36{\cdot}13 . \supset \vdash : y \in \overleftarrow{P_{\mathrm{fn}}}\text{'}x . \supset . x \{P_{\mathrm{fn}} \sqsubset (\iota\text{'}x \cup \overleftarrow{P_{\mathrm{fn}}}\text{'}x)\} y .$

$[*10{\cdot}24] \quad \supset . (\exists z) . z \{P_{\mathrm{fn}} \sqsubset (\iota\text{'}x \cup \overleftarrow{P_{\mathrm{fn}}}\text{'}x)\} y \quad (3)$

$\vdash . (2) . (3) . \supset \vdash : \mathrm{Hp} . \supset . \iota\text{'}x \cup \overleftarrow{P_{\mathrm{fn}}}\text{'}x \subset C\text{'}\{P_{\mathrm{fn}} \sqsubset (\iota\text{'}x \cup \overleftarrow{P_{\mathrm{fn}}}\text{'}x)\} .$

$[*37{\cdot}41] \quad \supset . \iota\text{'}x \cup \overleftarrow{P_{\mathrm{fn}}}\text{'}x = C\text{'}\{P_{\mathrm{fn}} \sqsubset (\iota\text{'}x \cup \overleftarrow{P_{\mathrm{fn}}}\text{'}x)\} \quad (4)$

$\vdash . (1) . (4) . \supset \vdash . \mathrm{Prop}$

***260·32.** $\vdash : P_{po} \in \mathrm{Ser} \,.\supset .$

$$P_{fn} \pounds (\iota\text{'}x \cup \overleftarrow{P_{fn}}\text{'}x) = P_{po} \pounds (\iota\text{'}x \cup \overleftarrow{P_{fn}}\text{'}x) \,.\, P_{fn} \pounds (\iota\text{'}x \cup \overleftarrow{P_{fn}}\text{'}x) \in \mathrm{Ser}$$

Dem.

$\vdash . *260{\cdot}12 \,.\supset \vdash . P_{fn} \pounds (\iota\text{'}x \cup \overleftarrow{P_{fn}}\text{'}x) \subset P_{po} \pounds (\iota\text{'}x \cup \overleftarrow{P_{fn}}\text{'}x)$ (1)

$\vdash . *260{\cdot}3 \,.\, *200{\cdot}35 \,.\supset$

$\vdash : \mathrm{Hp} \,.\, x \sim\in \mathrm{D}\text{'}P_1 \,.\supset . P_{fn} \pounds (\iota\text{'}x \cup \overleftarrow{P_{fn}}\text{'}x) = \dot{\Lambda} = P_{po} \pounds (\iota\text{'}x \cup \overleftarrow{P_{fn}}\text{'}x)$ (2)

$\vdash . *201{\cdot}521 \,.\, *260{\cdot}27 \,.\supset$

$\vdash : \mathrm{Hp} \,.\, x \in \mathrm{D}\text{'}P_1 \,.\supset . P_{fn} \pounds (\iota\text{'}x \cup \overleftarrow{P_{fn}}\text{'}x) = (P_1)_{po} \pounds \overleftarrow{(P_1)_*}\text{'}x \,.$

$[*202{\cdot}14 . *260{\cdot}22] \supset . P_{fn} \pounds (\iota\text{'}x \cup \overleftarrow{P_{fn}}\text{'}x) \in \mathrm{connex} \,.$

$[*260{\cdot}202] \qquad \supset . P_{fn} \pounds (\iota\text{'}x \cup \overleftarrow{P_{fn}}\text{'}x) \in \mathrm{Ser} \,.$ (3)

$[(1) . *260{\cdot}31 . *204{\cdot}41] \supset . P_{fn} \pounds (\iota\text{'}x \cup \overleftarrow{P_{fn}}\text{'}x) = P_{po} \pounds (\iota\text{'}x \cup \overleftarrow{P_{fn}}\text{'}x)$ (4)

$\vdash . (2) . (3) . (4) . \supset \vdash . \mathrm{Prop}$

***260·33.** $\vdash : P_{po} \in \mathrm{Ser} \,.\, x \in \mathrm{D}\text{'}P_1 \,.\, P_1 = R \,.\supset .$

$$P_{fn} \pounds (\iota\text{'}x \cup \overleftarrow{P_{fn}}\text{'}x) = (\overleftarrow{R_*}\text{'}x) \upharpoonleft R_{po} = \{(\overleftarrow{R_*}\text{'}x) \upharpoonleft R\}_{po} = \{R \upharpoonright (\overleftarrow{R_{po}}\text{'}x)\}_{po}$$

Dem.

$\vdash . *260{\cdot}27{\cdot}31 \,.\supset \vdash : \mathrm{Hp} \,.\supset . P_{fn} \pounds (\iota\text{'}x \cup \overleftarrow{P_{fn}}\text{'}x) = R_{po} \pounds \overleftarrow{R_*}\text{'}x$

$[*96{\cdot}16 . *91{\cdot}602] \qquad = (\overleftarrow{R_*}\text{'}x) \upharpoonleft R_{po}$ (1)

$[*96{\cdot}13] \qquad = \{(\overleftarrow{R_*}\text{'}x) \upharpoonleft R\}_{po}$ (2)

$[*96{\cdot}2 . *260{\cdot}22] \qquad = \{R \upharpoonright (\overleftarrow{R_{po}}\text{'}x)\}_{po}$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash . \mathrm{Prop}$

***260·34.** $\vdash : \mathrm{Hp}\, *260{\cdot}33 \,.\supset . \{P_{fn} \pounds (\iota\text{'}x \cup \overleftarrow{P_{fn}}\text{'}x)\}_1 = (\overleftarrow{R_*}\text{'}x) \upharpoonleft R = R \upharpoonright \overleftarrow{R_{po}}\text{'}x$

Dem.

$\vdash . *260{\cdot}33 \,.\, *121{\cdot}254 \,.\supset$

$\vdash : \mathrm{Hp} \,.\supset . \{P_{fn} \pounds (\iota\text{'}x \cup \overleftarrow{P_{fn}}\text{'}x)\}_1 = \{(\overleftarrow{R_*}\text{'}x) \upharpoonleft R\}_1 = \{R \upharpoonright \overleftarrow{R_{po}}\text{'}x\}_1$ (1)

$\vdash . (1) . *121{\cdot}31 \,.\, *260{\cdot}22 \,.\supset \vdash . \mathrm{Prop}$

The following propositions are concerned with the result of confining the field of P_{fn} to a single family.

***260·4.** $\vdash : P_{po} \in \mathrm{Ser} \,.\supset . P_{fn} \pounds \overleftrightarrow{P_{fn}}\text{'}x \in \mathrm{Ser} \,.$

$$C\text{'}(P_{fn} \pounds \overleftrightarrow{P_{fn}}\text{'}x) = \overleftrightarrow{P_{fn}}\text{'}x = (\overleftrightarrow{P_1})_*\text{'}x \,.\, \overleftrightarrow{P_{fn}}\text{'}x \sim\in 1$$

Dem.

$\vdash . *260{\cdot}27 \,.\, *97{\cdot}17 \,.\supset \vdash : \mathrm{Hp} \,.\supset . P_{fn} \pounds \overleftrightarrow{P_{fn}}\text{'}x = (P_1)_{po} \pounds (\overleftrightarrow{P_1})_*\text{'}x \,.$

$[*202{\cdot}15 . *260{\cdot}22] \qquad \supset . P_{fn} \pounds \overleftrightarrow{P_{fn}}\text{'}x \in \mathrm{connex} \,.$

$[*260{\cdot}202 . *204{\cdot}42] \qquad \supset . P_{fn} \pounds \overleftrightarrow{P_{fn}}\text{'}x \in \mathrm{Ser}$ (1)

$\vdash . *97{\cdot}18 . \supset \vdash . C'(P_{fn} \sqsubset \overleftrightarrow{P}_{fn}'x) = \overleftrightarrow{P}_{fn}'x$ (2)

$\vdash . (2) . *260{\cdot}202 . *200{\cdot}12 . \supset \vdash : \mathrm{Hp} . \supset . \overleftrightarrow{P}_{fn}'x \sim \epsilon 1$ (3)

$\vdash . *260{\cdot}27 . *97{\cdot}17 . \supset \vdash : \mathrm{Hp} . \supset . \overleftrightarrow{P}_{fn}'x = (\overleftrightarrow{P}_1)_*'x$ (4)

$\vdash . (1) . (2) . (3) . (4) . \supset \vdash . \mathrm{Prop}$

***260·41.** $\vdash : P_{po} \epsilon \mathrm{Ser} . R = P_1 . \supset .$

$$P_{fn} \sqsubset \overleftrightarrow{P}_{fn}'x = R_{po} \sqsubset \overleftrightarrow{R}_*'x = (\overleftrightarrow{R}_*'x) \rceil R_{po} = R_{po} \upharpoonright \overleftrightarrow{R}_*'x$$

Dem.

$\vdash . *260{\cdot}27 . *97{\cdot}17 . \supset \vdash : \mathrm{Hp} . \supset . P_{fn} \sqsubset \overleftrightarrow{P}_{fn}'x = R_{po} \sqsubset \overleftrightarrow{R}_*'x$ (1)

$\vdash . *97{\cdot}13 . \supset \vdash : \mathrm{Hp} . y \epsilon \overleftrightarrow{R}_*'x . y R_{po} z . \supset . z \epsilon \breve{R}_{po}``\overrightarrow{R}_*'x \cup \breve{R}_{po}``\overleftarrow{R}_*'x .$

$[*92{\cdot}311 . *260{\cdot}22] \quad \supset . z \epsilon \overrightarrow{R}_*'x \cup \overleftarrow{R}_*'x .$

$[*97{\cdot}13 . *36{\cdot}13] \quad \supset . y (R_{po} \sqsubset \overleftrightarrow{R}_*'x) z$ (2)

$\vdash . *35{\cdot}21{\cdot}441 . \supset \vdash . R_{po} \sqsubset \overleftrightarrow{R}_*'x \Subset (\overleftrightarrow{R}_*'x) \rceil R_{po}$ (3)

$\vdash . (2) . (3) . \quad \supset \vdash : \mathrm{Hp} . \supset . R_{po} \sqsubset \overleftrightarrow{R}_*'x = (\overleftrightarrow{R}_*'x) \rceil R_{po}$ (4)

Similarly $\quad \vdash : \mathrm{Hp} . \supset . R_{po} \sqsubset \overleftrightarrow{R}_*'x = R_{po} \upharpoonright \overleftrightarrow{R}_*'x$ (5)

$\vdash . (1) . (4) . (5) . \supset \vdash . \mathrm{Prop}$

***260·42.** $\vdash : \mathrm{Hp}\ *260{\cdot}41 . \supset . P_{fn} \sqsubset \overleftrightarrow{P}_{fn}'x = (\overleftrightarrow{R}_*'x \rceil R)_{po} = (R \upharpoonright \overleftrightarrow{R}_*'x)_{po}$

Dem.

$\vdash . *92{\cdot}32 . *260{\cdot}22 . \supset \vdash : \mathrm{Hp} . \supset . \breve{R}``\overleftrightarrow{R}_*'x \subset \overleftrightarrow{R}_*'x .$

$[*96{\cdot}111] \quad \supset . (\overleftrightarrow{R}_*'x) \rceil R_{po} = \{(\overleftrightarrow{R}_*'x) \rceil R\}_{po}$ (1)

Similarly $\quad \vdash : \mathrm{Hp} . \supset . R_{po} \upharpoonright \overleftrightarrow{R}_*'x = \{R \upharpoonright \overleftrightarrow{R}_*'x\}_{po}$ (2)

$\vdash . (1) . (2) . *260{\cdot}41 . \supset \vdash . \mathrm{Prop}$

***260·43.** $\vdash : P_{po} \epsilon \mathrm{Ser} . \supset .$

$$\{P_{fn} \sqsubset \overleftrightarrow{P}_{fn}'x\}_1 = P_1 \sqsubset \overleftrightarrow{P}_{fn}'x = (\overleftrightarrow{P}_{fn}'x) \rceil P_1 = P_1 \upharpoonright (\overleftrightarrow{P}_{fn}'x)$$

Dem.

$\vdash . *260{\cdot}42 . *121{\cdot}254 . \supset$

$\vdash : \mathrm{Hp} . R = P_1 . \supset . \{P_{fn} \sqsubset \overleftrightarrow{P}_{fn}'x\}_1 = \{(\overleftrightarrow{R}_*'x) \rceil R\}_1$

$[*121{\cdot}31 . *260{\cdot}22] \quad = (\overleftrightarrow{R}_*'x) \rceil R$

$[*97{\cdot}17 . *260{\cdot}27] \quad = (\overleftrightarrow{P}_{fn}'x) \rceil P_1$ (1)

Similarly $\quad \vdash : \mathrm{Hp} . \supset . \{P_{fn} \sqsubset \overleftrightarrow{P}_{fn}'x\}_1 = P_1 \upharpoonright \overleftrightarrow{P}_{fn}'x$ (2)

$\vdash . (1) . (2) . *35{\cdot}11 . \supset \vdash : \mathrm{Hp} . \supset . \{P_{fn} \sqsubset \overleftrightarrow{P}_{fn}'x\}_1 = P_1 \sqsubset \overleftrightarrow{P}_{fn}'x$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash . \mathrm{Prop}$

Observe that the two series $P_{fn} \sqsubset \overleftrightarrow{P}_{fn}'x$ and $P_{fn} \sqsubset \overleftrightarrow{P}_{fn}'y$ are either identical or have no common terms in their fields. This results immediately from *97·16, since the fields of the two series are $(\overleftrightarrow{P}_1)_*'x$ and $(\overleftrightarrow{P}_1)_*'y$.

*261. FINITE AND INFINITE SERIES.

Summary of *261.

In this number we define finite and infinite series, and we show that, where well-ordered series are concerned, there is only one kind of finitude, *i.e.* there is not the distinction, which exists in cardinals, between "inductive" and "non-reflexive." We also give various equivalent forms of the distinction between finite and infinite series, and some of the simpler properties of each. The propositions of this number are numerous and important.

We define an infinite series as one whose field is a reflexive class, and a finite series as one which is not infinite. Thus we put

$$\text{Ser infin} = \text{Ser} \cap \breve{C}``\text{Cls refl} \quad \text{Df},$$

$$\Omega \text{ infin} = \Omega \cap \breve{C}``\text{Cls refl} \quad \text{Df},$$

$$\text{Ser fin} = \text{Ser} - \text{Ser infin} \quad \text{Df},$$

$$\Omega \text{ fin} = \Omega - \Omega \text{ infin} \quad \text{Df}.$$

We also put, to begin with,

$$\Omega \text{ induct} = \Omega \cap \breve{C}``\text{Cls induct} \quad \text{Df},$$

but in the course of this number we prove

***261·42.** $\vdash . \Omega \text{ fin} = \Omega \text{ induct}$

so that the symbol "Ω induct" is not required after the present number.

After some preliminary propositions, we proceed (*261·2 ff.) to various criteria of finitude and infinity. We have

***261·25.** $\vdash :. P \in \text{Ser} . \supset :$

$$C`P \in \text{Cls induct} - \iota`\Lambda . \equiv . P = P_{\text{fn}} . \text{E} ! B`P . \text{E} ! B`\breve{P}$$

The condition $P = P_{\text{fn}}$ insures that every interval is finite, but this still leaves it possible for our series to be a progression, or its converse, or the converse of a progression followed by a progression (*i.e.* the type of the negative and positive finite integers in order of magnitude). The third of these

possibilities is excluded by either $\mathrm{E}\,!\,B\text{'}P$ or $\mathrm{E}\,!\,B\text{'}\breve{P}$; the second is excluded by $\mathrm{E}\,!\,B\text{'}P$, and the first by $\mathrm{E}\,!\,B\text{'}\breve{P}$. We have

***261·212.** $\vdash :. P \in \Omega . \supset : \text{Cl}\text{'}P_1 = \text{Cl}\text{'}P . \equiv . P = (P_1)_{\text{po}} . \equiv . P = P_{\text{fn}}$

"$\text{Cl}\text{'}P_1 = \text{Cl}\text{'}P$" means that every term except the first has an immediate predecessor. We have

***261·26.** $\vdash : P \in \text{Ser} . \alpha \subset C\text{'}P . \exists ! \alpha . \alpha \in \text{Cls induct} . \supset . \mathrm{E}\,!\,\text{min}_P\text{'}\alpha . \mathrm{E}\,!\,\text{max}_P\text{'}\alpha$

and

***261·27.** $\vdash :. P \in \text{Ser} : \alpha \subset C\text{'}P . \exists ! \alpha . \supset_\alpha . \mathrm{E}\,!\,\text{min}_P\text{'}\alpha . \mathrm{E}\,!\,\text{max}_P\text{'}\alpha : \supset .$
$P = P_{\text{fn}} . C\text{'}P \in \text{Cls induct}$

whence we obtain

***261·28.** $\vdash :: P \in \text{Ser} . \supset :.$
$\alpha \subset C\text{'}P . \exists ! \alpha . \supset_\alpha . \mathrm{E}\,!\,\text{min}_P\text{'}\alpha . \mathrm{E}\,!\,\text{max}_P\text{'}\alpha : \equiv . C\text{'}P \in \text{Cls induct}$

I.e. a series whose field is inductive is one in which every existent sub-class of the field has both a minimum and a maximum.

From the above, together with an inductive proof that every inductive class which is not a unit class is the field of some series, we obtain

***261·29.** $\vdash . \text{Cls induct} =$
$1 \cup C\text{''}\hat{P}\{P \in \text{Ser} : \alpha \subset C\text{'}P . \exists ! \alpha . \supset_\alpha . \mathrm{E}\,!\,\text{min}_P\text{'}\alpha . \mathrm{E}\,!\,\text{max}_P\text{'}\alpha\}$
$= 1 \cup C\text{''}(\Omega \cap \text{Cnv}\text{''}\Omega)$

The above proposition is interesting as giving an alternative method of treating inductive classes. Instead of the definitions adopted in *120, we might have taken the above proposition as the definition of inductive classes, putting

$$\text{NC induct} = \text{Nc}\text{''}\text{Cls induct} \quad \text{Df.}$$

We should thus wholly avoid the use of mathematical induction in definitions; hence if such avoidance were in any way desirable, it could be secured by dealing with series before introducing the distinction of finite and infinite, and then defining inductive classes as the fields of series which are well-ordered backwards as well as forwards. The inductive properties of such classes would then be deduced from *261·27, together with *260·27, in virtue of which we have

$$P \in \Omega \cap \text{Cnv}\text{''}\Omega . \supset . P = (P_1)_{\text{po}},$$

whence, by *91·62,

$$\vdash :: P \in \Omega \cap \text{Cnv}\text{''}\Omega . \supset :. xPy . \equiv : \breve{P}_1\text{''}\mu \subset \mu . \breve{P}_1\text{'}x \in \mu . \supset_\mu . y \in \mu.$$

In virtue of this proposition, if γ is the field of a well-ordered series P whose converse is well-ordered, then any property which is inherited with respect to P_1 belongs to all the successors of x (where $x \in \gamma$) if it belongs to the immediate successor of x. Hence mathematical induction follows.

From the above we obtain at once

***261·31.** $\vdash :. P \epsilon \mathrm{Ser} . \supset : C'P \epsilon \mathrm{Cls\,induct} . \equiv . P, \breve{P} \epsilon \Omega$

I.e. series whose fields are inductive are the same as inductive well-ordered series, and are also the same as well-ordered series whose converses are well-ordered. Hence also we obtain

***261·33.** $\vdash : P, Q \epsilon \Omega . Q \subseteq \breve{P} . \supset . Q \epsilon \Omega\,\mathrm{induct}$

I.e. a *descending* well-ordered series of terms chosen out of a well-ordered series must be finite. This proposition, which is due to Cantor, has been used by him in many proofs.

We have

***261·35.** $\vdash :. P \epsilon \Omega . \supset : C'P \epsilon \mathrm{Cls\,induct} - \iota'\Lambda . \equiv . \text{Ꭰ}'P_1 = \text{Ꭰ}'P . \mathrm{E} ! B'\breve{P}$

In *253·51 and following propositions we have already had the hypothesis $\text{Ꭰ}'P_1 = \text{Ꭰ}'P . \mathrm{E} ! B'\breve{P}$, which now turns out to be equivalent to the hypothesis that our series is finite and not null. Thus we have

***261·36.** $\vdash :. P \epsilon \Omega . \supset : C'P \epsilon \mathrm{Cls\,induct} - \iota'\Lambda . \equiv . \mathrm{Nr}'P \neq \dot{1} \dotplus \mathrm{Nr}'P$

*261·4 and following propositions are concerned in proving that a well-ordered series which is not inductive always contains progressions, and in deducing consequences from this. We have

***261·4.** $\vdash : P \epsilon \Omega - \Omega\,\mathrm{induct} . \supset . \{\overleftarrow{(P_1)_*}'B'P\} \upharpoonright P_1 \epsilon \mathrm{Prog}$

The above proposition is very important, for many reasons. One of its most important consequences is that, if P is a well-ordered series which is not inductive, its field contains an $\aleph_0$, and is therefore a reflexive class (*261·401). Hence a class which can be well-ordered is either inductive or reflexive (*261·43), and a well-ordered series is either inductive or infinite according to the definitions given above (*261·41). Hence where well-ordered series are concerned, the two ways of defining finite and infinite (namely those in *120 and *124) give equivalent results. This cannot (so far as is known) be proved for classes in general without assuming the multiplicative axiom.

From the above-mentioned propositions it follows that an infinite well-ordered series is one in which P_1 limited to the posterity of $B'P$ with respect to $\dot{P_1}$ is a progression in the sense of *122 (*261·44), and that any class contained in a well-ordered series is either inductive or reflexive (*261·46).

The number ends with some propositions in ordinal arithmetic. We prove that P^Q is well-ordered if P is well-ordered and Q is a finite well-ordered series (*261·62); that if R is a finite well-ordered series, and P is less than Q (in the sense of *254), then P^R is less than Q^R; and that a finite well-ordered series is less than an infinite one (*261·65).

***261·01.** $\text{Ser infin} = \text{Ser} \cap \breve{C}``\text{Cls refl}$ Df

***261·02.** $\Omega\ \text{infin} = \Omega \cap \breve{C}``\text{Cls refl}$ Df

***261·03.** $\text{Ser fin} = \text{Ser} - \text{Ser infin}$ Df

***261·04.** $\Omega\ \text{fin} = \Omega - \Omega\ \text{infin}$ Df

***261·05.** $\Omega\ \text{induct} = \Omega \cap \breve{C}``\text{Cls induct}$ Df

***261·1.** $\vdash : P \in \text{Ser infin} . \equiv . P \in \text{Ser} . C`P \in \text{Cls refl}$ [(*261·01)]

***261·11.** $\vdash : P \in \Omega\ \text{infin} . \equiv . P \in \Omega . C`P \in \text{Cls refl}$ [(*261·02)]

***261·12.** $\vdash : P \in \text{Ser fin} . \equiv . P \in \text{Ser} - \text{Ser infin} . \equiv . P \in \text{Ser} . C`P \sim\in \text{Cls refl}$
[(*261·03)]

***261·13.** $\vdash : P \in \Omega\ \text{fin} . \equiv . P \in \Omega - \Omega\ \text{infin} . \equiv . P \in \Omega . C`P \sim\in \text{Cls refl}$
[(*261·04)]

***261·14.** $\vdash : P \in \Omega\ \text{induct} . \equiv . P \in \Omega . C`P \in \text{Cls induct}$ [(*261·05)]

***261·15.** $\vdash : P \in \text{Ser infin} . P \ \text{smor}\ Q . \supset . Q \in \text{Ser infin}$

Dem.

$\vdash . \text{*261·1} . \supset \vdash : \text{Hp} . \supset . P \in \text{Ser} . C`P \in \text{Cls refl} . P \ \text{smor}\ Q .$
[*204·21.*151·18] $\supset . Q \in \text{Ser} . C`P \in \text{Cls refl} . C`P \ \text{sm}\ C`Q .$
[*124·18] $\supset . Q \in \text{Ser} . C`Q \in \text{Cls refl} .$
[*261·1] $\supset . Q \in \text{Ser infin} : \supset \vdash . \text{Prop}$

***261·151.** $\vdash : P \in \text{Ser infin} . \supset . \text{Nr}`P \subset \text{Ser infin}$ [*261·15]

***261·152.** $\vdash : P \in \text{Ser infin} . \equiv . \text{N}_0\text{r}`P \subset \text{Ser infin} . \equiv . \exists ! \text{N}_0\text{r}`P \cap \text{Ser infin}$
[*261·151 . *155·12]

***261·153.** $\vdash : P \in \text{Ser infin} . \equiv . (\exists Q) . P \ \text{smor}\ Q . Q \in \text{Ser infin}$
[*261·15 . *151·13]

***261·16.** $\vdash : P \in \Omega\ \text{infin} . P \ \text{smor}\ Q . \supset . Q \in \Omega\ \text{infin}$
[Proof as in *261·15, using *261·11 . *251·111 . *151·18 . *124·18]

***261·161.** $\vdash : P \in \Omega\ \text{infin} . \supset . \text{Nr}`P \subset \Omega\ \text{infin}$ [*261·16]

***261·162.** $\vdash : P \in \Omega\ \text{infin} . \equiv . \text{N}_0\text{r}`P \subset \Omega\ \text{infin} . \equiv . \exists ! \text{N}_0\text{r}`P \cap \text{Ser infin}$
[*261·161 . *155·12]

***261·163.** $\vdash : P \in \Omega\ \text{infin} . \equiv . (\exists Q) . P \ \text{smor}\ Q . Q \in \Omega\ \text{infin}$ [*261·16 . *151·13]

***261·17.** $\vdash : P \in \text{Ser fin} . P \ \text{smor}\ Q . \supset . Q \in \text{Ser fin}$ [*261·15 . Transp]

***261·171.** $\vdash : P \in \text{Ser fin} . \supset . \text{Nr}`P \subset \text{Ser fin}$ [*261·17]

***261·172.** $\vdash : P \in \text{Ser fin} . \equiv . \text{N}_0\text{r}`P \subset \text{Ser fin} . \equiv . \exists ! \text{N}_0\text{r}`P \cap \text{Ser fin}$
[*261·171 . *155·12]

***261·173.** $\vdash : P \in \text{Ser fin} . \equiv . (\exists Q) . P \ \text{smor}\ Q . Q \in \text{Ser fin}$ [*261·17 . *151·13]

***261·18.** $\vdash : P \epsilon \Omega \text{ fin} . P \text{ smor } Q . \supset . Q \epsilon \Omega \text{ fin}$ [*261·16 . Transp]

***261·181.** $\vdash : P \epsilon \Omega \text{ fin} . \supset . \text{Nr}`P \subset \Omega \text{ fin}$ [*261·18]

***261·182.** $\vdash : P \epsilon \Omega \text{ fin} . \equiv . \text{N}_0\text{r}`P \subset \Omega \text{ fin} . \equiv . \exists ! \text{N}_0\text{r}`P \cap \Omega \text{ fin}$
[*261·181 . *155·12]

***261·183.** $\vdash : P \epsilon \Omega \text{ fin} . \equiv . (\exists Q) . P \text{ smor } Q . Q \epsilon \Omega \text{ fin}$ [*261·18 . *151·13]

***261·19.** $\vdash : P \epsilon \Omega \text{ induct} . P \text{ smor } Q . \supset . Q \epsilon \Omega \text{ induct}$
[Proof as in *261·16, using *120·214 instead of *124·18]

***261·191.** $\vdash : P \epsilon \Omega \text{ induct} . \supset . \text{Nr}`P \subset \Omega \text{ induct}$ [*261·19]

***261·192.** $\vdash : P \epsilon \Omega \text{ induct} . \equiv . \text{N}_0\text{r}`P \subset \Omega \text{ induct} . \equiv . \exists ! \text{N}_0\text{r}`P \cap \Omega \text{ induct}$
[*261·191 . *155·12]

***261·193.** $\vdash : P \epsilon \Omega \text{ induct} . \equiv . (\exists Q) . P \text{ smor } Q . Q \epsilon \Omega \text{ induct}$
[*261·19 . *151·13]

***261·2.** $\vdash : P_{po} \epsilon \text{ connex} . (B`P) P_{fn} (B`\breve{P}) . \supset . C`P \epsilon \text{ Cls induct}$

Dem.

$\vdash . *202·181 . \supset \vdash : \text{Hp} . \supset . C`P = P(B`P \mapsto B`\breve{P}) .$
$[*260·11 . \text{Hp}] \qquad \supset . C`P \epsilon \text{ Cls induct} : \supset \vdash . \text{Prop}$

***261·21.** $\vdash : P \epsilon \text{ connex} . P = P_{fn} . \text{E} ! B`P . \text{E} ! B`\breve{P} . \supset . C`P \epsilon \text{ Cls induct}$

Dem.

$\vdash . *202·103 . *93·101 . \supset \vdash : \text{Hp} . \supset . (B`P) P (B`\breve{P}) .$
$[\text{Hp}] \qquad \supset . (B`P) P_{fn} (B`\breve{P}) .$
$[*261·2] \qquad \supset . C`P \epsilon \text{ Cls induct} : \supset \vdash . \text{Prop}$

***261·211.** $\vdash : P \epsilon \text{ Ser} . \supset . \text{min}_P`\{\overrightarrow{P}`x - \overleftarrow{(P_1)_{po}}`x\} \subset \text{Ɑ}`P - \text{Ɑ}`P_1$

Dem.

$\vdash . *91·511 . *121·305 . \supset$
$\vdash :. \text{Hp} . \supset : y \epsilon \overleftarrow{P}`x \cap \overleftarrow{(P_1)_{po}}`x . yP_1z . \supset . z \epsilon \overleftarrow{P}`x \cap \overleftarrow{(P_1)_{po}}`x :$
$[\text{Transp}] \supset : z \epsilon \overleftarrow{P}`x - \overleftarrow{(P_1)_{po}}`x . yP_1z . \supset . y \epsilon - \overleftarrow{P}`x \cup - \overleftarrow{(P_1)_{po}}`x \qquad (1)$
$\vdash . *91·502 . \supset \vdash :. z \epsilon \overleftarrow{P}`x - \overleftarrow{(P_1)_{po}}`x . \supset : z \epsilon \overleftarrow{P}`x - \overleftarrow{P_1}`x :$
$[*201·63] \qquad \supset : \text{Hp} . \supset . xP^2z \qquad (2)$
$\vdash . *201·63 . \supset \vdash : \text{Hp} . xP^2z . yP_1z . \supset . \sim(yPx) . y \neq x .$
$[*202·103] \qquad \supset . xPy \qquad (3)$
$\vdash . (2) . (3) . \supset \vdash : \text{Hp} . z \epsilon \overleftarrow{P}`x - \overleftarrow{(P_1)_{po}}`x . yP_1z . \supset . y \epsilon \overleftarrow{P}`x .$
$[(1)] \qquad \supset . y \epsilon \overleftarrow{P}`x - \overleftarrow{(P_1)_{po}}`x .$
$[*201·63] \qquad \supset . y \epsilon \overrightarrow{P}`z \cap \{\overleftarrow{P}`x - \overleftarrow{(P_1)_{po}}`x\} .$
$[*205·14] \qquad \supset . z \sim \epsilon \text{ min}_P`\{\overrightarrow{P}`x - \overleftarrow{(P_1)_{po}}`x\} \qquad (4)$
$\vdash . (4) . \text{Transp} . \supset$
$\vdash : \text{Hp} . z \epsilon \text{ min}_P`\{\overleftarrow{P}`x - \overleftarrow{(P_1)_{po}}`x\} . \supset . \sim(\exists y) . yP_1z : \supset \vdash . \text{Prop}$

***261·212.** $\vdash :. P \in \Omega . \supset : \text{Ɑ}`P_1 = \text{Ɑ}`P . \equiv . P = (P_1)_{po} . \equiv . P = P_{fn}$

Dem.

$\vdash . *121\cdot305 . \supset \vdash : \mathrm{Hp} . \supset . (P_1)_{po} \Subset P$ (1)

$\vdash . (1) . \quad \supset \vdash : \mathrm{Hp} . P \neq (P_1)_{po} . \supset . (\exists x, y) . xPy . \sim \{x (P_1)_{po} y\} .$

$[*32\cdot18] \quad \supset . (\exists x) . \exists ! \overleftarrow{P}`x - \overleftarrow{(P_1)_{po}}`x .$

$[*250\cdot121] \quad \supset . (\exists x) . E ! \min_P`\{\overleftarrow{P}`x - \overleftarrow{(P_1)_{po}}`x\} .$

$[*261\cdot211] \quad \supset . \exists ! \text{Ɑ}`P - \text{Ɑ}`P_1$ (2)

$\vdash . (2) . \mathrm{Transp} . \supset \vdash : \mathrm{Hp} . \text{Ɑ}`P = \text{Ɑ}`P_1 . \supset . P = (P_1)_{po}$ (3)

$\vdash . *91\cdot504 . \quad \supset \vdash : P = (P_1)_{po} . \supset . \text{Ɑ}`P = \text{Ɑ}`P_1$ (4)

$\vdash . (3) . (4) . \quad \supset \vdash :. \mathrm{Hp} . \supset : \text{Ɑ}`P_1 = \text{Ɑ}`P . \equiv . P = (P_1)_{po} .$

$[*260\cdot27] \quad \equiv . P = P_{fn} :. \supset \vdash . \mathrm{Prop}$

***261·22.** $\vdash : P \in \mathrm{Ser} . C`P \in \mathrm{Cls\ induct} . \supset . P = P_{fn} . \mathrm{D}`P = \mathrm{D}`P_1 . \text{Ɑ}`P = \text{Ɑ}`P_1$

Dem.

$\vdash . *260\cdot12 . *201\cdot18 . \supset \vdash : \mathrm{Hp} . \supset . P_{fn} \Subset P$ (1)

$\vdash . *121\cdot242 . \quad \supset \vdash : \mathrm{Hp} . xPy . \supset . x, y \in P(x \mapsto y) . x \neq y .$

$[*52\cdot41] \quad \supset . P(x \mapsto y) \sim \in 0 \cup 1$ (2)

$\vdash . *120\cdot481 . \quad \supset \vdash : \mathrm{Hp} . \supset . P(x \mapsto y) \in \mathrm{Cls\ induct}$ (3)

$\vdash . (2) . (3) . *260\cdot11 . \supset \vdash : \mathrm{Hp} . xPy . \supset . xP_{fn}y$ (4)

$\vdash . (1) . (4) . \quad \supset \vdash : \mathrm{Hp} . \supset . P = P_{fn} .$ (5)

$[*260\cdot3] \quad \supset . \mathrm{D}`P = \mathrm{D}`P_1 . \text{Ɑ}`P = \text{Ɑ}`P_1$ (6)

$\vdash . (5) . (6) . \supset \vdash . \mathrm{Prop}$

***261·23.** $\vdash : P \in \mathrm{Ser} . \mathrm{D}`P_1 = \mathrm{D}`P . \sim E ! B`\breve{P} . \dot{\exists} ! P . \supset . C`P \in \mathrm{Cls\ refl}$

Dem.

$\vdash . *91\cdot52 . \supset \vdash . \breve{P}_1``\overleftarrow{(P_1)_*}`x = \overleftarrow{(P_1)_{po}}`x$ (1)

$\vdash . *91\cdot54 . *260\cdot22 . \supset$

$\vdash : \mathrm{Hp} . x \in C`P . \supset . \overleftarrow{(P_1)_*}`x = \iota`x \cup \overleftarrow{(P_1)_{po}}`x . x \sim \in \overleftarrow{(P_1)_{po}}`x$ (2)

$\vdash . *93\cdot11 . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{D}`P_1 = C`P .$ (3)

$[*90\cdot18] \quad \supset . \overleftarrow{(P_1)_*}`x \subset \mathrm{D}`P_1$ (4)

$\vdash . *260\cdot22 . \supset \vdash : \mathrm{Hp} . \supset . P_1 \in 1 \rightarrow 1$ (5)

$\vdash . (1) . (2) . (4) . (5) . *73\cdot21 . *91\cdot74 . \supset$

$\vdash :. \mathrm{Hp} . \supset : x \in C`P . \supset . \overleftarrow{(P_1)_*}`x \ \mathrm{sm}\ \overleftarrow{(P_1)_{po}}`x . \overleftarrow{(P_1)_{po}}`x \subset \overleftarrow{(P_1)_*}`x .$

$\exists ! \overleftarrow{(P_1)_*}`x - \overleftarrow{(P_1)_{po}}`x .$

$[*124\cdot16] \quad \supset . \overleftarrow{(P_1)_*}`x \in \mathrm{Cls\ refl}$ (6)

$\vdash . (6) . (3) . (4) . \supset \vdash : \mathrm{Hp} . \supset . \exists ! \mathrm{Cls\ refl} \cap \mathrm{Cl}`C`P .$

$[*124\cdot141] \quad \supset . C`P \in \mathrm{Cls\ refl} : \supset \vdash . \mathrm{Prop}$

***261·24.** $\vdash : P \in \text{Ser} \,.\, C'P \in \text{Cls induct} - \iota'\Lambda \,.\, \supset \,.\, E\,!\,B'P \,.\, E\,!\,B'\breve{P}$

Dem.

$$\vdash \,.\, *261\cdot22 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, D'P = D'P_1 \,.$$
$$[*261\cdot23.\text{Transp}] \quad \supset \,.\, E\,!\,B'\breve{P} \qquad (1)$$
$$\vdash \,.\, (1)\frac{\breve{P}}{P} \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, E\,!\,B'P \qquad (2)$$
$$\vdash \,.\, (1) \,.\, (2) \,.\, \supset \vdash \,.\, \text{Prop}$$

***261·25.** $\vdash :. P \in \text{Ser} \,.\, \supset : C'P \in \text{Cls induct} - \iota'\Lambda \,.\, \equiv \,.\, P = P_{\text{fn}} \,.\, E\,!\,B'P \,.\, E\,!\,B'\breve{P}$ [*261·22·24·21]

When $P = P_{\text{fn}} \,.\, E\,!\,B'P \,.\, \sim E\,!\,B'\breve{P}$, P is a progression; when $P = P_{\text{fn}} \,.\, E\,!\,B'\breve{P} \,.\, \sim E\,!\,B'P$, P is a regression (*i.e.* the converse of a progression); and when

$$P = P_{\text{fn}} \,.\, \sim E\,!\,B'P \,.\, \sim E\,!\,B'\breve{P},$$

P is the sum of a regression and a progression. These propositions will be proved in the next number.

***261·26.** $\vdash : P \in \text{Ser} \,.\, \alpha \subset C'P \,.\, \exists\,!\,\alpha \,.\, \alpha \in \text{Cls induct} \,.\, \supset \,.\, E\,!\,\min_P{}'\alpha \,.\, E\,!\,\max_P{}'\alpha$

Dem.

$$\vdash \,.\, *205\cdot17 \,.\, \supset \vdash : \text{Hp} \,.\, \alpha \in 1 \,.\, \supset \,.\, E\,!\,\min_P{}'\alpha \,.\, E\,!\,\max_P{}'\alpha \qquad (1)$$
$$\vdash \,.\, *202\cdot55 \,.\, \supset \vdash : \text{Hp} \,.\, \alpha \sim\in 1 \,.\, \supset \,.\, \alpha = C'(P \upharpoonright \alpha) \,.$$
$$[*261\cdot24] \quad \supset \,.\, E\,!\,B'(P \upharpoonright \alpha) \,.\, E\,!\,B'\text{Cnv}'(P \upharpoonright \alpha) \,.$$
$$[*205\cdot42] \quad \supset \,.\, E\,!\,\min_P{}'\alpha \,.\, E\,!\,\max_P{}'\alpha \qquad (2)$$
$$\vdash \,.\, (1) \,.\, (2) \,.\, \supset \vdash \,.\, \text{Prop}$$

***261·27.** $\vdash :. P \in \text{Ser} : \alpha \subset C'P \,.\, \exists\,!\,\alpha \,.\, \supset_\alpha \,.\, E\,!\,\min_P{}'\alpha \,.\, E\,!\,\max_P{}'\alpha : \supset \,.$
$P = P_{\text{fn}} \,.\, C'P \in \text{Cls induct}$

Dem.

$$\vdash \,.\, *250\cdot121 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, P \in \Omega \,.$$
$$[*250\cdot21] \quad \supset \,.\, D'P = D'P_1$$
$$[*260\cdot3] \quad \supset \,.\, D'P = D'P_{\text{fn}} \qquad (1)$$
$$\vdash \,.\, (1) \,.\, \supset \vdash : \text{Hp} \,.\, xP_{\text{fn}}y \,.\, y \in D'P \,.\, \supset \,.\, y \in D'P_{\text{fn}} \,.\, xP_{\text{fn}}y \,.$$
$$[*260\cdot201] \quad \supset \,.\, y \in P_{\text{fn}}{}''\overleftarrow{P_{\text{fn}}}'x \,.$$
$$[*260\cdot12.*201\cdot18] \quad \supset \,.\, y \in P''\overleftarrow{P_{\text{fn}}}'x \,.$$
$$[*205\cdot111] \quad \supset \,.\, y \neq \max_P{}'\overleftarrow{P_{\text{fn}}}'x \qquad (2)$$
$$\vdash \,.\, (1) \,.\, (2) \,.\, \text{Transp} \,.\, \supset \vdash : \text{Hp} \,.\, x \in D'P \,.\, \supset \,.\, \max_P{}'\overleftarrow{P_{\text{fn}}}'x = B'\breve{P} \qquad (3)$$
$$\vdash \,.\, *250\cdot121\cdot13 \,.\, \supset \vdash : \text{Hp} \,.\, \exists\,!\,P \,.\, \supset \,.\, E\,!\,B'P \,.$$
$$[(3)] \quad \supset \,.\, (B'P)\,P_{\text{fn}}\,(B'\breve{P}) \,.$$

$[*260\cdot11]\quad \supset . P(B`P \mapsto B`\breve{P}) \in \mathrm{Cls\ induct}\, .$

$[*202\cdot181]\quad \supset . C`P \in \mathrm{Cls\ induct}$ (4)

$\vdash . *120\cdot212 . \supset \vdash : P = \dot{\Lambda} . \supset . C`P \in \mathrm{Cls\ induct}$ (5)

$\vdash . (4) . (5) . \quad \supset \vdash : \mathrm{Hp} . \supset . C`P \in \mathrm{Cls\ induct}\, .$ (6)

$[*261\cdot22]\quad \supset . P = P_{\mathrm{fn}}$ (7)

$\vdash . (6) . (7) . \supset \vdash . \mathrm{Prop}$

***261·28.** $\vdash :: P \in \mathrm{Ser} . \supset :.$

$\alpha \subset C`P . \exists ! \alpha . \supset_\alpha . E ! \min_P`\alpha . E ! \max_P`\alpha : \equiv . C`P \in \mathrm{Cls\ induct}$

$[*261\cdot26\cdot27]$

***261·281.** $\vdash : \gamma \in \mathrm{Cls\ induct} - 1 . \supset . \gamma \in C``\mathrm{Ser}$

Dem.

$\vdash . *204\cdot24 . \quad \supset \vdash . \Lambda \in C``\mathrm{Ser}$ (1)

$\vdash . *52\cdot22 . \quad \supset \vdash . \Lambda \cup \iota`x \in 1$ (2)

$\vdash . *52\cdot22 . \quad \supset \vdash : x = y . \supset . \iota`x \cup \iota`y \in 1$ (3)

$\vdash . *204\cdot25 . \quad \supset \vdash : x \neq y . \supset . \iota`x \cup \iota`y \in C``\mathrm{Ser}$ (4)

$\vdash . (3) . (4) . \quad \supset \vdash . \iota`x \cup \iota`y \in 1 \cup C``\mathrm{Ser}\, .$

$[*52\cdot1]\quad \supset \vdash : \gamma \in 1 . \supset . \gamma \cup \iota`y \in 1 \cup C``\mathrm{Ser}$ (5)

$\vdash . *51\cdot2 . \quad \supset \vdash : \gamma \in C``\mathrm{Ser} . y \in \gamma . \supset . \gamma \cup \iota`y \in C``\mathrm{Ser}$ (6)

$\vdash . *204\cdot51 . *161\cdot14 . \supset \vdash : \gamma \in C``\mathrm{Ser} . \exists ! \gamma . y \sim\in \gamma . \supset . \gamma \cup \iota`y \in C``\mathrm{Ser}$ (7)

$\vdash . (6) . (7) . \quad \supset \vdash : \gamma \in C``\mathrm{Ser} . \exists ! \gamma . \supset . \gamma \cup \iota`y \in C``\mathrm{Ser}$ (8)

$\vdash . (2) . (5) . (8) . \quad \supset \vdash : \gamma \in 1 \cup C``\mathrm{Ser} . \supset . \gamma \cup \iota`y \in 1 \cup C``\mathrm{Ser}$ (9)

$\vdash . (1) . (9) . *120\cdot26 . \supset \vdash : \gamma \in \mathrm{Cls\ induct} . \supset . \gamma \in 1 \cup C``\mathrm{Ser} : \supset \vdash . \mathrm{Prop}$

***261·29.** $\vdash . \mathrm{Cls\ induct} =$

$1 \cup C``\hat{P}\{P \in \mathrm{Ser} : \alpha \subset C`P . \exists ! \alpha . \supset_\alpha . E ! \min_P`\alpha . E ! \max_P`\alpha\}$

$= 1 \cup C``(\Omega \cap \mathrm{Cnv}``\Omega)$

Dem.

$\vdash . *261\cdot281 . \supset \vdash :. \gamma \in \mathrm{Cls\ induct} - 1 . \supset : (\exists P) : P \in \mathrm{Ser} . \gamma = C`P :$

$[*261\cdot28]$

$\supset : (\exists P) : P \in \mathrm{Ser} : \alpha \subset C`P . \exists ! \alpha . \supset_\alpha . E ! \min_P`\alpha . E ! \max_P`\alpha : \gamma = C`P :$

$[*37\cdot6]\quad \supset : \gamma \in C``\hat{P}\{P \in \mathrm{Ser} : \alpha \subset C`P . \exists ! \alpha . \supset_\alpha . E ! \min_P`\alpha . E ! \max_P`\alpha\}$ (1)

$\vdash . *261\cdot28 . \supset \vdash :. P \in \mathrm{Ser} :$

$\alpha \subset C`P . \exists ! \alpha . \supset_\alpha . E ! \min_P`\alpha . E ! \max_P`\alpha : \gamma = C`P : \supset . \gamma \in \mathrm{Cls\ induct} :.$

$[*37\cdot6] \supset \vdash : \gamma \in C``\hat{P}(P \in \mathrm{Ser} : \alpha \subset C`P . \exists ! \alpha . \supset_\alpha . E ! \min_P`\alpha . E ! \max_P`\alpha) . \supset .$

$\gamma \in \mathrm{Cls\ induct}$ (2)

$\vdash . *120\cdot213 . \supset \vdash . 1 \subset \mathrm{Cls\ induct}$ (3)

$\vdash . (1) . (2) . (3) . \supset$

$\vdash . \mathrm{Cls\ induct} = C``\hat{P}\{P \in \mathrm{Ser} : \alpha \subset C`P . \exists ! \alpha . \supset_\alpha . E ! \min_P`\alpha . E ! \max_P`\alpha\}$

$[*250\cdot121]\quad = C``(\Omega \cap \mathrm{Cnv}``\Omega) . \supset \vdash . \mathrm{Prop}$

The following four propositions are immediate consequences of the propositions already proved.

$*261 \cdot 3$. $\vdash :: P \in \mathrm{Ser} \,.\, \supset :.$

$$C'P \in \mathrm{Cls\ induct} \,.\, \equiv : P \in \Omega : \alpha \subset C'P \,.\, \exists ! \alpha \,.\, \supset_\alpha \,.\, E ! \max_P{}'\alpha$$

$[*261 \cdot 28 \,.\, *250 \cdot 121]$

$*261 \cdot 31$ $\vdash :. P \in \mathrm{Ser} \,.\, \supset : C'P \in \mathrm{Cls\ induct} \,.\, \equiv .\, P, \breve{P} \in \Omega$ $[*261 \cdot 3 \,.\, *250 \cdot 121]$

$*261 \cdot 32$. $\vdash .\, \mathrm{Ser} \cap \breve{C}``\mathrm{Cls\ induct} = \Omega \ \mathrm{induct} = \Omega \cap \mathrm{Cnv}``\Omega$ $[*261 \cdot 31 \cdot 14]$

On account of this proposition, we do not introduce the notation "Ser induct" for "$\mathrm{Ser} \cap \breve{C}``\mathrm{Cls\ induct}$," because a series whose field is inductive is a well-ordered series, and therefore the notation "Ω induct" gives all that is wanted.

$*261 \cdot 33$. $\vdash : P, Q \in \Omega \,.\, Q \mathrel{\subseteq\!\!\!\!\cdot\,} \breve{P} \,.\, \supset .\, Q \in \Omega \ \mathrm{induct}$

Dem.

$$
\begin{aligned}
&\vdash .\, *204 \cdot 2 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset .\, \breve{Q} \in \mathrm{Ser} \,.\, \breve{Q} \mathrel{\subseteq\!\!\!\!\cdot\,} P \,. \\
&[*250 \cdot 14] \qquad \supset .\, \breve{Q} \in \mathrm{Ser} \cap \mathrm{Bord} \,. \\
&[*250 \cdot 12] \qquad \supset .\, \breve{Q} \in \Omega \,. \\
&[*261 \cdot 32] \qquad \supset .\, Q \in \Omega \ \mathrm{induct} : \supset \vdash .\, \mathrm{Prop}
\end{aligned}
$$

This proposition (which is due to Cantor) is of great importance in the theory of well-ordered series. It shows that, however great a well-ordered series may be, any *descending* well-ordered series contained in it must be finite. (A *descending* series in a given series is a series contained in the converse of the given series.)

$*261 \cdot 34$. $\vdash : P \in \Omega \,.\, \text{Ɑ}'P_1 = \text{Ɑ}'P \,.\, E ! B'\breve{P} \,.\, \supset .\, C'P \in \mathrm{Cls\ induct}$

Dem.

$$
\begin{aligned}
&\vdash .\, *250 \cdot 23 \,.\, *214 \cdot 12 \,.\, \supset \vdash :. \mathrm{Hp} \,.\, \alpha \subset C'P \,.\, \supset : E ! \max_P{}'\alpha \,.\, \vee \,.\, E ! \mathrm{seq}_P{}'\alpha \quad (1) \\
&\vdash .\, *206 \cdot 181 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \alpha \subset C'P \,.\, \exists ! \alpha \,.\, E ! \mathrm{seq}_P{}'\alpha \,.\, \supset .\, \mathrm{seq}_P{}'\alpha \in \text{Ɑ}'P_1 \,. \\
&[*204 \cdot 7] \qquad \supset .\, E ! P_1{}'\mathrm{seq}_P{}'\alpha \,. \\
&[*206 \cdot 451] \qquad \supset .\, E ! \max_P{}'\alpha \quad (2) \\
&\vdash .\, (1) \,.\, (2) \,.\, \supset \vdash :. \mathrm{Hp} \,.\, \supset : \alpha \subset C'P \,.\, \exists ! \alpha \,.\, \supset_\alpha \,.\, E ! \max_P{}'\alpha : \\
&[*261 \cdot 3] \qquad \supset : C'P \in \mathrm{Cls\ induct} :. \supset \vdash .\, \mathrm{Prop}
\end{aligned}
$$

$*261 \cdot 35$. $\vdash :. P \in \Omega \,.\, \supset : C'P \in \mathrm{Cls\ induct} - \iota'\Lambda \,.\, \equiv .\, \text{Ɑ}'P_1 = \text{Ɑ}'P \,.\, E ! B'\breve{P}$

$[*261 \cdot 22 \cdot 24 \cdot 34]$

Observe that "$\text{Ɑ}'P_1 = \text{Ɑ}'P \,.\, E ! B'\breve{P}$" occurs as hypothesis in $*253 \cdot 51$ and some succeeding propositions. Thus this hypothesis is equivalent to the hypothesis that the field of P is an inductive existent class. It follows that

if P is an inductive well-ordered series, $\mathrm{Nr}`P_s = \mathrm{Nr}`P$, whereas if P is a well-ordered series which is not inductive, $\mathrm{Nr}`P_s = \mathrm{Nr}`P \dot{+} \dot{1}$; also that

***261·36.** $\vdash :. P \in \Omega . \supset : C`P \in \text{Cls induct} - \iota`\Lambda . \equiv . \mathrm{Nr}`P \neq \dot{1} \dot{+} \mathrm{Nr}`P$

[*253·573 . *261·35]

***261·37.** $\vdash :. P \in \Omega . \supset : C`P \in \text{Cls induct} . \equiv . \dot{1} \dot{+} \mathrm{Nr}`P = \mathrm{Nr}`P \dot{+} \dot{1}$

[*253·574 . *261·35 . *161·2·201]

***261·38.** $\vdash :. P \in \Omega . \supset : C`P \in \text{Cls induct} - \iota`\Lambda . \supset . \mathrm{Nr}`P_s = \mathrm{Nr}`P :$

$C`P \sim\in \text{Cls induct} - \iota`\Lambda . \supset . \mathrm{Nr}`P_s = \mathrm{Nr}`P \dot{+} \dot{1}$

[*253·56 . *261·35]

***261·4.** $\vdash : P \in \Omega - \Omega \text{ induct} . \supset . \{\overleftarrow{(P_1)_*}`B`P\} \upharpoonright P_1 \in \text{Prog}$

Dem.

$\vdash . *204·7 . \supset \vdash : \mathrm{Hp} . R = P_1 . \supset . R \in 1 \to 1$ (1)

$\vdash . *120·212 . \supset \vdash :. \mathrm{Hp} . \supset : \dot{\exists} ! P :$

[*250·13] $\supset : E ! B`P :$

[*250·21] $\supset : R = P_1 . \supset . B`P \in D`R$ (2)

$\vdash . *260·22 . \supset \vdash : \mathrm{Hp} . R = P_1 . \supset . R_{po} \subseteq J$ (3)

$\vdash . *93·103 . *202·52 . \supset$

$\vdash : P \in \Omega . R = P_1 . \exists ! \overleftarrow{R_*}`B`P - D`P . \supset . B`\breve{P} \in \overleftarrow{R_*}`B`P .$

[*93·101.*91·54] $\supset . (B`P) R_{po} (B`\breve{P}) .$

[*260·27] $\supset . (B`P) P_{fn} (B`\breve{P}) .$

[*261·2] $\supset . C`P \in \text{Cls induct}$ (4)

$\vdash . (4) . \text{Transp} . \supset \vdash : \mathrm{Hp} . R = P_1 . \supset . \overleftarrow{R_*}`B`P \subset D`P .$

[*250·21] $\supset . \overleftarrow{R_*}`B`P \subset D`R$ (5)

$\vdash . (1) . (2) . (3) . (5) . \supset \vdash : \mathrm{Hp} . R = P_1 . \supset .$

$R \in 1 \to 1 . B`P \in D`R . \sim \{(B`P) R_{po} (B`P)\} . \overleftarrow{R_*}`B`P \subset D`R .$

[*122·52] $\supset . (\overleftarrow{R_*}`B`P) \upharpoonright R \in \text{Prog} : \supset \vdash . \text{Prop}$

***261·401.** $\vdash : P \in \Omega - \Omega \text{ induct} . \supset . \exists ! \aleph_0 \cap \mathrm{Cl}`C`P . C`P \in \text{Cls refl}$

Dem.

$\vdash . *261·4 . *123·1 . \supset \vdash : \mathrm{Hp} . \supset . D`\{\overleftarrow{(P_1)_*}`B`P\} \upharpoonright P_1 \in \aleph_0$ (1)

$\vdash . *121·305 . \supset \vdash : \mathrm{Hp} . \supset . D`\{\overleftarrow{(P_1)_*}`B`P\} \upharpoonright P_1 \subset C`P$ (2)

$\vdash . (1) . (2) . \supset \vdash : \mathrm{Hp} . \supset . \exists ! \aleph_0 \cap \mathrm{Cl}`C`P .$ (3)

[*124·15] $\supset . C`P \in \text{Cls refl}$ (4)

$\vdash . (3) . (4) . \supset \vdash . \text{Prop}$

***261·41.** $\vdash . \Omega - \Omega \text{ induct} = \Omega \text{ infin}$ [*261·401 . *261·11 . *124·271]

***261·42.** $\vdash . \Omega \text{ fin} = \Omega \text{ induct}$ [*261·41 . Transp . *124·271]

We shall henceforth use "Ω fin" in preference to "Ω induct."

***261·43.** $\vdash . C``\Omega \subset \text{Cls induct} \cup \text{Cls refl}$ [*261·401·14]

***261·431.** $\vdash : P \in \Omega - \iota`\dot{\Lambda} . \supset .$

$$\{\overleftarrow{(P_1)_*}`B`P\} \upharpoonleft P_1 = P_1 \upharpoonright \overleftarrow{P_{\text{fn}}}`B`P = P_1 \restriction (\iota`B`P \cup \overleftarrow{P_{\text{fn}}}`B`P)$$
$$= (\iota`B`P \cup \overleftarrow{P_{\text{fn}}}`B`P) \upharpoonleft P_1$$

Dem.

$\vdash . *250·13·21 . \quad \supset \vdash : \text{Hp} . \supset . B`P \in \text{D}`P_1 .$ (1)

$[*260·31] \quad \supset . \iota`B`P \cup \overleftarrow{P_{\text{fn}}}`B`P = \overleftarrow{(P_1)_*}`B`P$ (2)

$\vdash . (1) . *260·27 . \quad \supset \vdash : \text{Hp} . \supset . \overleftarrow{P_{\text{fn}}}`B`P = \overleftarrow{(P_1)_{\text{po}}}`B`P .$

$[*260·34] \quad \supset . P_1 \upharpoonright \overleftarrow{P_{\text{fn}}}`B`P = \{\overleftarrow{(P_1)_*}`B`P\} \upharpoonleft P_1$ (3)

$[(2)] \quad = (\iota`B`P \cup \overleftarrow{P_{\text{fn}}}`B`P) \upharpoonleft P_1$ (4)

$\vdash . (3) . (4) . *35·11 . \supset \vdash : \text{Hp} . \supset . \{\overleftarrow{(P_1)_*}`B`P\} \upharpoonleft P_1 = P_1 \restriction (\iota`B`P \cup \overleftarrow{P_{\text{fn}}}`B`P)$ (5)

$\vdash . (3) . (4) . (5) . \supset \vdash . \text{Prop}$

***261·44.** $\vdash :. P \in \Omega . \supset : P_1 \upharpoonright \overleftarrow{P_{\text{fn}}}`B`P \in \text{Prog} . \equiv . P \in \Omega \text{ infin}$

Dem.

$\vdash . *123·1 . \supset \vdash : P \in \Omega . P_1 \upharpoonright \overleftarrow{P_{\text{fn}}}`B`P \in \text{Prog} . \supset . \exists ! \aleph_0 \cap \text{Cl}`C`P .$

$[*124·15] \quad \supset . C`P \in \text{Cls refl} .$

$[*261·1] \quad \supset . P \in \Omega \text{ infin}$ (1)

$\vdash . *261·4·431·41 . \supset \vdash : P \in \Omega \text{ infin} . \supset . P_1 \upharpoonright \overleftarrow{P_{\text{fn}}}`B`P \in \text{Prog}$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***261·45.** $\vdash . \Omega \text{ infin} = \Omega \cap \hat{P}\{P_1 \upharpoonright \overleftarrow{P_{\text{fn}}}`B`P \in \text{Prog}\}$ [*261·44]

***261·46.** $\vdash : P \in \Omega . \supset . \text{Cl}`C`P \subset \text{Cls induct} \cup \text{Cls refl}$

Dem.

$\vdash . *250·141 . *202·55 . \supset$

$\vdash : \text{Hp} . \alpha \subset C`P . \alpha \sim \in 1 . \supset . P \restriction \alpha \in \Omega . \alpha = C`(P \restriction \alpha) .$

$[*261·43] \quad \supset . \alpha \in \text{Cls induct} \cup \text{Cls refl}$ (1)

$\vdash . *120·213 . \supset \vdash : \alpha \in 1 . \supset . \alpha \in \text{Cls induct}$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***261·47.** $\vdash :. P \in \Omega . \alpha \subset C`P . \supset : \alpha \in \text{Cls induct} . \equiv . \alpha \sim \in \text{Cls refl}$
[*261·46 . *124·271]

***261·6.** $\vdash :. P \in \Omega . C`P \subset \Omega . \text{Nc}`C`P = \nu . \supset_P . \Pi`P \in \Omega :$
$\nu \in \text{Nc induct} - \iota`0 - \iota`1 : \supset :$
$Q \in \Omega . C`Q \subset \Omega . \text{Nc}`C`Q = \nu +_c 1 . \supset_Q . \Pi`Q \in \Omega$

Dem.

$\vdash . *204·272 . \supset \vdash : \text{Nc}`\text{D}`Q = 1 . Q \in \text{Ser} . \supset . Q \in 2_r .$

$[*56·112] \quad \supset . C`Q \in 2$ (1)

$\vdash . (1) . \text{Transp} . \supset \vdash : Q \in \Omega . C\text{‘}Q \subset \Omega . \text{Nc}\text{‘}C\text{‘}Q = \nu +_c 1 .$
$\nu \in \text{NC induct} - \iota\text{‘}0 - \iota\text{‘}1 . \supset . \text{D}\text{‘}Q \sim\in 1$ (2)

$\vdash . {*}261{\cdot}24 . \supset \vdash : \text{Hp}(2) . \supset . E ! B\text{‘}\breve{Q} .$

$[(2).{*}204{\cdot}461] \quad \supset . Q = Q \upharpoonright \text{D}\text{‘}Q \nrightarrow B\text{‘}\breve{Q} .$

$[{*}172{\cdot}32] \quad \supset . \Pi\text{‘}Q \text{ smor } \Pi\text{‘}(Q \upharpoonright \text{D}\text{‘}Q) \times B\text{‘}\breve{Q}$ (3)

$\vdash . {*}110{\cdot}63 . \supset \vdash : \text{Hp}(2) . \supset . \text{Nc}\text{‘}\text{D}\text{‘}Q +_c 1 = \nu +_c 1 .$

$[{*}120{\cdot}311] \quad \supset . \text{Nc}\text{‘}\text{D}\text{‘}Q = \nu$ (4)

$\vdash . (4) . \quad \supset \vdash :. \text{Hp}(2) : P \in \Omega . C\text{‘}P \subset \Omega . \text{Nc}\text{‘}C\text{‘}P = \nu . \supset_P . \Pi\text{‘}P \in \Omega : \supset .$
$\Pi\text{‘}(Q \upharpoonright \text{D}\text{‘}Q) \in \Omega .$

$[(3).{*}251{\cdot}55] \quad \supset . \Pi\text{‘}Q \in \Omega$ (5)

$\vdash . (5) . \text{Exp} . \supset$

$\vdash :. \text{Hp} . \supset : Q \in \Omega . C\text{‘}Q \subset \Omega . \text{Nc}\text{‘}C\text{‘}Q = \nu +_c 1 . \supset . \Pi\text{‘}Q \in \Omega :. \supset \vdash . \text{Prop}$

$*261{\cdot}61$. $\vdash : P \in \Omega \text{ fin} . C\text{‘}P \subset \Omega . \supset . \Pi\text{‘}P \in \Omega$

Dem.

$\vdash . {*}261{\cdot}6 . \quad \supset \vdash :: \phi\nu . \equiv_\nu : P \in \Omega . C\text{‘}P \subset \Omega . \text{Nc}\text{‘}C\text{‘}P = \nu . \supset_P . \Pi\text{‘}P \in \Omega :. \supset :.$
$\nu \in \text{Nc induct} - \iota\text{‘}0 - \iota\text{‘}1 . \supset : \phi\nu . \supset . \phi(\nu +_c 1)$ (1)

$\vdash . {*}200{\cdot}12 . \supset \vdash . \sim(\exists P) . P \in \Omega . C\text{‘}P \subset \Omega . \text{Nc}\text{‘}C\text{‘}P = 1 .$

$[{*}10{\cdot}53] \quad \supset \vdash : \text{Hp}(1) . \supset . \phi 1$ (2)

$\vdash . {*}172{\cdot}13 . {*}250{\cdot}4 . \quad \supset \vdash : \text{Hp}(1) . \supset . \phi 0$ (3)

$\vdash . {*}172{\cdot}23 . {*}251{\cdot}55 . \supset \vdash :. Y \neq Z . Y, Z \in \Omega . \supset : \Pi\text{‘}(Y \downarrow Z), \Pi\text{‘}(Z \downarrow Y) \in \Omega :$

$[{*}55{\cdot}54.{*}204{\cdot}13] \quad \supset : P \in \text{Ser} . C\text{‘}P = \iota\text{‘}Y \cup \iota\text{‘}Z . \supset . \Pi\text{‘}P \in \Omega$ (4)

$\vdash . (4) . {*}54{\cdot}101 . \supset \vdash : \text{Hp}(1) . \supset . \phi 2$ (5)

$\vdash . (2) . (3) . (5) . \quad \supset \vdash :. \text{Hp}(1) . \supset : \phi 0 : \nu \in \iota\text{‘}0 \cup \iota\text{‘}1 . \phi\nu . \supset . \phi(\nu +_c 1)$ (6)

$\vdash . (1) . (6) . \quad \supset \vdash :. \text{Hp}(1) . \supset : \nu \in \text{NC induct} . \phi\nu . \supset_\nu . \phi(\nu +_c 1) : \phi 0 :$

$[{*}120{\cdot}13] \quad \supset : \alpha \in \text{NC induct} . \supset_\alpha . \phi\alpha$ (7)

$\vdash . (7) . {*}13{\cdot}191 . \supset \vdash : P \in \Omega . C\text{‘}P \subset \Omega . \text{Nc}\text{‘}C\text{‘}P \in \text{NC induct} . \supset_P . \Pi\text{‘}P \in \Omega :$

$[{*}261{\cdot}14{\cdot}42] \quad \supset \vdash : P \in \Omega \text{ fin} . C\text{‘}P \subset \Omega . \supset_P . \Pi\text{‘}P \in \Omega : \supset \vdash . \text{Prop}$

$*261{\cdot}62$. $\vdash : P \in \Omega . Q \in \Omega \text{ fin} . \supset . P^Q \in \Omega$

Dem.

$\vdash . {*}251{\cdot}51 . \quad \supset \vdash : \text{Hp} . \dot{\exists} ! P . \supset . P \dot{\downarrow} {}^{;}Q \in \Omega$ (1)

$\vdash . {*}165{\cdot}26 . \quad \supset \vdash : \text{Hp} . \supset . C\text{‘}P \dot{\downarrow} {}^{;}Q \subset \Omega$ (2)

$\vdash . (1) . {*}165{\cdot}25 . {*}261{\cdot}18 . \supset \vdash : \text{Hp} . \dot{\exists} ! P . \supset . P \dot{\downarrow} {}^{;}Q \in \Omega \text{ fin}$ (3)

$\vdash . (1) . (2) . (3) . {*}261{\cdot}61 . \supset \vdash : \text{Hp} . \dot{\exists} ! P . \supset . \Pi\text{‘}P \dot{\downarrow} {}^{;}Q \in \Omega .$

$[{*}176{\cdot}181{\cdot}182] \quad \supset . P^Q \in \Omega$ (4)

$\vdash . {*}176{\cdot}151 . {*}250{\cdot}4 . \quad \supset \vdash : P = \Lambda . \supset . P^Q \in \Omega$ (5)

$\vdash . (4) . (5) . \supset \vdash . \text{Prop}$

***261·63.** $\vdash : \mathrm{E} ! B`R . P \mathrel{\mathsf{G}} Q . x \in C`Q \cap p`\overleftarrow{Q}``C`P . \supset .$

$$(\iota`x) \uparrow C`R \in C`Q^R \cap p`\overleftarrow{Q^R}``C`P^R$$

Dem.

$\vdash . *116{\cdot}12 . \quad \supset \vdash : \mathrm{Hp} . \supset . (\iota`x) \uparrow C`R \in (C`Q \uparrow C`R)_\Delta`C`R .$

$[*176{\cdot}14] \quad \supset . (\iota`x) \uparrow C`R \in C`Q^R \quad (1)$

$\vdash . *116{\cdot}12 . *93{\cdot}11 . \supset \vdash :. \mathrm{Hp} . S \in (C`P \uparrow C`R)_\Delta`C`R . T = (\iota`x) \uparrow C`R . \supset :$

$(S`B`R) Q (T`B`R) : \sim (\exists y) . yR (B`R) :$

$[*10{\cdot}53] \quad \supset : (S`B`R) Q (T`B`R) : yR (B`R) . y \neq B`R . \supset_y . S`y = T`y :$

$[*176{\cdot}19.(1)] \supset : S (Q^R) T \quad (2)$

$\vdash . (2) . *176{\cdot}16 . \supset \vdash :. \mathrm{Hp} . \supset : S \in C`P^R . \supset . S (Q^R) \{(\iota`x) \uparrow C`R\} \quad (3)$

$\vdash . (1) . (3) . \supset \vdash . \mathrm{Prop}$

***261·64.** $\vdash : R \in \Omega \,\mathrm{fin} - \iota`\dot{\Lambda} . P \,\mathrm{less}\, Q . \supset . P^R \,\mathrm{less}\, Q^R$

Dem.

$\vdash . *254{\cdot}55 . \supset \vdash : \mathrm{Hp} . \supset . (\exists P') . P' \,\mathrm{smor}\, P . P' \mathrel{\mathsf{G}} Q . \exists ! C`Q \cap p`\overleftarrow{Q}``C`P' .$

$[*261{\cdot}63 . *250{\cdot}13] \quad \supset . (\exists P') . P' \,\mathrm{smor}\, P . P' \mathrel{\mathsf{G}} Q . \exists ! C`Q^R \cap p`\overleftarrow{Q^R}``C`(P')^R .$

$[*176{\cdot}35{\cdot}22] \quad \supset . (\exists M) . M \,\mathrm{smor}\, P^R . M \mathrel{\mathsf{G}} Q^R . \exists ! C`Q^R \cap p`\overleftarrow{Q^R}``C`M .$

$[*254{\cdot}55 . *261{\cdot}62] \quad \supset . P^R \,\mathrm{less}\, Q^R : \supset \vdash . \mathrm{Prop}$

***261·65.** $\vdash : P \in \Omega \,\mathrm{infin} . Q \in \Omega \,\mathrm{fin} . \supset . Q \,\mathrm{less}\, P$

Dem.

$\vdash . *261{\cdot}11{\cdot}14{\cdot}42 . \supset \vdash : \mathrm{Hp} . \supset . C`P \in \mathrm{Cls\,refl} . C`Q \in \mathrm{Cls\,induct} .$

$[*124{\cdot}26] \quad \supset . \mathrm{Nc}`C`P > \mathrm{Nc}`C`Q .$

$[*255{\cdot}75] \quad \supset . Q \,\mathrm{less}\, P : \supset \vdash . \mathrm{Prop}$

$*262$. FINITE ORDINALS.

Summary of $*262$.

Finite ordinals are defined as the ordinals of finite well-ordered series; infinite ordinals are defined as the ordinals of infinite well-ordered series. In virtue of $*261{\cdot}42$, finite ordinals are those whose members have fields which are inductive, and are also those whose members have fields which are not reflexive. Finite ordinals have the formal properties which cardinals have but which relation-numbers and ordinals in general do not have, *i.e.* their sums and products are commutative, and the distributive law holds in the form

$$\mu \dot{\times} (\nu \dot{+} \varpi) = (\mu \dot{\times} \nu) \dot{+} (\mu \dot{\times} \varpi),$$

as well as in the form

$$(\nu \dot{+} \varpi) \dot{\times} \mu = (\nu \dot{\times} \mu) \dot{+} (\varpi \dot{\times} \mu),$$

which was proved generally in $*184{\cdot}35$.

The distinguishing properties of finite ordinals are most readily established by means of their correspondence with inductive cardinals. In general, two well-ordered series whose fields have the same cardinal need not be ordinally similar, but when the cardinal of the fields is inductive, the two series must be ordinally similar. Hence the ordinal of a *finite* well-ordered series is determined by the cardinal of the field of the series. We put generally

$$\mu_r = \Omega \cap \breve{C}\text{“}\mu \quad \text{Df.}$$

The result is that, if μ is an inductive cardinal, μ_r is the ordinal of all those series whose fields have μ members. Thus there is a one-one correspondence of inductive cardinals and finite ordinals; and in virtue of this correspondence, the formal properties of finite ordinals can be deduced from those of inductive cardinals.

It will be observed that, according to the definitions already given,

$$\vdash . \, 0_r = \Omega \cap \breve{C}\text{“}\Lambda \text{ by } *250{\cdot}43,$$

$$\vdash . \, 2_r = \Omega \cap \breve{C}\text{“}2 \text{ by } *250{\cdot}44.$$

Hence the notations 0_r, 2_r are particular cases of the general notation μ_r. But in virtue of $*200 \cdot 12$, we have, by the definition of μ_r,

$$\vdash . 1_r = \Lambda,$$

so that 1_r does not take its place in the series of finite ordinals.

Our definitions in this number are

$$\text{NO fin} = \text{N}_0\text{r}``\Omega \text{ fin} \quad \text{Df},$$

$$\text{NO infin} = \text{N}_0\text{r}``\Omega \text{ infin} \quad \text{Df},$$

$$\mu_r = \Omega \cap \breve{C}``\mu \quad \text{Df}.$$

It will be observed that for the sake of convenience we have defined NO fin and NO infin so as to exclude Λ. The definition of μ_r is chiefly useful when μ is an inductive cardinal.

The number begins with various elementary propositions, partly embodying the definitions, partly concerned with μ_r. We have

$*262 \cdot 12$. $\vdash : P \in \mu_r . \equiv . P \in \Omega . C`P \in \mu$

$*262 \cdot 18$. $\vdash : \mu \in \text{NC} . \exists ! \mu_r . \supset . \mu = C``\mu_r$

This proposition does not require that μ_r should be a relation-number. If μ is a reflexive cardinal, μ_r is not a relation-number unless it is null, because series of many different relation-numbers can be made with a given cardinal number of terms. When μ is a cardinal, "$\exists ! \mu_r$" means that classes having μ terms can be well-ordered.

$*262 \cdot 19$. $\vdash :. \mu, \nu \in \text{NC} . \exists ! \mu_r . \supset : \mu = \nu . \equiv . \mu_r = \nu_r$

Thus the relation of μ to μ_r is one-one so long as μ is the cardinal number of a class which can be well-ordered.

We next prove that if μ is an inductive cardinal other than Λ or 1, μ_r is a finite ordinal, and that every finite ordinal is of the form μ_r for an appropriate μ. We have

$*262 \cdot 21$. $\vdash : \mu \in \text{NC induct} - \iota`\Lambda - \iota`1 . \supset . \exists ! \mu_r$

$*262 \cdot 24$. $\vdash : \mu \in \text{NC induct} - \iota`\Lambda - \iota`1 . \supset . \mu_r \in \text{NO fin}$

We prove this by means of an inductive proof that two series are similar if their fields are inductive and similar.

$*262 \cdot 26$. $\vdash : \alpha \in \text{NO fin} . \equiv . (\exists \mu) . \mu \in \text{N}_0\text{C induct} - \iota`1 . \alpha = \mu_r$

Hence we easily obtain the properties of finite ordinals from those of the corresponding cardinals. Assuming that μ, ν are inductive cardinals other than 1, we have

$*262 \cdot 33$. $\mu_r \dot{+} \nu_r = (\mu +_0 \nu)_r$

$*262 \cdot 35$. $\mu_r \dot{+} \dot{1} = (\mu +_0 1)_r$, if $\mu \neq 0$,

$*262 \cdot 43$. $\mu_r \dot{\times} \nu_r = (\mu \times_0 \nu)_r$

***262·53.** $\mu_r \exp_r \nu_r = (\mu^\nu)_r$, if $\nu \neq 0$,

***262·7.** $\mu > \nu \,.\equiv.\, \mu_r > \nu_r$

Hence if α, β, γ are finite ordinals,

***262·6.** $\alpha \dot{+} \beta = \beta \dot{+} \alpha$

***262·61.** $\alpha \dot{\times} \beta = \beta \dot{\times} \alpha$

***262·62.** $\alpha \dot{\times} (\beta \dot{+} \gamma) = (\alpha \dot{\times} \beta) \dot{+} (\alpha \dot{\times} \gamma)$

***262·63.** $(\alpha \dot{\times} \beta) \exp_r \gamma = (\alpha \exp_r \gamma) \dot{\times} (\beta \exp_r \gamma)$

Thus the arithmetic of finite ordinals obeys the same formal laws as the arithmetic of inductive cardinals.

***262·01.** $\text{NO fin} = \text{N}_0\text{r}``\Omega \text{ fin}$ Df

***262·02.** $\text{NO infin} = \text{N}_0\text{r}``\Omega \text{ infin}$ Df

***262·03.** $\mu_r = \Omega \cap \breve{C}``\mu$ Df

***262·1.** $\vdash : \alpha \in \text{NO fin} \,.\equiv.\, (\exists P) \,.\, P \in \Omega \text{ fin} \,.\, \alpha = \text{N}_0\text{r}`P$ [(*262·01)]

***262·11.** $\vdash : \alpha \in \text{NO infin} \,.\equiv.\, (\exists P) \,.\, P \in \Omega \text{ infin} \,.\, \alpha = \text{N}_0\text{r}`P$ [(*262·02)]

***262·111.** $\vdash :. \alpha \in \text{NO fin} \,.\equiv: \alpha \in \text{N}_0\text{O} : \alpha \neq \dot{1} \dot{+} \alpha \,.\vee.\, \alpha = 0_r :$
$\equiv : \alpha \in \text{NO} : \alpha \neq \dot{1} \dot{+} \alpha \,.\vee.\, \alpha = 0_r$

Dem.

$\vdash . *262·1 . \supset$

$\vdash :. \alpha \in \text{NO fin} \,.\equiv: \alpha \in \text{N}_0\text{O} : (\exists P) \,.\, P \in \Omega \text{ fin} \,.\, \alpha = \text{Nr}`P :$

[*261·36] $\equiv : \alpha \in \text{N}_0\text{O} : (\exists P) : \text{Nr}`P \neq \dot{1} \dot{+} \text{Nr}`P \,.\vee.\, P = \dot{\Lambda} : \alpha = \text{Nr}`P :$

[(*255·03)] $\equiv : \alpha \in \text{N}_0\text{O} : \alpha \neq \dot{1} \dot{+} \alpha \,.\vee.\, \alpha = 0_r :$ (1)

[*180·4.*155·5] $\equiv : \alpha \in \text{NO} : \alpha \neq \dot{1} \dot{+} \alpha \,.\vee.\, \alpha = 0_r$ (2)

$\vdash . (1) . (2) . \supset \vdash .$ Prop

***262·112.** $\vdash : \alpha \in \text{NO infin} \,.\equiv.\, \alpha \in \text{N}_0\text{O} - \iota`0_r \,.\, \dot{1} \dot{+} \alpha = \alpha$
[*262·111 . Transp . *261·13]

***262·12.** $\vdash : P \in \mu_r \,.\equiv.\, P \in \Omega \,.\, C`P \in \mu$ [(*262·03)]

***262·13.** $\vdash : \text{Nr}`P \in \text{NO fin} \,.\equiv.\, P \in \Omega \text{ fin} \,.\equiv.\, P \in \Omega \,.\, C`P \in \text{Cls induct}$

Dem.

$\vdash . *262·1 . \supset \vdash : \text{Nr}`P \in \text{NO fin} \,.\equiv.\, (\exists Q) \,.\, Q \in \Omega \text{ fin} \,.\, \text{Nr}`P = \text{N}_0\text{r}`Q \,.$

[*152·35.*155·13] $\equiv.\, (\exists Q) \,.\, Q \in \Omega \text{ fin} \,.\, P \text{ smor } Q \,.$

[*261·183] $\equiv.\, P \in \Omega \text{ fin} \,.$ (1)

[*261·42·14] $\equiv.\, P \in \Omega \,.\, C`P \in \text{Cls induct}$ (2)

$\vdash . (1) . (2) . \supset \vdash .$ Prop

***262·14.** $\vdash : \mathrm{Nr}`P \in \mathrm{NO}\ \mathrm{infin} . \equiv . P \in \Omega\ \mathrm{infin} . \equiv . P \in \Omega . C`P \in \mathrm{Cls\ refl}$

[Proof as in *262·13]

***262·15.** $\vdash :. \alpha \in \mathrm{N_0O} . \supset : \alpha \in \mathrm{NO\ fin} . \equiv . C``\alpha \in \mathrm{NC\ induct}$

Dem.

$\vdash . *262·13 . *120·21 . \supset$

$\vdash : \mathrm{N_0r}`P \in \mathrm{NO\ fin} . \equiv . P \in \Omega . \mathrm{N_0c}`C`P \in \mathrm{NC\ induct}$ (1)

$\vdash . (1) . *251·1 . \supset$

$\vdash :. \mathrm{N_0r}`P \in \mathrm{NO} . \supset : \mathrm{N_0r}`P \in \mathrm{NO\ fin} . \equiv . \mathrm{N_0c}`C`P \in \mathrm{NC\ induct} .$

[*152·7] $\equiv . C``\mathrm{N_0r}`P \in \mathrm{NC\ induct}$ (2)

$\vdash . (2) . *155·2 . \supset \vdash . \mathrm{Prop}$

***262·16.** $\vdash :. \alpha \in \mathrm{N_0O} . \supset :$

$\alpha \in \mathrm{NO\ infin} . \equiv . C``\alpha \sim\in \mathrm{NC\ induct} . \equiv . C``\alpha \in \mathrm{NC\ refl}$

[Proof as in *262·15]

***262·17.** $\vdash : P \in \Omega . \supset . P \in (\mathrm{Nc}`C`P)_r$

Dem.

$\vdash . *100·3 . \supset \vdash . C`P \in \mathrm{Nc}`C`P$ (1)

$\vdash . (1) . *262·12 . \supset \vdash . \mathrm{Prop}$

***262·18.** $\vdash : \mu \in \mathrm{NC} . \exists ! \mu_r . \supset . \mu = C``\mu_r$

Dem.

$\vdash . *262·12 . \quad \supset \vdash . C``\mu_r \subset \mu$ (1)

$\vdash . *262·12 . \quad \supset \vdash : \alpha \in \mu . P \in \mu_r . \supset . \alpha, C`P \in \mu$ (2)

$\vdash . (2) . *100·5 . \supset \vdash : \mathrm{Hp} . \alpha \in \mu . P \in \mu_r . \supset . \alpha\ \mathrm{sm}\ C`P .$

[*73·1] $\supset . (\exists S) . S \in 1 \to 1 . \alpha = \mathrm{D}`S . C`P = \mathrm{G}`S .$

[*151·1.*150·23] $\supset . (\exists S) . S;P\ \mathrm{smor}\ P . C`S;P = \alpha .$

[*251·111.*262·12] $\supset . (\exists S) . S;P \in \Omega . C`S;P = \alpha .$

[*262·12.Hp] $\supset . (\exists S) . S;P \in \mu_r . C`S;P = \alpha .$

[*37·6] $\supset . \alpha \in C``\mu_r$ (3)

$\vdash . (3) . *10·23 . \supset \vdash : \mathrm{Hp} . \supset . \mu \subset C``\mu_r$ (4)

$\vdash . (1) . (4) . \supset \vdash . \mathrm{Prop}$

***262·19.** $\vdash :. \mu, \nu \in \mathrm{NC} . \exists ! \mu_r . \supset : \mu = \nu . \equiv . \mu_r = \nu_r$

Dem.

$\vdash . *262·12 . \supset \vdash : \mu = \nu . \supset . \mu_r = \nu_r$ (1)

$\vdash . *262·18 . \supset \vdash : \mathrm{Hp} . \mu_r = \nu_r . \supset . \mu = C``\nu_r$

[*262·18] $= \nu$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***262·2.** $\vdash . \text{Cls induct} - 1 = C``(\Omega \cap \text{Cnv}``\Omega)$

Dem.

$\vdash . *261·29 . \supset \vdash . \text{Cls induct} - 1 = C``(\Omega \cap \text{Cnv}``\Omega) - \text{l}$

$[*200·12] \qquad = C``(\Omega \cap \text{Cnv}``\Omega) . \supset \vdash . \text{Prop}$

***262·21.** $\vdash : \mu \,\epsilon\, \text{NC induct} - \iota`\Lambda - \iota`1 . \supset . \exists ! \mu_r$

Dem.

$\vdash . *120·2 . *100·43 . \supset \vdash : \text{Hp} . \supset . (\exists \alpha) . \alpha \,\epsilon\, \mu . \alpha \,\epsilon\, \text{Cls induct} . \alpha \sim\epsilon\, 1 .$

$[*262·2] \qquad \supset . (\exists \alpha, P) . \alpha \,\epsilon\, \mu . P \,\epsilon\, \Omega . C`P = \alpha .$

$[*262·12] \qquad \supset . \exists ! \mu_r : \supset \vdash . \text{Prop}$

***262·211.** $\vdash : \alpha \,\epsilon\, \text{Cls induct} - 1 . \supset . \exists ! (\text{Nc}`\alpha)_r \cap t_{00}`\alpha$

Dem.

$\vdash . *262·21 . *103·12 . \supset \vdash : \text{Hp} . \supset . \exists ! (\text{N}_0\text{c}`\alpha)_r . \alpha \,\epsilon\, \text{N}_0\text{c}`\alpha .$

$[*262·12] \qquad \supset . (\exists P) . P \,\epsilon\, (\text{Nc}`\alpha)_r . C`P \,\epsilon\, \text{N}_0\text{c}`\alpha . \alpha \,\epsilon\, \text{N}_0\text{c}`\alpha .$

$[*63·13] \qquad \supset . (\exists P) . P \,\epsilon\, (\text{Nc}`\alpha)_r . C`P \,\epsilon\, t`\alpha .$

$[*64·24 . *35·9] \qquad \supset . (\exists P) . P \,\epsilon\, (\text{Nc}`\alpha)_r . P \,\epsilon\, t`(\alpha \uparrow \alpha) .$

$[*64·11] \qquad \supset . \exists ! (\text{Nc}`\alpha)_r \cap t_{00}`\alpha : \supset \vdash . \text{Prop}$

***262·212.** $\vdash : \mu \neq 0 . \mu \neq 1 . P \,\epsilon\, (\mu +_c 1)_r . \supset . P \upharpoonright \text{Ɑ}`P \,\epsilon\, \mu_r$

Dem.

$\vdash . *262·12 . \qquad \supset \vdash : \text{Hp} . \supset . C`P \,\epsilon\, \mu +_c 1 . P \,\epsilon\, \Omega . \qquad (1)$

$[*110·4] \qquad \supset . \mu \,\epsilon\, \text{NC} - \iota`\Lambda \qquad (2)$

$\vdash . *93·103 . *250·13 . \supset \vdash : \text{Hp} . \supset . C`P = \iota`B`P \cup \text{Ɑ}`P . B`P \sim\epsilon\, \text{Ɑ}`P .$

$[*110·63] \qquad \supset . \text{Nc}`C`P = \text{Nc}`\text{Ɑ}`P +_c 1 .$

$[(1).(2)] \qquad \supset . \mu +_c 1 = \text{Nc}`\text{Ɑ}`P +_c 1 .$

$[*120·311.(1)] \qquad \supset . \mu = \text{Nc}`\text{Ɑ}`P . P \,\epsilon\, \Omega .$

$[*202·55 . *250·141] \qquad \supset . \mu = \text{Nc}`C`(P \upharpoonright \text{Ɑ}`P) . P \upharpoonright \text{Ɑ}`P \,\epsilon\, \Omega .$

$[*262·12 . *100·3 . (2)] \qquad \supset . P \upharpoonright \text{Ɑ}`P \,\epsilon\, \mu_r : \supset \vdash . \text{Prop}$

***262·213.** $\vdash :. \mu \neq 0 . \mu \neq 1 : P, Q \,\epsilon\, \mu_r . \supset_{P,Q} . P \text{ smor } Q : \supset :$

$P, Q \,\epsilon\, (\mu +_c 1)_r . \supset_{P,Q} . P \text{ smor } Q$

Dem.

$\vdash . *262·212·12 . *120·124 . \supset$

$\vdash : \text{Hp} . P, Q \,\epsilon\, (\mu +_c 1)_r . \supset . P \upharpoonright \text{Ɑ}`P, Q \upharpoonright \text{Ɑ}`Q \,\epsilon\, \mu_r . P, Q \,\epsilon\, \Omega - \iota`\dot{\Lambda} .$

$[*11·1 . \text{Hp}] \qquad \supset . P \upharpoonright \text{Ɑ}`P \text{ smor } Q \upharpoonright \text{Ɑ}`Q . P, Q \,\epsilon\, \Omega - \iota`\dot{\Lambda} .$

$[*250·17] \qquad \supset . P \text{ smor } Q : \supset \vdash . \text{Prop}$

***262·22.** $\vdash : \mu \in \mathrm{NC\ induct} \,.\, P, Q \in \mu_r \,.\, \supset \,.\, P \mathrm{\ smor\ } Q$

Dem.

$\vdash \,.\, *153·101 \,.\, *262·12 \,.\, \supset \vdash : P, Q \in 0_r \,.\, \supset \,.\, P \mathrm{\ smor\ } Q$ (1)

$\vdash \,.\, *200·12 \,.\quad \supset \vdash \,.\, 1_r = \Lambda \,.$

$[*10·53] \quad \supset \vdash : P, Q \in 1_r \,.\, \supset \,.\, P \mathrm{\ smor\ } Q$ (2)

$\vdash \,.\, *153·202 \,.\quad \supset \vdash : P, Q \in 2_r \,.\, \supset \,.\, P \mathrm{\ smor\ } Q$ (3)

$\vdash \,.\, (2) \,.\, (3) \,.\, *2·02 \,.\quad \supset \vdash :.\, \mu = 0 \,.\, \vee \,.\, \mu = 1 :$

$P, Q \in \mu_r \,.\, \supset_{P,Q} \,.\, P \mathrm{\ smor\ } Q : \supset : P, Q \in (\mu +_c 1)_r \,.\, \supset_{P,Q} \,.\, P \mathrm{\ smor\ } Q$ (4)

$\vdash \,.\, (4) \,.\, *262·213 \,.\, \supset$

$\vdash :.\, P, Q \in \mu_r \,.\, \supset_{P,Q} \,.\, P \mathrm{\ smor\ } Q : \supset : P, Q \in (\mu +_c 1)_r \,.\, \supset_{P,Q} \,.\, P \mathrm{\ smor\ } Q$ (5)

$\vdash \,.\, (5) \,.\, (1) \,.\, \mathrm{Induct} \,.\, \supset \vdash \,.\, \mathrm{Prop}$

***262·23.** $\vdash :.\, P, Q \in \Omega \mathrm{\ fin} \,.\, \supset : C`P \mathrm{\ sm\ } C`Q \,.\, \equiv \,.\, P \mathrm{\ smor\ } Q$

Dem.

$\vdash \,.\, *262·17·13 \,.\, \supset$

$\vdash : \mathrm{Hp} \,.\, C`P \mathrm{\ sm\ } C`Q \,.\, \supset \,.\, P, Q \in (\mathrm{Nc}`C`P)_r \,.\, \mathrm{Nc}`C`P \in \mathrm{NC\ induct} \,.$

$[*262·22] \quad \supset \,.\, P \mathrm{\ smor\ } Q$ (1)

$\vdash \,.\, (1) \,.\, *151·18 \,.\, \supset \vdash \,.\, \mathrm{Prop}$

The above is the fundamental proposition in the theory of finite ordinals, since it enables us to reduce relations among finite ordinals to relations among the corresponding cardinals.

***262·24.** $\vdash : \mu \in \mathrm{NC\ induct} - \iota`\Lambda - \iota`1 \,.\, \supset \,.\, \mu_r \in \mathrm{NO\ fin}$

Dem.

$\vdash \,.\, *262·21 \,.\quad \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, \exists ! \, \mu_r$ (1)

$\vdash \,.\, *262·22 \,.\quad \supset \vdash : \mathrm{Hp} \,.\, P \in \mu_r \,.\, \supset \,.\, \mu_r \subset \mathrm{Nr}`P$ (2)

$\vdash \,.\, *262·12 \,.\, *151·18 \,.\, \supset \vdash : P \in \mu_r \,.\, \supset \,.\, \mathrm{Nr}`P \subset \mu_r$ (3)

$\vdash \,.\, (2) \,.\, (3) \,.\quad \supset \vdash : \mathrm{Hp} \,.\, P \in \mu_r \,.\, \supset \,.\, \mu_r = \mathrm{Nr}`P$ (4)

$\vdash \,.\, (1) \,.\, (4) \,.\quad \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, \mu_r \in \mathrm{NR} - \iota`\Lambda$ (5)

$\vdash \,.\, *262·12 \,.\quad \supset \vdash : \mathrm{Hp} \,.\, P \in \mu_r \,.\, \supset \,.\, C`P \in \mathrm{Cls\ induct} \,.$

$[*262·13.(4).(5)] \quad \supset \,.\, \mu_r \in \mathrm{NO\ fin}$ (6)

$\vdash \,.\, (1) \,.\, (6) \,.\, \supset \vdash \,.\, \mathrm{Prop}$

***262·241.** $\vdash :.\, \mu \in \mathrm{NC\ induct} \,.\, P \in \Omega \,.\, \supset : \mu_r = \mathrm{Nr}`P \,.\, \equiv \,.\, \mu = \mathrm{Nc}`C`P$

Dem.

$\vdash \,.\, *100·3 \,.\quad \supset \vdash : \mathrm{Hp} \,.\, \mu = \mathrm{Nc}`C`P \,.\, \supset \,.\, C`P \in \mu \,.$

$[*262·12] \quad \supset \,.\, P \in \mu_r \,.$

$[*152·45.*262·24] \quad \supset \,.\, \mu_r = \mathrm{Nr}`P$ (1)

$\vdash \,.\, *152·3 \,.\, *262·18 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \mu_r = \mathrm{Nr}`P \,.\, \supset \,.\, \mu = C``\mathrm{Nr}`P$

$[*152·7] \quad \supset \,.\, \mu = \mathrm{Nc}`C`P$ (2)

$\vdash \,.\, (1) \,.\, (2) \,.\, \supset \vdash \,.\, \mathrm{Prop}$

$*262\cdot25$. $\vdash : (\exists\mu) . \mu \in \text{NC induct} - \iota`1 - \iota`\Lambda . \alpha = \mu_r . \equiv . \alpha \in \text{NO fin}$

Dem.

$\vdash . *262\cdot1\cdot13 . \supset$

$\vdash : \alpha \in \text{NO fin} . \supset . (\exists P) . P \in \Omega\ \text{fin} . \alpha = \text{Nr}`P . \text{Nc}`C`P \in \text{NC induct} .$

$[*262\cdot241] \quad \supset . (\exists P) . P \in \Omega\ \text{fin} . \alpha = \text{Nr}`P . (\text{Nc}`C`P)_r = \text{Nr}`P .$

$\text{Nc}`C`P \in \text{NC induct} .$

$[*13\cdot172] \quad \supset . (\exists P) . \alpha = (\text{Nc}`C`P)_r . \text{Nc}`C`P \in \text{NC induct} .$

$[*200\cdot12 . *262\cdot1 . *155\cdot13] \supset . (\exists\mu) . \mu \in \text{NC induct} - \iota`1 - \iota`\Lambda . \alpha = \mu_r \quad (1)$

$\vdash . *264\cdot24 . \supset \vdash : (\exists\mu) . \mu \in \text{NC induct} - \iota`1 - \iota`\Lambda . \alpha = \mu_r . \supset . \alpha \in \text{NO fin} \quad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

$*262\cdot26$. $\vdash : \alpha \in \text{NO fin} . \equiv . (\exists\mu) . \mu \in \text{N}_0\text{C induct} - \iota`1 . \alpha = \mu_r$

$[*262\cdot25 . *103\cdot13\cdot34]$

$*262\cdot27$. $\vdash : \alpha, \beta \in \text{NO fin} . \supset . \alpha \dot{+} \beta \in \text{NO fin}$

Dem.

$\vdash . *180\cdot21 . \supset \vdash : \text{Hp} . P \in \alpha . Q \in \beta . \supset . P + Q \in \alpha \dot{+} \beta \quad (1)$

$\vdash . *251\cdot24 . \supset \vdash : \text{Hp} . \supset . \alpha \dot{+} \beta \in \text{NO} \quad (2)$

$\vdash . *180\cdot111 . \supset \vdash : \text{Hp}\,(1) . \supset . \text{Nc}`C`(P + Q) = \text{Nc}`(C`P + C`Q)$

$[*110\cdot3] \quad = \text{Nc}`C`P +_c \text{Nc}`C`Q \quad (3)$

$\vdash . *262\cdot13 . \supset \vdash : \text{Hp}\,(1) . \supset . \text{Nc}`C`P, \text{Nc}`C`Q \in \text{NC induct} .$

$[*120\cdot45] \quad \supset . \text{Nc}`C`P +_c \text{Nc}`C`Q \in \text{NC induct} \quad (4)$

$\vdash . (1) . (2) . *155\cdot26 . *251\cdot122 . \supset$

$\vdash : \text{Hp}\,(1) . \supset . P + Q \in \Omega . \alpha \dot{+} \beta = \text{N}_0\text{r}`(P + Q) \quad (5)$

$\vdash . (3) . (4) . \quad \supset \vdash : \text{Hp}\,(1) . \supset . C`(P + Q) \in \text{Cls induct} \quad (6)$

$\vdash . (5) . (6) . *262\cdot1 . *261\cdot42 . \supset \vdash : \text{Hp}\,(1) . \supset . \alpha \dot{+} \beta \in \text{NO fin} \quad (7)$

$\vdash . *262\cdot1 . *155\cdot13 . \quad \supset \vdash : \text{Hp} . \supset . \exists ! \alpha . \exists ! \beta \quad (8)$

$\vdash . (7) . (8) . \supset \vdash . \text{Prop}$

$*262\cdot271$. $\vdash : \alpha, \beta \in \text{NO fin} . \supset . \alpha \dot{\times} \beta \in \text{NO fin}$

[Proof as in $*262\cdot27$, using $*184\cdot12 . *166\cdot12 . *251\cdot55 . *120\cdot5$]

$*262\cdot272$. $\vdash : \alpha, \beta \in \text{NO fin} . \supset . \alpha \exp_r \beta \in \text{NO fin}$

[Proof as in $*262\cdot27$, using $*186\cdot1 . *176\cdot14 . *261\cdot62 . *120\cdot52$]

$*262\cdot31$. $\vdash : \mu, \nu \in \text{NC} . \supset . \mu_r \dot{+} \nu_r \subset (\mu +_c \nu)_r$

Dem.

$\vdash . *180\cdot2 . \supset$

$\vdash :. Z \in \mu_r \dot{+} \nu_r . \equiv : (\exists P, Q) . \mu_r = \text{N}_0\text{r}`P . \nu_r = \text{N}_0\text{r}`Q . Z \,\text{smor}\, (P + Q) : \quad (1)$

$[*180\cdot111 . *151\cdot18] \supset : (\exists P, Q) . \mu_r = \text{N}_0\text{r}`P . \nu_r = \text{N}_0\text{r}`Q . C`Z \,\text{sm}\, (C`P + C`Q) :$

$[*155\cdot12] \quad \supset : (\exists P, Q) . P \in \mu_r . Q \in \nu_r . C`Z \,\text{sm}\, (C`P + C`Q) :$

$[*262\cdot12] \quad \supset : (\exists P, Q) . C`P \in \mu . C`Q \in \nu . C`Z \,\text{sm}\, (C`P + C`Q) :$

$[*110\cdot21] \quad \supset : \text{Hp} . \supset . C`Z \in \mu +_c \nu \quad (2)$

$\vdash . (1) . *262{\cdot}12 . *155{\cdot}12 . \supset$

$\vdash : Z \in \mu_r \dot{+} \nu_r . \supset . (\exists P, Q) . P, Q \in \Omega . Z \,\mathrm{smor}\, (P + Q) .$

$[*251{\cdot}25.*180{\cdot}11{\cdot}12.(*180{\cdot}01)] \quad \supset . Z \in \Omega \qquad (3)$

$\vdash . (2) . (3) . *262{\cdot}12 . \supset \vdash :. \mathrm{Hp} . \supset : Z \in \mu_r \dot{+} \nu_r . \supset . Z \in (\mu +_c \nu)_r :. \supset \vdash . \mathrm{Prop}$

***262·32.** $\vdash : \mu, \nu \in \mathrm{NC\,induct} . P \in \mu_r . Q \in \nu_r . \supset . P + Q \in \mu_r \dot{+} \nu_r$

Dem.

$\vdash . *200{\cdot}12 . *262{\cdot}12 . \supset \vdash : \mathrm{Hp} . \supset . \mu, \nu \in - \iota‘1 - \iota‘\Lambda .$

$[*262{\cdot}24] \quad \supset . \mu_r, \nu_r \in \mathrm{NO} .$

$[*180{\cdot}21] \quad \supset . P + Q \in \mu_r \dot{+} \nu_r : \supset \vdash . \mathrm{Prop}$

***262·33.** $\vdash : \mu, \nu \in \mathrm{NC\,induct} - \iota‘1 . \supset . \mu_r \dot{+} \nu_r = (\mu +_c \nu)_r$

Dem.

$\vdash . *262{\cdot}12 . \supset \vdash :. \mu = \Lambda . \vee . \nu - \Lambda : \supset : \mu_r = \Lambda . \vee . \nu_r = \Lambda :$

$[*180{\cdot}4] \quad \supset : \mu_r \dot{+} \nu_r = \Lambda \qquad (1)$

$\vdash . *110{\cdot}4 . \supset \vdash :. \mu = \Lambda . \vee . \nu = \Lambda : \supset . \mu +_c \nu = \Lambda .$

$[*262{\cdot}12] \quad \supset . (\mu +_c \nu)_r = \Lambda \qquad (2)$

$\vdash . *262{\cdot}32 . \supset \vdash : \mathrm{Hp} . P \in \mu_r . Q \in \nu_r . \supset . P + Q \in \mu_r \dot{+} \nu_r . \qquad (3)$

$[*180{\cdot}42.*152{\cdot}45] \quad \supset . \mu_r \dot{+} \nu_r = \mathrm{Nr}‘(P + Q) \qquad (4)$

$\vdash . (3) . *262{\cdot}31 . \supset \vdash : \mathrm{Hp}\,(3) . \supset . P + Q \in (\mu +_c \nu)_r .$

$[*120{\cdot}45.*262{\cdot}24] \quad \supset . P + Q \in (\mu +_c \nu)_r . (\mu +_c \nu)_r \in \mathrm{NR} .$

$[*152{\cdot}45] \quad \supset . (\mu +_c \nu)_r = \mathrm{Nr}‘(P + Q) \qquad (5)$

$\vdash . (4) . (5) . *10{\cdot}23 . *262{\cdot}21 . \supset \vdash : \mathrm{Hp} . \exists ! \mu . \exists ! \nu . \supset . \mu_r \dot{+} \nu_r = (\mu +_c \nu)_r \qquad (6)$

$\vdash . (1) . (2) . (6) . \supset \vdash . \mathrm{Prop}$

The above proposition still holds (as we shall now prove) when one of μ and ν is equal to 1, but not both. When both are equal to 1, $\mu_r \dot{+} \nu_r = \Lambda$, while $(\mu +_c \nu)_r = 2_r$.

***262·34.** $\vdash : \mu \in \mathrm{NC} - \iota‘0 . \supset . \mu_r \dot{+} \dot{1} \subset (\mu +_c 1)_r$

Dem.

$\vdash . *181{\cdot}2 . \supset \vdash :. Z \in \mu_r \dot{+} \dot{1} . \equiv : (\exists P, x) . \mu_r = \mathrm{N_0r}‘P . Z \,\mathrm{smor}\, (P \dot{+}\!\!\rightarrow x) \qquad (1)$

$\vdash . *181{\cdot}6 . *152{\cdot}7 . \supset \vdash : \dot{\exists} ! P . \supset . \mathrm{Nc}‘C‘(P \dot{+}\!\!\rightarrow x) = \mathrm{Nc}‘C‘P +_c 1 \qquad (2)$

$\vdash . (1) . (2) . \supset$

$\vdash :. \mathrm{Hp} . \supset : Z \in \mu_r \dot{+} \dot{1} . \supset . (\exists P) . \mu_r = \mathrm{N_0r}‘P . \mathrm{Nc}‘C‘Z = \mathrm{Nc}‘C‘P +_c 1 .$

$[*262{\cdot}241{\cdot}12] \quad \supset . (\exists P) . \mu_r = \mathrm{N_0r}‘P . \mathrm{Nc}‘C‘Z = \mu +_c 1 .$

$[*100{\cdot}3] \quad \supset . C‘Z \in \mu +_c 1 \qquad (3)$

$\vdash . (1) . *262{\cdot}12 . *155{\cdot}12 . \supset \vdash : Z \in \mu_r \dot{+} \dot{1} . \supset . (\exists P) . P \in \Omega . \mu_r = \mathrm{N_0r}‘P .$

$[*251{\cdot}1{\cdot}132] \quad \supset . \mu_r \dot{+} \dot{1} \in \mathrm{NO} .$

$[*251{\cdot}122] \quad \supset . Z \in \Omega \qquad (4)$

$\vdash . (3) . (4) . *262{\cdot}12 . \supset \vdash :. \mathrm{Hp} . \supset : Z \in \mu_r \dot{+} \dot{1} . \supset . Z \in (\mu +_c 1)_r :. \supset \vdash . \mathrm{Prop}$

***262·341.** $\vdash : \mu \in \text{NC induct} \,.\, P \in \mu_r \,.\, \supset \,.\, P \dot{+}\!\!\rightarrow x \in \mu_r \dot{+} \dot{1}$

Dem.

$\vdash . *200·12 . *262·12 . \supset \vdash : \text{Hp} . \supset . \mu \in - \iota`1 - \iota`\Lambda .$

$[*262·24] \quad \supset . \mu_r \in \text{NO} .$

$[*181·21] \quad \supset . P \dot{+}\!\!\rightarrow x \in \mu_r \dot{+} \dot{1} : \supset \vdash . \text{Prop}$

***262·35.** $\vdash : \mu \in \text{NC induct} - \iota`0 - \iota`1 . \supset . \mu_r \dot{+} \dot{1} = (\mu +_c 1)_r$

Dem.

$\vdash . *262·12 . \quad \supset \vdash : \mu = \Lambda . \supset . \mu_r = \Lambda .$

$[*181·4] \quad \supset . \mu_r \dot{+} \dot{1} = \Lambda$ (1)

$\vdash . *110·4 . \quad \supset \vdash : \mu = \Lambda . \supset . \mu +_c 1 = \Lambda .$

$[*262·12] \quad \supset . (\mu +_c 1)_r = \Lambda$ (2)

$\vdash . *262·341 . \quad \supset \vdash : \text{Hp} . P \in \mu_r . \supset . P \dot{+}\!\!\rightarrow x \in \mu_r \dot{+} \dot{1} .$ (3)

$[*181·42 . *152·45] \quad \supset . \mu_r \dot{+} \dot{1} = \text{Nr}`(P \dot{+}\!\!\rightarrow x)$ (4)

$\vdash . (3) . *262·34 . \supset \vdash : \text{Hp} . P \in \mu_r . \supset . P \dot{+}\!\!\rightarrow x \in (\mu +_c 1)_r .$

$[*120·45 . *262·24] \quad \supset . P \dot{+}\!\!\rightarrow x \in (\mu +_c 1)_r . (\mu +_c 1)_r \in \text{NR} .$

$[*152·45] \quad \supset . (\mu +_c 1)_r = \text{Nr}`(P \dot{+}\!\!\rightarrow x)$ (5)

$\vdash . (4) . (5) . \quad \supset \vdash : \text{Hp} . \exists ! \mu_r . \supset . \mu_r \dot{+} \dot{1} = (\mu +_c 1)_r :$

$[*262·21] \quad \supset \vdash : \text{Hp} . \exists ! \mu . \supset . \mu_r \dot{+} \dot{1} = (\mu +_c 1)_r$ (6)

$\vdash . (1) . (2) . (6) . \supset \vdash . \text{Prop}$

***262·36.** $\vdash : \mu \in \text{NC induct} - \iota`0 - \iota`1 . \supset . \dot{1} \dot{+} \mu_r = (1 +_c \mu)_r$

[Proof as in *262·35, by means of analogues of *262·34·341]

***262·41.** $\vdash : \mu, \nu \in \text{NC} . \supset . \mu_r \dot{\times} \nu_r \subset (\mu \times_c \nu)_r$

[Proof as in *262·31, using *184·1·5 . *113·21]

***262·42.** $\vdash : \mu, \nu \in \text{NC induct} . P \in \mu_r . Q \in \nu_r . \supset . P \times Q \in \mu_r \dot{\times} \nu_r$

[Proof as in *262·32, using *184·12]

***262·43.** $\vdash : \mu, \nu \in \text{NC induct} - \iota`1 . \supset . \mu_r \dot{\times} \nu_r = (\mu \times_c \nu)_r$

[Proof as in *262·33, using *184·11 . *113·204 . *184·15 . *120·5]

***262·51.** $\vdash : \mu \in \text{NC} . \nu \in \text{NC induct} . \supset . \mu_r \exp_r \nu_r \subset (\mu^\nu)_r$

Dem.

$\vdash . *186·5 . \quad \supset \vdash : \mu_r, \nu_r \in \text{N}_0\text{R} . \nu \neq 0 . R \in \mu_r \exp_r \nu_r . \supset . C`R \in (C``\mu_r)^{C``\nu_r}$ (1)

$\vdash . *186·11 . \supset \vdash : R \in \mu_r \exp_r \nu_r . \supset . \exists ! \mu_r . \exists ! \nu_r$ (2)

$\vdash . (1) . (2) . *262·18 . \supset \vdash : \text{Hp} . \nu \neq 0 . R \in \mu_r \exp_r \nu_r . \supset . C`R \in \mu^\nu$ (3)

$\vdash . *262·12 . \quad \supset \vdash . \mu_r \subset \Omega .$

$[(2) . *251·1 . *186·11] \quad \supset \vdash : R \in \mu_r \exp_r \nu_r . \supset . \mu_r \in \text{NO}$ (4)

$\vdash . *262·24 . \quad \supset \vdash : \text{Hp} . \nu \neq 1 . \nu \neq \Lambda . \supset . \nu_r \in \text{NO fin}$ (5)

$\vdash . (2) . (4) . (5) . *261·62 . \supset \vdash : \text{Hp} . \nu \neq 1 . R \in \mu_r \exp_r \nu_r . \supset . R \in \Omega$ (6)

$\vdash . (2) . *200·12 . \supset \vdash : R \in \mu_r \exp_r \nu_r . \supset . \nu \neq 1$ (7)

$\vdash . (3) . (6) . (7) . \supset \vdash :. \text{Hp} . \supset : R \in \mu_r \exp_r \nu_r . \supset . R \in \Omega . C`R \in \mu^\nu .$

$[*262·12] \quad \supset . R \in (\mu^\nu)_r :. \supset \vdash . \text{Prop}$

$*262\cdot52$. $\vdash : \mu, \nu \in \text{NC induct} \,.\, P \in \mu_r \,.\, Q \in \nu_r \,.\, \supset \,.\, (P \exp Q) \in (\mu_r \exp_r \nu_r)$

Dem.

$\vdash \,.\, *200\cdot12 \,.\, *262\cdot12 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, \mu, \nu \in -\iota\text{'}1 - \iota\text{'}\Lambda \,.$

$[*262\cdot24] \quad \supset \,.\, \mu_r, \nu_r \in \text{NO} \,.$

$[*186\cdot13 \,.\, *152\cdot45] \quad \supset \,.\, (P \exp Q) \in (\mu_r \exp_r \nu_r) : \supset \vdash \,.\, \text{Prop}$

$*262\cdot53$. $\vdash : \mu, \nu \in \text{NC induct} - \iota\text{'}1 \,.\, \nu \neq 0 \,.\, \supset \,.\, \mu_r \exp_r \nu_r = (\mu^\nu)_r$

Dem.

$\vdash \,.\, *262\cdot12 \,.\, *186\cdot11 \,.\, \supset \vdash :. \mu = \Lambda \,.\vee.\, \nu = \Lambda : \supset \,.\, \mu_r \exp_r \nu_r = \Lambda$ (1)

$\vdash \,.\, *116\cdot204 \,.\, *262\cdot12 \,.\, \supset \vdash :. \mu = \Lambda \,.\vee.\, \nu = \Lambda : \supset \,.\, (\mu^\nu)_r = \Lambda$ (2)

$\vdash \,.\, *262\cdot52 \,.\, \supset \vdash : \text{Hp} \,.\, P \in \mu_r \,.\, Q \in \nu_r \,.\, \supset \,.\, (P \exp Q) \in (\mu_r \exp_r \nu_r) \,.$ (3)

$[*186\cdot13 \,.\, *152\cdot45] \quad \supset \,.\, \text{Nr}\text{'}(P \exp Q) = \mu_r \exp_r \nu_r$ (4)

$\vdash \,.\, (3) \,.\, *262\cdot51 \,.\, \quad \supset \vdash : \text{Hp}(3) \,.\, \supset \,.\, (P \exp Q) \in (\mu^\nu)_r$ (5)

$\vdash \,.\, (5) \,.\, *120\cdot52 \,.\, \quad \supset \vdash : \text{Hp}(3) \,.\, \supset \,.\, \mu^\nu \in \text{NC induct}$ (6)

$\vdash \,.\, (5) \,.\, \quad \supset \vdash : \text{Hp}(3) \,.\, \supset \,.\, \exists ! (\mu^\nu)_r \,.$

$[*200\cdot12 \,.\, *262\cdot12] \quad \supset \,.\, \mu^\nu \neq 1$ (7)

$\vdash \,.\, (6) \,.\, (7) \,.\, *262\cdot24 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, (\mu^\nu)_r \in \text{NO}$ (8)

$\vdash \,.\, (5) \,.\, (8) \,.\, *152\cdot45 \,.\, \supset \vdash : \text{Hp}(3) \,.\, \supset \,.\, \text{Nr}\text{'}(P \exp Q) = (\mu^\nu)_r \,.$

$[(4)] \quad \supset \,.\, \mu_r \exp_r \nu_r = (\mu^\nu)_r$ (9)

$\vdash \,.\, (9) \,.\, *262\cdot21 \,.\, \quad \supset \vdash : \text{Hp} \,.\, \exists ! \mu \,.\, \exists ! \nu \,.\, \supset \,.\, \mu_r \exp_r \nu_r = (\mu^\nu)_r$ (10)

$\vdash \,.\, (1) \,.\, (2) \,.\, (10) \,.\, \supset \vdash \,.\, \text{Prop}$

We are now in a position to establish the commutative property of addition and multiplication of finite ordinals. This is effected by means of $*262\cdot33$ and $*262\cdot43$.

$*262\cdot6$. $\vdash : \alpha, \beta \in \text{NO fin} \,.\, \supset \,.\, \alpha \dot{+} \beta = \beta \dot{+} \alpha$

Dem.

$\vdash \,.\, *262\cdot26 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, (\exists \mu, \nu) \,.\, \mu, \nu \in \text{NC induct} - \iota\text{'}1 \,.\, \alpha = \mu_r \,.\, \beta = \nu_r \,.$

$[*13\cdot12] \supset \,.\, (\exists \mu, \nu) \,.\, \mu, \nu \in \text{NC induct} - \iota\text{'}1 \,.\, \alpha \dot{+} \beta = \mu_r \dot{+} \nu_r \,.\, \alpha = \mu_r \,.\, \beta = \nu_r \,.$

$[*262\cdot33] \supset \,.\, (\exists \mu, \nu) \,.\, \mu, \nu \in \text{NC induct} - \iota\text{'}1 \,.\, \alpha \dot{+} \beta = (\mu +_c \nu)_r \,.\, \alpha = \mu_r \,.\, \beta = \nu_r$

$[*110\cdot51] \supset \,.\, (\exists \mu, \nu) \,.\, \mu, \nu \in \text{NC induct} - \iota\text{'}1 \,.\, \alpha \dot{+} \beta = (\nu +_c \mu)_r \,.\, \alpha = \mu_r \,.\, \beta = \nu_r \,.$

$[*262\cdot33] \supset \,.\, (\exists \mu, \nu) \,.\, \mu, \nu \in \text{NC induct} - \iota\text{'}1 \,.\, \alpha \dot{+} \beta = \nu_r \dot{+} \mu_r \,.\, \alpha = \mu_r \,.\, \beta = \nu_r \,.$

$[*13\cdot22] \supset \,.\, \alpha \dot{+} \beta = \beta \dot{+} \alpha : \supset \vdash \,.\, \text{Prop}$

$*262\cdot61$. $\vdash : \alpha, \beta \in \text{NO fin} \,.\, \supset \,.\, \alpha \dot{\times} \beta = \beta \dot{\times} \alpha$

[Proof as in $*262\cdot6$, using $*262\cdot43$ and $*113\cdot27$]

$*262\cdot62$. $\vdash : \alpha, \beta, \gamma \in \text{NO fin} \,.\, \supset \,.\, \alpha \dot{\times} (\beta \dot{+} \gamma) = (\alpha \dot{\times} \beta) \dot{+} (\alpha \dot{\times} \gamma)$

Dem.

$\vdash \,.\, *262\cdot27\cdot61 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, \alpha \dot{\times} (\beta \dot{+} \gamma) = (\beta \dot{+} \gamma) \dot{\times} \alpha$

$[*184\cdot35] \quad = (\beta \dot{\times} \alpha) \dot{+} (\gamma \dot{\times} \alpha)$

$[*262\cdot61] \quad = (\alpha \dot{\times} \beta) \dot{+} (\alpha \dot{\times} \gamma) : \supset \vdash \,.\, \text{Prop}$

***262·63.** $\vdash : \alpha, \beta, \gamma \in \text{NO fin} . \supset . (\alpha \dot{\times} \beta) \exp_r \gamma = (\alpha \exp_r \gamma) \dot{\times} (\beta \exp_r \gamma)$

Dem.

$\vdash . *262{\cdot}26 . \supset$

$\vdash : \text{Hp} . \supset . (\exists \mu, \nu, \varpi) . \mu, \nu, \varpi \in \text{NC induct} - \iota`1 . \alpha = \mu_r . \beta = \nu_r . \gamma = \varpi_r \quad (1)$

$\vdash . *262{\cdot}43 . \supset$

$\vdash : \mu, \nu, \varpi \in \text{NC induct} - \iota`1 . \supset . (\mu_r \dot{\times} \nu_r) \exp_r \varpi_r = (\mu \times_c \nu)_r \exp_r \varpi_r \quad (2)$

$\vdash . *113{\cdot}602 . \supset \vdash : \mu = 0 . \nu = 0 . \supset . \mu \times_c \nu \neq 1 \quad (3)$

$\vdash . *117{\cdot}631 . \supset \vdash : \mu, \nu \in \text{NC} - \iota`0 - \iota`1 . \supset . \mu \times_c \nu \neq 1 \quad (4)$

$\vdash . (3) . (4) . \quad \supset \vdash : \text{Hp}(2) . \supset . \mu \times_c \nu \neq 1 \quad (5)$

$\vdash . *120{\cdot}5 . \quad \supset \vdash : \text{Hp}(2) . \supset . \mu \times_c \nu \in \text{NC induct} \quad (6)$

$\vdash . (5) . (6) . *262{\cdot}53 . \supset \vdash : \text{Hp}(2) . \varpi \neq 0_r . \supset . (\mu \times_c \nu)_r \exp_r \varpi_r = \{(\mu \times_c \nu)^\varpi\}_r$

$[*116{\cdot}55] \qquad = (\mu^\varpi \times_c \nu^\varpi)_r \quad (7)$

$\vdash . *117{\cdot}652 . \supset \vdash : \text{Hp}(7) . \mu \neq 0_r . \supset . \mu^\varpi \geqslant \mu \times_c \varpi$

$[*117{\cdot}631] \qquad \supset . \mu^\varpi \neq 1 \quad (8)$

$\vdash . *116{\cdot}311 . \supset \vdash : \text{Hp}(7) . \mu = 0_r . \supset . \mu^\varpi \neq 1 \quad (9)$

$\vdash . (8) . (9) . \quad \supset \vdash : \text{Hp}(7) . \supset . \mu^\varpi \neq 1 \quad (10)$

Similarly $\quad \vdash : \text{Hp}(7) . \supset . \nu^\varpi \neq 1 \quad (11)$

$\vdash . (10) . (11) . *120{\cdot}52 . *262{\cdot}43 . \supset \vdash : \text{Hp}(7) . \supset . (\mu^\varpi \times_c \nu^\varpi)_r = (\mu^\varpi)_r \dot{\times} (\nu^\varpi)_r$

$[*262{\cdot}53] \qquad = (\mu_r \exp_r \varpi_r) \dot{\times} (\nu_r \exp_r \varpi_r) \quad (12)$

$\vdash . (2) . (7) . (12) . \supset$

$\vdash : \text{Hp}(7) . \supset . (\mu_r \dot{\times} \nu_r) \exp_r \varpi_r = (\mu_r \exp_r \varpi_r) \dot{\times} (\nu_r \exp_r \varpi_r) \quad (13)$

$\vdash . (1) . (13) . *262{\cdot}19 . \supset$

$\vdash : \text{Hp} . \gamma \neq 0_r . \supset . (\alpha \dot{\times} \beta) \exp_r \gamma = (\alpha \exp_r \gamma) \dot{\times} (\beta \exp_r \gamma) \quad (14)$

$\vdash . *186{\cdot}2 . *184{\cdot}16 . \supset$

$\vdash : \text{Hp} . \gamma = 0_r . \supset . (\alpha \dot{\times} \beta) \exp_r \gamma = 0_r . (\alpha \exp_r \gamma) \dot{\times} (\beta \exp_r \gamma) = 0_r \quad (15)$

$\vdash . (14) . (15) . \supset \vdash . \text{Prop}$

***262·64.** $\vdash : \alpha \in \text{NO fin} . \supset . \alpha \dot{+} \dot{1} = \dot{1} \dot{+} \alpha$

Dem.

$\vdash . *262{\cdot}35{\cdot}36{\cdot}26 . *110{\cdot}51 . \supset \vdash : \text{Hp} . \alpha \neq 0_r . \supset . \alpha \dot{+} \dot{1} = \dot{1} \dot{+} \alpha \quad (1)$

$\vdash . *161{\cdot}2{\cdot}201 . \qquad \supset \vdash : \alpha = 0_r . \supset . \alpha \dot{+} \dot{1} = 0_r . \dot{1} \dot{+} \alpha = 0_r \quad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***262·65.** $\vdash : \alpha, \beta \in \text{NO fin} . \beta \neq 0_r . \supset . \alpha \dot{\times} (\beta \dot{+} \dot{1}) = (\alpha \dot{\times} \beta) \dot{+} \alpha$

Dem.

$\vdash . *262{\cdot}61 . \supset \vdash : \text{Hp} . \supset . \alpha \dot{\times} (\beta \dot{+} \dot{1}) = (\beta \dot{+} \dot{1}) \dot{\times} \alpha$

$[*184{\cdot}41] \qquad = (\beta \dot{\times} \alpha) \dot{+} \alpha$

$[*262{\cdot}61] \qquad = (\alpha \dot{\times} \beta) \dot{+} \alpha : \supset \vdash . \text{Prop}$

***262·66.** $\vdash : \alpha, \beta \in \text{NO fin} . \beta \neq 0_r . \supset . \alpha \dot{\times} (1 \dot{+} \beta) = \alpha \dot{+} (\alpha \dot{\times} \beta)$

[Proof as in *262·65]

***262·7.** $\vdash :. \mu, \nu \in \text{NC induct} - \iota`1 . \supset : \mu > \nu . \equiv . \mu_r \gtrdot \nu_r$

Dem.

$\vdash . *262·21 . *117·12 . \supset \vdash : \text{Hp} . \mu > \nu . \supset . \exists ! \mu_r . \exists ! \nu_r .$

$[*262·18] \quad \supset . \mu = C``\mu_r . \nu = C``\nu_r . \quad (1)$

$[*255·76 . *262·24] \quad \supset . \mu_r \gtrdot \nu_r \quad (2)$

$\vdash . *120·441 . \supset \vdash : \text{Hp} . \sim(\mu > \nu) . \supset . \mu \leqslant \nu \quad (3)$

$\vdash . (1) . \quad \supset \vdash : \text{Hp} . \mu < \nu . \supset . \mu_r \lessdot \nu_r \quad (4)$

$\vdash . *262·21 . \supset \vdash : \text{Hp} . \mu = \text{sm}``\nu . \supset . (\exists P) . \mu = \text{N}_0\text{c}`C`P . \mu = \text{sm}``\nu .$

$[*103·4] \quad \supset . (\exists P) . \mu = \text{N}_0\text{c}`C`P . \nu = \text{Nc}`C`P .$

$[*262·241] \quad \supset . (\exists P) . \mu_r = \text{N}_0\text{r}`P . \nu_r = \text{Nr}`P .$

$[*155·4] \quad \supset . \mu_r = \text{smor}``\nu \quad (5)$

$\vdash . (4) . (5) . *117·104 . \supset \vdash : \text{Hp} . \mu \leqslant \nu . \supset . \mu_r \lessdot\!\!= \nu_r \quad (6)$

$\vdash . (3) . (6) . *255·483 . \supset \vdash : \text{Hp} . \sim(\mu > \nu) . \supset . \sim(\mu_r \gtrdot \nu_r) \quad (7)$

$\vdash . (2) . (7) . \supset \vdash . \text{Prop}$

***262·71.** $\vdash : \alpha \in \text{NO fin} - \iota`0_r . \supset . (\exists \beta) . \beta \in \text{NO fin} - \iota`0_r \cup \iota`\dot{1} . \alpha = \beta \dot{+} \dot{1}$

Dem.

$\vdash . *262·11 . *261·24 . \supset \vdash : \text{Hp} . \supset . \exists ! \alpha \cap \text{G}`(B \mid \text{Cnv}) \quad (1)$

$\vdash . (1) . *204·483 . (*181·04) . \supset \vdash . \text{Prop}$

***262·8.** $\vdash : \alpha, \beta \in \text{NO} . \gamma \in \text{NO fin} . \alpha \lessdot \beta . \supset . \alpha \exp_r \gamma \lessdot \beta \exp_r \gamma \quad [*261·64]$

***262·81.** $\vdash : \alpha \ \beta \in \text{N}_0\text{O} . \gamma \in \text{NO fin} . \alpha \exp_r \gamma = \beta \exp_r \gamma . \supset . \alpha = \text{smor}``\beta$

Dem.

$\vdash . *262·8 . \text{Transp} . *255·42 . \supset \vdash : \text{Hp} . \supset . \sim(\alpha \lessdot \beta) . \sim(\alpha \gtrdot \beta) .$

$[*255·112] \quad \supset . \alpha = \text{smor}``\beta : \supset \vdash . \text{Prop}$

***262·82.** $\vdash : \alpha \in \text{NO fin} . \beta \in \text{NO infin} . \supset . \alpha \lessdot \beta \quad [*261·65]$

***262·83.** $\vdash : \alpha \in \text{N}_0\text{O} - \iota`0_r . \beta, \gamma \in \text{NO fin} . \beta \lessdot \gamma . \supset . \alpha \exp_r \beta \lessdot \alpha \exp_r \gamma$

Dem.

$\vdash . *255·33 . \supset \vdash :. \text{Hp} . \supset : (\exists \varpi) . \varpi \in \text{NO} - \iota`0_r . \gamma = \beta \dot{+} \varpi . \vee . \beta \neq 0_r . \gamma = \beta \dot{+} \dot{1} \quad (1)$

$\vdash . *254·51 . \supset \vdash : Q \in P . \supset . \sim(P \text{ less } Q) \quad (2)$

$\vdash . (2) . *255·1 . \supset \vdash : \gamma = \beta \dot{+} \varpi . \supset . \sim(\gamma \lessdot \varpi) \quad (3)$

$\vdash . (3) . *262·82 . \text{Transp} . \supset \vdash : \text{Hp} . \gamma = \beta \dot{+} \varpi . \supset . \varpi \in \text{NO fin} \quad (4)$

$\vdash . *186·14 . \supset \vdash : \text{Hp}(4) . \varpi \neq 0_r . \beta \neq 0_r . \supset . \alpha \exp_r \gamma = (\alpha \exp_r \beta) \dot{\times} (\alpha \exp_r \varpi) \quad (5)$

$\vdash . *262·71·272 . \supset \vdash : \text{Hp}(5) . \supset . (\exists \delta) . \delta \in \text{NR} - \iota`0_r \cup \iota`\dot{1} . \alpha \exp_r \beta = \delta \dot{+} \dot{1} .$

$[(5).(4).*255·573] \quad \supset . \alpha \exp_r \gamma \gtrdot \alpha \exp_r \beta \quad (6)$

$\vdash . *255·51 . \supset \vdash : \text{Hp}(4) . \varpi \neq 0_r . \beta = 0_r . \supset . \alpha \exp_r \gamma \gtrdot \alpha \exp_r \beta \quad (7)$

$\vdash . *186·22 . \supset \vdash : \text{Hp} . \beta \neq 0_r . \gamma = \beta \dot{+} \dot{1} . \supset . \alpha \exp_r \gamma = (\alpha \exp_r \beta) \dot{\times} \beta .$

$[*262·71 . *255·573] \quad \supset . \alpha \exp_r \gamma \gtrdot \alpha \exp_r \beta \quad (8)$

$\vdash . (1) . (6) . (7) . (8) . \supset \vdash : \text{Hp} . \supset . \alpha \exp_r \gamma \gtrdot \alpha \exp_r \beta : \supset \vdash . \text{Prop}$

***262·84.** $\vdash : P \in \Omega - \iota`\dot{\Lambda} . Q, R \in \Omega \text{ fin} . Q \text{ less } R . \supset . P^Q \text{ less } P^R \quad [*262·83]$

$*263$. PROGRESSIONS.

Summary of $*263$.

If R is a progression in the sense defined in $*122$, *i.e.* a one-one relation whose field is the posterity of its first term, then R_{po} is a serial relation, and the series generated by R_{po} is of the type which Cantor calls ω, *i.e.* the smallest of infinite series. It is easy to prove that all progressions are ordinally similar, and that, if all inductive cardinals exist, the series of inductive cardinals in order of magnitude is of the type ω. Thus ω is an ordinal number, which is not null if the axiom of infinity holds.

Most of the properties of ω are easily deduced from the corresponding properties of "Prog," which have been proved in $*122$. The definition is

$$\omega = \hat{P}\{(\exists R) \,.\, R \in \mathrm{Prog} \,.\, P = R_{po}\} \quad \mathrm{Df.}$$

The axiom of infinity implies that "less to greater" with its field confined to inductive cardinals is a member of ω, or, what comes to the same thing but is easier to prove, that $\{(\mathrm{NC\,induct}) \upharpoonright (+_c 1)\}_{po}$ is a member of ω ($*263 \cdot 12$). Thus the axiom of infinity for the type of x implies the existence of ω in the type $t^{33}{}^{\backprime}x$ ($*263 \cdot 132$); and generally the existence of ω in any type of relations is equivalent to the existence of $\aleph_0$ in the type of their fields ($*263 \cdot 131$), because $\aleph_0 = \mathrm{D}^{\backprime\backprime}\omega = C^{\backprime\backprime}\omega$ ($*263 \cdot 101$).

By using the fact that in a progression R (in the sense of $*122$) all the terms are values of ν_R, where every inductive cardinal occurs as a value of ν (which was proved in $*122$), we easily deduce that if there are progressions they are the series that are ordinally similar to the series of inductive cardinals ($*263 \cdot 161$). Hence both "Prog" and ω are relation-numbers ($*263 \cdot 162 \cdot 19$). Moreover, by $*122 \cdot 21 \cdot 23$, ω consists of well-ordered series ($*263 \cdot 11$). Hence ω is an ordinal number ($*263 \cdot 2$).

We next prove that progressions are infinite series ($*263 \cdot 23$), and that a series contained in a progression is finite if it has a maximum ($*263 \cdot 27$), and is a progression if it has no maximum ($*263 \cdot 26$). It follows that, assuming the existence of progressions or the axiom of infinity, ω is the smallest ordinal which is greater than all the finite ordinals ($*263 \cdot 31 \cdot 32$). Connected with this is the fact that the predecessors of any term in a progression are an inductive class ($*263 \cdot 412$).

*263·44·48 give various formulae for ω, any one of which might be taken as the definition. We have

***263·44.** $\vdash . \omega = \Omega - \iota`\Lambda \cap \hat{P}(\text{G}`P_1 = \text{G}`P . \sim E ! B`\breve{P})$

I.e. progressions are existent well-ordered series in which every term except the first has an immediate predecessor, and there is no last term.

***263·46.** $\vdash . \omega = \Omega \cap \hat{P}(E ! B`P_1 . \sim E ! B`\breve{P})$

I.e. progressions are well-ordered series in which there is only one term having an immediate successor but no immediate predecessor, and there is no last term.

***263·47.** $\vdash . \omega = \Omega \cap \hat{P}\{\alpha \subset C`P . \supset_\alpha : \alpha \in \text{Cls induct} . \equiv . \exists ! C`P \cap p`\overleftarrow{P}``\alpha\}$

I.e. a progression is a well-ordered series in which any sub-class α stops short of some point of the series if α is inductive, but not otherwise. This proposition will be useful in the next section.

***263·49.** $\vdash . \Omega \text{ fin} \cup \omega = \Omega \cap \hat{P}(\text{G}`P_1 = \text{G}`P) = \Omega \cap \hat{P}(P = P_{\text{fn}})$

I.e. finite well-ordered series and progressions together are those well-ordered series in which every term except the first has an immediate predecessor, and are also those in which every interval is an inductive class.

From *261·45 it follows that, if P is an infinite well-ordered series, P confined to the terms at a finite distance from $B`P$ is a progression, *i.e.*

***263·5.** $\vdash : P \in \Omega \text{ infin} . \supset . P \upharpoonright (\iota`B`P \cup \overleftarrow{P}_{\text{fn}}`B`P) \in \omega$

Hence it follows at once that an infinite ordinal is at least as great as ω, and therefore infinite ordinals other than ω are greater than ω, *i.e.*

***263·54.** $\vdash : \alpha \in \text{NO infin} - \iota`\omega . \supset . \alpha > \omega$

The remaining propositions of this number are occupied in proving $\omega \dot{\times} 2_r = \omega$ (*263·63) and $\omega \dot{\times} \alpha = \omega$ if α is finite and not zero (*263·66). It is not the case that $2_r \dot{\times} \omega = \omega$ or $\alpha \dot{\times} \omega = \omega$.

Cantor has varied his definitions of multiplication as regards the order of the factors. Originally, he adopted the same rule as we have adopted, but in later works he inverted the rule, so that what we call $2_r \dot{\times} \omega$ he calls $\omega \dot{\times} 2_r$, and vice versa. Thus with his definitions in his later works, $2_r \dot{\times} \omega = \omega$ but $\omega \dot{\times} 2_r \neq \omega$. We have reverted to his earlier practice, for various reasons, but chiefly in order to have $\text{Nr}`\Pi`(P \downarrow Q) = \text{Nr}`P \dot{\times} \text{Nr}`Q$ (cf. *172). Whichever rule we adopt, there are some inconveniences, so that the question as to which is chosen is not of great importance.

$*263\cdot01.\quad \omega = \hat{P}\{(\exists R) . R \in \text{Prog} . P = R_{\text{po}}\}$ Df

$*263\cdot02.\quad N = \hat{\mu}\hat{\nu}\{\mu \in \text{NC induct} . \nu = (\mu +_c 1) \cap t_0{}^{\backprime}\mu\}$ Dft [$*263$]

The above temporary definition of N is the same as that in $*123$.

$*263\cdot1.\quad \vdash : P \in \omega . \equiv . (\exists R) . R \in \text{Prog} . P = R_{\text{po}}$ [($*263\cdot01$)]

$*263\cdot101.\ \vdash . \aleph_0 = \text{D}^{\backprime\backprime}\omega = C^{\backprime\backprime}\omega$ [$*123\cdot1 . *122\cdot141 . *91\cdot504$]

$*263\cdot11.\quad \vdash . \omega \subset \Omega$

Dem.

$\vdash . *122\cdot23\cdot141 . *263\cdot1 . \supset \vdash : P \in \omega . \alpha \subset C^{\backprime}P . \exists ! \alpha . \supset . E ! \min_P{}^{\backprime}\alpha$ (1)

$\vdash . (1) . *250\cdot125 . \supset \vdash . \text{Prop}$

$*263\cdot12.\quad \vdash : \text{Infin ax} . \supset . N_{\text{po}} \in \omega$ [$*123\cdot25 . *263\cdot1$]

$*263\cdot13.\quad \vdash : \exists ! \aleph_0(x) . \equiv . \exists ! \omega \cap t^{11}{}^{\backprime}x$

Dem.

$\vdash . *263\cdot101 . (*65\cdot02) . \supset$

$\vdash : \exists ! \aleph_0(x) . \equiv . (\exists P) . P \in \omega . C^{\backprime}P \in t^{\backprime}t^{\backprime}x .$

$[*64\cdot57 . *63\cdot5] \equiv . (\exists P) . P \in \omega . P \in t^{11}{}^{\backprime}x : \supset \vdash . \text{Prop}$

$*263\cdot131.\ \vdash : \exists ! (\aleph_0)_\alpha . \equiv . \exists ! \omega \cap t_{00}{}^{\backprime}\alpha$ [Proof as in $*263\cdot13$]

$*263\cdot132.\ \vdash : \text{Infin ax}(x) . \equiv . \exists ! \omega \cap t^{33}{}^{\backprime}x .$

Dem.

$\vdash . *125\cdot23 . *263\cdot13 . \supset \vdash : \text{Infin ax}(x) . \equiv . \exists ! \omega \cap t^{11}{}^{\backprime}t^{2}{}^{\backprime}x .$

$[(*64\cdot011\cdot014)] \qquad \equiv . \exists ! \omega \cap t^{33}{}^{\backprime}x : \supset \vdash . \text{Prop}$

This proposition asserts that, if the number of individuals of the same type as x is not an inductive number, then there is a progression whose terms are of the type of $t^{2}{}^{\backprime}x$. This progression will be that of the inductive cardinals which are applicable to classes whose terms are of the same type as x.

$*263\cdot14.\quad \vdash : R \in \text{Prog} . P = R_{\text{po}} . \supset . P = P_{\text{fn}} = R_{\text{fn}} . R = P_1$

Dem.

$\vdash . *121\cdot254 . \supset \vdash : \text{Hp} . \supset . P_1 = R_1 .$

$[*121\cdot31 . *122\cdot1\cdot16] \quad \supset . P_1 = R .$ (1)

$[\text{Hp}] \quad \supset . (P_1)_{\text{po}} = P .$

$[*260\cdot27 . *263\cdot11] \quad \supset . P_{\text{fn}} = P .$ (2)

$[*260\cdot15 . \text{Hp}] \quad \supset . R_{\text{fn}} = P$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash . \text{Prop}$

***263·141.** $\vdash : P \epsilon \omega . \supset . P_1 \epsilon \text{Prog} . P = (P_1)_{\text{fn}} = (P_1)_{\text{po}}$

Dem.

$\vdash . *263·1 . \supset \vdash : \text{Hp} . \supset . (\exists R) . R \epsilon \text{Prog} . P = R_{\text{po}} .$

$[*263·14] \quad \supset . (\exists R) . R \epsilon \text{Prog} . P_1 = R . P = R_{\text{fn}} . P = R_{\text{po}} .$

$[*13·195] \quad \supset . P_1 \epsilon \text{Prog} . P = (P_1)_{\text{fn}} = (P_1)_{\text{po}} : \supset \vdash . \text{Prop}$

The above proposition shows that every interval $P(x \mapsto y)$ in a progression is an inductive class.

***263·142** $\vdash : R, S \epsilon \text{Prog} . R_{\text{po}} = S_{\text{po}} . \supset . R = S$

Dem.

$\vdash . *263·14 . \supset \vdash : \text{Hp} . \supset . R = (S_{\text{po}})_1$

$[*263·14] \quad = S : \supset \vdash . \text{Prop}$

***263·143.** $\vdash : P, Q \epsilon \omega . P_1 = Q_1 . \supset . P = Q$

Dem.

$\vdash . *263·1 . \supset \vdash : \text{Hp} . \supset . (\exists R, S) . R, S \epsilon \text{Prog} . P = R_{\text{po}} . Q = S_{\text{po}} . P_1 = Q_1 .$

$[*263·14] \supset . (\exists R, S) . R, S \epsilon \text{Prog} . P = R_{\text{po}} . Q = S_{\text{po}} . R = P_1 . S = Q_1 . P_1 = Q_1 .$

$[*13·17] \quad \supset . (\exists R, S) . R, S \epsilon \text{Prog} . P = R_{\text{po}} . Q = S_{\text{po}} . R = S .$

$[*13·17] \quad \supset . P = Q : \supset \vdash . \text{Prop}$

***263·15.** $\vdash : R \epsilon \text{Prog} . S = \hat{x}\hat{\nu} \{\nu \epsilon \text{NC induct} . x = (\nu +_c 1)_R\} . \supset . S \epsilon R \,\overline{\text{smor}}\, N$

Dem.

$\vdash . *123·3 . \supset \vdash : \text{Hp} . \supset . S \epsilon 1 \rightarrow 1 . \text{D}'S = \text{D}'R . \text{Ɑ}'S = \text{NC induct} \quad (1)$

$\vdash . *123·21 . \supset \vdash . \text{NC induct} = C'N \quad (2)$

$\vdash . *110·56·643 . \supset \vdash : \text{Hp} . (\mu +_c 1) N (\nu +_c 1) . \supset . \nu +_c 1 = \mu +_c 2 \quad (3)$

$\vdash . (3) . \supset \vdash :. \text{Hp} . \supset :$

$x (S ; N) y . \equiv . (\exists \mu) . \mu \epsilon \text{NC induct} . x = (\mu +_c 1)_R . y = (\mu +_c 2)_R .$

$[*121·332·131] \quad \equiv . (\exists \mu) . \mu \epsilon \text{NC induct} . (B'R) R_\mu x . (B'R)(R_\mu | R) y .$

$[*122·341.*121·342] \quad \equiv . xRy \quad (4)$

$\vdash . (1) . (2) . (4) . \supset \vdash . \text{Prop}$

***263·151.** $\vdash : R \epsilon \text{Prog} . \supset . R \text{ smor } N \quad [*263·15]$

***263·152.** $\vdash : R \epsilon \text{Prog} . Q \text{ smor } R . \supset . Q \epsilon \text{Prog} \quad [*123·32]$

***263·16.** $\vdash : R \epsilon \text{Prog} . \supset . \text{Prog} = \text{Nr}'R = \text{Nr}'N \quad [*263·151·152]$

***263·161.** $\vdash : \exists ! \text{Prog} . \supset . \text{Prog} = \text{Nr}'N \quad [*263·16]$

***263·162.** $\vdash . \text{Prog} \epsilon \text{NR} \quad [*263·161 . *154·242]$

***263·17.** $\vdash : P \epsilon \omega . \supset . \omega = \text{Nr}'P = \text{Nr}'N_{po}$

Dem.

$\vdash . *263·1 . \supset \vdash : \text{Hp} . \supset . (\exists R) . R \epsilon \text{Prog} . P = R_{po} .$

$[*263·151] \qquad \supset . (\exists R) . R \text{ smor } N . P = R_{po} .$

$[*151·56] \qquad \supset . P \text{ smor } N_{po} .$ (1)

$[*152·321] \qquad \supset . \text{Nr}'P = \text{Nr}'N_{po}$ (2)

$\vdash . *151·59 . \supset \vdash : P \epsilon \omega . Q \text{ smor } P . \supset . Q_1 \text{ smor } P_1 .$

$[*263·141·152] \qquad \supset . Q_1 \epsilon \text{Prog}$ (3)

$\vdash . *150·83 . \supset \vdash : P \epsilon \omega . S \epsilon Q \,\overline{\text{smor}}\, P . \supset . (Q_1)_{po} = S;(P_1)_{po}$

$[*263·141] \qquad = S;P$

$[*151·11] \qquad = Q$ (4)

$\vdash . (3) . (4) . *263·1 . \supset \vdash : P \epsilon \omega . Q \text{ smor } P . \supset . Q \epsilon \omega$ (5)

$\vdash . (1) . \qquad \supset \vdash : P, Q \epsilon \omega . \supset . P \text{ smor } Q$ (6)

$\vdash . (5) . (6) . \supset \vdash : P \epsilon \omega . \supset . \omega = \text{Nr}'P$ (7)

$\vdash . (7) . (2) . \supset \vdash . \text{Prop}$

***263·18.** $\vdash : \exists ! \omega . \supset . \omega = \text{Nr}'N_{po}$ [*263·17]

***263·19.** $\vdash . \omega \epsilon \text{NR}$ [*263·18 . *154·242]

***263·2.** $\vdash . \omega \epsilon \text{NO}$ [*263·19·11 . *256·54]

***263·22.** $\vdash : P \epsilon \omega . \supset . \text{G}'P \subset \text{D}'P . \sim \text{E} ! \breve{B}'P . \text{E} ! B'P$

[*122·141 . *263·1 . *122·11]

***263·23.** $\vdash . \omega \subset \Omega \text{ infin}$

Dem.

$\vdash . *261·35 . \text{Transp} . *263·11·22 . \supset \vdash : P \epsilon \omega . \supset . C'P \sim\epsilon \text{ Cls induct} - \iota'\Lambda$ (1)

$\vdash . *263·22 . \supset \vdash : P \epsilon \omega . \supset . \exists ! C'P$ (2)

$\vdash . (1) . (2) . \supset \vdash : P \epsilon \omega . \supset . C'P \sim\epsilon \text{ Cls induct} .$

$[*261·14·41 . *263·11] \qquad \supset . P \epsilon \Omega \text{ infin} : \supset \vdash . \text{Prop}$

***263·24.** $\vdash : \exists ! \omega . \supset . \omega \epsilon \text{NO infin}$ [*262·14 . *263·17·23]

***263·26.** $\vdash : P \epsilon \omega . \exists ! \alpha \cap C'P . \sim \text{E} ! \max_P{}'\alpha . \supset . P \upharpoonright \alpha \epsilon \omega$

Dem.

$\vdash . *263·1 . *205·123 . \supset$

$\vdash : \text{Hp} . \supset . (\exists R) . R \epsilon \text{Prog} . P = R_{po} . \exists ! \alpha \cap C'R . \alpha \cap C'R \subset R_{po}``\alpha .$

$[*122·442·45] \supset . (\exists R) . R \epsilon \text{Prog} . P = R_{po} . P \upharpoonright \alpha \dot{-} (P \upharpoonright \alpha)^2 \epsilon \text{Prog} .$

$P \upharpoonright \alpha = \{P \upharpoonright \alpha \dot{-} (P \upharpoonright \alpha)^2\}_{po} .$

$[*263·1] \qquad \supset . P \upharpoonright \alpha \epsilon \omega : \supset \vdash . \text{Prop}$

***263·27.** $\vdash : P \epsilon \omega . \text{E} ! \max_P{}'\alpha . \supset . P \upharpoonright \alpha \epsilon \Omega \text{ fin}$

Dem.

$\vdash . *122·43 . *263·1 . \qquad \supset \vdash : \text{Hp} . \supset . \alpha \cap C'P \epsilon \text{ Cls induct} .$

$[*37·41 . *120·481] \qquad \supset . C'(P \upharpoonright \alpha) \epsilon \text{ Cls induct}$ (1)

$\vdash . *263·11 . *250·141 . \supset \vdash : \text{Hp} . \supset . P \upharpoonright \alpha \epsilon \Omega$ (2)

$\vdash . (1) . (2) . *261·14·42 . \supset \vdash . \text{Prop}$

***263·28.** $\vdash : P \in \omega . \supset . \mathrm{Ser} \cap \mathrm{Rl}`P \subset \omega \cup \Omega\,\mathrm{fin}$ [*204·421 . *263·26·27]

***263·29.** $\vdash : P \in \omega . Q \in \Omega\,\mathrm{fin} . \supset . Q \,\mathrm{less}\, P$ [*261·65 . *263·23]

***263·3.** $\vdash : P \in \omega . \supset . \overrightarrow{\mathrm{less}}`P = \Omega\,\mathrm{fin}$

Dem.

$\vdash . *254{\cdot}1 . *263{\cdot}17 . \supset$

$\vdash : P \in \omega . Q \,\mathrm{less}\, P . \supset . \exists ! \mathrm{Nr}`Q \cap \mathrm{Rl}`P . Q \sim\in \omega . Q \in \Omega .$

[*263·17] $\supset . (\exists R) . R \in \mathrm{Nr}`Q \cap \mathrm{Rl}`P . R \sim\in \omega .$

[*263·28] $\supset . (\exists R) . R \in \mathrm{Nr}`Q \cap \Omega\,\mathrm{fin} .$

[*261·183] $\supset . Q \in \Omega\,\mathrm{fin}$ (1)

$\vdash . (1) . *263{\cdot}29 . \supset \vdash . \mathrm{Prop}$

***263·31.** $\vdash :. \exists ! \omega . \supset : \alpha \lessdot \omega . \equiv . \alpha \in \mathrm{NO\,fin}$

Dem.

$\vdash . *255{\cdot}17 . *263{\cdot}17 . \supset \vdash :. P \in \omega . \supset : \mathrm{Nr}`Q \lessdot \omega . \equiv . Q \,\mathrm{less}\, P .$

[*263·3] $\equiv . Q \in \Omega\,\mathrm{fin} .$

[*262·13] $\equiv . \mathrm{Nr}`Q \in \mathrm{NO\,fin} :$

[*152·4] $\supset : \alpha \in \mathrm{NR} . \alpha \lessdot \omega . \equiv . \alpha \in \mathrm{NR} . \alpha \in \mathrm{NO\,fin} :$

[*255·12.*262·1.*152·4] $\supset : \alpha \lessdot \omega . \equiv . \alpha \in \mathrm{NO\,fin} :. \supset \vdash . \mathrm{Prop}$

***263·32.** $\vdash :. \mathrm{Infin\,ax} . \supset : \alpha \lessdot \omega . \equiv . \alpha \in \mathrm{NO\,fin}$ [*263·31·12]

***263·33.** $\vdash : \alpha \lessdot \omega . \supset . \alpha \in \mathrm{NO\,fin}$

Dem.

$\vdash . *255{\cdot}1 . *155{\cdot}13 . \supset \vdash : \mathrm{Hp} . \supset . \exists ! \omega$ (1)

$\vdash . (1) . *263{\cdot}31 . \supset \vdash . \mathrm{Prop}$

***263·34.** $\vdash . \dot{1} \dotplus \omega = \omega$

Dem.

$\vdash . *262{\cdot}112 . *263{\cdot}24 . \supset \vdash : \mathrm{Hp} . \exists ! \omega . \supset . \dot{1} \dotplus \omega = \omega$ (1)

$\vdash . *181{\cdot}4 . \quad \supset \vdash : \omega = \Lambda . \supset . \dot{1} \dotplus \omega = \Lambda$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***263·35.** $\vdash : \alpha \in \mathrm{NO\,fin} . \supset . \alpha \dotplus \omega = \omega$

Dem.

$\vdash . *180{\cdot}61 . *263{\cdot}18 . \supset \vdash : \exists ! \omega . \supset . 0_r \dotplus \omega = \omega$ (1)

$\vdash . *180{\cdot}4 . \quad \supset \vdash : \omega = \Lambda . \supset . 0_r \dotplus \omega = \Lambda$ (2)

$\vdash . (1) . (2) . \quad \supset \vdash . 0_r \dotplus \omega = \omega$ (3)

$\vdash . *181{\cdot}57 . *263{\cdot}34 . \supset \vdash . 2_r \dotplus \omega = \dot{1} \dotplus \omega$

[*263·34] $= \omega$ (4)

$\vdash . *262{\cdot}36 . \supset \vdash : \mu \in \mathrm{NC\,induct} - \iota`0 - \iota`1 . \supset . (\mu +_c 1)_r \dotplus \omega = \mu_r \dotplus \dot{1} \dotplus \omega$

[*263·34.*181·58] $= \mu_r \dotplus \omega$ (5)

$\vdash . (5) . \supset \vdash : \mu \in \mathrm{NC\,induct} - \iota`0 - \iota`1 . \mu_r \dotplus \omega = \omega . \supset . (\mu +_c 1)_r \dotplus \omega = \omega$ (6)

$\vdash . (4) . (6) . \mathrm{Induct} . \supset \vdash : \mu \in \mathrm{NC\,induct} - \iota`0 - \iota`1 . \supset . \mu_r \dotplus \omega = \omega$ (7)

$\vdash . (3) . (7) . \quad \supset \vdash : \mu \in \mathrm{NC\,induct} - \iota`1 . \supset . \mu_r \dotplus \omega = \omega :$

[*262·26] $\supset \vdash : \alpha \in \mathrm{NO\,fin} . \supset . \alpha \dotplus \omega = \omega : \supset \vdash . \mathrm{Prop}$

$*263\cdot4.\quad \vdash : P \in \omega . \supset . \mathrm{D}`P_s \subset \Omega\,\mathrm{fin} . \mathrm{Nr}``\mathrm{D}`P_s = \mathrm{NO\,fin}$

Dem.

$\vdash . *254\cdot182 . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{D}`P_s \subset \overrightarrow{\mathrm{less}}`P .$

$[*263\cdot3] \qquad \supset . \mathrm{D}`P_s \subset \Omega\,\mathrm{fin} \quad (1)$

$\vdash . *263\cdot31 . \supset \vdash :. \mathrm{Hp} . \supset : \alpha < \mathrm{Nr}`P . \equiv . \alpha \in \mathrm{NO\,fin} :$

$[*256\cdot11] \qquad \supset : \alpha \in \mathrm{Nr}``\mathrm{D}`P_s . \equiv . \alpha \in \mathrm{NO\,fin} \quad (2)$

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

$*263\cdot401.\quad \vdash : P \in \omega . \alpha \in \mathrm{sect}`P - \iota`\Lambda - \iota`C`P . \supset . \mathrm{E} ! \max_P`\alpha$

Dem.

$\vdash . *250\cdot65 . \supset \vdash : \mathrm{Hp} . \supset . P \upharpoonright \alpha \sim \in \mathrm{Nr}`P .$

$[*263\cdot17] \qquad \supset . P \upharpoonright \alpha \sim \in \omega .$

$[*263\cdot26.\mathrm{Transp}] \qquad \supset . \mathrm{E} ! \max_P`\alpha : \supset \vdash . \mathrm{Prop}$

$*263\cdot402.\quad \vdash : P \in \omega . \supset . \mathrm{sect}`P - \iota`\Lambda - \iota`C`P = \overrightarrow{P_*}``C`P$

Dem.

$\vdash . *205\cdot131\cdot22 . *263\cdot401 . \supset$

$\vdash : \mathrm{Hp} . \alpha \in \mathrm{sect}`P - \iota`\Lambda - \iota`C`P . \supset . \alpha \cup P``\alpha = \overrightarrow{P}`\max_P`\alpha \cup \iota`\max_P`\alpha .$

$[*211\cdot1.*91\cdot54] \qquad \supset . \alpha = \overrightarrow{P_*}`\max_P`\alpha .$

$[*205\cdot111] \qquad \supset . \alpha \in \overrightarrow{P_*}``C`P \quad (1)$

$\vdash . *211\cdot3\cdot13 . \quad \supset \vdash . \overrightarrow{P_*}``C`P \subset \mathrm{sect}`P \quad (2)$

$\vdash . *90\cdot12 . \quad \supset \vdash . \overrightarrow{P_*}``C`P \subset -\iota`\Lambda \quad (3)$

$\vdash . *205\cdot197 . \quad \supset \vdash : \mathrm{Hp} . x \in C`P . \supset . \mathrm{E} ! \max_P`\overrightarrow{P_*}`x .$

$[*263\cdot22] \qquad \supset . \overrightarrow{P_*}`x \neq C`P \quad (4)$

$\vdash . (2) . (3) . (4) . \supset \vdash : \mathrm{Hp} . \supset . \overrightarrow{P_*}``C`P \subset \mathrm{sect}`P - \iota`\Lambda - \iota`C`P \quad (5)$

$\vdash . (1) . (5) . \supset \vdash . \mathrm{Prop}$

$*263\cdot41.\quad \vdash : P \in \omega . \supset . P_s \upharpoonright \mathrm{D}`P_s = P \upharpoonright ; \overrightarrow{P_*} ; P$

Dem.

$\vdash . *213\cdot11\cdot141\cdot151 . \supset$

$\vdash :. \mathrm{Hp} . \supset : Q (P_s \upharpoonright \mathrm{D}`P_s) R . \equiv . (\exists \alpha, \beta) . \alpha, \beta \in \mathrm{sect}`P - \iota`\Lambda - \iota`C`P . \alpha \subset \beta . \alpha \neq \beta .$

$Q = P \upharpoonright \alpha . R = P \upharpoonright \beta .$

$[*263\cdot402]$

$\equiv . (\exists x, y) . x, y \in C`P . \overrightarrow{P_*}`x \subset \overrightarrow{P_*}`y . \overrightarrow{P_*}`x \neq \overrightarrow{P_*}`y . Q = P \upharpoonright \overrightarrow{P_*}`x . R = P \upharpoonright \overrightarrow{P_*}`y .$

$[*200\cdot391]$

$\equiv . (\exists x, y) . x, y \in C`P . \overrightarrow{P_*}`x \subset \overrightarrow{P_*}`y . x \neq y . Q = P \upharpoonright \overrightarrow{P_*}`x . R = P \upharpoonright \overrightarrow{P_*}`y .$

$[*204\cdot32.*90\cdot12]$

$\equiv . (\exists x, y) . x P_* y . x \neq y . \overrightarrow{P_*}`x \subset \overrightarrow{P_*}`y . Q = P \upharpoonright \overrightarrow{P_*}`x . R = P \upharpoonright \overrightarrow{P_*}`y .$

$[*201\cdot14\cdot15] \equiv . (\exists x, y) . x P_* y . x \neq y . Q = P \upharpoonright \overrightarrow{P_*}`x . R = P \upharpoonright \overrightarrow{P_*}`y .$

$[*201\cdot18] \quad \equiv . (\exists x, y) . x P y . Q = P \upharpoonright \overrightarrow{P_*}`x . R = P \upharpoonright \overrightarrow{P_*}`y .$

$[*150\cdot1] \quad \equiv . Q (P \upharpoonright ; \overrightarrow{P_*} ; P) R :. \supset \vdash . \text{Prop}$

$*263\cdot411$. $\vdash : P \in \omega . \supset . C``\mathrm{D}`P_s = \overrightarrow{P_*}``\text{ɑ}`P \cup \iota`\Lambda$

Dem.

$\vdash . *213\cdot141 . *263\cdot402 . \supset$

$\vdash : \text{Hp} . \supset . C``\mathrm{D}`P_s = C``P \upharpoonright``\overrightarrow{P_*}``C`P$

$[*93\cdot103] \qquad = C``P \upharpoonright``\overrightarrow{P_*}``\text{ɑ}`P \cup \iota`C`P \upharpoonright \overrightarrow{P_*}`B`P$

$[*201\cdot521 . *202\cdot55] = \overrightarrow{P_*}``\text{ɑ}`P \cup \iota`C`P \upharpoonright \overrightarrow{P_*}`B`P$

$[*201\cdot521 . *200\cdot35] = \overrightarrow{P_*}``\text{ɑ}`P \cup \iota`\Lambda : \supset \vdash . \text{Prop}$

$*263\cdot412$. $\vdash : P \in \omega . \supset . \overrightarrow{P}`x, \overrightarrow{P_*}`x \in \text{Cls induct}$

Dem.

$\vdash . *205\cdot197 . \supset \vdash : \text{Hp} . x \in C`P . \supset . \mathrm{E} ! \max_P`\overrightarrow{P_*}`x .$

$[*263\cdot27 . *202\cdot55 . *120\cdot213] \quad \supset . \overrightarrow{P_*}`x \in \text{Cls induct} . \qquad (1)$

$[*120\cdot481] \quad \supset . \overrightarrow{P}`x \in \text{Cls induct} \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

$*263\cdot42$. $\vdash : P \in \omega . \supset . \text{sgm}`P = \Lambda \downarrow (C`P)$

Dem.

$\vdash . *212\cdot21 . *211\cdot12 . \supset$

$\vdash :. \text{Hp} . \supset : \alpha (\text{sgm}`P) \beta . \equiv . \alpha = P``\alpha . \beta = P``\beta . \alpha \subset \beta . \alpha \neq \beta \qquad (1)$

$\vdash . (1) . *211\cdot1 . *205\cdot123 . \supset$

$\vdash : \text{Hp} . \alpha (\text{sgm}`P) \beta . \supset . \alpha, \beta \in \text{sect}`P . \sim \mathrm{E} ! \max_P`\alpha . \sim \mathrm{E} ! \max_P`\beta .$

$[*263\cdot401] \quad \supset . \alpha, \beta \in \iota`\Lambda \cup \iota`C`P \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash : \text{Hp} . \alpha (\text{sgm}`P) \beta . \supset . \alpha = \Lambda . \beta = C`P \qquad (3)$

$\vdash . *37\cdot29 . \supset \vdash : \alpha = \Lambda . \supset . \alpha = P``\alpha \qquad (4)$

$\vdash . *263\cdot22 . \supset \vdash : \text{Hp} . \beta = C`P . \supset . \beta = P``\beta \qquad (5)$

$\vdash . (1) . (4) . (5) . \supset \vdash : \text{Hp} . \alpha = \Lambda . \beta = C`P . \supset . \alpha (\text{sgm}`P) \beta \qquad (6)$

$\vdash . (3) . (6) . \supset \vdash . \text{Prop}$

$*263\cdot43$. $\vdash : P \in \omega . \supset . \text{ɑ}`P_1 = \text{ɑ}`P$

Dem.

$\vdash . *263\cdot141 . \supset \vdash : \text{Hp} . \supset . \text{ɑ}`P = \text{ɑ}`(P_1)_{\text{po}}$

$[*91\cdot504] \qquad = \text{ɑ}`P_1 : \supset \vdash . \text{Prop}$

$*263·431$. $\vdash : P \in \Omega - \iota\text{‘}\dot{\Lambda} \,.\, \text{Ɑ}\text{‘}P_1 = \text{Ɑ}\text{‘}P \,.\, \sim E\,!\,B\text{‘}\breve{P} \,.\supset .\, P \in \omega$

Dem.

$\vdash .\, *261·35 \,.\, \text{Transp} \,.\supset \vdash : \text{Hp} \,.\supset .\, P \in \Omega\ \text{infin} \,.$

$[*261·44] \qquad \supset .\, P_1 \upharpoonright \overleftarrow{P}_{\text{fn}}\text{‘}B\text{‘}P \in \text{Prog} \,.$

$[*261·212] \qquad \supset .\, P_1 \upharpoonright \overleftarrow{P}\text{‘}B\text{‘}P \in \text{Prog} \,.$

$[*202·524] \qquad \supset .\, P_1 \in \text{Prog} \qquad (1)$

$\vdash .\, *261·212 \,. \qquad \supset \vdash : \text{Hp} \,.\supset .\, P = (P_1)_{\text{po}} \qquad (2)$

$\vdash .\, (1) \,.\, (2) \,.\, *263·1 \,.\supset \vdash .\, \text{Prop}$

$*263·44$. $\vdash .\, \omega = \Omega - \iota\text{‘}\dot{\Lambda} \cap \hat{P}(\text{Ɑ}\text{‘}P_1 = \text{Ɑ}\text{‘}P \,.\, \sim E\,!\,B\text{‘}\breve{P}) \qquad [*263·43·22·431]$

$*263·45$. $\vdash .\, \omega = \Omega - \iota\text{‘}\dot{\Lambda} \cap \hat{P}(P = P_{\text{fn}} \,.\, \sim E\,!\,B\text{‘}\breve{P}) \qquad [*261·212 \,.\, *263·44]$

$*263·46$. $\vdash .\, \omega = \Omega \cap \hat{P}(E\,!\,B\text{‘}P_1 \,.\, \sim E\,!\,B\text{‘}\breve{P})$

Dem.

$\vdash .\, *121·305 \,.\, *93·101 \,.\supset$

$\vdash : P \in \Omega \,.\, \sim E\,!\,B\text{‘}\breve{P} \,.\, \text{Ɑ}\text{‘}P_1 \neq \text{Ɑ}\text{‘}P \,.\supset .\, \exists\,!\,\text{Ɑ}\text{‘}P - \text{Ɑ}\text{‘}P_1 \,.\, \text{Ɑ}\text{‘}P = D\text{‘}P - \iota\text{‘}B\text{‘}P \,.$

$[*250·21] \qquad \supset .\, \exists\,!\,D\text{‘}P_1 - \text{Ɑ}\text{‘}P_1 - \iota\text{‘}B\text{‘}P \,.$

$[*93·101] \qquad \supset .\, \exists\,!\,\overrightarrow{B}\text{‘}P_1 - \iota\text{‘}B\text{‘}P \qquad (1)$

$\vdash .\, *121·305 \,.\, *250·21 \,.\supset \vdash : P \in \Omega - \iota\text{‘}\dot{\Lambda} \,.\supset .\, B\text{‘}P \in \overrightarrow{B}\text{‘}P_1 \qquad (2)$

$\vdash .\, (1) \,.\, (2) \,. \qquad \supset \vdash : P \in \Omega \,.\, \sim E\,!\,B\text{‘}\breve{P} \,.\, \text{Ɑ}\text{‘}P_1 \neq \text{Ɑ}\text{‘}P \,.\supset .\, \overrightarrow{B}\text{‘}P_1 \sim\in 1 \,.$

$[*53·3] \qquad \supset .\, \sim E\,!\,B\text{‘}P_1 \qquad (3)$

$\vdash .\, (3) \,.\, \text{Transp} \,.\supset \vdash : P \in \Omega \,.\, E\,!\,B\text{‘}P_1 \,.\, \sim E\,!\,B\text{‘}\breve{P} \,.\supset .\, \text{Ɑ}\text{‘}P_1 = \text{Ɑ}\text{‘}P \,.$

$[*263·44] \qquad \supset .\, P \in \omega \qquad (4)$

$\vdash .\, *250·21 \,.\, *263·44 \,.\supset \vdash : P \in \omega \,.\supset .\, \overrightarrow{B}\text{‘}P_1 = \overrightarrow{B}\text{‘}P \,.$

$[*250·13] \qquad \supset .\, E\,!\,B\text{‘}P_1 \qquad (5)$

$\vdash .\, (5) \,.\, *263·44 \,. \qquad \supset \vdash : P \in \omega \,.\supset .\, E\,!\,B\text{‘}P_1 \,.\, \sim E\,!\,B\text{‘}\breve{P} \qquad (6)$

$\vdash .\, (4) \,.\, (6) \,.\supset \vdash .\, \text{Prop}$

$*263·47$. $\vdash .\, \omega = \Omega \cap \hat{P}\{\alpha \subset C\text{‘}P \,.\supset_\alpha : \alpha \in \text{Cls induct} \,.\equiv .\, \exists\,!\,C\text{‘}P \cap p\text{‘}\overleftarrow{P}\text{‘‘}\alpha\}$

Dem.

$\vdash .\, *254·52 \,.\supset \vdash : P \in \omega \,.\, \alpha \subset C\text{‘}P \,.\, \exists\,!\,C\text{‘}P \cap p\text{‘}\overleftarrow{P}\text{‘‘}\alpha \,.\supset .\, (P \lfloor \alpha)\ \text{less}\ P \,.$

$[*263·3] \qquad \supset .\, P \lfloor \alpha \in \Omega\ \text{fin} \,.$

$[*261·42·14] \qquad \supset .\, C\text{‘}(P \lfloor \alpha) \in \text{Cls induct} \,.$

$[*202·55 . *120·213] \qquad \supset .\, \alpha \in \text{Cls induct} \qquad (1)$

$\vdash .\, *261·26 \,.\supset \vdash : P \in \omega \,.\, \alpha \subset C\text{‘}P \,.\, \alpha \in \text{Cls induct} \,.\, \exists\,!\,\alpha \,.\supset .\, E\,!\,\max_P\text{‘}\alpha \,.$

$[*263·22] \qquad \supset .\, \exists\,!\,\overleftarrow{P}\text{‘}\max_P\text{‘}\alpha \,.$

$[*205·65 . *40·69] \qquad \supset .\, \exists\,!\,C\text{‘}P \cap p\text{‘}\overleftarrow{P}\text{‘‘}\alpha \qquad (2)$

$\vdash . (1) . (2) . *40\cdot2 . \supset$

$\vdash :. P \epsilon \omega . \alpha \subset C'P . \supset : \alpha \epsilon \text{Cls induct} . \equiv . \exists ! C'P \cap p'\overleftarrow{P}``\alpha$ (3)

$\vdash . *40\cdot2 . *120\cdot212 . \supset$

$\vdash :: P \epsilon \Omega :. \alpha \subset C'P . \supset_\alpha : \alpha \epsilon \text{Cls induct} . \equiv . \exists ! C'P \cap p'\overleftarrow{P}``\alpha :. \supset . \dot{\exists} ! P$ (4)

$\vdash . (4) . *200\cdot51 . \supset \vdash : \text{Hp}(4) . \supset . C'P \sim\epsilon \text{Cls induct}$ (5)

$\vdash . *250\cdot16 . \supset$

$\vdash : \text{Hp}(4) . \exists ! \text{G}'P - \text{G}'P_1 . \supset . \overrightarrow{P}'\text{min}_P{}'(\text{G}'P - \text{G}'P_1) \epsilon \text{Cls induct} .$

[$*261\cdot26$] $\supset . \text{E} ! \max_P{}'\overrightarrow{P}'\text{min}_P{}'(\text{G}'P - \text{G}'P_1) .$

[$*205\cdot252$] $\supset . \text{min}_P{}'(\text{G}'P - \text{G}'P_1) \epsilon \text{G}'P_1$ (6)

$\vdash . (6) . \text{Transp} . \quad \supset \vdash : \text{Hp}(4) . \supset . \text{G}'P_1 = \text{G}'P$ (7)

$\vdash . (5) . (7) . *261\cdot34 . \supset \vdash : \text{Hp}(4) . \supset . \sim \text{E} ! B'\breve{P}$ (8)

$\vdash . (4) . (7) . (8) . \quad \supset \vdash : \text{Hp}(4) . \supset . P \epsilon \omega$ (9)

$\vdash . (3) . (9) . \supset \vdash . \text{Prop}$

$*263\cdot48$. $\vdash . \omega = \Omega \cap \hat{P} \{\alpha \subset C'P . \supset_\alpha : \alpha \sim\epsilon \text{Cls refl} . - . \exists ! C'P \cap p'\overleftarrow{P}``\alpha\}$ [$*263\cdot47 . *261\cdot47$]

$*263\cdot49$. $\vdash . \Omega \,\text{fin} \cup \omega = \Omega \cap \hat{P}(\text{G}'P_1 = \text{G}'P) = \Omega \cap \hat{P}(P = P_{\text{fn}})$

Dem.

$\vdash . *261\cdot22 . *263\cdot44 . \supset \vdash : P \epsilon \Omega \,\text{fin} \cup \omega . \supset . \text{G}'P_1 = \text{G}'P$ (1)

$\vdash . *261\cdot34 . *263\cdot44 . \supset \vdash : P \epsilon \Omega . \text{G}'P_1 = \text{G}'P . \supset . P \epsilon \Omega \,\text{fin} \cup \omega$ (2)

$\vdash . (1) . (2) . \quad \supset \vdash . \Omega \,\text{fin} \cup \omega = \Omega \cap \hat{P}(\text{G}'P_1 = \text{G}'P)$

[$*261\cdot212$] $= \Omega \cap \hat{P}(P = P_{\text{fn}}) . \supset \vdash . \text{Prop}$

$*263\cdot491$. $\vdash : P \epsilon \Omega \,\text{fin} \cup \omega . \supset . P = (P_1)_{\text{po}} . P_\sigma = (P_1)_\sigma$

Dem.

$\vdash . *263\cdot49 . *261\cdot212 . \supset \vdash : \text{Hp} . \supset . P = (P_1)_{\text{po}} .$ (1)

[$*91\cdot602 . *121\cdot103$] $\supset . P(x \mapsto y) = P_1(x \mapsto y) .$

[$*121\cdot11$] $\supset . P_\sigma = (P_1)_\sigma$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

$*263\cdot5$. $\vdash : P \epsilon \Omega \,\text{infin} . \supset . P \,\text{⊑}\, (\iota'B'P \cup \overleftarrow{P}_{\text{fn}}{}'B'P) \epsilon \omega$

Dem.

$\vdash . *261\cdot45 . \quad \supset \vdash : \text{Hp} . \supset . P_1 \upharpoonright \overleftarrow{P}_{\text{fn}}{}'B'P \epsilon \text{Prog}$ (1)

$\vdash . *260\cdot33\cdot27 . \supset \vdash : \text{Hp} . \supset . (P_1 \upharpoonright \overleftarrow{P}_{\text{fn}}{}'B'P)_{\text{po}} = P_{\text{fn}} \,\text{⊑}\, (\iota'B'P \cup \overleftarrow{P}_{\text{fn}}{}'B'P)$

[$*260\cdot32$] $= P \,\text{⊑}\, (\iota'B'P \cup \overleftarrow{P}_{\text{fn}}{}'B'P)$ (2)

$\vdash . (1) . (2) . *263\cdot1 . \supset \vdash . \text{Prop}$

***263·51.** $\vdash : P \epsilon \Omega \,\text{infin} . \supset .$

$$\iota \text{`} B \text{`} P \cup \overleftarrow{P}_{\text{fn}} \text{`} B \text{`} P \epsilon \text{D`}(P_\epsilon \dot{\frown} I) . \iota \text{`} B \text{`} P \cup \overleftarrow{P}_{\text{fn}} \text{`} B \text{`} P \epsilon \text{C`sgm`} P$$

Dem.

$\vdash . *263\cdot5\cdot22 . \supset \vdash : \text{Hp} . \supset . \sim \text{E} ! \max_P \text{`}(\iota \text{`} B \text{`} P \cup \overleftarrow{P}_{\text{fn}} \text{`} B \text{`} P)$ (1)

$\vdash . *260\cdot11 . \supset \vdash : \text{Hp} . y \epsilon \text{C`} P - \overleftarrow{P}_{\text{fn}} \text{`} B \text{`} P . x \epsilon \overleftarrow{P}_{\text{fn}} \text{`} B \text{`} P . \supset .$

$P(B\text{`}P \mapsto y) \sim \epsilon \text{Cls induct} . P(B\text{`}P \mapsto x) \epsilon \text{Cls induct} .$

[*120·49] $\supset . \text{Nc`} P(B\text{`}P \mapsto y) > \text{Nc`} P(B\text{`}P \mapsto x)$

[*117·222.Transp] $\supset . \sim (yPx)$ (2)

$\vdash . (2) . \text{Transp} . \supset \vdash : \text{Hp} . \supset . P \text{``} \overleftarrow{P}_{\text{fn}} \text{`} B \text{`} P \subset \overrightarrow{B} \text{`} P \cup \overleftarrow{P}_{\text{fn}} \text{`} B \text{`} P$ (3)

$\vdash . (3) . *93\cdot101 . \supset \vdash : \text{Hp} . \supset . P \text{``}(\iota \text{`} B \text{`} P \subset \overleftarrow{P}_{\text{fn}} \text{`} B \text{`} P) \subset \iota \text{`} B \text{`} P \cup \overleftarrow{P}_{\text{fn}} \text{`} B \text{`} P$ (4)

$\vdash . (1) . (4) . *211\cdot41 . \supset \vdash : \text{Hp} . \supset . \iota \text{`} B \text{`} P \cup \overleftarrow{P}_{\text{fn}} \text{`} B \text{`} P \epsilon \text{D`}(P_\epsilon \dot{\frown} I) .$ (5)

[*212·152] $\supset . \iota \text{`} B \text{`} P \cup \overleftarrow{P}_{\text{fn}} \text{`} B \text{`} P \epsilon \text{C`sgm`} P$ (6)

$\vdash . (5) . (6) . \supset \vdash . \text{Prop}$

***263·52.** $\vdash : P \epsilon \Omega \,\text{infin} - \omega . \supset . (\exists x) . x \epsilon \text{C`} P . \overleftarrow{P}_{\text{fn}} \text{`} B \text{`} P \cup \iota \text{`} B \text{`} P = \overrightarrow{P} \text{`} x$

Dem.

$\vdash . *263\cdot49 . \text{Transp} . \supset \vdash : \text{Hp} . \supset . \exists ! \text{C`} P - \text{C`} P_1 .$

[*260·27] $\supset . \exists ! \text{C`} P - \overleftarrow{P}_{\text{fn}} \text{`} B \text{`} P .$

[*250·121] $\supset . \text{E} ! \min_P \text{`}(\text{C`} P - \overleftarrow{P}_{\text{fn}} \text{`} B \text{`} P) .$

[*263·51.*206·25.*211·726] $\supset . (\exists x) . x \epsilon \text{C`} P . \overleftarrow{P}_{\text{fn}} \text{`} B \text{`} P \cup \iota \text{`} B \text{`} P = \overrightarrow{P} \text{`} x :$

$\supset \vdash . \text{Prop}$

***263·53.** $\vdash : P \epsilon \Omega \,\text{infin} - \omega . \supset . \text{Nr`} P > \omega$

Dem.

$\vdash . *253\cdot13 . *263\cdot52 . \supset \vdash : \text{Hp} . \supset . P \upharpoonright (\overleftarrow{P}_{\text{fn}} \text{`} B \text{`} P \cup \iota \text{`} B \text{`} P) \epsilon \text{D`} P_s .$

[*263·5] $\supset . \exists ! \omega \cap \text{D`} P_s .$

[*255·17.*263·18] $\supset . \text{Nr`} P > \omega : \supset \vdash . \text{Prop}$

The above proposition shows that ω is the smallest of infinite ordinals. The same fact is otherwise expressed by the following proposition.

***263·54.** $\vdash : \alpha \epsilon \text{NO infin} - \iota \text{`} \omega . \supset . \alpha > \omega$ [*263·53]

***263·55.** $\vdash : P \epsilon \omega . \supset . P_s \epsilon \omega \dot{+} \dot{1} . \text{s`} P \epsilon \omega \dot{+} \dot{1}$

Dem.

$\vdash . *253\cdot511 . *263\cdot44 . \supset \vdash : \text{Hp} . \supset . P_s \epsilon \omega \dot{+} \dot{1}$ (1)

$\vdash . *252\cdot372 . *263\cdot44 . \supset \vdash : \text{Hp} . \supset . \text{s`} P \epsilon \omega \dot{+} \dot{1}$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

The following propositions are lemmas for proving $\omega \dot{\times} 2_r = \omega$ (*263·63).

***263·6.** $\vdash :: P \in \mathrm{Ser} . x \neq y . M = P \times (x \downarrow y) . \supset :. RM_1 S . \equiv :$
$(\exists u) . u \in C\text{‘}P . R = x \downarrow u . S = y \downarrow u . \vee . (\exists u, v) . uP_1 v . R = y \downarrow u . S = x \downarrow v$

Dem.

$\vdash . *166·111 . \supset \vdash :. \mathrm{Hp} . uPv . R = x \downarrow u : S = x \downarrow v . \vee . S = y \downarrow v : \supset .$
$RM(y \downarrow u) . (y \downarrow u) MS .$
$[*201·63 . *204·55] \quad \supset . \sim (RM_1 S)$ (1)
Similarly $\vdash :. \mathrm{Hp} . uPv . R = y \downarrow u . S = y \downarrow v . \supset . \sim (RM_1 S)$ (2)
$\vdash . *166·111 . \supset$
$\vdash : \mathrm{Hp} . uPw . wPv . R = y \downarrow u . S = x \downarrow v . \supset . RM(x \downarrow w) . (x \downarrow w) MS .$
$[*201·63 . *204·55] \quad \supset . \sim (RM_1 S)$ (3)
$\vdash .\,(1) . (2) . (3) . \mathrm{Transp} . *166·111 . \supset$
$\vdash :. \mathrm{Hp} . RM_1 S . \supset : (\exists u) . R = x \downarrow u . S = y \downarrow u . u \in C\text{‘}P . \vee .$
$(\exists u, v) . uP_1 v . R = y \downarrow u . S = x \downarrow v$ (4)
$\vdash . *166·111 . \supset \vdash : \mathrm{Hp} . R = x \downarrow u . S = y \downarrow u . RM(x \downarrow v) . \supset . SM(x \downarrow v)$ (5)
$\vdash . *166·111 . \supset \vdash :. \mathrm{Hp} . R = x \downarrow u . S = y \downarrow u . RM(y \downarrow v) . \supset : u = v . \vee . uPv :$
$[*166·111] \quad \supset : y \downarrow v = S . \vee . SM(y \downarrow v)$ (6)
$\vdash . *166·111 . \supset$
$\vdash : \mathrm{Hp} . R = y \downarrow u . S = x \downarrow v . uP_1 v . RM(y \downarrow w) . \supset . SM(y \downarrow w)$ (7)
$\vdash . *166·111 . \supset \vdash :. \mathrm{Hp} . R = y \downarrow u . S = x \downarrow v . uP_1 v . RM(x \downarrow w) . \supset :$
$x \downarrow w = S . \vee . SM(x \downarrow w)$ (8)
$\vdash . (5) . (6) . (7) . (8) . \supset \vdash :. \mathrm{Hp} : u \in C\text{‘}P . R = x \downarrow u . S = y \downarrow u . \vee .$
$uP_1 v . R = y \downarrow u . S = x \downarrow v : \supset . RM_1 S$ (9)
$\vdash . (4) . (9) . \supset \vdash . \mathrm{Prop}$

***263·61.** $\vdash : P \in \mathrm{Ser} . x \neq y . M = P \times (x \downarrow y) . \supset . \mathrm{G}\text{‘}M_1 = y \downarrow \text{“}C\text{‘}P \cup x \downarrow \text{“}\mathrm{G}\text{‘}P_1$
[*263·6]

***263·62.** $\vdash : P \in \omega . x \neq y . \supset . P \times (x \downarrow y) \in \omega$

Dem.

$\vdash . *263·61·43 . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{G}\text{‘}\{P \times (x \downarrow y)\}_1 = y \downarrow \text{“}C\text{‘}P \cup x \downarrow \text{“}\mathrm{G}\text{‘}P$
$[*166·111] \quad = \mathrm{G}\text{‘}\{P \times (x \downarrow y)\}$ (1)
$\vdash . *251·55 . \quad \supset \vdash : \mathrm{Hp} . \supset . P \times (x \downarrow y) \in \Omega$ (2)
$\vdash . *166·14 . \quad \supset \vdash : \mathrm{Hp} . \supset . P \times (x \downarrow y) \in - \iota\text{‘}\Lambda$ (3)
$\vdash . *166·16 . *263·22 . \supset \vdash : \mathrm{Hp} . \supset . \overrightarrow{B}\text{‘}\mathrm{Cnv}\text{‘}\{P \times (x \downarrow y)\} = \Lambda$ (4)
$\vdash . (1) . (2) . (3) . (4) . *263·44 . \supset \vdash : \mathrm{Hp} . \supset . P \times (x \downarrow y) \in \omega : \supset \vdash . \mathrm{Prop}$

***263·63.** $\vdash . \omega \dot{\times} 2_r = \omega$

Dem.

$\vdash . *263·62·17 . \quad \supset \vdash : P \in \omega . Q \in 2_r . \supset . \mathrm{Nr}\text{‘}(P \times Q) = \omega$ (1)
$\vdash . *184·13 . *263·17 . \supset \vdash : P \in \omega . Q \in 2_r . \supset . \mathrm{Nr}\text{‘}(P \times Q) = \omega \dot{\times} 2_r$ (2)
$\vdash . (1) . (2) . \quad \supset \vdash : \exists ! \omega . \exists ! 2_r . \supset . \omega \dot{\times} 2_r = \omega$ (3)
$\vdash . *184·11 . \quad \supset \vdash : \omega = \Lambda . \supset . \omega \dot{\times} 2_r = \Lambda$ (4)

$\vdash . *123 \cdot 14 . *263 \cdot 101 . \supset \vdash : \exists ! \omega . \supset . \exists ! 2 .$

$[*262 \cdot 21] \qquad \supset . \exists ! 2_r \qquad (5)$

$\vdash . (3) . (4) . (5) . \supset \vdash . \text{Prop}$

The following propositions are lemmas for proving $*263 \cdot 66$.

$*263 \cdot 64$. $\vdash : P, Q \epsilon \text{Ser} . x \epsilon C'P . zQ_1 w . M = P \times Q . \supset . (z \downarrow x) M_1 (w \downarrow x)$

Dem.

$\vdash . *166 \cdot 111 . \supset \vdash : \text{Hp} . \supset . (z \downarrow x) M (w \downarrow x) \qquad (1)$

$\vdash . *166 \cdot 111 . \supset \vdash :. \text{Hp} . (z \downarrow x) M (u \downarrow y) . \supset : xPy . \vee . x = y . zQu :$

$[*204 \cdot 71] \qquad \supset : xPy . \vee . x = y . u = w . \vee . x = y . wQy :$

$[*166 \cdot 111] \qquad \supset : (w \downarrow x) M (u \downarrow y) . \vee . (w \downarrow x) = (u \downarrow y) \qquad (2)$

$\vdash . (2) . *204 \cdot 55 . \supset \vdash : \text{Hp} (2) . \supset . \sim \{(u \downarrow y) M (w \downarrow x)\} \qquad (3)$

$\vdash . (1) . (3) . *201 \cdot 63 . \supset \vdash . \text{Prop}$

$*263 \cdot 641$. $\vdash : P, Q \epsilon \text{Ser} . z = B'\breve{Q} . w = B'Q . xP_1 y . M = P \times Q . \supset .$

$(z \downarrow x) M_1 (w \downarrow y)$

Dem.

$\vdash . *166 \cdot 111 . \supset \vdash : \text{Hp} . \supset . (z \downarrow x) M (w \downarrow y) \qquad (1)$

$\vdash . *166 \cdot 111 . \supset \vdash :. \text{Hp} . (z \downarrow x) M (u \downarrow v) . \supset : xPv :$

$[*204 \cdot 71] \qquad \supset : v = y . \vee . yPv \qquad (2)$

$\vdash . (2) . *166 \cdot 111 . \supset$

$\vdash :. \text{Hp} . (z \downarrow x) M (u \downarrow v) . \supset : u \downarrow v = w \downarrow y . \vee . (w \downarrow y) M (u \downarrow v) :$

$[*204 \cdot 55] \qquad \supset : \sim \{(u \downarrow v) M (w \downarrow y)\} \qquad (3)$

$\vdash . (1) . (3) . *201 \cdot 63 . \supset \vdash . \text{Prop}$

$*263 \cdot 642$. $\vdash : P, Q \epsilon \text{Ser} . M = P \times Q . \supset . (C'P \times \text{G}'Q_1) \subset \text{G}'M_1 \quad [*263 \cdot 64]$

$*263 \cdot 643$. $\vdash : P, Q \epsilon \text{Ser} . \text{E} ! B'Q . \text{E} ! B'\breve{Q} . M = P \times Q . \supset . (B'Q) \downarrow "\text{G}'P_1 \subset \text{G}'M_1$ $[*263 \cdot 64]$

$*263 \cdot 65$. $\vdash : P \epsilon \omega . Q \epsilon \Omega \text{ fin} - \iota'\Lambda . \supset . P \times Q \epsilon \omega$

Dem.

$\vdash . *251 \cdot 55 . \supset \vdash : \text{Hp} . \supset . P \times Q \epsilon \Omega \qquad (1)$

$\vdash . *166 \cdot 14 . \supset \vdash : \text{Hp} . \supset . P \times Q \epsilon - \iota'\Lambda \qquad (2)$

$\vdash . *263 \cdot 642 \cdot 643 . *261 \cdot 24 . \supset$

$\vdash : \text{Hp} . \supset . (C'P \times \text{G}'Q_1) \cup (B'Q) \downarrow "\text{G}'P_1 \subset \text{G}'(P_1 \times Q)_1 .$

$[*263 \cdot 49] \qquad \supset . (C'P \times \text{G}'Q) \cup (B'Q) \downarrow "\text{G}'P \subset \text{G}'(P \times Q)_1 .$

$[*166 \cdot 12 \cdot 16] \qquad \supset . C'(P \times Q) - \overrightarrow{B}'(P \times Q) \subset \text{G}'(P \times Q)_1 .$

$[*93 \cdot 101 . *201 \cdot 63] \supset . \text{G}'(P \times Q) = \text{G}'(P \times Q)_1 \qquad (3)$

$\vdash . *166 \cdot 16 . *263 \cdot 22 . \supset \vdash : \text{Hp} . \supset . \overrightarrow{B}'\text{Cnv}'(P \times Q) = \Lambda \qquad (4)$

$\vdash . (1) . (2) . (3) . (4) . *263 \cdot 44 . \supset \vdash . \text{Prop}$

$*263 \cdot 66$. $\vdash : \alpha \epsilon \text{NO fin} - \iota'0_r . \supset . \omega \dot{\times} \alpha = \omega \quad [*263 \cdot 65]$

The proof proceeds as in $*263 \cdot 63$.

$*264$. DERIVATIVES OF WELL-ORDERED SERIES.

Summary of $*264$.

The principal purpose of the present number is to show that every infinite well-ordered series is the sum of a series of progressions followed by a finite tag, which may be null. For this purpose, we proceed as follows: If x is any member of $C\text{‘}P$, it must belong to the family, with respect to P_1, of some member of $C\text{‘}P - \text{Ɑ}\text{‘}P_1$, unless $x = B\text{‘}\breve{P}$ and $B\text{‘}\breve{P} \sim\epsilon\, \text{Ɑ}\text{‘}P_1$. Assuming that we have either $\sim E\,!\, B\text{‘}\breve{P}$ or $B\text{‘}\breve{P} \,\epsilon\, \text{Ɑ}\text{‘}P_1$, and assuming further that P is an infinite well-ordered series other than a progression, it follows that every member of $C\text{‘}P$ belongs to the family, with respect to P_1, of some member of $C\text{‘}\nabla\text{‘}P$, because, by $*216{\cdot}611$, $C\text{‘}\nabla\text{‘}P = \text{D}\text{‘}P_1 - \text{Ɑ}\text{‘}P_1$ in the circumstances contemplated ($*264{\cdot}15$). Now P limited to any one family with respect to P_1 is a progression, unless that family includes $B\text{‘}\breve{P}$; and if it includes $B\text{‘}\breve{P}$, it is finite. Hence our proposition follows.

An important consequence of the above proposition is that every cardinal which is not inductive and is applicable to classes that can be well-ordered is a multiple of $\aleph_0$ ($*264{\cdot}48$).

For the purposes of this number we need a notation for the series of series each of which consists of the family of some member of $C\text{‘}\nabla\text{‘}P$. We therefore put

$$P_{\text{pr}} = P \upharpoonright ; \overleftarrow{(P_1)_*} ; \nabla\text{‘}P \quad \text{Dft } [*264].$$

Here "pr" is intended to suggest "progression." When $P \,\epsilon\, \Omega \text{ infin} - \omega$, P_{pr} is the series of progressions (possibly ending in a finite tag) whose sum is P (or $P \upharpoonright \text{D}\text{‘}P$, in one case). Before using this definition, some preliminary considerations are necessary. $\nabla\text{‘}P$ is the series of limit-points of P, including $B\text{‘}P$. In order that $\nabla\text{‘}P$ may exist, there must be at least one limit-point besides $B\text{‘}P$. Now the limit-points of a series are $C\text{‘}P - \text{Ɑ}\text{‘}P_1$, *i.e.* the limit-points other than $B\text{‘}P$ are $\text{Ɑ}\text{‘}P - \text{Ɑ}\text{‘}P_1$ ($*216{\cdot}21$). Hence when $B\text{‘}P$ exists and $\text{Ɑ}\text{‘}P - \text{Ɑ}\text{‘}P_1$ exists, $\nabla\text{‘}P$ exists. Hence by $*263{\cdot}49$,

$*264{\cdot}13$. $\vdash :. P \,\epsilon\, \Omega \,.\supset :\, \exists\,!\, \nabla\text{‘}P \,.\equiv .\, P \,\epsilon\, \Omega \text{ infin} - \omega$

I.e. a well-ordered series whose derivative exists is one which is infinite and not a progression. We have similarly

*264·14. $\vdash : P \epsilon \Omega \operatorname{infin} - \omega . \supset . C'\nabla'P = C'P - \mathrm{Cl}'P_1$

and

*264·12. $\vdash : P \epsilon \Omega . \supset . \mathrm{Cl}'\nabla'P = \mathrm{Cl}'P - \mathrm{Cl}'P_1$

We next proceed (*264·2—·261) to study the posterity of a term x with respect to P_1, *i.e.* the series $P \upharpoonright \overleftarrow{(P_1)_*}'x$. We show that if this series has a last term, it is finite (*264·21), and ends with $B'\breve{P}$ (*264·2), while if not, and if $x \epsilon C'P_1$, *i.e.* if x has either an immediate successor or an immediate predecessor, the series is a progression (*264·22). Hence we have

*264·23. $\vdash :. P \epsilon \Omega . x \epsilon C'\nabla'P \cap C'P_1 . \supset :$

$$\mathrm{E} ! \max_P{}'\overleftarrow{(P_1)_*}'x . \equiv . x = B'\mathrm{Cnv}'\nabla'P . \mathrm{E} ! B'\breve{P}$$

Moreover, if $x \epsilon C'P_1$, the ancestry of x with respect to P_1 must end with a member of the derivative of P, *i.e.*

*264·233. $\vdash : P \epsilon \Omega \operatorname{infin} - \omega . x \epsilon C'P_1 . \supset . \min_P{}'\overrightarrow{(P_1)_*}'x \epsilon C'\nabla'P$

We thus arrive at the result that if P has a last term, so has $\nabla'P$ (*264·24), and if x is any member of the derivative except the last, the series $P \upharpoonright \overleftarrow{(P_1)_*}'x$ is a progression (*264·25), while if x is the last term of the derivative, and the series P has a last term, then $P \upharpoonright \overleftarrow{(P_1)_*}'x$ is finite (*264·252). Moreover the supposition that P ends with a member of the derivative is equivalent to the supposition that P ends with a term which has no immediate predecessor (*264·26).

We now proceed (*264·3—·403) to consider the relation P_{pr} defined above. If we take any term y in a well-ordered series, there is some term x belonging to $C'P - \mathrm{Cl}'P_1$ such that the family of y with respect to P_1 is the posterity of x. This results from *264·233 above. Thus we may divide the field of P into mutually exclusive stretches, each of which is the posterity of some member of $C'P - \mathrm{Cl}'P_1$ with respect to P_1. The series of series thus obtained is P_{pr}. There is an exceptional case, when the series ends in a term having no immediate predecessor, for then the posterity of this term with respect to P_1 is null, and therefore P_{pr} omits this term. Otherwise, we shall have $\Sigma'P_{\mathrm{pr}} = P$; *i.e.* we have

*264·39. $\vdash : P \epsilon \Omega \operatorname{infin} - \omega . \sim (B'\breve{P} \epsilon C'\nabla'P) . \supset . \Sigma'P_{\mathrm{pr}} = P$

*264·391. $\vdash : P \epsilon \Omega . B'\breve{P} \epsilon C'\nabla'P . \supset . \Sigma'P_{\mathrm{pr}} = P \upharpoonright \mathrm{D}'P$

Moreover we have

*264·36. $\vdash : P \epsilon \Omega . \supset . P_{\mathrm{pr}} \operatorname{smor} \nabla'P . P_{\mathrm{pr}} \epsilon \mathrm{Rel}^2 \operatorname{excl}$

From what was proved earlier we know that, assuming $P \epsilon \Omega$, we have $\text{D}`P_{\text{pr}} \subset \omega$ (*264·401); if $P$ has no last term, $C`P_{\text{pr}} \subset \omega$; if P is infinite and has a last term, $B`\breve{P}_{\text{pr}}$ is finite, and if the last term of $P$ belongs to $C`\nabla`P$, $B`\breve{P}_{\text{pr}} = \dot{\Lambda}$. Hence, using *251·63, which assures us that, in virtue of *264·36 above, if $C`P_{\text{pr}} \subset \omega$, $\Sigma`P_{\text{pr}}$ is a multiple of ω, we find (*264·44) that every well-ordered series has an ordinal number of the form $(\alpha \dot{\times} \omega) \dot{+} \beta$, where α and β may be any ordinals, including 0_r and $\dot{1}$ (putting $\dot{1} \dot{\times} \alpha = \alpha$ to avoid exceptional cases). The above account omits the exceptional cases, which require special treatment and render the proof long; but in the end the above simple result is obtained.

Since a multiple of $\aleph_0$ is not increased by the addition of an inductive cardinal, it follows (*264·44) that the cardinal number of the field of an infinite well-ordered series is always a multiple of $\aleph_0$ (*264·47). Hence if all classes can be well-ordered, all cardinals which are not inductive are multiples of $\aleph_0$. In virtue of Zermelo's theorem, the same result follows if the multiplicative axiom is true.

***264·01.** $P_{\text{pr}} = P \upharpoonright ; \overleftarrow{\overline{(P_1)_*}} ; \nabla`P$ Dft [*264]

***264·11.** $\vdash :. P \epsilon \Omega . \supset : \dot{\exists} ! \text{sgm}`P . \equiv . P \epsilon \Omega \text{ infin}$

Dem.

$\vdash . *263{\cdot}51 . \quad \supset \vdash : P \epsilon \Omega \text{ infin} . \supset . \dot{\exists} ! \text{sgm}`P \qquad (1)$

$\vdash . *212{\cdot}152 . *211{\cdot}41 . \supset \vdash : P \epsilon \Omega . \dot{\exists} ! \text{sgm}`P . \supset . \exists ! \text{sect}`P - \iota`\Lambda - \text{Cl}`\text{max}_P .$

$[*261{\cdot}28.\text{Transp}] \quad \supset . P \epsilon \Omega \text{ infin} \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***264·12.** $\vdash : P \epsilon \Omega . \supset . \text{Cl}`\nabla`P = \text{Cl}`P - \text{Cl}`P_1$

Dem.

$\vdash . *216{\cdot}61 . \quad \supset \vdash : \text{Hp} . \dot{\exists} ! P . \supset . \text{Cl}`\nabla`P = \text{Cl}`P - \text{Cl}`P_1 \qquad (1)$

$\vdash . *216{\cdot}612 . *33{\cdot}241 . \supset \vdash : P = \dot{\Lambda} . \supset . \text{Cl}`\nabla`P = \Lambda . \text{Cl}`P - \text{Cl}`P_1 = \Lambda \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***264·13.** $\vdash :. P \epsilon \Omega . \supset : \dot{\exists} ! \nabla`P . \equiv . P \epsilon \Omega \text{ infin} - \omega$

Dem.

$\vdash . *264{\cdot}12 . \supset \vdash :. \text{Hp} . \supset : \dot{\exists} ! \nabla`P . \equiv . \exists ! \text{Cl}`P - \text{Cl}`P_1 .$

$[*263{\cdot}49] \quad \equiv . P \epsilon \Omega \text{ infin} - \omega : \supset \vdash . \text{Prop}$

***264·14.** $\vdash : P \epsilon \Omega \text{ infin} - \omega . \supset . C`\nabla`P = C`P - \text{Cl}`P_1 \quad [*264{\cdot}13 . *216{\cdot}611]$

***264·15.** $\vdash :. P \epsilon \Omega \text{ infin} - \omega : \sim E ! B`\breve{P} . \text{v} . B`\breve{P} \epsilon \text{Cl}`P_1 : \supset . C`\nabla`P = \overrightarrow{B}`P_1$

Dem.

$\vdash . *264{\cdot}14 . *93{\cdot}103 . \supset \vdash : \text{Hp} . \sim E ! B`\breve{P} . \supset . C`\nabla`P = C`P - \text{Cl}`P_1 . C`P = \text{D}`P .$

$[*93{\cdot}101 . *250{\cdot}21] \quad \supset . C`\nabla`P = \overrightarrow{B}`P_1 \qquad (1)$

$\vdash . *93 \cdot 101 . \quad \supset \vdash : B'\breve{P} \epsilon \text{Œ}'P_1 . \supset . C'P - \text{Œ}'P_1 \subset \text{D}'P$ (2)

$\vdash . (2) . *264 \cdot 14 . \supset \vdash : \text{Hp} . B'\breve{P} \epsilon \text{Œ}'P_1 . \supset . C'\nabla'P = \text{D}'P - \text{Œ}'P_1$

$[*93 \cdot 101 . *250 \cdot 21] \qquad = \overrightarrow{B}'P_1$ (3)

$\vdash . (1) . (3) . \supset \vdash . \text{Prop}$

***264·2.** $\vdash : P \epsilon \Omega . \text{E} ! \max_P'\overleftarrow{(P_1)_*}'x . \supset . \max_P'\overleftarrow{(P_1)_*}'x = B'\breve{P}$

Dem.

$\vdash . *206 \cdot 42 \cdot 46 . \supset \vdash : \text{Hp} . \supset . \overrightarrow{\text{seq}_P}'\overleftarrow{(P_1)_*}'x = \overleftarrow{P_1}'\max_P'\overleftarrow{(P_1)_*}'x .$

$[*90 \cdot 16] \qquad \supset . \overrightarrow{\text{seq}_P}'\overleftarrow{(P_1)_*}'x \subset \overleftarrow{(P_1)_*}'x .$

$[*206 \cdot 2] \qquad \supset . \overrightarrow{\text{seq}_P}'\overleftarrow{(P_1)_*}'x = \Lambda .$

$[*250 \cdot 126] \qquad \supset . \max_P'\overleftarrow{(P_1)_*}'x = B'\breve{P} : \supset \vdash . \text{Prop}$

***264·21.** $\vdash : P \epsilon \Omega . \text{E} ! \max_P'\overleftarrow{(P_1)_*}'x . \supset .$

$P \upharpoonright \overleftarrow{(P_1)_*}'x \epsilon \Omega \text{ fin} . P (x \mapsto B'\breve{P}) \epsilon \text{Cls induct}$

Dem.

$\vdash . *200 \cdot 35 . \supset \vdash : \text{Hp} . \overleftarrow{(P_1)_*}'x = \iota'x . \supset . P \upharpoonright \overleftarrow{(P_1)_*}'x = \dot{\Lambda}$ (1)

$\vdash . *260 \cdot 27 . \supset \vdash : \text{Hp} . \overleftarrow{(P_1)_*}'x \neq \iota'x . \supset . x P_{\text{fn}} \max_P'\overleftarrow{(P_1)_*}'x .$

$[*260 \cdot 11] \qquad \supset . P \{x \mapsto \max_P'\overleftarrow{(P_1)_*}'x\} \epsilon \text{Cls induct} .$ (2)

$[*205 \cdot 2] \qquad \supset . C'P \upharpoonright \overleftarrow{(P_1)_*}'x \epsilon \text{Cls induct}$ (3)

$\vdash . (1) . (2) . (3) . *264 \cdot 2 . \supset \vdash . \text{Prop}$

***264·22.** $\vdash : P \epsilon \Omega . \sim \text{E} ! \max_P'\overleftarrow{(P_1)_*}'x . x \epsilon C'P_1 . \supset . P \upharpoonright \overleftarrow{(P_1)_*}'x \epsilon \omega$

Dem.

$\vdash . *260 \cdot 32 \cdot 34 \cdot 27 . \supset \vdash : \text{Hp} . \supset . \{P \upharpoonright \overleftarrow{(P_1)_*}'x\}_1 = \{\overleftarrow{(P_1)_*}'x\} \upharpoonleft P_1 .$ (1)

$[*122 \cdot 52] \qquad \supset . \{P \upharpoonright \overleftarrow{(P_1)_*}'x\}_1 \epsilon \text{Prog}$ (2)

$\vdash . (1) . *260 \cdot 33 . \quad \supset \vdash : \text{Hp} . \supset . [\{P \upharpoonright \overleftarrow{(P_1)_*}'x\}_1]_{\text{po}} = P \upharpoonright \overleftarrow{(P_1)_*}'x$ (3)

$\vdash . (2) . (3) . *263 \cdot 1 . \supset \vdash . \text{Prop}$

***264·221.** $\vdash : P \epsilon \Omega . x (\nabla'P) y . \supset . P (x - y) \sim \epsilon \text{Cls induct}$

Dem.

$\vdash . *207 \cdot 34 . *216 \cdot 6 . \supset \vdash : \text{Hp} . \supset . x P^2 y . y = \text{lt}_P'\overrightarrow{P}'y .$

$[*207 \cdot 25] \qquad \supset . x P^2 y . y = \text{lt}_P'(\overleftarrow{P}'x \cap \overrightarrow{P}'y) .$

$[*207 \cdot 13] \qquad \supset . x P^2 y . \sim \text{E} ! \max_P'(\overleftarrow{P}'x \cap \overrightarrow{P}'y) .$

$[*261 \cdot 26] \qquad \supset . \overleftarrow{P}'x \cap \overrightarrow{P}'y \sim \epsilon \text{Cls induct} : \supset \vdash . \text{Prop}$

***264·222.** $\vdash : P \epsilon \Omega . \overleftarrow{P}'x \epsilon \text{Cls induct} . \supset . x \sim \epsilon \text{D}'\nabla'P \quad [*264 \cdot 221 . \text{Transp}]$

***264·223.** $\vdash : P \epsilon \Omega \,.\, P(x - y) \sim \epsilon \text{ Cls induct} \,.\supset .\, \exists \,!\, \text{G}'\nabla' P \cap P(x \dashv y)$

Dem.

$\vdash .\, *261\cdot3 \,.\supset \vdash : \text{Hp} \,.\supset .\, (\exists \alpha) \,.\, \alpha \subset P(x-y) \,.\, \exists \,!\, \alpha \,.\, \sim \text{E} \,!\, \text{max}_P{}'\alpha \,.$

$[*250\cdot122] \quad \supset .\, (\exists \alpha) \,.\, \alpha \subset P(x-y) \,.\, \exists \,!\, \alpha \,.\, \text{E} \,!\, \text{lt}_P{}'\alpha \,.$

$[*206\cdot213] \quad \supset .\, (\exists \alpha) \,.\, \alpha \subset P(x-y) \,.\, \exists \,!\, \alpha \,.\, \text{lt}_P{}'\alpha \,\epsilon\, P(x \dashv y) \,.$

$[*206\cdot181] \quad \supset .\, \exists \,!\, \text{D}'\text{lt}_P \cap \text{G}'P \cap P(x \dashv y) \,.$

$[*216\cdot602] \quad \supset .\, \exists \,!\, \text{G}'\nabla' P \cap P(x \dashv y) : \supset \vdash .\, \text{Prop}$

***264·224.** $\vdash : P \epsilon \Omega \,.\, x = B'\text{Cnv}'\nabla' P \,.\, \text{E} \,!\, B'\breve{P} \,.\supset .\, \overleftarrow{P}'x \,\epsilon\, \text{Cls induct}$

Dem.

$\vdash .\, *264\cdot223 \,.\, \text{Transp} \,.\supset \vdash : \text{Hp} \,.\supset .\, P(x - B'\breve{P}) \,\epsilon\, \text{Cls induct} : \supset \vdash .\, \text{Prop}$

***264·225.** $\vdash :.\, P \epsilon \Omega \,.\, x \,\epsilon\, C'P_1 \,.\supset : \text{E} \,!\, \text{max}_P{}'\overleftarrow{(P_1)_*}'x \,.\equiv .\, \overleftarrow{(P_1)_*}'x \,\epsilon\, \text{Cls induct}$ $[*264\cdot21\cdot22]$

***264·23.** $\vdash :.\, P \epsilon \Omega \,.\, x \,\epsilon\, C'\nabla' P \cap C'P_1 \,.\supset :$

$$\text{E} \,!\, \text{max}_P{}'\overleftarrow{(P_1)_*}'x \,.\equiv .\, x = B'\text{Cnv}'\nabla' P \,.\, \text{E} \,!\, B'\breve{P}$$

Dem.

$\vdash .\, *264\cdot2 \,. \quad \supset \vdash : \text{Hp} \,.\, \text{E} \,!\, \text{max}_P{}'\overleftarrow{(P_1)_*}'x \,.\supset .\, \text{E} \,!\, B'\breve{P}$ (1)

$\vdash .\, *264\cdot21\cdot222 \,.\supset \vdash : \text{Hp}(1) \,.\supset .\, x \sim \epsilon\, \text{D}'\nabla' P \,.$

$[*93\cdot103] \quad \supset .\, x = B'\text{Cnv}'\nabla' P$ (2)

$\vdash .\, *264\cdot224 \,. \quad \supset \vdash : \text{Hp} \,.\, x = B'\text{Cnv}'\nabla' P \,.\, \text{E} \,!\, B'\breve{P} \,.\supset .\, \overleftarrow{P}'x \,\epsilon\, \text{Cls induct} \,.$

$[*120\cdot481\cdot251] \quad \supset .\, \overleftarrow{(P_1)_*}'x \,\epsilon\, \text{Cls induct} \,.$

$[*90\cdot12.\text{Hp}.*261\cdot26] \quad \supset .\, \text{E} \,!\, \text{max}_P{}'\overleftarrow{(P_1)_*}'x$ (3)

$\vdash .\, (1) \,.\, (2) \,.\, (3) \,.\supset \vdash .\, \text{Prop}$

***264·231.** $\vdash : P \epsilon \Omega \,.\, x \,\epsilon\, C'\nabla' P - C'P_1 \,.\supset .\, x = B'\text{Cnv}'\nabla' P = B'\breve{P}$

Dem.

$\vdash .\, *250\cdot21 \,.\supset \vdash : \text{Hp} \,.\supset .\, x \sim \epsilon\, \text{D}'P \,.$

$[*93\cdot103] \quad \supset .\, x = B'\breve{P} \,.$ (1)

$[*216\cdot6] \quad \supset .\, x \sim \epsilon\, \text{D}'\nabla' P \,.$

$[*93\cdot103] \quad \supset .\, x = B'\text{Cnv}'\nabla' P$ (2)

$\vdash .\, (1) \,.\, (2) \,.\supset \vdash .\, \text{Prop}$

***264·232.** $\vdash :.\, P \epsilon \Omega \,.\, x \,\epsilon\, C'\nabla' P \,.\supset :$

$$\overleftarrow{(P_1)_*}'x \,\epsilon\, \text{Cls induct} \,.\equiv .\, x = B'\text{Cnv}'\nabla' P \,.\, \text{E} \,!\, B'\breve{P}$$

This proposition differs from *264·23 by not assuming that $x \,\epsilon\, C'P_1$. If $B'\breve{P}$ has no immediate predecessor, $B'\breve{P} \,\epsilon\, C'\nabla' P - C'P_1$, so that $B'\breve{P}$ satisfies the hypothesis of *264·232, but not that of *264·23.

Dem.

$\vdash . *90\cdot13 . \quad \supset \vdash : \text{Hp} . \overleftarrow{(P_1)_*}\text{‘}x = \Lambda . \supset . x \sim\epsilon C\text{‘}P_1 .$

$[*264\cdot231] \qquad \supset . x = B\text{‘}\text{Cnv}\text{‘}\nabla\text{‘}P . \text{E} ! B\text{‘}\breve{P} \qquad (1)$

$\vdash . *120\cdot212 . \supset \vdash : \text{Hp}(1) . \supset . \overleftarrow{(P_1)_*}\text{‘}x \epsilon \text{Cls induct} \qquad (2)$

$\vdash . *264\cdot225 . \supset$

$\vdash :. \text{Hp} . \exists ! \overleftarrow{(P_1)_*}\text{‘}x . \supset : \overleftarrow{(P_1)_*}\text{‘}x \epsilon \text{Cls induct} . \equiv . \text{E} ! \max_P\text{‘}\overleftarrow{(P_1)_*}\text{‘}x .$

$[*264\cdot23] \qquad \equiv . x = B\text{‘}\text{Cnv}\text{‘}\nabla\text{‘}P . \text{E} ! B\text{‘}\breve{P} \qquad (3)$

$\vdash . (1) . (2) . (3) . \supset \vdash . \text{Prop}$

***264·233.** $\vdash : P \epsilon \Omega \text{ infin} - \omega . x \epsilon C\text{‘}P_1 . \supset . \min_P\text{‘}\overrightarrow{(P_1)_*}\text{‘}x \epsilon C\text{‘}\nabla\text{‘}P$

Dem.

$\vdash . *250\cdot121 . \quad \supset \vdash : \text{Hp} . \supset . \text{E} ! \min_P\text{‘}\overrightarrow{(P_1)_*}\text{‘}x \qquad (1)$

$\vdash . *90\cdot172 . \quad \supset \vdash : \text{Hp} . y (P_1)_* x . z P_1 y . \supset . z \epsilon \overrightarrow{(P_1)_*}\text{‘}x \cap \overrightarrow{P}\text{‘}y .$

$[*205\cdot14] \qquad \supset . y \neq \min_P\text{‘}\overrightarrow{(P_1)_*}\text{‘}x \qquad (2)$

$\vdash . (2) . \text{Transp} . \supset \vdash : \text{Hp} . y = \min_P\text{‘}\overrightarrow{(P_1)_*}\text{‘}x . \supset . y \sim\epsilon \text{G}\text{‘}P_1 .$

$[*264\cdot14] \qquad \supset . y \epsilon C\text{‘}\nabla\text{‘}P \qquad (3)$

$\vdash . (1) . (3) . \supset \vdash . \text{Prop}$

***264·24.** $\vdash : P \epsilon \Omega \text{ infin} . \text{E} ! B\text{‘}\breve{P} . \supset . \text{E} ! B\text{‘}\text{Cnv}\text{‘}\nabla\text{‘}P$

Dem.

$\vdash . *264\cdot12 . \supset \vdash : \text{Hp} . B\text{‘}\breve{P} \sim\epsilon C\text{‘}P_1 . \supset . B\text{‘}\breve{P} \epsilon C\text{‘}\nabla\text{‘}P .$

$[*216\cdot6] \qquad \supset . B\text{‘}\breve{P} = B\text{‘}\text{Cnv}\text{‘}\nabla\text{‘}P \qquad (1)$

$\vdash . *264\cdot233 . *263\cdot22 . \supset \vdash : \text{Hp} . B\text{‘}\breve{P} \epsilon C\text{‘}P_1 . \supset . \min_P\text{‘}\overrightarrow{(P_1)_*}\text{‘}B\text{‘}\breve{P} \epsilon C\text{‘}\nabla\text{‘}P \qquad (2)$

$\vdash . *205\cdot55 . \supset \vdash : \text{Hp}(2) . x = \min_P\text{‘}\overrightarrow{(P_1)_*}\text{‘}B\text{‘}\breve{P} . \supset . B\text{‘}\breve{P} = \max_P\text{‘}\overleftarrow{(P_1)_*}\text{‘}x .$

$[*264\cdot23.(2)] \qquad \supset . x = B\text{‘}\text{Cnv}\text{‘}\nabla\text{‘}P \qquad (3)$

$\vdash . (1) . (3) . \supset \vdash . \text{Prop}$

***264·25.** $\vdash : P \epsilon \Omega . x \epsilon \text{D}\text{‘}\nabla\text{‘}P . \supset . P \upharpoonright \overleftarrow{(P_1)_*}\text{‘}x \epsilon \omega$

Dem.

$\vdash . *264\cdot232 . *250\cdot21 . \supset \vdash : \text{Hp} . \supset . \overleftarrow{(P_1)_*}\text{‘}x \sim\epsilon \text{Cls induct} . x \epsilon \text{D}\text{‘}P_1 .$

$[*264\cdot225] \qquad \supset . \sim \text{E} ! \max_P\text{‘}\overleftarrow{(P_1)_*}\text{‘}x . x \epsilon \text{D}\text{‘}P_1 .$

$[*264\cdot22] \qquad \supset . P \upharpoonright \overleftarrow{(P_1)_*}\text{‘}x \epsilon \omega : \supset \vdash . \text{Prop}$

***264·251.** $\vdash : P \epsilon \Omega . \sim \text{E} ! B\text{‘}\breve{P} . x \epsilon C\text{‘}\nabla\text{‘}P . \supset . P \upharpoonright \overleftarrow{(P_1)_*}\text{‘}x \epsilon \omega$

Dem.

$\vdash . *250\cdot21 . \supset \vdash : \text{Hp} . \supset . x \epsilon \text{D}\text{‘}P_1 .$

$[*264\cdot23.\text{Hp}] \qquad \supset . \sim \text{E} ! \max_P\text{‘}\overleftarrow{(P_1)_*}\text{‘}x . x \epsilon \text{D}\text{‘}P_1 .$

$[*264\cdot22] \qquad \supset . P \upharpoonright \overleftarrow{(P_1)_*}\text{‘}x \epsilon \omega : \supset \vdash . \text{Prop}$

***264·252.** $\vdash : P \in \Omega \,.\, \mathrm{E}\,!\, B\text{‘}\breve{P} \,.\, x = B\text{‘}\mathrm{Cnv}\text{‘}\nabla\text{‘}P \,.\, \supset .\, P \upharpoonright \overleftarrow{(P_1)_*}\text{‘}x \in \Omega\ \mathrm{fin}$

Dem.

$\vdash .\, *264\cdot23 \,.\, \supset \vdash : \mathrm{Hp} \,.\, x \in C\text{‘}P_1 \,.\, \supset .\, \mathrm{E}\,!\, \max_P\text{‘}\overleftarrow{(P_1)_*}\text{‘}x \,.$

$[*264\cdot21] \qquad \supset .\, P \upharpoonright \overleftarrow{(P_1)_*}\text{‘}x \in \Omega\ \mathrm{fin}$ (1)

$\vdash .\, *90\cdot14 \,.\, \supset \vdash : x \sim\in C\text{‘}P_1 \,.\, \supset .\, P \upharpoonright \overleftarrow{(P_1)_*}\text{‘}x = \dot{\Lambda}$ (2)

$\vdash .\, (1) \,.\, (2) \,.\, \supset \vdash .\, \mathrm{Prop}$

***264·26.** $\vdash :.\, P \in \Omega \,.\, \supset : B\text{‘}\breve{P} \in C\text{‘}\nabla\text{‘}P \,.\, \equiv .\, \mathrm{E}\,!\, B\text{‘}\breve{P} \,.\, B\text{‘}\breve{P} \sim\in \text{Ɑ}\text{‘}P_1$

Dem.

$\vdash .\, *14\cdot21 \,.\, \supset \vdash : B\text{‘}\breve{P} \in C\text{‘}\nabla\text{‘}P \,.\, \supset .\, \mathrm{E}\,!\, B\text{‘}\breve{P}$ (1)

$\vdash .\, *264\cdot12 \,.\, \supset \vdash : \mathrm{Hp} \,.\, B\text{‘}\breve{P} \in C\text{‘}\nabla\text{‘}P \,.\, \supset .\, B\text{‘}\breve{P} \sim\in \text{Ɑ}\text{‘}P_1$ (2)

$\vdash .\, *264\cdot12 \,.\, \supset \vdash : \mathrm{Hp} \,.\, B\text{‘}\breve{P} \sim\in \text{Ɑ}\text{‘}P_1 \,.\, \supset .\, B\text{‘}\breve{P} \in C\text{‘}\nabla\text{‘}P$ (3)

$\vdash .\, (1) \,.\, (2) \,.\, (3) \,.\, \supset \vdash .\, \mathrm{Prop}$

***264·261.** $\vdash :.\, P \in \Omega \,.\, \supset : \sim(B\text{‘}\breve{P} \in C\text{‘}\nabla\text{‘}P) \,.\, \equiv .\, C\text{‘}P = C\text{‘}P_1$

Dem.

$\vdash .\, *264\cdot26 \,.\, \supset \vdash :: \mathrm{Hp} \,.\, \supset :.\, \sim(B\text{‘}\breve{P} \in C\text{‘}\nabla\text{‘}P) \,.\, \equiv : \sim \mathrm{E}\,!\, B\text{‘}\breve{P} \,.\, \mathrm{v} \,.\, B\text{‘}\breve{P} \in \text{Ɑ}\text{‘}P_1 :$

$[*202\cdot52] \qquad \equiv : \overrightarrow{B}\text{‘}\breve{P} \subset \text{Ɑ}\text{‘}P_1 :$

$[*250\cdot21] \qquad \equiv : C\text{‘}P \subset C\text{‘}P_1 :$

$[*121\cdot322] \qquad \equiv : C\text{‘}P = C\text{‘}P_1 :: \supset \vdash .\, \mathrm{Prop}$

***264·3.** $\vdash : QP_{\mathrm{pr}}R \,.\, \equiv .\, (\exists x, y) \,.\, x\,(\nabla\text{‘}P)\,y \,.\, Q = P \upharpoonright \overleftarrow{(P_1)_*}\text{‘}x \,.\, R = P \upharpoonright \overleftarrow{(P_1)_*}\text{‘}y$
$[(*264\cdot01)]$

***264·31.** $\vdash :.\, P \in \mathrm{Ser} \,.\, \supset : QP_{\mathrm{pr}}R \,.\, \equiv .$

$(\exists x, y) \,.\, x, y \in C\text{‘}P - \text{Ɑ}\text{‘}P_1 \,.\, xPy \,.\, Q = P \upharpoonright \overleftarrow{(P_1)_*}\text{‘}x \,.\, R = P \upharpoonright \overleftarrow{(P_1)_*}\text{‘}y$
$[*207\cdot35 \,.\, *264\cdot3 \,.\, *216\cdot6]$

***264·32.** $\vdash .\, C\text{‘}P_{\mathrm{pr}} = P \upharpoonright \text{‘‘}\overleftarrow{(P_1)_*}\text{‘‘}C\text{‘}\nabla\text{‘}P \quad [*150\cdot22 \,.\, (*264\cdot01)]$

***264·321.** $\vdash : P \in \mathrm{Ser} \,.\, x \in C\text{‘}\nabla\text{‘}P \,.\, \supset .\, \overleftarrow{(P_1)_*}\text{‘}x \sim\in 1$

Dem.

$\vdash .\, *216\cdot611 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset .\, x \in C\text{‘}P - \text{Ɑ}\text{‘}P_1$ (1)

$\vdash .\, *90\cdot14 \,.\, \supset \vdash : x \sim\in C\text{‘}P_1 \,.\, \supset .\, \overleftarrow{(P_1)_*}\text{‘}x = \Lambda$ (2)

$\vdash .\, *121\cdot305 \,.\, \supset \vdash : \mathrm{Hp} \,.\, x \in \mathrm{D}\text{‘}P_1 \,.\, \supset .\, \exists\,!\, \overleftarrow{(P_1)_*}\text{‘}x - \iota\text{‘}x \,.$

$[*90\cdot12] \qquad \supset .\, \overleftarrow{(P_1)_*}\text{‘}x \sim\in 1$ (3)

$\vdash .\, (1) \,.\, (2) \,.\, (3) \,.\, \supset \vdash .\, \mathrm{Prop}$

***264·33.** $\vdash : P \in \mathrm{Ser} \,.\, \supset .\, C\text{‘‘}C\text{‘}P_{\mathrm{pr}} = \overleftarrow{(P_1)_*}\text{‘‘}C\text{‘}\nabla\text{‘}P$
$[*264\cdot321 \,.\, *202\cdot55 \,.\, *264\cdot32]$

***264·34.** $\vdash : P \in \Omega \,.\, x, y \in C`P \,.\, P \upharpoonright \overleftarrow{(P_1)_*}`x = P \upharpoonright \overleftarrow{(P_1)_*}`y \,.\, \supset \,.\, x = y$

Dem.

$\vdash \,.\, *264·321 \,.\, *202·55 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, \overleftarrow{(P_1)_*}`x = \overleftarrow{(P_1)_*}`y$ (1)

$\vdash \,.\, (1) \,.\, *90·12 \,.\, \supset \vdash : \text{Hp} \,.\, x \in C`P_1 \,.\, \supset \,.\, x (P_1)_* y \,.\, y (P_1)_* x \,.$

$[*260·22 \,.\, *91·541] \quad \supset \,.\, x = y$ (2)

$\vdash \,.\, *250·21 \,.\, \supset \vdash : \text{Hp} \,.\, x \sim \in C`P_1 \,.\, \supset \,.\, x = B`\breve{P}$ (3)

$\vdash \,.\, (1) \,.\, *90·12·14 \,.\, \supset \vdash : \text{Hp} \,.\, x \sim \in C`P_1 \,.\, \supset \,.\, y \sim \in C`P_1 \,.$

$[*250·21] \quad \supset \,.\, y = B`\breve{P}$ (4)

$\vdash \,.\, (3) \,.\, (4) \,.\, \supset \vdash : \text{Hp} \,.\, x \sim \in C`P_1 \,.\, \supset \,.\, x = y$ (5)

$\vdash \,.\, (2) \,.\, (5) \,.\, \supset \vdash \,.\, \text{Prop}$

***264·341.** $\vdash : P \in \text{Ser} \,.\, x, y \in C`\nabla`P \,.\, x (P_1)_* y \,.\, \supset \,.\, x = y$

Dem.

$\vdash \,.\, *216·611 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, y \sim \in \text{Ɑ}`P_1 \,.$

$[*91·504] \quad \supset \,.\, \sim \{x (P_1)_{po} y\} \,.$

$[*91·54] \quad \supset \,.\, x = y : \supset \vdash \,.\, \text{Prop}$

***264·35.** $\vdash : P \in \text{Ser} \,.\, x, y \in C`\nabla`P \,.\, \exists ! \overleftarrow{(P_1)_*}`x \cap \overleftarrow{(P_1)_*}`y \,.\, \supset \,.\, x = y$

Dem.

$\vdash \,.\, *96·302 \,.\, \supset \vdash :. \text{Hp} \,.\, \supset : x (P_1)_* y \,.\, \vee \,.\, y (P_1)_* x :$

$[*264·341] \quad \supset : x = y :. \supset \vdash \,.\, \text{Prop}$

***264·36.** $\vdash : P \in \Omega \,.\, \supset \,.\, P_{pr} \text{ smor } \nabla`P \,.\, P_{pr} \in \text{Rel}^2 \text{ excl} \quad [*264·34·35]$

The following propositions lead up to *264·39·391.

***264·37.** $\vdash : P \in \Omega \text{ infin} - \omega \,.\, \supset \,.\, \dot{s}`C`P_{pr} = P_{fn}$

Dem.

$\vdash \,.\, *264·32 \,.\, \supset \vdash :. \text{Hp} \,.\, \supset : x (\dot{s}`C`P_{pr}) y \,.\, \equiv \,.\, (\exists a) \,.\, a \in C`\nabla`P \,.\, x, y \in \overleftarrow{(P_1)_*}`a \,.\, xPy \,.$

$[*260·32·27] \quad \equiv \,.\, (\exists a) \,.\, a \in C`\nabla`P \,.\, x, y \in \overleftarrow{(P_1)_*}`a \,.\, xP_{fn}y \,.$

$[*264·233·35] \quad \equiv \,.\, (\exists a) \,.\, a = \min_P`\overrightarrow{(P_1)_*}`x = \min_P`\overrightarrow{(P_1)_*}`y \,.\, xP_{fn}y \,.$

$[*13·195] \quad \equiv \,.\, \min_P`\overrightarrow{(P_1)_*}`x = \min_P`\overrightarrow{(P_1)_*}`y \,.\, xP_{fn}y$ (1)

$\vdash \,.\, *260·27 \,.\, \supset \vdash : \text{Hp} \,.\, xP_{fn}y \,.\, \supset \,.\, \overrightarrow{(P_1)_*}`x \subset \overrightarrow{(P_1)_*}`y \,.$

$[*205·5] \quad \supset \,.\, \min_P`\overrightarrow{(P_1)_*}`x = \min_P`\overrightarrow{(P_1)_*}`y$ (2)

$\vdash \,.\, (1) \,.\, (2) \,.\, \supset \vdash :. \text{Hp} \,.\, \supset : x (\dot{s}`C`P_{pr}) y \,.\, \equiv \,.\, xP_{fn}y :. \supset \vdash \,.\, \text{Prop}$

***264·371.** $\vdash : P \in \text{Ser} \,.\, a (\nabla`P) b \,.\, \supset \,.\, \overleftarrow{(P_1)_*}`a \subset \overrightarrow{P}`b$

Dem.

$\vdash \,.\, *216·6 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, a \in \overrightarrow{P}`b$ (1)

$\vdash \,.\, *204·71 \,.\, \supset \vdash : \text{Hp} \,.\, xPb \,.\, xP_1y \,.\, \sim (yPb) \,.\, \supset \,.\, y = b \,.$

$[*33·14] \quad \supset \,.\, b \in \text{Ɑ}`P_1$ (2)

$\vdash \,.\, (2) \,.\, \text{Transp} \,.\, *216·611 \,.\, \supset \vdash :. \text{Hp} \,.\, \supset : xPb \,.\, xP_1y \,.\, \supset \,.\, yPb$ (3)

$\vdash \,.\, (1) \,.\, (3) \,.\, *90·112 \,.\, \supset \vdash :. \text{Hp} \,.\, \supset : a (P_1)_* x \,.\, \supset \,.\, xPb :. \supset \vdash \,.\, \text{Prop}$

$*264{\cdot}372.\ \vdash : P \in \mathrm{Ser} \,.\supset .\, F;P_{\mathrm{pr}} \mathrel{\dot\subset} P \dot{-} P_{\mathrm{fn}}$

Dem.

$\vdash .\, *264{\cdot}3{\cdot}321 \,.\, *202{\cdot}55 \,.\supset$

$\vdash :.\, \mathrm{Hp} \,.\supset : x\,(F;P_{\mathrm{pr}})\,y \,.\equiv .\, (\exists a, b) \,.\, a\,(\nabla`P)\,b \,.\, x \in \overleftarrow{(P_1)_*}`a \,.\, y \in \overleftarrow{(P_1)_*}`b \,.$ (1)

$[*264{\cdot}371] \qquad \supset .\, xPy$ (2)

$\vdash .\, *264{\cdot}35 \,.\supset \vdash : \mathrm{Hp} \,.\, a\,(\nabla`P)\,b \,.\, x \in \overleftarrow{(P_1)_*}`a \,.\, y \in \overleftarrow{(P_1)_*}`b \,.\supset .\, y \sim\in \overleftarrow{(P_1)_*}`a \,.$

$[*90{\cdot}17] \qquad \supset .\, y \sim\in \overleftarrow{(P_1)_*}`x \,.$

$[*260{\cdot}27] \qquad \supset .\, \sim(xP_{\mathrm{fn}}y)$ (3)

$\vdash .\, (1) \,.\, (2) \,.\, (3) \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, F;P_{\mathrm{pr}} \mathrel{\dot\subset} P \dot{-} P_{\mathrm{fn}} :\supset \vdash .\, \mathrm{Prop}$

$*264{\cdot}373.\ \vdash : P \in \Omega \,.\sim(B`\breve{P} \in C`\nabla`P) \,.\supset .\, P \dot{-} P_{\mathrm{fn}} \mathrel{\dot\subset} F;P_{\mathrm{pr}}$

Dem.

$\vdash .\, *264{\cdot}261{\cdot}233 \,.\, *263{\cdot}49 \,.\supset$

$\vdash : \mathrm{Hp} \,.\, x\,(P \dot{-} P_{\mathrm{fn}})\,y \,.\supset .\, \mathrm{min}_P`\overrightarrow{(P_1)_*}`x,\ \mathrm{min}_P`\overrightarrow{(P_1)_*}`y \in C`\nabla`P$ (1)

$\vdash .\, *96{\cdot}301 \,.\supset \vdash :.\, \mathrm{Hp} \,.\, \mathrm{min}_P`\overrightarrow{(P_1)_*}`x = \mathrm{min}_P`\overrightarrow{(P_1)_*}`y \,.\supset : x\,(P_1)_*\,y \,.\vee .\, y\,(P_1)_*\,x :$

$[*260{\cdot}27] \qquad \supset : x = y \,.\vee .\, xP_{\mathrm{fn}}y \,.\vee .\, yP_{\mathrm{fn}}x$ (2)

$\vdash .\, (2) \,.\, \mathrm{Transp} \,.\quad \supset \vdash : \mathrm{Hp}\,(1) \,.\supset .\, \mathrm{min}_P`\overrightarrow{(P_1)_*}`x \neq \mathrm{min}_P`\overrightarrow{(P_1)_*}`y$ (3)

$\vdash .\, (1) \,.\, *264{\cdot}371 \,.\supset \vdash : \mathrm{Hp} \,.\, \mathrm{min}_P`\overrightarrow{(P_1)_*}`y\, P\, \mathrm{min}_P`\overrightarrow{(P_1)_*}`x \,.\supset .\, yPx$ (4)

$\vdash .\, (4) \,.\, \mathrm{Transp} \,.\quad \supset \vdash : \mathrm{Hp}\,(1) \,.\supset .\sim\{\mathrm{min}_P`\overrightarrow{(P_1)_*}`y\, P\, \mathrm{min}_P`\overrightarrow{(P_1)_*}`x\}$ (5)

$\vdash .\, (3) \,.\, (5) \,.\supset \vdash : \mathrm{Hp}\,(1) \,.\supset .\, \mathrm{min}_P`\overrightarrow{(P_1)_*}`x\, P\, \mathrm{min}_P`\overrightarrow{(P_1)_*}`y$ (6)

$\vdash .\, (1) \,.\, (6) \,.\supset \vdash : \mathrm{Hp}\,(1) \,.\supset .\, (\exists a, b) \,.\, a\,(\nabla`P)\,b \,.\, x \in \overleftarrow{(P_1)_*}`a \,.\, y \in \overleftarrow{(P_1)_*}`b \,.$

$[*264{\cdot}3{\cdot}321 \,.\, *202{\cdot}55] \qquad \supset .\, x\,(F;P_{\mathrm{pr}})\,y :\supset \vdash .\, \mathrm{Prop}$

$*264{\cdot}38.\ \vdash : P \in \Omega \,.\sim(B`\breve{P} \in C`\nabla`P) \,.\supset .\, F;P_{\mathrm{pr}} = P \dot{-} P_{\mathrm{fn}}$ $[*264{\cdot}372{\cdot}373]$

$*264{\cdot}381.\ \vdash : P \in \Omega \,.\, B`\breve{P} \in C`\nabla`P \,.\supset .\, F;P_{\mathrm{pr}} = P \upharpoonright \mathrm{D}`P \dot{-} P_{\mathrm{fn}}$

Dem.

$\vdash .\, *264{\cdot}33 \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, s`C``C`P_{\mathrm{pr}} \subset C`P_1 \,.$

$[*264{\cdot}26 \,.\, *42{\cdot}2] \qquad \supset .\, B`\breve{P} \sim\in C`F;P_{\mathrm{pr}} \,.$

$[*264{\cdot}372] \qquad \supset .\, F;P_{\mathrm{pr}} \mathrel{\dot\subset} P \upharpoonright \mathrm{D}`P \dot{-} P_{\mathrm{fn}}$ (1)

$\vdash .\, *250{\cdot}21 \,.\supset \vdash : \mathrm{Hp} \,.\, x\,(P \upharpoonright \mathrm{D}`P \dot{-} P_{\mathrm{fn}})\,y \,.\supset .\, x, y \in C`P_1 \,.$

$[*264{\cdot}233 \,.\, *263{\cdot}49] \supset .\, \mathrm{min}_P`\overrightarrow{(P_1)_*}`x,\ \mathrm{min}_P`\overrightarrow{(P_1)_*}`y \in C`\nabla`P$ (2)

Thence as in the proof of $*264{\cdot}373$,

$\vdash : \mathrm{Hp} \,.\, x\,(P \upharpoonright \mathrm{D}`P \dot{-} P_{\mathrm{fn}})\,y \,.\supset .\, x\,(F;P_{\mathrm{pr}})\,y$ (3)

$\vdash .\, (1) \,.\, (3) \,.\supset \vdash .\, \mathrm{Prop}$

***264·39.** $\vdash : P \in \Omega \text{ infin} - \omega \,.\, \sim(B\text{‘}\breve{P} \in C\text{‘}\nabla\text{‘}P) \,.\supset .\, \Sigma\text{‘}P_{\text{pr}} = P$
[*264·37·38 . *260·12 . *162·1]

***264·391.** $\vdash : P \in \Omega \,.\, B\text{‘}\breve{P} \in C\text{‘}\nabla\text{‘}P \,.\supset .\, \Sigma\text{‘}P_{\text{pr}} = P \upharpoonright \text{D‘}P$

Dem.

$\vdash .\, *264\cdot13 \,.\supset \vdash : \text{Hp} \,.\supset .\, P \in \Omega \text{ infin} - \omega$ (1)

$\vdash .\, *260\cdot27 \,.\supset \vdash : \text{Hp} \,.\supset .\, P_{\text{fn}} = P_{\text{fn}} \upharpoonright C\text{‘}P_1$

[*264·26] $= P_{\text{fn}} \upharpoonright \text{D‘}P$ (2)

$\vdash .\, (1) \,.\, (2) \,.\, *264\cdot37 \,.\, *260\cdot12 \,.\supset \vdash : \text{Hp} \,.\supset .\, \dot{s}\text{‘}C\text{‘}P_{\text{pr}} = P_{\text{fn}} \,.\, P_{\text{fn}} \subseteq P \upharpoonright \text{D‘}P$ (3)

$\vdash .\, (3) \,.\, *264\cdot381 \,.\supset \vdash .\, \text{Prop}$

***264·4.** $\vdash : P \in \Omega \,.\, \sim\text{E}\,!\, B\text{‘}\breve{P} \,.\supset .\, C\text{‘}P_{\text{pr}} \subset \omega$ [*264·251·32]

***264·401.** $\vdash : P \in \Omega \,.\supset .\, \text{D‘}P_{\text{pr}} \subset \omega$

Dem.

$\vdash .\, *151\cdot5 \,.\, *264\cdot34 \,.\supset \vdash : \text{Hp} \,.\supset .\, \text{D‘}P_{\text{pr}} = P \upharpoonright \text{“}\overleftarrow{(P_1)_*}\text{“D‘}\nabla\text{‘}P$ (1)

$\vdash .\, (1) \,.\, *264\cdot25 \,.\supset \vdash .\, \text{Prop}$

***264·402.** $\vdash : P \in \Omega \text{ infin} \,.\, \text{E}\,!\, B\text{‘}\breve{P} \,.\supset .\, B\text{‘}\breve{P}_{\text{pr}} \in \Omega \text{ fin}$

Dem.

$\vdash .\, *264\cdot24 \,.\supset \vdash : \text{Hp} \,.\supset .\, \text{E}\,!\, B\text{‘Cnv‘}\nabla\text{‘}P \,.$

[*151·5.*264·34] $\supset .\, B\text{‘}\breve{P}_{\text{pr}} = P \upharpoonright \overleftarrow{(P_1)_*}\text{‘}B\text{‘Cnv‘}\nabla\text{‘}P \,.$

[*264·252] $\supset .\, B\text{‘}\breve{P}_{\text{pr}} \in \Omega \text{ fin} : \supset \vdash .\, \text{Prop}$

***264·403.** $\vdash : P \in \Omega \,.\, B\text{‘}\breve{P} \in C\text{‘}\nabla\text{‘}P \,.\supset .\, B\text{‘}\breve{P}_{\text{pr}} = \dot{\Lambda}$

Dem.

$\vdash .\, *264\cdot26\cdot231 \,.\supset \vdash : \text{Hp} \,.\supset .\, B\text{‘}\breve{P} \sim\in C\text{‘}P_1 \,.\, B\text{‘}\breve{P} = B\text{‘Cnv‘}\nabla\text{‘}P \,.$

[*90·14] $\supset .\, \overleftarrow{(P_1)_*}\text{‘}B\text{‘Cnv‘}\nabla\text{‘}P = \Lambda \,.$

[*151·5.*264·34] $\supset .\, B\text{‘}\breve{P}_{\text{pr}} = \dot{\Lambda} : \supset \vdash .\, \text{Prop}$

The following propositions deal with the various different cases that arise. Their net result is expressed in *264·44.

***264·41.** $\vdash : P \in \Omega \text{ infin} - \omega \,.\, \sim\text{E}\,!\, B\text{‘}\breve{P} \,.\supset .\, \text{Nr‘}P = \text{Nr‘}\nabla\text{‘}P \dot{\times} \omega$

Dem.

$\vdash .\, *264\cdot36\cdot4 \,.\supset \vdash : \text{Hp} \,.\supset .\, P_{\text{pr}} \in \text{Rel}^2\text{excl} \cap \text{Nr‘}\nabla\text{‘}P \,.\, C\text{‘}P_{\text{pr}} \subset \omega \,.$

[*251·63] $\supset .\, \Sigma\text{‘}P_{\text{pr}} \in \text{Nr‘}\nabla\text{‘}P \dot{\times} \omega \,.$

[*264·39] $\supset .\, P \in \text{Nr‘}\nabla\text{‘}P \dot{\times} \omega : \supset \vdash .\, \text{Prop}$

***264·42.** $\vdash : P \epsilon \Omega \,.\, B\text{‘}\breve{P} \sim\epsilon C\text{‘}\nabla\text{‘}P \,.\, \nabla\text{‘}P \epsilon 2_r \,.\, \supset \,.\, \text{Nr}\text{‘}P = \omega \dotplus \text{Nr}\text{‘}B\text{‘}\breve{P}_{\text{pr}}$

Dem.

$\vdash \,.\, *264·36 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, P_{\text{pr}} = (B\text{‘}P_{\text{pr}}) \downarrow (B\text{‘}\breve{P}_{\text{pr}}) \,.$

$[*162·3 \,.\, *264·39·13] \quad \supset \,.\, P = B\text{‘}P_{\text{pr}} \dagger B\text{‘}\breve{P}_{\text{pr}} \,.$

$[*264·36·401] \quad \supset \,.\, \text{Nr}\text{‘}P = \omega \dotplus B\text{‘}\breve{P}_{\text{pr}} : \supset \vdash \,.\, \text{Prop}$

***264·421.** $\vdash : P \epsilon \Omega \,.\, B\text{‘}\breve{P} \epsilon C\text{‘}\nabla\text{‘}P \,.\, \nabla\text{‘}P \epsilon 2_r \,.\, \supset \,.\, \text{Nr}\text{‘}P = \omega \dotplus \dot{1}$

Dem.

$\vdash \,.\, *264·36 \,.\, \supset : \text{Hp} \,.\, \supset \,.\, P_{\text{pr}} = (B\text{‘}P_{\text{pr}}) \downarrow (B\text{‘}\breve{P}_{\text{pr}}) \,.$

$[*162·3 \,.\, *264·391·13] \quad \supset \,.\, P \upharpoonright \text{D}\text{‘}P = B\text{‘}P_{\text{pr}} \dagger B\text{‘}\breve{P}_{\text{pr}}$

$[*264·403 \,.\, *160·21] \quad = B\text{‘}P_{\text{pr}} \,.$

$[*264·401] \quad \supset \,.\, P \upharpoonright \text{D}\text{‘}P \epsilon \omega \,.$

$[*204·461] \quad \supset \,.\, P \epsilon \omega \dotplus \dot{1} : \supset \vdash \,.\, \text{Prop}$

***264·422.** $\vdash : P \epsilon \Omega \text{ infin} - \omega \,.\, B\text{‘}\breve{P} \sim\epsilon C\text{‘}\nabla\text{‘}P \,.\, \nabla\text{‘}P \sim\epsilon 2_r \,.\, \supset \,.$

$$\text{Nr}\text{‘}P = \{\text{Nr}\text{‘}(\nabla\text{‘}P) \upharpoonright (\text{D}\text{‘}\nabla\text{‘}P) \dot{\times} \omega\} \dotplus \text{Nr}\text{‘}B\text{‘}\breve{P}_{\text{pr}}$$

Dem.

$\vdash \,.\, *264·36 \,.\, *204·272 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, \text{D}\text{‘}P_{\text{pr}} \sim\epsilon 1 \,.$

$[*204·461 \,.\, *264·24·36] \quad \supset \,.\, P_{\text{pr}} = P_{\text{pr}} \upharpoonright \text{D}\text{‘}P_{\text{pr}} \rightarrowtail B\text{‘}\breve{P}_{\text{pr}} \,.$

$[*162·43 \,.\, *264·39] \quad \supset \,.\, P = \Sigma\text{‘}(P_{\text{pr}} \upharpoonright \text{D}\text{‘}P_{\text{pr}}) \dagger B\text{‘}\breve{P}_{\text{pr}} \quad (1)$

$\vdash \,.\, *264·36·401 \,.\, *251·63 \,.\, \supset$

$\vdash : \text{Hp} \,.\, \supset \,.\, \text{Nr}\text{‘}\Sigma\text{‘}(P_{\text{pr}} \upharpoonright \text{D}\text{‘}P_{\text{pr}}) = \text{Nr}\text{‘}(\nabla\text{‘}P) \upharpoonright (\text{D}\text{‘}\nabla\text{‘}P) \dot{\times} \omega \quad (2)$

$\vdash \,.\, (1) \,.\, (2) \,.\, *264·36 \,.\, \supset \vdash \,.\, \text{Prop}$

***264·423.** $\vdash : P \epsilon \Omega \,.\, B\text{‘}\breve{P} \epsilon C\text{‘}\nabla\text{‘}P \,.\, \nabla\text{‘}P \sim\epsilon 2_r \,.\, \supset \,.$

$$\text{Nr}\text{‘}P = \{\text{Nr}\text{‘}(\nabla\text{‘}P) \upharpoonright (\text{D}\text{‘}\nabla\text{‘}P) \dot{\times} \omega\} \dotplus \dot{1}$$

Dem.

As in *264·422,

$\vdash : \text{Hp} \,.\, \supset \,.\, P_{\text{pr}} = P_{\text{pr}} \upharpoonright \text{D}\text{‘}P_{\text{pr}} \rightarrowtail B\text{‘}\breve{P}_{\text{pr}} \,.$

$[*162·43 \,.\, *264·391] \supset \,.\, P \upharpoonright \text{D}\text{‘}P = \Sigma\text{‘}(P_{\text{pr}} \upharpoonright \text{D}\text{‘}P_{\text{pr}}) \dagger B\text{‘}\breve{P}_{\text{pr}}$

$[*264·403] \quad = \Sigma\text{‘}(P_{\text{pr}} \upharpoonright \text{D}\text{‘}P_{\text{pr}}) \quad (1)$

$\vdash \,.\, *264·36·401 \,.\, *251·63 \,.\, \supset$

$\vdash : \text{Hp} \,.\, \supset \,.\, \text{Nr}\text{‘}\Sigma\text{‘}(P_{\text{pr}} \upharpoonright \text{D}\text{‘}P_{\text{pr}}) = \text{Nr}\text{‘}(\nabla\text{‘}P) \upharpoonright (\text{D}\text{‘}\nabla\text{‘}P) \dot{\times} \omega \quad (2)$

$\vdash \,.\, *204·461 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, \text{Nr}\text{‘}P = \text{Nr}\text{‘}(P \upharpoonright \text{D}\text{‘}P) \dotplus \dot{1} \quad (3)$

$\vdash \,.\, (1) \,.\, (2) \,.\, (3) \,.\, \supset \vdash \,.\, \text{Prop}$

$*264{\cdot}429$. $\dot{1} \dot{\times} \alpha = \alpha$ Df

This definition is merely intended to enable us to include $\dot{1}$ with ordinals in general formulae.

$*264{\cdot}44$. $\vdash : P \epsilon \Omega . \supset . (\exists \alpha, \beta) . \alpha \epsilon \mathrm{NO} \cup \iota`\dot{1} . \beta \epsilon \mathrm{NO\,fin} \cup \iota`\dot{1} . \mathrm{Nr}`P = (\alpha \dot{\times} \omega) \dot{+} \beta$

Dem.

$\vdash . *160{\cdot}22 . *166{\cdot}13 . \supset \vdash : P \epsilon \Omega\,\mathrm{fin} . \supset . \mathrm{Nr}`P = (0_r \dot{\times} \omega) \dot{+} \mathrm{Nr}`P$ (1)

$\vdash . *160{\cdot}21 . \quad \supset \vdash : P = \omega . \supset . \mathrm{Nr}`P = (\dot{1} \dot{\times} \omega) \dot{+} 0_r$ (2)

$\vdash . *264{\cdot}41 . *160{\cdot}21 . \supset$

$\vdash : P \epsilon \Omega\,\mathrm{infin} - \omega . \sim \mathrm{E} ! B`\breve{P} . \supset . (\exists \alpha) . \alpha \epsilon \mathrm{NO} . \mathrm{Nr}`P = (\alpha \dot{\times} \omega) \dot{+} 0_r$ (3)

$\vdash . *264{\cdot}42{\cdot}402 . \supset$

$\vdash : P \epsilon \Omega . B`\breve{P} \sim \epsilon C`\nabla`P . \nabla`P \epsilon 2_r . \supset . (\exists \beta) . \beta \epsilon \mathrm{NO\,fin} . \mathrm{Nr}`P = (\dot{1} \dot{\times} \omega) \dot{+} \beta$ (4)

$\vdash . *264{\cdot}421 . \quad \supset \vdash : P \epsilon \Omega . B`\breve{P} \epsilon C`\nabla`P . \nabla`P \epsilon 2_r . \supset . \mathrm{Nr}`P = (\dot{1} \dot{\times} \omega) \dot{+} \dot{1}$ (5)

$\vdash . *264{\cdot}422{\cdot}402 . \supset \vdash : P \epsilon \Omega\,\mathrm{infin} - \omega . B`\breve{P} \sim \epsilon C`\nabla`P . \nabla`P \sim \epsilon 2_r . \supset .$

$(\exists \alpha, \beta) . \alpha \epsilon \mathrm{NO} . \beta \epsilon \mathrm{NO\,fin} . \mathrm{Nr}`P = (\alpha \dot{\times} \omega) \dot{+} \beta$ (6)

$\vdash . *264{\cdot}423 . \supset \vdash : P \epsilon \Omega . B`\breve{P} \epsilon C`\nabla`P . \nabla`P \sim \epsilon 2_r . \supset .$

$(\exists \alpha) . \alpha \epsilon \mathrm{NO} . \mathrm{Nr}`P = (\alpha \dot{\times} \omega) \dot{+} \dot{1}$ (7)

$\vdash . (1) . (2) . (3) . (4) . (5) . (6) . (7) . \supset \vdash . \mathrm{Prop}$

The following propositions apply the above results to the cardinal number of the field of a well-ordered series.

$*264{\cdot}45$. $\vdash : P \epsilon \Omega . \nabla`P \epsilon 2_r . \supset . \mathrm{Nc}`C`P = \aleph_0$

Dem.

$\vdash . *264{\cdot}42{\cdot}402 . *180{\cdot}71 . *152{\cdot}7 . \supset$

$\vdash : \mathrm{Hp} . B`\breve{P} \sim \epsilon C`\nabla`P . \supset . (\exists \mu) . \mu \epsilon \mathrm{NC\,induct} . \mathrm{Nc}`C`P = C``\omega +_c \mu .$

$[*263{\cdot}101 . *123{\cdot}41] \quad \supset . \mathrm{Nc}`C`P = \aleph_0$ (1)

$\vdash . *264{\cdot}421 . *181{\cdot}62 . \supset \vdash : \mathrm{Hp} . B`\breve{P} \epsilon C`\nabla`P . \supset . \mathrm{Nc}`C`P = C``\omega +_c 1$

$[*263{\cdot}101 . *123{\cdot}4] \quad = \aleph_0$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

$*264{\cdot}451$. $\vdash : P \epsilon \Omega\,\mathrm{infin} - \omega . \sim \mathrm{E} ! B`\breve{P} . \supset . \mathrm{Nc}`C`P = \mathrm{Nc}`C`\nabla`P \times_c \aleph_0$

Dem.

$\vdash . *264{\cdot}41 . *184{\cdot}5 . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{Nc}`C`P = \mathrm{Nc}`C`\nabla`P \times_c C``\omega$

$[*263{\cdot}101] \quad = \mathrm{Nc}`C`\nabla`P \times_c \aleph_0 : \supset \vdash . \mathrm{Prop}$

$*264{\cdot}452$. $\vdash : P \epsilon \Omega\,\mathrm{infin} - \omega . \nabla`P \sim \epsilon 2_r . B`\breve{P} \sim \epsilon C`\nabla`P . \supset .$

$\mathrm{Nc}`C`P = \mathrm{Nc}`\mathrm{D}`\nabla`P \times_c \aleph_0$

Dem.

$\vdash . *264{\cdot}422 . *184{\cdot}5 . *180{\cdot}71 . \supset$

$\vdash : \mathrm{Hp} . \supset . (\exists \mu) . \mu \epsilon \mathrm{NC\,induct} . \mathrm{Nc}`C`P = (\mathrm{Nc}`\mathrm{D}`\nabla`P \times_c \aleph_0) +_c \mu$ (1)

$\vdash . *123\cdot43 . *117\cdot62 . \supset \vdash : \text{Hp} . \mu \in \text{NC induct} . \supset . \mu < \text{Nc}`\text{D}`\nabla`P \times_c \aleph_0 .$

$[*117\cdot561] \supset . (\text{Nc}`\text{D}`\nabla`P \times_c \aleph_0) +_c \mu \leqslant (\text{Nc}`\text{D}`\nabla`P \times_c \aleph_0) +_c (\text{Nc}`\text{D}`\nabla`P \times_c \aleph_0)$

$[*123\cdot421 . *113\cdot43] \qquad \leqslant \text{Nc}`\text{D}`\nabla`P \times_c \aleph_0 \quad (2)$

$\vdash . (1) . (2) . *117\cdot6\cdot25 . \supset \vdash : \text{Hp} . \supset . \text{Nc}`C`P = \text{Nc}`\text{D}`\nabla`P \times_c \aleph_0 : \supset \vdash . \text{Prop}$

***264·453.** $\vdash : P \in \Omega \text{ infin} - \omega . \text{E} ! B`\breve{P} . \nabla`P \sim\in 2_r . \supset . \text{Nc}`C`P = \text{Nc}`\text{D}`\nabla`P \times_c \aleph_0$

Dem.

As in *264·452,

$\vdash . *264\cdot423 . \supset \vdash : \text{Hp} . B`\breve{P} \in C`\nabla`P . \supset . \text{Nc}`C`P = \text{Nc}`\text{D}`\nabla`P \times_c \aleph_0 \quad (1)$

$\vdash . (1) . *264\cdot452 . \supset \vdash . \text{Prop}$

***264·46.** $\vdash : P \in \Omega \text{ infin} - \omega . \supset . \text{Nc}`C`P = \text{Nc}`C`\nabla`P \times_c \aleph_0$

Dem.

$\vdash . *123\cdot421 . *264\cdot45 . \supset \vdash : \text{Hp} . \nabla`P \in 2_r . \supset . \text{Nc}`C`P = \text{Nc}`C`\nabla`P \times_c \aleph_0 \quad (1)$

$\vdash . *264\cdot453 . \supset$

$\vdash : \text{Hp} . \text{E} ! B`\breve{P} . \nabla`P \sim\in 2_r . \text{Nc}`C`\nabla`P = \mu +_c 1 . \supset . \text{Nc}`C`P = \mu \times_c \aleph_0$

$[*123\cdot421 . *113\cdot43] = (\mu \times_c \aleph_0) +_c (\mu \times_c \aleph_0) \quad (2)$

$\vdash . *117\cdot571\cdot6 . \supset$

$\vdash : \text{Hp} . \supset . \mu \times_c \aleph_0 \leqslant (\mu +_c 1) \times_c \aleph_0 . (\mu +_c 1) \times_c \aleph_0 \leqslant (\mu \times_c \aleph_0) +_c (\mu \times_c \aleph_0) \quad (3)$

$\vdash . (2) . (3) . \supset \vdash : \text{Hp} . \supset . \text{Nc}`C`P = (\mu +_c 1) \times_c \aleph_0$

$[\text{Hp}] \qquad = \text{Nc}`C`\nabla`P \times_c \aleph_0 \quad (4)$

$\vdash . (1) . (4) . *264\cdot451 . \supset \vdash . \text{Prop}$

***264·47.** $\vdash : P \in \Omega \text{ infin} . \supset . (\exists \mu) . \mu \in \text{NC} - \iota`0 . \text{Nc}`C`P = \mu \times_c \aleph_0$ [*264·46]

***264·48.** $\vdash : \alpha \in C``\Omega - \text{Cls induct} . \supset . \text{Nc}`\alpha \in \text{D}`(\times_c \aleph_0)$ [*264·47]

*265. THE SERIES OF ALEPHS.

Summary of *265.

In the present number, we shall confine ourselves to the most elementary properties of the ordinals and cardinals considered. The most important propositions to be proved are the existence-theorems. These all depend upon the axiom of infinity; moreover, as the numbers concerned grow greater, the existence-theorems require continually higher types.

In virtue of the definition in *262, $(\aleph_0)_r$ is the class of well-ordered series whose fields have $\aleph_0$ terms. This is not an ordinal number, but the logical sum of a certain class of ordinal numbers, namely of $\mathrm{Nr}``(\aleph_0)_r$.

ω_1 is the smallest ordinal whose field has more than $\aleph_0$ terms. We do not, however, take this as the definition of ω_1: we define ω_1 as the class of relations P such that the relations less than P (in the sense of *254) are those well-ordered series which are finite or have $\aleph_0$ terms in their fields, *i.e.*

$$\omega_1 = \hat{P}\{\overrightarrow{\mathrm{less}}`P = (\aleph_0)_r \cup \Omega\ \mathrm{fin}\} \quad \mathrm{Df.}$$

By *254·401 it follows immediately that if $P \in \omega_1$, P is a well-ordered series and ω_1 is its ordinal number (*265·11). Hence ω_1 is an ordinal number (*265·12), though we need the axiom of infinity to show that ω_1 exists.

Assuming the axiom of infinity, the existence-theorem for ω_1 is derived from the series of ordinals which are finite or belong to series of $\aleph_0$ terms For notational convenience, we temporarily define this series as N; thus

$$N = (\lessdot) \upharpoonright \{\mathrm{NO\ fin} \cup \mathrm{Nr}``(\aleph_0)_r\} \quad \mathrm{Dft}\ [*265].$$

It is also convenient temporarily to write M for "$\lessdot$": thus

$$M = \lessdot \quad \mathrm{Dft}\ [*265].$$

It is easy to prove that if $\aleph_0$ exists, N is an ω_1 (*265·25). Hence we obtain the existence-theorem for ω_1 in either of the forms:

***265·27.** $\vdash : \exists ! \aleph_0 \cap t`\alpha . \supset . \exists ! \omega_1 \cap t^{11}`t_{00}`\alpha$

***265·28.** $\vdash : \mathrm{Infin\ ax}\,(x) . \supset . \exists ! \omega_1 \cap t^{11}`t^{33}`x$

It is easy to prove that ω_1 is greater than the ordinal number of any series of $\aleph_0$ terms (*265·3), and that if ω_1 exists,

$$\overrightarrow{M}`\omega_1 = \mathrm{NO\ fin} \cup \mathrm{Nr}``(\aleph_0)_r \quad (*265{\cdot}35),$$

i.e. the ordinals less than ω_1 are those that apply to series of $\aleph_0$ terms or of a finite number of terms.

We define $\aleph_1$ as $C``\omega_1$, *i.e.* as the class of those classes which can be arranged in a series whose ordinal number is ω_1. It follows from $*152 \cdot 71$ that $\aleph_1$ so defined is a cardinal number ($*265 \cdot 33$), and that if $\aleph_0$ exists, $\aleph_1 > \aleph_0$ ($*265 \cdot 34$).

In a precisely analogous fashion we can put

$$\omega_2 = \hat{P}\{\overrightarrow{\text{less}}`P = (\aleph_1)_r \cup (\aleph_0)_r \cup \Omega \text{ fin}\} \quad \text{Df},$$

$$\aleph_2 = C``\omega_2 \quad \text{Df}.$$

Theorems similar to those mentioned above can be proved for ω_2 and $\aleph_2$ by similar methods. We can proceed to ω_ν and $\aleph_\nu$, where ν is any ordinal number. But our methods of proving existence-theorems fail if ν is not finite, since at each stage the existence-theorem is proved in a higher type and we know of no meaning that can be assigned to types whose order is not finite.

It is easy to prove that the sum of two ordinals which are less than ω_1 is less than ω_1. Much of the accepted theory of $(\aleph_0)_r$ and ω_1 depends upon the proposition that the limit of any progression of ordinals less than ω_1 is less than ω_1, so that in the series N, every progression has a limit within the series. This proposition—or at any rate the current proof of it—depends upon the multiplicative axiom. The proof, in outline, is as follows:

It is easy to prove that an ordinal which has $\aleph_0$ predecessors must be a member of $\text{Nr}``(\aleph_0)_r$, *i.e.* must be, in Cantor's language, an ordinal of the second class. Now consider any progression P contained in N, *i.e.* consider a series $\alpha_1, \alpha_2, \ldots \alpha_\nu, \ldots$ of increasing ordinals of the second class. The interval between any two consecutive terms of this series is either finite or has $\aleph_0$ terms. Hence $N``C`P$, *i.e.* the class of ordinals preceding the limit of our series, is the sum of $\aleph_0$ classes each of which is finite or has $\aleph_0$ terms. It is then argued that, because $\aleph_0 \times_c \aleph_0 = \aleph_0$, the whole class $N``C`P$ must consist of $\aleph_0$ terms. This conclusion, however, except in special cases, requires the multiplicative axiom, since it depends upon $*113 \cdot 32$, *i.e.*

$$\vdash :. \text{Mult ax} . \supset : \mu, \nu \in \text{NC} . \kappa \in \nu \cap \text{Cl.excl}`\mu . \supset . s`\kappa \in \mu \times_c \nu.$$

It follows that, unless for those who regard the multiplicative axiom as certain, it cannot be regarded as proved that ω_1 is not the limit of a progression of smaller ordinals. With this, much of the recognized theory of ordinals of the second class becomes doubtful. For example, Cantor proceeds to define a host of ordinals of the second class as the limits of given series of such ordinals. It is probable that, in regard to all the ordinals which he has defined in this way, a proof that they belong to the second class can be found, by actually arranging the finite integers in a series of the specified type. But the mere fact that they are limits of progressions of numbers of

the second class does not, of itself, suffice to prove that they are of the second class.

As another example we may mention the very interesting work of Hausdorff*, much of which is based upon the proposition that a term which is the limit of an ω_1 chosen out of a given series cannot be the limit of an ω chosen out of the same series. This proposition is a consequence of the proposition that ω_1 is not the limit of a progression of smaller ordinals, and must therefore be regarded as doubtful. Hausdorff constructs by means of it many remarkable series, for example, compact series in which no progression or regression has a limit. The existence of such series appears, however, to be open to question, unless the multiplicative axiom is assumed.

It is not improbable that a proof, independent of the multiplicative axiom, can be found for the proposition that ω_1 is not the limit of a progression; but until such a proof is forthcoming, the proposition cannot be regarded as certain.

***265·01.** $\omega_1 = \hat{P}\{\overrightarrow{\text{less}}\text{'}P = (\aleph_0)_r \cup \Omega \text{ fin}\}$ Df

***265·02.** $\aleph_1 = C\text{''}\omega_1$ Df

***265·03.** $\omega_2 = \hat{P}\{\overrightarrow{\text{less}}\text{'}P = (\aleph_1)_r \cup (\aleph_0)_r \cup \Omega \text{ fin}\}$ Df

***265·04.** $\aleph_2 = C\text{''}\omega_2$ Df

etc.

***265·05.** $M = \lessdot$ Dft [*265]

This definition is revived from *256.

***265·06.** $N = M \upharpoonright \{\text{NO fin} \cup \text{Nr}\text{''}(\aleph_0)_r\}$ Dft [*265]

The existence-theorem for ω_1 is derived from N, since, if $\aleph_0$ exists, $N \in \omega_1$.

***265·1.** $\vdash :. P \in \omega_1 . \equiv : Q \text{ less } P . \equiv_Q . Q \in \Omega . C\text{'}Q \in \text{Cls induct} \cup \aleph$ [(*265·01)]

***265·11.** $\vdash : P \in \omega_1 . \supset . \omega_1 = \text{Nr}\text{'}P . P \in \Omega$

Dem.

$\vdash . *265\cdot1 . \quad \supset \vdash : \text{Hp} . \supset . \dot{\Lambda} \text{ less } P .$

$[*254\cdot1] \quad \supset . P \in \Omega \quad (1)$

$\vdash . *254\cdot401 . (1) . (*265\cdot01) . \supset \vdash : \text{Hp} . Q \in \omega_1 . \supset . Q \text{ smor } P \quad (2)$

$\vdash . *254\cdot401 . (1) . (*265\cdot01) . \supset \vdash : \text{Hp} . Q \text{ smor } P . \supset . \overrightarrow{\text{less}}\text{'}Q = (\aleph_0)_r \cup \Omega \text{ fin} .$

$[(*265\cdot01)] \quad \supset . Q \in \omega_1 \quad (3)$

$\vdash . (1) . (2) . (3) . \supset \vdash . \text{Prop}$

* *Untersuchungen über Ordnungstypen.* Berichte der mathematisch-physischen Klasse der Königlich Sächsischen Gesellschaft der Wissenschaften zu Leipzig, Feb. 1906 and Feb. 1907.

***265·12.** $\vdash . \omega_1 \epsilon \mathrm{NO}$ [*265·11 . *256·54]

***265·13.** $\vdash : \alpha \epsilon \mathrm{NO}\ \mathrm{infin} . \supset . M \upharpoonright \overrightarrow{M}‘\alpha \epsilon \alpha$

Dem.

$\vdash . *256·202 . \supset \vdash : P \epsilon \Omega\ \mathrm{infin} . \supset . \mathrm{Nr}‘M \upharpoonright (\overrightarrow{M}‘\mathrm{Nr}‘P) = \mathrm{Nr}‘(P \upharpoonright \mathrm{C}‘P)$

[*262·112] $= \mathrm{Nr}‘P$ (1)

$\vdash . (1) . *262·11 . \supset \vdash .$ Prop

***265·2.** $\vdash . \mathrm{C}‘N = \mathrm{NO}\ \mathrm{fin} - \iota‘0_r \cup \mathrm{Nr}‘‘(\aleph_0)_r = \overleftarrow{N}‘0_r$ [*255·51]

***265·21.** $\vdash : \exists ! \aleph_0 . \alpha \epsilon \mathrm{NO}\ \mathrm{fin} \cup \mathrm{Nr}‘‘(\aleph_0)_r . \supset .$

$M \upharpoonright \overrightarrow{M}‘\alpha\ \mathrm{less}\ N . \alpha M (\mathrm{Nr}‘N) . \alpha \subset \overrightarrow{\mathrm{less}}‘N$

Dem.

$\vdash . *253·13 . *265·2 . \supset \vdash : \mathrm{Hp} . \alpha \epsilon \mathrm{NO}\ \mathrm{fin} \cup \mathrm{Nr}‘‘(\aleph_0)_r . \supset . M \upharpoonright \overrightarrow{M}‘\alpha \epsilon \mathrm{D}‘N_s .$

[*254·182] $\supset . M \upharpoonright \overrightarrow{M}‘\alpha\ \mathrm{less}\ N$ (1)

$\vdash . (1) . *265·13 . \supset \vdash : \mathrm{Hp} . \alpha \epsilon \mathrm{Nr}‘‘(\aleph_0)_r . \supset . \alpha M (\mathrm{Nr}‘N)$ (2)

$\vdash . (2) . *263·31·101 . \supset \vdash : \mathrm{Hp} . \alpha \epsilon \mathrm{NO}\ \mathrm{fin} . \supset . \alpha M \omega . \omega M (\mathrm{Nr}‘N) .$

[*256·1] $\supset . \alpha M (\mathrm{Nr}‘N)$ (3)

$\vdash . (2) . (3) . \supset \vdash : \mathrm{Hp} . \alpha \epsilon \mathrm{NO}\ \mathrm{fin} \cup \mathrm{Nr}‘‘(\aleph_0)_r . \supset . \alpha M (\mathrm{Nr}‘N) .$ (4)

[*255·17] $\supset . \alpha \subset \overrightarrow{\mathrm{less}}‘N$ (5)

$\vdash . (1) . (4) . (5) . \supset \vdash .$ Prop

***265·22.** $\vdash : \exists ! \aleph_0 . \supset . \Omega\ \mathrm{fin} \cup (\aleph_0)_r \subset \overrightarrow{\mathrm{less}}‘N$ [*265·21]

***265·23.** $\vdash : P \epsilon \mathrm{D}‘N_s . \supset . (\exists \alpha) . \alpha \epsilon \mathrm{NO}\ \mathrm{fin} \cup \mathrm{Nr}‘‘(\aleph_0)_r . P = M \upharpoonright \overrightarrow{M}‘\alpha . \mathrm{Nr}‘P = \alpha$
[*265·2 . *253·13 . *265·13 . *262·7 . *120·429]

***265·24.** $\vdash : P \epsilon \mathrm{D}‘N_s . \supset . P \epsilon \Omega\ \mathrm{fin} \cup (\aleph_0)_r$ [*265·23]

***265·25.** $\vdash : \exists ! \aleph_0 . \supset . N \epsilon \omega_1$

Dem.

$\vdash . *254·41·12 . \supset \vdash : P\ \mathrm{less}\ N . \supset . (\exists Q) . Q \epsilon \mathrm{D}‘N_s . P\ \mathrm{smor}\ Q .$

[*265·24.*261·18.*151·18] $\supset . P \epsilon \Omega\ \mathrm{fin} \cup (\aleph_0)_r$ (1)

$\vdash . (1) . *265·22 . \supset \vdash : \mathrm{Hp} . \supset . \overrightarrow{\mathrm{less}}‘N = \Omega\ \mathrm{fin} \cup (\aleph_0)_r .$

[*265·1] $\supset . N \epsilon \omega_1 : \supset \vdash .$ Prop

***265·26.** $\vdash : \alpha \epsilon \aleph_0 . \supset . \mathrm{N_0r} ‘(\mathrm{less} \upharpoonright \breve{C}‘‘\mathrm{Cl}‘\alpha) \epsilon \omega_1 . \mathrm{N_0r} ‘(\mathrm{less} \upharpoonright \breve{C}‘‘\mathrm{Cl}‘\alpha) = N$

Dem.

$\vdash . *254·431 . *150·37 . \supset$

$\vdash . \mathrm{N_0r} ‘(\mathrm{less} \upharpoonright \breve{C}‘‘\mathrm{Cl}‘\alpha) = (\mathrm{N_0r} ‘\mathrm{less}) \upharpoonright \mathrm{N_0r}‘‘(\Omega \cap \breve{C}‘‘\mathrm{Cl}‘\alpha)$ (1)

$\vdash . *123·16 . \supset \vdash : \alpha \epsilon \aleph_0 . \supset . \mathrm{N_0r}‘‘(\Omega \cap \breve{C}‘‘\mathrm{Cl}‘\alpha) \subset \mathrm{NO}\ \mathrm{fin} \cup \mathrm{N_0r}‘‘(\aleph_0)_r$ (2)

$\vdash . *123{\cdot}14 . *262{\cdot}18{\cdot}21 . \supset \vdash : \alpha \in \aleph_0 . \mu \in \text{NC induct} - \iota`1 . \supset . \exists ! \mu_r \cap \breve{C}``\text{Cl}`\alpha :$

$[*262{\cdot}25] \quad \supset \vdash : \alpha \in \aleph_0 . \nu \in \text{NO fin} . \supset . \exists ! \nu \cap \breve{C}``\text{Cl}`\alpha .$

$[*152{\cdot}45] \quad \supset . \nu \in \text{N}_0\text{r}``\breve{C}``\text{Cl}`\alpha \quad (3)$

$\vdash . *152{\cdot}7 . \supset \vdash : P \in (\aleph_0)_r . \alpha \in \aleph_0 . \supset . \alpha \in C``\text{N}_0\text{r}`P .$

$[*60{\cdot}34] \quad \supset . \text{Nr}`P \in \text{N}_0\text{r}``\breve{C}``\text{Cl}`\alpha \quad (4)$

$\vdash . (3) . (4) . \supset \vdash : \alpha \in \aleph_0 . \supset . \text{NO fin} \cup \text{Nr}``(\aleph_0)_r \subset \text{N}_0\text{r}``(\breve{C}``\text{Cl}`\alpha \cap \Omega) \quad (5)$

$\vdash . (2) . (5) . \supset \vdash : \alpha \in \aleph_0 . \supset . \text{NO fin} \cup \text{Nr}``(\aleph_0)_r = \text{N}_0\text{r}``(\breve{C}``\text{Cl}`\alpha \cap \Omega) \quad (6)$

$\vdash . (1) . (6) . (*255{\cdot}01 . *265{\cdot}05{\cdot}06) . \supset \vdash : \alpha \in \aleph_0 . \supset . \text{N}_0\text{r}`(\text{less} \upharpoonright \breve{C}``\text{Cl}`\alpha) = N .$

$[*265{\cdot}25] \quad \supset . \text{N}_0\text{r}`(\text{less} \upharpoonright \breve{C}``\text{Cl}`\alpha) \in \omega_1 : \supset \vdash . \text{Prop}$

***265·27.** $\vdash : \exists ! \aleph_0 \cap t`\alpha . \supset . \exists ! \omega_1 \cap t^{11}`t_{00}`\alpha$

Dem.

$\vdash . *64{\cdot}55 . \supset \vdash : \beta \in t`\alpha . C`P \subset \beta . \supset . P \in t_{00}`\alpha \quad (1)$

$\vdash . (1) . \quad \supset \vdash : \beta \in t`\alpha . \supset . \breve{C}``\text{Cl}`\beta \subset t_{00}`\alpha .$

$[*155{\cdot}12 . *63{\cdot}5] \quad \supset . \text{N}_0\text{r}``\breve{C}``\text{Cl}`\beta \subset t`t_{00}`\alpha .$

$[*64{\cdot}57] \quad \supset . \text{N}_0\text{r}`(\text{less} \upharpoonright \breve{C}``\text{Cl}`\beta) \in t^{11}`t_{00}`\alpha \quad (2)$

$\vdash . (2) . *265{\cdot}26 . \supset \vdash . \text{Prop}$

***265·28.** $\vdash : \text{Infin ax}\,(x) . \supset . \exists ! \omega_1 \cap t^{11}`t^{35}`x$

Dem.

$\vdash . *123{\cdot}37 . \supset \vdash : \text{Hp} . \supset . \exists ! \aleph_0 \cap t`t^{34}`x .$

$[*265{\cdot}27] \quad \supset . \exists ! \omega_1 \cap t^{11}`t_{00}`t^{34}`x .$

$[*64{\cdot}312] \quad \supset . \exists ! \omega_1 \cap t^{11}`t^{35}`x : \supset \vdash . \text{Prop}$

Propositions concerning $\aleph_2$ and ω_2, and generally $\aleph_\nu$ and ω_ν, where ν is an inductive cardinal, are proved precisely as the above propositions are proved. There is not, however, so far as we know, any proof of the existence of Alephs and Omegas with infinite suffixes, owing to the fact that the type increases with each successive existence-theorem, and that infinite types appear to be meaningless.

***265·3.** $\vdash : \alpha \in \text{Nr}``(\aleph_0)_r . \supset . \alpha < \omega_1 \quad [*265{\cdot}22{\cdot}25]$

***265·31.** $\vdash : \exists ! \aleph_0 . \supset . \aleph_1 \geqslant \aleph_0$

Dem.

$\vdash . *265{\cdot}25 . \quad \supset \vdash : \text{Hp} . \supset . C`N \in \aleph_1 \quad (1)$

$\vdash . *265{\cdot}2 . \quad \supset \vdash . \text{NO fin} - \iota`0_r \subset C`N \quad (2)$

$\vdash . *262{\cdot}19{\cdot}21 . *123{\cdot}27 . \supset \vdash : \text{Hp} . \supset . \text{NO fin} - \iota`0_r \in \aleph_0 \quad (3)$

$\vdash . (2) . (3) . \quad \supset \vdash : \text{Hp} . \supset . \text{Nc}`C`N \geqslant \aleph_0 \quad (4)$

$\vdash . (1) . (4) . \supset \vdash . \text{Prop}$

***265·32.** $\vdash : \exists ! \aleph_0 . \supset . \aleph_0 \neq \aleph_1 . \aleph_0 \cap \aleph_1 = \Lambda$

Dem.

$\vdash . *265·3 . \supset \vdash : P \in \Omega . C'P \in \aleph_0 . \supset . P \sim \in \omega_1 .$

$[(*265·02)] \quad \supset . C'P \sim \in \aleph_1 \quad (1)$

$\vdash . (1) . *262·18 . (*265·02) . \supset \vdash . \aleph_0 \cap \aleph_1 = \Lambda . \supset \vdash . \text{Prop}$

***265·33.** $\vdash . \aleph_1 \in \text{NC} \quad [*152·71 . *265·12]$

***265·34.** $\vdash : \exists ! \aleph_0 . \supset . \aleph_1 > \aleph_0 \quad [*265·31·32·33 . *255·74]$

***265·35.** $\vdash : \exists ! \omega_1 . \supset . \overrightarrow{M}'\omega_1 = \text{NO fin} \cup \text{Nr}``(\aleph_0)_r$

Dem.

$\vdash . *265·3 . *263·31 . \supset \vdash : \text{Hp} . \supset . \text{NO fin} \cup \text{Nr}``(\aleph_0)_r \subset \overrightarrow{M}'\omega_1 \quad (1)$

$\vdash . *265·11 . \supset \vdash : P \in \omega_1 . \text{Nr}'Q \in \overrightarrow{M}'\omega_1 . \supset . Q \text{ less } P .$

$[*265·1] \quad \supset . \text{Nr}'Q \in \text{NO fin} \cup \text{Nr}``(\aleph_0)_r \quad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***265·351.** $\vdash : P \in \omega_1 . \equiv . \exists ! \omega_1 . \text{Nr}``\text{D}'P_s = \text{NO fin} \cup \text{Nr}``(\aleph_0)_r$

Dem.

$\vdash . *256·11 . *265·35 . \supset$

$\vdash : \exists ! \omega_1 . \text{Nr}``\text{D}'P_s = \text{NO fin} \cup \text{Nr}``(\aleph_0)_r . \equiv . \exists ! \omega_1 . \overrightarrow{M}'\text{Nr}'P = \overrightarrow{M}'\omega_1 .$

$[*256·1.*204·34] \quad \equiv . P \in \omega_1 : \supset \vdash . \text{Prop}$

***265·352.** $\vdash : P \in \omega_1 . \supset . \text{Nr}``\text{D}'P_s = \overrightarrow{M}'\omega_1 \quad [*265·35·351]$

***265·36.** $\vdash : \alpha, \beta \in \text{Nr}``(\aleph_0)_r . \supset . \alpha \dot{+} \beta \in \text{Nr}``(\aleph_0)_r$

Dem.

$\vdash . *180·71 . \supset \vdash : \text{Hp} . \supset . C``(\alpha \dot{+} \beta) = C``\alpha +_c C``\beta$

$[*262·12] \quad = \aleph_0 +_c \aleph_0$

$[*123·421] \quad = \aleph_0 .$

$[*262·12] \quad \supset . \alpha \dot{+} \beta \in \text{Nr}``(\aleph_0)_r : \supset \vdash . \text{Prop}$

***265·361.** $\vdash . \alpha, \beta \in \text{NO fin} \cup \text{Nr}``(\aleph_0)_r . \supset . \alpha \dot{+} \beta \in \text{NO fin} \cup \text{Nr}``(\aleph_0)_r$

[Proof as in *265·36, using *120·45 and *123·41]

***265·4.** $\vdash : P \in \omega_1 . \alpha \subset C'P . P_*``\alpha \in \text{Cls induct} \cup \aleph_0 . \supset . \exists ! p'\overleftarrow{P}``\alpha$

Dem.

$\vdash . *265·1 . \supset \vdash : \text{Hp} . \supset . (P \upharpoonright P_*``\alpha) \text{ less } P .$

$[*254·51] \quad \supset . P_*``\alpha \neq C'P .$

$[*202·504] \quad \supset . \exists ! p'\overleftarrow{P}``\alpha : \supset \vdash . \text{Prop}$

***265·401.** $\vdash : P \in \omega_1 . \alpha \subset C'P . P``\alpha \in \text{Cls induct} \cup \aleph_0 . \supset . \exists ! p'\overleftarrow{P}``\alpha$

Dem.

$\vdash . *205·131 . \supset \vdash : \text{Hp} . \supset . P_*``\alpha = P``\alpha \cup \overrightarrow{\max}_P{}'\alpha .$

$[*205·3.*120·251.*123·4] \supset . P_*``\alpha \in \text{Cls induct} \cup \aleph_0 .$

$[*265·4] \quad \supset . \exists ! p'\overleftarrow{P}``\alpha : \supset \vdash . \text{Prop}$

***265·41.** $\vdash : P \epsilon \omega_1 . \supset . \overrightarrow{P}``C`P \subset \aleph_0 \cup \text{Cls induct} . \overrightarrow{P_*}``C`P \subset \aleph_0 \cup \text{Cls induct}$

Dem.

$\vdash . *254\cdot182 . \quad \supset \vdash :. \text{Hp} . \supset : x \epsilon C`P . \supset . (P \upharpoonright \overrightarrow{P}`x) \text{ less } P .$

$[*265\cdot1] \quad \supset . \overrightarrow{P}`x \epsilon \aleph_0 \cup \text{Cls induct} \quad (1)$

$\vdash . (1) . *120\cdot251 . *123\cdot4 . \supset \vdash :. \text{Hp} . \supset : x \epsilon C`P . \supset . \overrightarrow{P_*}`x \epsilon \aleph_0 \cup \text{Cls induct} \quad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***265·42.** $\vdash : P \epsilon \omega_1 . \supset . \text{G}`P \subset \text{D}`P$

Dem.

$\vdash . *265\cdot4\cdot41 . \supset \vdash : \text{Hp} . x \epsilon \text{G}`P . \supset . \exists ! p`\overleftarrow{P}``\iota`x .$

$[*53\cdot01\cdot31] \quad \supset . x \epsilon \text{D}`P : \supset \vdash . \text{Prop}$

***265·43.** $\vdash : P \epsilon \omega_1 . x \epsilon C`P . \supset . P \upharpoonright \overleftarrow{P_{\text{fn}}}`x \epsilon \omega . \text{E} ! \text{lt}_P`\overleftarrow{P_{\text{fn}}}`x$

Dem.

$\vdash . *264\cdot2 . *265\cdot42 . \supset \vdash : \text{Hp} . \supset . \sim \text{E} ! \max_P`\overleftarrow{P_{\text{fn}}}`x . \quad (1)$

$[*264\cdot22] \quad \supset . P \upharpoonright \overleftarrow{P_{\text{fn}}}`x \epsilon \omega \quad (2)$

$\vdash . (2) . *265\cdot41 . *123\cdot421 . \supset \vdash : \text{Hp} . \supset . P``\overleftarrow{P_{\text{fn}}}`x \epsilon \aleph_0 .$

$[*265\cdot401] \quad \supset . \exists ! p`\overleftarrow{P}``\overleftarrow{P_{\text{fn}}}`x .$

$[(1).*250\cdot123] \quad \supset . \text{E} ! \text{lt}_P`\overleftarrow{P_{\text{fn}}}`x \quad (3)$

$\vdash . (2) . (3) . \supset \vdash . \text{Prop}$

***265·431.** $\vdash : P \epsilon \omega_1 . Q \Subset P . x \epsilon C`Q . \overleftarrow{Q}`x \subset \overleftarrow{P_{\text{fn}}}`x . \supset . \exists ! p`\overleftarrow{P}``C`Q$

Dem.

$\vdash . *265\cdot43 . \supset \vdash : \text{Hp} . \supset . C`Q \subset \overrightarrow{P}`\text{lt}_P`\overleftarrow{P_{\text{fn}}}`x : \supset \vdash . \text{Prop}$

***265·44.** $\vdash : P \epsilon \omega_1 . x \epsilon C`P . \supset . P \upharpoonright \overleftarrow{P_*}`x \epsilon \omega_1$

Dem.

$\vdash . *253\cdot13 . \supset \vdash : \text{Hp} . \supset . \text{D}`(P \upharpoonright \overleftarrow{P_*}`x)_s = \hat{R}\{(\exists y) . xP_*y . R = P \upharpoonright P(x \mapsto y)\} \quad (1)$

$\vdash . *254\cdot101 . \supset \vdash : \text{Hp} . xP_*y . \supset . \text{Nr}`P \upharpoonright P(x \mapsto y) \leqslant \text{Nr}`P \upharpoonright \overrightarrow{P}`y .$

$[*265\cdot352] \quad \supset . \text{Nr}`P \upharpoonright P(x \mapsto y) \epsilon \overrightarrow{M}`\omega_1 \quad (2)$

$\vdash . *265\cdot352 . \supset \vdash : \text{Hp} . \supset . \text{Nr}`P \upharpoonright \overrightarrow{P}`x \epsilon \overrightarrow{M}`\omega_1 \quad (3)$

$\vdash . (3) . *265\cdot361\cdot35 . \supset$

$\vdash : \text{Hp} . \alpha \epsilon \overrightarrow{M}`\omega_1 . \supset . \text{Nr}`P \upharpoonright \overrightarrow{P}`x \dot{+} \alpha \epsilon \overrightarrow{M}`\omega_1 .$

$[*265\cdot351] \quad \supset . (\exists y) . \text{Nr}`P \upharpoonright \overrightarrow{P}`x \dot{+} \alpha = \text{Nr}`P \upharpoonright \overrightarrow{P}`y .$

$[*253\cdot47\cdot11] \quad \supset . (\exists y) . xP_*y . \text{Nr}`P \upharpoonright \overrightarrow{P}`x \dot{+} \alpha = \text{Nr}`P \upharpoonright \overrightarrow{P}`y .$

$[*204{\cdot}45]\quad \supset . (\exists y) . xP_{*}y . \mathrm{Nr}`P \restriction \overrightarrow{P}`x \dot{+} \alpha = \mathrm{Nr}`P \restriction \overrightarrow{P}`x + \mathrm{Nr}`P \restriction P(x \mapsto y) .$

$[*255{\cdot}564]\quad \supset . (\exists y) . xP_{*}y . \alpha = \mathrm{Nr}`P \restriction (x \mapsto y) .$

$[(1)]\quad \supset . \alpha \in \mathrm{Nr}``\mathrm{D}`(P \restriction \overleftarrow{P_{*}}`x)_{s}$ (4)

$\vdash . (2) . (4) . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{Nr}``\mathrm{D}`(P \restriction \overleftarrow{P_{*}}`x)_{s} = \overrightarrow{M}`\omega_1 .$

$[*265{\cdot}35{\cdot}351]\quad \supset . P \restriction \overleftarrow{P_{*}}`x \in \omega_1 : \supset \vdash . \mathrm{Prop}$

$*265{\cdot}441$. $\vdash : P \in \mathrm{Ser} . Q, R \in \omega \cap \mathrm{Rl}`P . R \mathrel{G} Q . \supset .$

$P``C`R = P``C`Q . Q``C`R = C`Q$

Dem.

$\vdash . *263{\cdot}27 . \mathrm{Transp} . \supset \vdash : \mathrm{Hp} . \supset . \sim \mathrm{E}!\, \mathrm{max}_Q`C`R .$

$[*205{\cdot}123]\quad \supset . C`R \subset Q``C`R .$ (1)

$[*37{\cdot}2]\quad \supset . P``C`R \subset P``Q``C`R$

$[*37{\cdot}15{\cdot}2]\quad \subset P``C`Q$ (2)

$\vdash . *263{\cdot}47 . \mathrm{Transp} . \supset \vdash : \mathrm{Hp} . \supset . p`\overleftarrow{Q}``C`R = \Lambda .$

$[(1).*202{\cdot}51]\quad \supset . C`Q = Q``C`R .$ (3)

$[*201{\cdot}5.\mathrm{Hp}]\quad \supset . P``C`Q \subset P``C`R .$

$[(2)]\quad \supset . P``C`R = P``C`Q$ (4)

$\vdash . (3) . (4) . \supset \vdash . \mathrm{Prop}$

$*265{\cdot}45$. $\vdash :. P \in \omega_1 . Q \mathrel{G} P : x \in C`Q . \supset_x . \exists !\, \overleftarrow{Q}`x - \overleftarrow{P_{\mathrm{fn}}}`x : Q \in \omega .$

$S = \hat{x}\hat{y}\{x \in C`Q . y = \mathrm{min}_Q`(\overleftarrow{Q}`x - \overleftarrow{P_{\mathrm{fn}}}`x)\} . R = S \restriction \overleftarrow{S_{*}}`B`Q : \supset .$

$R_{\mathrm{po}} \in \omega . R_{\mathrm{po}} \mathrel{G} Q . P``C`R_{\mathrm{po}} = P``C`Q$

Dem.

$\vdash . *32{\cdot}181 . \supset \vdash : \mathrm{Hp} . \supset . S \mathrel{G} Q .$ (1)

$[*91{\cdot}59.*201{\cdot}18]\quad \supset . R_{\mathrm{po}} \mathrel{G} Q$ (2)

$\vdash . *263{\cdot}11 . \supset \vdash :. \mathrm{Hp} . \supset : x \in C`Q . \supset_x . \mathrm{E}!\, \breve{S}`x :$

$[*71{\cdot}571]\quad \supset : S \in \mathrm{Cls} \rightarrow 1 . C`Q \subset \mathrm{D}`S :$

$[(1)]\quad \supset : S \in \mathrm{Cls} \rightarrow 1 . \mathrm{Q}`S \subset \mathrm{D}`S :$

$[*122{\cdot}51.*96{\cdot}21]\quad \supset : R \in \mathrm{Prog} :$

$[*263{\cdot}1]\quad \supset : R_{\mathrm{po}} \in \omega$ (3)

$\vdash . (2) . (3) . *265{\cdot}441 . \supset \vdash : \mathrm{Hp} . \supset . P``C`R = P``C`Q$ (4)

$\vdash . (2) . (3) . (4) . \supset \vdash . \mathrm{Prop}$

$*265{\cdot}451$. $\vdash :. \mathrm{Hp}\, *265{\cdot}45 . \supset : x \in C`R . \supset . P(x \mapsto \breve{R}_1`x) \in \aleph_0$

Dem.

$\vdash . *265{\cdot}45 . *263{\cdot}14 . \supset \vdash :. \mathrm{Hp} . \supset : x \in C`R . \supset . \breve{R}_1`x = \breve{S}`x .$

$[\mathrm{Hp}]\quad \supset . \breve{R}_1`x \in \overleftarrow{P}`x - \overleftarrow{P_{\mathrm{fn}}}`x .$

$[*260{\cdot}131]\quad \supset . P(x \mapsto \breve{R}_1`x) \sim \in \mathrm{Cls\,induct}$ (1)

$\vdash . *265{\cdot}41 . \supset \vdash :. \mathrm{Hp} . \supset : x \in C`R . \supset . P(x \mapsto \breve{R}_1`x) \in \aleph_0 \cup \mathrm{Cls\,induct}$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

$*265 \cdot 452$. $\vdash : \mathrm{Hp}\, *265 \cdot 45 \,.\, \exists \,!\, P(x \mapsto \breve{R}_1 `x) \cap P(y \mapsto \breve{R}_1 `y) \,.\supset .\, x = y$

Dem.

$$\begin{array}{ll} \vdash . *201 \cdot 18 . \supset \vdash :. \mathrm{Hp} . \supset : xP(\breve{R}_1 `y) . yP(\breve{R}_1 `x) : & \\ [*14 \cdot 21] & \supset : x, y \in C`R . xP(\breve{R}_1 `y) . yP(\breve{R}_1 `x) : \\ [*204 \cdot 41 . *265 \cdot 45] & \supset : xR_{\mathrm{po}}(\breve{R}_1 `y) . yR_{\mathrm{po}}(\breve{R}_1 `x) : \\ [*204 \cdot 71] & \supset : x = y . \vee . xR_{\mathrm{po}}y : y = x . \vee . yR_{\mathrm{po}}x : \\ [*4 \cdot 41] & \supset : x = y . \vee . xR_{\mathrm{po}}y . yR_{\mathrm{po}}x : \\ [*204 \cdot 13 . *265 \cdot 45] & \supset : x = y :. \supset \vdash . \mathrm{Prop} \end{array}$$

$*265 \cdot 453$. $\vdash : \mathrm{Hp}\, *265 \cdot 45 \,.\, \kappa = \hat{\alpha}\{(\exists x) . x \in C`R . \alpha = P(x \mapsto \breve{R}_1 `x)\} . \supset .$

$\kappa \in \aleph_0 \cap \mathrm{Cl\ excl}`\aleph_0 \,.\, s`\kappa = P``C`R \cap \breve{P}_*``C`R$ [$*265 \cdot 451 \cdot 452$]

$*265 \cdot 454$. $\vdash :. \mathrm{Hp}\, *265 \cdot 453 : \kappa \in \aleph_0 \cap \mathrm{Cl\ excl}`\aleph_0 \,.\supset_\kappa .\, s`\kappa \in \aleph_0 : \supset .$

$P``C`R \cap \breve{P}_*``C`R \in \aleph_0$ [$*265 \cdot 453$]

$*265 \cdot 46$. $\vdash :. P \in \omega_1 \,.\, Q \in \omega \cap \mathrm{Rl}`P : x \in C`Q \,.\supset_x .\, \exists \,!\, \overleftarrow{Q}`x - \overleftarrow{P}_{\mathrm{fn}}`x :$

$\kappa \in \aleph_0 \cap \mathrm{Cl\ excl}`\aleph_0 \,.\supset_\kappa .\, s`\kappa \in \aleph_0 : \supset .\, P``C`Q \in \aleph_0$

[$*265 \cdot 41 \cdot 454 \,.\, *123 \cdot 421$]

$*265 \cdot 461$. $\vdash : \mathrm{Hp}\, *265 \cdot 46 \,.\supset .\, \exists \,!\, p`\overleftarrow{P}``C`Q$ [$*265 \cdot 46 \cdot 401$]

$*265 \cdot 47$. $\vdash :. P \in \omega_1 \,.\, Q \in \omega \cap \mathrm{Rl}`P : \kappa \in \aleph_0 \cap \mathrm{Cl\ excl}`\aleph_0 \,.\supset_\kappa .\, s`\kappa \in \aleph_0 : \supset .$

$\exists \,!\, p`\overleftarrow{P}``C`Q$ [$*265 \cdot 461 \cdot 431$]

$*265 \cdot 48$. $\vdash :. \kappa \in \aleph_0 \cap \mathrm{Cl\ excl}`\aleph_0 \,.\supset_\kappa .\, s`\kappa \in \aleph_0 : \supset : P \in \omega_1 \,.\, Q \in \omega \cap \mathrm{Rl}`P \,.\supset .\, E \,!\, \mathrm{lt}_P`C`Q$

[$*265 \cdot 47 \,.\, *250 \cdot 123$]

$*265 \cdot 481$. $\vdash : \mathrm{Mult\ ax} \,.\supset .\, \mathrm{Hp}\, *265 \cdot 48$ [$*113 \cdot 32 \,.\, *123 \cdot 52$]

$*265 \cdot 49$. $\vdash :. \mathrm{Mult\ ax} \,.\supset : P \in \omega_1 \,.\, Q \in \omega \cap \mathrm{Rl}`P \,.\supset .\, E \,!\, \mathrm{lt}_P`C`Q$ [$*265 \cdot 48 \cdot 481$]

This proposition shows that, assuming the multiplicative axiom, any progression of ordinals of the second class (*i.e.* consisting of series having $\aleph_0$ terms) has a limit in the second class, because $N \in \omega_1$.

$*265 \cdot 5$. $\vdash : P \in \omega_1 \,.\, Q \in \omega \,.\, C`Q \subset C`P \,.\, \sim E \,!\, \max_P`C`Q \,.$

$R = \hat{x}\hat{y}\{x \in C`Q \,.\, y = \min_Q`(\overleftarrow{P}`x \cap \overleftarrow{Q}`x)\} \,.\, S = R \upharpoonright \overleftarrow{R}_*`B`Q \,.\supset .$

$S_{\mathrm{po}} \in \omega \,.\, S_{\mathrm{po}} \Subset P \,.\, P``C`S_{\mathrm{po}} = P``C`Q$

Dem.

$$\begin{array}{lll} \vdash . *205 \cdot 11 . & \supset \vdash : \mathrm{Hp} . \supset . R \Subset P . R \Subset Q . & (1) \\ [*201 \cdot 18] & \supset . S_{\mathrm{po}} \Subset P . S_{\mathrm{po}} \Subset Q & (2) \\ \vdash . *205 \cdot 197 . & \supset \vdash : \mathrm{Hp} . x \in C`Q . \overleftarrow{Q}_*`x \subset \overrightarrow{P}_*`x . \supset . x = \max_P`\overleftarrow{Q}_*`x & (3) \\ \vdash . *263 \cdot 412 . *261 \cdot 26 . & \supset \vdash : \mathrm{Hp} . x \in \mathrm{G}`Q . \supset . E \,!\, \max_P`\overrightarrow{Q}`x & (4) \end{array}$$

$\vdash . (3) . (4) . *205{\cdot}193 . \supset \vdash : \mathrm{Hp} . x \epsilon C`Q . \overleftarrow{Q_*}`x \subset \overrightarrow{P_*}`x . \supset . \mathrm{E} ! \max_P`C`Q$ (5)

$\vdash . (5) . \mathrm{Transp} . \quad \supset \vdash :. \mathrm{Hp} . \supset : x \epsilon C`Q . \supset . \exists ! \overleftarrow{Q_*}`x - \overrightarrow{P_*}`x .$

$[*91{\cdot}542 . *202{\cdot}103] \quad \supset . \exists ! \overleftarrow{Q}`x \cap \overleftarrow{P}`x .$

$[*250{\cdot}121] \quad \supset . \mathrm{E} ! \breve{R}`x$ (6)

$\vdash . (1) . (6) . *122{\cdot}51 . \supset \vdash : \mathrm{Hp} . \supset . S \epsilon \mathrm{Prog} .$

$[*263{\cdot}1] \quad \supset . S_{\mathrm{po}} \epsilon \omega$ (7)

$\vdash . (2) . (7) . *265{\cdot}441 . \supset \vdash : \mathrm{Hp} . \supset . P``C`S_{\mathrm{po}} = P``C`Q$ (8)

$\vdash . (2) . (7) . (8) . \supset \vdash . \mathrm{Prop}$

***265·51.** $\vdash : \mathrm{Hp}\, *265{\cdot}48 . P \epsilon \omega_1 . \alpha \epsilon \aleph_0 \cap \mathrm{Cl}`C`P . \sim \mathrm{E} ! \max_P`\alpha . \supset . \mathrm{E} ! \mathrm{lt}_P`\alpha$

Dem.

$\vdash . *265{\cdot}5 . \supset \vdash : \mathrm{Hp} . \supset . (\exists S) . S \epsilon \omega \cap \mathrm{Rl}`P . P``C`S = \alpha$ (1)

$\vdash . (1) . *265{\cdot}48 . \supset \vdash . \mathrm{Prop}$

The following propositions follow easily.

***265·52.** $\vdash :. \mathrm{Hp}\, *265{\cdot}48 . P \epsilon \omega_1 . \supset :$

$\alpha \cap C`P \epsilon \aleph_0 \cup \mathrm{Cls\ induct} . \equiv . \exists ! C`P \cap p`\overleftarrow{P}``(\alpha \cap C`P)$ [*265·51·41]

***265·53.** $\vdash :: \mathrm{Hp}\, *265{\cdot}48 . \supset :. P \epsilon \omega_1 . \equiv :$

$P \epsilon \Omega : \alpha \cap C`P \epsilon \aleph_0 \cup \mathrm{Cls\ induct} . \equiv_\alpha . \exists ! C`P \cap p`\overleftarrow{P}``(\alpha \cap C`P)$

***265·54.** $\vdash : P \epsilon \omega_1 . \supset . \mathrm{G}`\nabla`P \subset \mathrm{lt}_P``C``(\omega \cap \mathrm{Rl}`P)$ [*265·5]

I.e. every limit-point in an ω_1 is the limit of a progression, which is what (following Hausdorff) may be conveniently called an ω-limit.

***265·55.** $\vdash : P \epsilon \omega_1 . \supset . \mathrm{G}`\nabla`P = \mathrm{lt}_P``C``(\omega \cap \mathrm{Rl}`P)$ [*265·54 . *216·602]

This proposition does not, like *265·48, assert that every progression in P has a limit, and therefore it does not require the hypothesis of *265·48.

SECTION F.

COMPACT SERIES, RATIONAL SERIES, AND CONTINUOUS SERIES.

Summary of Section F.

A *compact* series is one in which there is a term between any two, *i.e.* in which $P \subset P^2$, where P is the generating relation. We may call any relation P compact when $P \subset P^2$; then a *transitive* compact relation will be one for which $P = P^2$. Hence a serial relation P is compact whenever $P = P^2$. Compact series in general have certain properties, some of which have been already proved; but the majority of the interesting propositions in this subject come from adding some other condition besides compactness. Thus series having *Dedekindian continuity*, which have many important properties, are such as are compact and Dedekindian. *Rational* series (*i.e.* such as are ordinally similar to the series of all rational numbers, positive and negative, or, what is equivalent, to the series of rational proper fractions) are defined as such as are compact, without beginning or end, and consisting of $\aleph_0$ terms. Such series, also, have many important properties. A *continuous* series (in Cantor's sense) is a Dedekindian series containing a rational series in such a way that there are terms of the rational series between any two terms of the given series. This species of compact series also has many important properties. It consists of all series ordinally similar to the series of real numbers including 0 and ∞.

$*270$. COMPACT SERIES.

Summary of $*270$.

The propositions of the present number are mostly either obvious or repetitions of previously proved propositions. The latter are repeated here for convenience of reference.

We put $\text{comp} = \hat{P}(P \subseteq P^2)$ Df,

so that the class of compact series is $\text{Ser} \cap \text{comp}$. We have

$*270\cdot11$. $\vdash :. P \in \text{comp} . \equiv : xPy . \supset_{x,y} . \exists ! \overleftarrow{P}\text{`}x \cap \overrightarrow{P}\text{`}y$

$*270\cdot34$. $\vdash : P \in \text{trans} \cap \text{comp} . \supset . \text{s`}P = \text{sgm`}P$

The proposition $\text{s`}P_* = \text{sgm`}P_*$, which was proved in $*212$, is a particular case of the above.

$*270\cdot41$. $\vdash : P \in \text{Ser} \cap \text{comp} . \supset . \text{Nr`}P \subset \text{Ser} \cap \text{comp}$

I.e. a series which is similar to a compact series is a compact series.

$*270\cdot56$. $\vdash : P \in \text{Ser} . Q \in \Omega . \sim E ! B\text{`}\breve{P} . \sim E ! B\text{`}\breve{Q} . \supset . P^Q \in \text{Ser} \cap \text{comp}$

This proposition gives us a means of manufacturing compact series of various types, such as $\omega \exp_r \omega$, $\omega \exp_r \omega_1$, etc.

$*270\cdot01$. $\text{comp} = \hat{P}(P \subseteq P^2)$ Df

Here "comp" is an abbreviation for "compact." "Compact" series are the same as the series which Cantor calls "überall dicht."

$*270\cdot1$. $\vdash : P \in \text{comp} . \equiv . P \subseteq P^2$ [($*270\cdot01$)]

$*270\cdot11$. $\vdash :. P \in \text{comp} . \equiv : xPy . \supset_{x,y} . \exists ! \overleftarrow{P}\text{`}x \cap \overrightarrow{P}\text{`}y$ [$*270\cdot1$]

$*270\cdot12$. $\vdash : P \in \text{comp} . \equiv . \breve{P} \in \text{comp}$ [$*270\cdot11$]

$*270\cdot13$. $\vdash : P \in \text{trans} \cap \text{comp} . \equiv . P = P^2$ [$*270\cdot1 . *201\cdot1$]

$*270\cdot14$. $\vdash : P \in \text{Ser} \cap \text{comp} . \equiv . P \in \text{Rl`}J \cap \text{connex} . P = P^2 . \equiv . P \in \text{Ser} . P = P^2$ [$*270\cdot13$]

$*270\cdot15$ $\vdash : P \in \text{Ser} \cap \text{comp} . \equiv . P \in \text{Ser} . P_1 = \dot{\Lambda}$ [$*201\cdot65 . *270\cdot14$]

***270·2.** $\vdash : P \epsilon \text{comp} . \supset . \sim \exists ! \overrightarrow{\max_P}`\overrightarrow{P}`x$ [*205·25 . *270·1]

***270·201.** $\vdash : P \epsilon \text{comp} . \supset . \sim \exists ! \overrightarrow{\min_P}`\text{Q}`P . \sim \exists ! \overrightarrow{\max_P}`\text{D}`P$

Dem.

$\vdash . *37·25 . \quad \supset \vdash . \overrightarrow{\min_P}`\text{Q}`P = \breve{P}``\text{D}`P - (\breve{P}^2)``\text{D}`P \qquad (1)$

$\vdash . (1) . *270·1 . \supset \vdash : \text{Hp} . \supset . \overrightarrow{\min_P}`\text{Q}`P = \Lambda \qquad (2)$

Similarly $\quad \vdash : \text{Hp} . \supset . \overrightarrow{\max_P}`\text{D}`P = \Lambda \qquad (3)$

$\vdash . (2) . (3) . \supset \vdash .$ Prop

***270·202.** $\vdash : P \epsilon \text{comp} . \supset . \sim \exists ! \overrightarrow{\min_P}`\breve{P}``\alpha . \sim \exists ! \overrightarrow{\max_P}`P``\alpha$
[Proof as in *270·201]

***270·203.** $\vdash : P \epsilon \text{comp} . \supset . \sim \exists ! \overrightarrow{\text{seq}_P}`\iota`x$ [*206·42 . *270·1]

***270·204.** $\vdash :. P \epsilon \text{Ser} \cap \text{comp} . \text{E} ! \text{seq}_P`\alpha . \supset . \sim \text{E} ! \max_P`\alpha$
[*206·451 . *270·15]

***270·205.** $\vdash : P \epsilon \text{Ser} \cap \text{comp} . \supset . \text{lt}_P = \text{seq}_P$ [*207·1 . *270·204]

***270·21.** $\vdash : P \epsilon \text{Rl}`J \cap \text{comp} . x \epsilon C`P . \supset . x \, \text{lt}_P (\overrightarrow{P}`x)$ [*207·31 . *270·1]

***270·211.** $\vdash : P \epsilon \text{Rl}`J \cap \text{comp} . \supset . \text{D}`\text{lt}_P = C`P$ [*270·21]

Thus if a relation is compact and contained in diversity, every member of its field is a limit-point.

***270·212.** $\vdash : P \epsilon \text{connex} . \text{D}`\text{lt}_P = C`P . \supset . P \epsilon \text{comp}$

Dem.

$\vdash . *207·34 . \supset \vdash : \text{Hp} . \supset . C`P \subset - \text{Q}`(P \dot{-} P^2) .$

[*33·251] $\quad \supset . \text{Q}`(P \dot{-} P^2) = \Lambda .$

[*270·1] $\quad \supset . P \epsilon \text{comp} : \supset \vdash .$ Prop

***270·22.** $\vdash :. P \epsilon \text{Rl}`J \cap \text{connex} . \supset : P \epsilon \text{comp} . \equiv . \text{D}`\text{lt}_P = C`P . \equiv . \text{Q}`P \subset \text{D}`\text{lt}_P$
[*270·211·212 . *207·18]

***270·23.** $\vdash : P \epsilon \text{comp} - \iota`\dot{\Lambda} . \supset . P \sim \epsilon \text{Bord}$

Dem.

$\vdash . *270·201 . \supset \vdash : \text{Hp} . \supset . (\exists \alpha) . \alpha \subset C`P . \exists ! \alpha . \sim \exists ! \overrightarrow{\min_P}`\alpha .$

[*250·101] $\quad \supset . P \sim \epsilon \text{Bord} : \supset \vdash .$ Prop

***270·24.** $\vdash : P \epsilon \text{Ser} \cap \text{comp} - \iota`\dot{\Lambda} . \supset . C`P \sim \epsilon \text{Cls induct}$

Dem.

$\vdash . *270·23 . \supset \vdash : \text{Hp} . \supset . P \sim \epsilon \Omega .$

[*261·31] $\quad \supset . C`P \sim \epsilon \text{Cls induct} : \supset \vdash .$ Prop

***270·3.** $\vdash : P \epsilon \text{Ser} \cap \text{comp} . \supset . \text{sect}`P - \text{D}`P_\epsilon = P_*``C`P$
[*211·351 . *270·15]

$*270\cdot31$. $\vdash : P \epsilon \text{trans} \cap \text{comp} . \supset . \text{D}`P_\epsilon = \text{D}`(P_\epsilon \dot{\cap} I)$ [$*211\cdot51 . *270\cdot14$]

$*270\cdot32$. $\vdash : P \epsilon \text{trans} \cap \text{comp} . \supset . \overrightarrow{P}`x \epsilon \text{D}`(P_\epsilon \dot{\cap} I)$ [$*211\cdot452 . *270\cdot1$]

$*270\cdot321$. $\vdash : \overrightarrow{P}``C`P \subset \text{D}`(P_\epsilon \dot{\cap} I) . \supset . P \epsilon \text{comp}$ [$*211\cdot451 . *270\cdot1$]

$*270\cdot322$. $\vdash :. P \epsilon \text{trans} . \supset : \overrightarrow{P}``C`P \subset \text{D}`(P_\epsilon \dot{\cap} I) . \equiv . P \epsilon \text{comp}$
[$*270\cdot32\cdot321$]

$*270\cdot33$. $\vdash :. P \epsilon \text{Ser} . \supset : P \epsilon \text{comp} . \equiv . \text{G}`\text{max}_P \cap \text{G}`\text{seq}_P = \Lambda$
[$*211\cdot551 . *270\cdot14$]

$*270\cdot34$. $\vdash : P \epsilon \text{trans} \cap \text{comp} . \supset . \text{s}`P = \text{sgm}`P$ [$*270\cdot31 . (*212\cdot01\cdot02)$]

$*270\cdot35$. $\vdash :. P \epsilon \text{trans} \cap \text{connex} \cap \text{comp} . \supset : P \epsilon \text{Ded} . \equiv . \text{G}`\text{max}_P = - \text{G}`\text{seq}_P$
[$*214\cdot4 . *270\cdot13$]

$*270\cdot351$. $\vdash :. P \epsilon \text{Ser} . \supset : P \epsilon \text{comp} \cap \text{Ded} . \equiv . \text{G}`\text{max}_P = - \text{G}`\text{seq}_P$
[$*214\cdot41 . *270\cdot14$]

A series which is compact and Dedekindian is one which has Dedekindian continuity. Thus the above proposition states that a series which has Dedekindian continuity is a series such that every class has either a maximum or a sequent, but not both.

$*270\cdot352$. $\vdash : P \epsilon \text{Ser} \cap \text{comp} \cap \text{Ded} . \alpha \epsilon \text{sect}`P . \supset . \text{limax}_P`\alpha = \text{limin}_P`(C`P - \alpha)$
[$*214\cdot42$]

$*270\cdot36$. $\vdash : P \epsilon \text{Rl}`J \cap \text{comp} . \supset . \delta_P`C`P = \text{G}`P . \nabla`P = P$
[$*216\cdot2 . *270\cdot211 . (*216\cdot05)$]

$*270\cdot4$. $\vdash : P \epsilon \text{comp} . \supset . \text{Nr}`P \subset \text{comp}$

Dem.

$\vdash . *201\cdot2 . \quad \supset \vdash : S \epsilon P \,\overline{\text{smor}}\, Q . \supset . (S;Q)^2 = S;Q^2 . P = S;Q$ (1)

$\vdash . (1) . *270\cdot1 . \supset \vdash : P \epsilon \text{comp} . S \epsilon P \,\overline{\text{smor}}\, Q . \supset . S;Q \mathrel{\text{G}} S;Q^2 .$

[$*150\cdot31$] $\supset . \breve{S};S;Q \mathrel{\text{G}} \breve{S};S;Q^2 .$

[$*151\cdot252$] $\supset . Q \mathrel{\text{G}} Q^2 : \supset \vdash . \text{Prop}$

$*270\cdot401$. $\vdash : P \epsilon \text{comp} . \equiv . \text{N}_0\text{r}`P \subset \text{comp}$ [$*270\cdot4 . *155\cdot12$]

$*270\cdot41$. $\vdash : P \epsilon \text{Ser} \cap \text{comp} . \supset . \text{Nr}`P \subset \text{Ser} \cap \text{comp}$ [$*270\cdot4 . *204\cdot22$]

$*270\cdot411$. $\vdash : P \epsilon \text{Ser} \cap \text{comp} . \equiv . \text{N}_0\text{r}`P \subset \text{Ser} \cap \text{comp}$ [$*270\cdot41 . *155\cdot12$]

$*270\cdot42$. $\vdash : P \epsilon \text{comp} . \supset . P \upharpoonright \overleftarrow{P_*}`x, P \upharpoonright \overrightarrow{P_*}`x \epsilon \text{comp}$

Dem.

$\vdash . *270\cdot11 . \supset \vdash : \text{Hp} . y, z \epsilon \overleftarrow{P_*}`x . yPz . \supset . (\exists w) . yPw . wPz .$

[$*90\cdot16$] $\supset . (\exists w) . w \epsilon \overleftarrow{P_*}`x . yPw . wPz$ (1)

$\vdash . (1) . *270\cdot11 . \supset \vdash : \text{Hp} . \supset . P \upharpoonright \overleftarrow{P_*}`x \epsilon \text{comp}$ (2)

Similarly $\vdash : \text{Hp} . \supset . P \upharpoonright \overrightarrow{P_*}`x \epsilon \text{comp}$ (3)

$\vdash . (2) . (3) . \supset \vdash . \text{Prop}$

***270·5.** $\vdash : P, Q \in \mathrm{Ser} \cap \mathrm{comp} \,.\, C`P \cap C`Q = \Lambda \,.\, \sim(\mathrm{E} \,!\, B`\breve{P} \,.\, \mathrm{E} \,!\, B`Q) \,.\, \supset .$
$P \mathbin{\text{⤉}} Q \in \mathrm{Ser} \cap \mathrm{comp}$

Dem.

$\vdash . *160{\cdot}51 . \quad \supset \vdash : \mathrm{Hp} . \supset . (P \mathbin{\text{⤉}} Q)^2 = P^2 \cup Q^2 \cup \mathrm{D}`P \uparrow C`Q \cup C`P \uparrow \text{Ɑ}`Q$
$[*93{\cdot}103.\mathrm{Hp}] \qquad = P^2 \cup Q^2 \cup C`P \uparrow C`Q \quad (1)$
$\vdash . (1) . *270{\cdot}1 . \supset \vdash : \mathrm{Hp} . \supset . P \mathbin{\text{⤉}} Q \subseteq (P \mathbin{\text{⤉}} Q)^2 \quad (2)$
$\vdash . (2) . *204{\cdot}5 . \supset \vdash . \mathrm{Prop}$

***270·51.** $\vdash : P \in \mathrm{Ser} \cap \mathrm{comp} \,.\, C`P \subset \mathrm{Ser} \cap \mathrm{comp} \,.\, P \in \mathrm{Rel}^2 \,\mathrm{excl} \,.\, \supset .$
$\Sigma`P \in \mathrm{Ser} \cap \mathrm{comp}$

Dem.

$\vdash . *204{\cdot}52 . \supset \vdash : \mathrm{Hp} . \supset . \Sigma`P \in \mathrm{Ser} \quad (1)$
$\vdash . *162{\cdot}1 . \supset$
$\vdash . (\Sigma`P)^2 = (\dot{s}`C`P)^2 \cup (F{;}P)^2 \cup (\dot{s}`C`P) \,|\, (F{;}P) \cup (F{;}P) \,|\, (\dot{s}`C`P) \quad (2)$
$\vdash . *270{\cdot}1 . \supset \vdash : \mathrm{Hp} . x (\dot{s}`C`P) y . \supset . (\exists Q) . Q \in C`P . x Q^2 y .$
$[*41{\cdot}13] \qquad \supset . x (\dot{s}`C`P)^2 y \quad (3)$
$\vdash . *270{\cdot}1 . \supset \vdash : \mathrm{Hp} . x (F{;}P) y . \supset . x (F{;}P^2) y .$
$[*163{\cdot}12.*201{\cdot}2] \qquad \supset . x (F{;}P)^2 y \quad (4)$
$\vdash . (2) . (3) . (4) . *162{\cdot}1 . \supset \vdash : \mathrm{Hp} . \supset . \Sigma`P \subseteq (\Sigma`P)^2 \quad (5)$
$\vdash . (1) . (5) . \supset \vdash . \mathrm{Prop}$

The hypothesis of *270·51 is in excess of what is required for the conclusion, which only requires, in place of $P \in \mathrm{comp}$, that there should be no two consecutive relations in $C`P$ of which the first has a last term while the second has a first term. This is proved in the following proposition.

***270·52** $\vdash : P \in \mathrm{Ser} \cap \mathrm{Rel}^2 \,\mathrm{excl} \,.\, C`P \subset \mathrm{Ser} \cap \mathrm{comp} \,.$
$B``\breve{P}_1``(C`P \cap \mathrm{Cnv}``\text{Ɑ}`B) = \Lambda \,.\, \supset .\, \Sigma`P \in \mathrm{Ser} \cap \mathrm{comp}$

Dem.

$\vdash . *270{\cdot}1 . *163{\cdot}12 . \supset \vdash : \mathrm{Hp} . \supset . \dot{s}`C`P \subseteq (\dot{s}`C`P)^2 \quad (1)$
$\vdash . *201{\cdot}63 . \qquad \supset \vdash : \mathrm{Hp} . \supset . F{;}P = F{;}P_1 \cup F{;}P^2 \quad (2)$
$\vdash . *93{\cdot}103 . \qquad \supset \vdash :. \mathrm{Hp} . Q P_1 R . \supset : \mathrm{D}`Q = C`Q . \mathrm{v} . \text{Ɑ}`R = C`R \quad (3)$
$\vdash . (3) . \supset \vdash :. \mathrm{Hp} . x (F{;}P_1) y . \supset :$
$(\exists Q, R) : x \in \mathrm{D}`Q . y \in C`R . \mathrm{v} . x \in C`Q . y \in \text{Ɑ}`R : Q P_1 R :$
$[*33{\cdot}13{\cdot}131{\cdot}17]$
$\supset : (\exists Q, R, z) : x Q z . z \in C`Q . y \in C`R . \mathrm{v} . x \in C`Q . z \in C`R . z R y : Q P_1 R :$
$[*150{\cdot}52.*201{\cdot}63] \ \supset : x \{(\dot{s}`C`P) \,|\, (F{;}P)\} y . \mathrm{v} . x \{(F{;}P) \,|\, (\dot{s}`C`P)\} y :$
$[*162{\cdot}1] \qquad \supset : x (\Sigma`P)^2 y \quad (4)$
$\vdash . *163{\cdot}12 . *201{\cdot}2 . \supset \vdash : \mathrm{Hp} . \supset . F{;}P^2 = (F{;}P)^2 \quad (5)$
$\vdash . (2) . (5) . *162{\cdot}1 . \supset \vdash : \mathrm{Hp} . \supset . F{;}P \subseteq (\Sigma`P)^2 \quad (6)$
$\vdash . (1) . (6) . *162{\cdot}1 . \supset \vdash : \mathrm{Hp} . \supset . \Sigma`P \subseteq (\Sigma`P)^2 \quad (7)$
$\vdash . (4) . (7) . *204{\cdot}52 . \supset \vdash . \mathrm{Prop}$

***270·521.** $\vdash :. P \epsilon \mathrm{Ser} \cap \mathrm{Rel}^2\,\mathrm{excl} \,.\, C\text{‘}P \subset \mathrm{Ser} \cap \mathrm{comp} :$

$C\text{‘}P \cap \mathrm{Cnv}\text{‘‘}\text{Ɑ}\text{‘}B = \Lambda \,.\, \mathrm{v} \,.\, C\text{‘}P \cap \text{Ɑ}\text{‘}B = \Lambda : \supset . \breve{\Sigma}\text{‘}P \epsilon \mathrm{Ser} \cap \mathrm{comp}$ [*270·52]

***270·53.** $\vdash : P \epsilon \mathrm{Ser} \,.\, Q \epsilon \mathrm{Ser} \cap \mathrm{comp} \,.\, \sim(\mathrm{E}\,!\,B\text{‘}Q \,.\, \mathrm{E}\,!\,B\text{‘}\breve{Q}) \,.\supset .\, P \times Q \epsilon \mathrm{Ser} \cap \mathrm{comp}$

Dem.

$\vdash . *166\cdot1 \,.\supset \vdash .\, P \times Q = \Sigma\text{‘}Q \underset{;}{\downarrow} ;P$ (1)

$\vdash . *165\cdot21 \,.\supset \vdash .\, Q \underset{;}{\downarrow} ;P \epsilon \mathrm{Rel}^2\,\mathrm{excl}$ (2)

$\vdash . *165\cdot25 . *204\cdot21 \,.\supset \vdash : \mathrm{Hp} \,.\, \dot{\exists}\,!\,P \,.\supset .\, Q \underset{;}{\downarrow} ;P \epsilon \mathrm{Ser}$ (3)

$\vdash . *165\cdot26 . *270\cdot4 \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, C\text{‘}Q \underset{;}{\downarrow} ;P \subset \mathrm{Ser} \cap \mathrm{comp}$ (4)

$\vdash . *151\cdot5 . *165\cdot26 \,.\supset \vdash : \mathrm{Hp} \,.\sim \mathrm{E}\,!\,B\text{‘}Q \,.\supset .\, C\text{‘}Q \underset{;}{\downarrow} ;P \cap \text{Ɑ}\text{‘}B = \Lambda$ (5)

$\vdash . *151\cdot5 . *165\cdot26 \,.\supset \vdash : \mathrm{Hp} \,.\sim \mathrm{E}\,!\,B\text{‘}\breve{Q} \,.\supset .\, C\text{‘}Q \underset{;}{\downarrow} ;P \cap \mathrm{Cnv}\text{‘‘}\text{Ɑ}\text{‘}B = \Lambda$ (6)

$\vdash . (1) . (2) . (3) . (4) . (5) . (6) . *270\cdot521 \,.\supset$

$\vdash : \mathrm{Hp} \,.\, \dot{\exists}\,!\,P \,.\supset .\, P \times Q \epsilon \mathrm{Ser} \cap \mathrm{comp}$ (7)

$\vdash . *166\cdot13 \,.\supset \vdash : P = \dot{\Lambda} \,.\supset .\, P \times Q \epsilon \mathrm{Ser} \cap \mathrm{comp}$ (8)

$\vdash . (7) . (8) \,.\supset \vdash .$ Prop

***270·54.** $\vdash : P \epsilon \mathrm{Ser} \cap \mathrm{comp} \,.\sim \mathrm{E}\,!\,B\text{‘}\breve{P} \,.\, x \sim\epsilon C\text{‘}P \,.\supset .\, P \mathbin{+\!\!\rightarrow} x \epsilon \mathrm{Ser} \cap \mathrm{comp}$

Dem.

$\vdash . *204\cdot51 \,.\quad \supset \vdash : \mathrm{Hp} \,.\supset .\, P \mathbin{+\!\!\rightarrow} x \epsilon \mathrm{Ser}$ (1)

$\vdash . *161\cdot1 \,.\quad \supset \vdash : \mathrm{Hp} \,.\supset .\, (P \mathbin{+\!\!\rightarrow} x)^2 = P^2 \cup \mathrm{D}\text{‘}P \uparrow \iota\text{‘}x$

[*93·103] $\qquad = P^2 \cup C\text{‘}P \uparrow \iota\text{‘}x$ (2)

$\vdash . (2) . *270\cdot1 \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, P \mathbin{+\!\!\rightarrow} x \Subset (P \mathbin{+\!\!\rightarrow} x)^2$ (3)

$\vdash . (1) . (3) \,.\supset \vdash .$ Prop

***270·541.** $\vdash : P \epsilon \mathrm{Ser} \cap \mathrm{comp} \,.\sim \mathrm{E}\,!\,B\text{‘}P \,.\, x \sim\epsilon C\text{‘}P \,.\supset .\, x \mathbin{\leftarrow\!\!+} P \epsilon \mathrm{Ser} \cap \mathrm{comp}$

[Proof as in *270·54]

***270·55.** $\vdash : P \epsilon \Omega \,.\, C\text{‘}P \subset \mathrm{Ser} \,.\sim \mathrm{E}\,!\,B\text{‘}\breve{P} \,.\, C\text{‘}P \cap \mathrm{Cnv}\text{‘‘}\text{Ɑ}\text{‘}B = \Lambda \,.\supset .$

$\Pi\text{‘}P \epsilon \mathrm{Ser} \cap \mathrm{comp}$

Dem.

$\vdash . *251\cdot3 \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, \Pi\text{‘}P \epsilon \mathrm{Ser}$ (1)

$\vdash . *250\cdot21 . *93\cdot103 \,.\supset$

$\vdash : \mathrm{Hp} \,.\, Q \epsilon C\text{‘}P \,.\, M \epsilon F_\Delta\text{‘}C\text{‘}P \,.\supset .\, (\exists x) . (M\text{‘}\breve{P}_1\text{‘}Q)(\breve{P}_1\text{‘}Q)\, x$ (2)

$\vdash . *200\cdot43 \,.\supset$

$\vdash : \mathrm{Hp}(2) . (M\text{‘}\breve{P}_1\text{‘}Q)(\breve{P}_1\text{‘}Q)x \,.\, L = M \upharpoonright (-\iota\text{‘}\breve{P}_1\text{‘}Q) \cup x \downarrow (\breve{P}_1\text{‘}Q) \,.\supset .\, M(\Pi\text{‘}P)L$ (3)

$\vdash . *200\cdot43 \,.\supset$

$\vdash : \mathrm{Hp}(3) \,.\, N \epsilon F_\Delta\text{‘}C\text{‘}P \,.\, (M\text{‘}Q)\, Q\, (N\text{‘}Q) \,.\, M \upharpoonright \overrightarrow{P}\text{‘}Q = N \upharpoonright \overrightarrow{P}\text{‘}Q \,.\supset .\, L(\Pi\text{‘}P)N$ (4)

$\vdash . (2) . (3) . (4) \,.\supset$

$\vdash : \mathrm{Hp} \,.\, M, N \epsilon F_\Delta\text{‘}C\text{‘}P \,.\, Q \epsilon C\text{‘}P \,.\, (M\text{‘}Q)\, Q\, (N\text{‘}Q) \,.\, M \upharpoonright \overrightarrow{P}\text{‘}Q = N \upharpoonright \overrightarrow{P}\text{‘}Q \,.\supset .$

$(\exists L) \,.\, M(\Pi\text{‘}P)L \,.\, L(\Pi\text{‘}P)N$ (5)

$\vdash . (5) . *200\cdot43 \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, \Pi\text{‘}P \Subset (\Pi\text{‘}P)^2$ (6)

$\vdash . (1) . (6) \,.\supset \vdash .$ Prop

$*270\cdot56.\quad \vdash : P \in \mathrm{Ser} \,.\, Q \in \Omega \,.\, \sim \mathrm{E}\,!\, B`\breve{P} \,.\, \sim \mathrm{E}\,!\, B`\breve{Q} \,.\, \supset .\, P^Q \in \mathrm{Ser} \cap \mathrm{comp}$

Dem.

$\vdash .\, *176\cdot151 \,.\quad \supset \vdash : P = \dot{\Lambda} \,.\, \supset .\, P^Q \in \mathrm{Ser} \cap \mathrm{comp}$ (1)

$\vdash .\, *176\cdot181\cdot182 \,.\quad \supset \vdash .\, P^Q \,\mathrm{smor}\, \Pi`P \underset{\cdot\cdot}{\downarrow} ;Q$ (2)

$\vdash .\, *165\cdot25 \,.\, *251\cdot121 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \dot{\exists}\,!\, P \,.\, \supset .\, P \underset{\cdot\cdot}{\downarrow} ;Q \in \Omega$ (3)

$\vdash .\, *165\cdot26 \,.\, *204\cdot21 \,.\quad \supset \vdash : \mathrm{Hp} \,.\, \supset .\, C`P \underset{\cdot\cdot}{\downarrow} ;Q \subset \mathrm{Ser}$ (4)

$\vdash .\, *165\cdot25 \,.\, *151\cdot5 \,.\quad \supset \vdash : \mathrm{Hp} \,.\, \dot{\exists}\,!\, P \,.\, \supset .\, \sim \mathrm{E}\,!\, B`\mathrm{Cnv}`P \underset{\cdot\cdot}{\downarrow} ;Q$ (5)

$\vdash .\, *165\cdot26 \,.\, *151\cdot5 \,.\quad \supset \vdash : \mathrm{Hp} \,.\, \supset .\, C`P \underset{\cdot\cdot}{\downarrow} ;Q \cap \mathrm{Cnv}``\mathrm{Cl}`B = \Lambda$ (6)

$\vdash .\, (3) \,.\, (4) \,.\, (5) \,.\, (6) \,.\, *270\cdot55 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \dot{\exists}\,!\, P \,.\, \supset .\, \Pi`P \underset{\cdot\cdot}{\downarrow} ;Q \in \mathrm{Ser} \cap \mathrm{comp} \,.$

$[(2) . *270\cdot41] \qquad \supset .\, P^Q \in \mathrm{Ser} \cap \mathrm{comp}$ (7)

$\vdash .\, (1) \,.\, (7) \,.\, \supset \vdash .\,$ Prop

By means of the above proposition, compact series can be manufactured by taking series of such types as $\omega \exp_r \omega$, $\omega \exp_r \omega_1$, $\omega_1 \exp_r \omega$, etc. Any power $\alpha \exp_r \beta$ consists of compact series, if β is an ordinal having no immediate predecessor, and α is any serial number having no immediate predecessor (*i.e.* not formed by adding $\dot{1}$ to a serial number).

$*271$. MEDIAN CLASSES IN SERIES.

Summary of $*271$.

We shall call a class α a "median" class in P if $\alpha \subset C'P$ and there is a member of α between any two terms of which one has the relation P to the other. When this is the case, we have

$$xPy \,.\supset_{x,y} .\, (\exists z) \,.\, z \in \alpha \,.\, xPz \,.\, zPy,$$

i.e.
$$P \subseteq P \upharpoonright \alpha \,|\, P.$$

Thus P cannot contain any median class unless P is compact. Conversely, if P is compact, $C'P$ is a median class. Hence relations containing median classes are the same as compact relations. Median classes are important in dealing with rational and continuous series: the rationals are a median class in the series of real numbers, and the series which Cantor calls continuous are characterized by the fact that, in addition to being Dedekindian, they contain a median class which forms a series of the same type as the rationals. If P is a compact series, the class $\overrightarrow{P}``\text{Cl}'P$ is a median class in the series $\varsigma'P$ ($*271\cdot31$). This fact is used in proving that the series of segments of a rational series is a continuous series.

Our definition is

$$\text{med} = \hat{\alpha}\hat{P}\,(\alpha \subset C'P \,.\, P \subseteq P \upharpoonright \alpha \,|\, P) \quad \text{Df.}$$

Thus $\overrightarrow{\text{med}}'P$ will be the median classes of P, and "$P \in \text{Cl}'\text{med}$" means that there are median classes of P. We have $\text{Cl}'\text{med} = \text{comp}$ ($*271\cdot18$); also

$*271\cdot15$. $\vdash : \alpha \,\text{med}\, P \,.\supset .\, P, P \upharpoonright\!\!\!\!\upharpoonright \alpha \in \text{comp}$

$*271\cdot16$. $\vdash : (\alpha \cap C'P) \,\text{med}\, P \,.\equiv .\, (\alpha \cap \text{D}'P) \,\text{med}\, P \,.\equiv .\, (\alpha \cap \text{Cl}'P) \,\text{med}\, P \,.$
$\equiv .\, (\alpha \cap \text{D}'P \cap \text{Cl}'P) \,\text{med}\, P$

If P is a series, and $\alpha \subset C'P$, α is a median class when, and only when, its derivative is $\text{Cl}'P$, *i.e.*

$*271\cdot2$. $\vdash :.\, P \in \text{Ser} \,.\, \alpha \subset C'P \,.\supset : \alpha \,\text{med}\, P \,.\equiv .\, \text{Cl}'P = \delta_P{}'\alpha$

An important proposition is

$*271\cdot39$. $\vdash : P, Q \in \text{Ser} \cap \text{Ded} \,.\, \alpha \,\text{med}\, P \,.\, \beta \,\text{med}\, Q \,.\, (P \upharpoonright\!\!\!\!\upharpoonright \alpha) \,\text{smor}\, (Q \upharpoonright\!\!\!\!\upharpoonright \beta) \,.\supset .$
$P \,\text{smor}\, Q$

I.e. if P and Q are Dedekindian series, and α, β are median classes of P and Q respectively, then if $P \upharpoonright\!\!\!\!\upharpoonright \alpha$ and $Q \upharpoonright\!\!\!\!\upharpoonright \beta$ are similar, so are P and Q. This

proposition is proved by showing that P is similar to the series of segments of $P \upharpoonright \alpha$, the correlator being lt_P with its converse domain limited (*271·37). Another important proposition is

*271·4. $\vdash : S \in P \,\overline{\mathrm{smor}}\, Q \,.\, \beta \,\mathrm{med}\, Q \,.\, \supset .\, (S``\beta) \,\mathrm{med}\, P$

I.e. a correlator of P with Q correlates median classes with median classes.

The above two propositions are used in *275·3·31, which prove that two series which are continuous (in Cantor's sense) are similar, and that a series similar to a continuous series is continuous.

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*271·01. $\mathrm{med} = \hat{\alpha}\hat{P}(\alpha \subset C`P \,.\, P \in P \upharpoonright \alpha | P)$ Df

*271·1. $\vdash :. \alpha \,\mathrm{med}\, P \,.\equiv : \alpha \subset C`P \,.\, P \in P \upharpoonright \alpha | P : \equiv :$

$\alpha \subset C`P : xPy \,.\supset_{x,y} .\, \exists ! \alpha \cap \overleftarrow{P}`x \cap \overrightarrow{P}`y$ [(*271·01)]

*271·11. $\vdash : \alpha \,\mathrm{med}\, P \,.\equiv .\, \alpha \,\mathrm{med}\, \breve{P}$ [*271·1]

*271·13. $\vdash : \alpha \,\mathrm{med}\, P \,.\, \beta \subset C`P \,.\supset .\, (\alpha \cup \beta) \,\mathrm{med}\, P$ [*271·1]

*271·14. $\vdash : \alpha \,\mathrm{med}\, P \,.\supset .\, C`P \upharpoonright \alpha \,\mathrm{med}\, (P \upharpoonright \alpha)$

Dem.

$\vdash .\, *271·1 \,.\supset$

$\vdash :. \alpha \,\mathrm{med}\, P \,.\supset : x, y \in \alpha \,.\, xPy \,.\supset_{x,y} .\, (\exists z) \,.\, z \in \alpha \,.\, xPz \,.\, zPy \,.$

[*35·102] $\supset_{x,y} .\, (\exists z) \,.\, z \in \alpha \,.\, x(P \upharpoonright \alpha) z \,.\, z(P \upharpoonright \alpha) y :$

[*35·102.*271·1] $\supset : C`P \upharpoonright \alpha \,\mathrm{med}\, (P \upharpoonright \alpha) :. \supset \vdash .\, \mathrm{Prop}$

*271·15. $\vdash : \alpha \,\mathrm{med}\, P \,.\supset .\, P, P \upharpoonright \alpha \in \mathrm{comp}$

Dem.

$\vdash .\, *271·1 \,.\quad \supset \vdash : \mathrm{Hp} \,.\supset .\, P \in P^2 \,.$

[*270·1] $\supset .\, P \in \mathrm{comp}$ (1)

$\vdash .\, (1) \,.\, *271·14 \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, P \upharpoonright \alpha \in \mathrm{comp}$ (2)

$\vdash .\, (1) \,.\, (2) \,.\supset \vdash .\, \mathrm{Prop}$

*271·16. $\vdash : (\alpha \cap C`P) \,\mathrm{med}\, P \,.\equiv .\, (\alpha \cap D`P) \,\mathrm{med}\, P \,.\equiv .\, (\alpha \cap \mathrm{Cl}`P) \,\mathrm{med}\, P \,.$

$\equiv .\, (\alpha \cap D`P \cap \mathrm{Cl}`P) \,\mathrm{med}\, P$

Dem.

$\vdash .\, *271·1 \,.\, *33·15 \,.\supset$

$\vdash :. (\alpha \cap C`P) \,\mathrm{med}\, P \,.\equiv : xPy \,.\supset_{x,y} .\, \exists ! \alpha \cap D`P \cap \overleftarrow{P}`x \cap \overrightarrow{P}`y :$

[*271·1] $\equiv : (\alpha \cap D`P) \,\mathrm{med}\, P$ (1)

$\vdash .\, *271·1 \,.\, *33·151 \,.\supset \vdash : (\alpha \cap C`P) \,\mathrm{med}\, P \,.\equiv .\, (\alpha \cap \mathrm{Cl}`P) \,\mathrm{med}\, P$ (2)

$\vdash .\, *271·1 \,.\, *33·15·151 \,.\supset$

$\vdash :. (\alpha \cap C`P) \,\mathrm{med}\, P \,.\equiv : xPy \,.\supset_{x,y} .\, \exists ! \alpha \cap D`P \cap \mathrm{Cl}`P \cap \overleftarrow{P}`x \cap \overrightarrow{P}`y :$

[*271·1] $\equiv : (\alpha \cap D`P \cap \mathrm{Cl}`P) \,\mathrm{med}\, P$ (3)

$\vdash .\, (1) \,.\, (2) \,.\, (3) \,.\supset \vdash .\, \mathrm{Prop}$

***271·17.** $\vdash : P \in \text{comp} . \supset . C\text{‘}P, \text{D‘}P, \text{Ɑ‘}P \in \overrightarrow{\text{med}}\text{‘}P$

Dem.

$\vdash . *35\cdot452 . *270\cdot1 . \supset \vdash : P \in \text{comp} . \supset . P \subseteq P \upharpoonright \text{Ɑ‘}P \,|\, P .$

$[*271\cdot1] \qquad \supset . \text{Ɑ‘}P \in \overrightarrow{\text{med}}\text{‘}P . \qquad (1)$

$[*271\cdot13] \qquad \supset . C\text{‘}P \in \overrightarrow{\text{med}}\text{‘}P . \qquad (2)$

$[*271\cdot16] \qquad \supset . \text{D‘}P \in \overrightarrow{\text{med}}\text{‘}P \qquad (3)$

$\vdash . (1) . (2) . (3) . \supset \vdash . \text{Prop}$

***271·18.** $\vdash . \text{Ɑ‘med} = \text{comp} \quad [*271\cdot15\cdot17]$

***271·2.** $\vdash :. P \in \text{Ser} . \alpha \subset C\text{‘}P . \supset : \alpha \,\text{med}\, P . \equiv . \text{Ɑ‘}P = \delta_P\text{‘}\alpha \quad [*216\cdot13 . *271\cdot1]$

***271·3.** $\vdash : P \in \text{Rl‘}J \cap \text{trans} . \alpha \,\text{med}\, P . \supset . \overrightarrow{P}\text{“}\alpha \,\text{med}\, (\varsigma\text{‘}P)$

Dem.

$\vdash . *271\cdot15 . *270\cdot34 . \supset \vdash : \text{Hp} . \supset . \varsigma\text{‘}P = \text{sgm‘}P .$

$[*212\cdot11] \qquad \supset . \varsigma\text{‘}P = \hat{\beta}\hat{\gamma}\{\beta, \gamma \in \text{D‘}(P_\epsilon \dot{\cap} I) . \exists ! \gamma - \beta\} \quad (1)$

$\vdash . (1) . *211\cdot12 . \supset \vdash : \text{Hp} . \beta (\varsigma\text{‘}P) \gamma . \supset . \exists ! \gamma - \beta . P\text{“}\gamma = \gamma . P\text{“}\beta = \beta .$

$[*37\cdot1] \qquad \supset . (\exists x, y) . x \in \gamma - \beta . xPy . y \in \gamma .$

$[*271\cdot1] \qquad \supset . (\exists x, y, z) . x \in \gamma - \beta . xPz . zPy . z \in \alpha . y \in \gamma .$

$[*201\cdot12] \qquad \supset . (\exists x, y, z) . x \in \gamma - \beta . xPz . zPy . z \in \alpha . y \in \gamma . \sim (yPz) .$

$[*32\cdot18] \qquad \supset . (\exists z) . z \in \alpha . \exists ! \overrightarrow{P}\text{‘}z - \beta . \exists ! \gamma - \overrightarrow{P}\text{‘}z .$

$[(1) . *270\cdot322] \qquad \supset . (\exists z) . z \in \alpha . \beta (\varsigma\text{‘}P) (\overrightarrow{P}\text{‘}z) . (\overrightarrow{P}\text{‘}z) (\varsigma\text{‘}P) \gamma \qquad (2)$

$\vdash . (2) . *271\cdot1 . \supset \vdash . \text{Prop}$

***271·31.** $\vdash : P \in \text{Rl‘}J \cap \text{trans} \cap \text{comp} . \supset . \overrightarrow{P}\text{“Ɑ‘}P \,\text{med}\, (\varsigma\text{‘}P) \quad [*271\cdot3\cdot17]$

The following propositions lead up to the proposition

***271·37.** $\vdash : P \in \text{Ser} \cap \text{Ded} . \alpha \,\text{med}\, P . \supset . \text{lt}_P \upharpoonright C\text{‘}\varsigma\text{‘}(P \upharpoonleft \alpha) \in P \,\overline{\text{smor}}\, \{\varsigma\text{‘}(P \upharpoonleft \alpha)\}$

whence, if α is a median class of P, P is similar to the series of segments of $P \upharpoonleft \alpha$. This proposition is used in proving that every continuous series is similar to the series of segments of a rational series.

***271·32.** $\vdash : P \in \text{Ser} . R = P \upharpoonleft \alpha . \beta \in \text{D‘}R_\epsilon . \text{E} ! \,\text{lt}_P\text{‘}\beta . \supset . \beta = R\text{“}\beta = \alpha \cap \overrightarrow{P}\text{‘lt}_P\text{‘}\beta$

Dem.

$\vdash . *205\cdot9 . \supset \vdash : \text{Hp} . \alpha \cap C\text{‘}P \sim \in 1 . \supset . \overrightarrow{\text{max}}_R\text{‘}\beta = \overrightarrow{\text{max}}_P(\alpha \cap \beta)$

$[*37\cdot413 . *211\cdot11] \qquad = \overrightarrow{\text{max}}_P\text{‘}\beta$

$[*207\cdot13] \qquad = \Lambda \qquad (1)$

$\vdash . (1) . *200\cdot35 . \supset \vdash : \text{Hp} . \supset . \overrightarrow{\text{max}}_R\text{‘}\beta = \Lambda .$

$[*211\cdot42\cdot12] \qquad \supset . \beta = R\text{“}\beta \qquad (2)$

$\vdash . *207\cdot231 . \quad \supset \vdash : \text{Hp} . \supset . P\text{“}\beta = \overrightarrow{P}\text{‘lt}_P\text{‘}\beta .$

$[*37\cdot413] \qquad \supset . R\text{“}\beta = \alpha \cap \overrightarrow{P}\text{‘lt}_P\text{‘}\beta \qquad (3)$

$\vdash . (2) . (3) . \supset \vdash . \text{Prop}$

***271·321.** $\vdash : P \epsilon \text{Ser} \,.\, R = P \upharpoonright \alpha \,.\, \supset \,.\, \text{lt}_P \upharpoonright \text{D}'R_\epsilon \,\epsilon\, 1 \to 1$

Dem.

$\vdash \,.\, *271{\cdot}32 \,.\, \supset \vdash : \text{Hp} \,.\, \beta, \gamma \,\epsilon\, \text{D}'R_\epsilon \,.\, \text{lt}_P{}'\beta = \text{lt}_P{}'\gamma \,.\, \supset \,.\, \beta = \gamma : \supset \vdash \,.\, \text{Prop}$

***271·322.** $\vdash : P \epsilon \text{Ser} \,.\, R = P \upharpoonright \alpha \,.\, \supset \,.\, \text{lt}_P \,;\, \varsigma' R \subseteq P$

Dem.

$\vdash \,.\, *212{\cdot}23 \,.\, \supset \vdash :. \text{Hp} \,.\, \supset : x (\text{lt}_P \,;\, \varsigma' R)\, y \,.\, \equiv .$

$(\exists \beta, \gamma) \,.\, \beta, \gamma \,\epsilon\, \text{D}'R_\epsilon \,.\, \beta \subset \gamma \,.\, \beta \neq \gamma \,.\, x = \text{lt}_P{}'\beta \,.\, y = \text{lt}_P{}'\gamma \,.$

[*207·231] $\supset \,.\, (\exists \beta, \gamma) \,.\, \beta, \gamma \,\epsilon\, \text{D}'R_\epsilon \,.\, \beta \subset \gamma \,.\, \beta \neq \gamma \,.\, \overrightarrow{P}'x = P``\beta \,.\, \overrightarrow{P}'y = P``\gamma \,.$

[*37·2.*271·321] $\supset \,.\, \overrightarrow{P}'x \subset \overrightarrow{P}'y \,.\, x \neq y \,.$

[*204·33] $\supset \,.\, xPy :. \supset \vdash \,.\, \text{Prop}$

***271·33.** $\vdash : P \,\epsilon\, \text{trans} \,.\, \alpha \,\text{med}\, P \,.\, \supset \,.\, \overrightarrow{P}'x = P``(\alpha \cap \overrightarrow{P}'x)$

Dem.

$\vdash \,.\, *201{\cdot}501 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, P``\overrightarrow{P}'x \subset \overrightarrow{P}'x \,.$

[*37·2] $\supset \,.\, P``(\alpha \cap \overrightarrow{P}'x) \subset \overrightarrow{P}'x$ (1)

$\vdash \,.\, *271{\cdot}1 \,.\, \supset \vdash :. \text{Hp} \,.\, \supset : yPx \,.\, \supset \,.\, (\exists z) \,.\, yPz \,.\, z \,\epsilon\, \alpha \,.\, zPx \,.$

[*37·1] $\supset \,.\, y \,\epsilon\, P``(\alpha \cap \overrightarrow{P}'x)$ (2)

$\vdash \,.\, (1) \,.\, (2) \,.\, \supset \vdash \,.\, \text{Prop}$

***271·331.** $\vdash : \text{Hp}\, *271{\cdot}33 \,.\, R = P \upharpoonright \alpha \,.\, \supset \,.\, \alpha \cap \overrightarrow{P}'x = R``(\alpha \cap \overrightarrow{P}'x)$

Dem.

$\vdash \,.\, *271{\cdot}33 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, \alpha \cap \overrightarrow{P}'x = \alpha \cap P``(\alpha \cap \overrightarrow{P}'x)$

[*37·413] $= R``(\alpha \cap \overrightarrow{P}'x) : \supset \vdash \,.\, \text{Prop}$

***271·332.** $\vdash : P \,\epsilon\, \text{Ser} \,.\, \alpha \,\text{med}\, P \,.\, x \,\epsilon\, C'P \,.\, \supset \,.\, x = \text{lt}_P{}'(\alpha \cap \overrightarrow{P}'x)$

Dem.

$\vdash \,.\, *271{\cdot}331 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, \alpha \cap \overrightarrow{P}'x \subset P``(\alpha \cap \overrightarrow{P}'x) \,.$

[*205·123] $\supset \,.\, \overrightarrow{\text{max}_P}{}'(\alpha \cap \overrightarrow{P}'x) = \Lambda$ (1)

$\vdash \,.\, (1) \,.\, *271{\cdot}33 \,.\, \supset$

$\vdash : \text{Hp} \,.\, \supset \,.\, x \,\epsilon\, C'P \,.\, \overrightarrow{P}'x = P``(\alpha \cap \overrightarrow{P}'x) \,.\, \sim \text{E} \,!\, \text{max}_P{}'(\alpha \cap \overrightarrow{P}'x) \,.$

[*207·521] $\supset \,.\, x = \text{lt}_P{}'(\alpha \cap \overrightarrow{P}'x) : \supset \vdash \,.\, \text{Prop}$

***271·34.** $\vdash : P \,\epsilon\, \text{Ser} \,.\, \alpha \,\text{med}\, P \,.\, \supset \,.\, P = \text{lt}_P \,;\, \varsigma'(P \upharpoonright \alpha)$

Dem.

$\vdash \,.\, *271{\cdot}331 \,.\, *211{\cdot}11 \,.\, \supset \vdash : \text{Hp} \,.\, R = P \upharpoonright \alpha \,.\, \supset \,.\, \alpha \cap \overrightarrow{P}'x \,\epsilon\, \text{D}'R_\epsilon$ (1)

$\vdash \,.\, *204{\cdot}33 \,.\, \supset \vdash : \text{Hp} \,.\, xPy \,.\, \supset \,.\, \alpha \cap \overrightarrow{P}'x \subset \alpha \cap \overrightarrow{P}'y$ (2)

$\vdash \,.\, *271{\cdot}332 \,.\, \supset \vdash : \text{Hp} \,.\, xPy \,.\, \supset \,.\, x = \text{lt}_P{}'(\alpha \cap \overrightarrow{P}'x) \,.\, y = \text{lt}_P{}'(\alpha \cap \overrightarrow{P}'y) \,.$ (3)

[*204·1] $\supset \,.\, \alpha \cap \overrightarrow{P}'x \neq \alpha \cap \overrightarrow{P}'y$ (4)

$\vdash . (1) . (2) . (4) . *212\cdot23 . \supset$

$\vdash :. \mathrm{Hp} . R = P \mathbin{\underset{\smile}{\restriction}} \alpha . \supset : xPy . \supset . (\alpha \cap \overrightarrow{P}\text{‘}x) (\mathfrak{s}\text{‘}R) (\alpha \cap \overrightarrow{P}\text{‘}y)$ (5)

$\vdash . (3) . (5) . \supset \vdash :. \mathrm{Hp} . \supset : xPy . \supset . x \{\mathrm{lt}_P ; \mathfrak{s}\text{‘}(P \mathbin{\underset{\smile}{\restriction}} \alpha)\} y$ (6)

$\vdash . (6) . *271\cdot322 . \supset \vdash . \mathrm{Prop}$

***271·35.** $\vdash : \alpha \,\mathrm{med}\, P . \supset . \mathrm{D}\text{‘}(P \mathbin{\underset{\smile}{\restriction}} \alpha)_\epsilon \subset - \mathrm{C\!\!I}\text{‘}\mathrm{max}_P$

Dem.

$\vdash . *37\cdot413 . *211\cdot11 . \supset$

$\vdash :. \beta \in \mathrm{D}\text{‘}(P \mathbin{\underset{\smile}{\restriction}} \alpha)_\epsilon . \supset : (\exists \rho) . \beta = \alpha \cap P\text{“}(\rho \cap \alpha) :$ (1)

[*37·1] $\supset : (\exists \rho) : x \in \beta . \supset_x . (\exists y) . y \in \rho \cap \alpha . xPy$ (2)

$\vdash . (2) . *271\cdot1 . \supset$

$\vdash :. \mathrm{Hp} . \beta \in \mathrm{D}\text{‘}(P \mathbin{\underset{\smile}{\restriction}} \alpha)_\epsilon . \supset : (\exists \rho) : x \in \beta . \supset_x . (\exists y, z) . xPz . z \in \alpha . zPy . y \in \rho \cap \alpha .$

[(1)] $\supset_x . (\exists z) . xPz . z \in \beta .$

[*37·1] $\supset_x . x \in P\text{“}\beta$ (3)

$\vdash . (3) . *205\cdot123 . \supset \vdash : \mathrm{Hp} . \beta \in \mathrm{D}\text{‘}(P \mathbin{\underset{\smile}{\restriction}} \alpha)_\epsilon . \supset . \overrightarrow{\mathrm{max}}_P\text{‘}\beta = \Lambda : \supset \vdash . \mathrm{Prop}$

***271·36.** $\vdash : P \in \mathrm{Ded} . \alpha \,\mathrm{med}\, P . \supset . \mathrm{D}\text{‘}(P \mathbin{\underset{\smile}{\restriction}} \alpha)_\epsilon \subset \mathrm{C\!\!I}\text{‘}\mathrm{lt}_P$ [*271·35 . *214·101]

***271·37.** $\vdash : P \in \mathrm{Ser} \cap \mathrm{Ded} . \alpha \,\mathrm{med}\, P . \supset . \mathrm{lt}_P \restriction C\text{‘}\mathfrak{s}\text{‘}(P \mathbin{\underset{\smile}{\restriction}} \alpha) \in P \,\overline{\mathrm{smor}}\, \{\mathfrak{s}\text{‘}(P \mathbin{\underset{\smile}{\restriction}} \alpha)\}$
[*271·321·34·36 . *151·22]

***271·38.** $\vdash : P \in \mathrm{Ser} \cap \mathrm{Ded} . \alpha \,\mathrm{med}\, P . \supset . P \,\mathrm{smor}\, \{\mathfrak{s}\text{‘}(P \mathbin{\underset{\smile}{\restriction}} \alpha)\}$ [*271·37]

***271·39.** $\vdash : P, Q \in \mathrm{Ser} \cap \mathrm{Ded} . \alpha \,\mathrm{med}\, P . \beta \,\mathrm{med}\, Q . (P \mathbin{\underset{\smile}{\restriction}} \alpha) \,\mathrm{smor}\, (Q \mathbin{\underset{\smile}{\restriction}} \beta) . \supset .$
$P \,\mathrm{smor}\, Q$

Dem.

$\vdash . *212\cdot72 . \supset \vdash : \mathrm{Hp} . \supset . \{\mathfrak{s}\text{‘}(P \mathbin{\underset{\smile}{\restriction}} \alpha)\} \,\mathrm{smor}\, \{\mathfrak{s}\text{‘}(P \mathbin{\underset{\smile}{\restriction}} \beta)\}$ (1)

$\vdash . *271\cdot38 . \supset \vdash : \mathrm{Hp} . \supset . P \,\mathrm{smor}\, \{\mathfrak{s}\text{‘}(P \mathbin{\underset{\smile}{\restriction}} \alpha)\} . Q \,\mathrm{smor}\, \{\mathfrak{s}\text{‘}(Q \mathbin{\underset{\smile}{\restriction}} \beta)\}$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

This proposition is used in proving that all continuous series are similar, by means of the fact that such series contain rational series as medians, and that all rational series are similar.

***271·4.** $\vdash : S \in P \,\mathrm{smor}\, Q . \beta \,\mathrm{med}\, Q . \supset . (S\text{“}\beta) \,\mathrm{med}\, P$

Dem.

$\vdash . *35\cdot354 . *74\cdot14 . \supset \vdash : \mathrm{Hp} . \supset . Q \restriction \beta \,|\, \breve{S} = Q \,|\, \breve{S} \restriction S\text{“}\beta .$

[*150·1] $\supset . S ; (Q \restriction \beta) = (S;Q) \restriction S\text{“}\beta .$

[*151·11] $\supset . \{S;(Q \restriction \beta)\} \,|\, (S;Q) = (P \restriction S\text{“}\beta) \,|\, P$ (1)

$\vdash . *72\cdot6 .$ $\supset \vdash : \mathrm{Hp} . \supset . (Q \restriction \beta) \,|\, \breve{S} \,|\, S = Q \restriction \beta .$

[*150·1] $\supset . \{S;(Q \restriction \beta)\} \,|\, (S;Q) = S \,|\, Q \restriction \beta \,|\, Q \,|\, \breve{S}$ (2)

$\vdash . (2) . *271\cdot1 .$ $\supset \vdash : \mathrm{Hp} . \supset . S \,|\, Q \,|\, \breve{S} \in \{S;(Q \restriction \beta)\} \,|\, (S;Q) .$

[*151·11.(1)] $\supset . P \in (P \restriction S\text{“}\beta) \,|\, P .$

[*271·1] $\supset . (S\text{“}\beta) \,\mathrm{med}\, P : \supset \vdash . \mathrm{Prop}$

$*272$. SIMILARITY OF POSITION.

Summary of $*272$.

If P, Q are two serial relations, and T is a correlator which correlates some terms of $C`P$ with some terms of $C`Q$, we say that two terms x and y, of which x belongs to $C`P$ and $y$ to $C`Q$, have similar positions with respect to T if y comes after the correlates of all members of $\mathrm{D}`T$ which $x$ comes after, and $y$ comes before the correlates of all members of $\mathrm{D}`T$ which x comes before. This notion is useful for inductive definitions of correlations. If we start by correlating any two terms x_1, y_1, and take another term x_2 coming (say) after x_1, a term y_2 having similarity of position with respect to $x_1 \downarrow y_1$ must come after y_1. Suppose now we take x_3 between x_1 and x_2. Then a term y_3 having similarity of position with respect to $x_1 \downarrow y_1 \cup x_2 \downarrow y_2$ must come between y_1 and y_2; and so on. A correlation T constructed in this way will be such that $T ; Q \Subset P \,.\, \breve{T} ; P \Subset Q$. If the whole of $C`P$ and $C`Q$ can be obtained by prolonging the construction long enough, T will at last become a correlator of P and Q. This is the principle of Cantor's proof that any two rational series are similar.

As a rule, when the notion of similarity of position is useful, the relation T will be one-one, but this is not assumed in the definition. We write "$xT_{PQ}y$" for "x and y have similar positions in P and Q respectively with respect to T," or, as we may express it more shortly, "the P-position of x is T-similar to the Q-position of y." The definition is

$$T_{PQ} = \hat{x}\hat{y}\,\{x \in C`P \,.\, y \in C`Q \,.\, \mathrm{D}`T \cap \overrightarrow{P}`x \subset T``\overrightarrow{Q}`y \,.\, \mathrm{D}`T \cap \overleftarrow{P}`x \subset T``\overleftarrow{Q}`y \,.$$
$$\mathrm{D}`T \cap \iota`x \subset \overrightarrow{T}`y\} \quad \mathrm{Df.}$$

This definition states that the predecessors of x which have T-correlates are to be correlated with predecessors of y, the successors of x which have T-correlates are to be correlated with successors of y, and if x itself has a T-correlate, y is to be a T-correlate of x.

When T is a many-one relation, the definition becomes somewhat simpler. We then have

$*272{\cdot}13$. $\vdash :: T \in \mathrm{Cls} \rightarrow 1 \,.\, \supset :.\, xT_{PQ}y \,.\, \equiv :$

$$x \in C`P \,.\, y \in C`Q : z \in \mathrm{D}`T \cap \overrightarrow{P}`x \,.\, \supset_z .\, \breve{T}`zQy : z \in \mathrm{D}`T \cap \overleftarrow{P}`x \,.\, \supset_z .\, yQ\breve{T}`z :$$
$$x \in \mathrm{D}`T \,.\, \supset .\, y = \breve{T}`x$$

We have

***272·16.** $\vdash . (\mathrm{D}`T) \upharpoonleft T_{PQ} \subseteq T$

That is, a term which has a correlate cannot have similarity of position with any term except one with which it is correlated. A member of $C`P \cap \mathrm{D}`T$ will have similarity of position with its correlate (assuming $T \epsilon \mathrm{Cls} \to 1$) if $P \upharpoonleft \mathrm{D}`T \subseteq T ; Q . \breve{T}``C`P \subset C`Q$ (*272·18).

Under ordinary circumstances, a term which is not a member of $\mathrm{D}`T$ cannot have similarity of position with any member of $\mathrm{G}`T$ (*272·2). When T is many-one and its domain is contained in $C`P$, and P and Q are series, and x has no T-correlate, we have (*272·21)

$$xT_{PQ}y . \equiv : x \epsilon C`P . y \epsilon C`Q : z \epsilon \mathrm{D}`T \cap \overrightarrow{P}`x . \equiv_z . \breve{T}`zQy,$$

i.e. in this case, x and y have similar positions if the predecessors of x which have correlates are the terms whose correlates precede y. In this case, if $x \epsilon C`P$, we have (*272·212)

$$\overleftarrow{T_{PQ}}`x = C`Q \cap \hat{y} (\mathrm{D}`T \cap \overrightarrow{P}`x = T``\overrightarrow{Q}`y) = C`Q \cap \hat{y} (\mathrm{D}`T \cap \overleftarrow{P}`x = T``\overleftarrow{Q}`y).$$

We next investigate the condition for $C`P = \mathrm{D}`T_{PQ}$, *i.e.* the condition required in order that every member of $C`P$ may have similarity of position with some member of $C`Q$. A sufficient condition is

$$P, Q \epsilon \mathrm{Ser} . Q \epsilon \mathrm{comp} . T \epsilon \mathrm{Cls} \to 1 . \mathrm{D}`T \epsilon \mathrm{Cls\,induct} . P \upharpoonleft \mathrm{D}`T \subseteq T ; Q . \dot{\exists} ! Q . \breve{T}``C`P \subset \mathrm{D}`Q \cap \mathrm{G}`Q$$

as is proved in *272·34.

We next consider the reversibility of T_{PQ}, *i.e.* the condition that the converse of T_{PQ} should be $(\breve{T})_{QP}$. A sufficient condition is

$$P, Q \epsilon \mathrm{Ser} . T \epsilon 1 \to 1 . \mathrm{D}`T \subset C`P . \mathrm{G}`T \subset C`Q \quad (*272·42).$$

Finally, we have two propositions on the addition of another couple $x \downarrow y$ to T. With the above-mentioned hypothesis of *272·42, if $xT_{PQ}y$ and $T ; Q \subseteq P$, putting $W = T \cup x \downarrow y$, we shall have $P \upharpoonleft \mathrm{D}`W = W ; Q$ (*272·51), so that the hypothesis we had for T still holds for W.

The propositions of this number are in the nature of lemmas for Cantor's proof that any two rational series are similar, which is given in *273.

***272·01.** $T_{PQ} = \hat{x}\hat{y} \{x \epsilon C`P . y \epsilon C`Q . \mathrm{D}`T \cap \overrightarrow{P}`x \subset T``\overrightarrow{Q}`y . \mathrm{D}`T \cap \overleftarrow{P}`x \subset T``\overleftarrow{Q}`y . \mathrm{D}`T \cap \iota`x \subset \overrightarrow{T}`y\}$ Df

***272·1.** $\vdash : xT_{PQ}y . \equiv . x \epsilon C`P . y \epsilon C`Q . \mathrm{D}`T \cap \overrightarrow{P}`x \subset T``\overrightarrow{Q}`y . \mathrm{D}`T \cap \overleftarrow{P}`x \subset T``\overleftarrow{Q}`y . \mathrm{D}`T \cap \iota`x \subset \overrightarrow{T}`y$ [(*272·01)]

***272·11.** $\vdash : x \in C'P . \supset .$

$$\overleftarrow{T}_{PQ}'x = C'Q \cap p'\breve{Q}``\overleftarrow{T}``(\text{D}'T \cap \overrightarrow{P}'x) \cap p'Q``\overleftarrow{T}``(\text{D}'T \cap \overleftarrow{P}'x) \cap p'\overleftarrow{T}``(\text{D}'T \cap \iota'x)$$

Dem.

$\vdash . *272·1 . \supset \vdash : \text{Hp} . \supset .$

$$\overleftarrow{T}_{PQ}'x = C'Q \cap \hat{y}\{z \in \text{D}'T \cap \overrightarrow{P}'x . \supset_z . zT \mid Qy : z \in \text{D}'T \cap \overleftarrow{P}'x . \supset_z . zT \mid \breve{Q}y : z \in \text{D}'T \cap \iota'x . \supset_z . zTy\}$$

$$[*40·51·53] \quad = C'Q \cap p'\breve{Q}``\overleftarrow{T}``(\text{D}'T \cap \overrightarrow{P}'x) \cap p'Q``\overleftarrow{T}``(\text{D}'T \cap \overleftarrow{P}'x) \cap p'\overleftarrow{T}``(\text{D}'T \cap \iota'x) : \supset \vdash . \text{Prop}$$

***272·111.** $\vdash : x \in C'P . \supset .$

$$\overleftarrow{T}_{PQ}'x = C'Q \cap p'\{\breve{Q}``\overleftarrow{T}``(\text{D}'T \cap \overrightarrow{P}'x) \cup Q``\overleftarrow{T}``(\text{D}'T \cap \overleftarrow{P}'x) \cup \overleftarrow{T}``(\text{D}'T \cap \iota'x)\}$$

[*272·11 . *40·18]

***272·12.** $\vdash :: xT_{PQ}y . \equiv :. x \in C'P . y \in C'Q :. z \in \text{D}'T . \supset_z : zPx . \supset . zT \mid Qy : z\breve{P}x . \supset . zT \mid \breve{Q}y : z = x . \supset . zTy$ [*272·1]

***272·13.** $\vdash :: T \in \text{Cls}\rightarrow 1 . \supset :. xT_{PQ}y . \equiv : x \in C'P . y \in C'Q :$

$$z \in \text{D}'T \cap \overrightarrow{P}'x . \supset_z . \breve{T}'zQy : z \in \text{D}'T \cap \overleftarrow{P}'x . \supset_z . yQ\breve{T}'z : x \in \text{D}'T . \supset . y = \breve{T}'x$$

[*272·12 . *71·701]

***272·131.** $\vdash : T \in \text{Cls}\rightarrow 1 . x \in C'P . \supset .$

$$\overleftarrow{T}_{PQ}'x = C'Q \cap p'\{\overleftarrow{Q}``\breve{T}``\overrightarrow{P}'x \cup \overrightarrow{Q}``\breve{T}``\overleftarrow{P}'x \cup \overleftarrow{T}``(\text{D}'T \cap \iota'x)\}$$

[*272·111 . *71·613]

***272·14.** $\vdash : x \in C'P - \text{D}'T . \supset .$

$$\overleftarrow{T}_{PQ}'x = C'Q \cap p'\breve{Q}``\overleftarrow{T}``(\text{D}'T \cap \overrightarrow{P}'x) \cap p'Q``\overleftarrow{T}``(\text{D}'T \cap \overleftarrow{P}'x)$$

[*272·111 . *40·18]

***272·141.** $\vdash : x \in C'P - \text{D}'T . \supset .$

$$\overleftarrow{T}_{PQ}'x = C'Q \cap \hat{y}\,(\text{D}'T \cap \overrightarrow{P}'x \subset T``\overrightarrow{Q}'y . \text{D}'T \cap \overleftarrow{P}'x \subset T``\overleftarrow{Q}'y)$$

[*272·1]

***272·15.** $\vdash : T \in \text{Cls}\rightarrow 1 . x \in C'P - \text{D}'T . \supset .$

$$\overleftarrow{T}_{PQ}'x = C'Q \cap p'\overleftarrow{Q}``\breve{T}``\overrightarrow{P}'x \cap p'\overrightarrow{Q}``\breve{T}``\overleftarrow{P}'x$$

[*272·131 . *40·18]

***272·16.** $\vdash . (\text{D}'T) \upharpoonright T_{PQ} \in T$

Dem.

$\vdash . *272·12 . \supset \vdash : x \in \text{D}'T . xT_{PQ}y . \supset . xTy : \supset \vdash . \text{Prop}$

***272·161.** $\vdash : T \epsilon \text{Cls} \to 1 \,.\, P \upharpoonright \text{D}`T \mathrel{\Subset} T \,;\, Q \,.\, \supset \,.\, (\text{D}`T) \rceil T_{PQ} = C`P \rceil T \upharpoonright C`Q$

Dem.

$\vdash \,.\, *150·41 \,.\quad \supset \vdash : \text{Hp} \,.\, z \,\epsilon\, \text{D}`T \,.\, zPx \,.\, xTy \,.\, \supset \,.\, \breve{T}`zQy \qquad (1)$

$\vdash \,.\, *150·41 \,.\quad \supset \vdash : \text{Hp} \,.\, z \,\epsilon\, \text{D}`T \,.\, xPz \,.\, xTy \,.\, \supset \,.\, yQ\breve{T}`z \qquad (2)$

$\vdash \,.\, (1) \,.\, (2) \,.\, *272·13 \,.\, \supset \vdash : \text{Hp} \,.\, xTy \,.\, x \,\epsilon\, C`P \,.\, y \,\epsilon\, C`Q \,.\, \supset \,.\, xT_{PQ}y \qquad (3)$

$\vdash \,.\, (3) \,.\, *272·16 \,.\, \supset \vdash \,.\, \text{Prop}$

***272·17.** $\vdash : T \epsilon \text{Cls} \to 1 \,.\, P \upharpoonright \text{D}`T \mathrel{\Subset} T \,;\, Q \,.\, \text{D}`T \subset C`P \,.\, \text{ᗡ}`T \subset C`Q \,.\, \supset \,.$

$T = (\text{D}`T) \rceil T_{PQ} \quad [*272·161]$

The hypothesis of *272·17 is satisfied in all the important uses of T_{PQ}.

***272·171.** $\vdash : \text{Hp } *272·17 \,.\, x \,\epsilon\, \text{D}`T \,.\, \supset \,.\, \overleftarrow{T_{PQ}}`x = \iota`\breve{T}`x \quad [*272·17]$

***272·18.** $\vdash : T \epsilon \text{Cls} \to 1 \,.\, P \upharpoonright \text{D}`T \mathrel{\Subset} T \,;\, Q \,.\, \breve{T}``C`P \subset C`Q \,.\, x \,\epsilon\, C`P \cap \text{D}`T \,.\, \supset \,.$

$\breve{T}_{PQ}`x = \breve{T}`x$

Dem.

$\vdash \,.\, *150·41 \,.\, \supset \vdash :. \text{Hp} \,.\, \supset : z \,\epsilon\, \text{D}`T \cap \overrightarrow{P}`x \,.\, \supset_z \,.\, (\breve{T}`z)\, Q\, (\breve{T}`x) \qquad (1)$

$\vdash \,.\, *150·41 \,.\, \supset \vdash :. \text{Hp} \,.\, \supset : z \,\epsilon\, \text{D}`T \cap \overleftarrow{P}`x \,.\, \supset_z \,.\, (\breve{T}`x)\, Q\, (\breve{T}`z) \qquad (2)$

$\vdash \,.\, *37·61 \,.\quad \supset \vdash : \text{Hp} \,.\, \supset \,.\, \breve{T}`x \,\epsilon\, C`Q \qquad (3)$

$\vdash \,.\, (1) \,.\, (2) \,.\, (3) \,.\, *272·13 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, xT_{PQ}(\breve{T}`x) \qquad (4)$

$\vdash \,.\, *272·13 \,.\, \supset \vdash : \text{Hp} \,.\, xT_{PQ}y \,.\, \supset \,.\, y = \breve{T}`x \qquad (5)$

$\vdash \,.\, (4) \,.\, (5) \,.\, \supset \vdash \,.\, \text{Prop}$

***272·2.** $\vdash : T \epsilon \text{Cls} \to 1 \,.\, \text{D}`T \subset C`P \,.\, P \,\epsilon\, \text{connex} \,.\, Q \mathrel{\Subset} J \,.\, x \sim\epsilon\, \text{D}`T \,.\, \supset \,.$

$\overleftarrow{T_{PQ}}`x \cap \text{ᗡ}`T = \Lambda$

Dem.

$\vdash \,.\, *272·13 \,.\, \supset \vdash : \text{Hp} \,.\, xT_{PQ}y \,.\, z \,\epsilon\, \text{D}`T \cap \overrightarrow{P}`x \,.\, \supset \,.\, \breve{T}`z \neq y \qquad (1)$

$\vdash \,.\, *272·13 \,.\, \supset \vdash : \text{Hp} \,.\, xT_{PQ}y \,.\, z \,\epsilon\, \text{D}`T \cap \overleftarrow{P}`x \,.\, \supset \,.\, \breve{T}`z \neq y \qquad (2)$

$\vdash \,.\, (1) \,.\, (2) \,.\, \supset \vdash : \text{Hp} \,.\, xT_{PQ}y \,.\, z \,\epsilon\, \text{D}`T \,.\, \supset \,.\, \breve{T}`z \neq y : \supset \vdash \,.\, \text{Prop}$

***272·201.** $\vdash : T \epsilon \text{Cls} \to 1 \,.\, \text{D}`T \subset C`P \,.\, P \,\epsilon\, \text{connex} \,.\, \exists ! \,\text{D}`T_{PQ} - \text{D}`T \,.\, \supset \,.$

$\text{ᗡ}`T \subset C`Q$

Dem.

$\vdash \,.\, *202·104 \,.\, \supset \vdash :. \text{Hp} \,.\, z \,\epsilon\, \text{D}`T \,.\, xT_{PQ}y \,.\, x \sim\epsilon\, \text{D}`T \,.\, \supset : zPx \,.\, \vee \,.\, xPz :$

$[*272·13] \qquad \supset : \breve{T}`zQy \,.\, \vee \,.\, yQ(\breve{T}`z) :$

$[*33·132] \qquad \supset : \breve{T}`z \,\epsilon\, C`Q :. \supset \vdash \,.\, \text{Prop}$

***272·21.** $\vdash :: T \epsilon \text{Cls} \to 1 \,.\, \text{D}`T \subset C`P \,.\, P, Q \,\epsilon\, \text{Ser} \,.\, x \sim\epsilon\, \text{D}`T \,.\, \supset :.$

$xT_{PQ}y \,.\, \equiv : x \,\epsilon\, C`P \,.\, y \,\epsilon\, C`Q : z \,\epsilon\, \text{D}`T \cap \overrightarrow{P}`x \,.\, \equiv_z \,.\, \breve{T}`zQy$

Dem.

$\vdash \,.\, *272·2 \,.\, \supset \vdash :. \text{Hp} \,.\, z \,\epsilon\, \text{D}`T \,.\, xT_{PQ}y \,.\, \supset : x \neq z \,.\, y \neq \breve{T}`z :$

$[*204·3 \,.\, *272·201] \qquad \supset : xPz \,.\, \equiv \,.\, \sim(zPx) : yQ(\breve{T}`z) \,.\, \equiv \,.\, \sim\{(\breve{T}`z)\,Qy\} \quad (1)$

$\vdash . (1) . *272\cdot13 . \supset \vdash ::. \mathrm{Hp} . \supset :: xT_{PQ}y . \equiv :.$

$x \epsilon C\text{‘}P . y \epsilon C\text{‘}Q :. z \epsilon \mathrm{D}\text{‘}T . \supset_z : zPx . \supset . \breve{T}\text{‘}zQy : \sim(zPx) . \supset . \sim(\breve{T}\text{‘}z)\, Qy$ (2)

$\vdash . (2) . \supset \vdash :: \mathrm{Hp} . \supset :. xT_{PQ}y . \equiv : x \epsilon C\text{‘}P . y \epsilon C\text{‘}Q : z \epsilon \mathrm{D}\text{‘}T . zPx . \equiv_z . \breve{T}\text{‘}zQy ::$

$\supset \vdash . \mathrm{Prop}$

***272·211.** $\vdash :: \mathrm{Hp}\, *272\cdot21 . \supset :. xT_{PQ}y . \equiv :$

$x \epsilon C\text{‘}P . y \epsilon C\text{‘}Q : z \epsilon \mathrm{D}\text{‘}T \cap \overleftarrow{P}\text{‘}x . \equiv_z . yQ(\breve{T}\text{‘}z)$ [Proof as in *272·21]

***272·212.** $\vdash : \mathrm{Hp}\, *272\cdot21 . x \epsilon C\text{‘}P . \supset .$

$\overleftarrow{T_{PQ}}\text{‘}x = C\text{‘}Q \cap \hat{y}\,(\mathrm{D}\text{‘}T \cap \overrightarrow{P}\text{‘}x = T\text{‘‘}\overrightarrow{Q}\text{‘}y) = C\text{‘}Q \cap \hat{y}\,(\mathrm{D}\text{‘}T \cap \overleftarrow{P}\text{‘}x = T\text{‘‘}\overleftarrow{Q}\text{‘}y)$

[*272·21·211]

***272·22.** $\vdash : T \epsilon \mathrm{Cls} \to 1 . P, Q \epsilon \mathrm{trans} . xT_{PQ}y . z, w \epsilon \mathrm{D}\text{‘}T . x \epsilon P(z - w) . \supset .$

$y \epsilon Q(\breve{T}\text{‘}z - \breve{T}\text{‘}w)$

Dem.

$\vdash . *272\cdot13 . \supset \vdash : \mathrm{Hp} . \supset . \breve{T}\text{‘}zQy . yQ\breve{T}\text{‘}w : \supset \vdash . \mathrm{Prop}$

***272·221.** $\vdash : T \epsilon \mathrm{Cls} \to 1 . P, Q \epsilon \mathrm{trans} . \exists ! \mathrm{D}\text{‘}T_{PQ} \cap P(z - w) . \supset . (\breve{T}\text{‘}z)\, Q\, (\breve{T}\text{‘}w)$

[*272·22] $z, w \epsilon \mathrm{D}\text{‘}T$

***272·23.** $\vdash :. T \epsilon \mathrm{Cls} \to 1 . P, Q \epsilon \mathrm{trans} :$

$z(P \upharpoonright \mathrm{D}\text{‘}T)w . \supset_{z,w} . \exists ! \mathrm{D}\text{‘}T_{PQ} \cap P(z - w) : \supset . P \upharpoonright \mathrm{D}\text{‘}T \subseteq T ; Q$

Dem.

$\vdash . *272\cdot221 . \supset \vdash :. \mathrm{Hp} . \supset : z(P \upharpoonright \mathrm{D}\text{‘}T)w . \supset . (\breve{T}\text{‘}z)\, Q\, (\breve{T}\text{‘}w) .$

[*150·41] $\supset . z(T;Q)w :. \supset \vdash . \mathrm{Prop}$

***272·24.** $\vdash : \mathrm{D}\text{‘}T \cap C\text{‘}P = \Lambda . \supset . T_{PQ} = C\text{‘}P \uparrow C\text{‘}Q$ [*272·1]

***272·3.** $\vdash : T \epsilon \mathrm{Cls} \to 1 . S \subseteq T . \supset . T_{PQ} \subseteq S_{PQ}$

Dem.

$\vdash . *272\cdot13 . \supset \vdash :. \mathrm{Hp} . xT_{PQ}y . \supset : z \epsilon \mathrm{D}\text{‘}T . zPx . \supset . \breve{T}\text{‘}zQy :$

[*72·9] $\supset : z \epsilon \mathrm{D}\text{‘}S . zPx . \supset . \breve{S}\text{‘}zQy$ (1)

Similarly $\vdash :. \mathrm{Hp} . xT_{PQ}y . \supset : z \epsilon \mathrm{D}\text{‘}S . xPz . \supset . yQ\breve{S}\text{‘}z$ (2)

$\vdash . *272\cdot13 . \supset \vdash :. \mathrm{Hp} . xT_{PQ}y . \supset : z \epsilon \mathrm{D}\text{‘}T . z = x . \supset . \breve{T}\text{‘}z = y :$

[*72·9] $\supset : z \epsilon \mathrm{D}\text{‘}S . z = x . \supset . \breve{S}\text{‘}z = y$ (3)

$\vdash . (1) . (2) . (3) . *272\cdot13 . \supset \vdash : \mathrm{Hp} . xT_{PQ}y . \supset xS_{PQ}y : \supset \vdash . \mathrm{Prop}$

The following propositions lead up to *272·34.

***272·31.** $\vdash : P, Q \epsilon \mathrm{Ser} . T \epsilon \mathrm{Cls} \to 1 . x \sim\epsilon \mathrm{D}\text{‘}T . z = \max_P\text{‘}(\mathrm{D}\text{‘}T \cap \overrightarrow{P}\text{‘}x) .$

$w = \min_P\text{‘}(\mathrm{D}\text{‘}T \cap \overleftarrow{P}\text{‘}x) . P \upharpoonright \mathrm{D}\text{‘}T \subseteq T;Q . \supset . \overleftarrow{T_{PQ}}\text{‘}x = Q(\breve{T}\text{‘}z - \breve{T}\text{‘}w)$

Dem.

$\vdash . *205\cdot21 . \supset \vdash : \mathrm{Hp} . u \epsilon \mathrm{D}\text{‘}T \cap \overrightarrow{P}\text{‘}x - \iota\text{‘}z . \supset . uPz .$

[*150·41.Hp] $\supset . \breve{T}\text{‘}uQ\breve{T}\text{‘}z$ (1)

$\vdash . (1) . \supset \vdash : \mathrm{Hp} . y \in Q(\breve{T}\text{‘}z - \breve{T}\text{‘}w) . u \in \mathrm{D}\text{‘}T \cap \overrightarrow{P}\text{‘}x . \supset . \breve{T}\text{‘}u Q y$ (2)

Similarly $\vdash : \mathrm{Hp} . y \in Q(\breve{T}\text{‘}z - \breve{T}\text{‘}w) . u \in \mathrm{D}\text{‘}T \cap \overleftarrow{P}\text{‘}x . \supset . y Q \breve{T}\text{‘}u$ (3)

$\vdash . (2) . (3) . *272\cdot13 . \supset \vdash : \mathrm{Hp} . y \in Q(\breve{T}\text{‘}z - \breve{T}\text{‘}w) . \supset . x T_{PQ} y$ (4)

$\vdash . *272\cdot22 . \quad \supset \vdash : \mathrm{Hp} . \supset . \overleftarrow{T_{PQ}}\text{‘}x \subset Q(\breve{T}\text{‘}z - \breve{T}\text{‘}w)$ (5)

$\vdash . (4) . (5) . \supset \vdash . \mathrm{Prop}$

***272·32.** $\vdash : P, Q \in \mathrm{Ser} . T \in \mathrm{Cls} \to 1 . \mathrm{D}\text{‘}T \subset \overrightarrow{P}\text{‘}x .$

$$P \upharpoonright \mathrm{D}\text{‘}T \Subset T;Q . z = \mathrm{max}_P\text{‘}\mathrm{D}\text{‘}T . \supset . \overleftarrow{T_{PQ}}\text{‘}x = \overleftarrow{Q}\text{‘}\breve{T}\text{‘}z$$

Dem.

$\vdash . *272\cdot13 . \supset \vdash :: \mathrm{Hp} . \supset :. x T_{PQ} y . \equiv : u \in \mathrm{D}\text{‘}T . \supset_u . \breve{T}\text{‘}u Q y$ (1)

$\vdash . *205\cdot21 . \supset \vdash : \mathrm{Hp} . u \in \mathrm{D}\text{‘}T - \iota\text{‘}z . \supset . u P z .$

$[*150\cdot41.\mathrm{Hp}] \quad \supset . \breve{T}\text{‘}u Q \breve{T}\text{‘}z$ (2)

$\vdash . (2) . \quad \supset \vdash :. \mathrm{Hp} . y \in \overleftarrow{Q}\text{‘}\breve{T}\text{‘}z . \supset : u \in \mathrm{D}\text{‘}T . \supset_u . \breve{T}\text{‘}u Q y :$

$[(1)] \quad \supset : x T_{PQ} y$ (3)

$\vdash . (1) . \quad \supset \vdash : \mathrm{Hp} . x T_{PQ} y . \supset . \breve{T}\text{‘}z Q y$ (4)

$\vdash . (3) . (4) . \supset \vdash . \mathrm{Prop}$

***272·321.** $\vdash : P, Q \in \mathrm{Ser} . T \in \mathrm{Cls} \to 1 . \mathrm{D}\text{‘}T \subset \overleftarrow{P}\text{‘}x .$

$$P \upharpoonright \mathrm{D}\text{‘}T \Subset T;Q . w = \mathrm{min}_P\text{‘}\mathrm{D}\text{‘}T . \supset . \overleftarrow{T_{PQ}}\text{‘}x = \overrightarrow{Q}\text{‘}\breve{T}\text{‘}w$$

[Proof as in *272·32]

***272·33.** $\vdash : P, Q \in \mathrm{Ser} . Q \in \mathrm{comp} . T \in \mathrm{Cls} \to 1 . \mathrm{D}\text{‘}T \in \mathrm{Cls\,induct} .$

$$P \upharpoonright \mathrm{D}\text{‘}T \Subset T;Q . \supset . (P\text{“}\mathrm{D}\text{‘}T \cap \breve{P}\text{“}\mathrm{D}\text{‘}T) - \mathrm{D}\text{‘}T \subset \mathrm{D}\text{‘}T_{PQ}$$

Dem.

$\vdash . *261\cdot26 . \supset \vdash : \mathrm{Hp} . \exists ! \mathrm{D}\text{‘}T \cap \overrightarrow{P}\text{‘}x . \supset . \mathrm{E} ! \mathrm{max}_P\text{‘}(\mathrm{D}\text{‘}T \cap \overrightarrow{P}\text{‘}x)$ (1)

$\vdash . *261\cdot26 . \supset \vdash : \mathrm{Hp} . \exists ! \mathrm{D}\text{‘}T \cap \overleftarrow{P}\text{‘}x . \supset . \mathrm{E} ! \mathrm{min}_P\text{‘}(\mathrm{D}\text{‘}T \cap \overleftarrow{P}\text{‘}x)$ (2)

$\vdash . *205\cdot11\cdot111 . \supset$

$\vdash : \mathrm{Hp} . x \sim\in \mathrm{D}\text{‘}T . z = \mathrm{max}_P\text{‘}(\mathrm{D}\text{‘}T \cap \overrightarrow{P}\text{‘}x) . w = \mathrm{min}_P\text{‘}(\mathrm{D}\text{‘}T \cap \overleftarrow{P}\text{‘}x) . \supset . z P w .$

$[*150\cdot41] \quad \supset . \breve{T}\text{‘}z Q \breve{T}\text{‘}w .$

$[*270\cdot11] \quad \supset . \exists ! Q(\breve{T}\text{‘}z - \breve{T}\text{‘}w) .$

$[*272\cdot31] \quad \supset . \exists ! \overleftarrow{T_{PQ}}\text{‘}x$ (3)

$\vdash . (1) . (2) . (3) . \supset$

$\vdash : \mathrm{Hp} . x \sim\in \mathrm{D}\text{‘}T . \exists ! \mathrm{D}\text{‘}T \cap \overrightarrow{P}\text{‘}x . \exists ! \mathrm{D}\text{‘}T \cap \overleftarrow{P}\text{‘}x . \supset . x \in \mathrm{D}\text{‘}T_{PQ} : \supset \vdash . \mathrm{Prop}$

$*272\cdot331$. $\vdash : \text{Hp}\,*272\cdot33 \,.\, \dot{\exists}\,!\,Q \,.\, \breve{T}``C`P \subset \text{D}`Q \,.\, \supset .\, C`P \cap p`\overleftarrow{P}``\text{D}`T \subset \text{D}`T_{PQ}$

Dem.

$\vdash .\, *261\cdot26 \,.\, \supset \vdash : \text{Hp} \,.\, \exists\,!\,\text{D}`T \cap C`P \,.\, \supset .\, \text{E}\,!\,\text{max}_P`\text{D}`T$ (1)

$\vdash .\, *272\cdot32 \,.\, \supset \vdash : \text{Hp} \,.\, x \in p`\overleftarrow{P}``\text{D}`T \,.\, z = \text{max}_P`\text{D}`T \,.\, \supset .\, \overleftarrow{T_{PQ}}`x = \overleftarrow{Q}`\breve{T}`z \,.$

$[*33\cdot4]$ $\supset .\, \exists\,!\,\overleftarrow{T_{PQ}}`x$ (2)

$\vdash .\, (1) \,.\, (2) \,.\, \supset \vdash : \text{Hp} \,.\, x \in p`\overleftarrow{P}``\text{D}`T \,.\, \exists\,!\,\text{D}`T \cap C`P \,.\, \supset .\, x \in \text{D}`T_{PQ}$ (3)

$\vdash .\, *35\cdot85 \,.\, *272\cdot24 \,.\, \supset \vdash : \text{Hp} \,.\, \text{D}`T \cap C`P = \Lambda \,.\, \supset .\, C`P \subset \text{D}`T_{PQ}$ (4)

$\vdash .\, (3) \,.\, (4) \,.\, \supset \vdash .\, \text{Prop}$

$*272\cdot332$. $\vdash : \text{Hp}\,*272\cdot33 \,.\, \dot{\exists}\,!\,Q \,.\, \breve{T}``C`P \subset \text{Ɑ}`Q \,.\, \supset .\, C`P \cap p`\overrightarrow{P}``\text{D}`T \subset \text{D}`T_{PQ}$
[Proof as in $*272\cdot331$]

$*272\cdot34$. $\vdash : \text{Hp}\,*272\cdot33 \,.\, \dot{\exists}\,!\,Q \,.\, \breve{T}``C`P \subset \text{D}`Q \cap \text{Ɑ}`Q \,.\, \supset .\, C`P = \text{D}`T_{PQ}$
$[*272\cdot33\cdot331\cdot332\cdot18 \,.\, *202\cdot505]$

The following propositions are lemmas for $*272\cdot42$.

$*272\cdot4$. $\vdash : P, Q \in \text{Ser} \,.\, T \in 1 \to 1 \,.\, \text{D}`T \subset C`P \,.\, \text{Ɑ}`T \subset C`Q \,.$
$x \sim\in \text{D}`T \,.\, xT_{PQ}y \,.\, \supset .\, y(\breve{T})_{QP}x$

Dem.

$\vdash .\, *272\cdot21 \,.\, \supset \vdash :.\, \text{Hp} \,.\, \supset : x \in C`P \,.\, y \in C`Q : z \in \text{D}`T \cap \overrightarrow{P}`x \,.\, \equiv_z .\, \breve{T}`zQy :$

$[*72\cdot243]$ $\supset : x \in C`P \,.\, y \in C`Q : (T`w)\,Px \,.\, \equiv_w .\, w \in \text{Ɑ}`T \,.\, wQy :$

$[*272\cdot21]$ $\supset : y(\breve{T})_{QP}x :.\, \supset \vdash .\, \text{Prop}$

$*272\cdot41$. $\vdash : P, Q \in \text{Ser} \,.\, T \in 1 \to 1 \,.\, \text{D}`T \subset C`P \,.\, \text{Ɑ}`T \subset C`Q \,.$
$x \in \text{D}`T \,.\, xT_{PQ}y \,.\, \supset .\, y(\breve{T})_{QP}x$

Dem.

$\vdash .\, *272\cdot13 \,.\, \supset \vdash ::\, \text{Hp} \,.\, \supset :.\, x \in C`P \,.\, y = \breve{T}`x :$
$z \in \text{D}`T \cap \overrightarrow{P}`x \,.\, \supset_z .\, \breve{T}`zQy : z \in \text{D}`T \cap \overleftarrow{P}`x \,.\, \supset_z .\, yQ(\breve{T}`z) :.$

$[*204\cdot3]$ $\supset :.\, x \in C`P \,.\, y = \breve{T}`x : z \in \text{D}`T \cap \overrightarrow{P}`x \,.\, \supset_z .\, \breve{T}`zQy :$
$z \in \text{D}`T - \iota`x - \overrightarrow{P}`x \,.\, \supset_z .\, \breve{T}`z \neq y \,.\, \sim\{(\breve{T}`z)Qy\} :.$

$[\text{Transp}]$ $\supset :.\, x \in C`P \,.\, y = \breve{T}`x :.\, z \in \text{D}`T - \iota`x \,.\, \supset_z : zPx \,.\, \equiv .\, (\breve{T}`z)Qy :.$

$[*204\cdot1]$ $\supset :.\, x \in C`P \,.\, y = \breve{T}`x :.\, z \in \text{D}`T \,.\, \supset_z : zPx \,.\, \equiv .\, (\breve{T}`z)\,Qy :.$

$[*72\cdot243]$ $\supset :.\, x \in C`P \,.\, y = \breve{T}`x :.\, (T`w)\,Px \,.\, \equiv_z .\, w \in \text{Ɑ}`T \,.\, wQy :.$

$[*71\cdot362]$ $\supset :.\, y \in C`Q \,.\, x = T`y :.\, (T`w)\,Px \,.\, \equiv_z .\, w \in \text{Ɑ}`T \,.\, wQy :.$

$[*14\cdot21 . *33\cdot43]$ $\supset :.\, y \in C`Q \,.\, x = T`y :.\, w \in \text{Ɑ}`T \,.\, \supset_w : (T`w)\,Px \,.\, \equiv .\, wQy :.$

$[*204\cdot3]$ $\supset :.\, y \in C`Q \,.\, x = T`y :.\, w \in \text{Ɑ}`T \cap \overrightarrow{Q}`y \,.\, \supset_w .\, T`wPx :$
$w \in \text{Ɑ}`T \cap \overleftarrow{Q}`y \,.\, \supset_w .\, xP(T`w) :.$

$[*272\cdot13]$ $\supset :.\, y(\breve{T})_{QP}x ::\, \supset \vdash .\, \text{Prop}$

***272·42.** $\vdash : P, Q \in \text{Ser} . T \in 1 \to 1 . \text{D}`T \subset C`P . \text{Ɑ}`T \subset C`Q . \supset . (\breve{T})_{QP} = \breve{T}_{PQ}$

Dem.

$\vdash . *272·4·41 . \supset \vdash : \text{Hp} . \supset . \breve{T}_{PQ} \Subset (\breve{T})_{QP}$ (1)

$\vdash . (1) \dfrac{\breve{T}, Q, P}{T, P, Q} . \supset \vdash : \text{Hp} . \supset . \text{Cnv}`(\breve{T})_{QP} \Subset T_{PQ}$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***272·43.** $\vdash : P, Q \in \text{Ser} \cap \text{comp} - \iota`\dot{\Lambda} . T \in 1 \to 1 . \text{D}`T \subset \text{D}`P \cap \text{Ɑ}`P .$
$\text{Ɑ}`T \subset \text{D}`Q \cap \text{Ɑ}`Q . P \upharpoonright \text{D}`T = T;Q . \text{D}`T \in \text{Cls induct} . \supset .$
$\text{D}`T_{PQ} = C`P . \text{Ɑ}`T_{PQ} = C`Q$

Dem.

$\vdash . *272·34 . \supset \vdash : \text{Hp} . \supset . \text{D}`T_{PQ} = C`P$ (1)

$\vdash . *150·36 . \supset \vdash . T;Q = T;Q \upharpoonright \text{Ɑ}`T . \breve{T};P = \breve{T};P \upharpoonright \text{D}`T$ (2)

$\vdash . (2) . \quad \vdash : \text{Hp} . \supset . P \upharpoonright \text{D}`T = T;Q \upharpoonright \text{Ɑ}`T .$

[*151·25] $\quad \supset . Q \upharpoonright \text{Ɑ}`T = \breve{T};P \upharpoonright \text{D}`T$

[(2)] $\quad = \breve{T};P$ (3)

$\vdash . *120·214 . \quad \supset \vdash : \text{Hp} . \supset . \text{Ɑ}`T \in \text{Cls induct}$ (4)

$\vdash . (3) . (4) . *272·34 . \supset \vdash : \text{Hp} . \supset . C`Q = \text{D}`(\breve{T})_{QP}$

[*272·42] $\quad = \text{Ɑ}`T_{PQ}$ (5)

$\vdash . (1) . (5) . \supset \vdash . \text{Prop}$

***272·5.** $\vdash : P, Q \in \text{Ser} . T \in \text{Cls} \to 1 . \text{D}`T \subset C`P . xT_{PQ}y . T;Q \Subset P . \supset .$
$(T \cup x \downarrow y);Q \Subset P$

Dem.

$\vdash . *150·75 . \supset$

$\vdash : \text{Hp} . \supset . (T \cup x \downarrow y);Q = T;Q \cup T``\overrightarrow{Q}`y \uparrow \iota`x \cup \iota`x \uparrow T``\overleftarrow{Q}`y$ (1)

$\vdash . *272·212 . \supset \vdash : \text{Hp} . x \sim \in \text{D}`T . \supset . T``\overrightarrow{Q}`y \subset \overrightarrow{P}`x . T``\overleftarrow{Q}`y \subset \overleftarrow{P}`x$ (2)

$\vdash . (1) . (2) . \quad \supset \vdash : \text{Hp} . x \sim \in \text{D}`T . \supset . (T \cup x \downarrow y);Q \Subset P$ (3)

$\vdash . *272·16 . \quad \supset \vdash : x \in \text{D}`T . \supset . T \cup x \downarrow y = T$ (4)

$\vdash . (3) . (4) . \supset \vdash . \text{Prop}$

***272·51.** $\vdash : P, Q \in \text{Ser} . T \in 1 \to 1 . \text{D}`T \subset C`P . \text{Ɑ}`T \subset C`Q .$
$xT_{PQ}y . P \upharpoonright \text{D}`T = T;Q . W = T \cup x \downarrow y . \supset . P \upharpoonright \text{D}`W = W;Q$

Dem.

$\vdash . *272·5 . \quad \supset \vdash : \text{Hp} . \supset . W;Q \Subset P$ (1)

$\vdash . *272·42 . \quad \supset \vdash : \text{Hp} . \supset . y(\breve{T})_{QP}x$ (2)

$\vdash . *150·36 . *151·26 . \supset \vdash : \text{Hp} . \supset . \breve{T};P = Q \upharpoonright \text{Ɑ}`T$ (3)

$\vdash . (2) . (3) . *272·5 . \quad \supset \vdash : \text{Hp} . \supset . \breve{W};P \Subset Q$ (4)

$\vdash . (1) . (4) . *150·36 . \supset \vdash : \text{Hp} . \supset . W;Q \Subset P \upharpoonright \text{D}`W . \breve{W};(P \upharpoonright \text{D}`W) \Subset Q .$

[*151·26] $\quad \supset . P \upharpoonright \text{D}`W = W;Q : \supset \vdash . \text{Prop}$

$*273$. RATIONAL SERIES.

Summary of $*273$.

A "rational series" is a series ordinally similar to the series of all positive and negative rational numbers in order of magnitude, or, what is equivalent, a series ordinally similar to the series of all rational proper fractions (0 excluded). This characteristic of rational series is not, however, the most convenient for purposes of definition. Following Cantor, we define a rational series as one which is compact, has no beginning or end, and has $\aleph_0$ terms in its field. Thus the field of a rational series can be arranged in a progression, and this is the source of the special properties which distinguish rational series from other compact series.

Rational proper fractions can be arranged in a progression in many ways, for example the following: If two fractions (in their lowest terms) have the same denominator, put the one with the smaller numerator first; if they have different denominators, put the one with the smaller denominator first. We thus obtain the series

$$\tfrac{1}{2}, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{4}, \tfrac{3}{4}, \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{3}{5}, \tfrac{4}{5}, \tfrac{1}{6}, \ldots.$$

This series is a progression, and contains all rational proper fractions.

Conversely, the natural numbers can be arranged in a rational series. Take, *e.g.*, the following arrangement: Express the numbers in the dyadic scale, so that every number is of the form

$$\Sigma\, 2^\mu \, (\mu \in \kappa),$$

where κ is a finite class of integers. The relation of the number to κ is one-one. Arrange the various κ's by the principle of first differences, *i.e.* form the series $M_{\mathrm{cl}} \upharpoonright (\mathrm{Cls\ induct} - \iota`\Lambda)$, where M is the relation "less than" among finite integers. The resulting series is a rational series; thus the integers are arranged in a rational series by virtue of their correlation with the classes κ. This arrangement places all the odd numbers before all the even numbers, all numbers of the form $4\nu + 2$ before all numbers of the form 4ν, and so on. If two numbers are expressed in the dyadic scale, their relative position in the series is determined by the first digit (starting from the right) which is not the same in the two numbers: the one in which this digit is 1 precedes the one in which it is 0.

The two chief propositions in regard to rational series are (1) that any two rational series are ordinally similar, (2) that if R is a progression, its finite existent sub-classes arranged by the principle of first differences form a rational series. The second of these propositions is proved by showing (a) that the finite existent sub-classes arranged by first differences form a compact series, (b) that the finite existent sub-classes arranged by *last* differences form a progression. By this means, given any progression, we can specify a relation which arranges its terms in a rational series. For if T is a correlator of our progression R with the progression

$$R_{1c} \upharpoonright (\text{Cls induct} - \iota`\Lambda),$$

then $$T \,;\, R_{cl} \upharpoonright (\text{Cls induct} - \iota`\Lambda)$$

is a rational series whose field is $C`R$. Hence rational series exist in any type in which progressions exist.

The arrangement of the finite sub-classes of a progression, with the resultant existence-theorem for rational series, will be dealt with in the following number. In the present number, we shall be concerned with the proof that any two rational series are ordinally similar.

The proof of the similarity of any two rational series is due to Cantor. It is long and rather complicated; in outline, it is as follows.

Let P, Q be two rational series, and R, S two progressions whose fields are $C`P$ and $C`Q$ respectively. Construct a series of correlations of parts of P with parts of Q on the following plan: Begin with $\dot\Lambda$, and if T is any correlation, let the next be

$$T \mathbin{\dot\cup} \text{seq}_R`\text{D}`T \downarrow \min_S`\overleftarrow{T_{PQ}}`\text{seq}_R`\text{D}`T.$$

Then the sum of all the correlations generated from $\dot\Lambda$ by this law of succession will be a correlation of P with Q. It will be seen that, if we put

$$W = \hat{X}\hat{T}\{X = \text{seq}_R`\text{D}`T \downarrow \min_S`\overleftarrow{T_{PQ}}`\text{seq}_R`\text{D}`T\},$$

the relation which is to be shown to be a correlator of P and Q is W_A, in the sense defined in $*259$. Thus we have to prove

$$W_A \in 1 \to 1 \,.\, \text{ᗡ}`W_A = C`Q \,.\, P = W_A \,;\, Q.$$

$W_A \in 1 \to 1$ results immediately from $*259{\cdot}15$.

$P \upharpoonright \text{D}`W_A = W_A \,;\, Q$ results immediately from $*259{\cdot}16$ and $*272{\cdot}51$.

Thus it remains to prove $\text{D}`W_A = C`P \,.\, \text{ᗡ}`W_A = C`Q$.

$\text{D}`W_A = C`P$ is easily proved. By induction, if T is one of the series of partial correlators, $\text{D}`T \in \text{Cls induct}$, and therefore $\text{E}\,!\,\text{seq}_R`\text{D}`T$, by $*263{\cdot}47$, and by $*272{\cdot}34$, $C`P = \text{D}`T_{PQ}$; hence $\exists\,!\,\overleftarrow{T_{PQ}}`\text{seq}_R`\text{D}`T$, and therefore, by $*250{\cdot}121$, $\text{E}\,!\,\min_S`\overleftarrow{T_{PQ}}`\text{seq}_R`\text{D}`T$. Hence T has a successor, which correlates

$\text{seq}_R\text{'}D\text{'}T$ with $\text{min}_S\text{'}\overleftarrow{T_{PQ}}\text{'seq}_R\text{'}D\text{'}T$. Hence the successor, in R, of every member of $C\text{'}R$ which has a correlate, has a correlate; hence by induction every member of $C\text{'}R$ (*i.e.* of $C\text{'}P$) has a correlate. Hence $D\text{'}W_A = C\text{'}P$.

The proof of $\text{Cl}\text{'}W_A = C\text{'}Q$ is more difficult. As before, let T be one of the series of partial correlators. We have to prove that there is a correlator which has $\text{seq}_S\text{'}\text{Cl}\text{'}T$ in its converse domain; when this is proved, the result follows by induction. To prove this, put

$$x = \text{min}_R\text{'}\overrightarrow{T_{PQ}}\text{'seq}_S\text{'}\text{Cl}\text{'}T.$$

x exists, in virtue of $*272\cdot43$. Also since $D\text{'}W_A = C\text{'}P$, it follows from $*259\cdot13$ that there is a partial correlator U such that

$$x = \text{seq}_R\text{'}D\text{'}U.$$

We then have to prove $\quad \text{seq}_S\text{'}\text{Cl}\text{'}T = \text{min}_S\text{'}\overleftarrow{U_{PQ}}\text{'}x.$

Put $y = \text{seq}_S\text{'}\text{Cl}\text{'}T$. Then $\overrightarrow{S}\text{'}y \subset \text{Cl}\text{'}T$. Hence, by $*272\cdot2$, $\overrightarrow{S}\text{'}y \cap \overleftarrow{U_{PQ}}\text{'}x = \Lambda$. Thus if $xU_{PQ}y$, it follows that $y = \text{min}_S\text{'}\overleftarrow{U_{PQ}}\text{'}x$. To prove $xU_{PQ}y$, observe that

$$T \subseteq U \,.\, U_{PQ} \subseteq T_{PQ} \,.\, P \upharpoonright D\text{'}U = U ; Q.$$

We have $u \in D\text{'}U \,.\, \supset \,.\, \sim(uT_{PQ}y)$, by $*272\cdot2$. Hence, by the definition of T_{PQ}, we have, if $u \in D\text{'}U$,

$$(\exists z) \,.\, z \in D\text{'}T \,.\, zPu \,.\, \sim(\breve{T}\text{'}zQy) \,.\, \text{v} \,.\, (\exists z) \,.\, z \in D\text{'}T \,.\, uPz \,.\, \sim(yQ\breve{T}\text{'}z).$$

In the first case, we have $(\exists z) \,.\, z \in D\text{'}T \,.\, zPu \,.\, \sim(zPx)$, because $xT_{PQ}y$ Hence, since $x \neq z$ because $x \sim \in D\text{'}T$,

$$(\exists z) \,.\, z \in D\text{'}T \,.\, zPu \,.\, xPz.$$

Similarly, in the second case,

$$(\exists z) \,.\, z \in D\text{'}T \,.\, uPz \,.\, zPx.$$

The second case is incompatible with xPu, and the first with uPx. Hence

$$xPu \,.\, \supset \,.\, (\exists z) \,.\, z \in D\text{'}T \,.\, xPz \,.\, zPu : uPx \,.\, \supset \,.\, (\exists z) \,.\, z \in D\text{'}T \,.\, uPz \,.\, zPx.$$

But, since $xT_{PQ}y$, $xPz \,.\, \supset \,.\, yQ(\breve{T}\text{'}z) \,.\, \supset \,.\, yQ(\breve{U}\text{'}z)$, because $T \subseteq U$, and since

$$P \upharpoonright D\text{'}U = U ; Q, \quad zPu \,.\, \supset \,.\, (\breve{U}\text{'}z)\, Q\, (\breve{U}\text{'}u).$$

Hence $xPu \,.\, \supset \,.\, yQ(\breve{U}\text{'}u)$, and similarly $uPx \,.\, \supset \,.\, (\breve{U}\text{'}u)\, Qy$. Hence $xU_{PQ}y$. Hence $y = \text{min}_S\text{'}\overleftarrow{U_{PQ}}\text{'}x$, and therefore y belongs to the converse domain of the next correlator after U. Hence every term of $C\text{'}Q$ belongs to the converse domain of some correlator, and therefore to $\text{Cl}\text{'}W_A$. Hence W_A correlates P and Q, and P and Q are ordinally similar.

$*273 \cdot 01.\quad \eta = \mathrm{Ser} \cap \mathrm{comp} \cap \breve{C}``\aleph_0 \cap \hat{P}\,(\mathrm{D}‘P = \reflectbox{D}‘P) \quad \mathrm{Df}$

Following Cantor, we use η for the class of rational series.

$*273 \cdot 02.\quad R_{SPQ}‘T = T \cup \mathrm{seq}_R‘\mathrm{D}‘T \downarrow \mathrm{min}_S‘\overleftarrow{T_{PQ}}‘\mathrm{seq}_R‘\mathrm{D}‘T \quad \mathrm{Dft}\ [*273]$

$*273 \cdot 03.\quad (RS)_{PQ} = \overrightarrow{(R_{SPQ})_*}‘\dot{\Lambda} \quad \mathrm{Dft}\ [*273]$

$*273 \cdot 04.\quad T_{RSPQ} = \dot{s}‘(RS)_{PQ} \quad \mathrm{Dft}\ [*273]$

T_{RSPQ} will be shown to be a correlator of P with Q when P and Q are rational series, and R and S are progressions whose fields are $C‘P$ and $C‘Q$ respectively.

$*273 \cdot 1.\quad \vdash : P \in \eta \,.\equiv .\, P \in \mathrm{Ser} \cap \mathrm{comp} \,.\, C‘P \in \aleph_0 \,.\, \mathrm{D}‘P = \reflectbox{D}‘P \quad [(*273 \cdot 01)]$

$*273 \cdot 11.\quad \vdash :. P \in \eta \,.\equiv :\, P \in \mathrm{Ser} \cap \mathrm{comp} \,.\, \mathrm{D}‘P = \reflectbox{D}‘P : (\exists R) \,.\, R \in \omega \,.\, C‘P = C‘R$
$[*273 \cdot 1 \,.\, *263 \cdot 101]$

$*273 \cdot 2.\quad \vdash : W = \hat{X}\hat{T}\{X = \mathrm{seq}_R‘\mathrm{D}‘T \downarrow \mathrm{min}_S‘\overleftarrow{T_{PQ}}‘\mathrm{seq}_R‘\mathrm{D}‘T\} \,.\supset .$
$R_{SPQ} = A_W \,.\, (RS)_{PQ} \subset (A_W * A)‘\dot{\Lambda} \,.\, T_{RSPQ} \,\dot{\subset}\, W_A \,.\, T_{RSPQ} \in (A_W * A)‘\dot{\Lambda}$
$[*257 \cdot 125 \,.\, *258 \cdot 242 \,.\, (*273 \cdot 02 \cdot 03 \cdot 04 \,.\, *259 \cdot 02 \cdot 03)]$

Here the temporary definitions of $*259$ are revived.

The second of the above inclusions might be changed into an equality, but it is not necessary for our purposes to prove this.

$*273 \cdot 21.\quad \vdash : \mathrm{Hp}\ *273 \cdot 2 \,.\supset .\, \mathrm{D}‘W_A \subset C‘R \,.\, \reflectbox{D}‘W_A \subset C‘S$

Dem.

$\vdash .\, *259 \cdot 13 \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, \mathrm{D}‘W_A = s‘\mathrm{D}``W``(A_W * A)‘\dot{\Lambda} \quad (1)$

$\vdash .\, *206 \cdot 18 \,.\supset \vdash : \mathrm{Hp} \,.\, X \in \mathrm{D}‘W \,.\supset .\, \mathrm{D}‘X \subset C‘R \quad (2)$

$\vdash .\, (1) \,.\, (2) \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, \mathrm{D}‘W_A \subset C‘R \quad (3)$

Similarly $\vdash : \mathrm{Hp} \,.\supset .\, \reflectbox{D}‘W_A \subset C‘S \quad (4)$

$\vdash .\, (3) \,.\, (4) \,.\supset \vdash .\, \mathrm{Prop}$

$*273 \cdot 211.\quad \vdash : \mathrm{Hp}\ *273 \cdot 2 \,.\, T \in \reflectbox{D}‘W \,.\supset .\, \mathrm{D}‘T \cap \mathrm{D}‘W‘T = \Lambda \quad [*206 \cdot 2]$

$*273 \cdot 212.\quad \vdash : \mathrm{Hp}\ *273 \cdot 2 \,.\supset .\, W_A \in \mathrm{Cls} \to 1 \,.\, \mathrm{D} \upharpoonright (A_W * A)‘\dot{\Lambda} \in 1 \to 1$
$[*273 \cdot 211 \,.\, *259 \cdot 141 \cdot 171]$

$*273 \cdot 22.\quad \vdash : \mathrm{Hp}\ *273 \cdot 2 \,.\, C‘P = C‘R \,.\, P \in \mathrm{connex} \,.\, Q \,\dot{\subset}\, J \,.\supset .$
$W_A \in 1 \to 1 \,.\, \reflectbox{D} \upharpoonright (A_W * A)‘\dot{\Lambda} \in 1 \to 1$

Dem.

$\vdash .\, *273 \cdot 211 \cdot 212 \cdot 21 \,.\, *206 \cdot 2 \,.\, (*259 \cdot 03) \,.\supset$

$\vdash :. \mathrm{Hp} \,.\supset : T \in (A_W * A)‘\dot{\Lambda} \cap \reflectbox{D}‘W \,.\supset .\, T \in \mathrm{Cls} \to 1 \,.\, \mathrm{D}‘T \subset C‘P \,.\, \mathrm{seq}_R‘\mathrm{D}‘T \sim\in \mathrm{D}‘T \,.$

$[*272 \cdot 2] \quad \supset .\, \mathrm{min}_S‘\overleftarrow{T_{PQ}}‘\mathrm{seq}_R‘\mathrm{D}‘T \sim\in \reflectbox{D}‘T \quad (1)$

$\vdash .(1) .\supset \vdash :. \mathrm{Hp} \,.\supset : T \in (A_W * A)‘\dot{\Lambda} \cap \reflectbox{D}‘W \,.\supset_T .\, \reflectbox{D}‘T \cap \reflectbox{D}‘W‘T = \Lambda :$

$[*259 \cdot 14 \cdot 17] \quad \supset : W_A \in 1 \to \mathrm{Cls} \,.\, \reflectbox{D} \upharpoonright (A_W * A) \in 1 \to 1 \quad (2)$

$\vdash .\, (2) \,.\, *273 \cdot 212 \,.\supset \vdash .\, \mathrm{Prop}$

***273·23.** $\vdash : \mathrm{Hp} {*}273{\cdot}2 \,.\, P, Q \,\epsilon\, \mathrm{Ser} \,.\, C‘P = C‘R \,.\, C‘Q = C‘S \,.\, T \,\epsilon\, (A_W {*} A)‘\dot{\Lambda} \,.\, \supset .$
$P \upharpoonright \mathrm{D}‘T = T ; Q$

Dem.

$\vdash . {*}272{\cdot}51 . {*}273{\cdot}21 . \supset \vdash : \mathrm{Hp} \,.\, T \,\epsilon\, \text{Ɑ}‘W \,.\, \supset . \, P \upharpoonright \mathrm{D}‘\breve{A}_W‘T = (\breve{A}_W‘T) ; Q$ (1)

$\vdash . (1) . {*}259{\cdot}16 . \supset \vdash . \mathrm{Prop}$

***273·24** $\vdash : T \,\epsilon\, (RS)_{PQ} \,.\, \supset . \, \mathrm{D}‘T, \text{Ɑ}‘T \,\epsilon\, \mathrm{Cls\ induct}$

Dem.

$\vdash . {*}120{\cdot}251 . \supset$

$\vdash :. \mathrm{Hp} \,.\, \supset : T \,\epsilon\, \mathrm{D}‘A_W \,.\, \mathrm{D}‘T \,\epsilon\, \mathrm{Cls\ induct} \,.\, \supset . \, \mathrm{D}‘\breve{A}_W‘T \,\epsilon\, \mathrm{Cls\ induct} :$

[*90·112] $\supset : \dot{\Lambda} \, (A_W)_{*} \, T \,.\, \supset . \, \mathrm{D}‘T \,\epsilon\, \mathrm{Cls\ induct} :$

[*273·2.(*273·03)] $\supset : T \,\epsilon\, (RS)_{PQ} \,.\, \supset . \, \mathrm{D}‘T \,\epsilon\, \mathrm{Cls\ induct}$ (1)

Similarly $\vdash :. \mathrm{Hp} \,.\, \supset : T \,\epsilon\, (RS)_{PQ} \,.\, \supset . \, \text{Ɑ}‘T \,\epsilon\, \mathrm{Cls\ induct}$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***273·25.** $\vdash : P, Q \,\epsilon\, \eta \,.\, C‘P = C‘R \,.\, C‘Q = C‘S \,.\, T \,\epsilon\, (RS)_{PQ} \,.\, \supset .$
$\mathrm{D}‘T_{PQ} = C‘P \,.\, \text{Ɑ}‘T_{PQ} = C‘Q$

Dem.

$\vdash . {*}273{\cdot}1 . \supset$

$\vdash : \mathrm{Hp} \,.\, \supset . \, P, Q \,\epsilon\, \mathrm{Ser} \cap \mathrm{comp} \,.\, C‘P = \mathrm{D}‘P = \text{Ɑ}‘P \,.\, C‘Q = \mathrm{D}‘Q = \text{Ɑ}‘Q$ (1)

$\vdash . {*}273{\cdot}1 . {*}263{\cdot}44 . \supset \vdash : \mathrm{Hp} \,.\, \supset . \, \dot{\exists} \,!\, P \,.\, \dot{\exists} \,!\, Q$ (2)

$\vdash . (1) . (2) . {*}273{\cdot}22{\cdot}23{\cdot}24 . {*}272{\cdot}43 . \supset \vdash . \mathrm{Prop}$

***273·26.** $\vdash :. P, Q \,\epsilon\, \eta \,.\, R, S \,\epsilon\, \omega \,.\, C‘P = C‘R \,.\, C‘Q = C‘S \,.\, \supset :$
$T \,\epsilon\, (RS)_{PQ} \,.\, \supset . \, \mathrm{E} \,!\, \mathrm{seq}_R‘\mathrm{D}‘T \,.\, \mathrm{E} \,!\, \mathrm{min}_S‘\overleftarrow{T_{PQ}}‘\mathrm{seq}_R‘\mathrm{D}‘T$

Dem.

$\vdash . {*}273{\cdot}21 . {*}263{\cdot}47 . {*}273{\cdot}24 . \supset \vdash : \mathrm{Hp} \,.\, T \,\epsilon\, (RS)_{PQ} \,.\, \supset . \, \exists \,!\, C‘R \cap p‘\overleftarrow{R}“\mathrm{D}‘T \,.$

[*250·122] $\supset . \, \mathrm{E} \,!\, \mathrm{seq}_R‘\mathrm{D}‘T$ (1)

$\vdash . (1) . {*}273{\cdot}25 . \supset \vdash : \mathrm{Hp} \,.\, \supset . \, \exists \,!\, \overleftarrow{T_{PQ}}‘\mathrm{seq}_R‘\mathrm{D}‘T \,.$

[*250·121.*272·1] $\supset . \, \mathrm{E} \,!\, \mathrm{min}_S‘\overleftarrow{T_{PQ}}‘\mathrm{seq}_R‘\mathrm{D}‘T$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***273·27.** $\vdash : \mathrm{Hp} \, {*}273{\cdot}2 \,.\, \mathrm{Hp} \, {*}273{\cdot}26 \,.\, \supset . \, (RS)_{PQ} \subset \text{Ɑ}‘W \,.\, (RS)_{PQ} \subset \mathrm{D}‘A_W$
[*273·26]

***273·271.** $\vdash : \mathrm{Hp} \, {*}273{\cdot}26 \,.\, T \,\epsilon\, (RS)_{PQ} \,.\, \supset . \, \mathrm{seq}_R‘\mathrm{D}‘T \,\epsilon\, \mathrm{D}‘T_{RSPQ}$

Dem.

$\vdash . {*}273{\cdot}2 . \supset \vdash : \mathrm{Hp} \,.\, \mathrm{Hp} \, {*}273{\cdot}2 \,.\, \supset . \, T \epsilon (RS)_{PQ} \cap \mathrm{D}‘A_W \,.\, \supset . \, \breve{A}_W‘T \,\epsilon\, (RS)_{PQ}$ (1)

$\vdash . {*}273{\cdot}2 . \supset$

$\vdash : \mathrm{Hp} \,.\, \mathrm{Hp} \, {*}273{\cdot}2 \,.\, T \,\epsilon\, (RS)_{PQ} \,.\, \mathrm{E} \,!\, \breve{A}_W‘T \,.\, \supset . \, \mathrm{seq}_R‘\mathrm{D}‘T \,\epsilon\, \mathrm{D}‘\breve{A}_W‘T$ (2)

$\vdash . (1) . (2) . {*}273{\cdot}27 . \supset$

$\vdash : \mathrm{Hp} \,.\, \mathrm{Hp} \, {*}273{\cdot}2 \,.\, \supset . \, \breve{A}_W‘T \,\epsilon\, (RS)_{PQ} \,.\, \mathrm{seq}_R‘\mathrm{D}‘T \,\epsilon\, \mathrm{D}‘\breve{A}_W‘T \,.$

[*273·2.(*273·04)] $\supset . \, \mathrm{seq}_R‘\mathrm{D}‘T \,\epsilon\, \mathrm{D}‘T_{RSPQ} : \supset \vdash . \mathrm{Prop}$

$*273\cdot272$. $\vdash : \mathrm{Hp}\,*273\cdot26 \,.\supset .\, \mathrm{D}``(RS)_{PQ} = \overrightarrow{R}``C`R$

Dem.

$\vdash .\, *206\cdot401 \,.\supset \vdash : \mathrm{Hp} \,.\, T \in (RS)_{PQ} \,.\, \mathrm{D}`T = \overrightarrow{R}`x \,.\, x \in C`R \,.\supset .\, x = \mathrm{seq}_R`\mathrm{D}`T \,.$

$[*204\cdot71 . *250\cdot21] \qquad \supset .\, \mathrm{D}`R_{SPQ}`T = \overrightarrow{R}`\breve{R}_1`x \qquad (1)$

$\vdash .\, *250\cdot13 \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, \mathrm{D}`\dot{\Lambda} = \overrightarrow{R}`B`R \qquad (2)$

$\vdash .\, (1) \,.\, (2) \,.\, *90\cdot131 \,.\supset \vdash :. \mathrm{Hp} \,.\supset : T (R_{SPQ})_* \dot{\Lambda} \,.\supset .\, \mathrm{D}`T \in \overrightarrow{R}``C`R :$

$[(*273\cdot03)] \qquad \supset : \mathrm{D}``(RS)_{PQ} \subset \overrightarrow{R}``C`R \qquad (3)$

$\vdash .\, (1) \,.\, (*273\cdot03) \,.\supset$

$\vdash :. \mathrm{Hp} \,.\supset : x \in C`R \,.\, \overrightarrow{R}`x \in \mathrm{D}``(RS)_{PQ} \,.\supset .\, \overrightarrow{R}`\breve{R}_1`x \in \mathrm{D}``(RS)_{PQ} \qquad (4)$

$\vdash .\, (2) \,. \qquad \supset \vdash : \mathrm{Hp} \,.\supset .\, \overrightarrow{R}`B`R \in \mathrm{D}``(RS)_{PQ} \qquad (5)$

$\vdash .\, (4) \,.\, (5) \,.\, *90\cdot112 \,.\supset \vdash :. \mathrm{Hp} \,.\supset : x \in \overleftarrow{(R_1)_*}`B`R \,.\supset .\, \overrightarrow{R}`x \in \mathrm{D}``(RS)_{PQ} \qquad (6)$

$\vdash .\, *263\cdot43 \,.\, *250\cdot21 \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, C`R = C`R_1 \,.\, B`R = B`R_1 \qquad (7)$

$\vdash .\, (6) \,.\, (7) \,.\, *263\cdot141 \,.\, *122\cdot1\cdot141 \,.\supset$

$\vdash :. \mathrm{Hp} \,.\supset : x \in C`R \,.\supset .\, \overrightarrow{R}`x \in \mathrm{D}``(RS)_{PQ} \qquad (8)$

$\vdash .\, (3) \,.\, (8) \,.\supset \vdash .\, \mathrm{Prop}$

$*273\cdot28$. $\vdash : \mathrm{Hp}\,*272\cdot26 \,.\supset .\, T_{RSPQ} \in 1 \rightarrow 1 \,.\, \mathrm{D}`T_{RSPQ} = C`P \,.\, P = T_{RSPQ} ; Q$

Dem.

$\vdash .\, *273\cdot2\cdot22 \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, T_{RSPQ} \in 1 \rightarrow 1 \qquad (1)$

$\vdash .\, *273\cdot272 \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, \mathrm{D}`T_{RSPQ} = s`\overrightarrow{R}``C`R$

$[*263\cdot22] \qquad = C`R \qquad (2)$

$\vdash .\, *273\cdot2\cdot23 \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, P \upharpoonright \mathrm{D}`T_{RSPQ} = T_{RSPQ} ; Q \,.$

$[(2)] \qquad \supset .\, P = T_{RSPQ} ; Q \qquad (3)$

$\vdash .\, (1) \,.\, (2) \,.\, (3) \,.\supset \vdash .\, \mathrm{Prop}$

In order to prove $T_{RSPQ} \in P \,\overline{\mathrm{smor}}\, Q$, it only remains to prove

$$\mathrm{ⴲ}`T_{RSPQ} = C`Q.$$

$*273\cdot3$. $\vdash :. \mathrm{Hp}\,*273\cdot2 \,.\, T, U \in (A_W * A)`\dot{\Lambda} \,.\supset : \mathrm{D}`T \subset \mathrm{D}`U \,.\equiv .\, T \Subset U$

Dem.

$\vdash .\, *33\cdot263 \,. \qquad \supset \vdash : T \Subset U \,.\supset .\, \mathrm{D}`T \subset \mathrm{D}`U \qquad (1)$

$\vdash .\, *259\cdot111 \,. \qquad \supset \vdash :. \mathrm{Hp} \,.\supset : T \Subset U \,.\vee.\, U \Subset T \qquad (2)$

$\vdash .\, *33\cdot263 \,. \qquad \supset \vdash : U \Subset T \,.\, \mathrm{D}`T \subset \mathrm{D}`U \,.\supset .\, \mathrm{D}`T = \mathrm{D}`U \qquad (3)$

$\vdash .\, (3) \,.\, *273\cdot212 \,.\supset \vdash : \mathrm{Hp} \,.\, U \Subset T \,.\, \mathrm{D}`T \subset \mathrm{D}`U \,.\supset .\, T = U \qquad (4)$

$\vdash .\, (2) \,.\, (4) \,. \qquad \supset \vdash : \mathrm{Hp} \,.\, \mathrm{D}`T \subset \mathrm{D}`U \,.\supset .\, T \Subset U \qquad (5)$

$\vdash .\, (1) \,.\, (5) \,.\supset \vdash .\, \mathrm{Prop}$

***273·31.** $\vdash : \text{Hp} *273\cdot26 \,.\, T \in (RS)_{PQ} \,.\, y \in C`S - \text{Q}`T \,.\, \overrightarrow{S}`y \subset \text{Q}`T \,.\, \supset .$

$(\exists x, U) \,.\, x = \min_R`\overrightarrow{T_{PQ}}`y \,.\, U \in (RS)_{PQ} \,.\, x = \text{seq}_R`\text{D}`U$

Dem.

$\vdash . *273\cdot25 . *250\cdot121 . \supset \vdash : \text{Hp} . \supset . (\exists x) . x = \min_R`\overrightarrow{T_{PQ}}`y$ (1)

$\vdash . *273\cdot272 . \supset \vdash : \text{Hp} . x = \min_R`\overrightarrow{T_{PQ}}`y . \supset . (\exists U) . U \in (RS)_{PQ} . \text{D}`U = \overrightarrow{R}`x .$

[*206·401] $\supset . (\exists U) . U \in (RS)_{PQ} . x = \text{seq}_R`\text{D}`U$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***273·32.** $\vdash : \text{Hp} *273\cdot31 . x = \min_R`\overrightarrow{T_{PQ}}`y . U \in (RS)_{PQ} . x = \text{seq}_R`\text{D}`U . \supset .$

$x U_{PQ} y . T \Subset U$

Dem.

$\vdash . *205\cdot14 . \supset \vdash :. \text{Hp} . uRx . \supset : \sim (uT_{PQ}y) :$

[*272·13] $\supset : (\exists z) : z \in \text{D}`T : zPu . \sim (\breve{T}`zQy) . \vee . uPz . \sim (yQ\breve{T}`z)$ (1)

$\vdash . *272\cdot2\cdot42 . \quad \supset \vdash : \text{Hp} . \supset . x \sim \in \text{D}`T .$ (2)

[*273·272] $\supset . \text{D}`T \subset \overrightarrow{R}`x$ (3)

$\vdash . *273\cdot272 . \quad \supset \vdash : \text{Hp} . \supset . \overrightarrow{R}`x = \text{D}`U$ (4)

$\vdash . (3) . (4) . *273\cdot3 . \supset \vdash : \text{Hp} . \supset . T \Subset U$ (5)

$\vdash . (1) . *272\cdot13 . \supset$

$\vdash :. \text{Hp} . uRx . \supset : (\exists z) : z \in \text{D}`T : zPu . \sim (zPx) . \vee . uPz . \sim (xPz)$ (6)

$\vdash . *204\cdot1 . \supset \vdash :. \text{Hp} . \supset : uPx . zPu . \supset . zPx : xPu . uPz . \supset . xPz$ (7)

$\vdash . (6) . (7) . (4) . \supset \vdash :. \text{Hp} . u \in \text{D}`U . \supset : uPx . \supset . (\exists z) . z \in \text{D}`T . uPz . \sim (xPz) :$

$xPu . \supset . (\exists z) . z \in \text{D}`T . zPu . \sim (zPx) :$

[(2)] $\supset : uPx . \supset . (\exists z) . z \in \text{D}`T . uPz . zPx :$

$xPu . \supset . (\exists z) . z \in \text{D}`T . zPu . xPz$ (8)

$\vdash . *272\cdot13 . *273\cdot23 . \supset$

$\vdash : \text{Hp} . u \in \text{D}`U . z \in \text{D}`T . uPz . zPx . \supset . (\breve{U}`u) Q (\breve{U}`z) . (\breve{T}`z) Qy .$

[(5)] $\supset . (\breve{U}`u) Qy$ (9)

Similarly $\vdash : \text{Hp} . u \in \text{D}`U . z \in \text{D}`T . zPu . xPz . \supset . yQ (\breve{U}`u)$ (10)

$\vdash . (8) . (9) . (10) . \supset$

$\vdash :. \text{Hp} . u \in \text{D}`U . \supset : uPx . \supset . (\breve{U}`u) Qy : xPu . \supset . yQ (\breve{U}`u)$ (11)

$\vdash . (11) . *272\cdot13 . \supset \vdash : \text{Hp} . \supset . xU_{PQ}y$ (12)

$\vdash . (5) . (12) . \supset \vdash . \text{Prop}$

$*273{\cdot}33.\quad \vdash : \mathrm{Hp}\,*273{\cdot}32 \,.\supset .\, y = \mathrm{min}_S{}^{\epsilon}\overleftarrow{U}_{PQ}{}^{\epsilon}x \,.\, x\,(R_{SPQ}{}^{\epsilon}U)\,y$

Dem.

$$\begin{array}{ll} \vdash .\, *273{\cdot}32 \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, \overrightarrow{S}{}^{\epsilon}y \subset \text{Ɑ}{}^{\epsilon}U \,. & \\ [*272{\cdot}2{\cdot}42] \qquad\qquad \supset .\, \overrightarrow{S}{}^{\epsilon}y \cap \overleftarrow{U}_{PQ}{}^{\epsilon}x = \Lambda & (1) \\ \vdash .\,(1)\,.\, *273{\cdot}32 \,.\, *205{\cdot}14 \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, y = \mathrm{min}_S{}^{\epsilon}\overleftarrow{U}_{PQ}{}^{\epsilon}x & (2) \\ \vdash .\,(2)\,.\,(*273{\cdot}02) \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, x\,(R_{SPQ}{}^{\epsilon}U)\,y : \supset \vdash .\, \mathrm{Prop} & \end{array}$$

$*273{\cdot}34.\quad \vdash : \mathrm{Hp}\,*273{\cdot}31 \,.\supset .\, y \in \text{Ɑ}{}^{\epsilon}T_{RSPQ}$

Dem.

$$\begin{array}{ll} \vdash .\, *273{\cdot}31{\cdot}33 \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, (\exists U)\,.\, U \in (RS)_{PQ} \,.\, y \in \text{Ɑ}{}^{\epsilon}R_{SQP}{}^{\epsilon}U \,. \\ [*90{\cdot}16.(*273{\cdot}03)] \qquad \supset .\, (\exists W)\,.\, W \in (RS)_{PQ} \,.\, y \in \text{Ɑ}{}^{\epsilon}W \,. \\ [(*273{\cdot}04)] \qquad \supset .\, y \in \text{Ɑ}{}^{\epsilon}T_{RSPQ} : \supset \vdash .\, \mathrm{Prop} \end{array}$$

$*273{\cdot}35.\quad \vdash : \mathrm{Hp}\,*273{\cdot}26 \,.\supset .\, \text{Ɑ}{}^{\epsilon}T_{RSPQ} = C{}^{\epsilon}Q$

Dem.

$$\begin{array}{ll} \vdash .\, *273{\cdot}34 \,.\supset \vdash : \mathrm{Hp} \,.\, y \in C{}^{\epsilon}S \,.\, \overrightarrow{S}{}^{\epsilon}y \subset \text{Ɑ}{}^{\epsilon}T_{RSPQ} \,.\supset .\, y \in \text{Ɑ}{}^{\epsilon}T_{RSPQ} & (1) \\ \vdash .\,(1)\,.\, *250{\cdot}34 \,.\supset \vdash .\, \mathrm{Prop} & \end{array}$$

$*273{\cdot}36.\quad \vdash : \mathrm{Hp}\,*273{\cdot}26 \,.\supset .\, T_{RSPQ} \in P\,\overline{\mathrm{smor}}\,Q \quad [*273{\cdot}28{\cdot}35]$

$*273{\cdot}4.\quad \vdash : P, Q \in \eta \,.\supset .\, P\,\mathrm{smor}\,Q$

Dem.

$$\begin{array}{l} \vdash .\, *273{\cdot}11 \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, (\exists R, S)\,.\, R, S \in \omega \,.\, C{}^{\epsilon}P = C{}^{\epsilon}R \,.\, C{}^{\epsilon}Q = C{}^{\epsilon}S \\ [*273{\cdot}36] \qquad \supset .\, (\exists R, S)\,.\, T_{RSPQ} \in P\,\overline{\mathrm{smor}}\,Q : \supset \vdash .\, \mathrm{Prop} \end{array}$$

$*273{\cdot}41.\quad \vdash : P \in \eta \,.\, P\,\mathrm{smor}\,Q \,.\supset .\, Q \in \eta$

Dem.

$$\begin{array}{ll} \vdash .\, *270{\cdot}41 \,. \qquad \supset \vdash : \mathrm{Hp} \,.\supset .\, Q \in \mathrm{Ser} \cap \mathrm{comp} & (1) \\ \vdash .\, *151{\cdot}18 \,.\, *123{\cdot}321 \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, C{}^{\epsilon}Q \in \aleph_0 & (2) \\ \vdash .\, *151{\cdot}5 \,. \qquad \supset \vdash : \mathrm{Hp} \,.\supset .\, \mathrm{D}{}^{\epsilon}Q = \text{Ɑ}{}^{\epsilon}Q & (3) \\ \vdash .\,(1)\,.\,(2)\,.\,(3)\,.\, *273{\cdot}1 \,.\supset \vdash .\, \mathrm{Prop} & \end{array}$$

$*273{\cdot}42.\quad \vdash : P \in \eta \,.\supset .\, \eta = \mathrm{Nr}{}^{\epsilon}P \quad [*273{\cdot}4{\cdot}41]$

$*273{\cdot}43.\quad \vdash .\, \eta \in \mathrm{NR} \qquad [*273{\cdot}42 \,.\, *256{\cdot}54]$

The following propositions are easy to prove:

$$\vdash : Q \in \mathrm{Ser} \cap \breve{C}{}^{\epsilon\epsilon}\aleph_0 \,.\, P \in \eta \,.\supset .\, Q \times P \in \eta,$$

whence $\vdash : \alpha \in \mathrm{NR} \cap \mathrm{Cl}{}^{\epsilon}\mathrm{Ser} \,.\, C{}^{\epsilon\epsilon}\alpha = \aleph_0 \,.\supset .\, \alpha \dot{\times} \eta = \eta\,;$

and

$$\vdash : P \in \eta \,.\, Q \in \mathrm{Ser} \cap \breve{C}{}^{\epsilon\epsilon}\aleph_0 \,.\, x \in C{}^{\epsilon}P \,.\supset .\, x \downarrow {}^{;}Q \in \mathrm{Nr}{}^{\epsilon}Q \cap \mathrm{Rl}{}^{\epsilon}(Q \times P) \,.\, Q \times P \in \eta,$$

whence, from the fact that all η's are similar,

$$\vdash : P \in \eta \,.\, Q \in \mathrm{Ser} \cap \breve{C}{}^{\epsilon\epsilon}\aleph_0 \,.\supset .\, \exists\,!\, \mathrm{Nr}{}^{\epsilon}Q \cap \mathrm{Rl}{}^{\epsilon}P.$$

Thus an η contains series of all the order-types composed of $\aleph_0$ terms.

$*274$. ON SERIES OF FINITE SUB-CLASSES OF A SERIES.

Summary of $*274$.

In the present number, we shall be concerned with the construction of a rational series consisting of the finite existent sub-classes of a progression. When the finite sub-classes of a progression (excluding Λ) are arranged by the principle of first differences, the result is a rational series. When they are arranged by the principle of last differences, the result is a progression. These two propositions, with the consequent existence-theorems, are to be proved in the present number.

We define "P_η" as P_{cl} with its field limited to finite existent classes. (For the definition of P_{cl}, see $*170{\cdot}01$.) In the present number, we shall be chiefly concerned with P_η when $P \epsilon \omega$, but it has interesting properties in many other cases.

Our definition is

$$P_\eta = P_{\text{cl}} \upharpoonright (\text{Cls induct} - \iota`\Lambda) \quad \text{Df.}$$

We shall be concerned in this number not only with P_η, but also with $P_{\text{1c}} \upharpoonright (\text{Cls induct} - \iota`\Lambda)$. This is $\text{Cnv}`(\breve{P})_\eta$. Thus if we put $\breve{P} = Q$, the hypothesis that $P \epsilon \Omega$ as used in studying $P_{\text{1c}} \upharpoonright (\text{Cls induct} - \iota`\Lambda)$ is equivalent to the hypothesis that $\breve{Q} \epsilon \Omega$ as used in studying $\text{Cnv}`Q_\eta$, *i.e.* $\breve{Q}_\eta$. Thus the study of P_{cl} and P_{1c} with their fields limited to inductive existent classes may be replaced by the study of P_η in the two cases where (1) $P \epsilon \Omega$, (2) $\breve{P} \epsilon \Omega$. The second case is the simpler, and is considered first. We have first, however, a collection of propositions which only assume that P is a series.

Since an inductive existent class in a series must have a maximum and a minimum, we have

$*274{\cdot}12$. $\vdash :: P \epsilon \text{Ser} . \supset :. \alpha P_\eta \beta . \equiv :$

$$\alpha, \beta \epsilon \text{Cl induct}`C`P - \iota`\Lambda : (\exists z) . z \epsilon \alpha - \beta . \alpha \cap \overrightarrow{P}`z = \beta \cap \overrightarrow{P}`z$$

We have

$*274{\cdot}17$. $\vdash : C`P \sim \epsilon 1 . \supset . C`P_\eta = \text{Cl induct}`C`P - \iota`\Lambda$

Whenever P is a series, P_η is a series (*274·18). If P has a last term, the class consisting of this last term only is the last term of P_η; if P has no last term, P_η has no last term (*274·191). If $C'P$ is an inductive existent class, the first term of P_η is $C'P$ (*274·194); if not, P_η has no first term (*274·195). Hence if P has no last term, P_η has no first or last term, and we have $\mathrm{D}'P_\eta = \mathrm{G}'P_\eta$ (*274·196). Thus of the characteristics used in defining η, we have $P_\eta \epsilon \mathrm{Ser}$ whenever $P \epsilon \mathrm{Ser}$, and $\mathrm{D}'P_\eta = \mathrm{G}'P_\eta$ whenever $\sim \mathrm{E} ! B'\breve{P}$

We next prove

***274·22.** $\vdash : \breve{P} \epsilon \Omega \,.\, \supset .\, \breve{P}_\eta \epsilon \Omega$

which, in virtue of what was said above, is equivalent to

$$P \epsilon \Omega \,.\, \supset .\, P_{\mathrm{lc}} \upharpoonright (\mathrm{Cls\ induct} - \iota'\Lambda) \epsilon \Omega,$$

that is: The principle of last differences applied to the inductive existent sub-classes of any well-ordered series gives a well-ordered series.

To prove *274·22, since we already know that P_η is a series, we only have to prove that every existent sub-class of $C'P_\eta$ has a maximum with respect to P_η. This is proved as follows.

Let κ be any existent sub-class of $\mathrm{Cl\,induct}'C'P - \iota'\Lambda$. Consider the minima of all the members of κ: these minima all exist, because κ is composed of inductive classes. Then in virtue of the nature of the principle of first differences, members of κ which have a later minimum come later than those that have an earlier minimum. Hence if we consider $\mathrm{min}_P``\kappa$, the classes whose minimum is the maximum of $\mathrm{min}_P``\kappa$ (which exists, because $\breve{P} \epsilon \Omega$) are later than any other members of κ. Put

$$x_1 = \mathrm{max}_P`\mathrm{min}_P``\kappa \,.\, \kappa_1 = \kappa \cap \overleftarrow{\mathrm{min}_P}`x_1.$$

Thus κ_1 consists of those members of κ which have the largest minimum, and members of κ_1 come later than any other members of κ. Similarly the latest members of κ_1 will be those that have the greatest second term. That is, if we take away the (common) first term from each member of κ_1, and if λ_1 is the resulting class of classes, we have to apply to λ_1 precisely the same process as we have already applied to κ. Thus we are led to put

$$x_1 = \mathrm{max}_P`\mathrm{min}_P``\kappa \,.\, \kappa_1 = \kappa \cap \overleftarrow{\mathrm{min}_P}`x_1 \,.\, \lambda_1 = (-\iota`x_1)``\kappa_1,$$
$$x_2 = \mathrm{max}_P`\mathrm{min}_P``\lambda_1 \,.\, \kappa_2 = \lambda_1 \cap \overleftarrow{\mathrm{min}_P}`x_2 \,.\, \lambda_2 = (-\iota`x_2)``\kappa_2,$$

and so on. The series $x_1, x_2, \ldots$ is an ascending series in P, and is therefore finite, by *261·33. It therefore has a last term, say x_ν. Then the class $\iota`x_1 \cup \iota`x_2 \cup \ldots \cup \iota`x_\nu$ is a member of κ, and is easily shown to be its maximum. Hence every existent sub-class κ of $C'P_\eta$ has a maximum, and therefore $\breve{P}_\eta \epsilon \Omega$.

In order to symbolize the above process, we put

$$P_m{}^\backprime\kappa = \max_P{}^\backprime\min_P{}^{\backprime\backprime}\kappa \qquad \text{Dft,}$$

$$\breve{T}_P{}^\backprime\kappa = (-\iota^\backprime P_m{}^\backprime\kappa)^{\backprime\backprime}(\kappa \cap \overleftarrow{\min_P}{}^\backprime P_m{}^\backprime\kappa) - \iota^\backprime\Lambda \qquad \text{Dft,}$$

$$M_P{}^\backprime\kappa = P_m{}^{\backprime\backprime}\overleftarrow{(\breve{T}_P)_*}{}^\backprime\kappa \qquad \text{Dft.}$$

Then $P_m{}^\backprime\kappa$ is what we called x_1, $\breve{T}_P{}^\backprime\kappa$ is what we called λ_1, $\overleftarrow{(\breve{T}_P)_*}{}^\backprime\kappa$ is the class $\kappa, \lambda_1, \lambda_2, \ldots \lambda_{\nu-1}$, and $M_P{}^\backprime\kappa$ is the class $x_1, x_2, x_3, \ldots x_\nu$. Thus what we have to prove is

$$M_P{}^\backprime\kappa = \max(P_\eta)^\backprime\kappa,$$

which is proved in $*274\cdot215$.

We prove next

$*274\cdot25.$ $\vdash : \breve{P} \epsilon \omega . \supset . \breve{P}_\eta \epsilon \omega$

For this purpose we use $*263\cdot44$, namely

$$\omega = \Omega - \iota^\backprime\Lambda \cap \hat{P}(\mathrm{G}^\backprime P_1 = \mathrm{G}^\backprime P . \sim \mathrm{E}\,!\, B^\backprime\breve{P}).$$

Thus it only remains to prove

$$\mathrm{D}^\backprime(P_\eta)_1 = \mathrm{D}^\backprime P_\eta . \sim \mathrm{E}\,!\, B^\backprime P_\eta.$$

$\sim \mathrm{E}\,!\, B^\backprime P_\eta$ follows from $*274\cdot195$, and $\mathrm{D}^\backprime(P_\eta)_1 = \mathrm{D}^\backprime P_\eta$ is proved without any difficulty; hence our proposition follows.

From $*274\cdot25\cdot17$, by substituting P for $\breve{P}$, we obtain

$*274\cdot26.$ $\vdash : P \epsilon \omega . \supset . P_{1c} \upharpoonright (\text{Cls induct} - \iota^\backprime\Lambda) \epsilon \omega .$

$$C^\backprime P_{1c} \upharpoonright (\text{Cls induct} - \iota^\backprime\Lambda) = \text{Cl induct}^\backprime C^\backprime P - \iota^\backprime\Lambda$$

whence it follows immediately that

$*274\cdot27.$ $\vdash : \alpha \epsilon \aleph_0 . \supset . \text{Cl induct}^\backprime\alpha \epsilon \aleph_0 . \text{Cl induct}^\backprime\alpha - \iota^\backprime\Lambda \epsilon \aleph_0$

I.e. a class of $\aleph_0$ terms contains $\aleph_0$ inductive sub-classes.

We now have to prove

$*274\cdot33.$ $\vdash : P \epsilon \omega . \supset . P_\eta \epsilon \eta$

In virtue of $*274\cdot17\cdot27$, we have $C^\backprime P_\eta \epsilon \aleph_0$; and by $*274\cdot18$, $P_\eta \epsilon \text{Ser}$. Thus it only remains to prove $P_\eta \epsilon \text{comp} . \mathrm{D}^\backprime P_\eta = \mathrm{G}^\backprime P_\eta$. The second of these results immediately from $*274\cdot196$. As for $P_\eta \epsilon$ comp, if $\alpha P_\eta \beta$, $\alpha \cup \beta \epsilon \text{Cls induct}$, and therefore $\exists\,!\, p^\backprime\overleftarrow{P}^{\backprime\backprime}(\alpha \cup \beta)$; but if $x \epsilon p^\backprime\overleftarrow{P}^{\backprime\backprime}(\alpha \cup \beta)$, we have $\alpha P_\eta (\beta \cup \iota^\backprime x) . (\beta \cup \iota^\backprime x) P_\eta \beta$; hence $P_\eta \subseteq P_\eta{}^2$. This completes the proof that $P_\eta \epsilon \eta$.

The proposition holds not only if $P \epsilon \omega$, but if P is any series which has no last term and whose field has $\aleph_0$ terms ($*274\cdot32$).

Finally, we deal with the existence of η ($*274\cdot4$—$\cdot46$). If $P \epsilon \omega$, P is similar to $P_{1c} \upharpoonright (\text{Cls induct} - \iota^\backprime\Lambda)$, by $*274\cdot26$; and if T is a correlator of

these two, $T \mathbin{;} P_\eta$ is an η whose field is $C\text{‘}P$ (*274·4). Hence the existence of η in any type is equivalent to the existence of ω in that type (*274·41). Hence we have merely to apply previous propositions on the existence of ω.

***274·01.** $P_\eta = P_{\mathrm{cl}} \upharpoonright (\mathrm{Cls\ induct} - \iota\text{‘}\Lambda)$ Df

***274·02.** $P_m\text{‘}\kappa = \max_P\text{‘}\min_P\text{“}\kappa$ Dft [*274]

***274·03.** $\breve{T}_P\text{‘}\kappa = (-\iota\text{‘}P_m\text{‘}\kappa)\text{“}(\kappa \cap \overleftarrow{\min_P}\text{‘}P_m\text{‘}\kappa) - \iota\text{‘}\Lambda$ Dft [*274]

***274·04.** $M_P\text{‘}\kappa = P_m\text{“}\overleftarrow{(T_P)_*}\text{‘}\kappa$ Dft [*274]

***274·1.** $\vdash : \alpha P_\eta \beta . \equiv . \alpha, \beta \in \mathrm{Cl\ induct}\text{‘}C\text{‘}P - \iota\text{‘}\Lambda . \exists ! \alpha - \beta - \breve{P}\text{“}(\beta - \alpha)$ [*170·1 . (*274·01)]

***274·11.** $\vdash : P \in \mathrm{Ser} . \alpha \in \mathrm{Cl\ induct}\text{‘}C\text{‘}P - \iota\text{‘}\Lambda . \supset . E ! \min_P\text{‘}\alpha . E ! \max_P\text{‘}\alpha$ [*261·26]

***274·111.** $\vdash : P \in \mathrm{Ser} . \sim E ! B\text{‘}\breve{P} . \alpha \in \mathrm{Cl\ induct}\text{‘}C\text{‘}P . \supset . \exists ! p\text{‘}\overleftarrow{P}\text{“}\alpha$

Dem.

$\vdash . *274·11 . \supset \vdash : \mathrm{Hp} . \exists ! \alpha . \supset . \max_P\text{‘}\alpha \in D\text{‘}P .$

[*205·65] $\supset . \exists ! p\text{‘}\overleftarrow{P}\text{“}\alpha$ (1)

$\vdash . (1) . *40·2 . \supset \vdash . \mathrm{Prop}$

***274·12.** $\vdash :: P \in \mathrm{Ser} . \supset :. \alpha P_\eta \beta . \equiv :$
$\alpha, \beta \in \mathrm{Cl\ induct}\text{‘}C\text{‘}P - \iota\text{‘}\Lambda : (\exists z) . z \in \alpha - \beta . \alpha \cap \overrightarrow{P}\text{‘}z = \beta \cap \overrightarrow{P}\text{‘}z$

Dem.

$\vdash . *170·2 . \supset$

$\vdash :. \alpha, \beta \in \mathrm{Cl\ induct}\text{‘}C\text{‘}P - \iota\text{‘}\Lambda : (\exists z) . z \in \alpha - \beta . \alpha \cap \overrightarrow{P}\text{‘}z = \beta \cap \overrightarrow{P}\text{‘}z : \supset . \alpha P_\eta \beta$ (1)

$\vdash . *274·11 . \supset \vdash : \mathrm{Hp} . \alpha P_\eta \beta . \supset . E ! \min_P\text{‘}\{\alpha - \beta - \breve{P}\text{“}(\beta - \alpha)\} .$

[*170·23.*205·192] $\supset . (\exists z) . z \in \alpha - \beta . \alpha \cap \overrightarrow{P}\text{‘}z = \beta \cap \overrightarrow{P}\text{‘}z$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***274·13.** $\vdash . P_{1\mathrm{c}} \upharpoonright (\mathrm{Cls\ induct} - \iota\text{‘}\Lambda) = \mathrm{Cnv}\text{‘}(\breve{P})_\eta$ [*170·101 . (*274·01)]

***274·14.** $\vdash :: P \in \mathrm{Ser} . \supset :. \alpha \{P_{1\mathrm{c}} \upharpoonright (\mathrm{Cls\ induct} - \iota\text{‘}\Lambda)\} \beta . \equiv :$
$\alpha, \beta \in \mathrm{Cl\ induct}\text{‘}C\text{‘}P - \iota\text{‘}\Lambda : (\exists z) . z \in \beta - \alpha . \alpha \cap \overleftarrow{P}\text{‘}z = \beta \cap \overleftarrow{P}\text{‘}z$
[*274·12·13]

***274·15.** $\vdash : \alpha, \beta \in \mathrm{Cl\ induct}\text{‘}C\text{‘}P - \iota\text{‘}\Lambda . \beta \subset \alpha . \beta \neq \alpha . \supset . \alpha P_\eta \beta$ [*170·16 . *274·1]

***274·151.** $\vdash : \alpha \in \mathrm{Cl\ induct}\text{‘}C\text{‘}P - 1 . x \in \alpha . \supset . \alpha P_\eta (\iota\text{‘}x)$ [*274·15]

***274·16.** $\vdash : \dot{\exists} ! P_\eta . \equiv . C`P \sim \epsilon\, 0 \cup 1$

Dem.

$\vdash . *274·1 . \supset \vdash : \dot{\exists} ! P_\eta . \supset . \exists ! C`P$ (1)

$\vdash . *274·151 . \supset \vdash : C`P \sim \epsilon\, 0 \cup 1 . \supset . \dot{\exists} ! P_\eta$ (2)

$\vdash . *60·38 . \supset \vdash : C`P \,\epsilon\, 1 . \supset . \sim (\exists \alpha, \beta) . \alpha, \beta \,\epsilon\, \mathrm{Cl}`C`P - \iota`\Lambda . \exists ! \alpha - \beta .$

$[*274·1] \qquad \supset . P_\eta = \dot{\Lambda}$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash . \mathrm{Prop}$

***274·17.** $\vdash : C`P \sim \epsilon\, 1 . \supset . C`P_\eta = \mathrm{Cl\,induct}`C`P - \iota`\Lambda$

Dem.

$\vdash . *274·151 . \supset \vdash . \dot{\mathrm{C}}\mathrm{l\,induct}`C`P - \iota`\Lambda - 1 \subset \mathrm{D}`P_\eta$ (1)

$\vdash . *274·151 . \supset \vdash : x \,\epsilon\, C`P . C`P \neq \iota`x . \supset . \iota`x \,\epsilon\, \mathrm{G}`P_\eta$ (2)

$\vdash . (2) . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{Cl}`C`P \cap 1 \subset \mathrm{G}`P_\eta$ (3)

$\vdash . (1) . (3) . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{Cl\,induct}`C`P - \iota`\Lambda \subset C`P_\eta$ (4)

$\vdash . (4) . *274·1 . \supset \vdash . \mathrm{Prop}$

***274·171.** $\vdash : P^2 \mathrel{\mathsf{G}} J . xPy . \supset . (\iota`x) P_\eta (\iota`y) \quad [*274·1]$

***274·18** $\vdash : P \,\epsilon\, \mathrm{Ser} . \supset . P_\eta \,\epsilon\, \mathrm{Ser}$

Dem.

$\vdash . *201·14 . \supset$

$\vdash :. \mathrm{Hp} . z \,\epsilon\, \alpha - \beta . w \,\epsilon\, \beta - \gamma . \alpha \cap \overrightarrow{P}`z = \beta \cap \overrightarrow{P}`z . \beta \cap \overrightarrow{P}`w = \gamma \cap \overrightarrow{P}`w . \supset :$

$zPw . \supset . z \,\epsilon\, \alpha - \gamma . \alpha \cap \overrightarrow{P}`z = \gamma \cap \overrightarrow{P}`z$ (1)

$\vdash . *201·14 . \supset \vdash :. \mathrm{Hp}(1) . \supset : wPz . \supset . w \,\epsilon\, \alpha - \gamma . \alpha \cap \overrightarrow{P}`w = \gamma \cap \overrightarrow{P}`w$ (2)

$\vdash . (1) . (2) . *202·103 . *274·12 . \supset \vdash : \mathrm{Hp} . \alpha P_\eta \beta . \beta P_\eta \gamma . \supset . \alpha P_\eta \gamma$ (3)

$\vdash . *274·11 . \supset$

$\vdash : \mathrm{Hp} . \alpha, \beta \,\epsilon\, \mathrm{Cl\,induct}`C`P - \iota`\Lambda . \alpha \neq \beta . \supset . (\exists z) . z = \min_P`\{(\alpha - \beta) \cup (\beta - \alpha)\} .$

$[*205·14] \qquad \supset . (\exists z) . z \,\epsilon\, (\alpha - \beta) \cup (\beta - \alpha) . \alpha \cap \overrightarrow{P}`z = \beta \cap \overrightarrow{P}`z .$

$[*274·12] \qquad \supset . \alpha (P_\eta \cup \breve{P}_\eta) \beta$ (4)

$\vdash . (3) . (4) . *170·17 . \supset \vdash . \mathrm{Prop}$

***274·19.** $\vdash : P \,\epsilon\, \mathrm{connex} . P^2 \mathrel{\mathsf{G}} J . \supset . \overrightarrow{B}`\breve{P}_\eta = \iota``\overrightarrow{B}`\breve{P}$

Dem.

$\vdash . *274·151 . \supset \vdash . \mathrm{Cl\,induct}`C`P - 1 \subset \mathrm{D}`P_\eta$ (1)

$\vdash . *274·171 . \supset \vdash : \mathrm{Hp} . \supset . \iota``\mathrm{D}`P \subset \mathrm{D}`P_\eta$ (2)

$\vdash . (1) . (2) . *274·17 . \supset \vdash : \mathrm{Hp} . \supset . \overrightarrow{B}`\breve{P}_\eta \subset \iota``\overrightarrow{B}`\breve{P}$ (3)

$\vdash . *202·524 . \supset$

$\vdash : \mathrm{Hp} . x \,\epsilon\, \overrightarrow{B}`\breve{P} . \beta \,\epsilon\, \mathrm{Cl}`C`P - \iota`\Lambda . x \sim \epsilon\, \beta . \supset . x \,\epsilon\, \breve{P}``(\beta - \iota`x)$ (4)

$\vdash . (4) . \supset$

$\vdash : \text{Hp} . x \in \overrightarrow{B}{}^{\backprime}\breve{P} . \supset . \sim(\exists\beta) . \beta \in \text{Cl induct}{}^{\backprime}C{}^{\backprime}P - \iota{}^{\backprime}\Lambda . \exists ! \iota{}^{\backprime}x - \beta - \breve{P}{}^{\backprime\backprime}(\beta - \iota{}^{\backprime}x) .$

[*274·1] $\supset . \iota{}^{\backprime}x \sim\in \text{D}{}^{\backprime}P_\eta$ (5)

$\vdash . (5) . *274\cdot17 . \supset \vdash : \text{Hp} . \supset . \iota{}^{\backprime\backprime}\overrightarrow{B}{}^{\backprime}\breve{P} \subset \overrightarrow{B}{}^{\backprime}\breve{P}_\eta$ (6)

$\vdash . (3) . (6) . \supset \vdash . \text{Prop}$

***274·191.** $\vdash :. P \in \text{connex} . P^2 \mathrel{\mathsf{G}} J . \supset : \text{E} ! B{}^{\backprime}\breve{P} . \supset . B{}^{\backprime}\breve{P}_\eta = \iota{}^{\backprime}B{}^{\backprime}\breve{P} :$

$\sim \text{E} ! B{}^{\backprime}\breve{P} . \supset . \overrightarrow{B}{}^{\backprime}\breve{P}_\eta = \Lambda$ [*274·19]

***274·192.** $\vdash :. P \in \text{connex} . P^2 \mathrel{\mathsf{G}} J . \supset : \text{E} ! B{}^{\backprime}\breve{P} . \equiv . \text{E} ! B{}^{\backprime}\breve{P}_\eta$ [*274·191]

***274·193.** $\vdash . \overrightarrow{B}{}^{\backprime}P_\eta = \iota{}^{\backprime}C{}^{\backprime}P \cap (\text{Cls induct} - \iota{}^{\backprime}\Lambda - 1)$

Dem.

$\vdash . *274\cdot15\cdot1 . \supset \vdash : C{}^{\backprime}P \in \text{Cls induct} - \iota{}^{\backprime}\Lambda - 1 . \supset . C{}^{\backprime}P \in \overrightarrow{B}{}^{\backprime}P_\eta$ (1)

$\vdash . *274\cdot16\cdot17 . \supset \vdash : C{}^{\backprime}P \sim\in (\text{Cls induct} - \iota{}^{\backprime}\Lambda - 1) . \supset . C{}^{\backprime}P \sim\in C{}^{\backprime}P_\eta$ (2)

$\vdash . *274\cdot15 . \supset \vdash : \alpha \in \text{Cl induct}{}^{\backprime}C{}^{\backprime}P - \iota{}^{\backprime}\Lambda . x \in C{}^{\backprime}P - \alpha . \supset . (\alpha \cup \iota{}^{\backprime}x) P_\eta \alpha$ (3)

$\vdash . (3) . \supset \vdash . \text{Cl induct}{}^{\backprime}C{}^{\backprime}P - \iota{}^{\backprime}\Lambda - \iota{}^{\backprime}C{}^{\backprime}P \subset \text{Ɑ}{}^{\backprime}P_\eta$ (4)

$\vdash . (4) . \text{Transp} . *274\cdot1 . \supset \vdash . \overrightarrow{B}{}^{\backprime}P_\eta \subset (\text{Cl induct}{}^{\backprime}C{}^{\backprime}P - \iota{}^{\backprime}\Lambda) \cap \iota{}^{\backprime}C{}^{\backprime}P$ (5)

$\vdash . (5) . *274\cdot16 . \supset \vdash . \overrightarrow{B}{}^{\backprime}P_\eta \subset (\text{Cls induct} - \iota{}^{\backprime}\Lambda - 1) \cap \iota{}^{\backprime}C{}^{\backprime}P$ (6)

$\vdash . (1) . (2) . (6) . \supset \vdash . \text{Prop}$

***274·194.** $\vdash : C{}^{\backprime}P \in \text{Cls induct} - \iota{}^{\backprime}\Lambda - 1 . \supset . B{}^{\backprime}P_\eta = C{}^{\backprime}P$ [*274·193]

***274·195.** $\vdash : C{}^{\backprime}P \sim\in \text{Cls induct} . \supset . \overrightarrow{B}{}^{\backprime}P_\eta = \Lambda$ [*274·193]

***274·196.** $\vdash : P \in \text{Ser} . \sim \text{E} ! B{}^{\backprime}\breve{P} . \supset . \text{D}{}^{\backprime}P_\eta = \text{Ɑ}{}^{\backprime}P_\eta$

Dem.

$\vdash . *274\cdot192 . \supset \vdash : \text{Hp} . \supset . \overrightarrow{B}{}^{\backprime}\breve{P}_\eta = \Lambda$ (1)

$\vdash . *274\cdot195\cdot16 . *261\cdot24 . \supset \vdash : \text{Hp} . \supset . \overrightarrow{B}{}^{\backprime}P_\eta = \Lambda$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

The following propositions give the proof of $\breve{P} \in \Omega . \supset . \breve{P}_\eta \in \Omega$ (*274·22).

***274·2.** $\vdash : \breve{P} \in \Omega . \kappa \subset C{}^{\backprime}P_\eta . \exists ! \kappa . \supset . \text{E} ! P_m{}^{\backprime}\kappa . P_m{}^{\backprime}\kappa \in \text{min}_P{}^{\backprime\backprime}\kappa$

[*274·1·11 . *250·121 . (*274·02)]

***274·201.** $\vdash : \beta \in \breve{T}_P{}^{\backprime}\kappa . \equiv . (\exists\alpha) . \alpha \in \kappa . \text{min}_P{}^{\backprime}\alpha = P_m{}^{\backprime}\kappa . \beta = \alpha - \iota{}^{\backprime}P_m{}^{\backprime}\kappa . \exists ! \beta$

[(*274·03)]

***274·202.** $\vdash : \text{E} ! P_m{}^{\backprime}\kappa . \supset . \text{E} ! \breve{T}_P{}^{\backprime}\kappa$ [(*274·03) . *14·21]

***274·203.** $\vdash :. \text{Hp} *274\cdot2 . \supset : \breve{T}_P{}^{\backprime}\kappa = \Lambda . \equiv . \kappa \cap \overleftarrow{\text{min}_P}{}^{\backprime}P_m{}^{\backprime}\kappa = \iota{}^{\backprime}\iota{}^{\backprime}P_m{}^{\backprime}\kappa$

Dem.

$\vdash . *274\cdot2\cdot202 . \supset$

$\vdash :: \text{Hp} . \supset :. \breve{T}_P{}^{\backprime}\kappa = \Lambda . \equiv : \sim(\exists\alpha, \beta) . \alpha \in \kappa . \text{min}_P{}^{\backprime}\alpha = P_m{}^{\backprime}\kappa . \beta = \alpha - \iota{}^{\backprime}P_m{}^{\backprime}\kappa . \exists ! \beta :$

[*13·191] $\equiv : \alpha \in \kappa \cap \overleftarrow{\text{min}_P}{}^{\backprime}P_m{}^{\backprime}\kappa . \supset_\alpha . \alpha - \iota{}^{\backprime}P_m{}^{\backprime}\kappa = \Lambda :$

[*274·2] $\equiv : \alpha \in \kappa \cap \overleftarrow{\text{min}_P}{}^{\backprime}P_m{}^{\backprime}\kappa . \equiv_\alpha . \alpha = \iota{}^{\backprime}P_m{}^{\backprime}\kappa :: \supset \vdash . \text{Prop}$

***274·204.** $\vdash : \kappa \subset C‘P_\eta \,.\, \kappa\,(T_P)_*\lambda \,.\supset.\, \lambda \subset C‘P_\eta$

Dem.

$\vdash .\, *120·481 \,.\, *274·201 \,.\supset \vdash : \kappa \subset \text{Cls induct} \,.\, E!\,\breve{T}_P‘\kappa \,.\supset.\, \breve{T}_P‘\kappa \subset \text{Cls induct}$ (1)

$\vdash .\, *274·201 \,.\supset \vdash : \kappa \subset \text{Cl}‘C‘P \,.\, E!\,\breve{T}_P‘\kappa \,.\supset.\, \breve{T}_P‘\kappa \subset \text{Cl}‘C‘P - \iota‘\Lambda$ (2)

$\vdash .\, (1) \,.\, (2) \,.\, *274·16 \,.\supset \vdash : \kappa \subset C‘P_\eta \,.\, E!\,\breve{T}_P‘\kappa \,.\supset.\, \breve{T}_P‘\kappa \subset C‘P_\eta$ (3)

$\vdash .\, (3) \,.\, \text{Induct} \,.\supset \vdash .\, \text{Prop}$

***274·205.** $\vdash : P \,\epsilon\, \text{Ser} \,.\, E!\,P_m‘\breve{T}_P‘\lambda \,.\supset.\, (P_m‘\lambda)\, P\, (P_m‘\breve{T}_P‘\lambda)$

Dem.

$\vdash .\, *274·201 \,.\, *205·21 \,.\supset \vdash : \text{Hp} \,.\, \beta \,\epsilon\, \breve{T}_P‘\lambda \,.\supset.\, \beta \subset \overleftarrow{P}‘P_m‘\lambda$ (1)

$\vdash .\, *205·11 \,.\, (*274·02) \,.\supset \vdash : \text{Hp} \,.\supset.\, P_m‘\breve{T}_P‘\lambda \,\epsilon\, s‘\breve{T}_P‘\lambda$ (2)

$\vdash .\, (1) \,.\, (2) \,.\supset \vdash .\, \text{Prop}$

***274·206.** $\vdash : \text{Hp}\, *274·205 \,.\, \kappa\,(T_P)_*\lambda \,.\supset.\, (P_m‘\kappa)\, P\, (P_m‘\breve{T}_P‘\lambda)$

Dem.

$\vdash .\, *14·21 \,.\, (*274·02) \,.\supset \vdash : E!\,P_m‘\breve{T}_P‘\lambda \,.\supset.\, E!\,P_m‘\lambda$ (1)

$\vdash .\, (1) \,.\, \text{Induct} \,. \quad \supset \vdash : \text{Hp} \,.\supset.\, E!\,P_m‘\kappa$ (2)

$\vdash .\, (2) \,.\, *274·205 \,.\, \text{Induct} \,.\supset \vdash .\, \text{Prop}$

***274·207.** $\vdash : \breve{P} \,\epsilon\, \Omega \,.\, \kappa\,(T_P)_*\lambda \,.\, P_m‘\lambda = \max_P‘M_P‘\kappa \,.\supset.$
$\sim E!\,P_m‘\breve{T}_P‘\lambda \,.\, \breve{T}_P‘\lambda = \Lambda$

Dem.

$\vdash .\, *274·205 \,.\, \text{Transp} \,.\supset \vdash : \text{Hp} \,.\supset.\, \sim E!\,P_m‘\breve{T}_P‘\lambda \,.$

[*274·204·2.Transp] $\supset .\, \breve{T}_P‘\lambda = \Lambda : \supset \vdash .\, \text{Prop}$

***274·208.** $\vdash :.\, \breve{P} \,\epsilon\, \Omega \,.\, \kappa \subset C‘P_\eta \,.\, \exists!\,\kappa \,.\supset :$
$\Lambda \,\epsilon\, \overleftarrow{(T_P)_*}‘\kappa : (\exists\lambda) \,.\, \kappa\,(T_P)_*\lambda \,.\, \lambda \cap \overleftarrow{\min}_P‘P_m‘\lambda = \iota‘\iota‘P_m‘\lambda \,.\, \breve{T}_P‘\lambda = \Lambda$

Dem.

$\vdash .\, *250·121 \,.\supset \vdash : \text{Hp} \,.\supset.\, E!\,\max_P‘M_P‘\kappa$ (1)

$\vdash .\, (1) \,.\, *274·207·203·204 \,.\supset \vdash .\, \text{Prop}$

***274·21.** $\vdash : \beta \,\epsilon\, \breve{T}_P‘\kappa \,.\supset.\, \beta \cup \iota‘P_m‘\kappa \,\epsilon\, \kappa$ [*274·201]

***274·211.** $\vdash : \kappa\,(T_P)_*\lambda \,.\, \beta \,\epsilon\, \lambda \,.\supset.\, \beta \cup P_m‘‘T_P(\kappa \mapsto \lambda) \,\epsilon\, \kappa$

Dem.

$\vdash .\, *274·21 \,.\supset \vdash :.\, \text{Hp} : \beta \,\epsilon\, \lambda \,.\supset_\beta.\, \beta \cup P_m‘‘T_P(\kappa \mapsto \lambda) \,\epsilon\, \kappa : \supset :$
$\gamma \,\epsilon\, \breve{T}_P‘\lambda \,.\supset_\gamma.\, \gamma \cup P_m‘‘T_P(\kappa \mapsto \breve{T}_P‘\lambda) \,\epsilon\, \kappa$ (1)

$\vdash .\, *274·21 \,.\, (1) \,.\, \text{Induct} \,.\supset \vdash .\, \text{Prop}$

***274·212.** $\vdash : \breve{P} \in \Omega \,.\, \kappa \subset C'P_\eta \,.\, \exists ! \kappa \,.\, \supset \,.\, M_P{}'\kappa \in \kappa$

Dem.

$\vdash \,.\, *274\cdot208\cdot211 \,.\, \supset$

$\vdash : \mathrm{Hp} \,.\, \supset \,.\, (\exists\lambda) \,.\, \kappa (T_P)_* \lambda \,.\, \breve{T}_P{}'\lambda = \Lambda \,.\, \iota' P_m{}'\lambda \in \lambda \,.\, \iota' P_m{}'\lambda \cup P_m{}''T_P(\kappa \vdash \lambda) \in \kappa \,.$

$[*121\cdot103] \supset \,.\, (\exists\lambda) \,.\, P_m{}''T_P(\kappa \mapsto \lambda) \in \kappa \,.\, P_m{}''T_P(\kappa \mapsto \lambda) = P_m{}''\overleftarrow{(T_P)_*}{}'\kappa :$

$\supset \vdash \,.\, \mathrm{Prop}$

***274·213.** $\vdash : P \in \mathrm{Ser} \,.\, \kappa \subset C'P_\eta \,.\, \alpha \in \kappa \,.\, \kappa (T_P)_* \lambda \,.\, \overrightarrow{P}{}'P_m{}'\lambda \cap M_P{}'\kappa \subset \alpha \,.\, \supset .$
$\alpha - (\overrightarrow{P}{}'P_m{}'\lambda \cap M_P{}'\kappa) \in \lambda$

Dem.

$\vdash \,.\, *274\cdot201 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \kappa = \lambda \,.\, \supset \,.\, \alpha - (\overrightarrow{P}{}'P_m{}'\lambda \cap M_P{}'\kappa) = \alpha \,.$

$[*13\cdot12] \quad \supset \,.\, \alpha - (\overrightarrow{P}{}'P_m{}'\lambda \cap M_P{}'\kappa) \in \kappa$ (1)

$\vdash \,.\, *274\cdot206 \,.\, \supset$

$\vdash :. \mathrm{Hp} : \beta \in \kappa \,.\, \overrightarrow{P}{}'P_m{}'\lambda \cap M_P{}'\kappa \subset \beta \,.\, \supset_\beta \,.\, \beta - (\overrightarrow{P}{}'P_m{}'\lambda \cap M_P{}'\kappa) \in \lambda : \supset :$
$\beta \in \kappa \,.\, \overrightarrow{P}{}'P_m{}'\breve{T}_P{}'\lambda \cap M_P{}'\kappa \subset \beta \,.\, \supset \,.\, \overrightarrow{P}{}'P_m{}'\lambda \cap M_P{}'\kappa \subset \beta \,.\, P_m{}'\lambda \in \beta \,.$
$\{\beta - (\overrightarrow{P}{}'P_m{}'\lambda \cap M_P{}'\kappa)\} \in \lambda \,.\, P_m{}'\lambda \in \{\beta - (\overrightarrow{P}{}'P_m{}'\lambda \cap M_P{}'\kappa)\} \,.$

$[*274\cdot201] \quad \supset \,.\, \{\beta - (\overrightarrow{P}{}'P_m{}'\lambda \cap M_P{}'\kappa) - \iota'P_m{}'\lambda\} \in \breve{T}_P{}'\lambda \,.$

$[*274\cdot206] \quad \supset \,.\, \{\beta - (\overrightarrow{P}{}'P_m{}'\breve{T}_P{}'\lambda \cap M_P{}'\kappa)\} \in \breve{T}_P{}'\lambda$ (2)

$\vdash \,.\, (1) \,.\, (2) \,.\, \mathrm{Induct} \,.\, \supset \vdash \,.\, \mathrm{Prop}$

***274·214.** $\vdash : \breve{P} \in \Omega \,.\, \kappa \subset C'P_\eta \,.\, \alpha \in \kappa - \iota' M_P{}'\kappa \,.\, \supset \,.\, \alpha P_\eta (M_P{}'\kappa)$

Dem.

$\vdash \,.\, *274\cdot212 \,.\, \quad \supset \vdash :. \mathrm{Hp} \,.\, \supset : M_P{}'\kappa \in \mathrm{Cls\,induct} :$ (1)

$[*170\cdot16] \quad \supset : M_P{}'\kappa \subset \alpha \,.\, \supset \,.\, \alpha P_\eta (M_P{}'\kappa)$ (2)

$\vdash \,.\, *274\cdot11 \,.\, (1) \,.\, \supset \vdash : \mathrm{Hp} \,.\, \exists ! M_P{}'\kappa - \alpha \,.\, \supset \,.\, \mathrm{E} ! \min_P{}'(M_P{}'\kappa - \alpha) \,.$

$[*205\cdot14.(*274\cdot04)] \supset \,.\, (\exists\lambda) \,.\, \kappa (T_P)_* \lambda \,.\, P_m{}'\lambda \sim\in \alpha \,.\, \overrightarrow{P}{}'P_m{}'\lambda \cap M_P{}'\kappa \subset \alpha \,.$

$[*274\cdot213] \quad \supset \,.\, (\exists\lambda) \,.\, \kappa (T_P)_* \lambda \,.\, P_m{}'\lambda \sim\in \alpha \,.\, \alpha - (\overrightarrow{P}{}'P_m{}'\lambda \cap M_P{}'\kappa) \in \lambda \,.$
$\overrightarrow{P}{}'P_m{}'\lambda \cap M_P{}'\kappa \subset \alpha \,.$

$[*274\cdot201] \quad \supset \,.\, (\exists\lambda, z) \,.\, \kappa (T_P)_* \lambda \,.\, z = \min_P{}'\{\alpha - (\overrightarrow{P}{}'P_m{}'\lambda \cap M_P{}'\kappa)\} \,.$
$z P (P_m{}'\lambda) \,.\, \overrightarrow{P}{}'P_m{}'\lambda \cap M_P{}'\kappa \subset \alpha \,.$

$[*31\cdot18] \quad \supset \,.\, (\exists z) \,.\, z \in \alpha - M_P{}'\kappa \,.\, M_P{}'\kappa \cap \overrightarrow{P}{}'z \subset \alpha \,.$

$[*170\cdot11] \quad \supset \,.\, \alpha P_\eta (M_P{}'\kappa)$ (3)

$\vdash \,.\, (2) \,.\, (3) \,.\, \supset \vdash \,.\, \mathrm{Prop}$

***274·215.** $\vdash : \breve{P} \in \Omega \,.\, \kappa \subset C'P_\eta \,.\, \exists ! \kappa \,.\, \supset \,.\, M_P{}'\kappa = \max (P_\eta)'\kappa$ [*274·212·214]

***274·22.** $\vdash : \breve{\dot{P}} \epsilon \Omega . \supset . \breve{P}_\eta \epsilon \Omega$

Dem.

$\vdash . *274·215 . \supset \vdash : \text{Hp} . \supset . \text{E} !! \max (P_\eta)``\text{Cl ex}`C`P_\eta .$

$[*250·125] \qquad \supset . \breve{P}_\eta \epsilon \Omega : \supset \vdash . \text{Prop}$

The following propositions constitute the proof of

$$\breve{P} \epsilon \omega . \supset . \breve{P}_\eta \epsilon \omega \ (*274·25).$$

***274·221.** $\vdash : P \epsilon \text{Ser} . \overleftarrow{P}`\max_P`\alpha \epsilon \text{Cls induct} . \alpha \epsilon \text{Cl induct}`C`P - \iota`\Lambda - \iota`\overrightarrow{B}`\breve{P} .$
$\beta = (\alpha - \iota`\max_P`\alpha) \cup \overleftarrow{P}`\max_P`\alpha . \supset . \alpha P_\eta \beta$

Dem.

$\vdash . *205·55 . \quad \supset \vdash : \text{Hp} . B`\breve{P} \epsilon \alpha . \supset . \exists ! \alpha - \iota`\max_P`\alpha \quad (1)$

$\vdash . *202·511 . \quad \supset \vdash : \text{Hp} . B`\breve{P} \sim \epsilon \alpha . \supset . B`\breve{P} \epsilon \overleftarrow{P}`\max_P`\alpha \quad (2)$

$\vdash . *93·101 . \quad \supset \vdash : \text{Hp} . \sim \text{E} ! B`\breve{P} . \supset . \exists ! \overleftarrow{P}`\max_P`\alpha \quad (3)$

$\vdash . (1) . (2) . (3) . \supset \vdash : \text{Hp} . \supset . \exists ! \beta \quad (4)$

$\vdash . *120·481·71 . \supset \vdash : \text{Hp} . \supset . \beta \epsilon \text{Cls induct} \quad (5)$

$\vdash . *205·21 . *200·361 . \supset \vdash : \text{Hp} . \supset . \beta \cap \overrightarrow{P}`\max_P`\alpha = \alpha \cap \overrightarrow{P}`\max_P`\alpha \quad (6)$

$\vdash . (4) . (5) . (6) . \supset \vdash : \text{Hp} . \supset . \alpha, \beta \epsilon \text{Cl induct}`C`P - \iota`\Lambda . \max_P`\alpha \epsilon \alpha - \beta .$
$\alpha \cap \overrightarrow{P}`\max_P`\alpha = \beta \cap \overrightarrow{P}`\max_P`\alpha .$

$[*274·12] \qquad \supset . \alpha P_\eta \beta : \supset \vdash . \text{Prop}$

***274·222.** $\vdash : \text{Hp} *274·221 . \alpha P_\eta \gamma . \max_P`\alpha \epsilon \gamma . \supset . \beta P_\eta \gamma$

Dem.

$\vdash . *274·12 . \supset \vdash : \text{Hp} . \supset . (\exists z) . z \epsilon \alpha - \gamma . z \neq \max_P`\alpha . \alpha \cap \overrightarrow{P}`z = \gamma \cap \overrightarrow{P}`z .$

$[*201·14 . *205·21 . \text{Hp}] \supset . (\exists z) . z \epsilon \beta - \gamma . \beta \cap \overrightarrow{P}`z = \gamma \cap \overrightarrow{P}`z .$

$[*274·12] \qquad \supset . \beta P_\eta \gamma : \supset \vdash . \text{Prop}$

***274·223.** $\vdash : \text{Hp} *274·221 . \alpha P_\eta \gamma . \max_P`\alpha \sim \epsilon \gamma . \gamma \neq \beta . \supset . \beta P_\eta \gamma$

Dem.

$\vdash . *274·12 . \supset \vdash :. \text{Hp} . \supset : (\exists z) . z \epsilon \alpha - \gamma - \iota`\max_P`\alpha . \alpha \cap \overrightarrow{P}`z = \gamma \cap \overrightarrow{P}`z . \vee .$
$\alpha \cap \overrightarrow{P}`\max_P`\alpha = \gamma \cap \overrightarrow{P}`\max_P`\alpha \quad (1)$

$\vdash . *201·14 . *205·21 . \supset$

$\vdash :. \text{Hp} : (\exists z) . z \epsilon \alpha - \gamma - \iota`\max_P`\alpha . \alpha \cap \overrightarrow{P}`z = \gamma \cap \overrightarrow{P}`z : \supset . \beta P_\eta \gamma \quad (2)$

$\vdash . *205·21 . \supset \vdash : \text{Hp} . \alpha \cap \overrightarrow{P}`\max_P`\alpha = \gamma \cap \overrightarrow{P}`\max_P`\alpha . \supset .$
$\alpha - \iota`\max_P`\alpha = \gamma \cap \overrightarrow{P}`\max_P`\alpha \quad (3)$

$\vdash . *202·101 . \supset \vdash : \text{Hp} . \supset . \gamma \subset \overrightarrow{P}`\max_P`\alpha \cup \overleftarrow{P}`\max_P`\alpha \quad (4)$

$\vdash . (3) . (4) . \supset \vdash :. \text{Hp} . \alpha \cap \overrightarrow{P}`\max_P`\alpha = \gamma \cap \overrightarrow{P}`\max_P`\alpha . \supset : \gamma \subset \beta :$

$[*170·16 . (*274·01)] \qquad \supset : \gamma \neq \beta . \supset . \beta P_\eta \gamma \quad (5)$

$\vdash . (1) . (2) . (5) . \supset \vdash . \text{Prop}$

***274·224.** $\vdash : \mathrm{Hp}\,{*}274{\cdot}221 \,.\, \alpha P_\eta \gamma \,.\, \beta \neq \gamma \,.\, \supset \,.\, \beta P_\eta \gamma$ [*274·222·223]

***274·23.** $\vdash : \mathrm{Hp}\,{*}274{\cdot}221 \,.\, \supset \,.\, \alpha\,(P_\eta)_1\,\beta$ [*274·221·224 . *204·72]

***274·25.** $\vdash : \breve{P} \in \omega \,.\, \supset \,.\, \breve{P}_\eta \in \omega$

Dem.

$\vdash \,.\, {*}274{\cdot}22{\cdot}16 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, \breve{P}_\eta \in \Omega - \iota\text{‘}\dot{\Lambda}$ (1)

$\vdash \,.\, {*}274{\cdot}191{\cdot}17 \,.\, \supset$

$\vdash : \mathrm{Hp} \,.\, \alpha \in \mathrm{D}\text{‘}P_\eta \,.\, \supset \,.\, \alpha \in \mathrm{Cl\ induct}\text{‘}C\text{‘}P - \iota\text{‘}\Lambda - \iota\text{‘}\overrightarrow{B}\text{‘}\breve{P}$ (2)

$\vdash \,.\, {*}263{\cdot}412 \,.\, {*}274{\cdot}11 \,.\, \supset$

$\vdash : \mathrm{Hp} \,.\, \alpha \in \mathrm{Cl\ induct}\text{‘}C\text{‘}P - \iota\text{‘}\Lambda \,.\, \supset \,.\, \overleftarrow{P}\text{‘}\mathrm{max}_P\text{‘}\alpha \in \mathrm{Cls\ induct}$ (3)

$\vdash \,.\, (2) \,.\, (3) \,.\, {*}274{\cdot}23 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \alpha \in \mathrm{D}\text{‘}P_\eta \,.\, \supset \,.\, \alpha \in \mathrm{D}\text{‘}(P_\eta)_1$ (4)

$\vdash \,.\, (1) \,.\, (4) \,.\, {*}274{\cdot}195 \,.\, {*}121{\cdot}323 \,.\, \supset$

$\vdash : \mathrm{Hp} \,.\, \supset \,.\, \breve{P}_\eta \in \Omega - \iota\text{‘}\dot{\Lambda} \,.\, \mathrm{D}\text{‘}P_\eta = \mathrm{D}\text{‘}(P_\eta)_1 \,.\, \sim \mathrm{E}\,!\, B\text{‘}P_\eta \,.$

$[{*}263{\cdot}44] \supset \,.\, \breve{P}_\eta \in \omega : \supset \vdash \,.\, \mathrm{Prop}$

***274·26.** $\vdash : P \in \omega \,.\, \supset \,.\, P_{1c} \upharpoonright (\mathrm{Cls\ induct} - \iota\text{‘}\Lambda) \in \omega \,.$
$C\text{‘}P_{1c} \upharpoonright (\mathrm{Cls\ induct} - \iota\text{‘}\Lambda) = \mathrm{Cl\ induct}\text{‘}C\text{‘}P - \iota\text{‘}\Lambda$

Dem.

$\vdash \,.\, {*}274{\cdot}13 \,.\, \supset \vdash : Q = \breve{P} \,.\, \supset \,.\, P_{1c} \upharpoonright (\mathrm{Cls\ induct} - \iota\text{‘}\Lambda) = \breve{Q}_\eta$ (1)

$\vdash \,.\, {*}274{\cdot}25 \,.\, \supset \vdash : P \in \omega \,.\, Q = \breve{P} \,.\, \supset \,.\, \breve{Q}_\eta \in \omega$ (2)

$\vdash \,.\, {*}274{\cdot}17 \,.\, \supset \vdash : P \in \omega \,.\, Q = \breve{P} \,.\, \supset \,.\, C\text{‘}\breve{Q}_\eta = \mathrm{Cl\ induct}\text{‘}C\text{‘}P - \iota\text{‘}\Lambda$ (3)

$\vdash \,.\, (1) \,.\, (2) \,.\, (3) \,.\, \supset \vdash \,.\, \mathrm{Prop}$

***274·27.** $\vdash : \alpha \in \aleph_0 \,.\, \supset \,.\, \mathrm{Cl\ induct}\text{‘}\alpha \in \aleph_0 \,.\, \mathrm{Cl\ induct}\text{‘}\alpha - \iota\text{‘}\Lambda \in \aleph_0$

Dem.

$\vdash \,.\, {*}263{\cdot}101 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, (\exists P) \,.\, P \in \omega \,.\, \alpha = C\text{‘}P \,.$

$[{*}274{\cdot}26] \quad \supset \,.\, (\exists M) \,.\, M \in \omega \,.\, \mathrm{Cl\ induct}\text{‘}\alpha - \iota\text{‘}\Lambda = C\text{‘}M \,.$

$[{*}263{\cdot}101] \quad \supset \,.\, \mathrm{Cl\ induct}\text{‘}\alpha - \iota\text{‘}\Lambda \in \aleph_0 \,.$ (1)

$[{*}123{\cdot}4] \quad \supset \,.\, \mathrm{Cl\ induct}\text{‘}\alpha \in \aleph_0$ (2)

$\vdash \,.\, (1) \,.\, (2) \,.\, \supset \vdash \,.\, \mathrm{Prop}$

The following propositions constitute the proof of

$$P \in \omega \,.\, \supset \,.\, P_\eta \in \eta \quad ({*}274{\cdot}33).$$

***274·3.** $\vdash : P \in \mathrm{Ser} \,.\, \alpha P_\eta \beta \,.\, x \in p\text{‘}\overleftarrow{P}\text{“}(\alpha \cup \beta) \,.\, \supset \,.\, \alpha P_\eta (\beta \cup \iota\text{‘}x) \,.\, (\beta \cup \iota\text{‘}x)\, P_\eta \beta$

Dem.

$\vdash \,.\, {*}200{\cdot}53 \,.\, \quad \supset \vdash : \mathrm{Hp} \,.\, z \in \alpha \,.\, \supset \,.\, \beta \cap \overrightarrow{P}\text{‘}z = (\beta \cup \iota\text{‘}x) \cap \overrightarrow{P}\text{‘}z$ (1)

$\vdash \,.\, {*}200{\cdot}5 \,.\, \quad \supset \vdash : \mathrm{Hp} \,.\, z \in \alpha - \beta \,.\, \supset \,.\, z \in \alpha - (\beta \cup \iota\text{‘}x)$ (2)

$\vdash \,.\, (1) \,.\, (2) \,.\, {*}274{\cdot}12 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, \alpha P_\eta (\beta \cup \iota\text{‘}x)$ (3)

$\vdash \,.\, {*}200{\cdot}5 \,.\, {*}170{\cdot}16 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, (\beta \cup \iota\text{‘}x)\, P_\eta \beta$ (4)

$\vdash \,.\, (3) \,.\, (4) \,.\, \supset \vdash \,.\, \mathrm{Prop}$

***274·31.** $\vdash : P \in \text{Ser} \,.\, \sim \text{E} \,!\, B\text{‘}\breve{P} \,.\, \supset \,.\, P_\eta \in \text{Ser} \cap \text{comp}$

Dem.

$\vdash . *274\cdot1 . *120\cdot71 . \supset \vdash : \alpha P_\eta \beta . \supset . \alpha \cup \beta \in \text{Cls induct} - \iota\text{‘}\Lambda$ (1)

$\vdash . (1) . *274\cdot11 . \quad \supset \vdash : \text{Hp} . \alpha P_\eta \beta . \supset . \text{E} \,!\, \text{max}_P\text{‘}(\alpha \cup \beta) .$

$[*93\cdot103] \quad \supset . \exists \,!\, \overleftarrow{P}\text{‘max}_P\text{‘}(\alpha \cup \beta) .$

$[*205\cdot67] \quad \supset . \exists \,!\, p\text{‘}\overleftarrow{P}\text{‘‘}(\alpha \cup \beta) .$

$[*274\cdot3] \quad \supset . \alpha P_\eta{}^2 \beta$ (2)

$\vdash . (2) . *274\cdot18 . \supset \vdash . \text{Prop}$

***274·32.** $\vdash : P \in \text{Ser} \cap \breve{C}\text{‘‘}\aleph_0 \,.\, \sim \text{E} \,!\, B\text{‘}\breve{P} \,.\, \supset . P_\eta \in \eta$

Dem.

$\vdash . *274\cdot31 . \quad \supset \vdash : \text{Hp} . \supset . P_\eta \in \text{Ser} \cap \text{comp}$ (1)

$\vdash . *274\cdot196 . \quad \supset \vdash : \text{Hp} . \supset . \text{D}\text{‘}P_\eta = \text{Ꮖ}\text{‘}P_\eta$ (2)

$\vdash . *274\cdot27\cdot17 . \supset \vdash : \text{Hp} . \supset . C\text{‘}P_\eta \in \aleph_0$ (3)

$\vdash . (1) . (2) . (3) . *273\cdot1 . \supset \vdash . \text{Prop}$

***274·33.** $\vdash : P \in \omega . \supset . P_\eta \in \eta \quad [*274\cdot32 . *263\cdot101\cdot11\cdot22]$

This is the principal proposition of the present number.

***274·34.** $\vdash : \alpha \in \aleph_0 . \supset . \exists \,!\, \eta \cap \overleftarrow{C}\text{‘}(\text{Cl induct‘}\alpha - \iota\text{‘}\Lambda)$

Dem.

$\vdash . *263\cdot101 . \supset \vdash : \text{Hp} . \supset . (\exists P) . P \in \omega . C\text{‘}P = \alpha .$

$[*274\cdot33\cdot17] \quad \supset . (\exists M) . M \in \eta . C\text{‘}M = \text{Cl induct‘}\alpha - \iota\text{‘}\Lambda : \supset \vdash . \text{Prop}$

The following propositions are concerned with the existence-theorem for η. They all follow from *274·33.

***274·4.** $\vdash : P \in \omega . T = \breve{\iota}\text{‘}P \,\overline{\text{smor}}\, \{P_{1c} \upharpoonright (\text{Cls induct} - \iota\text{‘}\Lambda)\} . \supset . T ; P_\eta \in \eta \cap \overleftarrow{C}\text{‘}C\text{‘}P$

Dem.

$\vdash . *274\cdot26\cdot17 . \quad \supset \vdash : \text{Hp} . \supset . \text{Ꮖ}\text{‘}T = C\text{‘}P_\eta$ (1)

$\vdash . (1) . *151\cdot11\cdot131 . \supset \vdash : \text{Hp} . \supset . T ; P_\eta \,\text{smor}\, P_\eta . C\text{‘}T ; P_\eta = C\text{‘}P .$

$[*274\cdot33 . *273\cdot41] \quad \supset . T ; P_\eta \in \eta . C\text{‘}T ; P_\eta = C\text{‘}P : \supset \vdash . \text{Prop}$

***274·41.** $\vdash : \exists \,!\, \omega \cap t\text{‘}P . \equiv . \exists \,!\, \eta \cap t\text{‘}P$

Dem.

$\vdash . *274\cdot4 . \quad \supset \vdash : Q \in \omega \cap t\text{‘}P . \supset . (\exists R) . R \in \eta . C\text{‘}R = C\text{‘}Q .$

$[*64\cdot24] \quad \supset . \exists \,!\, \eta \cap t\text{‘}P$ (1)

$\vdash . *273\cdot11 . \supset \vdash : R \in \eta \cap t\text{‘}P . \supset . (\exists Q) . Q \in \omega . C\text{‘}Q = C\text{‘}R .$

$[*64\cdot24] \quad \supset . \exists \,!\, \omega \cap t\text{‘}P$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***274·42.** $\vdash : \alpha \in \aleph_0 . \supset . \exists \,!\, \eta \cap \overleftarrow{C}\text{‘}\alpha \quad [*274\cdot4\cdot26 . *263\cdot17 . *250\cdot6 . *263\cdot101]$

***274·43.** $\vdash . \aleph_0 = C\text{‘‘}\eta \quad [*273\cdot1 . *274\cdot42]$

***274·44.** $\vdash : \exists \,!\, \aleph_0 \cap t\text{‘}\alpha . \equiv . \exists \,!\, \eta \cap t_{00}\text{‘}\alpha \quad [*263\cdot131 . *274\cdot41]$

***274·45.** $\vdash : \exists \,!\, \aleph_0 (x) . \equiv . \exists \,!\, \eta \cap t^{11}\text{‘}x \quad [*263\cdot13 . *274\cdot41]$

***274·46.** $\vdash : \text{Infin ax}(x) . \equiv . \exists \,!\, \eta \cap t^{33}\text{‘}x \quad [*263\cdot132 . *274\cdot41]$

$*275$. CONTINUOUS SERIES.

Summary of $*275$.

The definition of continuity to be given in this number is due to Cantor. A different and not equivalent definition was given by Dedekind: series which are continuous in Cantor's sense are also continuous in Dedekind's sense, but not vice versa. Cantor's definition has the advantage (among others) that two series which are continuous in his sense are ordinally similar, which is not necessarily the case with series that are continuous in Dedekind's sense. Dedekind's definition of "continuous series" is, in our language, "series which are compact and Dedekindian." Cantor's definition (after a certain amount of simplification) is "series which are Dedekindian and contain an $\aleph_0$ as a median class." In the case of the real numbers, the rationals are a median class of this sort.

An equivalent definition to the above is that a continuous series is a Dedekindian series whose converse domain is the derivative of a contained rational series ($*275{\cdot}13$).

Following Cantor, we shall use θ for the class of continuous series.

In what follows, we prove first that the series of segments of a rational series is a continuous series, *i.e.*

$*275{\cdot}21$. $\vdash : P \epsilon \eta . \supset . \text{ς}`P \epsilon \theta$

The contained $\aleph_0$ is $\overrightarrow{P}``C`P$. The proposition follows at once from $*271{\cdot}31$. On its importance, see remarks on $*275{\cdot}21$ below.

From this proposition, it follows that if η exists in any type, θ exists in the next type ($275{\cdot}22$), whence the existence of θ in sufficiently high types follows from the axiom of infinity ($*275{\cdot}25$).

To prove that any two continuous series are similar, we use $*271{\cdot}39$. By the definition, if P and Q are continuous, they contain respectively two median classes α and β, such that $P \restriction \alpha$ and $Q \restriction \beta$ are rational series. Hence by $*273{\cdot}4$, $P \restriction \alpha \,\text{smor}\, Q \restriction \beta$, and therefore $P \,\text{smor}\, Q$, by $*271{\cdot}39$. Also obviously $P \epsilon \theta . P \,\text{smor}\, Q . \supset . Q \epsilon \theta$. Hence

$*275{\cdot}32$. $\vdash : P \epsilon \theta . \supset . \theta = \text{Nr}`P$

and

$*275{\cdot}33$. $\vdash . \theta \epsilon \text{NR}$

***275·01.** $\theta = \text{Ser} \cap \text{Ded} \cap \breve{\text{med}}``\aleph_0 \quad \text{Df}$

***275·1.** $\vdash : P \in \theta . \equiv . P \in \text{Ser} \cap \text{Ded} . \exists ! \aleph_0 \cap \overrightarrow{\text{med}}`P$
[(*275·01)]

***275·11.** $\vdash :. P \in \theta . \equiv : P \in \text{Ser} \cap \text{Ded} : (\exists \alpha) . \alpha \in \aleph_0 . \delta_P`\alpha = \text{Ɑ}`P . \alpha \subset C`P$
[*275·1 . *271·2]

***275·12.** $\vdash :: P \in \theta . \equiv :. P \in \text{Ser} \cap \text{Ded} :. (\exists \alpha) : \alpha \in \aleph_0 :$
$xPy . \supset_{x,y} . \exists ! \alpha \cap P(x-y) : \alpha \subset C`P$ [*275·1 . *271·1]

***275·13.** $\vdash :. P \in \theta . \equiv : P \in \text{Ser} \cap \text{Ded} : (\exists R) . R \mathrel{\text{Ⴚ}} P . R \in \eta . \delta_P`C`R = \text{Ɑ}`P$

Dem.

$\vdash . *273·1 . *271·2 . \supset$

$\vdash : P \in \text{Ser} \cap \text{Ded} . R \mathrel{\text{Ⴚ}} P . R \in \eta . \delta_P`C`R = \text{Ɑ}`P . \supset . C`R \in \aleph_0 . C`R \in \overrightarrow{\text{med}}`P .$

[*275·1] $\supset . P \in \theta$ (1)

$\vdash . *271·16 . \supset \vdash : \alpha \,\text{med}\, P . \beta = \alpha \cap \text{D}`P \cap \text{Ɑ}`P . \supset . \beta \,\text{med}\, P .$ (2)

[*271·15] $\supset . P \text{£} \beta \in \text{comp}$ (3)

$\vdash . *123·17 . \supset \vdash : \text{Hp}(2) . P \in \text{Ser} . \alpha \in \aleph_0 \cap \text{Cl}`C`P . \supset . \beta \in \aleph_0 \cap \text{Cl}`C`P$ (4)

$\vdash . *271·1 . \supset \vdash : \beta \,\text{med}\, P . \supset . P``\beta = \text{D}`P . \breve{P}``\beta = \text{Ɑ}`P$ (5)

$\vdash . (5) . *37·41 . (2) . \quad \supset \vdash : \text{Hp}(2) . \supset . \text{D}`(P \text{£} \beta) = \beta . \text{Ɑ}`(P \text{£} \beta) = \beta$ (6)

$\vdash . (3) . (4) . (6) . *273·1 . \supset \vdash : \text{Hp}(4) . \supset . P \text{£} \beta \in \eta$ (7)

$\vdash . (2) . *271·2 . \supset \vdash : \text{Hp}(4) . \supset . \delta_P`C`(P \text{£} \beta) = \text{Ɑ}`P$ (8)

$\vdash . (7) . (8) . *275·1 . \supset \vdash : P \in \theta . \supset . (\exists \beta) . P \text{£} \beta \in \eta . \delta_P`C`(P \text{£} \beta) = \text{Ɑ}`P$ (9)

$\vdash . (1) . (9) . \supset \vdash . \text{Prop}$

***275·14.** $\vdash . \theta = \text{Cnv}``\theta$

Dem.

$\vdash . *214·14 . *271·11 . \supset \vdash : P \in \text{Ser} \cap \text{Ded} . \alpha \in \aleph_0 \cap \overrightarrow{\text{med}}`P . \equiv .$
$\breve{P} \in \text{Ser} \cap \text{Ded} . \alpha \in \aleph_0 \cap \overrightarrow{\text{med}}`\breve{P}$ (1)

$\vdash . (1) . *275·1 . \supset \vdash . \text{Prop}$

***275·2.** $\vdash : P \in \eta . \supset . \varsigma`P \in \text{Ser} \cap \text{Ded} . \overrightarrow{P}``C`P \in \aleph_0 . \overrightarrow{P}``C`P \in \overrightarrow{\text{med}}`\varsigma`P$

Dem.

$\vdash . *214·33 . \quad \supset \vdash : \text{Hp} . \supset . \varsigma`P \in \text{Ser} \cap \text{Ded}$ (1)

$\vdash . *204·35 . \quad \supset \vdash : \text{Hp} . \supset . \overrightarrow{P}``C`P \,\text{sm}\, C`P .$

[*273·1.*123·321] $\supset . \overrightarrow{P}``C`P \in \aleph_0$ (2)

$\vdash . *271·31 . *273·1 . \supset \vdash : \text{Hp} . \supset . \overrightarrow{P}``C`P \in \overrightarrow{\text{med}}`\varsigma`P$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash . \text{Prop}$

***275·21.** $\vdash : P \epsilon \eta . \supset . \varsigma`P \epsilon \theta$ [*275·2·1]

This proposition is of great importance, particularly in the theory of real numbers. We shall define the real numbers as segments of the series of rational numbers, in order to be sure of their existence. Thus if P is the series of rational numbers, $\varsigma`P$, which may be taken to be the series of real numbers, is continuous. If $P$ is the series of rational proper fractions, excluding 0, $\varsigma`P$ is the series of real proper fractions together with 0 and 1: this series is continuous in virtue of the above proposition.

The above proposition is also useful as enabling us to deduce the existence of θ from that of η, and thence from that of $\aleph_0$, and thence from the axiom of infinity. A rise of type is, however, required for the existence-theorems, which are given in the following propositions.

***275·22.** $\vdash : \exists ! \eta \cap t_{00}`\alpha . \supset . \exists ! \theta \cap t^{11}`\alpha$

Dem.

$\vdash . *64·55 . \supset \vdash : \exists ! \eta \cap t_{00}`\alpha . \supset . (\exists P) . P \epsilon \eta . C`P \subset t_0`\alpha .$

[*63·371] $\qquad \supset . (\exists P) . P \epsilon \eta . C`P \epsilon t`\alpha .$

[*275·21] $\qquad \supset . (\exists Q) . Q \epsilon \theta . C`Q \subset t`\alpha .$

[*64·57] $\qquad \supset . \exists ! \theta \cap t^{11}`x : \supset \vdash . \text{Prop}$

***275·23.** $\vdash : \exists ! \aleph_0 \cap t`\alpha . \supset . \exists ! \theta \cap t^{11}`\alpha$ [*274·44 . *275·22]

***275·24.** $\vdash : \exists ! \aleph_0 (x) . \supset . \exists ! \theta \cap t^{22}`x$ [*275·23 . *64·31·312 . (*65·02)]

***275·25.** $\vdash : \text{Infin ax} (x) . \supset . \exists ! \theta \cap t^{44}`x$

Dem.

$\vdash . *123·37 . \supset \vdash : \text{Hp} . \supset . \exists ! \aleph_0 (t^{2}`x) .$

[*275·24] $\qquad \supset . \exists ! \theta \cap t^{22}`t^{2}`x .$

[*64·312] $\qquad \supset . \exists ! \theta \cap t^{44}`x : \supset \vdash . \text{Prop}$

***275·3** $\vdash : P, Q \epsilon \theta . \supset . P \text{ smor } Q$

Dem.

$\vdash . *275·13 . \supset \vdash :. \text{Hp} . \supset : P, Q \epsilon \text{Ser} \cap \text{Ded} :$

$\qquad (\exists R, S) . R, S \epsilon \eta . R \mathrel{\mathsf{G}} P . S \mathrel{\mathsf{G}} Q . C`R \epsilon \overrightarrow{\text{med}}`P . C`S \epsilon \overrightarrow{\text{med}}`Q :$

[*204·41] $\supset : P, Q \epsilon \text{Ser} \cap \text{Ded} : (\exists \alpha, \beta) . \alpha \text{ med } P . \beta \text{ med } Q . P \upharpoonright \alpha, Q \upharpoonright \beta \epsilon \eta :$

[*273·4] $\supset : P, Q \epsilon \text{Ser} \cap \text{Ded} : (\exists \alpha, \beta) . \alpha \text{ med } P . \beta \text{ med } Q . (P \upharpoonright \alpha) \text{ smor } (Q \upharpoonright \beta) :$

[*271·39] $\supset : P \text{ smor } Q :. \supset \vdash . \text{Prop}$

***275·31.** $\vdash : P \epsilon \theta . P \text{ smor } Q . \supset . Q \epsilon \theta$

Dem.

$\vdash . *271·4 . \supset \vdash : P \text{ smor } Q . \exists ! \aleph_0 \cap \overrightarrow{\text{med}}`P . \supset . \exists ! \aleph_0 \cap \overrightarrow{\text{med}}`Q \qquad (1)$

$\vdash . *204·21 . *214·6 . \supset \vdash : P \epsilon \text{Ser} \cap \text{Ded} . P \text{ smor } Q . \supset . Q \epsilon \text{Ser} \cap \text{Ded} \qquad (2)$

$\vdash . (1) . (2) . *275·1 . \supset \vdash . \text{Prop}$

***275·32.** $\vdash : P \epsilon \theta . \supset . \theta = \text{Nr}`P$ [*275·3·31]

***275·33.** $\vdash . \theta \epsilon \text{NR}$ [*275·32 . *256·54]

$*276$. ON SERIES OF INFINITE SUB-CLASSES OF A SERIES.

Summary of $*276$.

The subject of the present number bears the same relation to θ as that of $*274$ bears to η. We shall consider, in the present number, the arrangement of all the *infinite* sub-classes of a series (together with Λ) by the principle of first differences, *i.e.* the relation

$$P_{\mathrm{cl}} \upharpoonright (-\,\mathrm{Cls\,induct} \cup \iota`\Lambda),$$

where P is the given series. This relation we will call P_θ. It consists of P_{cl} with its field limited to terms not belonging to $C`P_\eta$ ($*276\cdot12$). It will (under a certain hypothesis) contain a part similar to $\breve{P}_\eta$, namely $P_{\mathrm{cl}}$ with its field limited to complements of finite sub-classes of $C`P$. Hence if $P \in \omega$, P_θ will contain an η, whose field is composed of the complements of members of $C`P_\eta$ ($*276\cdot2$). The field of this η will be a median class of P_θ. We shall find, also, that $P_\theta \in \mathrm{Ser}$, if $P \in \Omega$ ($*276\cdot14$), and $P_\theta \in \mathrm{Ded}$, if $P \in \Omega\,\mathrm{infin}$ ($*276\cdot4$). Hence

$*276\cdot41$. $\vdash : P \in \omega \,.\supset .\, P_\theta \in \theta$

Also, since $P \in \omega \,.\supset .\, \mathrm{Cl}`C`P \in 2^{\aleph_0}$, and since $C`P_\eta \in \aleph_0$, we shall have $C`P_\theta \in 2^{\aleph_0}$ ($*276\cdot42$). This result is important, since it gives the proposition

$*276\cdot43$. $\vdash .\, C``\theta = 2^{\aleph_0}$

The proof that P_θ is Dedekindian if P is an infinite well-ordered series is somewhat complicated. We proceed by proving that every sub-class of $C`P_\theta$ has a lower limit or a minimum. In this proof, we observe first of all that

$$C`P = B`P_\theta \,.\, \Lambda = B`\breve{P}_\theta \quad (*276\cdot121).$$

Hence $C`P$ is the lower limit of the null-class, and $\Lambda$ is the minimum of $\iota`\Lambda$; also if κ is any existent sub-class of $C`P_\theta$, other than $\iota`\Lambda$, we have

$$\overrightarrow{\mathrm{limin}}\,(P_\theta)`\kappa = \overrightarrow{\mathrm{limin}}\,(P_\theta)`(\kappa - \iota`\Lambda).$$

Hence if we can prove

$$\kappa \subset C`P_\theta \,.\, \exists!\,\kappa \,.\, \Lambda \sim\in \kappa \,.\supset_\kappa .\, E!\,\mathrm{limin}\,(P_\theta)`\kappa \qquad (\mathrm{A}),$$

we shall have $\qquad \mathrm{Cl\,ex}`C`P_\theta \subset \mathrm{Cl}`\mathrm{limin}\,(P_\theta)$,

whence, by $*214 \cdot 12 \cdot 14$, we shall have $P_\theta \in \text{Ded}$. Thus we have to prove (A), *i.e.* $\kappa \subset D`P_\theta . \exists ! \kappa . \supset_\kappa . E ! \text{limin}(P_\theta)`\kappa$, which is $*276 \cdot 39$. To prove this proposition, consider $\text{min}_P`(s`\kappa - p`\kappa)$. This exists unless $\kappa \in 1$; it is the first term which belongs to some members of $\kappa$ but not to others. Those members of $\kappa$ to which it belongs precede (in the order $P_\theta$) those to which it does not belong. Let us call those to which it belongs $\breve{T}_P`\kappa$, so that

$$T_P = \hat{\kappa}\hat{\lambda}\{\lambda = \kappa \cap \overleftarrow{\epsilon}`\text{min}_P`(s`\kappa - p`\kappa)\}.$$

Put also $$P_m`\kappa = \text{min}_P`(s`\kappa - p`\kappa) \quad \text{Dft},$$

so that we may put $$\breve{T}_P`\kappa = \kappa \cap \overleftarrow{\epsilon}`P_m`\kappa \quad \text{Dft}.$$

Then if we put $A = \hat{\kappa}\hat{\lambda}(\lambda \subset \kappa . \lambda \neq \kappa)$, T_P and A fulfil the hypotheses of $*258$, and we have

$$A(T_P, \kappa) \in \Omega.$$

The series $A(T_P, \kappa)$ proceeds to smaller and smaller sub-classes of κ, of which any one, say λ, consists of terms which come earlier (in the order P_θ) than any other sub-class of κ not belonging to λ. By $*258 \cdot 231$, the series $A(T_P, \kappa)$ has an end, namely

$$p`(T_P * A)`\kappa.$$

If this is not null, it must consist of a single term, which will be the minimum of κ ($*276 \cdot 33$). But if it is null, we proceed as follows. Put

$$P_{tl}`\kappa = s`\hat{\gamma}\{(\exists\lambda) . \lambda \in (T_P * A)`\kappa . \gamma = p`\lambda \cap \overrightarrow{P}`P_m`\lambda\} \quad \text{Dft}.$$

Then $P_{tl}`\kappa$ will be the lower limit of κ.

In the first place, we easily prove that, since $p`(T_P * A)`\kappa = \Lambda$, if

$$\lambda \in (T_P * A)`\kappa - \iota`\Lambda,$$

$P_m`\lambda$ and $\breve{T}_P`\lambda$ both exist ($*276 \cdot 341$). Hence every member of κ has predecessors in κ, and κ has no minimum. In the second place, we show that

$$\lambda \{A(T_P, \kappa)\} \mu . \exists ! \mu . \supset . (P_m`\lambda) P (P_m`\mu) \quad (*276 \cdot 34 \cdot 342),$$

and that $$\alpha \in \lambda . \supset . p`\lambda \cap \overrightarrow{P}`P_m`\lambda = \alpha \cap \overrightarrow{P}`P_m`\lambda \quad (*276 \cdot 353).$$

Hence we find that

$$\begin{aligned} \lambda \{A(T_P, \kappa)\} \mu . \alpha \in \mu . \supset . \; & p`\lambda \cap \overrightarrow{P}`P_m`\lambda = p`\mu \cap \overrightarrow{P}`P_m`\lambda = \alpha \cap \overrightarrow{P}`P_m`\lambda . \\ \supset . \; & p`\lambda \cap \overrightarrow{P}`P_m`\lambda \subset p`\mu \cap \overrightarrow{P}`P_m`\mu . \\ & (p`\mu \cap \overrightarrow{P}`P_m`\mu) \cap \overrightarrow{P}`P_m`\lambda = p`\lambda \cap \overrightarrow{P}`P_m`\mu, \end{aligned}$$

whence it follows that

$$\lambda \in (T_P * A)`\kappa - \iota`\Lambda . \supset . p`\lambda \cap \overrightarrow{P}`P_m`\lambda = P_{tl}`\kappa \cap \overrightarrow{P}`P_m`\lambda,$$

whence, by what was stated above,

$$\lambda \in (T_P * A)`\kappa . \alpha \in \lambda . \supset . \alpha \cap \overrightarrow{P}`P_m`\lambda = P_{tl}`\kappa \cap \overrightarrow{P}`P_m`\lambda \quad (*276 \cdot 354).$$

Again, if $\alpha \epsilon \kappa$, the product of all the members of $(T_P * A)`\kappa$ to which $\alpha$ belongs is a member of $(T_P * A)`\kappa$ to which α belongs, but if we call this product λ, $P_m`\lambda \sim\epsilon\, \alpha$ (because, if $P_m`\lambda \epsilon \alpha$, $\alpha \epsilon \breve{T}_P`\lambda$, which is contrary to the definition of λ). Hence we have

$$\alpha \epsilon \kappa \,.\supset.\, (P_{\text{tl}}`\kappa) P_\theta \alpha \quad (*276\cdot36).$$

It only remains to prove

$$(P_{\text{tl}}`\kappa) P_\theta \beta \,.\supset.\, (\exists \alpha) \,.\, \alpha \epsilon \kappa \,.\, \alpha P_\theta \beta \quad (*276\cdot37).$$

By the hypothesis, and the definition of $P_{\text{tl}}`\kappa$, we have

$$(\exists z, \lambda) \,.\, \lambda \epsilon (T_P * A)`\kappa \,.\, z \epsilon p`\lambda \cap \overrightarrow{P}`P_m`\lambda - \beta \,.\, P_{\text{tl}}`\kappa \cap \overrightarrow{P}`z = \beta \cap \overrightarrow{P}`z.$$

Since this involves $\text{E}\,!\, P_m`\lambda$, it involves $\lambda \neq \Lambda$, hence, by what was stated above, it involves

$$(\exists z, \lambda, \alpha) \,.\, \lambda \epsilon (T_P * A)`\kappa \,.\, \alpha \epsilon \lambda \,.\, z \epsilon \alpha \cap \overrightarrow{P}`P_m`\lambda - \beta \,.\, P_{\text{tl}}`\kappa \cap \overrightarrow{P}`z = \beta \cap \overrightarrow{P}`z.$$

Hence we obtain $\qquad \beta \cap \overrightarrow{P}`z \subset P_{\text{tl}}`\kappa \cap \overrightarrow{P}`P_m`\lambda,$

and $\qquad P_{\text{tl}}`\kappa \cap \overrightarrow{P}`P_m`\lambda = \alpha \cap \overrightarrow{P}`P_m`\lambda,$

whence $\qquad \beta \cap \overrightarrow{P}`z \subset \alpha.$

Hence, by $*170\cdot11$, we have $\alpha P_\theta \beta$.

This completes the proof of $P_{\text{tl}}`\kappa = \text{tl}\,(P_\theta)`\kappa$ ($*276\cdot38$). Hence, combining the two cases, we find that κ has a minimum if $\exists\,!\, p`(T_P * A)`\kappa$, and a lower limit if $\sim\exists\,!\, p`(T_P * A)`\kappa$. Hence $\text{E}\,!\, \text{limin}\,(P_\theta)`\kappa$, in either case ($*276\cdot39$).

This completes the proof of $P_\theta \epsilon \text{Ded}$ if $P \epsilon \Omega\,\text{infin}$.

$*276\cdot01$. $P_\theta = P_{\text{cl}} \upharpoonright (-\,\text{Cls induct} \cup \iota`\Lambda) \quad \text{Df}$

$*276\cdot02$. $A = \hat{\alpha}\hat{\beta}\,(\beta \subset \alpha \,.\, \beta \neq \alpha) \quad \text{Dft [}*276\text{]}$

$*276\cdot03$. $P_m`\lambda = \text{min}_P`(s`\lambda - p`\lambda) \quad \text{Dft [}*276\text{]}$

$*276\cdot04$. $T_P = \hat{\lambda}\hat{\mu}\,\{\mu = \lambda \cap \overleftarrow{\epsilon}`P_m`\lambda\} \quad \text{Dft [}*276\text{]}$

$*276\cdot05$. $P_{\text{tl}}`\kappa = s`\hat{\gamma}\,\{(\exists \lambda) \,.\, \lambda \epsilon (T_P * A)`\kappa - \iota`\Lambda \,.\, \gamma = p`\lambda \cap \overrightarrow{P}`P_m`\lambda\} \quad \text{Dft [}*276\text{]}$

$*276\cdot1$. $\vdash : \alpha P_\theta \beta \,.\equiv.\, \alpha, \beta \epsilon (\text{Cl}`C`P - \text{Cls induct}) \cup \iota`\Lambda \,.\, \exists\,!\, \alpha - \beta - \breve{P}``(\beta - \alpha)$
[$*170\cdot1 \,.\, (*276\cdot01)$]

$*276\cdot11$ $\vdash :: P \epsilon \Omega \,.\supset :.\, \alpha P_\theta \beta \,.\equiv :\, \alpha, \beta \epsilon (\text{Cl}`C`P - \text{Cls induct}) \cup \iota`\Lambda :$
$(\exists z) \,.\, z \epsilon \alpha - \beta \,.\, \alpha \cap \overrightarrow{P}`z = \beta \cap \overrightarrow{P}`z \quad [*251\cdot35 \,.\, (*276\cdot01)]$

$*276\cdot12$. $\vdash : C`P \sim\epsilon\, 1 \,.\supset.\, P_\theta = P_{\text{cl}} \upharpoonright (-\,C`P_\eta) \quad [*274\cdot17 \,.\, *276\cdot1 \,.\, *170\cdot1]$

$*276\cdot121$. $\vdash : C`P \sim\epsilon\, \text{Cls induct} \,.\supset.$
$B`\breve{P}_\theta = \Lambda \,.\, B`P_\theta = C`P \,.\, C`P_\theta = (\text{Cl}`C`P - \text{Cls induct}) \cup \iota`\Lambda$
[$*170\cdot31\cdot32\cdot38 \,.\, (*276\cdot01)$]

$*276\cdot122$. $\vdash : C'P \sim\epsilon\, 0 \cup 1 . \supset . C'P_\eta \cup C'P_\theta = \mathrm{Cl}'C'P$ $[*276\cdot121 . *274\cdot17]$

$*276\cdot123$. $\vdash : C'P \sim\epsilon\, \text{Cls induct} . \equiv . \dot{\exists} ! P_\theta$ $[*276\cdot1\cdot121]$

$*276\cdot13$. $\vdash : C'P \sim\epsilon\, 0 \cup 1 . \supset . \mathrm{Nc}'C'P_\eta +_c \mathrm{Nc}'C'P_\theta = 2^{\mathrm{Nc}'C'P}$
$[*276\cdot122 . *116\cdot72]$

$*276\cdot14$. $\vdash : P \epsilon \Omega . \supset . P_\theta \epsilon \mathrm{Ser}$ $[*251\cdot36 . (*276\cdot01)]$

$*276\cdot2$. $\vdash : P \epsilon \omega . \supset . (C'P -)``(\mathrm{Cl\,induct}'C'P - \iota'\Lambda) \epsilon \aleph_0 \cap \overrightarrow{\mathrm{med}}'P_\theta$

Dem.

$\vdash . *24\cdot492 . \supset \vdash . (C'P -)``(\mathrm{Cl\,induct}'C'P - \iota'\Lambda)\,\mathrm{sm}\,(\mathrm{Cl\,induct}'C'P - \iota'\Lambda)$ (1)

$\vdash . (1) . *274\cdot27 . \supset \vdash : \mathrm{Hp} . \supset . (C'P -)``(\mathrm{Cl\,induct}'C'P - \iota'\Lambda) \epsilon \aleph_0$ (2)

$\vdash . *200\cdot361 . *263\cdot47 . \supset$

$\vdash : \mathrm{Hp} . \alpha P_\theta \beta . z \epsilon \alpha - \beta . \alpha \cap \overrightarrow{P}'z = \beta \cap \overrightarrow{P}'z . \gamma = (\alpha \cap \overrightarrow{P_*}'z) \cup \overleftarrow{P}'\mathrm{min}_P'(\alpha \cap \overleftarrow{P}'z) .$

$\supset . \mathrm{min}_P'(\alpha \cap \overleftarrow{P}'z) \epsilon \alpha - \gamma . \alpha \cap \overrightarrow{P}'\mathrm{min}_P'(\alpha \cap \overleftarrow{P}'z) = \gamma \cap \overrightarrow{P}'\mathrm{min}_P'(\alpha \cap \overleftarrow{P}'z) .$

$z \epsilon \gamma - \beta . \gamma \cap \overrightarrow{P}'z = \alpha \cap \overrightarrow{P}'z = \beta \cap \overrightarrow{P}'z . \gamma \sim\epsilon\, \text{Cls induct} .$

$[*276\cdot11]$ $\supset . \alpha P_\theta \gamma . \gamma P_\theta \beta$ (3)

$\vdash . *263\cdot47 . \supset \vdash : \mathrm{Hp}(3) . \supset . C'P - \gamma \epsilon\, \text{Cls induct}$ (4)

$\vdash . (3) . (4) . *276\cdot11 . \supset$

$\vdash : \mathrm{Hp} . \alpha P_\theta \beta . \supset . (\exists \gamma) . C'P - \gamma \epsilon\, \text{Cls induct} . \alpha P_\theta \gamma . \gamma P_\theta \beta$ (5)

$\vdash . *120\cdot71 . \mathrm{Transp} . \supset$

$\vdash : \mathrm{Hp} . \alpha \epsilon\, \mathrm{Cl\,induct}'C'P - \iota'\Lambda . \supset . (C'P - \alpha) \sim\epsilon\, \text{Cls induct}$ (6)

$\vdash . (6) . *276\cdot121 . \supset \vdash : \mathrm{Hp} . \supset . (C'P -)``(\mathrm{Cl\,induct}'C'P - \iota'\Lambda) \subset C'P_\theta$ (7)

$\vdash . (2) . (5) . *271\cdot1 . (7) . \supset \vdash . \mathrm{Prop}$

The following propositions constitute the proof of

$$P \epsilon \Omega\, \mathrm{infin} . \supset . P_\theta \epsilon\, \mathrm{Ded} \quad (*276\cdot4).$$

$*276\cdot3$. $\vdash :. \mathrm{E} ! P_m'\lambda . \supset : \alpha \epsilon \breve{T}_P'\lambda . \equiv . \alpha \epsilon \lambda . P_m'\lambda \epsilon \alpha : P_m'\lambda = \mathrm{min}_P'(s'\lambda - p'\lambda)$
$[(*276\cdot03\cdot04)]$

$*276\cdot301$. $\vdash : P \epsilon \Omega . \lambda \subset \mathrm{Cl}'C'P - \iota'\Lambda . \lambda \sim\epsilon\, 0 \cup 1 . \supset . \mathrm{E} ! P_m'\lambda . \mathrm{E} ! \breve{T}_P'\lambda$

Dem.

$\vdash . *40\cdot12\cdot13 .$ $\supset \vdash :. p'\lambda = s'\lambda . \supset : \alpha, \beta \epsilon \lambda . \supset_{\alpha,\beta} . \alpha = \beta$ (1)

$\vdash . (1) . \mathrm{Transp} . *40\cdot23 . \supset \vdash : \mathrm{Hp} . \supset . \exists ! s'\kappa - p'\kappa .$

$[*250\cdot121]$ $\supset . \mathrm{E} ! \mathrm{min}_P'(s'\kappa - p'\kappa) : \supset \vdash . \mathrm{Prop}$

$*276\cdot302$. $\vdash : \mathrm{E} ! P_m'\lambda . \supset . P_m'\lambda \epsilon p'\breve{T}_P'\lambda - p'\lambda$ $[*276\cdot3]$

$*276\cdot303$. $\vdash . T_P \in A . (T_P)_{\mathrm{po}} \in A$

Dem.

$\vdash . *276\cdot3 .$ $\supset \vdash : \mu \breve{T}_P \lambda . \supset . \mu \subset \lambda$ (1)

$\vdash . *276\cdot302 . \supset \vdash : \mu \breve{T}_P \lambda . \supset . \mu \neq \lambda$ (2)

$\vdash . (1) . (2) . *201\cdot18 . \supset \vdash . \mathrm{Prop}$

∗276·304. $\vdash : \mu \{A(T_P, \kappa)\} \lambda . \supset . \mu \subset \lambda . p`\lambda \subset p`\mu . \mu \neq \lambda . p`\lambda \neq p`\mu$
[∗276·302·303]

∗276·305. $\vdash . A(T_P, \kappa) \epsilon \Omega$ [∗258·201 . ∗276·303]

∗276·31. $\vdash : P \epsilon \Omega . \exists ! \lambda . \lambda \subset \mathrm{Cl}`C`P - \iota`\Lambda . \lambda \sim \epsilon \mathrm{D}`\breve{T}_P . \supset .$
$\lambda \epsilon 1 . s`\lambda = p`\lambda = \iota`\lambda$ [∗276·301 . Transp]

∗276·32. $\vdash :. P \epsilon \Omega . \lambda \sim \epsilon 0 \cup 1 . \lambda \subset \mathrm{D}`P_\theta . \supset :$
$P_m`\lambda \epsilon p`\breve{T}_P`\lambda - p`\lambda : \alpha \epsilon \lambda . \supset_\alpha . \alpha \cap \overrightarrow{P}`P_m`\lambda = p`\lambda \cap \overrightarrow{P}`P_m`\lambda$

Dem.

$\vdash . ∗276·301 . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{E} ! \breve{T}_P`\lambda . \mathrm{E} ! P_m`\lambda .$ (1)

[∗276·302] $\supset . P_m`\lambda \epsilon p`\breve{T}_P`\lambda - p`\lambda$ (2)

$\vdash . (1) . ∗276·3 . \supset \vdash : \mathrm{Hp} . \supset . \overrightarrow{P}`P_m`\lambda \cap s`\lambda = \overrightarrow{P}`P_m`\lambda \cap p`\lambda$ (3)

$\vdash . (2) . (3) . \supset \vdash .$ Prop

∗276·321. $\vdash : \mathrm{Hp}\, ∗276·32 . \alpha \epsilon \breve{T}_P`\lambda . \beta \epsilon \lambda - \breve{T}_P`\lambda . \supset . \alpha P_\theta \beta$

Dem.

$\vdash . ∗276·3·32 . \supset \vdash : \mathrm{Hp} . \supset . P_m`\lambda \epsilon \alpha - \beta . \alpha \cap \overrightarrow{P}`P_m`\lambda = \beta \cap \overrightarrow{P}`P_m`\lambda .$

[∗276·11] $\supset . \alpha P_\theta \beta : \supset \vdash .$ Prop

∗276·322. $\vdash : \mathrm{Hp}\, ∗276·32 . \mu \epsilon (T_P ∗ A)`\lambda . \alpha \epsilon \mu . \beta \epsilon \lambda - \mu . \supset . \alpha P_\theta \beta$

Dem.

$\vdash . ∗40·23 . \supset \vdash :. \rho \subset (T_P ∗ A)`\lambda . \exists ! \rho : \mu \epsilon \rho . \alpha \epsilon \mu . \beta \epsilon \lambda - \mu . \supset_{\mu, \alpha, \beta} . \alpha P_\theta \beta : \supset :$
$\alpha \epsilon p`\rho . \beta \epsilon \lambda - p`\rho . \supset_{\alpha, \beta} . \alpha P_\theta \beta$ (1)

$\vdash . (1) . ∗276·321 . ∗258·241 . \supset \vdash .$ Prop

∗276·33. $\vdash : \mathrm{Hp}\, ∗276·32 . \exists ! p`(T_P ∗ A)`\lambda . \supset . \breve{\iota}`p`(T_P ∗ A)`\lambda = \min (P_\theta)`\lambda$

Dem.

$\vdash . ∗276·31 . ∗258·231 . \supset \vdash : \mathrm{Hp} . \supset . p`(T_P ∗ A)`\lambda \epsilon 1$ (1)

$\vdash . (1) . ∗276·322 . \supset \vdash : \mathrm{Hp} . \alpha \epsilon \lambda - p`(T_P ∗ A)`\lambda . \supset . \{\breve{\iota}`p`(T_P ∗ A)`\lambda\} P_\theta \alpha$ (2)

$\vdash . (1) . (2) . \supset \vdash .$ Prop

∗276·331. $\vdash : \mathrm{Hp}\, ∗276·32 . \exists ! p`(T_P ∗ A)`\lambda . \supset . \mathrm{E} ! \min (P_\theta)`\lambda$ [∗276·33]

∗276·34. $\vdash : \mathrm{Hp}\, ∗276·32 . \mu \breve{T}_P \lambda . \mu \epsilon \mathrm{D}`T_P . \supset . (P_m`\lambda) P (P_m`\mu)$

Dem.

$\vdash . ∗276·3 . \supset \vdash : \mathrm{Hp} . \supset . P_m`\lambda = \min_P`(s`\lambda - p`\lambda)$ (1)

$\vdash . ∗276·3·304 . \supset \vdash : \mathrm{Hp} . \supset . P_m`\mu \epsilon (s`\lambda - p`\lambda)$ (2)

$\vdash . ∗276·302 . \supset \vdash : \mathrm{Hp} . \supset . P_m`\lambda \epsilon p`\mu . P_m`\mu \sim \epsilon p`\mu .$

[∗13·12] $\supset . P_m`\lambda \neq P_m`\mu$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash .$ Prop

***276·341.** $\vdash :. \mathrm{Hp} \ast 276{\cdot}32 \,.\, p{`}(T_P \ast A){`}\lambda = \Lambda \,.\, \supset :$

$P_m{``}(T_P \ast A){`}\lambda \subset P{``}P_m{``}(T_P \ast A){`}\lambda \,.\, P_m{``}(T_P \ast A){`}\lambda \sim\in \mathrm{Cls\ induct} :$

$\mu \in (T_P \ast A){`}\lambda - \iota{`}\Lambda \,.\, \supset_\mu \,.\, \mathrm{E}\,!\,\breve{T}_P{`}\mu \,.\, \mathrm{E}\,!\,P_m{`}\mu$

Dem.

$\vdash . \ast 258{\cdot}231 \,.\, \ast 276{\cdot}301 \,.\, \supset$

$\vdash :. \mathrm{Hp} \,.\, \supset : \mu \in (T_P \ast A){`}\lambda - \iota{`}p{`}(T_P \ast A){`}\lambda \,.\, \supset \,.\, \mathrm{E}\,!\,\breve{T}_P{`}\mu \,.\, \mathrm{E}\,!\,P_m{`}\mu :$

$[\ast 276{\cdot}34.\mathrm{Hp}] \quad \supset : \mu \in (T_P \ast A){`}\lambda \,.\, \mathrm{E}\,!\,P_m{`}\mu \,.\, \supset \,.\, (P_m{`}\mu)\, P\, (P_m{`}\breve{T}_P{`}\mu) \qquad (1)$

$\vdash . (1) \,.\, \ast 261{\cdot}26 \,.\, \mathrm{Transp} \,.\, \supset \vdash . \mathrm{Prop}$

***276·342.** $\vdash : \mathrm{Hp} \ast 276{\cdot}341 \,.\, \lambda \{A\,(T_P, \kappa)\}\, \mu \,.\, \mathrm{E}\,!\,P_m{`}\mu \,.\, \supset \,.\, (P_m{`}\lambda)\, P\, (P_m{`}\mu)$

Dem.

$\vdash . \ast 276{\cdot}3 \,.\, \supset$

$\vdash :: \mathrm{Hp} : \rho \subset (T_P \ast A){`}\kappa \,.\, \exists\,!\,\rho \,.\, \exists\,!\,p{`}\rho : \supset :. P_m{`}p{`}\rho \in s{`}p{`}\rho - p{`}p{`}\rho :.$

$[\ast 40{\cdot}1{\cdot}11] \quad \supset :. (\exists\alpha) \,.\, \alpha \in p{`}\rho \,.\, P_m{`}p{`}\rho \in \alpha : (\exists\alpha) \,.\, \alpha \in p{`}\rho \,.\, P_m{`}p{`}\rho \sim\in \alpha :.$

$[\ast 40{\cdot}1.\ast 11{\cdot}26] \supset :. \lambda \in \rho \,.\, \supset_\lambda : (\exists\alpha) \,.\, \alpha \in \lambda \,.\, P_m{`}p{`}\rho \in \alpha : (\exists\alpha) \,.\, \alpha \in \lambda \,.\, P_m{`}p{`}\rho \sim\in \alpha :.$

$[\ast 40{\cdot}1{\cdot}11] \quad \supset :. \lambda \in \rho \,.\, \supset_\lambda \,.\, P_m{`}p{`}\rho \in (s{`}\lambda - p{`}\lambda) \qquad (1)$

$\vdash . (1) \,.\, \ast 276{\cdot}302 \,.\, \supset \vdash :. \mathrm{Hp}\,(1) \,.\, \supset :$

$\breve{T}_P{`}\lambda \in \rho \,.\, \lambda \in \rho \,.\, \supset_\lambda \,.\, P_m{`}\lambda \in p{`}\breve{T}_P{`}\lambda \,.\, P_m{`}p{`}\rho \sim\in p{`}\breve{T}_P{`}\lambda :$

$[\ast 13{\cdot}12] \quad \supset : \breve{T}_P{`}\lambda \in \rho \,.\, \lambda \in \rho \,.\, \supset_\lambda \,.\, P_m{`}\lambda \neq P_m{`}p{`}\rho \qquad (2)$

$\vdash . (1) \,.\, (2) \,.\, \ast 276{\cdot}3 \,.\, \supset \vdash : \mathrm{Hp}\,(1) \,.\, \breve{T}_P{`}\lambda \in \rho \,.\, \lambda \in \rho \,.\, \supset \,.\, (P_m{`}\lambda)\, P\, (P_m{`}p{`}\rho) \qquad (3)$

$\vdash . (3) \,.\, \ast 276{\cdot}34 \,.\, \ast 258{\cdot}241 \,.\, \supset \vdash . \mathrm{Prop}$

***276·35.** $\vdash :. P \in \Omega \,.\, \kappa \subset \mathrm{D}{`}P_\theta \,.\, \exists\,!\,\kappa \,.\, p{`}(T_P \ast A){`}\kappa = \Lambda \,.\, \supset :$

$\lambda \in (T_P \ast A){`}\kappa - \iota{`}\Lambda \,.\, \supset \,.\, P_m{`}\lambda \in p{`}\breve{T}_P{`}\lambda \cap \overrightarrow{P}{`}P_m{`}\breve{T}_P{`}\lambda$

Dem.

$\vdash . \ast 276{\cdot}341 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \lambda \in (T_P \ast A){`}\kappa - \iota{`}\Lambda \,.\, \supset \,.\, \mathrm{E}\,!\,\breve{T}_P{`}\lambda \,.$

$[\ast 276{\cdot}302{\cdot}34] \qquad \supset \,.\, P_m{`}\lambda \in p{`}\breve{T}_P{`}\lambda \cap \overrightarrow{P}{`}P_m{`}\breve{T}_P{`}\lambda : \supset \vdash . \mathrm{Prop}$

***276·351.** $\vdash : \mathrm{Hp} \ast 276{\cdot}35 \,.\, \supset \,.\, P_m{``}(T_P \ast A){`}\kappa \subset P_{\mathrm{tl}}{`}\kappa$

Dem.

$\vdash . \ast 276{\cdot}3 \,. \qquad \supset \vdash . \sim \mathrm{E}\,!\,P_m{`}\Lambda \qquad (1)$

$\vdash . \ast 276{\cdot}35 \,.\, (\ast 276{\cdot}05) \,.\, \supset \vdash : \mathrm{Hp} \,.\, \lambda \in (T_P \ast A){`}\kappa - \iota{`}\Lambda \,.\, \supset \,.\, P_m{`}\lambda \in P_{\mathrm{tl}}{`}\kappa \qquad (2)$

$\vdash . (1) \,.\, (2) \,.\, \supset \vdash . \mathrm{Prop}$

***276·352.** $\vdash : \mathrm{Hp} \ast 276{\cdot}35 \,.\, \supset \,.\, P_{\mathrm{tl}}{`}\kappa \sim\in \mathrm{Cls\ induct} \quad [\ast 276{\cdot}351{\cdot}341]$

***276·353.** $\vdash : \mathrm{Hp} \ast 276{\cdot}35 \,.\, \lambda \in (T_P \ast A){`}\kappa \,.\, \lambda \{A\,(T_P, \kappa)\}\, \mu \,.\, \alpha \in \mu \,.\, \supset \,.$

$p{`}\lambda \cap \overrightarrow{P}{`}P_m{`}\lambda = p{`}\mu \cap \overrightarrow{P}{`}P_m{`}\lambda = \alpha \cap \overrightarrow{P}{`}P_m{`}\lambda$

Dem.

$\vdash . \ast 276{\cdot}304 \,. \qquad \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, \alpha \in \lambda \qquad (1)$

$\vdash . \ast 276{\cdot}35{\cdot}31 \,.\, \mathrm{Transp} \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, \mathrm{E}\,!\,P_m{`}\lambda \,.\, \lambda \sim\in 0 \cup 1 \qquad (2)$

$\vdash . (1) \,.\, (2) \,.\, \ast 276{\cdot}32 \,. \qquad \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, p{`}\lambda \cap \overrightarrow{P}{`}P_m{`}\lambda = \alpha \cap \overrightarrow{P}{`}P_m{`}\lambda \qquad (3)$

$\vdash . (3) . \supset \vdash :. \text{Hp} . \supset : \beta \epsilon \mu . \supset_\beta . \alpha \cap \overrightarrow{P}`P_m`\lambda = \beta \cap \overrightarrow{P}`P_m`\lambda$ (4)

$\vdash . (4) . \supset \vdash : \text{Hp} . \supset . \alpha \cap \overrightarrow{P}`P_m`\lambda = p`\mu \cap \overrightarrow{P}`P_m`\lambda$ (5)

$\vdash . (3) . (5) . \supset \vdash . \text{Prop}$

***276·354.** $\vdash : \text{Hp} *276·35 . \lambda \epsilon (T_P * A)`\kappa . \alpha \epsilon \lambda . \supset .$

$$P_{\text{tl}}`\kappa \cap \overrightarrow{P}`P_m`\lambda = p`\lambda \cap \overrightarrow{P}`P_m`\lambda = \alpha \cap \overrightarrow{P}`P_m`\lambda$$

Dem.

$\vdash . *276·353 . \supset \vdash : \text{Hp} . \exists ! \mu . \lambda \{A (T_P, \kappa)\} \mu . \supset .$

$$p`\mu \cap \overrightarrow{P}`P_m`\lambda = p`\lambda \cap \overrightarrow{P}`P_m`\lambda .$$

[*22·47] $\supset . (p`\mu \cap \overrightarrow{P}`P_m`\mu) \cap \overrightarrow{P}`P_m`\lambda \subset p`\lambda \cap \overrightarrow{P}`P_m`\lambda$ (1)

$\vdash . *276·353 . \supset \vdash : \text{Hp} . \mu \{A (T_P, \kappa)\} \lambda . \supset . p`\mu \cap \overrightarrow{P}`P_m`\mu = p`\lambda \cap \overrightarrow{P}`P_m`\mu$

[*276·342] $\subset p`\lambda \cap \overrightarrow{P}`P_m`\lambda$ (2)

$\vdash . (1) . (2) . *276·305 . \supset$

$\vdash : \text{Hp} . \mu \epsilon (T_P * A)`\kappa - \iota`\Lambda . \supset . (p`\mu \cap \overrightarrow{P}`P_m`\mu) \cap \overrightarrow{P}`P_m`\lambda \subset p`\lambda \cap \overrightarrow{P}`P_m`\lambda$ (3)

$\vdash . (3) . *276·32 . (*276·05) . \supset \vdash . \text{Prop}$

***276·355.** $\vdash : \text{Hp} *276·35 . \alpha \epsilon \kappa . \supset . (\exists \lambda) . \lambda \epsilon (T_P * A)`\kappa . \alpha \epsilon \lambda . P_m`\lambda \sim \epsilon \alpha$

Dem.

$\vdash . *40·1 . \supset \vdash :. \text{Hp} . \supset : (\exists \lambda) . \lambda \epsilon (T_P * A)`\kappa . \alpha \sim \epsilon \lambda :$

[*276·305] $\supset : (\exists \lambda) : \lambda \epsilon (T_P * A)`\kappa . \alpha \sim \epsilon \lambda : \mu \{A (T_P, \kappa)\} \lambda . \supset_\mu . \alpha \epsilon \mu$ (1)

$\vdash . *40·1 . \supset \vdash :. \mu \{A (T_P, \kappa)\} \lambda . \supset_\mu . \alpha \epsilon \mu : \lambda = p`\overleftarrow{A (T_P, \kappa)}`\lambda : \supset . \alpha \epsilon \lambda$ (2)

$\vdash . (1) . (2) . \text{Transp} . \supset$

$\vdash : \text{Hp} . \supset . (\exists \lambda, \mu) . \mu, \lambda \epsilon (T_P * A)`\kappa . \lambda = \breve{T}_P`\mu . \alpha \epsilon \mu . \alpha \sim \epsilon \lambda .$

[*276·3] $\supset . (\exists \mu) . \mu \epsilon (T_P * A)`\kappa . \alpha \epsilon \mu . P_m`\mu \sim \epsilon \alpha : \supset \vdash . \text{Prop}$

***276·36.** $\vdash : \text{Hp} *276·35 . \alpha \epsilon \kappa . \supset . (P_{\text{tl}}`\kappa) P_\theta \alpha$

Dem.

$\vdash . *276·351·355·354 . \supset$

$\vdash : \text{Hp} . \supset . (\exists \lambda) . \lambda \epsilon (T_P * A)`\kappa . P_m`\lambda \epsilon P_{\text{tl}}`\kappa - \alpha . P_{\text{tl}}`\kappa \cap \overrightarrow{P}`P_m`\lambda = \alpha \cap \overrightarrow{P}`P_m`\lambda .$

[*276·352] $\supset . (P_{\text{tl}}`\kappa) P_\theta \alpha : \supset \vdash . \text{Prop}$

***276·361.** $\vdash : \text{Hp} *276·35 . \supset . \kappa \subset \overleftarrow{P}_\theta`P_{\text{tl}}`\kappa$ [*276·36]

***276·37.** $\vdash : \text{Hp} *276·35 . (P_{\text{tl}}`\kappa) P_\theta \beta . \supset . (\exists \alpha) . \alpha \epsilon \kappa . \alpha P_\theta \beta$

Dem.

$\vdash . *276·11 . \supset \vdash : \text{Hp} . \supset . (\exists z) . z \epsilon P_{\text{tl}}`\kappa - \beta . P_{\text{tl}}`\kappa \cap \overrightarrow{P}`z = \beta \cap \overrightarrow{P}`z .$

[(*276·05)] $\supset . (\exists z, \lambda) . \lambda \epsilon (T_P * A)`\kappa . z \epsilon p`\lambda \cap \overrightarrow{P}`P_m`\lambda - \beta .$

$$P_{\text{tl}}`\kappa \cap \overrightarrow{P}`z = \beta \cap \overrightarrow{P}`z .$$

[*276·354] $\supset . (\exists z, \lambda, \alpha) . \lambda \epsilon (T_P * A)`\kappa . \alpha \epsilon \lambda . z \epsilon \alpha - \beta .$

$$\overrightarrow{P}`z \subset \overrightarrow{P}`P_m`\lambda . \alpha \cap \overrightarrow{P}`P_m`\lambda = \beta \cap \overrightarrow{P}`P_m`\lambda .$$

[Fact.*276·304] $\supset . (\exists z, \alpha) . \alpha \epsilon \kappa . z \epsilon \alpha - \beta . \beta \cap \overrightarrow{P}`z \subset \alpha .$

[*170·11] $\supset . (\exists \alpha) . \alpha \epsilon \kappa . \alpha P_\theta \beta : \supset \vdash . \text{Prop}$

***276·38.** $\vdash : P \in \Omega \,.\, \kappa \subset \mathrm{D}^{\backprime}P_\theta \,.\, \exists ! \,\kappa \,.\, p^{\backprime}(T_P \ast A)^{\backprime}\kappa = \Lambda \,.\, \supset \,.\, P_{\mathrm{tl}}{}^{\backprime}\kappa = \mathrm{tl}\,(P_\theta)^{\backprime}\kappa$
[*276·361·37]

***276·381.** $\vdash : P \in \Omega \,.\, \kappa \subset \mathrm{D}^{\backprime}P_\theta \,.\, \exists ! \,\kappa \,.\, p^{\backprime}(T_P \ast A)^{\backprime}\kappa = \Lambda \,.\, \supset \,.\, \mathrm{E} ! \,\mathrm{tl}\,(P_\theta)^{\backprime}\kappa$
[*276·38]

***276·39.** $\vdash : P \in \Omega \,.\, \kappa \subset \mathrm{D}^{\backprime}P_\theta \,.\, \exists ! \,\kappa \,.\, \supset \,.\, \mathrm{E} ! \,\mathrm{limin}\,(P_\theta)^{\backprime}\kappa$ [*276·331·381]

In the following proposition, the only reason why P has to be infinite is in order that P_θ may exist; for "Ded" was so defined as to exclude $\dot{\Lambda}$.

***276·4.** $\vdash : P \in \Omega\,\mathrm{infin} \,.\, \supset \,.\, P_\theta \in \mathrm{Ded}$

Dem.

$\vdash .\, \ast 276{\cdot}121 .\, \ast 207{\cdot}3 .\, \ast 205{\cdot}18 .\, \supset \vdash : \mathrm{Hp} .\, \supset .\, \mathrm{limin}_P{}^{\backprime}\Lambda = C^{\backprime}P \,.\, \mathrm{limin}_P{}^{\backprime}\iota^{\backprime}\Lambda = \Lambda$ (1)

$\vdash .\, \ast 206{\cdot}7 .\quad \supset \vdash : \mathrm{Hp} \,.\, \kappa \subset C^{\backprime}P_\theta \,.\, \Lambda \in \kappa \,.\, \kappa \neq \iota^{\backprime}\Lambda \,.\, \supset .$

$\overrightarrow{\mathrm{prec}}\,(P_\theta)^{\backprime}\kappa = \overrightarrow{\mathrm{prec}}\,(P_\theta)^{\backprime}(\kappa - \iota^{\backprime}\Lambda)$ (2)

$\vdash .\, \ast 205{\cdot}192 .\, \supset \vdash : \mathrm{Hp}\,(2) .\, \supset .\, \overrightarrow{\mathrm{min}}\,(P_\theta)^{\backprime}\kappa = \overrightarrow{\mathrm{min}}\,(P_\theta)^{\backprime}(\kappa - \iota^{\backprime}\Lambda)$ (3)

$\vdash .\, (2) .\, (3) .\quad \supset \vdash : \mathrm{Hp}\,(2) .\, \supset .\, \overrightarrow{\mathrm{limin}}\,(P_\theta)^{\backprime}\kappa = \overrightarrow{\mathrm{limin}}\,(P_\theta)^{\backprime}(\kappa - \iota^{\backprime}\Lambda) \,.$

[*276·39] $\supset .\, \mathrm{E} ! \,\mathrm{limin}\,(P_\theta)^{\backprime}\kappa$ (4)

$\vdash .\, (1) .\, (4) .\, \ast 276{\cdot}39 .\, \supset \vdash :.\, \mathrm{Hp} .\, \supset : \kappa \subset C^{\backprime}P_\theta \,.\, \supset_\kappa .\, \mathrm{E} ! \,\mathrm{limin}\,(P_\theta)^{\backprime}\kappa :$

[*214·12·14] $\supset : P_\theta \in \mathrm{Ded} :.\, \supset \vdash .\, \mathrm{Prop}$

***276·41.** $\vdash : P \in \omega \,.\, \supset \,.\, P_\theta \in \theta$ [*276·2·4·14 . *275·1]

***276·42.** $\vdash : P \in \omega \,.\, \supset \,.\, C^{\backprime}P_\theta \in 2^{\aleph_0}$

Dem.

$\vdash .\, \ast 276{\cdot}13 .\, \ast 274{\cdot}27 .\, \supset \vdash : \mathrm{Hp} .\, \supset .\, \mathrm{Nc}^{\backprime}C^{\backprime}P_\theta +_{\mathrm{c}} \aleph_0 = 2^{\aleph_0}$ (1)

$\vdash .\, \ast 276{\cdot}2 .\quad \supset \vdash : \mathrm{Hp} .\, \supset .\, (\exists \mu) .\, \mathrm{Nc}^{\backprime}C^{\backprime}P_\theta = \mu +_{\mathrm{c}} \aleph_0 \,.$

[*123·421] $\supset .\, \mathrm{Nc}^{\backprime}C^{\backprime}P_\theta +_{\mathrm{c}} \aleph_0 = \mathrm{Nc}^{\backprime}C^{\backprime}P_\theta$ (2)

$\vdash .\, (1) .\, (2) .\, \supset \vdash .\, \mathrm{Prop}$

***276·43.** $\vdash .\, C^{\backprime\backprime}\theta = 2^{\aleph_0}$

Dem.

$\vdash .\, \ast 276{\cdot}42{\cdot}41 .\, \supset \vdash : \exists ! \,\omega .\, \supset .\, \exists ! \,C^{\backprime\backprime}\theta \cap 2^{\aleph_0} \,.$

[*100·42.*275·33.*152·71] $\supset .\, C^{\backprime\backprime}\theta = 2^{\aleph_0}$ (1)

$\vdash .\, \ast 275{\cdot}11 .\, \ast 263{\cdot}101 .\, \supset \vdash : \omega = \Lambda .\, \supset .\, \theta = \Lambda$ (2)

$\vdash .\, \ast 263{\cdot}101 .\, \ast 116{\cdot}204 .\, \supset \vdash : \omega = \Lambda .\, \supset .\, 2^{\aleph_0} = \Lambda$ (3)

$\vdash .\, (2) .\, (3) .\quad \supset \vdash : \omega = \Lambda .\, \supset .\, C^{\backprime\backprime}\theta = 2^{\aleph_0}$ (4)

$\vdash .\, (1) .\, (4) .\, \supset \vdash .\, \mathrm{Prop}$

The propositions proved in the present number are capable of being to some extent generalized. Also we can prove

$$\vdash .\, \theta = (\omega \,\mathrm{exp}_r\, \omega) \dotplus \dot{1}.$$

For this purpose, we prove first that if P, Q are well-ordered series, P^Q is Dedekindian (except that if $\sim \mathrm{E} ! B`\breve{P}$, P^Q has no last term); *i.e.* we prove

$$P, Q \epsilon \Omega \,.\, \supset : \lambda \subset C`P^Q \,.\, \exists ! \lambda \,.\, \supset_\lambda \,.\, \mathrm{E} ! \operatorname{limin}(P^Q)`\lambda.$$

For this purpose, assuming $\lambda \subset C`P^Q \,.\, \exists ! \lambda$, put

$$Q_m`\lambda = \min_Q`\hat{y}\,(\dot{s}`\overrightarrow{\lambda}`y \sim \epsilon\, 0 \cup 1),$$
$$\breve{T}_P`\lambda = \lambda \cap \hat{M}\,\{M`Q_m`\lambda = \min_P`\dot{s}`\overrightarrow{\lambda}`Q_m`\lambda\},$$
$$A = \hat{\lambda}\hat{\mu}\,(\mu \subset \lambda \,.\, \mu \neq \lambda),$$
$$(PQ)`\lambda = \dot{s}`\hat{N}\,\{(\exists\mu) \,.\, \mu \,\epsilon\, (T_P * A)`\lambda \,.\, N = (\dot{p}`\mu) \upharpoonright \overrightarrow{Q}`Q_m`\mu\}.$$

We can then show, by steps closely analogous to those in the proof of $P_\theta \epsilon \mathrm{Ded}$, that we have

$$\exists ! p`(T_P * A)`\lambda \,.\, \supset \,.\, \breve{\iota}`p`(T_P * A)`\lambda = \min(P^Q)`\lambda,$$
$$\sim \exists ! p`(T_P * A)`\lambda \,.\, \supset \,.\, (PQ)`\lambda = \operatorname{prec}(P^Q)`\lambda,$$

whence, in either case, $\mathrm{E} ! \operatorname{limin}(P^Q)`\lambda$.

Hence we have

$$\vdash : P, Q \epsilon \Omega \,.\, \mathrm{E} ! B`\breve{P} \,.\, \supset \,.\, P^Q \epsilon \mathrm{Ded},$$
$$\vdash : P, Q \epsilon \Omega \,.\, \sim \mathrm{E} ! B`\breve{P} \,.\, Z \sim \epsilon\, C`P^Q \,.\, \supset \,.\, P^Q \dashrightarrow Z \,\epsilon\, \mathrm{Ded}.$$

We have therefore $\quad \vdash . (\omega \exp_r \omega) \dot{+} \dot{1} \subset \mathrm{Ded}.$

We now have to prove

$$Q \,\epsilon\, (\omega \exp_r \omega) \dot{+} \dot{1} \,.\, \supset \,.\, \exists ! \aleph_0 \cap \overrightarrow{\mathrm{med}}`Q.$$

For this purpose, it will be sufficient to prove

$$P \,\epsilon\, \omega \,.\, \supset \,.\, \exists ! \aleph_0 \cap \overrightarrow{\mathrm{med}}`(P^P).$$

The $\aleph_0$ in question will be the class of those members of $C`(P^P)$ in which, from a certain point onward, the correlate of every member of $C`P$ is $B`P$. We have

$$M(P^P)N \,.\, \equiv : M, N \,\epsilon\, (C`P \uparrow C`P)_\Delta`C`P :$$
$$(\exists x) \,.\, x \,\epsilon\, C`P \,.\, M \upharpoonright \overrightarrow{P}`x = N \upharpoonright \overrightarrow{P}`x \,.\, (M`x)\, P\, (N`x).$$

Now consider the relation

$$L = M \upharpoonright \overrightarrow{P_*}`x \cup y \downarrow \breve{P}_1`x \cup (\iota`B`P) \uparrow \overleftarrow{P}`\breve{P}_1`x,$$

where $(M`\breve{P}_1`x)\, Py$.

Then $M(P^P)L \,.\, L(P^P)N$. Also L has $B`P$ for the correlate of every term after $\breve{P}_1`x$. Hence it is determined by the correlates of the terms up to and including $\breve{P}_1`x$. Thus, putting $z = \breve{P}_1`x$, we have to consider the class of relations

$$\mu = \hat{X}\,\{(\exists z) \,.\, z \,\epsilon\, \mathrm{G}`P \,.\, X \,\epsilon\, 1 \rightarrow \mathrm{Cls} \,.\, \mathrm{G}`X = \overrightarrow{P_*}`z \,.\, \mathrm{D}`X \subset C`P\}.$$

If $X \epsilon \mu$, $X \cup (\iota`B`P) \uparrow \overleftarrow{P}`\max_P`\mathrm{C\!\!\!I}`X$ is a member of $C`P^P$. We have therefore only to show that $\mu \epsilon \aleph_0$.

To show that $\mu \epsilon \aleph_0$, we observe that if $X \epsilon \mu$, $\mathrm{D}`X$ and $\mathrm{C\!\!\!I}`X$ are both inductive classes; hence each has a maximum. Let X and X' be two members of μ, and let us put

$$x = \max_P`\mathrm{D}`X \,.\, x' = \max_P`\mathrm{D}`X' \,.\, y = \max_P`\mathrm{C\!\!\!I}`X \,.\, y' = \max_P`\mathrm{C\!\!\!I}`X'.$$

If $x = \mu_P$ and $y = \nu_P$, put $x +_P y = (\mu +_c \nu)_P$. Then put X before X' if $(x +_P y) P (x' +_P y')$, or if $x +_P y = x' +_P y' \,.\, yPy'$. But if $x +_P y = x' +_P y'$ and $y = y'$, *i.e.* if $x = x' \,.\, y = y'$, take the immediate predecessors of x, y, x', y' in $\mathrm{D}`X$, $\mathrm{C\!\!\!I}`X$, $\mathrm{D}`X'$, $\mathrm{C\!\!\!I}`X'$ respectively, and apply the same tests to them, and so on, until we come to a difference. In this way, we obtain an arrangement by last differences (in a slightly extended sense), and this arrangement is easily shown to be an ω. Hence $\mu \epsilon \aleph_0$. Hence the class

$$\nu = \hat{\gamma}\{(\exists X) \,.\, X \epsilon \mu \,.\, \gamma = X \cup (\iota`B`P) \uparrow \overleftarrow{P}`\max_P`\mathrm{C\!\!\!I}`X\}$$

is an $\aleph_0$, and we have already shown that it is a median class of $C`P^P$. Hence

$$\vdash : P \epsilon \omega \,.\, \supset \,.\, \exists ! \aleph_0 \cap \overrightarrow{\mathrm{med}}`(P^P).$$

The same class will be a median class of $P^P \not\mapsto Z$, if $Z \sim \epsilon C`P^P$. Hence

$$\vdash : P \epsilon \omega \,.\, Z \sim \epsilon C`P^P \,.\, \supset \,.\, \exists ! \aleph_0 \cap \overrightarrow{\mathrm{med}}`(P^P \not\mapsto Z).$$

Hence, by what was proved earlier,

$$\vdash : P \epsilon \omega \,.\, Z \sim \epsilon C`P^P \,.\, \supset \,.\, (P^P \not\mapsto Z) \epsilon \theta,$$

i.e. $\vdash \,.\, (\omega \exp_r \omega) \dot{+} \dot{1} = \theta.$

PART VI.

QUANTITY.

SUMMARY OF PART VI.

THE purpose of this Part is to explain the kinds of applications of numbers which may be called *measurement*. For this purpose, we have first to consider generalizations of number. The numbers dealt with hitherto have been only integers (cardinal or ordinal); accordingly, in Section A, we consider positive and negative integers, ratios, and real numbers. (Complex numbers are dealt with later, under geometry, because they do not form a one-dimensional series.)

In Section B, we deal with what may be called "kinds" of quantity: thus *e.g.* masses, spatial distances, velocities, each form one kind of quantity. We consider each kind of quantity as what may be called a "vector-family," *i.e.* a class of one-one relations all having the same converse domain, and all having their domain contained in their converse domain. In such a case as spatial distances, the applicability of this view is obvious; in such a case as masses, the view becomes applicable by considering *e.g.* one gramme as + one gramme, *i.e.* as the relation of a mass m to a mass m' when m exceeds m' by one gramme. What is commonly called simply one gramme will then be the mass which has the relation + one gramme to the zero of mass. The reasons for treating quantities as vectors will be explained in Section B. Various different kinds of vector-families will be considered, the object being to obtain families whose members are capable of measurement either by means of ratios or by means of real numbers.

Section C is concerned with measurement, *i.e.* with the discovery of ratios, or of the relations expressed by real numbers, between the members of a vector-family. A family of vectors is measurable if it contains a member T (the unit) such that any other member S has to T a relation which is either a ratio or a real number. It will be shown that certain sorts of vector-families are in this sense measurable, and that measurement so defined has the mathematical properties which we expect it to possess.

Section D deals with *cyclic* families of vectors, such as angles or elliptic straight lines. The theory of measurement as applied to such families presents peculiar features, owing to the fact that any number of complete revolutions may be added to a vector without altering it. Thus there is not a single ratio of two vectors, but many ratios, of which we select one as the *principal* ratio.

SECTION A.

GENERALIZATION OF NUMBER.

Summary of Section A.

In this section, we first define the series of positive and negative integers. If μ is a cardinal, the corresponding positive and negative integers are the relations $+_c \mu$ and $-_c \mu$, or rather $(+_c \mu) \upharpoonright (\text{NC induct} - \iota`\Lambda)$ and $(-_c \mu) \upharpoonright (\text{NC induct} - \iota`\Lambda)$. (It will be observed that a positive integer must not be confounded with the corresponding signless integer, for while the former is a relation, the latter is a class of classes.) We next proceed to numerically-defined powers of relations, *i.e.* to R^ν, where ν is an inductive cardinal. We have already defined R^2 and R^3, but for the definition of ratio it is important to define R^ν generally. If $R \in 1 \to 1 \,.\, R_{po} \subset J$, we shall have $R^\nu = R_\nu$, and if $R \in \text{Ser}$, we shall have $(R_1)^\nu = R_\nu$. But these equations do not hold in general, and in particular if $R \subset I$ and $\nu \neq 0$, $R^\nu = R$ but $R_\nu = \dot{\Lambda}$. After a number devoted to relative primes, we proceed to the definition of signless ratios, thence to the multiplication and addition of signless ratios, thence to negative ratios, and thence to the generalized addition and multiplication which includes negative ratios. (In the case of ratios, signless ratios are identical with positive ratios. This is possible because signless ratios, unlike signless integers, are already relations.) We then proceed to the definition of real numbers, positive and negative, and to the addition and multiplication of real numbers. At each stage, we prove the commutative, associative, and distributive laws, and whatever else may seem necessary, for the particular kind of addition and multiplication in question.

Great difficulties are caused, in this section, by the existence-theorems and the question of types. These difficulties disappear if the axiom of infinity is assumed, but it seems improper to make the theory of (say) 2/3 depend upon the assumption that the number of objects in the universe is not finite. We have, accordingly, taken pains not to make this assumption, except where, as in the theory of real numbers, it is really essential, and not merely convenient. When the axiom of infinity is required, it is always explicitly stated in the hypothesis, so that our propositions, as enunciated, are true even if the axiom of infinity is false.

$*300$. POSITIVE AND NEGATIVE INTEGERS, AND NUMERICAL RELATIONS.

Summary of $*300$.

In this number, we introduce three definitions. We first define "U" as meaning the relation which holds between $\mu +_c \nu$ and μ whenever μ and ν are existent inductive cardinals of the same type, and $\nu \neq 0$, and $\mu +_c \nu$ exists in this type. Thus U is the relation "greater than" confined to existent inductive cardinals of the same type. The definition is:

$*300{\cdot}01$. $U = (+_c 1)_{po} \upharpoonright (\text{NC induct} - \iota`\Lambda)$ Df

Then if μ is an inductive cardinal which exists in the type in question, U_μ and $\breve{U}_\mu$ are the corresponding positive and negative integers, where "U_μ" has the meaning defined in $*121$. It will be observed that $0 U_\mu \mu$, so that U_μ exists, when μ exists in the type in question. We prove ($*300{\cdot}15$) that U is a series, and ($*300{\cdot}14$) that its field consists of all existent inductive cardinals of the type in question, its domain consists of all its field except 0, and its converse domain of all its field except the greatest (if any). If the axiom of infinity holds, $C`U$ consists of all inductive cardinals.

It will be observed that U arranges the inductive cardinals in *descending* order of magnitude. The reason for choosing this order instead of the converse is that U is less required in its serial use than as leading to the functional relations U_μ. As explained at the end of Part I, Section D, there is a broad difference between functional and serial relations, and this produces, where one relation (or its derivatives) is to have both uses, a certain conflict of convenience as to the *sense* in which the relation is to be taken. Considered as arranging the integers in a series, U would naturally be defined so as to arrange them in *ascending* order of magnitude, as was done with "N" in $*123$. But considered as functional relations, it is more convenient and more natural to take (say) $+_c 1$ as the relation to start with, and $-_c 1$ as its converse. Thus we want $\mu U_1 \nu$ when $\mu = \nu +_c 1$, *i.e.* we want $U_1`\nu = \nu +_c 1$; and this requires the definition of U given above.

We prove in this number ($*300{\cdot}23$) that $\breve{U}$ is well-ordered, and ($*300{\cdot}21{\cdot}22$) is either finite or a progression. We also prove ($*300{\cdot}17{\cdot}18$) that, if μ is any

typically indefinite inductive cardinal, μ and $\mu +_c 1$ will belong to $C'U$ if U is taken in a sufficiently high type.

Our other two definitions in this number define two classes of relations which are of vital importance in the theory of ratio. We define *numerical relations*, which are called "Rel num," as one-one relations whose powers are all contained in diversity, *i.e.* we put

∗300·02. $\text{Rel num} = (1 \to 1) \cap \hat{R}(\text{Pot}`R \subset \text{Rl}`J)$ Df

We thus have (∗300·3)

$$\vdash : R \in \text{Rel num} . \equiv . R \in 1 \to 1 . R_{po} \subseteq J.$$

It will be remembered that the hypothesis $R \in (\text{Cls} \to 1) \cup (1 \to \text{Cls}) . R_{po} \subseteq J$ played a great part in ∗121, and in all later work which depended upon ∗121. When both R and $\breve{R}$ fulfil this hypothesis, we have $R \in \text{Rel num}$, and vice versa. We prove (∗300·44) that if σ is an inductive cardinal not zero, and P is a series, then P_σ is a numerical relation, and so is $\breve{P}_\sigma$. If P is an endless well-ordered series, finid$`\breve{P}$ (*i.e.* the class of relations $\breve{P}_\sigma$) is what (in Section B) we shall call a vector-family: $\breve{P}_\sigma$ is the vector which carries a term σ steps along the series.

In order to be able to deal with zero, we have to consider the application of ratios, not only to such relations as are numerical in the above sense, but also to relations contained in identity, because a relation contained in identity may be regarded as a zero vector, so that (*e.g.*) if P is a series, $I \upharpoonright C`P$ will have a zero ratio to P_σ if σ is an inductive cardinal other than 0.

We therefore introduce a class "Rel num id" consisting of numerical relations together with such as are contained in identity: these may be called *numerical or identical relations*. They may be defined as one-one relations whose powers, other than R_0, are contained in diversity, because, if $R \subseteq I$, there are no powers other than R_0. Thus we put

∗300·03. $\text{Rel num id} = (1 \to 1) \cap \hat{R}(\text{Potid}`R - \iota`R_0 \subset \text{Rl}`J)$ Df

and we then prove

∗300·33. $\vdash . \text{Rel num id} = \text{Rl}`I \cup \text{Rel num}$

For the application of ratio, it is important to know under what circumstances there exists a numerical relation R such that R_σ is not null. We prove (∗300·45) that if σ is an inductive cardinal, and P is a series of $\sigma +_c 1$ terms, then $(B`P) P_\sigma (B`\breve{P})$. Now we also prove (∗300·44) that if P is a series, and $R = P_1$, $P_\sigma = R_\sigma$ and R is a numerical relation. Hence it follows, by ∗262·211, that if $\sigma \neq 0$ and α is a class of $\sigma +_c 1$ terms, there is

a numerical relation R whose field is of the same type as α and for which R_σ exists. Remembering *300·14 (quoted above), this proposition is:

***300·46.** $\vdash : \sigma \in \text{G}'U - \iota'0 \,.\supset.$

$(\exists P, R) \,.\, P \in (\sigma +_c 1)_r \,.\, R = P_1 \,.\, R \in \text{Rel num} \,.\, t'C'R = t_0'\sigma \,.\, (B'R) \, R_\sigma \, (B'\breve{R})$

We have conversely (*300·47)

$\vdash : R \in \text{Rel num} \,.\, \dot{\exists} \,!\, R_\sigma \,.\supset.\, \sigma \in \text{NC ind} \,.\, \exists \,!\, (\sigma +_c 1) \cap t'C'R \,.\, \sigma \cap t'C'R \in \text{G}'U,$

where "NC ind" has the meaning defined in *126, *i.e.* "$\sigma \in$ NC ind" means that σ is a typically indefinite cardinal.

The number ends by propositions proving (*300·52) that U_μ is a numerical relation, that (*300·57)

$$\dot{\exists} \,!\, (U_\xi)_\nu \,\dot{\cap}\, (U_\eta)_\mu \,.\supset.\, \xi \times_c \nu \in C'U \,.\, \xi \times_c \nu = \eta \times_c \mu,$$

and analogous theorems.

***300·01.** $U = (+_c 1)_{po} \upharpoonright (\text{NC induct} - \iota'\Lambda)$ Df

***300·02.** $\text{Rel num} = (1 \rightarrow 1) \cap \hat{R}\,(\text{Pot}'R \subset \text{Rl}'J)$ Df

***300·03.** $\text{Rel num id} = (1 \rightarrow 1) \cap \hat{R}\,(\text{Potid}'R - \iota'R_0 \subset \text{Rl}'J)$ Df

***300·1.** $\vdash : \mu U \nu \,.\equiv.\, \mu \,(+_c 1)_{po}\, \nu \,.\, \mu, \nu \in \text{NC induct} - \iota'\Lambda$ [(*300·01)]

***300·11.** $\vdash :. \mu U \nu \,.\equiv:$

$\mu, \nu \in \text{NC induct} - \iota'\Lambda : (\exists \lambda) \,.\, \lambda \in \text{NC induct} - \iota'0 \,.\, \mu = \nu +_c \lambda :$

$\equiv : \mu, \nu \in \text{NC induct} - \iota'\Lambda : (\exists \lambda) \,.\, \lambda \neq 0 \,.\, \mu = \nu +_c \lambda :$

$\equiv : \mu, \nu \in \text{NC induct} - \iota'\Lambda : (\exists \lambda) \,.\, \lambda \in \text{NC} - \iota'0 \,.\, \mu = \nu +_c \lambda$

[*300·1 . *120·42·428·462·452 . *110·4]

***300·12.** $\vdash : \mu U \nu \,.\equiv.\, \mu, \nu \in \text{NC induct} - \iota'\Lambda \,.\, \nu < \mu \,.$

$\equiv .\, \mu, \nu \in \text{NC induct} \,.\, \nu < \mu \,.$

$\equiv .\, \mu \in \text{NC induct} \,.\, \nu < \mu$

[*300·11 . *117·3 . *120·42 . *117·26 . *110·6 . *117·15 . *120·48]

***300·13.** $\vdash .\, U \subseteq J$ [*300·12 . *117·42]

***300·14.** $\vdash .\, C'U = \text{NC induct} - \iota'\Lambda \,.\, \text{D}'U = \text{NC induct} - \iota'\Lambda - \iota'0 \,.$

$\text{G}'U = \text{NC induct} \cap \hat{\nu}\,(\exists \,!\, \nu +_c 1) = \hat{\nu}\,(\nu +_c 1 \in \text{NC induct} - \iota'\Lambda) \,.$

$B'\breve{U} = 0$

[*300·12 . *117·511 . *120·122 . *101·241 . *120·429·422]

***300·15.** $\vdash .\, U \in \text{Ser}$ [*300·13 . *120·441]

***300·16.** $\vdash : \alpha \in \text{Cls induct} \,.\supset.\, \text{N}_0\text{c}'\alpha \in C'U \cap t^{2}{}'\alpha \,.\, \text{N}_0\text{c}'\alpha \in C'(U \upharpoonright t^{2}{}'\alpha)$

Dem.

$\vdash .\, *120{\cdot}21 \,.\supset \vdash : \text{Hp} \,.\supset.\, \text{N}_0\text{c}'\alpha \in \text{NC induct}$ (1)

$\vdash .\, *103{\cdot}13 \,.\supset \vdash .\, \text{N}_0\text{c}'\alpha \neq \Lambda$ (2)

$\vdash .\, *103{\cdot}11 \,.\supset \vdash .\, \text{N}_0\text{c}'\alpha \in t^{2}{}'\alpha$ (3)

$\vdash .\, (1) \,.\, (2) \,.\, (3) \,.\, *300{\cdot}14 \,.\supset \vdash .\, \text{Prop}$

$*300 \cdot 17$. $\vdash : \mu \in \text{NC ind} . \supset . (\exists \alpha) . \mu \cap t`\alpha \in C`U . \mu \in C`(U \upharpoonright t^{2}`\alpha)$

Dem.

$\vdash . *126 \cdot 1 . \supset \vdash : \text{Hp} . \supset . (\exists \alpha) . \alpha \in \text{Cls induct} . \mu = \text{Nc}`\alpha . \exists ! \mu .$

$[*103 \cdot 34] \quad \supset . (\exists \alpha) . \alpha \in \text{Cls induct} . \mu \cap t`\alpha = \text{N}_0\text{c}`\alpha \quad (1)$

$\vdash . (1) . *300 \cdot 16 . \supset \vdash : \text{Hp} . \supset . (\exists \alpha) . \mu \cap t`\alpha \in C`U . \quad (2)$

$[*65 \cdot 13] \quad \supset . (\exists \alpha) . \mu \in C`U . \mu \subset t`\alpha .$

$[*63 \cdot 5] \quad \supset . (\exists \alpha) . \mu \in C`U . \mu \in t^{2}`\alpha \quad (3)$

$\vdash . (2) . (3) . \supset \vdash . \text{Prop}$

$*300 \cdot 18$. $\vdash : \mu \in \text{NC ind} . \supset .$
$(\exists \sigma) . 2^{\mu} \in C`(U \upharpoonright t^{2}`\sigma) . (\mu +_c 1) \cap t`\sigma \in C`U . \mu \in \text{Cl}`(U \upharpoonright t^{2}`\sigma)$

$[*126 \cdot 13 \cdot 15 . *300 \cdot 17 \cdot 14]$

$*300 \cdot 181$. $\vdash : \mu \in \text{NC ind} . \mu \cap t`\alpha \in C`U . \supset .$
$2^{\mu} \cap t^{2}`\alpha \in C`U . (\mu +_c 1) \cap t^{2}`\alpha \in C`U . \mu \cap t^{2}`\alpha \in \text{Cl}`U$

$[*126 \cdot 23 . *300 \cdot 14]$

$*300 \cdot 2$. $\vdash : \text{Infin ax} . \supset . \breve{U} = \dot{N}_{\text{po}}$

Here $\dot{N}$ has the meaning defined in $*263 \cdot 02$.

Dem.

$\vdash . *300 \cdot 1 . *125 \cdot 1 . \supset \vdash :. \text{Hp} . \supset : \mu U \nu . \equiv . \mu, \nu \in \text{NC induct} . \mu \, (+_c 1)_{\text{po}} \, \nu .$

$[*120 \cdot 1 . *91 \cdot 574] \quad \equiv . \nu \, (+_c 1)_{*} \, 0 . \mu \, (+_c 1)_{\text{po}} \, \nu .$

$[*96 \cdot 13] \quad \equiv . \mu \, \{(+_c 1) \upharpoonright \overrightarrow{(+_c 1)_{*}}`0\}_{\text{po}} \, \nu .$

$[(*263 \cdot 02 . *120 \cdot 01)] \quad \equiv . \nu \dot{N}_{\text{po}} \mu :. \supset \vdash . \text{Prop}$

$*300 \cdot 21$. $\vdash : \text{Infin ax} . \supset . \breve{U} \in \omega \quad [*300 \cdot 2 . *263 \cdot 12]$

$*300 \cdot 22$. $\vdash : \sim \text{Infin ax} . \supset . \breve{U} \in \Omega \text{ induct}$

Dem.

$\vdash . *125 \cdot 16 \cdot 24 . \text{Transp} . \supset \vdash : \text{Hp} . \supset . C`U \in \text{Cls induct} \quad (1)$

$\vdash . (1) . *300 \cdot 15 . *261 \cdot 32 . \supset \vdash . \text{Prop}$

$*300 \cdot 23$. $\vdash . \breve{U} \in \Omega \quad [*300 \cdot 21 \cdot 22]$

$*300 \cdot 231$. $\vdash : \mu U_1 \nu . \equiv . \mu, \nu \in \text{NC induct} - \iota`\Lambda . \mu = \nu +_c 1 .$
$\equiv . \mu \in \text{NC induct} - \iota`\Lambda . \mu = \nu +_c 1 .$
$\equiv . \mu \in \text{NC induct} - \iota`\Lambda - \iota`0 . \nu = \mu -_c 1 .$
$\equiv . \nu \in \text{NC induct} - \iota`\Lambda . \nu = \mu -_c 1$

Dem.

$\vdash . *300 \cdot 15 \cdot 12 . *201 \cdot 63 . \supset$

$\vdash :. \mu U_1 \nu . \equiv : \mu, \nu \in \text{NC induct} - \iota`\Lambda . \nu < \mu : \sim (\exists \lambda) . \nu < \lambda . \lambda < \mu :$

$[*120 \cdot 429] \equiv : \mu, \nu \in \text{NC induct} - \iota`\Lambda . \nu < \mu : \nu +_c 1 \geqslant \mu . \mu \geqslant \nu +_c 1 :$

$[*117 \cdot 25] \quad \equiv : \mu, \nu \in \text{NC induct} - \iota`\Lambda . \mu = \nu +_c 1 \quad (1)$

$\vdash . (1) . *120 \cdot 422 \cdot 424 \cdot 423 . \supset \vdash . \text{Prop}$

***300·232**. $\vdash : \mu \epsilon \text{NC induct} . \supset .$

$$U_\mu = (+_c \mu) \upharpoonright (\text{NC induct} - \iota`\Lambda) . \breve{U}_\mu = (-_c \mu) \upharpoonright (\text{NC induct} - \iota`\Lambda)$$

For the definition of U_μ, see *121·02.

Dem.

$\vdash . *121·302 . *300·15 . \supset \vdash : \rho U_0 \sigma . \equiv . \sigma \epsilon C`U . \rho = \sigma .$

$[*300·14.*110·6] \quad \equiv . \rho, \sigma \epsilon \text{NC induct} - \iota`\Lambda . \rho = \sigma +_c 0 \quad (1)$

$\vdash . *260·22·28 . *121·332 . \supset$

$\vdash : U_\mu = (+_c \mu) \upharpoonright (\text{NC induct} - \iota`\Lambda) . \supset . U_{\mu +_c 1} = (+_c \mu) \upharpoonright (\text{NC induct} - \iota`\Lambda) | U_1$

$[*300·231] \quad = (+_c \mu) \upharpoonright (\text{NC induct} - \iota`\Lambda) | (+_c 1) \upharpoonright (\text{NC induct} - \iota`\Lambda)$

$[*120·45·452] \quad = \{+_c (\mu +_c 1)\} \upharpoonright (\text{NC induct} - \iota`\Lambda) \quad (2)$

$\vdash . (1) . (2) . \text{Induct} . \supset \vdash . \text{Prop}$

***300·24**. $\vdash : \mu \epsilon \text{NC induct} . \supset . \text{D}`U_\mu = \overrightarrow{U_*}`\mu = \text{NC induct} \cap \hat{\nu} (\nu \geqslant \mu)$
$[*300·232 . *117·31 . *120·45]$

***300·25**. $\vdash : \mu \epsilon \text{NC induct} . \supset .$

$$\overrightarrow{B}`\breve{U}_\mu = \overleftarrow{\dot{U}}`\mu = \text{NC induct} \cap \hat{\nu} (\nu < \mu) = \breve{U} (0 \vdash \mu)$$

$[*300·232·24·12]$

***300·26**. $\vdash : \mu \epsilon C`U . \equiv . \mu U_\mu 0 . \equiv . \dot{\exists} ! U_\mu \upharpoonright (C`U) \quad [*300·232·14 . *110·6]$

Here the μ in "U_μ" is of higher type than the μ in "$\mu \epsilon C`U$," because the interval $U(0 \mapsto \mu)$ is composed of members each of which is of the same type as μ.

***300·3**. $\vdash : R \epsilon \text{Rel num} . \equiv . R \epsilon 1 \to 1 . R_{\text{po}} \mathrel{\dot{\subset}} J . \equiv . R \epsilon 1 \to 1 . \text{Pot}`R \subset \text{Rl}`J$
$[(*300·02)]$

***300·31**. $\vdash : R \epsilon \text{Rel num id} . \equiv . R \epsilon 1 \to 1 . \text{Potid}`R - \iota`R_0 \subset \text{Rl}`J$
$[(*300·03)]$

***300·311**. $\vdash : R \mathrel{\dot{\subset}} I . \equiv . R_0 = R . \equiv . R = I \upharpoonright C`R$

Dem.

$\vdash . *201·13·18 . \supset \vdash :. R \mathrel{\dot{\subset}} I . \supset : x \epsilon C`R . \supset . \overleftarrow{R_*}`x \cap \overrightarrow{R_*}`x = \iota`x \quad (1)$

$\vdash . (1) . *121·11 . \supset \vdash : R \mathrel{\dot{\subset}} I . \supset . I \upharpoonright C`R \mathrel{\dot{\subset}} R_0 .$

$[*121·3] \quad \supset . R_0 = I \upharpoonright C`R .$

$[*72·92] \quad \supset . R_0 = R = I \upharpoonright C`R \quad (2)$

$\vdash . *121·3 . \quad \supset \vdash : R_0 = R . \supset . R \mathrel{\dot{\subset}} I \quad (3)$

$\vdash . (2) . (3) . \supset \vdash . \text{Prop}$

***300·312**. $\vdash : R \mathrel{\dot{\subset}} I . \supset . \text{Potid}`R = \iota`R = \iota`R_0 \quad [*300·311 . *50·72 . \text{Induct}]$

***300·313**. $\vdash : R \epsilon \text{Rel num id} . \supset . R_* \mathrel{\dot{-}} R_0 \mathrel{\dot{\subset}} J \quad [*300·31 . *91·55]$

***300·32.** $\vdash : R \in \text{Rel num id} \,.\, \supset \,.\, R_0 = I \upharpoonright C`R$

Dem.

$\vdash \,.\, *91{\cdot}35 \,.\, \supset \vdash \,.\, I \upharpoonright C`R \in \text{Potid}`R - \text{Rl ex}`J$ (1)

$\vdash \,.\, (1) \,.\, *300{\cdot}31 \,.\, \supset \vdash \,.\, \text{Prop}$

***300·321.** $\vdash : R \in \text{Rel num id} \,.\, R \neq R_0 \,.\, \supset \,.\, R \subseteq J \,.\, \dot{\exists}\, ! \, R$ [*300·31]

***300·322.** $\vdash : R \subseteq J \,.\, \supset \,.\, R_{po} \,\dot{\cap}\, R_0 = \dot{\Lambda}$

Dem.

$\vdash \,.\, *121{\cdot}3 \,.\, \supset \vdash : x R_{po} y \,.\, x \neq y \,.\, \supset \,.\, \sim (x R_0 y)$ (1)

$\vdash \,.\, *50{\cdot}24 \,.\, \supset \vdash :.\, \text{Hp} \,.\, \supset : \sim (xRx) :$ (2)

[*91·57] $\supset : x R_{po} x \,.\, \supset \,.\, x\, (R_{po} \,|\, R)\, x \,.$

[*121·103.(2)] $\supset \,.\, R\,(x \mapsto x) \neq \iota`x \,.$

[*121·11] $\supset \,.\, \sim (x R_0 x)$ (3)

$\vdash \,.\, (1) \,.\, (3) \,.\, \supset \vdash \,.\, \text{Prop}$

***300·323.** $\vdash : R \in \text{Rel num id} \,.\, R \neq R_0 \,.\, \supset \,.\, R_{po} \subseteq J$

Dem.

$\vdash \,.\, *300{\cdot}321{\cdot}322 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, R_{po} \,\dot{\cap}\, R_0 = \dot{\Lambda} \,.$

[*300·32] $\supset \,.\, R_{po} \,\dot{\cap}\, I \upharpoonright C`R = \dot{\Lambda} : \supset \vdash \,.\, \text{Prop}$

***300·324.** $\vdash :.\, R \in \text{Rel num id} \,.\, \supset : R \subseteq I \,.\, \mathsf{v} \,.\, R \in \text{Rel num}$

Dem.

$\vdash \,.\, *300{\cdot}311{\cdot}323 \,.\, \supset \vdash :.\, \text{Hp} \,.\, \supset : R \subseteq I \,.\, \mathsf{v} \,.\, R_{po} \subseteq J$ (1)

$\vdash \,.\, *300{\cdot}32 \,.\, \supset \vdash : R \in \text{Rel num id} \,.\, R_{po} \subseteq J \,.\, \supset \,.\, \text{Potid}`R - \iota`R_0 = \text{Pot}`R$ (2)

$\vdash \,.\, (2) \,.\, *300{\cdot}31 \,.\, \supset \vdash : R \in \text{Rel num id} \,.\, R_{po} \subseteq J \,.\, \supset \,.\, \text{Pot}`R \subset \text{Rl}`J$ (3)

$\vdash \,.\, (1) \,.\, (3) \,.\, *300{\cdot}3 \,.\, \supset \vdash \,.\, \text{Prop}$

***300·325.** $\vdash : R \subseteq I \,.\, \supset \,.\, R \in \text{Rel num id}$

Dem.

$\vdash \,.\, *300{\cdot}312 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, \text{Potid}`R - \iota`R_0 = \Lambda$ (1)

$\vdash \,.\, (1) \,.\, *300{\cdot}31 \,.\, \supset \vdash \,.\, \text{Prop}$

***300·326.** $\vdash : R \in \text{Rel num} \,.\, \supset \,.\, R \in \text{Rel num id}$

Dem.

$\vdash \,.\, *121{\cdot}3 \,.\, *300{\cdot}3 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, R_0 \sim\in \text{Pot}`R$ (1)

$\vdash \,.\, *121{\cdot}302 \,.\, *300{\cdot}3 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, R_0 = I \upharpoonright C`R$ (2)

$\vdash \,.\, (1) \,.\, (2) \,.\, *91{\cdot}35 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, \text{Potid}`R - \iota`R_0 = \text{Pot}`R$ (3)

$\vdash \,.\, (3) \,.\, *300{\cdot}3{\cdot}31 \,.\, \supset \vdash \,.\, \text{Prop}$

***300·33.** $\vdash \,.\, \text{Rel num id} = \text{Rl}`I \cup \text{Rel num}$ [*300·324·325·326]

***300·34.** $\vdash \,.\, \dot{\Lambda} \in \text{Rel num}$ [*300·3 . *72·1]

***300·4.** $\vdash \,.\, \text{Rel num} = \text{Cnv}``\text{Rel num}$ [*300·3 . *91·522]

***300·41.** $\vdash \,.\, \text{Rel num id} = \text{Cnv}``\text{Rel num id}$ [*300·31 . *91·521]

***300·42.** $\vdash : R \in \text{Rel num} . \supset . \text{Pot}`R \subset \text{Rel num}$

Dem.

$\vdash . *91·6 . *92·102 . \supset$

$\vdash : R \in \text{Rel num} . P \in \text{Pot}`R . \supset . P \in 1 \rightarrow 1 . \text{Pot}`P \subset \text{Rl}`J .$

$[*300·3] \qquad \supset . P \in \text{Rel num} : \supset \vdash . \text{Prop}$

***300·43.** $\vdash : R \in \text{Rel num id} . \supset . \text{Potid}`R \subset \text{Rel num id}$

Dem.

$\vdash . *300·325·312 . \quad \supset \vdash : R \mathrel{G} I . \supset . \text{Potid}`R \subset \text{Rel num id} \qquad (1)$

$\vdash . *300·325 . \quad \supset \vdash . I \upharpoonright C`R \in \text{Rel num id} \qquad (2)$

$\vdash . (2) . *300·42·326 . \supset \vdash : R \in \text{Rel num} . \supset . \text{Potid}`R \subset \text{Rel num id} \quad (3)$

$\vdash . (1) . (3) . *300·33 . \supset \vdash . \text{Prop}$

***300·44.** $\vdash :. P \in \text{Ser} . \sigma \in \text{NC ind} . \supset :$

$$P_\sigma, \breve{P}_\sigma \in \text{Rel num id} : \sigma \neq 0 . \supset . P_\sigma = (P_1)_\sigma . P_\sigma, \breve{P}_\sigma \in \text{Rel num}$$

Dem.

$\vdash . *121·302 . *300·325 . \supset \vdash : \text{Hp} . \sigma = 0 . \supset . P_\sigma, \breve{P}_\sigma \in \text{Rel num id} \quad (1)$

$\vdash . *260·28 . \quad \supset \vdash : \text{Hp} . \sigma \neq 0 . \supset . P_\sigma = (P_1)_\sigma \qquad (2)$

$\vdash . *300·3 . *260·22 . \quad \supset \vdash :. \text{Hp} . \supset : P_1 \in \text{Rel num} :$

$[*121·5 . *300·42] \qquad \supset : \sigma \neq 0 . \supset . (P_1)_\sigma \in \text{Rel num} .$

$[(2) . *300·4] \qquad \supset . P_\sigma, \breve{P}_\sigma \in \text{Rel num} \quad (3)$

$\vdash . (1) . (2) . (3) . \supset \vdash . \text{Prop}$

***300·45.** $\vdash : \sigma \in \text{NC ind} . P \in (\sigma +_c 1)_r . \supset . (B`P) P_\sigma (B`\breve{P})$

For the definition of $(\sigma +_c 1)_r$, see *262·03.

Dem.

$\vdash . *262·12 . \supset \vdash : \text{Hp} . \supset . P \in \Omega . C`P \in \sigma +_c 1 .$

$[*202·181 . *261·24] \quad \supset . (B`P) P_\sigma (B`\breve{P}) : \supset \vdash . \text{Prop}$

***300·46.** $\vdash : \sigma \in \text{Cl}`U - \iota`0 . \supset .$

$$(\exists P, R) . P \in (\sigma +_c 1)_r . R = P_1 . R \in \text{Rel num} . t`C`R = t_0`\sigma . (B`R) R_\sigma (B`\breve{R})$$

Dem.

$\vdash . *300·14 . \supset \vdash : \text{Hp} . \supset . (\exists \alpha) . \alpha \in \text{Cls induct} . t`\alpha = t_0`\sigma . \alpha \in \sigma +_c 1 .$

$[*262·211] \qquad \supset . (\exists P) . P \in (\sigma +_c 1)_r . t`C`P = t_0`\sigma .$

$[*300·45] \qquad \supset . (\exists P) . P \in (\sigma +_c 1)_r . t`C`P = t_0`\sigma . (B`P) P_\sigma (B`\breve{P}) .$

$[*300·44 . *261·22] \qquad \supset . (\exists P, R) . P \in (\sigma +_c 1)_r . R = P_1 . R \in \text{Rel num} .$

$$t`C`R = t_0`\sigma . (B`R) R_\sigma (B`\breve{R}) : \supset \vdash . \text{Prop}$$

***300·47.** $\vdash : R \in \text{Rel num} . \dot{\exists} ! R_\sigma . \supset .$

$\sigma \in \text{NC ind} . \exists ! (\sigma +_c 1) \cap t`C`R . \sigma \cap t`C`R \in \text{Cl}`U .$

Dem.

$\vdash . *121\cdot11 . \supset \vdash : \text{Hp} . \supset . (\exists x, y) . R(x \mapsto y) \in \sigma +_c 1 .$

$[*121\cdot46] \quad \supset . \sigma +_c 1 \in \text{NC ind} . \exists ! (\sigma +_c 1) \cap t`C`R .$

$[*120\cdot422 . *300\cdot14] \quad \supset . \sigma \in \text{NC ind} . \exists ! (\sigma +_c 1) \cap t`C`R .$

$\sigma \cap t`C`R \in \text{Cl}`U : \supset \vdash . \text{Prop}$

***300·48.** $\vdash : R \subseteq I . \nu \neq 0 . \supset . R_\nu = \dot{\Lambda}$

Dem.

$\vdash . *300\cdot312\cdot311 . *91\cdot55 . \supset \vdash : R \subseteq I . \supset . R_* = I \upharpoonright C`R \quad (1)$

$\vdash . (1) . *121\cdot103 . \supset \vdash : R \subseteq I . \supset . R(x \mapsto y) = C`R \cap \iota`x \cap \iota`y \quad (2)$

$\vdash . (2) . *121\cdot11 . \supset \vdash :. R \subseteq I . \supset : xR_\nu y . \equiv . C`R \cap \iota`x \cap \iota`y \in \nu +_c 1 .$

$[*117\cdot222] \quad \supset . \nu +_c 1 \leqslant \text{Nc}`\iota`x .$

$[*117\cdot54 . *120\cdot124] \quad \supset . \nu +_c 1 = 1 .$

$[*110\cdot641 . *120\cdot311] \quad \supset . \nu = 0 \quad (3)$

$\vdash . (3) . \text{Transp} . \supset \vdash . \text{Prop}$

***300·481.** $\vdash : R \in \text{Rel num id} . \nu \neq 0 . \supset . (R_0)_\nu = \dot{\Lambda} . (R_\nu)_0 \subseteq R_0$

Dem.

$\vdash . *300\cdot32\cdot48 . \supset \vdash : \text{Hp} . \supset . (R_0)_\nu = \dot{\Lambda} \quad (1)$

$\vdash . *300\cdot43\cdot32 . \supset \vdash : \text{Hp} . \supset . (R_\nu)_0 = I \upharpoonright C`R_\nu .$

$[*121\cdot322 . *300\cdot32] \quad \supset . (R_\nu)_0 \subseteq R_0 \quad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***300·49.** $\vdash : R \in \text{Rel num} . \dot{\Lambda} \sim\in \text{Pot}`R . \supset . C`R \sim\in \text{Cls induct}$

Dem.

$\vdash . *121\cdot5 . \supset \vdash :. \text{Hp} . \supset : \nu \in \text{NC induct} . \supset . \dot{\exists} ! R_\nu .$

$[*121\cdot11] \quad \supset . \exists ! (\nu +_c 1) \cap \text{Cl}`C`R :. \supset \vdash . \text{Prop}$

***300·491.** $\vdash : (\exists R) . R \in \text{Rel num} . \dot{\Lambda} \sim\in \text{Pot}`R . \supset . \text{Infin ax} \quad [*300\cdot49]$

***300·5.** $\vdash . U_1 \in \text{Rel num} \quad [*300\cdot15\cdot44]$

***300·51.** $\vdash . U_0 \in \text{Rel num id} \quad [*300\cdot15\cdot44]$

***300·511.** $\vdash . U_\sigma = (U_1)_\sigma \quad [*300\cdot21\cdot22 . *263\cdot491]$

***300·52.** $\vdash : \mu \in \text{NC ind} - \iota`0 . \supset . U_\mu \in \text{Rel num} \quad [*300\cdot15\cdot44]$

***300·53.** $\vdash . (\times_c 1) \upharpoonright C`U \in \text{Rel num id} \quad [*300\cdot325 . *113\cdot621]$

***300·54.** $\vdash : \text{Infin ax} . \mu \in \text{D}`U - \iota`1 . \supset . (\times_c \mu) \upharpoonright \text{D}`U \in \text{Rel num}$

Dem.

$\vdash . *120\cdot51 . \quad \supset \vdash : \text{Hp} . \supset . (\times_c \mu) \upharpoonright \text{D}`U \in 1 \to 1 \quad (1)$

$\vdash . *126\cdot51 . *113\cdot621 . \supset \vdash :. \text{Hp} . \supset : \rho \{(\times_c \mu) \upharpoonright \text{D}`U\} \sigma . \supset . \rho > \sigma :$

$[*117\cdot47\cdot42] \quad \supset : \{(\times_c \mu) \upharpoonright \text{D}`U\}_{\text{po}} \subseteq J \quad (2)$

$\vdash . (1) . (2) . *300\cdot3 . \supset \vdash . \text{Prop}$

***300·55.** $\vdash : \dot{\exists} ! R_\rho \dot{\wedge} R_\sigma . \supset . \exists ! (\rho +_c 1) \cap t'C'R . \rho = \sigma$ [*121·11 . *120·31]

***300·551.** $\vdash : \dot{\exists} ! R_\rho \dot{\wedge} R_\sigma . \equiv . \dot{\exists} ! R_\rho . \rho = \sigma$ [*300·55]

***300·552.** $\vdash : R \in \text{Rel num} . \supset . (R_\xi)_\nu \subset R_{\xi \times_c \nu}$

Dem.

$\vdash . *121{\cdot}36 . \supset \vdash : \text{Hp} . \xi, \nu \in \text{NC ind} - \iota'0 . \supset . (R_\xi)_\nu = R_{\xi \times_c \nu}$ (1)

$\vdash . *300{\cdot}481 . \supset \vdash : \text{Hp} . \xi = 0 . \nu \neq 0 . \supset . (R_\xi)_\nu = \dot{\Lambda}$ (2)

$\vdash . *300{\cdot}32{\cdot}311 . *113{\cdot}602 . \supset \vdash : \text{Hp} . \xi = 0 . \nu = 0 . \supset . (R_\xi)_\nu = R_{\xi \times_c \nu}$ (3)

$\vdash . *300{\cdot}481 . *113{\cdot}602 . \supset \vdash : \text{Hp} . \xi \neq 0 . \nu = 0 . \supset . (R_\xi)_\nu \subset R_{\xi \times_c \nu}$ (4)

$\vdash . *300{\cdot}47 . \supset \vdash : \text{Hp} . \sim(\xi, \nu \in \text{NC ind}) . \supset . (R_\xi)_\nu = \dot{\Lambda}$ (5)

$\vdash . (1) . (2) . (3) . (4) . (5) . \supset \vdash . \text{Prop}$

***300·56.** $\vdash : R \in \text{Rel num} . \dot{\exists} ! (R_\xi)_\nu \dot{\wedge} (R_\eta)_\mu . \supset .$
$\xi \times_c \nu = \eta \times_c \mu . (\xi \times_c \nu) \cap t'C'R \in \text{Cl}'U$

Dem.

$\vdash . *300{\cdot}552 . \supset \vdash : \text{Hp} . \supset . \dot{\exists} ! R_{\xi \times_c \nu} \dot{\wedge} R_{\eta \times_c \mu}$ (1)

$\vdash . (1) . *300{\cdot}55 . \supset \vdash . \text{Prop}$

***300·57.** $\vdash : \dot{\exists} ! (U_\xi)_\nu \dot{\wedge} (U_\eta)_\mu . \supset . \xi \times_c \nu \in C'U . \xi \times_c \nu = \eta \times_c \mu$

Dem.

$\vdash . *300{\cdot}5{\cdot}511{\cdot}56{\cdot}552 . \supset \vdash : \text{Hp} . \supset . \xi \times_c \nu = \eta \times_c \mu . \dot{\exists} ! U_{\xi \times_c \nu}$ (1)

$\vdash . (1) . *300{\cdot}26 . \supset \vdash . \text{Prop}$

By *300·56, we have, with the above hypothesis, $(\xi \times_c \nu) \cap t'C'U \in \text{Cl}'U$. But here the U in $\text{Cl}'U$ is of higher type than the U in $(\xi \times_c \nu) \cap t'C'U$ or in the hypothesis. In the type of the U in the hypothesis, we have $\xi \times_c \nu \in C'U$, not necessarily $\xi \times_c \nu \in \text{Cl}'U$.

***300·571.** $\vdash :. \xi, \eta \in \text{D}'U . \supset : \dot{\exists} ! (U_\xi)_\nu \dot{\wedge} (U_\eta)_\mu . \equiv . \xi \times_c \nu \in C'U . \xi \times_c \nu = \eta \times_c \mu$

Dem.

$\vdash . *300{\cdot}26 . \supset \vdash : \xi \times_c \nu \in C'U . \xi \times_c \nu = \eta \times_c \mu . \supset . (\xi \times_c \nu) \{U_{\xi \times_c \nu} \dot{\wedge} U_{\eta \times_c \mu}\} 0$ (1)

$\vdash . *121{\cdot}36 . \supset \vdash : \text{Hp} . \text{Hp}(1) . \mu \neq 0 . \nu \neq 0 . \supset . U_{\xi \times_c \nu} = (U_\xi)_\nu . U_{\eta \times_c \mu} = (U_\eta)_\mu$ (2)

$\vdash . *300{\cdot}32 . \supset \vdash : \text{Hp} . \text{Hp}(1) . \nu = 0 . \supset . (U_\xi)_\nu = I \restriction C'U_\xi .$

[*300·26] $\supset . 0 \{(U_\xi)_\nu\} 0$ (3)

Similarly $\vdash : \text{Hp} . \text{Hp}(1) . \mu = 0 . \supset . 0 \{(U_\eta)_\mu\} 0$ (4)

$\vdash . *113{\cdot}602 . \supset \vdash : \text{Hp} . \text{Hp}(1) . \nu = 0 . \supset . \mu = 0$ (5)

$\vdash . (1) . (2) . (3) . (4) . (5) . \supset \vdash : \text{Hp} . \text{Hp}(1) . \supset . \dot{\exists} ! (U_\xi)_\nu \dot{\wedge} (U_\eta)_\mu$ (6)

$\vdash . (6) . *300{\cdot}57 . \supset \vdash . \text{Prop}$

***300·572.** $\vdash :. \xi \in \text{D}'U . \supset : \dot{\exists} ! (U_\xi)_\nu . \equiv . \xi \times_c \nu \in C'U$ $\left[*300{\cdot}571 \dfrac{\xi, \nu}{\eta, \mu}\right]$

Summary of *301.

In this number, we have to exhibit the powers of a relation R, *i.e.* the various members of Potid$`R$, as of the form R^σ, where σ is an inductive cardinal. We have already had $R^2 = R \mid R$ and $R^3 = R^2 \mid R$. What we need is a definition which shall give

$$R^{\sigma +_c 1} = R^\sigma \mid R.$$

Now R^σ is a function of R and σ; thus we have to exhibit R^σ in the form $S`\sigma$, where S will be a function of R. That is, we have to define the relation S as a relation of R^σ to σ, and S must be such that, if it holds between R^σ and σ, it holds between $R^{\sigma +_c 1}$ and $\sigma +_c 1$. Thus we may take S as a sum of couples, such that if one couple is $R^\sigma \downarrow \sigma$, the next is $(R^\sigma \mid R) \downarrow (\sigma +_c 1)$, *i.e.* such that, if one couple is $Q \downarrow \sigma$, the next is $(Q \mid R) \downarrow (\sigma +_c 1)$. Now

$$(Q \mid R) \downarrow (\sigma +_c 1) = \{(\mid R) \parallel (-_c 1)\}`(Q \downarrow \sigma).$$

Hence, since we want to have $R^0 = I \upharpoonright C`R$, our class of couples is

$$\hat{M}\,[M\,\{(\mid R) \parallel (-_c 1)\}_* \,\{(I \upharpoonright C`R) \downarrow 0\}].$$

Calling this class num (R), we may therefore put

$$R^\sigma = \{\dot{s}`\text{num}\,(R)\}`\sigma \quad \text{Df.}$$

If we put $(\mid R) \parallel (-_c 1) = R_p$, the above definitions are

$$\text{num}\,(R) = \overrightarrow{(R_p)_*}`\{(I \upharpoonright C`R) \downarrow 0\} \quad \text{Dft},$$

$$R^\sigma = \{\dot{s}`\text{num}\,(R)\}`\sigma \qquad \text{Df.}$$

But the above definition of R_p requires some modification before it can be considered quite correct. With the above definition, we have

$$R_p`(Q \downarrow \sigma) = (Q \mid R) \downarrow (\sigma +_c 1) \qquad (1).$$

Now since num (R) is defined by means of $(R_p)_*$, and since the definition of R_* contains the hypothesis $\breve{R}``\mu \subset \mu$, it follows that, if num (R) is to be significant, the relation $-_c 1$ which appears in the definition of R_p must be homogeneous, so that, in (1), σ and $\sigma +_c 1$ must be of the same type. Hence σ, though typically ambiguous, cannot be typically indefinite;

therefore, if the axiom of infinity is not true, we shall sooner or later arrive at $\sigma = \Lambda$ as we travel up the inductive cardinals. In that case, we shall have

$$R^{\sigma -_c 1} \downarrow (\sigma -_c 1) \epsilon \operatorname{num}(R) . (R^{\sigma -_c 1} | R) \downarrow \Lambda \epsilon \operatorname{num}(R),$$
$$(R^{\sigma -_c 1} | R | R) \downarrow \Lambda \epsilon \operatorname{num}(R), \text{ etc.}$$

Now if (for example) R is a cyclic relation, such as that of an angle of a polygon to the next angle to the left, we shall not have

$$R^{\sigma -_c 1} = R^{\sigma -_c 1} | R \text{ or } R^{\sigma -_c 1} | R = R^{\sigma -_c 1} | R | R.$$

Hence $\dot{s}$'num (R) will fail to be one-many, and R^σ will fail to exist. Hence it becomes desirable to restrict σ to cardinals which exist in some assigned type, *i.e.* to replace $-_c 1$ by $(-_c 1) \upharpoonright$ (NC induct $- \iota$'Λ), *i.e.* by $\breve{U}_1$.

Thus we now put $\qquad R_p = (|R) \| \breve{U}_1$ Dft.

But even this definition is not quite complete, because the type of U is not assigned. It makes some difference how the type of U is assigned, for if we take as the type of C'U a type lower than that of t'N_0c't'R, we may find that our numbers become Λ before we have ceased to obtain fresh powers of R.

For example, suppose the total number of individuals were four, and that these were a, x, y, z. Let us write $x \downarrow (a, y, \ldots)$ for $x \downarrow a \cup x \downarrow y \cup \ldots$. Then consider the relation $R = x \downarrow (a, y) \cup a \downarrow y \cup y \downarrow (x, z)$. Then

$$R^2 = x \downarrow (x, y, z) \cup a \downarrow (x, z) \cup y \downarrow (a, y),$$
$$R^3 = x \downarrow (a, y, x, z) \cup a \downarrow (a, y) \cup y \downarrow (x, y, z),$$
$$R^4 = x \downarrow (y, x, z, a) \cup a \downarrow (y, x, z) \cup y \downarrow (a, y, x, z),$$
$$R^5 = x \downarrow (a, x, y, z) \cup a \downarrow (a, x, y, z) \cup y \downarrow (a, x, y, z).$$

After this, $R^6 = R^5 | R = R^5 | R^2 =$ etc. But up to R^5, each power of R is different from all its predecessors. If we take t'C'$U = t$'N_0c't'C'R, C'U consists only of the numbers 0, 1, 2, 3, 4, and is thus inadequate to deal with R^5. Hence the type in which we take U must be a sufficiently high type, which must increase with the type of R. Hence we take C'U in the type of t'N_0c't'R, *i.e.* in the type of t^3'R. This is secured by writing $U \upharpoonright t^3$'R in place of U in the definition of R_p. Hence the final definitions for R^σ are:

***301·01.** $R_p = (|R) \| (\breve{U}_1 \upharpoonright t^3\text{'}R)$ Dft [*301]

***301·02.** $\operatorname{num}(R) = \overrightarrow{(R_p)_*}\text{'}\{(I \upharpoonright C\text{'}R) \downarrow (0 \cap t^2\text{'}R)\}$ Dft [*301]

***301·03.** $R^\sigma = \{\dot{s}\text{'}\operatorname{num}(R)\}\text{'}\sigma$ Df

The two temporary definitions *301·01·02 are only to extend to the present number.

With the above definitions we have

***301·16.** $\vdash : \mu \,\epsilon\, C`U \cap t^3`R \,.\equiv.\, E\,!\,R^\mu$

***301·2.** $\vdash . R^0 = I \upharpoonright C`R \,.\, R^1 = R$

***301·21.** $\vdash : \nu \,\epsilon\, \text{Ɑ}`U \cap t^3`R \,.\supset.\, R^{\nu +_c 1} = R^\nu \,|\, R$

***301·23.** $\vdash : \mu +_0 \nu \,\epsilon\, C`U \cap t^3`R \,.\supset.\, R^{\mu +_c \nu} = R^\mu \,|\, R^\nu = R^\nu \,|\, R^\mu$

***301·26.** $\vdash : P \,\epsilon\, \text{Potid}`R \,.\equiv.\, (\exists\sigma) \,.\, P = R^\sigma$

I.e. the powers of R are the various relations R^σ. This proposition might have been not universally true if we had taken U in a lower type.

***301·3.** $\vdash : R \,\mathsf{G}\, I \,.\, \sigma \,\epsilon\, C`U \cap t^3`R \,.\supset.\, R^\sigma = R = R_0 = I \upharpoonright C`R$

It is largely for the sake of this proposition that we require powers of relations in dealing with ratio, rather than finid$`R$. For we have $R \,\mathsf{G}\, I \,.\, \sigma \neq 0 \,.\supset.\, R_\sigma = \dot{\Lambda}$, so that $R_\sigma$ does not give what is wanted if $R \,\mathsf{G}\, I$. On the other hand (\*301·41), if $R \,\epsilon\, \text{Rel num}$, we have $R^\sigma = R_\sigma$ if $\sigma \,\epsilon\, C`U \cap t^3`R$. Thus as applied to numerical relations, R_σ may always replace R^σ.

We have, whatever R may be,

***301·504.** $\vdash : \mu, \nu \,\epsilon\, C`U \cap t^2`C`R \,.\, \nu \neq 0 \,.\supset.\, (R^\mu)^\nu = R^{\mu \times_c \nu}$

The importance of this number will appear in connection with ratios.

***301·01.** $R_p = (\,|\,R) \,\|\, (\breve{U}_1 \upharpoonright t^3`R)$ Dft [*301]

***301·02.** $\text{num}\,(R) = \overrightarrow{(R_p)_*}`\{(I \upharpoonright C`R) \downarrow (0 \cap t^2`R)\}$ Dft [*301]

***301·03.** $R^\sigma = \{\dot{s}`\text{num}\,(R)\}`\sigma$ Df

***301·1.** $\vdash : \sigma \,\epsilon\, \text{Ɑ}`(U \upharpoonright t^3`R) \,.\supset.\, R_p`(Q \downarrow \sigma) = (Q \,|\, R) \downarrow \{(\sigma +_c 1) \cap t^2`R\}$
[*55·61 . (*301·01)]

***301·101.** $\vdash : \sigma \,\epsilon\, \text{Ɑ}`(U \upharpoonright t^3`R) \,.\equiv.\, \sigma \,\epsilon\, \text{Ɑ}`U \cap t^3`R \,.\equiv.\, \sigma \,\epsilon\, \text{Ɑ}`U \,.\, \sigma \subset t^2`R$
[*63·5]

***301·102.** $\vdash : \sigma \,\epsilon\, \text{Ɑ}`(U \upharpoonright t^3`R) \,.\equiv.$
$(\exists\lambda) \,.\, \lambda \,\epsilon\, \text{Cls induct} \,.\, \exists\,!\,-\lambda \,.\, R \,\epsilon\, t_0`\lambda \,.\, \sigma = N_0c`\lambda$
[*300·14 . *103·11]

***301·103.** $\vdash : \sigma \,\epsilon\, \text{Ɑ}`(U \upharpoonright t^3`R) \,.\equiv.$
$(\exists\lambda) \,.\, \lambda \,\epsilon\, \text{Cls induct} \,.\, \exists\,!\,-\lambda \,.\, R \,\epsilon\, \lambda \,.\, \sigma = N_0c`\lambda$
[*301·102 . *73·71·72]

***301·104.** $\vdash : \sigma \,\epsilon\, \text{Ɑ}`(U \upharpoonright t^3`R) \,.\equiv.\, (\sigma +_c 1) \cap t^2`R \,\epsilon\, \text{NC induct} - \iota`\Lambda$
[*301·101 . *300·14]

***301·105.** $\vdash : \sigma \,\epsilon\, \text{Ɑ}`(U \upharpoonright t^3`R) \,.\equiv.\, (\exists\lambda) \,.\, \lambda \,\epsilon\, \text{Cls induct} \,.\, R \,\epsilon\, \lambda \,.\, \sigma +_0 1 = N_0c`\lambda$
[*301·104]

***301·106.** $\vdash : \sigma \in \mathrm{G}`(U \upharpoonleft t^3`R) . \equiv . (\exists\lambda) . \lambda \in \text{Cls induct} . R \in t_0`\lambda . \sigma +_c 1 = \mathrm{N_0c}`\lambda$
[*301·104]

***301·107.** $\vdash : \sigma \in \mathrm{G}`(U \upharpoonleft t^3`R) . \equiv . \sigma \in \text{NC ind} . R \in s`(\sigma +_c 1)$
[*301·106 . *126·1]

***301·11.** $\vdash : \sigma \in \mathrm{G}`(U \upharpoonleft t^3`R) . \equiv . \mathrm{E} ! R_p`(Q \downarrow \sigma)$ [*301·1]

***301·12.** $\vdash : M \in \text{num}(R) . \supset . (\exists P, \sigma) . P \in \text{Potid}`R . \sigma \in C`U \cap t^3`R . M = P \downarrow \sigma$
[*95·22]

***301·13.** $\vdash : P \downarrow 0 \in \text{num}(R) . \supset . P = I \upharpoonright C`R$

Dem.

$\vdash . *90{\cdot}31 . (*301{\cdot}02) . \supset$

$\vdash : P \downarrow \mu \in \text{num}(R) - \iota`\{(I \upharpoonright C`R) \downarrow 0\} . \supset .$

$\qquad (P \downarrow \mu) \{(R_p)_* \mid R_p\} \{(I \upharpoonright C`R) \downarrow 0\} .$

$[*30{\cdot}33 . *301{\cdot}1] \supset . (P \downarrow \mu) (R_p)_* (R \downarrow 1) .$

$[*95{\cdot}22] \qquad \supset . \mu U_* 1 .$

$[*300{\cdot}24] \qquad \supset . \mu \neq 0 \qquad (1)$

$\vdash . (1) . \text{Transp} . \supset \vdash . \text{Prop}$

***301·14.** $\vdash : P \downarrow \mu, Q \downarrow \mu \in \text{num}(R) . \supset . P = Q$

Dem.

$\vdash . *120{\cdot}124 . *90{\cdot}31 . \supset$

$\vdash : \{S \downarrow (\mu +_c 1)\} (R_p)_* \{(I \upharpoonright C`R) \downarrow 0\} . \supset .$

$\quad \{S \downarrow (\mu +_c 1)\} \{R_p \mid (R_p)_*\} \{(I \upharpoonright C`R) \downarrow 0\} \qquad (1)$

$\vdash . (1) . (*301{\cdot}02) . *301{\cdot}12 . *300{\cdot}14 . \supset$

$\vdash : S \downarrow (\mu +_c 1) \in \text{num}(R) . \supset . S \downarrow (\mu +_c 1) \in R_p``\text{num}(R) . \exists ! \mu +_c 1 .$

[*301·1]

$\quad \supset . (\exists P, \nu) . P \downarrow \nu \in \text{num}(R) . S \downarrow (\mu +_c 1) = (P \mid R) \downarrow (\nu +_c 1) . \exists ! \mu +_c 1 .$

[*55·202 . *120·311]

$\quad \supset . (\exists P) . P \downarrow \mu \in \text{num}(R) . S \downarrow (\mu +_c 1) = (P \mid R) \downarrow (\mu +_c 1) \qquad (2)$

$\vdash . (2) . \supset \vdash :. P \downarrow \mu, Q \downarrow \mu \in \text{num}(R) . \supset_{P,Q} . P = Q : \supset :$

$\qquad S \downarrow (\mu +_c 1), T \downarrow (\mu +_c 1) \in \text{num}(R) . \supset_{S,T} . S = T \qquad (3)$

$\vdash . (3) . *301{\cdot}12{\cdot}13 . \text{Induct} . \supset \vdash . \text{Prop}$

***301·141.** $\vdash . \mathrm{G}`\dot{s}`\text{num}(R) = C`U \cap t^3`R$

Dem.

$\vdash . *301{\cdot}1 . \supset$

$\vdash : \sigma \in \mathrm{G}`U \cap t^3`R . \sigma \in \mathrm{G}`\dot{s}`\text{num}(R) . \supset . (\sigma +_c 1) \in \mathrm{G}`\dot{s}`\text{num}(R) \qquad (1)$

$\vdash . (1) . *300{\cdot}14 . \text{Induct} . \supset \vdash . \text{Prop}$

***301·15.** $\vdash . \dot{s}`\mathrm{num}(R) \in 1 \to \mathrm{Cls}$

Dem.

$\vdash . *301{\cdot}14 . \supset \vdash : M, N \in \mathrm{num}(R) . \exists ! \mathrm{D}`M \cap \mathrm{D}`N . \supset . M = N$ (1)

$\vdash . (1) . *72{\cdot}32 . \supset \vdash . \mathrm{Prop}$

***301·16.** $\vdash : \mu \in C`U \cap t^3`R . \equiv . \mathrm{E} ! R^\mu$ [*301·141·15 . (*301·03)]

***301·2.** $\vdash . R^0 = I \upharpoonright C`R . R^1 = R$ [*301·13·16·1 . (*301·03)]

***301·201.** $\vdash : \nu \in C`U \cap t^3`R . \supset . (R^\nu \downarrow \nu) \in \mathrm{num}(R)$

Dem.

$\vdash . *301{\cdot}16 . (*301{\cdot}03) . \supset \vdash : \mathrm{Hp} . \supset . R^\nu \{\dot{s}`\mathrm{num}(R)\} \nu .$

[*41·11] $\supset . (\exists M) . M \in \mathrm{num}(R) . R^\nu M \nu .$

[*301·12] $\supset . (\exists M, P, \sigma) . M \in \mathrm{num}(R) . M = P \downarrow \sigma . R^\nu M \nu .$

[*55·13] $\supset . (R^\nu \downarrow \nu) \in \mathrm{num}(R) : \supset \vdash . \mathrm{Prop}$

***301·21.** $\vdash : \nu \in \mathrm{D}`U \cap t^3`R . \supset . R^{\nu +_c 1} = R^\nu | R$

Dem.

$\vdash . *301{\cdot}1{\cdot}201 . \supset \vdash : \mathrm{Hp} . \supset . R^{\nu +_c 1} \downarrow (\nu +_c 1), (R^\nu | R) \downarrow (\nu +_c 1) \in \mathrm{num}(R) .$

[*301·14] $\supset . R^{\nu +_c 1} = R^\nu | R : \supset \vdash . \mathrm{Prop}$

***301·22.** $\vdash : \mathrm{E} ! R^\nu . \supset . R^\nu \in \mathrm{Potid}`R$ [*301·201·12·16]

***301·23.** $\vdash : \mu +_c \nu \in C`U \cap t^3`R . \supset . R^{\mu +_c \nu} = R^\mu | R^\nu = R^\nu | R^\mu$
[*301·21 . Induct]

***301·24.** $\vdash :. \sigma \in \mathrm{NC\,ind} : \mu \leqslant \sigma . \nu < \mu . \supset_{\mu, \nu} . R^\mu \neq R^\nu : \supset .$
$\hat{P} \{(\exists \mu) . \mu \leqslant \sigma . P = R^\mu\} \in \sigma +_c 1$

Dem.

$\vdash . *120{\cdot}442 . \supset \vdash : \mathrm{Hp} . \mu \leqslant \sigma . \nu \leqslant \sigma . R^\mu = R^\nu . \supset . \mu = \nu$ (1)

$\vdash . (1) . *73{\cdot}14 . *301{\cdot}15 . \supset$

$\vdash : \mathrm{Hp} . \supset . \mathrm{Nc}`\hat{P} \{(\exists \mu) . \mu \leqslant \sigma . P = R^\mu\} = \mathrm{Nc}`\hat{\mu} (\mu \leqslant \sigma)$ (2)

$\vdash . (2) . *120{\cdot}57 . \supset \vdash . \mathrm{Prop}$

***301·241.** $\vdash : \mathrm{Hp}\, *301{\cdot}24 . \supset . \sigma \cap t^{2}`R \in \mathrm{D}`(U \upharpoonright t^3`R) . R^{\sigma +_c 1} = R^\sigma | R$
[*301·24·104·21]

***301·242.** $\vdash : \sigma \in C`U \cap t^3`R . \mu \leqslant \sigma . \nu < \mu . R^\mu = R^\nu . \supset . R^\sigma | R = R^{\sigma -_c \mu +_c \nu +_c}$

Dem.

$\vdash . *120{\cdot}412{\cdot}416 . \supset \vdash : \mathrm{Hp} . \supset . \sigma = (\sigma -_c \mu) +_c \mu .$

[*301·23] $\supset . R^\sigma = R^{\sigma -_c \mu} | R^\mu .$

[Hp.*301·21] $\supset . R^\sigma | R = R^{\sigma -_c \mu} | R^{\nu +_c 1}$

[*301·23] $= R^{\sigma -_c \mu +_c \nu +_c 1} : \supset \vdash . \mathrm{Prop}$

***301·25.** $\vdash : (\exists \sigma) . P = R^\sigma . \supset . (\exists \tau) . P | R = R^\tau$ [*301·16·241·242]

***301·26.** $\vdash : P \epsilon \text{Potid}`R . \equiv . (\exists \sigma) . P = R^\sigma$

Dem.

$\vdash . *301·25·2 . \text{Induct} . \supset \vdash : P \epsilon \text{Potid}`R . \supset . (\exists \sigma) . P = R^\sigma$ (1)

$\vdash . (1) . *301·22 . \supset \vdash . \text{Prop}$

***301·3.** $\vdash : R \subseteq I . \sigma \epsilon C`U \cap t^3`R . \supset . R^\sigma = R = R_0 = I \upharpoonright C`R$
[*300·312 . *301·16·26]

***301·31.** $\vdash : R \subseteq I . \sigma \neq 0 . \supset . R_\sigma = \dot{\Lambda}$ [*300·48]

The above proposition is the same as *300·48, but is repeated here to show the relations of R_σ and R^σ.

***301·32.** $\vdash :. R \subseteq I . \dot{\exists} ! R . \supset : \dot{\exists} ! R_\sigma . \equiv . \sigma = 0$ [*300·311 . *301·31]

***301·4.** $\vdash : R \epsilon \text{Rel num} . \sigma \epsilon C`U \cap t^3`R . \supset . R_\sigma = R^\sigma$

Dem.

$\vdash . *301·2 . *121·302 . \supset \vdash : \text{Hp} . \supset . R_0 = R^0$ (1)

$\vdash . *301·21 . *121·332 . \supset$

$\vdash :. \text{Hp} . \sigma \epsilon C`U \cap t^3`R . \supset : R_\sigma = R^\sigma . \supset . R_{\sigma +_c 1} = R^{\sigma +_c 1}$ (2)

$\vdash . (1) . (2) . \text{Induct} . \supset \vdash . \text{Prop}$

***301·41.** $\vdash : R, S \epsilon \text{Rel num} . \dot{\exists} ! R^\mu \dot{\cap} R^\nu . \supset . \mu = \nu . \exists ! (\mu +_c 1) \cap t`C`R$
[*301·4·16 . *300·55]

***301·5.** $\vdash : \mu \times_c \nu \epsilon C`U \cap t^3`R . \mu \neq 0 . \nu \neq 0 . \supset . (R^\mu)^\nu = R^{\mu \times_c \nu}$

Dem.

$\vdash . *117·62·32 . \quad \supset \vdash : \text{Hp} . \supset . \mu, \nu \epsilon C`U \cap t^3`R$ (1)

$\vdash . (1) . *301·16·2 . \supset \vdash : \text{Hp} . \supset . (R^\mu)^1 = R^{\mu \times_c 1}$ (2)

$\vdash . *301·23 . \quad \supset \vdash : \nu +_c 1 \epsilon C`U \cap t^3`R . \supset . (R^\mu)^{\nu +_c 1} = (R^\mu)^\nu | R^\mu$ (3)

$\vdash . (3) . *301·23 . \supset$

$\vdash : (\mu \times_c \nu) +_c \mu \epsilon C`U \cap t^3`R . (R^\mu)^\nu = R^{\mu \times_c \nu} . \supset . (R^\mu)^{\nu +_c 1} = R^{(\mu \times_c \nu) +_c \mu}$ (4)

$\vdash . (4) . *113·671 . \supset$

$\vdash :. (R^\mu)^\nu = R^{\mu \times_c \nu} . \supset : \mu \times_c (\nu +_c 1) \epsilon C`U \cap t^3`R . \supset . (R^\mu)^{\nu +_c 1} = R^{\mu \times_c (\nu +_c 1)}$ (5)

$\vdash . *117·571·32 . \supset \vdash : \mu \times_c (\nu +_c 1) \epsilon C`U \cap t^3`R . \supset . \mu \times_c \nu \epsilon C`U \cap t^3`R$ (6)

$\vdash . (5) . (6) . \supset \vdash :. \mu \times_c \nu \epsilon C`U \cap t^3`R . \supset . (R^\mu)^\nu = R^{\mu \times_c \nu} : \supset :$

$\mu \times_c (\nu +_c 1) \epsilon C`U \cap t^3`R . \supset . (R^\mu)^{\nu +_c 1} = R^{\mu \times_c (\nu +_c 1)}$ (7)

$\vdash . (1) . (2) . (7) . \text{Induct} . \supset \vdash . \text{Prop}$

***301·501.** $\vdash : \mu = 0 . \nu \epsilon C`U \cap t^3`R . \supset . (R^\mu)^\nu = R^{\mu \times_c \nu}$ [*301·2·3]

***301·502.** $\vdash : \mu, \nu \epsilon C`U \cap t^2`C`R . \supset . \mu \times_c \nu \epsilon C`U \cap t^3`R . (\mu \times_c \nu) \cap t^2`R \epsilon C`U$

Dem.

$\vdash . *300·14 . *120·5 . \supset \vdash : \text{Hp} . \exists ! (\mu \times_c \nu) \cap t^2`R . \supset . (\mu \times_c \nu) \cap t^2`R \epsilon C`U$ (1)

$\vdash . *300·14 . \quad \supset \vdash : \text{Hp} . \supset . (\exists \alpha, \beta) . \alpha \epsilon \mu . \beta \epsilon \nu . \alpha, \beta \epsilon t`C`R .$

[*113·251] $\quad \supset . (\exists \alpha, \beta) . \alpha \times \beta \epsilon \mu \times_c \nu . \alpha, \beta \epsilon t`C`R .$

[*113·17.*64·61] $\quad \supset . (\exists \alpha, \beta) . \alpha \times \beta \epsilon (\mu \times_c \nu) \cap t^2`R$ (2)

$\vdash . (1) . (2) . *65·13 . \supset \vdash . \text{Prop}$

***301·503.** $\vdash : \nu \in \mathrm{NC\,ind} \,.\, \nu \cap t`C`R \in C`U \upharpoonright (t^{2}`C`R) \,.\, \supset \,.\, \nu \cap t^{2}`R \in C`(U \upharpoonright t^{3}`R)$

Dem.

$\vdash \,.\, *300·14 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, (\exists \alpha) \,.\, \alpha \in \nu \cap t`C`R \,.$

[*106·2] $\supset \,.\, (\exists x, \alpha) \,.\, \downarrow x``\alpha \in \nu \cap t^{2}`R$ (1)

$\vdash \,.\, (1) \,.\, *300·14 \,.\, \supset \vdash \,.\, \mathrm{Prop}$

***301·504.** $\vdash : \mu, \nu \in C`U \cap t^{2}`C`R \,.\, \nu \neq 0 \,.\, \supset \,.\, (R^{\mu})^{\nu} = R^{\mu \times_c \nu}$

[*301·5·501·502·503]

***301·505.** $\vdash :. \xi \in \mathrm{D}`U \,.\, \supset : \dot{\exists} \,!\, \{(+_c \xi) \upharpoonright C`U\}^{\nu} \,.\, \equiv \,.\, \xi \times_c \nu \in C`U$

Dem.

$\vdash \,.\, *120·452 \,.\, \supset \vdash : \dot{\exists} \,!\, \{(+_c \xi) \upharpoonright C`U\}^{\nu} \,.\, \equiv \,.\, \dot{\exists} \,!\, \{(+_c \xi) \upharpoonright C`U\}^{\nu} \,.\, \xi \in C`U \,.$

[*300·232] $\equiv \,.\, \dot{\exists} \,!\, (U_{\xi})^{\nu} \,.\, \xi \in C`U$ (1)

$\vdash \,.\, (1) \,.\, *300·52 \,.\, *301·4 \,.\, \supset$

$\vdash :. \mathrm{Hp} \,.\, \supset : \dot{\exists} \,!\, \{(+_c \xi) \upharpoonright C`U\}^{\nu} \,.\, \equiv \,.\, \dot{\exists} \,!\, (U_{\xi})_{\nu} \,.\, \xi \in C`U \,.$

[*300·572] $\equiv \,.\, \xi \times_c \nu \in C`U :. \supset \vdash \,.\, \mathrm{Prop}$

***301·51.** $\vdash :. \xi, \eta \in \mathrm{D}`U \,.\, \supset : \dot{\exists} \,!\, \{(+_c \xi) \upharpoonright C`U\}^{\nu} \mathbin{\dot\cap} \{(+_c \eta) \upharpoonright C`U\}^{\mu} \,.\, \equiv \,.$
$\xi \times_c \nu \in C`U \,.\, \xi \times_c \nu = \eta \times_c \mu$

Dem.

$\vdash \,.\, *301·505 \,.\, *300·232 \,.\, *301·4 \,.\, \supset$

$\vdash :. \mathrm{Hp} \,.\, \supset : \dot{\exists} \,!\, \{(+_c \xi) \upharpoonright C`U\}^{\nu} \mathbin{\dot\cap} \{(+_c \eta) \upharpoonright C`U\}^{\mu} \,.\, \equiv \,.\, \dot{\exists} \,!\, (U_{\xi})_{\nu} \mathbin{\dot\cap} (U_{\eta})_{\mu} \,.$

[*300·571] $\equiv \,.\, \xi \times_c \nu \in C`U \,.\, \xi \times_c \nu = \eta \times_c \mu :. \supset \vdash \,.\, \mathrm{Prop}$

***301·52.** $\vdash : \nu \in \mathrm{D}`U \cap t^{3}`R \,.\, \supset \,.\, (\times_c \mu)^{\nu} = \times_c (\mu^{\nu})$

Dem.

$\vdash \,.\, *301·2 \,.\, *113·204 \,.\, *116·204·321 \,.\, \supset \vdash \,.\, (\times_c \mu)^{1} = \times_c (\mu^{1})$ (1)

$\vdash \,.\, *301·21 \,.\, \supset \vdash : \nu \in \mathrm{C\!\!\!\!\;I}`U \cap t^{3}`R \,.\, \supset \,.\, (\times_c \mu)^{\nu +_c 1} = (\times_c \mu)^{\nu} \,|\, (\times_c \mu)$ (2)

$\vdash \,.\, (2) \,.\, \supset \vdash : \nu \in \mathrm{C\!\!\!\!\;I}`U \cap t^{3}`R \,.\, (\times_c \mu)^{\nu} = \times_c (\mu^{\nu}) \,.\, \supset \,.\, (\times_c \mu)^{\nu +_c 1} = \times_c (\mu^{\nu}) \,|\, (\times_c \mu)$

[*116·52·321] $= \times_c (\mu^{\nu +_c 1})$ (3)

$\vdash \,.\, (1) \,.\, (3) \,.\, \mathrm{Induct} \,.\, \supset \vdash \,.\, \mathrm{Prop}$

$*302$. ON RELATIVE PRIMES.

Summary of $*302$.

The present number is merely preparatory for the definition and discussion of ratios. We want, of course, to give a definition of ratio which shall ensure that $\mu/\nu = (\mu \times_c \tau)/(\nu \times_c \tau)$. Hence in defining μ/ν in any given type, we cannot exact that μ and ν themselves should exist in that type, but only that, if ρ/σ is the same ratio in its lowest terms, ρ and σ should exist in that type. Hence, if we are not to assume the axiom of infinity, it is necessary to deal with relative primes before defining ratios.

The theory of relative primes is concerned with typically indefinite inductive cardinals (NC ind). It will be observed that we have three different sorts of inductive cardinals, namely NC ind, NC induct, and $C'U$. NC ind consists of typically indefinite cardinals, NC induct consists of all cardinals of some one type, and $C'U$ consists of all *existent* cardinals of some one type. If the axiom of infinity holds, we have $C'U = \text{NC induct}$, and $\text{NC ind} = \text{sm}``\text{NC induct}$. But neither of these is true if the axiom of infinity does not hold. It will be found that, where we require typically definite cardinals, it is $C'U$ or $\text{Cl}'U$ or $\text{D}'U$ that we require, not NC induct; that is to say, we almost always want to exclude Λ, and sometimes we want to exclude the greatest existent cardinal of the type in question, or to exclude 0. Thus "NC induct" will seldom occur in what follows. The cases in which $C'U$ or $\text{D}'U$ or $\text{Cl}'U$ occurs are of two sorts: (1) where we are proving typically definite existent-theorems, (2) where we are concerned with *series*, as *e.g.* in $*300$, where we considered the series of existent cardinals, or in $*304$ below, where we shall consider the series of ratios. Wherever *series* are concerned, we must have typical definiteness, because the definition of "$P \epsilon \text{Ser}$" involves $C'P$, and therefore only a *homogeneous* relation can be serial. This is a particular instance of the fact that when we require numbers as apparent variables (as *e.g.* in the theory of real numbers), typical definiteness becomes essential. Many propositions containing the hypothesis "$\mu \epsilon \text{NC ind}$" (where μ is a real variable) do not allow of μ being turned into an apparent variable, because this requires that μ should be fixed in one type, and our original proposition may demand that the

type in which μ is fixed should be a function of μ. For example, $*300\cdot17$ states

$$\vdash : \mu \epsilon \text{NC ind} . \supset . (\exists \alpha) . \mu \epsilon C'(U \upharpoonright t^{2'}\alpha).$$

If we fix the type of μ, we thereby also fix the type of α, and the proposition becomes false unless the axiom of infinity is true. In fact, the proposition demands that, the greater μ becomes, the higher must the type of α become. "NC ind" is not a strictly correct idea, and the primitive proposition $*9\cdot13$ does not apply without reserve to propositions in which it occurs. We have introduced it because it immensely simplifies the expression of many propositions, and because it is easy to avoid the errors to which it might give rise if used without remembering that it is a concession to convenience.

It will be found that, when we are not concerned with existence-theorems, or with numbers as apparent variables, "NC ind" is almost always the notion required. This applies to all cases where we are only concerned with addition, multiplication, subtraction and division; it applies to ratios except when ratios are considered as forming a series, or when we are investigating their existence. As regards the use of an "NC ind" as an apparent variable, there is a distinction between "all values" and "some value." If we have "$\rho \epsilon \text{NC ind}$," "$(\exists \rho)$" will often be legitimate when "(ρ)" is not. The reason of this is that, if we are to fix upon one typically indefinite cardinal, it will be possible to assign one definite type in which it exists; *e.g.* there are at least two classes four classes of classes, sixteen classes of classes of classes, and so on. But if we are making a statement about *all* typically indefinite inductive cardinals, it will not be true unless there is a type such that our statement holds of all inductive cardinals in this type.

In virtue of $*300\cdot17$, if we have "$\rho \epsilon \text{NC ind}$," we may replace it by "$\rho \epsilon C'U$" if we may take U in as high a type as we please, or if, on account of the rest of our proposition, ρ cannot be greater than some assigned inductive cardinal so long as the hypothesis of our proposition is true.

The above remarks will enable the reader to test the uses of typically indefinite inductive cardinals as apparent variables, and the passage from propositions concerning NC ind to propositions concerning $C'U$.

We define ρ as prime to σ when both are typically indefinite cardinals and 1 is their only common factor, *i.e.* we put

$*302\cdot01$. $\text{Prm} = \hat{\rho}\hat{\sigma} \{\rho, \sigma \epsilon \text{NC ind} : \rho = \xi \times_0 \tau . \sigma = \eta \times_0 \tau . \supset_{\xi, \eta, \tau} . \tau = 1\}$ Df

In this definition, ξ, η, τ may be taken to be typically indefinite cardinals; because, when $\rho = \xi \times_0 \tau . \sigma = \eta \times_0 \tau$, we must have $\xi \leqslant \rho . \eta \leqslant \sigma . \tau \leqslant \rho . \tau \leqslant \sigma$, so that ξ, η, τ cannot grow indefinitely (with a given ρ and σ) while the hypothesis remains true.

We define "$(\rho, \sigma) \text{Prm}_\tau (\mu, \nu)$" as meaning that ρ is prime to σ, that τ is not zero, and $\mu = \rho \times_0 \tau . \nu = \sigma \times_0 \tau$, *i.e.* ρ/σ is μ/ν in its lowest terms, and τ is the highest common factor of μ and ν. The definition is:

***302·02.** $(\rho, \sigma) \operatorname{Prm}_\tau (\mu, \nu) . = .$

$$\rho \operatorname{Prm} \sigma . \tau \in \mathrm{NC\,ind} - \iota`0 . \mu = \rho \times_c \tau . \nu = \sigma \times_c \tau \quad \mathrm{Df}$$

We then put further

***302·03.** $(\rho, \sigma) \operatorname{Prm} (\mu, \nu) . = . (\exists \tau) . (\rho, \sigma) \operatorname{Prm}_\tau (\mu, \nu) \quad \mathrm{Df}$

Here again there is no objection to τ as an apparent variable, because τ must be not greater than μ and ν. "$(\rho, \sigma) \operatorname{Prm} (\mu, \nu)$" secures that ρ/σ is μ/ν in its lowest terms.

We also define, in this number, the lowest common multiple and the highest common factor.

Our definition of "Prm" is so framed that every inductive cardinal is prime to 1 (*302·12), that 1 is the only number which is prime to itself (*302·13), and the only number which is prime to 0 (*302·14).

After a number of preliminary propositions, we arrive at the result that if μ and ν are not both zero, and ξ and η are not both zero, the existence of a couple ρ, σ which is prime both to μ, ν and to ξ, η is equivalent to $\mu \times_c \eta = \nu \times_c \xi$, *i.e.*

***302·34.** $\vdash :. \mu, \nu, \xi, \eta \in \mathrm{NC\,ind} . \sim(\mu = \nu = 0) . \sim(\xi = \eta = 0) . \supset :$

$$\mu \times_c \eta = \nu \times_c \xi . \equiv . (\exists \rho, \sigma) . (\rho, \sigma) \operatorname{Prm} (\mu, \nu) . (\rho, \sigma) \operatorname{Prm} (\xi, \eta)$$

We have also

***302·36.** $\vdash : \mu, \nu \in \mathrm{NC\,ind} . \sim(\mu = \nu = 0) . \equiv . (\exists \rho, \sigma) . (\rho, \sigma) \operatorname{Prm} (\mu, \nu)$

***302·38.** $\vdash : (\rho, \sigma) \operatorname{Prm} (\mu, \nu) . (\xi, \eta) \operatorname{Prm} (\mu, \nu) . \supset . \rho = \xi . \sigma = \eta$

I.e. there is only one way of reducing a fraction to its lowest terms.

We prove also (*302·15) that if μ, ν are typically indefinite cardinals, which both exist in the type of λ (*i.e.* $\mu_\lambda, \nu_\lambda \in C`U$), then

$$(\rho, \sigma) \operatorname{Prm} (\mu, \nu) . \equiv . (\rho, \sigma) \operatorname{Prm} (\mu_\lambda, \nu_\lambda).$$

This enables us, when we wish, to substitute typically definite cardinals for the typically indefinite μ and ν.

***302·01.** $\operatorname{Prm} = \hat{\rho}\hat{\sigma} \{\rho, \sigma \in \mathrm{NC\,ind} : \rho = \xi \times_c \tau . \sigma = \eta \times_c \tau . \supset_{\xi, \eta, \tau} . \tau = 1\} \quad \mathrm{Df}$

***302·02.** $(\rho, \sigma) \operatorname{Prm}_\tau (\mu, \nu) . = .$

$$\rho \operatorname{Prm} \sigma . \tau \in \mathrm{NC\,ind} - \iota`0 . \mu = \rho \times_c \tau . \nu = \sigma \times_c \tau \quad \mathrm{Df}$$

Here μ, ν are to be typically indefinite in the same way as $\rho \times_c \tau$ and $\sigma \times_c \tau$. Thus if, in some one type, $\rho \times_c \tau$ and $\sigma \times_c \tau$ are both null, that does not justify us in writing $(\rho, \sigma) \operatorname{Prm}_\tau (\Lambda, \Lambda)$, because there are other types in which $\rho \times_c \tau$ and $\sigma \times_c \tau$ are not null. On this subject, cf. *126.

***302·03.** $(\rho, \sigma) \operatorname{Prm} (\mu, \nu) . = . (\exists \tau) . (\rho, \sigma) \operatorname{Prm}_\tau (\mu, \nu) \quad \mathrm{Df}$

***302·04.** $\operatorname{hcf} (\mu, \nu) = (\imath \tau) \{(\exists \rho, \sigma) . (\rho, \sigma) \operatorname{Prm}_\tau (\mu, \nu)\} \quad \mathrm{Df}$

***302·05.** $\operatorname{lcm} (\mu, \nu) = (\imath \xi) \{(\exists \rho, \sigma, \tau) . (\rho, \sigma) \operatorname{Prm}_\tau (\mu, \nu) . \xi = \rho \times_c \sigma \times_c \tau\} \quad \mathrm{Df}$

***302·1.** $\vdash :. \rho \operatorname{Prm} \sigma . \equiv : \rho, \sigma \in \mathrm{NC\,ind} : \rho = \xi \times_c \tau . \sigma = \eta \times_c \tau . \supset_{\xi,\eta,\tau} . \tau = 1$
[(*302·01)]

***302·11.** $\vdash : \rho \operatorname{Prm} \sigma . \equiv . \sigma \operatorname{Prm} \rho$ [*302·1]

***302·12.** $\vdash : \rho \operatorname{Prm} 1 . \equiv . \rho \in \mathrm{NC\,ind}$ [*302·1 . *117·631·61]

***302·13.** $\vdash : \rho \operatorname{Prm} \rho . \equiv . \rho = 1$

Dem.

$\vdash . *302·12 . \supset \vdash : \rho = 1 . \supset . \rho \operatorname{Prm} \rho$ (1)

$\vdash . *302·1 . \supset \vdash :. \rho \operatorname{Prm} \rho . \supset : \rho = 1 \times_c \rho . \supset . \rho = 1 :$

[*113·621] $\supset : \rho = 1$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***302·14.** $\vdash : 0 \operatorname{Prm} \mu . \equiv . \mu = 1$

Dem.

$\vdash . *302·12 . \supset \vdash : \mu = 1 . \supset . 0 \operatorname{Prm} \mu$ (1)

$\vdash . *302·1 . \supset \vdash :. 0 \operatorname{Prm} \mu . \supset : 0 = 0 \times_c \mu . \mu = 1 \times_c \mu . \supset . \mu = 1 :$

[*113·601·621] $\supset : \mu = 1$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***302·15.** $\vdash :. \mu, \nu \in \mathrm{NC\,ind} . \mu_\lambda, \nu_\lambda \in C`U . \supset :$

$(\rho, \sigma) \operatorname{Prm} (\mu, \nu) . \equiv . (\rho, \sigma) \operatorname{Prm} (\mu_\lambda, \nu_\lambda)$

Dem.

$\vdash . *126·101 . *300·14 . \supset$

$\vdash :. \mathrm{Hp} . \supset : \rho \operatorname{Prm} \sigma . \tau \in \mathrm{NC\,ind} - \iota`0 . \mu = \rho \times_c \tau . \nu = \sigma \times_c \tau . \equiv .$

$\rho \operatorname{Prm} \sigma . \tau \in \mathrm{NC\,ind} - \iota`0 . \mu_\lambda = \rho \times_c \tau . \nu_\lambda = \sigma \times_c \tau$ (1)

$\vdash . (1) . (*302·02·03) . \supset \vdash . \mathrm{Prop}$

***302·2.** $\vdash : \mu, \nu \in C`U . \sim(\mu = \nu = 0) . \kappa = \hat{\tau}\{(\exists \rho, \sigma) . \mu = \rho \times_c \tau . \nu = \sigma \times_c \tau\} . \supset .$

$\mathrm{E} ! \max (\breve{U})`\kappa . \max (\breve{U})`\kappa \in \mathrm{D}`U$

Dem.

$\vdash . *113·621 . \supset \vdash : \mathrm{Hp} . \supset . 1 \in \kappa$ (1)

$\vdash . *117·62 . *113·602 . \mathrm{Transp} . \supset$

$\vdash :. \mathrm{Hp} . \tau \in \kappa . \supset : \tau \leqslant \mu . \vee . \tau \leqslant \nu$ (2)

$\vdash . (1) . (2) . *300·21·22 . *261·26 . *300·26 . \supset \vdash . \mathrm{Prop}$

In the above proposition we write "$\max (\breve{U})`\kappa$" rather than "$\min (U)`\kappa$," because, since U arranges the natural numbers in *descending* order, "$\min (U)`\kappa$" is the *greatest* number which is a member of $\kappa$, and therefore it is less confusing to speak of this number as "$\max (\breve{U})`\kappa$."

***302·21.** $\vdash : \text{Hp} *302\cdot2 \,.\, \tau = \max(\breve{U})\text{‘}\kappa \,.\, \mu = \rho \times_c \tau \,.\, \nu = \sigma \times_c \tau \,.\supset.$
$$(\rho, \sigma) \,\text{Prm}_\tau\, (\mu, \nu)$$

Dem.

$\vdash . *13\cdot12 . \supset \vdash : \text{Hp} \,.\, \rho = \rho' \times_c \tau' \,.\, \sigma = \sigma' \times_c \tau' \,.\supset.$
$$\mu = \rho' \times_c \tau' \times_c \tau \,.\, \nu = \sigma' \times_c \tau' \times_c \tau \,.$$
$[*113\cdot602.\text{Transp.Hp}] \quad \supset . \tau' \times_c \tau \neq 0 \,.\, \tau' \times_c \tau \leqslant \tau \,.$
$[*120\cdot511.*117\cdot62] \quad \supset . \tau' = 1 \qquad (1)$
$\vdash . (1) . *302\cdot1 . \supset \vdash : \text{Hp} . \supset . \rho \,\text{Prm}\, \sigma \qquad (2)$
$\vdash . (2) . *302\cdot2 . (*302\cdot02) . \supset \vdash . \text{Prop}$

***302·22.** $\vdash :. \mu, \nu \in \text{NC ind} \,.\, \sim(\mu = \nu = 0) \,.\supset : (\exists \rho, \sigma, \tau) \,.\, (\rho, \sigma) \,\text{Prm}_\tau\, (\mu, \nu) :$
$$(\exists \rho, \sigma) \,.\, (\rho, \sigma) \,\text{Prm}\, (\mu, \nu)$$
$[*302\cdot2\cdot21 \,.\, *300\cdot17 \,.\, (*302\cdot03)]$

***302·23.** $\vdash :. \mu, \nu \in \text{D‘}U \,.\supset : (\exists \rho, \sigma) : \rho, \sigma \in \text{D‘}U \,.\, \mu \times_c \sigma = \nu \times_c \rho :$
$$\xi, \eta \in \text{D‘}U \,.\, \mu \times_c \eta = \nu \times_c \xi \,.\supset_{\xi,\eta} .\, \xi \geqslant \rho \,.\, \eta \geqslant \sigma$$

Dem.

$\vdash . *300\cdot23 \,.\, *113\cdot27 . \supset$
$\vdash : \text{Hp} \,.\, \kappa = \text{D‘}U \cap \hat{\rho}\{(\exists \sigma) \,.\, \mu \times_c \sigma = \nu \times_c \rho\} \,.\supset .\, \text{E}\,!\min(\breve{U})\text{‘}\kappa \qquad (1)$
$\vdash . (1) . *300\cdot12 . \supset$
$\vdash :. \text{Hp} . \supset : (\exists \rho, \sigma) : \rho, \sigma \in \text{D‘}U \,.\, \mu \times_c \sigma = \nu \times_c \rho :$
$$\xi, \eta \in \text{D‘}U \,.\, \mu \times_c \eta = \nu \times_c \xi \,.\supset_{\xi,\eta} .\, \xi \geqslant \rho \qquad (2)$$
$\vdash . *120\cdot51 . \supset$
$\vdash : \text{Hp} \,.\, \rho, \sigma \in \text{D‘}U \,.\, \mu \times_c \sigma = \nu \times_c \rho \,.\, \mu \times_c \eta = \nu \times_c \xi \,.\supset .\, \rho \times_c \eta = \sigma \times_c \xi \qquad (3)$
$\vdash . *117\cdot571 . \supset \vdash : \text{Hp}(3) \,.\, \xi, \eta \in \text{D‘}U \,.\, \xi \geqslant \rho \,.\supset .\, \xi \times_c \sigma \geqslant \rho \times_c \sigma \qquad (4)$
$\vdash . *126\cdot51 . \quad \supset \vdash : \text{Hp}(4) \,.\, \sigma > \eta \,.\supset .\, \rho \times_c \sigma > \rho \times_c \eta \qquad (5)$
$\vdash . (4) . (5) . \quad \supset \vdash :. \text{Hp}(4) . \supset : \sigma > \eta \,.\supset .\, \xi \times_c \sigma > \rho \times_c \eta :$
$[\text{Transp}] \quad \supset : \xi \times_c \sigma = \rho \times_c \eta \,.\supset .\, \eta \geqslant \sigma \qquad (6)$
$\vdash . (2) . (3) . (6) . \supset \vdash . \text{Prop}$

***302·24.** $\vdash :. \mu, \nu, \rho, \sigma \in \text{NC ind} - \iota\text{‘}0 \,.\, \mu \times_c \sigma = \nu \times_c \rho :$
$$\mu \times_c \eta = \nu \times_c \xi \,.\, \xi, \eta \in \text{D‘}U \,.\supset_{\xi,\eta} .\, \xi \geqslant \rho \,.\, \eta \geqslant \sigma : \supset .\, \rho \,\text{Prm}\, \sigma$$

Dem.

$\vdash . *302\cdot1 . \supset$
$\vdash : \rho, \sigma \in \text{D‘}U \,.\, \sim(\rho \,\text{Prm}\, \sigma) \,.\supset .\, (\exists \xi, \eta, \tau) \,.\, \tau \neq 1 \,.\, \rho = \xi \times_c \tau \,.\, \sigma = \eta \times_c \tau$
$[*113\cdot203\cdot602.*120\cdot511.*117\cdot62]$
$\supset . (\exists \xi, \eta, \tau) \,.\, \xi, \eta, \tau \in \text{D‘}U - \iota\text{‘}1 \,.\, \xi < \rho \,.\, \eta < \sigma \,.\, \rho = \xi \times_c \tau \,.\, \sigma = \eta \times_c \tau \qquad (1)$
$\vdash . *120\cdot51 . \supset \vdash : \mu, \nu, \rho, \sigma \in \text{D‘}U \,.\, \mu \times_c \sigma = \nu \times_c \rho \,.\, \rho = \xi \times_c \tau \,.\, \sigma = \eta \times_c \tau \,.\supset.$
$$\mu \times_c \eta = \nu \times_c \xi \qquad (2)$$
$\vdash . (1) . (2) . \supset \vdash : \mu, \nu, \rho, \sigma \in \text{D‘}U \,.\, \mu \times_c \sigma = \nu \times_c \rho \,.\, \sim(\rho \,\text{Prm}\, \sigma) \,.\supset.$
$$(\exists \xi, \eta) \,.\, \mu \times_c \eta = \nu \times_c \xi \,.\, \xi, \eta \in \text{D‘}U \,.\, \xi < \rho \,.\, \eta < \sigma \qquad (3)$$
$\vdash . (3) . \text{Transp} . *300\cdot17 . \supset \vdash . \text{Prop}$

***302·25.** $\vdash : \rho, \xi \in \mathrm{D}`U . \supset . (\exists \alpha, \beta) . \alpha \in C`U . \beta < \xi . \rho = (\alpha \times_0 \xi) +_0 \beta$

Dem.

$\vdash . *117·62 . *120·428 . \supset \vdash : \mathrm{Hp} . \supset . \rho < (\rho +_0 1) \times_0 \xi$ (1)

$\vdash . (1) . *300·23 . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{E} ! \min (\breve{U})`\hat{\alpha} \{\alpha \in C`U . \rho < (\alpha +_0 1) \times_0 \xi\} .$

[*120·414·416] $\supset . (\exists \alpha) . \alpha \in C`U . \rho < (\alpha +_0 1) \times_0 \xi . \rho \geqslant \alpha \times_0 \xi .$

[*117·31.*120·452] $\supset . (\exists \alpha, \beta) . \alpha, \beta \in C`U . \rho < (\alpha +_0 1) \times_0 \xi . \rho = (\alpha \times_0 \xi) +_0 \beta .$

[*113·671] $\supset . (\exists \alpha, \beta) . \alpha, \beta \in C`U . \rho < (\alpha \times_0 \xi) +_0 \xi . \rho = (\alpha \times_0 \xi) +_0 \beta .$

[*120·442.*117·561.Transp]

$\supset . (\exists \alpha, \beta) . \alpha \in C`U . \beta < \xi . \rho = (\alpha \times_0 \xi) +_0 \beta : \supset \vdash .$ Prop

***302·26.** $\vdash : \mathrm{Hp} *302·24 . \supset . (\rho, \sigma) \mathrm{Prm} (\mu, \nu)$

Dem.

$\vdash . *302·25 . \supset$

$\vdash : \mathrm{Hp} . \supset . (\exists \alpha, \beta, \gamma, \delta) . \mu = (\alpha \times_0 \rho) +_0 \beta . \nu = (\gamma \times_0 \sigma) +_0 \delta . \beta < \rho . \delta < \sigma$ (1)

$\vdash . *113·43 . \supset$

$\vdash : \mu = (\alpha \times_0 \rho) +_0 \beta . \nu = (\gamma \times_0 \sigma) +_0 \delta . \beta < \rho . \delta < \sigma . \mu \times_0 \sigma = \nu \times_0 \rho . \supset .$

$(\alpha \times_0 \rho \times_0 \sigma) +_0 (\beta \times_0 \sigma) = (\gamma \times_0 \rho \times_0 \sigma) +_0 (\delta \times_0 \rho) . \beta < \rho . \delta < \sigma .$ (2)

[*117·31.*120·452.*113·671]

$\supset . \alpha \times_0 \rho \times_0 \sigma < (\gamma +_0 1) \times_0 \rho \times_0 \sigma . \gamma \times_0 \rho \times_0 \sigma < (\alpha +_0 1) \times_0 \rho \times_0 \sigma .$

[*126·51] $\supset . \alpha < \gamma +_0 1 . \gamma < \alpha +_0 1 .$

[*120·429·442.*117·25] $\supset . \alpha = \gamma$ (3)

$\vdash . (2) . (3) . *120·41 . \supset \vdash : \mathrm{Hp} (2) . \supset . \beta \times_0 \sigma = \delta \times_0 \rho . \beta < \rho . \delta < \sigma .$

[Hp] $\supset . \beta = 0 . \delta = 0$ (4)

$\vdash . (3) . (4) . \supset \vdash : \mathrm{Hp} (2) . \supset . \mu = \alpha \times_0 \rho . \nu = \alpha \times_0 \sigma$ (5)

$\vdash . (1) . (5) . *302·24 . \supset \vdash .$ Prop

***302·27.** $\vdash : \mu, \nu, \rho, \sigma, \xi, \eta \in \mathrm{NC\,ind} - \iota`0 . \mu \times_0 \sigma = \nu \times_0 \rho . \mu \times_0 \eta = \nu \times_0 \xi . \supset .$
$\xi \times_0 \sigma = \eta \times_0 \rho$

Dem.

$\vdash . *113·27 . \supset \vdash : \mathrm{Hp} . \supset . \xi \times_0 \nu \times_0 \sigma = \eta \times_0 \mu \times_0 \sigma$

[Hp] $= \eta \times_0 \nu \times_0 \rho .$

[*126·41] $\supset . \xi \times_0 \sigma = \eta \times_0 \rho : \supset \vdash .$ Prop

***302·28.** $\vdash : \mathrm{Hp} *302·24 . \xi, \eta \in \mathrm{NC\,ind} - \iota`0 . \mu \times_0 \eta = \nu \times_0 \xi . \supset .$
$(\rho, \sigma) \mathrm{Prm} (\xi, \eta)$ [*302·26·27 . *300·17]

***302·29.** $\vdash : \mathrm{Hp} *302·28 . \xi \mathrm{Prm} \eta . \supset . \xi = \rho . \eta = \sigma$

Dem.

$\vdash . *302·28·1 . \supset$

$\vdash :. \mathrm{Hp} . \supset : (\exists \alpha) . \xi = \alpha \times_0 \rho . \eta = \alpha \times_0 \sigma : \xi = \alpha \times_0 \rho . \eta = \alpha \times_0 \sigma . \supset_\alpha . \alpha = 1 :$

[*14·122] $\supset : \xi = 1 \times_0 \rho . \eta = 1 \times_0 \sigma :. \supset \vdash .$ Prop

∗302·3. $\vdash : \mu, \nu, \xi, \eta \in \mathrm{NC\,ind} - \iota`0 . \mu \times_0 \eta = \nu \times_0 \xi . \supset .$
$(\exists \rho, \sigma) . (\rho, \sigma) \,\mathrm{Prm}\, (\mu, \nu) . (\rho, \sigma) \,\mathrm{Prm}\, (\xi, \eta)$

Dem.

$\vdash . ∗302·23·24 . \supset$

$\vdash :. \mathrm{Hp} . \supset : (\exists \rho, \sigma) : \rho \,\mathrm{Prm}\, \sigma . \rho, \sigma \in \mathrm{NC\,ind} - \iota`0 . \mu \times_0 \sigma = \nu \times_0 \rho :$
$\alpha, \beta \in \mathrm{D}`U . \mu \times_0 \beta = \nu \times_0 \alpha . \supset_{\alpha, \beta} . \alpha \geqslant \rho . \beta \geqslant \sigma :$

[∗302·26·28] $\supset : (\exists \rho, \sigma) . (\rho, \sigma) \,\mathrm{Prm}\, (\mu, \nu) . (\rho, \sigma) \,\mathrm{Prm}\, (\xi, \eta) :. \supset \vdash . \mathrm{Prop}$

∗302·31. $\vdash : (\rho, \sigma) \,\mathrm{Prm}\, (\mu, \nu) . \mu \,\mathrm{Prm}\, \nu . \supset . \mu = \rho . \nu = \sigma$

Dem.

$\vdash . ∗302·1 . (∗302·02·03) . \supset$

$\vdash :. \mathrm{Hp} . \supset : (\exists \tau) . \mu = \rho \times_0 \tau . \nu = \sigma \times_0 \tau : \mu = \rho \times_0 \tau . \nu = \sigma \times_0 \tau . \supset_\tau . \tau = 1 :$

[∗14·122] $\supset . \mu = \rho \times_0 1 . \nu = \sigma \times_0 1 :. \supset \vdash . \mathrm{Prop}$

∗302·32. $\vdash : \xi \,\mathrm{Prm}\, \eta . \mu \,\mathrm{Prm}\, \nu . \xi \times_0 \nu = \eta \times_0 \mu . \supset . \xi = \mu . \eta = \nu$

Dem.

$\vdash . ∗302·3·31 . \supset$

$\vdash : \mathrm{Hp} . \supset . (\exists \rho, \sigma) . \rho \,\mathrm{Prm}\, \sigma . \xi = \rho . \mu = \rho . \eta = \sigma . \nu = \sigma : \supset \vdash . \mathrm{Prop}$

∗302·33. $\vdash :. \mu, \nu, \xi, \eta \in \mathrm{NC\,ind} - \iota`0 . \supset :$
$\mu \times_0 \eta = \nu \times_0 \xi . \equiv . (\exists \rho, \sigma) . (\rho, \sigma) \,\mathrm{Prm}\, (\mu, \nu) . (\rho, \sigma) \,\mathrm{Prm}\, (\xi, \eta)$

Dem.

$\vdash . \mathrm{Id} . (∗302·02·03) . \supset \vdash : (\rho, \sigma) \,\mathrm{Prm}\, (\mu, \nu) . (\rho, \sigma) \,\mathrm{Prm}\, (\xi, \eta) . \supset .$
$(\exists \tau, \tau') . \tau, \tau' \in \mathrm{D}`U . \mu = \rho \times_0 \tau . \nu = \sigma \times_0 \tau . \xi = \rho \times_0 \tau' . \eta = \sigma \times_0 \tau' .$

[∗113·27] $\supset . (\exists \tau, \tau') . \mu \times_0 \eta = \rho \times_0 \sigma \times_0 \tau \times_0 \tau' = \nu \times_0 \xi$ (1)

$\vdash . (1) . ∗302·3 . \supset \vdash . \mathrm{Prop}$

∗302·34. $\vdash :. \mu, \nu, \xi, \eta \in \mathrm{NC\,ind} . \sim(\mu = \nu = 0) . \sim(\xi = \eta = 0) . \supset :$
$\mu \times_0 \eta = \nu \times_0 \xi . \equiv . (\exists \rho, \sigma) . (\rho, \sigma) \,\mathrm{Prm}\, (\mu, \nu) . (\rho, \sigma) \,\mathrm{Prm}\, (\xi, \eta)$

Dem.

$\vdash . ∗113·602 . \supset \vdash : \mathrm{Hp} . \mu = 0 . \nu \neq 0 . \supset . \xi = 0 . \eta \neq 0$ (1)

$\vdash . ∗113·602·621 . \supset$

$\vdash : \mu = 0 . \nu \neq 0 . \xi = 0 . \eta \neq 0 . \supset . \mu = 0 \times_0 \nu . \nu = 1 \times_0 \nu . \xi = 0 \times_0 \eta . \eta = 1 \times_0 \eta .$

[∗302·14] $\supset . (0, 1) \,\mathrm{Prm}\, (\mu, \nu) . (0, 1) \,\mathrm{Prm}\, (\xi, \eta)$ (2)

$\vdash . (1) . (2) . \supset \vdash : \mathrm{Hp} . \mu = 0 . \nu \neq 0 . \supset .$
$(\exists \rho, \sigma) . (\rho, \sigma) \,\mathrm{Prm}\, (\mu, \nu) . (\rho, \sigma) \,\mathrm{Prm}\, (\xi, \eta)$ (3)

Similarly $\vdash : \mathrm{Hp} . \nu = 0 . \mu \neq 0 . \supset .$
$(\exists \rho, \sigma) . (\rho, \sigma) \,\mathrm{Prm}\, (\mu, \nu) . (\rho, \sigma) \,\mathrm{Prm}\, (\xi, \eta)$ (4)

$\vdash . (3) . (4) . ∗302·33 . \supset \vdash . \mathrm{Prop}$

∗302·35. $\vdash :. \mu, \nu \in \mathrm{NC\,ind} . \sim(\mu = \nu = 0) . \rho \,\mathrm{Prm}\, \sigma . \supset :$
$\mu \times_0 \sigma = \nu \times_0 \rho . \equiv . (\rho, \sigma) \,\mathrm{Prm}\, (\mu, \nu)$ [∗302·34·14·31]

***302·36.** $\vdash : \mu, \nu \in \mathrm{NC\,ind} \,.\, \sim(\mu = \nu = 0) \,.\equiv.\, (\exists \rho, \sigma) \,.\, (\rho, \sigma) \,\mathrm{Prm}\, (\mu, \nu)$

Dem.

$\vdash . *302{\cdot}14 . \supset \vdash :. (\rho, \sigma) \,\mathrm{Prm}\, (\mu, \nu) . \supset :$

$\rho, \sigma \in \mathrm{NC\,ind} \,.\, \sim(\rho = \sigma = 0) : (\exists \tau) \,.\, \tau \in \mathrm{NC\,ind} - \iota'0 \,.\, \mu = \rho \times_0 \tau \,.\, \nu = \sigma \times_0 \tau :$

$[*120{\cdot}5 . *113{\cdot}602] \supset : \mu, \nu \in \mathrm{NC\,ind} \,.\, \sim(\mu = \nu = 0)$ (1)

$\vdash . (1) . *302{\cdot}22 . \supset \vdash . \mathrm{Prop}$

***302·37.** $\vdash : (\rho, \sigma) \,\mathrm{Prm}\, (\mu, \nu) . \equiv .$

$\mu, \nu \in \mathrm{NC\,ind} \,.\, \sim(\mu = 0 \,.\, \nu = 0) \,.\, \rho \,\mathrm{Prm}\, \sigma \,.\, \mu \times_0 \sigma = \nu \times_0 \rho$ [*302·35·36]

***302·38.** $\vdash : (\rho, \sigma) \,\mathrm{Prm}\, (\mu, \nu) \,.\, (\xi, \eta) \,\mathrm{Prm}\, (\mu, \nu) . \supset . \rho = \xi \,.\, \sigma = \eta$

Dem.

$\vdash . *302{\cdot}37 . \supset \vdash : \mathrm{Hp} . \supset . \rho \,\mathrm{Prm}\, \sigma \,.\, \xi \,\mathrm{Prm}\, \eta \,.\, \mu \times_0 \sigma = \nu \times_0 \rho \,.\, \mu \times_0 \eta = \nu \times_0 \xi \,.$

$\sim(\mu = 0 \,.\, \nu = 0)$ (1)

$\vdash . (1) . *302{\cdot}14 . *113{\cdot}602 . \supset \vdash : \mathrm{Hp} \,.\, \mu = 0 . \supset . \rho = 0 \,.\, \xi = 0 \,.\, \sigma = 1 \,.\, \eta = 1$ (2)

$\vdash . (1) . *302{\cdot}14 . *113{\cdot}602 . \supset \vdash : \mathrm{Hp} \,.\, \nu = 0 . \supset . \rho = 1 \,.\, \xi = 1 \,.\, \sigma = 0 \,.\, \eta = 0$ (3)

$\vdash . *302{\cdot}27 . \supset \vdash : \mathrm{Hp} \,.\, \mu \neq 0 \,.\, \nu \neq 0 . \supset . \rho \times_0 \eta = \sigma \times_0 \xi \,.$

$[(1) . *302{\cdot}32] \qquad \supset . \rho = \xi \,.\, \sigma = \eta$ (4)

$\vdash . (2) . (3) . (4) . \supset \vdash . \mathrm{Prop}$

***302·39.** $\vdash : (\rho, \sigma) \,\mathrm{Prm}\, (\mu, \nu) . \supset . \mu \geqslant \rho \,.\, \nu \geqslant \sigma$

Dem.

$\vdash . *302{\cdot}23{\cdot}36 . \supset \vdash :. \mu, \nu \in \mathrm{D}'U . \supset :$

$(\exists \rho, \sigma) : (\rho, \sigma) \,\mathrm{Prm}\, (\mu, \nu) : \xi, \eta \in \mathrm{D}'U \,.\, \mu \times_0 \eta = \nu \times_0 \xi . \supset_{\xi, \eta} . \xi \geqslant \rho \,.\, \eta \geqslant \sigma :$

$[*113{\cdot}27] \supset : (\exists \rho, \sigma) \,.\, (\rho, \sigma) \,\mathrm{Prm}\, (\mu, \nu) \,.\, \mu \geqslant \rho \,.\, \nu \geqslant \sigma :$

$[*302{\cdot}38] \supset : (\rho, \sigma) \,\mathrm{Prm}\, (\mu, \nu) . \supset . \mu \geqslant \rho \,.\, \nu \geqslant \sigma$ (1)

$\vdash . *302{\cdot}37{\cdot}14 . \supset \vdash : \mu = 0 \,.\, (\rho, \sigma) \,\mathrm{Prm}\, (\mu, \nu) . \supset . \nu \neq 0 \,.\, \rho = 0 \,.\, \sigma = 1$ (2)

Similarly $\qquad \vdash : \nu = 0 \,.\, (\rho, \sigma) \,\mathrm{Prm}\, (\mu, \nu) . \supset . \mu \neq 0 \,.\, \rho = 1 \,.\, \sigma = 0$ (3)

$\vdash . (2) . (3) . \supset \qquad \vdash :. (\rho, \sigma) \,\mathrm{Prm}\, (\mu, \nu) : \mu = 0 \,.\vee.\, \nu = 0 : \supset . \mu \geqslant \rho \,.\, \nu \geqslant \sigma$ (4)

$\vdash . (1) . (4) . *302{\cdot}36 . *300{\cdot}17 . \supset \vdash . \mathrm{Prop}$

***302·4.** $\vdash : \mu, \nu \in \mathrm{NC\,ind} \,.\, \sim(\mu = \nu = 0) . \supset . \mathrm{E}!\,\mathrm{hcf}\,(\mu, \nu)$

Dem.

$\vdash . *302{\cdot}22 . \supset \vdash : \mathrm{Hp} . \supset . (\exists \rho, \sigma, \tau) \,.\, (\rho, \sigma) \,\mathrm{Prm}_\tau\, (\mu, \nu)$ (1)

$\vdash . *302{\cdot}38 . (*302{\cdot}02{\cdot}03) . \supset$

$\vdash : (\rho, \sigma) \,\mathrm{Prm}_\tau (\mu, \nu) \,.\, (\xi, \eta) \,\mathrm{Prm}_\varpi (\mu, \nu) . \supset . \rho = \xi \,.\, \sigma = \eta \,.\, \mu = \rho \times_0 \tau \,.\, \mu = \xi \times_0 \varpi \,.$

$[*126{\cdot}41] \qquad \supset . \tau = \varpi$ (2)

$\vdash . (1) . (2) . (*302{\cdot}04) . \supset \vdash . \mathrm{Prop}$

***302·41.** $\vdash : \mu, \nu \in \mathrm{NC\,ind} \,.\, \sim(\mu = \nu = 0) . \supset . \mathrm{E}!\,\mathrm{lcm}\,(\mu, \nu)$

[Proof as in *302·4]

$*302\cdot42$. $\vdash : \mu, \nu \in \text{NC ind} . \sim(\mu = \nu = 0) . \supset . \text{hcf}(\mu, \nu) \times_c \text{lcm}(\mu, \nu) = \mu \times_c \nu$

Dem.

$\vdash . *302\cdot4\cdot41 . (*302\cdot04\cdot05) . \supset \vdash : \text{Hp} . \supset .$

$(\exists \rho, \sigma, \tau) . \mu = \rho \times_c \tau . \nu = \sigma \times_c \tau . \text{hcf}(\mu, \nu) = \tau . \text{lcm}(\mu, \nu) = \rho \times_c \sigma \times_c \tau .$

$[*113\cdot27 . *116\cdot34] \supset . (\exists \rho, \sigma, \tau) . \mu \times_c \nu = \rho \times_c \sigma \times_c \tau^2 .$

$\text{hcf}(\mu, \nu) \times_c \text{lcm}(\mu, \nu) = \rho \times_c \sigma \times_c \tau^2 : \supset \vdash . \text{Prop}$

$*302\cdot43$. $\vdash : (\rho, \sigma) \text{Prm}(\mu, \nu) . \supset . \rho \times_c \text{hcf}(\mu, \nu) = \mu . \sigma \times_c \text{hcf}(\mu, \nu) = \nu$

$[*302\cdot4 . (*302\cdot02\cdot04)]$

$*302\cdot44$. $\vdash : (\rho, \sigma) \text{Prm}(\mu, \nu) . \supset . \rho \times_c \nu = \text{lcm}(\mu, \nu) = \sigma \times_c \mu$

$[*302\cdot41 . (*302\cdot02\cdot05)]$

$*302\cdot45$. $\vdash : (\rho, \sigma) \text{Prm}(\mu, \nu) . \xi, \eta \in \text{NC ind} . \sim(\xi = \eta = 0) . \mu \times_c \eta = \nu \times_c \xi . \supset .$

$\text{lcm}(\xi, \eta) = \rho \times_c \xi = \sigma \times_c \eta$

Dem.

$\vdash . *302\cdot37 . \supset \vdash : \text{Hp} . \supset . (\rho, \sigma) \text{Prm}(\xi, \eta)$ (1)

$\vdash . (1) . *302\cdot44 . \supset \vdash . \text{Prop}$

*303. RATIOS.

Summary of *303.

In this number, we give the definition and elementary properties of the ratio μ/ν. Most of the important applications of ratios are to numerical or identical relations, *i.e.* to relations which may, in a certain sense, be called *vectors*. Neglecting identical relations for the moment, let us consider numerical relations, and to fix our ideas, let us take distances on a line. A distance on a line is a one-one relation whose converse domain (and its domain too) is the whole line. If we call two such distances R and S, we may say that they have the ratio μ/ν if, starting from some point x, ν repetitions of R take us to the same point y as we reach by μ repetitions of S, *i.e.* if $xR^\nu y \,.\, xS^\mu y$. Thus R and S will have the ratio μ/ν if $\dot{\exists} ! R^\nu \dot{\cap} S^\mu$. In order, however, to insure that $\mu/\nu = \rho/\sigma$ if $(\rho, \sigma) \operatorname{Prm} (\mu, \nu)$, it is necessary, in general, to substitute $\dot{\exists} ! R^\sigma \dot{\cap} S^\rho$ for $\dot{\exists} ! R^\nu \dot{\cap} S^\mu$. (In the above case of distances on a line, the two are equivalent.) Thus we shall say that R has the ratio μ/ν to S if $(\exists \rho, \sigma) \,.\, (\rho, \sigma) \operatorname{Prm} (\mu, \nu) \,.\, \dot{\exists} ! R^\sigma \dot{\cap} S^\rho$.

If we apply the above definition to identical relations, we find that, if $R \mathrel{\dot{\subset}} I \,.\, S \mathrel{\dot{\subset}} I$, R has the ratio μ/ν to S provided $\dot{\exists} ! R \dot{\cap} S$, *i.e.* provided $\exists ! C\text{‘}R \cap C\text{‘}S$. This application is required for dealing with zero quantities and zero ratios.

Thus we are led to the following definition of ratios:

***303·01.** $\mu/\nu = \hat{R}\hat{S} \{(\exists \rho, \sigma) \,.\, (\rho, \sigma) \operatorname{Prm} (\mu, \nu) \,.\, \dot{\exists} ! R^\sigma \dot{\cap} S^\rho\}$ Df

This definition, as it stands, requires justification in two respects: (1) we commonly think of ratios as applying to magnitudes other than relations, (2) we should not commonly include as examples of ratio certain cases included in the above definition. These two points must now be considered.

(1) In applying our theory to (say) the ratio of two masses, we note that the idea of quantity (say, of mass) in any usage depends upon a comparison of different quantities. The "vector quantity" R, which relates a quantity m_1 with a quantity m_2, is the relation arising from the existence of some definite physical process of addition by which a body of mass m_1 will be transformed into another body of mass m_2. Thus σ such steps, symbolized by R^σ,

represents the addition of the mass $\sigma(m_2 - m_1)$. Similarly if S is the relation between M_2 and M_1 which arises from the process of addition turning a body of mass M_1 into another body of mass M_2, then S^ρ symbolizes the addition of the mass $\rho(M_2 - M_1)$. Now $\exists ! R^\sigma \dot{\cap} S^\rho$ means that there are a pair of masses m and m', such that $mR^\sigma m'$ and $mS^\rho m'$. In other words, if we take a body A of mass m and transform it so as to turn it into another of mass $m + \sigma(m_2 - m_1)$, we obtain a body of the same mass m' as if we proceeded to transform A into a body of mass $m + \rho(M_2 - M_1)$. Hence $\sigma(m_2 - m_1) = \rho(M_2 - M_1)$; that is $(m_2 - m_1)/(M_2 - M_1) = \rho/\sigma$. But in our symbolism the addition of $m_2 - m_1$ is represented by the vector quantity R, and that of $M_2 - M_1$ by the vector quantity S; so in our symbolism R has to S the ratio of ρ to σ.

Thus to say that an entity possesses μ units of quantity means that, taking U to represent the unit vector quantity, U^μ relates the zero of quantity—whatever that may mean in reference to that kind of quantity—with the quantity possessed by that entity.

It can be claimed for this method of symbolizing the ideas of quantity (α) that it is always a possible method of procedure whatever view be taken of it as a representation of first principles, and (β) that it directly represents the principle "No *quantity* of any kind without a comparison of different quantities of that kind."

Furthermore analogously to our treatment of cardinal and ordinal numbers, we can define any definite quantity of some kind, say any definite quantity of mass, as being merely the class of all "bodies of equal mass" with some given body. The zero mass will be the class of all bodies of zero mass; and if there are no bodies with the properties that a body of zero mass should have, this class reduces to Λ in the appropriate type.

Thus the application of our symbolism to concrete cases demands the existence of a determinate test of "equality of quantity" of different entities, and a determinate process of "addition of quantity." The formal properties which the process of addition must possess are discussed in the numbers concerned with vector families.

(2) Having now shown that cases apparently excluded by our definition of ratio can be included, we have to show that no harm is done by our inclusion of cases which would naturally be excluded. In order that ratio may agree with our expectations it is necessary that the two relations R and S, whose ratio we are considering, should have the same converse domain. Otherwise we get such cases as the following: Let P, Q be two series, and suppose* $B'P = B'Q$, $5_P = 6_Q$, $11_P = 9_Q$, $13_P = 25_Q$, but that P and Q have no other terms in common. Then we shall have, if $R = P_1 . S = Q_1$,

$$(B'P)\, R^4\, 5_P \,.\, (B'P)\, S^5\, 5_P,$$

* For notation, cf. *121.

whence it follows that R has to S the ratio 5/4, *i.e.* we have $R(5/4)S$. But we shall also have $R(8/10)S$ and $R(24/12)S$, *i.e.* $R(4/5)S$ and $R(2/1)S$. Thus our definition does not make different ratios incompatible. In practical applications, however, when R and S are confined to one vector-family, different ratios do become incompatible, as will be proved at the beginning of Section C. And so long as we are not concerned with the applications which constitute measurement, the important thing about our definition of ratio is that it should yield the usual arithmetical properties, in particular the fundamental property

$$\mu/\nu = \rho/\sigma \,.\equiv.\, \mu \times_0 \sigma = \nu \dot{\times}_0 \rho,$$

which is proved, with our definition, in *303·39. Thus any further restriction in the definition would constitute an unnecessary complication.

In virtue of our definition of μ/ν, $\mu/\nu = \dot{\Lambda}$ if μ and ν are not both inductive cardinals, or if $\mu = \nu = 0$ (*303·11·14). We have (*303·13) $\vdash . \mu/\nu = \mathrm{Cnv}`(\nu/\mu)$, *i.e.* the converse of a ratio is its reciprocal. If $\mu = 0$, and $R(\mu/\nu)S$, R must have a part in common with identity (which we may express by saying that R is a zero vector), and S may be any numerical or identical relation whose field has a member which has the relation R to itself (*303·15). Thus if ν, σ are inductive cardinals other than 0, $0/\nu = 0/\sigma$. The common value of ratios whose numerator is 0 is the zero ratio, which we call 0_q (where "q" is intended to suggest "quantity"). The definition of 0_q is

***303·02.** $0_q = \dot{s}`0/``\mathrm{NC\,induct}$ Df

In like manner, if μ and ρ are inductive cardinals other than 0, we have $\mu/0 = \rho/0$. The common value of such ratios we call ∞_q, putting

***303·03.** $\infty_q = \dot{s}`/0``\mathrm{NC\,induct}$ Df

The properties of ratios require various existence-theorems, and in establishing existence-theorems without assuming the axiom of infinity, the question of types requires considerable care. We have

***303·211.** $\vdash : (\rho, \sigma)\,\mathrm{Prm}\,(\mu, \nu) \,.\supset.\, \mu/\nu = \rho/\sigma$

so that the existence of μ/ν does not depend upon μ and ν, but upon ρ and σ, where ρ/σ is μ/ν in its lowest terms. We may, therefore, in considering existence-theorems, confine ourselves, in the first instance, to *prime* ratios.

To prove the existence of $(\rho/\sigma) \upharpoonright t`R$, when $\rho\,\mathrm{Prm}\,\sigma$, we take two relations R and S both contained in identity. These have the ratio ρ/σ provided their fields have a member in common and $\mathrm{E}\,!\,R^\sigma . \mathrm{E}\,!\,S^\rho$. By *301·16, this requires $\rho, \sigma \in C``(U \upharpoonright t^{3}`R)$. Thus we have

***303·25.** $\vdash :. \rho\,\mathrm{Prm}\,\sigma \,.\supset:$

$$\dot{\exists}\,!\,(\rho/\sigma) \upharpoonright t`R \,.\equiv.\, \rho, \sigma \in C``(U \upharpoonright t^{3}`R) \,.\equiv.\, \rho(R), \dot{\sigma}(R) \in C`U$$

But this existence-theorem, which is obtained by supposing R and S contained in identity, is not much use in practice: what we require is the existence of a ratio between *numerical* relations. For this purpose, assuming $\rho \geqslant \sigma$ and $\sigma \neq 0$, let λ be a class of such a type that $\mathrm{Nc}'t'\lambda \geqslant \rho +_0 1$. (Such a class can always be found in some type, by $*300{\cdot}18$.) Then we have $\rho_\lambda \in \mathrm{G}'U$, and we can construct a series Q such that $C'Q$ is of the same type as λ and $\mathrm{Nc}'C'Q = \rho +_0 1$. (This is proved in $*262{\cdot}211$.) We can then choose out of Q a series P having the same beginning and end, and consisting of $\sigma +_0 1$ terms. We then have

$$(B'Q)(Q_1)^\rho(B'\breve{Q}) \,.\, (B'Q)(P_1)^\sigma(B'\breve{Q}).$$

Hence P_1 and Q_1 have the ratio ρ/σ. A similar argument applies if $\sigma \geqslant \rho$ and $\rho \neq 0$. Thus we arrive at the proposition:

$*303{\cdot}322.\ \vdash : \rho \,\mathrm{Prm}\, \sigma \,.\, \rho_\lambda, \sigma_\lambda \in \mathrm{D}'U \cap \mathrm{G}'U \,.\, \supset .\, \dot{\exists} ! (\rho/\sigma) \upharpoonright (\mathrm{Rel\,num} \cap t_{00}'\lambda)$

I.e. if ρ is prime to σ and neither is 0, and $\rho +_0 1$, $\sigma +_0 1$ both exist in the type of λ, then there are numerical relations having the ratio ρ/σ and having their fields of the same type as λ.

The case when either ρ or σ is 0 requires separate treatment. If R has to S the ratio $0/\sigma$, R must be partly contained in identity ($*303{\cdot}15$); hence we have to find a hypothesis for $\dot{\exists} ! (0/\sigma) \upharpoonright \mathrm{Rel\,num}$, since $\dot{\exists} ! (0/\sigma) \upharpoonright \mathrm{Rel\,num}$ is impossible. Since $0/\sigma = 0/1$, we only require the existence of 2 in the appropriate type, *i.e.* we have

$*303{\cdot}63.\ \vdash : \exists ! 2_\lambda \,.\, \supset .\, \dot{\exists} ! 0_q \upharpoonright (\mathrm{Rel\,num} \cap t_{00}'\lambda)$

It will be remembered that $\exists ! 2_\lambda$ is demonstrable except in the lowest type.

In the above propositions, μ and ν and ρ and σ have been typically indefinite. Ratios of typically definite inductive cardinals are dealt with by means of $*302{\cdot}15$, which gives at once

$*303{\cdot}27.\ \vdash : \mu, \nu \in \mathrm{NC\,ind} \,.\, \mu_\lambda, \nu_\lambda \in C'U \,.\, \supset .\, \mu/\nu = \mu_\lambda/\nu_\lambda$

I.e. a ratio may, without changing its value, have its numerator and denominator specified as belonging to any type in which both exist. This enables us to take ρ and σ as typically definite cardinals in $*303{\cdot}322$, thus obtaining the proposition

$*303{\cdot}332.\ \vdash :. \rho \,\mathrm{Prm}\, \sigma \,.\, \supset : \dot{\exists} ! (\rho/\sigma) \upharpoonright (\mathrm{Rel\,num} \cap t_{11}'\rho) \,.\, \equiv .\, \rho, \sigma \in \mathrm{D}'U \cap \mathrm{G}'U$

The above existence-theorems are useful in proving

$$\alpha/\beta = \gamma/\delta \,.\, \equiv .\, \alpha \times_0 \delta = \beta \times_0 \gamma.$$

We proceed as follows: We first show ($*303{\cdot}34$) that, if ρ, σ are inductive cardinals other than 0, and $\rho +_0 1$, $\sigma +_0 1$ exist in the type of λ, we can find numerical relations R and S such that $\dot{\exists} ! R^\sigma \dot{\cap} S^\rho$, but $\eta > \sigma \,.\, \supset .\, \sim\dot{\exists} ! R^\eta$.

This is done by taking two series P and Q having the same beginning and end, and having $C'P \epsilon \sigma +_0 1 . C'Q \epsilon \rho +_0 1$. Then if $R = P_1$ and $S = Q_1$, we have

$$(B'P) R^\sigma (B'\breve{P}) . (B'P) S^\rho (B'\breve{P}) : \eta > \sigma . \supset . R^\eta = \Lambda,$$

whence the result. From this proposition it follows immediately that if $\rho \operatorname{Prm} \sigma . \xi \operatorname{Prm} \eta . \eta > \sigma$, and if $\rho_\lambda, \sigma_\lambda \epsilon \mathrm{D}'U \cap \mathrm{Cl}'U$, we can find an R and an S such that $R (\rho/\sigma) S . \sim \{R (\xi/\eta) S\}$. A similar argument applies if $\eta < \sigma$ or $\xi > \rho$ or $\xi < \rho$. Hence we find, by transposition,

∗303·341. $\vdash : \rho_\lambda, \sigma_\lambda \epsilon \mathrm{D}'U \cap \mathrm{Cl}'U . \rho \operatorname{Prm} \sigma . \xi \operatorname{Prm} \eta . (\rho/\sigma) \upharpoonright t_{00}'\lambda = (\xi/\eta) \upharpoonright t_{00}'\lambda . \supset . \rho = \xi . \sigma = \eta$

From this point on, the argument offers no difficulty. For if we have

$$\alpha/\beta = \gamma/\delta . (\rho, \sigma) \operatorname{Prm} (\alpha, \beta) . (\xi, \eta) \operatorname{Prm} (\gamma, \delta),$$

we have, by ∗303·341·211, $\rho = \xi . \sigma = \eta$. Hence, by ∗302·32, we have $\alpha \times_0 \delta = \beta \times_0 \gamma$. What is approximately the converse, *i.e.*

∗303·23. $\vdash : \mu, \nu, \xi, \eta \epsilon \mathrm{NC} \operatorname{ind} .$
$\sim (\mu = \nu = 0) . \sim (\xi = \eta = 0) . \mu \times_0 \eta = \nu \times_0 \xi . \supset . \mu/\nu = \xi/\eta$

follows at once from ∗303·211 and ∗302·3. Hence, after dealing with special cases, we find

∗303·38. $\vdash :. \alpha, \beta, \gamma, \delta \epsilon \mathrm{NC} \operatorname{ind} :$
$\alpha_\lambda, \beta_\lambda \epsilon \mathrm{Cl}'U . \mathsf{v} . \gamma_\lambda, \delta_\lambda \epsilon \mathrm{Cl}'U : \sim (\alpha = \beta = 0) . \sim (\gamma = \delta = 0) : \supset :$
$(\alpha/\beta) \upharpoonright t_{00}'\lambda = (\gamma/\delta) \upharpoonright t_{00}'\lambda . \equiv . \alpha \times_0 \delta = \beta \times_0 \gamma$

It will be observed that α/β is typically indefinite, like $\mathrm{Nc}'\xi$. But in order to insure that $\alpha/\beta = \gamma/\delta$ however the type may be determined, it is only necessary to insure that this equation holds in a type in which $(\alpha/\beta) \upharpoonright \operatorname{Rel} \operatorname{num}$ exists. When we write simply "$\alpha/\beta = \gamma/\delta$," we shall mean that this equation holds however the type may be determined; in other words, that it holds in a type in which $(\alpha/\beta) \upharpoonright \operatorname{Rel} \operatorname{num}$ exists. (There always is such a type, if $\alpha, \beta \epsilon \mathrm{NC} \operatorname{ind} - \iota'0$, in virtue of ∗303·322 and ∗300·18.) Thus we have

∗303·391. $\vdash :. \alpha, \beta \epsilon \mathrm{NC} \operatorname{ind} . \alpha_\lambda, \beta_\lambda \epsilon \mathrm{Cl}'U . \sim (\alpha = \beta = 0) . \supset :$
$(\alpha/\beta) \upharpoonright t_{00}'\lambda = (\gamma/\delta) \upharpoonright t_{00}'\lambda . \equiv . \alpha/\beta = \gamma/\delta . \equiv . \alpha \times_0 \delta = \beta \times_c \gamma$

and, in virtue of ∗303·38, we have

∗303·39. $\vdash :. \alpha, \beta, \gamma, \delta \epsilon \mathrm{NC} \operatorname{ind} . \sim (\alpha = \beta = 0) . \sim (\gamma = \delta = 0) . \supset :$
$\alpha/\beta = \gamma/\delta . \equiv . \alpha \times_0 \delta = \beta \times_0 \gamma$

This proposition is, of course, essential to the justification of our definition of ratios.

The remaining propositions of ∗303 consist (1) of applications of the theory of ratio to powers of a given numerical relation, (2) of properties of 0_q and ∞_q, (3) of a few properties of the class of ratios. This last set of propositions depends upon two new definitions, which must be briefly explained.

We have already explained that μ/ν is typically indefinite. Thus if we call μ/ν a "ratio," ratios are, like "NC ind," not strictly a class, because every class must be confined within some one type. Nevertheless it is convenient, just as in the case of NC ind, to treat ratios as if they formed a class; and, with similar precautions, we can avoid the errors into which we might be led by treating them as a proper class. We therefore put

***303·04.** $\text{Rat} = \hat{X}\{(\exists\mu, \nu) . \mu, \nu \in \text{NC ind} . \nu \neq 0 . X = \mu/\nu\}$ Df

(The condition $\nu \neq 0$ is only introduced because it is usually convenient to exclude ∞_q.) It will be observed that μ/ν is still typically indefinite if μ and ν are typically definite. This results from *303·27. But we often want typically definite ratios. We want these defined in types in which there are numerical relations having the ratios in question. Hence we put

***303·05.** $\text{Rat def} = \hat{X}\{(\exists\mu, \nu) . \mu, \nu \in \text{D}`U \cap \text{G}`U . X = (\mu/\nu) \upharpoonright t_{11}`\mu\}$ Df

Here "def" stands for "definite," and μ, ν are typically definite inductive cardinals. The desired properties of "Rat def" result from *303·322. It should be observed that, besides consisting of typically definite ratios, "Rat def" differs from "Rat" by the exclusion of 0_q. This is merely for reasons of convenience.

The properties of "Rat" and "Rat def" follow immediately from previous propositions. We have

***303·721.** $\vdash : X \in \text{Rat} - \iota`0_q . \supset . (\exists\mu) . X \upharpoonright t_{11}`\mu \in \text{Rat def}$

***303·73.** $\vdash : X \in \text{Rat def} . \supset . \dot{\exists} ! X \upharpoonright \text{Rel num}$

By *303·322; and by *303·391,

***303·76.** $\vdash :. X, Y \in \text{Rat} . X \upharpoonright t_{11}`\rho \in \text{Rat def} . \supset : X \upharpoonright t_{11}`\rho = Y \upharpoonright t_{11}`\rho . \equiv . X = Y$

If the axiom of infinity holds, every member of "Rat" except 0_q becomes a member of "Rat def" as soon as it is made typically definite. Hence

***303·78.** $\vdash : \text{Infin ax} . \supset . \text{Rat def} = \text{Rat} - \iota`0_q$

The uses of "Rat" and "Rat def" differ just as the uses of "NC ind" and "NC induct" differ. The distinction is only important so long as the axiom of infinity is not assumed.

***303·01.** $\mu/\nu = \hat{R}\hat{S}\{(\exists\rho, \sigma) . (\rho, \sigma) \text{Prm} (\mu, \nu) . \dot{\exists} ! R^\sigma \dot{\cap} S^\rho\}$ Df

In the above definition, ρ, σ, μ, ν are typically ambiguous, but ρ, σ must (by *301·16) exist in the type of $t`R$, while μ, ν need not do so; μ, ν cannot however, be null in *all* types, by *300·17.

***303·02.** $0_q = \dot{s}`0/``\text{NC induct}$ Df

***303·03.** $\infty_q = \dot{s}`/0``\text{NC induct}$ Df

***303·04.** $\text{Rat} = \hat{X}\{(\exists \mu, \nu) . \mu, \nu \in \text{NC ind} . \nu \neq 0 . X = \mu/\nu\}$ Df

***303·05.** $\text{Rat def} = \hat{X}\{(\exists \mu, \nu) . \mu, \nu \in \text{D}`U \cap \text{G}`U . X = (\mu/\nu) \upharpoonright t_{11}`\mu\}$ Df

***303·1.** $\vdash : R(\mu/\nu) S . \equiv . (\exists \rho, \sigma) . (\rho, \sigma) \text{Prm} (\mu, \nu) . \dot{\exists} ! R^\sigma \dot{\cap} S^\rho$ [(*303·01)]

***303·11.** $\vdash : \sim(\mu, \nu \in \text{NC ind}) . \supset . \mu/\nu = \dot{\Lambda}$ [*303·1 . *302·36]

***303·13.** $\vdash . \mu/\nu = \text{Cnv}`(\nu/\mu)$ [*303·1 . *302·11]

***303·14.** $\vdash . 0/0 = \dot{\Lambda}$ [*303·1 . *302·36]

***303·15.** $\vdash : R(0/\nu) S . \equiv . \nu \in \text{NC ind} - \iota`0 . \dot{\exists} ! R \dot{\cap} I \upharpoonright C`S .$
$\equiv . \nu \in \text{NC ind} - \iota`0 . \exists ! C`S \cap \hat{x}(xRx)$

Dem.

$\vdash .$ *302·14·38 . *303·1 . $\supset$

$\vdash : R(0/\nu) S . \equiv . \nu \in \text{NC ind} - \iota`0 . \dot{\exists} ! R^1 \dot{\cap} S^0 .$

[*301·2] $\equiv . \nu \in \text{NC ind} - \iota`0 . \dot{\exists} ! R \dot{\cap} I \upharpoonright C`S : \supset \vdash .$ Prop

***303·151.** $\vdash :. R, S \in \text{Rel num id} . \supset : R(0/\nu) S . \equiv .$
$\nu \in \text{NC ind} - \iota`0 . R \in \text{Rl}`I . S \in \text{Rel num id} . \exists ! C`R \cap C`S$
[*303·15 . *300·324·3]

***303·16.** $\vdash : R(\mu/0) S . \equiv . \mu \in \text{NC ind} - \iota`0 . \dot{\exists} ! S \dot{\cap} I \upharpoonright C`R .$
$\equiv . \mu \in \text{NC ind} - \iota`0 . \exists ! C`R \cap \hat{x}(xSx)$ [*303·15·13]

***303·161.** $\vdash :. R, S \in \text{Rel num id} . \supset : R(\mu/0) S . \equiv .$
$\mu \in \text{NC ind} - \iota`0 . R \in \text{Rel num id} . S \in \text{Rl}`I . \exists ! C`R \cap C`S$
[*303·151·13]

***303·17.** $\vdash :. \mu, \nu \in \text{NC ind} - \iota`0 . R, S \in \text{Rel num id} . R(\mu/\nu) S . \supset :$
$R, S \in \text{Rl}`I . \vee . R, S \in \text{Rel num}$

Dem.

$\vdash .$ *303·1 . *113·602 . $\supset$

$\vdash :: \text{Hp} . \supset :. R, S \in \text{Rel num id} : (\exists \rho, \sigma) . \rho, \sigma \in \text{NC ind} - \iota`0 . \dot{\exists} ! R^\sigma \dot{\cap} S^\rho :.$

[*300·33.*301·3]

$\supset :. S \in \text{Rel num id} :. R \in \text{Rl}`I : (\exists \rho) . \rho \in \text{NC ind} - \iota`0 . \dot{\exists} ! R \dot{\cap} S^\rho : \vee :$
$R \in \text{Rel num} : (\exists \rho, \sigma) . \rho, \sigma \in \text{NC ind} - \iota`0 . \dot{\exists} ! R^\sigma \dot{\cap} S^\rho :.$

[*300·3] $\supset :. S \in \text{Rel num id} :. R \in \text{Rl}`I . \dot{\exists} ! I \dot{\cap} S_{po} . \vee . R \in \text{Rel num} . \dot{\exists} ! J \dot{\cap} S_{po} :.$

[*300·3·33] $\supset :. R, S \in \text{Rl}`I . \vee . R, S \in \text{Rel num} :: \supset \vdash .$ Prop

***303·18.** $\vdash :. \mu, \nu \in \mathrm{D}`U \upharpoonright t^{3}`R \,.\, R, S \in \mathrm{Rl}`I \,.\, \supset :$
$$R(\mu/\nu)S \,.\equiv .\, R(0/\nu)S \,.\equiv .\, R(\mu/0)S \,.\equiv .\, \exists !\, C`R \cap C`S$$
[*303·1·151·16 . *301·3]

***303·181.** $\vdash : \dot{\exists} !\, (\mu/\nu) \,.\equiv .\, (\exists \rho, \sigma) \,.\, (\rho, \sigma) \,\mathrm{Prm}\, (\mu, \nu)$

Dem.

$\vdash . *303·1 . \supset \vdash : \dot{\exists} !\, (\mu/\nu) \,.\supset .\, (\exists \rho, \sigma) \,.\, (\rho, \sigma) \,\mathrm{Prm}\, (\mu, \nu)$ (1)

$\vdash . *301·3 . *300·325·17 . \supset \vdash : (\rho, \sigma) \,\mathrm{Prm}\, (\mu, \nu) \,.\supset .\, (\exists x) \,.\, (x \downarrow x)(\mu/\nu)(x \downarrow x)$ (2)

$\vdash . (1) . (2) . \supset \vdash .$ Prop

In the above proposition, if μ/ν is typically indefinite, so that "$\dot{\exists} !\, \mu/\nu$" only asserts existence in a sufficiently high type, ρ, σ may also be typically indefinite. But if μ/ν is to be taken in a definite type, ρ and σ must be taken in the corresponding type, and must not be null in that type. This is proved later.

***303·182.** $\vdash :. 0/0 = \mu/\nu \,.\equiv : \sim(\mu, \nu \in \mathrm{NC\,ind}) \,.\vee .\, \mu = \nu = 0$

Here the equation $0/0 = \mu/\nu$ is assumed to hold in a sufficiently high type.

Dem.

$\vdash . *303·14 . \supset \vdash :. 0/0 = \mu/\nu \,.\supset : \mu/\nu = \dot{\Lambda} :$

[*303·181 . *302·36] $\supset : \sim(\mu, \nu \in \mathrm{NC\,ind} - \iota`0) \,.\vee .\, \mu = \nu = 0$ (1)

$\vdash . (1) . *303·11·14 . \supset \vdash .$ Prop

***303·19.** $\vdash : R(\mu/\nu)S \,.\equiv .\, \breve{R}(\mu/\nu)\breve{S}$ [*303·1 . *121·26]

***303·2.** $\vdash :. (\rho, \sigma) \,\mathrm{Prm}\, (\mu, \nu) \,.\supset : R(\mu/\nu)S \,.\equiv .\, \dot{\exists} !\, R^{\sigma} \dot{\cap} S^{\rho}$

Dem.

$\vdash . *303·1 . \qquad \supset \vdash : \mathrm{Hp} \,.\, \dot{\exists} !\, R^{\sigma} \dot{\cap} S^{\rho} \,.\supset .\, R(\mu/\nu)S$ (1)

$\vdash . *302·38 . *303·1 . \supset \vdash : \mathrm{Hp} \,.\, R(\mu/\nu)S \,.\supset .\, \dot{\exists} !\, R^{\sigma} \dot{\cap} S^{\rho}$ (2)

$\vdash . (1) . (2) . \supset \vdash .$ Prop

***303·21.** $\vdash :. \rho \,\mathrm{Prm}\, \sigma \,.\supset : R(\rho/\sigma)S \,.\equiv .\, \dot{\exists} !\, R^{\sigma} \dot{\cap} S^{\rho}$ [*302·31 . *303·1]

***303·211.** $\vdash : (\rho, \sigma) \,\mathrm{Prm}\, (\mu, \nu) \,.\supset .\, \mu/\nu = \rho/\sigma$ [*303·2·21]

***303·22.** $\vdash : \rho \,\mathrm{Prm}\, \sigma \,.\, \mu, \nu \in \mathrm{NC\,ind} \,.\, \sim(\mu = \nu = 0) \,.\, \mu \times_c \sigma = \nu \times_c \rho \,.\supset .\, \mu/\nu = \rho/\sigma$
[*302·37 . *303·211]

***303·23.** $\vdash : \mu, \nu, \xi, \eta \in \mathrm{NC\,ind} \,.\, \sim(\mu = \nu = 0) \,.\, \sim(\xi = \eta = 0) \,.\, \mu \times_c \eta = \nu \times_c \xi \,.\supset .$
$\mu/\nu = \xi/\eta$ [*302·3 . *303·211]

***303·24.** $\vdash : \mu, \nu \in \mathrm{NC\,ind} \,.\, \sim(\mu = \nu = 0) \,.\supset .\, (\exists \rho, \sigma) \,.\, \rho \,\mathrm{Prm}\, \sigma \,.\, \mu/\nu = \rho/\sigma$
[*303·211 . *302·22]

The following propositions give typically definite existence-theorems for ratios.

***303·25.** $\vdash :. \rho \operatorname{Prm} \sigma . \supset : \dot{\exists} ! (\rho/\sigma) \upharpoonright t\text{‘}R . \equiv . \rho, \sigma \in C\text{‘}(U \upharpoonright t^{3}\text{‘}R) . \equiv . \rho(R), \sigma(R) \in C\text{‘}U$

I.e. if $\rho \operatorname{Prm} \sigma$, there are relations of the same type as R and having the ratio ρ/σ when, and only when, the number of relations of the same type as R is at least as great as ρ and at least as great as σ.

Dem.

$\vdash . *303\cdot21 . \supset \vdash :. \mathrm{Hp} . \supset : \dot{\exists} ! (\rho/\sigma) \upharpoonright t\text{‘}R . \supset . (\exists S, T) . \mathrm{E} ! S^{\sigma} . \mathrm{E} ! T^{\rho} . S, T \in t\text{‘}R .$

$[*301\cdot16] \qquad \supset . \rho, \sigma \in C\text{‘}U \upharpoonright t^{3}\text{‘}R \qquad (1)$

$\vdash . *301\cdot16\cdot3 . \supset \vdash :. \mathrm{Hp} . \supset :$

$\rho, \sigma \in C\text{‘}U \upharpoonright t^{3}\text{‘}R . x \in t_0\text{‘}C\text{‘}R . \supset . (x \downarrow x)^{\rho} = (x \downarrow x)^{\sigma} = x \downarrow x \qquad (2)$

$\vdash . (2) . *303\cdot21 . \supset$

$\vdash :. \mathrm{Hp} . \supset : \rho, \sigma \in C\text{‘}U \upharpoonright t^{3}\text{‘}R . x \in t_0\text{‘}C\text{‘}R . \supset . (x \downarrow x)(\rho/\sigma)(x \downarrow x) \qquad (3)$

$\vdash . (1) . (3) . *63\cdot18 . \supset \vdash . \mathrm{Prop}$

***303·251.** $\vdash : \mu, \nu \in C\text{‘}U \upharpoonright t^{3}\text{‘}R . \sim (\mu = \nu = 0) . \supset . \dot{\exists} ! (\mu/\nu) \upharpoonright t\text{‘}R$

Dem.

$\vdash . *302\cdot36\cdot39 . \supset \vdash : \mathrm{Hp} . \supset . (\exists \rho, \sigma) . (\rho, \sigma) \operatorname{Prm} (\mu, \nu) . \mu \geqslant \rho . \nu \geqslant \sigma .$

$[*117\cdot32] \qquad \supset . (\exists \rho, \sigma) . (\rho, \sigma) \operatorname{Prm} (\mu, \nu) . \rho, \sigma \in C\text{‘}U \upharpoonright t^{3}\text{‘}R .$

$[*303\cdot211\cdot25] \qquad \supset . \dot{\exists} ! (\mu/\nu) \upharpoonright t\text{‘}R : \supset \vdash . \mathrm{Prop}$

***303·252.** $\vdash : \mu, \nu \in \mathrm{NC\,ind} \cap C\text{‘}U \upharpoonright t^{2}\text{‘}C\text{‘}R . \sim (\mu = \nu = 0) . \supset . \dot{\exists} ! (\mu/\nu) \upharpoonright t\text{‘}R$

Dem.

$\vdash . *64\cdot51\cdot55 . \supset \vdash : \mu = \mathrm{Nc}\text{‘}\alpha . \alpha \in t\text{‘}C\text{‘}R . x \in t_0\text{‘}C\text{‘}R . \supset . \downarrow x\text{‘‘}\alpha \in \mu \cap t^{2}\text{‘}R \qquad (1)$

$\vdash . (1) . *300\cdot14 . \supset \vdash : \mathrm{Hp} . \supset . \mu, \nu \in C\text{‘}U \upharpoonright t^{3}\text{‘}R \qquad (2)$

$\vdash . (2) . *303\cdot251 . \supset \vdash . \mathrm{Prop}$

In the above proof, μ, ν are assumed to be typically indefinite. If they are typically definite, $\mathrm{sm}\text{‘‘}\mu$ and $\mathrm{sm}\text{‘‘}\nu$ must be substituted for μ and ν on the right-hand side of (1) and (2). The hypothesis "$\mu, \nu \in \mathrm{NC\,ind} \cap C\text{‘}U \upharpoonright t^{2}\text{‘}C\text{‘}R$" is a convenient abbreviation for

"$\mu, \nu \in \mathrm{NC\,ind} . \mu \cap t\text{‘}C\text{‘}R, \nu \cap t\text{‘}C\text{‘}R \in C\text{‘}U \upharpoonright t^{2}\text{‘}C\text{‘}R .$"

By *65·13,

$$\mu \cap t\text{‘}C\text{‘}R \in C\text{‘}U \upharpoonright t^{2}\text{‘}C\text{‘}R . \equiv . \mu \subset t\text{‘}C\text{‘}R . \mu \in C\text{‘}U \upharpoonright t^{2}\text{‘}C\text{‘}R . \equiv . \mu \in C\text{‘}U \upharpoonright t^{2}\text{‘}C\text{‘}R.$$

But "$\mu \in C\text{‘}U \upharpoonright t^{2}\text{‘}C\text{‘}R$" requires that μ should be typically definite, whereas "$\mu \in \mathrm{NC\,ind}$" requires that μ should be typically indefinite. Hence the hypothesis of *303·252 is only defensible as an abbreviation, meaning "$\mu, \nu \in \mathrm{NC\,ind}$, and if μ, ν are given the suitable typical definition, they become members of $C\text{‘}U \upharpoonright t^{2}\text{‘}C\text{‘}R$."

***303·253.** $\vdash : \mu, \nu \in \mathrm{NC\,ind} \cap C\text{‘}U \upharpoonright t^{2}\text{‘}\lambda . \sim (\mu = \nu = 0) . \supset . \dot{\exists} ! (\mu/\nu) \upharpoonright t_{00}\text{‘}\lambda$
[*303·252]

***303·254.** $\vdash : \mu, \nu \in \mathrm{NC\,ind} . \mu_{\lambda}, \nu_{\lambda} \in C\text{‘}U . \sim (\mu = \nu = 0) . \supset . \dot{\exists} ! (\mu/\nu) \upharpoonright t_{00}\text{‘}\lambda$
[*303·253 . (*65·01)]

***303·26.** $\vdash : \mu, \nu \in \mathrm{NC\,ind} . \sim(\mu = \nu = 0) . \supset . (\exists\lambda) . \dot{\exists} ! (\mu/\nu) \upharpoonleft t_{00}`\lambda$
[*303·254 . *300·17]

***303·27.** $\vdash : \mu, \nu \in \mathrm{NC\,ind} . \mu_\lambda, \nu_\lambda \in C`U . \supset . \mu/\nu = \mu_\lambda/\nu_\lambda$ [*302·15 . *303·1]

***303·3.** $\vdash : \rho \,\mathrm{Prm}\, \sigma . \dot{\exists} ! P^{\rho \times_c \sigma} . \supset . P^\rho (\rho/\sigma) P^\sigma$

Dem.

$\vdash . *301{\cdot}16 . *14{\cdot}21 . \supset \vdash : \mathrm{Hp} . \supset . \rho \times_c \sigma \in C`U \cap t^{3}`R$ (1)

$\vdash . (1) . *301{\cdot}5 . \supset \vdash : \mathrm{Hp} . \rho \neq 0 . \sigma \neq 0 . \supset . (P^\rho)^\sigma = P^{\rho \times_c \sigma} = (P^\sigma)^\rho .$

[*303·21] $\supset . P^\rho (\rho/\sigma) P^\sigma$ (2)

$\vdash . *301{\cdot}2 . \supset \vdash : \mathrm{Hp} . \rho = 0 . \supset . P^\rho = I \upharpoonright C`P = P^{\rho \times_c \sigma} . \dot{\exists} ! I \upharpoonright C`P$ (3)

$\vdash . *302{\cdot}14 . \supset \vdash : \mathrm{Hp} . \rho = 0 . \supset . \sigma = 1 .$

[*301·2] $\supset . P^\sigma = P$ (4)

$\vdash . (3) . (4) . \supset \vdash : \mathrm{Hp} . \rho = 0 . \supset . \dot{\exists} ! (P^\rho)^\sigma \dot{\cap} (P^\sigma)^\rho .$

[*303·21] $\supset . P^\rho (\rho/\sigma) P^\sigma$ (5)

Similarly $\vdash : \mathrm{Hp} . \sigma = 0 . \supset . P^\rho (\rho/\sigma) P^\sigma$ (6)

$\vdash . (2) . (5) . (6) . \supset \vdash . \mathrm{Prop}$

***303·31.** $\vdash : \rho \,\mathrm{Prm}\, \sigma . \rho \neq 0 . \sigma \neq 0 . (\rho \times_c \sigma) \cap t`\lambda \in \mathrm{Cl}`U . \supset .$
$(\exists P) . P \in \mathrm{Rel\,num} \cap t_{00}`\lambda . P^\rho (\rho/\sigma) P^\sigma$

Dem.

$\vdash . *300{\cdot}46 . *301{\cdot}4 . \supset \vdash : \mathrm{Hp} . \supset . (\exists P) . P \in \mathrm{Rel\,num} . (B`P) P^{\rho \times_c \sigma} (B`\breve{P})$ (1)

$\vdash . (1) . *303{\cdot}3 . \supset \vdash . \mathrm{Prop}$

***303·311.** $\vdash : \rho_\lambda, \sigma_\lambda \in \mathrm{Cl}`U - \iota`0 . \rho \geqslant \sigma . \supset . (\exists P, Q) . P \in (\rho +_c 1)_r . Q \in (\sigma +_c 1)_r .$
$P, Q \in t_{00}`\lambda . Q \subseteq P . B`P = B`Q . B`\breve{P} = B`\breve{Q}$

Dem.

$\vdash . *262{\cdot}21 . \supset \vdash : \mathrm{Hp} . \supset . \exists ! (\rho +_c 1)_r \cap t_{00}`\lambda$ (1)

$\vdash . *117{\cdot}22 . \supset \vdash : \mathrm{Hp} . P \in (\rho +_c 1)_r . \supset . (\exists \alpha) . \alpha \subset C`P . \alpha \in \sigma +_c 1$ (2)

$\vdash . *261{\cdot}26 . *205{\cdot}732 . \supset$

$\vdash : \mathrm{Hp} . P \in (\rho +_c 1)_r . \alpha \subset C`P . \alpha \in \sigma +_c 1 .$

$\beta = (\alpha - \iota`\mathrm{min}_P`\alpha - \iota`\mathrm{max}_P`\alpha) \cup \iota`B`P \cup \iota`B`\breve{P} . \supset . \beta \in \sigma +_c 1 .$

[*250·141.*202·55] $\supset . P \upharpoonleft \beta \in (\sigma +_c 1)_r$ (3)

$\vdash . (1) . (2) . (3) . *205{\cdot}55 . \supset \vdash . \mathrm{Prop}$

***303·32.** $\vdash : \rho \,\mathrm{Prm}\, \sigma . \rho \geqslant \sigma . \sigma \neq 0 . \rho_\lambda \in \mathrm{Cl}`U . \supset .$
$\dot{\exists} ! (\rho/\sigma) \upharpoonleft (\mathrm{Rel\,num} \cap t_{00}`\lambda) \dot{\cap} \hat{R}\hat{S}(R_{\mathrm{po}} \subseteq S_{\mathrm{po}})$

Dem.

$\vdash . *303{\cdot}311 . \supset \vdash : \mathrm{Hp} . \supset . (\exists P, Q) . P \in (\rho +_c 1)_r . Q \in (\sigma +_c 1)_r . P, Q \in t_{00}`\lambda .$
$Q \subseteq P . B`P = B`Q . B`\breve{P} = B`\breve{Q}$ (1)

$\vdash . *300{\cdot}44{\cdot}45 . *301{\cdot}4 . \supset$

$\vdash : \mathrm{Hp} . P \in (\rho +_c 1)_r . S = P_1 . \supset . S \in \mathrm{Rel\,num} . (B`P) S^\rho (B`\breve{P})$ (2)

Similarly

$\vdash : \text{Hp} . Q \epsilon (\sigma +_0 1)_r . R = Q_1 . \supset . R \epsilon \text{Rel num} . (B'Q) R^\sigma (B'\breve{Q})$ (3)

$\vdash . (1) . (2) . (3) . *261\cdot35\cdot212 . \supset$

$\vdash : \text{Hp} . \supset . (\exists R, S) . R, S \epsilon \text{Rel num} \cap t_{00}'\lambda . R_{po} \subseteq S_{po} . \dot{\exists} ! R^\sigma \dot{\wedge} S^\rho$ (4)

$\vdash . (4) . *303\cdot21 . \supset \vdash . \text{Prop}$

***303·321.** $\vdash : \rho \text{ Prm } \sigma . \rho \neq 0 . \sigma \neq 0 . \rho_\lambda, \sigma_\lambda \epsilon \text{G}'U . \supset . \dot{\exists} ! (\rho/\sigma) \upharpoonright (\text{Rel num} \cap t_{00}'\lambda)$
[*303·32·13]

***303·322.** $\vdash : \rho \text{ Prm } \sigma . \rho_\lambda, \sigma_\lambda \epsilon \text{D}'U \cap \text{G}'U . \supset . \dot{\exists} ! (\rho/\sigma) \upharpoonright (\text{Rel num} \cap t_{00}'\lambda)$
[*303·321]

***303·323.** $\vdash : \mu, \nu \epsilon \text{NC ind} - \iota'0 . \supset . (\exists \lambda) . \dot{\exists} ! (\mu/\nu) \upharpoonright (\text{Rel num} \cap t_{00}'\lambda)$
[*303·322]

***303·324.** $\vdash : \mu, \nu \epsilon \text{NC ind} . \mu_\lambda, \nu_\lambda \epsilon \text{D}'U . \sim (\mu \text{ Prm } \nu) . \supset .$
$\dot{\exists} ! (\mu/\nu) \upharpoonright (\text{Rel num} \cap t_{00}'\lambda)$

Dem.

$\vdash . *302\cdot22 . \supset \vdash : \text{Hp} . \supset .$

$(\exists \rho, \sigma, \tau) . \rho \text{ Prm } \sigma . \rho \neq 0 . \sigma \neq 0 . \tau \neq 0 . \tau \neq 1 . \mu = \rho \times_0 \tau . \nu = \sigma \times_0 \tau . \exists ! \mu_\lambda . \exists ! \nu_\lambda .$

[*303·2·21]

$\supset . (\exists \rho, \sigma) . \rho \text{ Prm } \sigma . \rho \neq 0 . \sigma \neq 0 . \mu/\nu = \rho/\sigma . \exists ! (\rho +_0 1)_\lambda . \exists ! (\sigma +_0 1)_\lambda .$

[*303·321] $\supset . \dot{\exists} ! (\mu/\nu) \upharpoonright \text{Rel num} : \supset \vdash . \text{Prop}$

In order to the existence of $(\mu/\nu) \upharpoonright \text{Rel num}$ in any given type, it is by no means *necessary* to have $\mu, \nu \epsilon \text{D}'U$ in the corresponding type. If $\rho \text{ Prm } \sigma . \rho, \sigma \epsilon \text{D}'U \cap \text{G}'U$, $(\rho \times_0 \tau)/(\sigma \times_0 \tau)$ will exist, however great τ may be, because $(\rho \times_0 \tau)/(\sigma \times_0 \tau) = \rho/\sigma$.

***303·33.** $\vdash : \dot{\exists} ! (\mu/\nu) \upharpoonright (\text{Rel num} \cap t_{00}'\lambda) . \equiv .$
$(\exists \rho, \sigma) . (\rho, \sigma) \text{ Prm } (\mu, \nu) . \rho_\lambda, \sigma_\lambda \epsilon \text{D}'U \cap \text{G}'U$

Dem.

$\vdash . *303\cdot322\cdot211 . \supset$

$\vdash : (\rho, \sigma) \text{ Prm } (\mu, \nu) . \rho_\lambda, \sigma_\lambda \epsilon \text{D}'U \cap \text{G}'U . \supset . \dot{\exists} ! (\mu/\nu) \upharpoonright (\text{Rel num} \cap t_{00}'\lambda)$ (1)

$\vdash . *303\cdot181\cdot15\cdot16\cdot211 . \supset$

$\vdash :. \dot{\exists} ! (\mu/\nu) \upharpoonright (\text{Rel num} \cap t_{00}'\lambda) . \supset : (\exists \rho, \sigma) . (\rho, \sigma) \text{ Prm } (\mu, \nu) . \rho \neq 0 . \sigma \neq 0 .$
$\dot{\exists} ! (\rho/\sigma) \upharpoonright (\text{Rel num} \cap t_{00}'\lambda) :$

[*303·21] $\supset : (\exists \rho, \sigma) . (\rho, \sigma) \text{ Prm } (\mu, \nu) . \rho \neq 0 . \sigma \neq 0 :$
$(\exists R, S) . R, S \epsilon \text{Rel num} \cap t_{00}'\lambda . \dot{\exists} ! R^\sigma \dot{\wedge} S^\rho :$

[*301·41] $\supset : (\exists \rho, \sigma) . (\rho, \sigma) \text{ Prm } (\mu, \nu) . \rho \neq 0 . \sigma \neq 0 .$
$\exists ! (\rho +_0 1) \cap t_0'\lambda . \exists ! (\sigma +_0 1) \cap t_0'\lambda$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***303·331.** $\vdash :. \rho \text{ Prm } \sigma . \supset : \dot{\exists} ! (\rho/\sigma) \upharpoonright (\text{Rel num} \cap t_{00}'\lambda) . \equiv . \rho_\lambda, \sigma_\lambda \epsilon \text{D}'U \cap \text{G}'U$
[*303·33 . *302·31]

***303·332.** $\vdash :. \rho \operatorname{Prm} \sigma . \supset : \dot{\exists} ! (\rho/\sigma) \upharpoonright (\text{Rel num} \cap t_{11}{}^{\backprime}\rho) . \equiv . \rho, \sigma \in \text{D}^{\backprime}U \cap \text{Ꭰ}^{\backprime}U$
[*303·331]

In this proposition, ρ, σ are typically definite cardinals, whereas in *303·331 they are typically indefinite.

***303·34.** $\vdash : \rho, \sigma \in \text{NC ind} . \rho_\lambda, \sigma_\lambda \in \text{D}^{\backprime}U \cap \text{Ꭰ}^{\backprime}U . \eta > \sigma . \supset .$

$$(\exists R, S) . R, S \in \text{Rel num} \cap t_{00}{}^{\backprime}\lambda . \dot{\exists} ! R^\sigma \dot{\cap} S^\rho . \sim \{\dot{\exists} ! R^\eta \dot{\cap} S^\xi\}$$

Note that $\sim \{\dot{\exists} ! R^\eta \dot{\cap} S^\xi\}$ does not imply $\text{E} ! R^\eta$ or $\text{E} ! S^\xi$.

Dem.

$\vdash . *303·311 . \supset \vdash : \text{Hp} . \supset . (\exists P, Q, R, S) . P \in (\rho +_c 1)_r . Q \in (\sigma +_c 1)_r .$

$$P, Q \in t_{00}{}^{\backprime}\lambda . B^{\backprime}P = B^{\backprime}Q . B^{\backprime}\breve{P} = B^{\backprime}\breve{Q} . R = P_1 . S = Q_1 \quad (1)$$

As in *303·32 Dem,

$\vdash . (1) . \supset \vdash : \text{Hp} . \supset . (\exists P, Q, R, S) . P \in (\rho +_c 1)_r . Q \in (\sigma +_c 1)_r . S = P_1 . R = \breve{Q}_1 .$

$$R, S \in \text{Rel num} . (B^{\backprime}P)(R^\sigma \dot{\cap} S^\rho)(B^{\backprime}\breve{P}) .$$

[*121·48.*202·181.*301·4.*300·44]

$$\supset . (\exists R, S) . R, S \in \text{Rel num} \cap t_{00}{}^{\backprime}\lambda . \dot{\exists} ! R^\sigma \dot{\cap} S^\rho . \sim (\dot{\exists} ! R^\eta) : \supset \vdash . \text{Prop}$$

***303·341.** $\vdash : \rho_\lambda, \sigma_\lambda \in \text{D}^{\backprime}U \cap \text{Ꭰ}^{\backprime}U . \rho \operatorname{Prm} \sigma . \xi \operatorname{Prm} \eta . (\rho/\sigma) \upharpoonright t_{00}{}^{\backprime}\lambda = (\xi/\eta) \upharpoonright t_{00}{}^{\backprime}\lambda . \supset .$

$$\rho = \xi . \sigma = \eta$$

Dem.

$\vdash . *303·34·21 . \supset \vdash : \rho_\lambda, \sigma_\lambda \in \text{D}^{\backprime}U \cap \text{Ꭰ}^{\backprime}U . \rho \operatorname{Prm} \sigma . \xi \operatorname{Prm} \eta . \eta > \sigma . \supset .$

$$(\rho/\sigma) \upharpoonright t_{00}{}^{\backprime}\lambda \neq (\xi/\eta) \upharpoonright t_{00}{}^{\backprime}\lambda \quad (1)$$

$\vdash . (1) . \text{Transp} . *302·1 . \supset \vdash : \text{Hp} . \supset . \eta \leqslant \sigma \quad (2)$

$\vdash . (2) . *303·13 . \quad \supset \vdash : \text{Hp} . \supset . \xi \leqslant \rho \quad (3)$

$\vdash . (2) . (3) . *117·32 . \quad \supset \vdash : \text{Hp} . \supset . \xi_\lambda, \sigma_\lambda \in \text{Ꭰ}^{\backprime}U \quad (4)$

$\vdash . *303·322 . \quad \supset \vdash : \text{Hp} . \supset . \dot{\exists} ! (\xi/\eta) \upharpoonright \text{Rel num} .$

$[*303·11·15·16] \quad \supset . \xi \neq 0 . \eta \neq 0 \quad (5)$

$\vdash . (2) . (4) . (5) . \quad \supset \vdash : \text{Hp} . \supset . \xi_\lambda, \eta_\lambda \in \text{D}^{\backprime}U \cap \text{Ꭰ}^{\backprime}U .$

$\left[(2) . (3) . \dfrac{\xi, \eta, \rho, \sigma}{\rho, \sigma, \xi, \eta}\right] \quad \supset . \sigma \leqslant \eta . \rho \leqslant \xi \quad (6)$

$\vdash . (2) . (3) . (6) . \supset \vdash . \text{Prop}$

***303·35.** $\vdash : 1_\lambda \in \text{Ꭰ}^{\backprime}U . \xi \operatorname{Prm} \eta . (0/1) \upharpoonright t_{00}{}^{\backprime}\lambda = (\xi/\eta) \upharpoonright t_{00}{}^{\backprime}\lambda . \supset . \xi = 0 . \eta = 1$

Dem.

$\vdash . *300·14 . \supset \vdash : \text{Hp} . \supset . (\exists x, y) . x \neq y . x, y \in t_0{}^{\backprime}\lambda .$

$[*303·15] \quad \supset . (\exists x, y) . x \neq y . (x \downarrow x)(0/1)(x \downarrow y) . x \downarrow x, x \downarrow y \in t_{00}{}^{\backprime}\lambda .$

$[\text{Hp}] \quad \supset . (\exists x, y) . x \neq y . (x \downarrow x)(\xi/\eta)(x \downarrow y) .$

$[*303·16·17.\text{Transp}] \quad \supset . \xi = 0 . \quad (1)$

$[*302·14] \quad \supset . \eta = 1 \quad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***303·36.** $\vdash :. \rho_\lambda, \sigma_\lambda \in \text{Ɑ}`U . \mathbf{v} . \xi_\lambda, \eta_\lambda \in \text{Ɑ}`U : \rho \operatorname{Prm} \sigma . \xi \operatorname{Prm} \eta : \supset :$

$$(\rho/\sigma) \upharpoonright t_{00}`\lambda = (\xi/\eta) \upharpoonright t_{00}`\lambda . \equiv . \rho = \xi . \sigma = \eta$$

Dem.

$\vdash . *300·14 . *302·14 . \supset$

$\vdash :. \rho_\lambda, \sigma_\lambda \in \text{Ɑ}`U . \rho \operatorname{Prm} \sigma . \sim(\rho_\lambda, \sigma_\lambda \in \text{D}`U) . \supset : \rho = 0 . \sigma = 1 . \mathbf{v} . \rho = 1 . \sigma = 0 :$

$[*303·35·13] \supset : \xi \operatorname{Prm} \eta . (\rho/\sigma) \upharpoonright t_{00}`\lambda = (\xi/\eta) \upharpoonright t_{00}`\lambda . \supset . \rho = \xi . \sigma = \eta$ (1)

$\vdash . (1) . *303·341 . \supset \vdash . \text{Prop}$

***303·37.** $\vdash :. \alpha, \beta \in \text{NC ind} \cap \text{Ɑ}`(U \upharpoonright t^2`\lambda) . \sim(\alpha = \beta = 0) . \mathbf{v} .$

$\gamma, \delta \in \text{NC ind} \cap \text{Ɑ}`(U \upharpoonright t^2`\lambda) . \sim(\gamma = \delta = 0) : \supset :$

$$(\alpha/\beta) \upharpoonright t_{00}`\lambda = (\gamma/\delta) \upharpoonright t_{00}`\lambda . \supset . \alpha \times_c \delta = \beta \times_c \gamma$$

Dem.

$\vdash . *302·36 . *303·211 . \supset \vdash : \alpha, \beta \in \text{NC ind} . \alpha_\lambda, \beta_\lambda \in \text{Ɑ}`U . \sim(\alpha = \beta = 0) . \supset .$

$(\exists \rho, \sigma) . (\rho, \sigma) \operatorname{Prm} (\alpha, \beta) . \rho/\sigma = \alpha/\beta$ (1)

$\vdash . (1) . *303·254·181 . \supset \vdash : \text{Hp}(1) . (\alpha/\beta) \upharpoonright t_{00}`\lambda = (\gamma/\delta) \upharpoonright t_{00}`\lambda . \supset .$

$(\exists \xi, \eta) . (\xi, \eta) \operatorname{Prm} (\gamma, \delta)$ (2)

$\vdash . (1) . (2) . *302·21·22 . *303·211 . \supset$

$\vdash : \text{Hp}(2) . \supset . (\exists \rho, \sigma, \xi, \eta) . (\rho, \sigma) \operatorname{Prm} (\alpha, \beta) . (\xi, \eta) \operatorname{Prm} (\gamma, \delta) . \rho, \sigma \in \text{Ɑ}`U .$

$\rho/\sigma = \alpha/\beta = \gamma/\delta = \xi/\eta .$

$[*303·36] \supset . (\exists \rho, \sigma) . (\rho, \sigma) \operatorname{Prm} (\alpha, \beta) . (\rho, \sigma) \operatorname{Prm} (\gamma, \delta) .$

$[*302·34] \supset . \alpha \times_c \delta = \beta \times_c \gamma$ (3)

Similarly

$\vdash : \gamma, \delta \in \text{NC ind} . \gamma_\lambda, \delta_\lambda \in \text{Ɑ}`U . \sim(\gamma = \delta = 0) . (\alpha/\beta) \upharpoonright t_{00}`\lambda = (\gamma/\delta) \upharpoonright t_{00}`\lambda . \supset .$

$\alpha \times_c \delta = \beta \times_c \gamma$ (4)

$\vdash . (3) . (4) . \supset \vdash . \text{Prop}$

***303·371.** $\vdash : \alpha, \beta, \gamma, \delta \in \text{NC ind} . \alpha_\lambda, \beta_\lambda, \gamma_\lambda, \delta_\lambda \in C`U . \sim(\alpha \operatorname{Prm} \beta . \gamma \operatorname{Prm} \delta) .$

$$(\alpha/\beta) \upharpoonright t_{00}`\lambda = (\gamma/\delta) \upharpoonright t_{00}`\lambda . \supset . \alpha \times_c \delta = \beta \times_c \gamma$$

[Proof as in *303·37]

***303·38.** $\vdash :. \alpha, \beta, \gamma, \delta \in \text{NC ind} : \alpha_\lambda, \beta_\lambda \in \text{Ɑ}`U . \mathbf{v} . \gamma_\lambda, \delta_\lambda \in \text{Ɑ}`U :$

$\sim(\alpha = \beta = 0) . \sim(\gamma = \delta = 0) : \supset :$

$$(\alpha/\beta) \upharpoonright t_{00}`\lambda = (\gamma/\delta) \upharpoonright t_{00}`\lambda . \equiv . \alpha \times_c \delta = \beta \times_c \gamma \quad [*303·37·23]$$

***303·381.** $\vdash :. \alpha, \beta, \gamma, \delta \in \text{NC ind} . \alpha_\lambda, \beta_\lambda, \gamma_\lambda, \delta_\lambda \in C`U . \sim(\alpha \operatorname{Prm} \beta . \gamma \operatorname{Prm} \delta) . \supset :$

$$(\alpha/\beta) \upharpoonright t_{00}`\lambda = (\gamma/\delta) \upharpoonright t_{00}`\lambda . \equiv . \alpha \times_c \delta = \beta \times_c \gamma \quad [*303·371·23]$$

***303·39.** $\vdash :. \alpha, \beta, \gamma, \delta \in \text{NC ind} . \sim(\alpha = \beta = 0) . \sim(\gamma = \delta = 0) . \supset :$

$$\alpha/\beta = \gamma/\delta . \equiv . \alpha \times_c \delta = \beta \times_c \gamma \quad [*303·38 . *300·18]$$

***303·391.** $\vdash :. \alpha, \beta \in \text{NC ind} . \alpha_\lambda, \beta_\lambda \in \text{Ɑ}`U . \sim(\alpha = \beta = 0) . \supset :$

$$(\alpha/\beta) \upharpoonright t_{00}`\lambda = (\gamma/\delta) \upharpoonright t_{00}`\lambda . \equiv . \alpha/\beta = \gamma/\delta . \equiv . \alpha \times_c \delta = \beta \times_c \gamma$$

[*303·38·254·11·14]

Thus when α/β is used as a typically indefinite symbol, we obtain the same results as if we supposed it defined as of a type $t_{00}`\lambda$, where $\alpha +_0 1$ and $\beta +_0 1$ both exist in the type of $\lambda$, *i.e.* $\text{Nc}`t_0`\lambda > \alpha \,.\, \text{Nc}`t_0`\lambda > \beta$.

***303·392.** $\vdash :. \alpha, \beta \in \text{C}`U \,.\, \sim(\alpha = \beta = 0) \,.\, \supset : (\alpha/\beta) \upharpoonright t_{11}`\alpha = (\gamma/\delta) \upharpoonright t_{11}`\alpha \,.\, \equiv .$
$\alpha/\beta = \gamma/\delta \,.\, \equiv \,.\, \alpha \times_0 \delta = \beta \times_0 \gamma$ [*303·391·27]

This proposition differs from *303·391 by the fact that α, β have become typically definite. It will be observed that even when α and β are typically definite, α/β, like $\alpha \times_0 \beta$, remains typically indefinite.

***303·4.** $\vdash :. \rho \,\text{Prm}\, \sigma \,.\, R \in \text{Rel num} \,.\, \supset : R_\rho (\rho/\sigma) R_\sigma \,.\, \equiv . \,\dot{\exists}\,! R_{\rho \times_c \sigma}$
[*303·3·21 . *301·4]

***303·41.** $\vdash :: \mu, \nu \in \text{NC ind} \,.\, \sim(\mu = 0 \,.\, \nu = 0) \,.\, \supset :.$
$R \in \text{Rel num} \,.\, \xi = \text{lcm}(\mu, \nu) \,.\, \supset : R_\mu (\mu/\nu) R_\nu \,.\, \equiv . \,\dot{\exists}\,! R_\xi$

Dem.

$\vdash . $ *303·2 . *300·44 . $\supset$

$\vdash :. \text{Hp} \,.\, \mu \neq 0 \,.\, \nu \neq 0 \,.\, R \in \text{Rel num} \,.\, (\rho, \sigma) \,\text{Prm}_r (\mu, \nu) \,.\, \supset :$
$R_\mu (\mu/\nu) R_\nu \,.\, \equiv . \,\dot{\exists}\,! R_{\mu \times_c \sigma} \cap R_{\nu \times_c \rho} \,.$

[*302·37] $\equiv . \,\dot{\exists}\,! R_{\mu \times_c \sigma}$ (1)

$\vdash . (1) . $ *302·44 . $\supset \vdash :. \text{Hp}(1) \,.\, \xi = \text{lcm}(\mu, \nu) \,.\, \supset : R_\mu (\mu/\nu) R_\nu \,.\, \equiv . \,\dot{\exists}\,! R_\xi$ (2)

$\vdash . (2) . $ *302·22 . $\supset$

$\vdash :. \text{Hp} \,.\, \mu \neq 0 \,.\, \nu \neq 0 \,.\, R \in \text{Rel num} \,.\, \xi = \text{lcm}(\mu, \nu) \,.\, \supset : R_\mu (\mu/\nu) R_\nu \,.\, \equiv . \,\dot{\exists}\,! R_\xi$ (3)

$\vdash . $ *302·44 . $\supset$

$\vdash :. \text{Hp} \,.\, \mu = 0 \,.\, R \in \text{Rel num} \,.\, \xi = \text{lcm}(\mu, \nu) \,.\, \supset : \xi = 0 :$

[*303·15] $\supset : R_\mu (\mu/\nu) R_\nu \,.\, \equiv . \,\dot{\exists}\,! R_\xi$ (4)

Similarly

$\vdash :. \text{Hp} \,.\, \nu = 0 \,.\, R \in \text{Rel num} \,.\, \xi = \text{lcm}(\mu, \nu) \,.\, \supset : R_\mu (\mu/\nu) R_\nu \,.\, \equiv . \,\dot{\exists}\,! R_\xi$ (5)

$\vdash . (3) . (4) . (5) . \supset \vdash . \text{Prop}$

***303·42.** $\vdash :. \text{Hp} $ *303·41 $. \, \xi = \text{lcm}(\mu, \nu) \,.\, \supset : U_\mu (\mu/\nu) U_\nu \,.\, \equiv . \,\text{lcm}(\mu, \nu) \in C`U$
[*303·41 . *300·26]

***303·43.** $\vdash :. \text{Infin ax} \,.\, \supset : \mu, \nu \in \text{NC ind} \,.\, \sim(\mu = \nu = 0) \,.\, \supset_{\mu, \nu} . \, U_\mu (\mu/\nu) U_\nu$
[*303·42 . *300·14]

***303·44.** $\vdash :. \text{Hp} $ *303·42 $. \, P \in \text{Ser} \,.\, \supset : P_\mu (\mu/\nu) P_\nu \,.\, \equiv . \,\dot{\exists}\,! P_\xi$
[*303·41 . *300·44]

***303·45.** $\vdash : P \in \Omega \,\text{infin} \,.\, \mu, \nu \in \text{NC ind} \,.\, \sim(\mu = 0 \,.\, \nu = 0) \,.\, \supset . \, P_\mu (\mu/\nu) P_\nu$
[*300·44 . *303·44]

***303·46.** $\vdash :. (\rho, \sigma) \,\text{Prm}(\mu, \nu) \,.\, \xi, \eta \in \text{NC ind} \,.\, R \in \text{Rel num} \,.\, \supset :$
$R_\xi (\mu/\nu) R_\eta \,.\, \equiv . \, \xi \times_0 \sigma = \eta \times_0 \rho \,.\, \dot{\exists}\,! R_{\xi \times_c \sigma}$

Dem.

$\vdash . $ *303·211 . $\supset$

$\vdash :. \text{Hp} \,.\, \supset : R_\xi (\mu/\nu) R_\eta \,.\, \equiv . \, R_\xi (\rho/\sigma) R_\eta \,.$

[*303·21] $\equiv . \,\dot{\exists}\,! R_{\xi \times_c \sigma} \,\dot{\cap}\, R_{\eta \times_c \rho} \,.$

[*300·55] $\equiv . \, \xi \times_0 \sigma = \eta \times_0 \rho \,.\, \dot{\exists}\,! R_{\xi \times_c \sigma} :. \supset \vdash . \text{Prop}$

***303·461.** $\vdash :. \mu, \nu, \xi, \eta \in \mathrm{NC\,ind} . \sim (\mu = \nu = 0) . \sim (\xi = \eta = 0) . R \in \mathrm{Rel\,num} . \supset :$
$R_\xi (\mu/\nu) R_\eta . \equiv . \xi \times_c \nu = \eta \times_c \mu . \dot{\exists} ! R_{\mathrm{lcm}(\xi,\eta)}$

Dem.

$\vdash . *302{\cdot}45 . \supset$
$\vdash : \mathrm{Hp} . (\rho, \sigma) \mathrm{Prm} (\xi, \eta) . \supset . \xi \times_c \sigma = \mathrm{lcm} (\xi, \eta)$ (1)
$\vdash . *302{\cdot}35 . \supset$
$\vdash : \mathrm{Hp} . (\rho, \sigma) \mathrm{Prm} (\mu, \nu) . \xi \times_c \sigma = \eta \times_c \rho . \supset . (\rho, \sigma) \mathrm{Prm} (\xi, \eta) .$ (2)
$[*302{\cdot}34]$ $\supset . \xi \times_c \nu = \eta \times_c \mu$ (3)
$\vdash . *302{\cdot}35{\cdot}37 . \supset$
$\vdash : \mathrm{Hp} . (\rho, \sigma) \mathrm{Prm} (\mu, \nu) . \xi \times_c \nu = \eta \times_c \mu . \supset . \xi \times_c \sigma = \eta \times_c \rho$ (4)
$\vdash . (1) . (2) . (3) . (4) . *303{\cdot}42 . \supset \vdash . \mathrm{Prop}$

***303·47.** $\vdash :. \mathrm{Hp}\, *303{\cdot}461 . \dot{\Lambda} \sim \in \mathrm{Pot}`R . \supset : R_\xi (\mu/\nu) R_\eta . \equiv . \xi \times_c \nu = \eta \times_c \mu$
$[*303{\cdot}461]$

***303·471.** $\vdash :. \mu, \nu, \xi, \eta \in \mathrm{NC\,ind} . \sim (\mu = \nu = 0) . \sim (\xi = \eta = 0) . P \in \Omega\,\mathrm{infin} . \supset :$
$P_\xi (\mu/\nu) P_\eta . \equiv . \xi \times_c \nu = \eta \times_c \mu$
$[*303{\cdot}47 . *300{\cdot}44]$

***303·48.** $\vdash :. \mu, \nu, \xi, \eta \in \mathrm{NC\,ind} . \sim (\mu = \nu = 0) . \sim (\xi = \eta = 0) . \supset :$
$U_\xi (\mu/\nu) U_\eta . \equiv . \xi \times_c \nu = \eta \times_c \mu . \mathrm{lcm} (\xi, \eta) \in C`U$
$[*303{\cdot}461 . *300{\cdot}26]$

***303·49.** $\vdash :: \mathrm{Infin\,ax} . \supset :. \mu, \nu, \xi, \eta \in \mathrm{NC\,ind} . \sim (\mu = \nu = 0) . \supset :$
$U_\xi (\mu/\nu) U_\eta . \equiv . \xi \times_c \nu = \eta \times_c \mu$

Dem.

$\vdash . *303{\cdot}15 . \supset \vdash :. \mu, \nu, \xi, \eta \in \mathrm{NC\,ind} . \mu = 0 . \nu \neq 0 . \supset :$
$U_\xi (\mu/\nu) U_\eta . \equiv . U_\xi \in \mathrm{Rl}`I . U_\eta \in \mathrm{Rel\,num\,id} .$
$[*120{\cdot}42]$ $\equiv . \xi = 0 .$
$[*113{\cdot}602]$ $\equiv . \xi \times_c \nu = \eta \times_c \mu$ (1)
Similarly
$\vdash :. \mu, \nu, \xi, \eta \in \mathrm{NC\,ind} . \mu \neq 0 . \nu = 0 . \supset : U_\xi (\mu/\nu) U_\eta . \equiv . \xi \times_c \nu = \eta \times_c \mu$ (2)
$\vdash . (1) . (2) . *303{\cdot}48 . \supset \vdash . \mathrm{Prop}$

***303·5.** $\vdash : \rho, \sigma \in \mathrm{NC\,ind} - \iota`0 . \exists ! (\rho +_c \sigma)_\lambda . \supset .$
$(\exists P, Q) . P \in (\rho +_c 1)_r . Q \in (\sigma +_c 1)_r . P, Q \in t_{00}`\lambda .$
$B`P = B`Q . B`\breve{P} = B`\breve{Q} . C`P \cap C`Q = \iota`B`P \cup \iota`B`\breve{P}$

Dem.

$\vdash . *110{\cdot}202 . *120{\cdot}417 . \supset$
$\vdash : \mathrm{Hp} . \supset . (\exists \alpha, \beta) . \alpha, \beta \in t_0`\lambda . \alpha \in \rho +_c 1 . \beta \in \sigma -_c 1 . \alpha \cap \beta = \Lambda$ (1)

$\vdash . *262 \cdot 2 . \supset$

$\vdash : \text{Hp} . \alpha, \beta \epsilon t_0{}^\backprime\lambda . \alpha \epsilon \rho +_0 1 . \beta \epsilon \sigma -_0 1 . \alpha \cap \beta = \Lambda . \sigma \neq 2 . \supset .$

$(\exists P, S) . P, S \epsilon \Omega \cap t_{00}{}^\backprime\lambda . C^\backprime P = \alpha . C^\backprime S = \beta . \alpha \cap \beta = \Lambda .$

$[*251 \cdot 131 \cdot 141] \supset . (\exists P, S, Q) . P, S, Q \epsilon \Omega \cap t_{00}{}^\backprime\lambda . C^\backprime P = \alpha . C^\backprime S = \beta .$

$Q = B^\backprime P \leftarrow\!\!\!\!\!+ S \rightarrow\!\!\!\!\!+ B^\backprime \breve{P} . C^\backprime P \cap C^\backprime Q = \iota^\backprime B^\backprime P \cup \iota^\backprime B^\backprime Q \quad (2)$

$\vdash . *262 \cdot 2 . \supset \vdash : \text{Hp} . \alpha, \beta \epsilon t_0{}^\backprime\lambda . \alpha \epsilon \rho +_0 1 . \beta = \iota^\backprime x . x \sim \epsilon \alpha . \sigma = 2 . \supset .$

$(\exists P, Q) . P, Q \epsilon t_{00}{}^\backprime\lambda . \dot{P} \epsilon \Omega . C^\backprime P = \alpha . Q = (B^\backprime P) \downarrow x \rightarrow\!\!\!\!\!+ B^\backprime \breve{P} \quad (3)$

$\vdash . (1) . (2) . (3) . \supset \vdash . \text{Prop}$

***303·51.** $\vdash :. \rho \,\text{Prm}\, \sigma . \rho \neq 0 . \sigma \neq 0 . \exists ! (\rho +_0 \sigma)_\lambda . \supset :$

$(\exists R, S) : R, S \epsilon \text{Rel num} \cap t_{00}{}^\backprime\lambda . R(\rho/\sigma)S : \xi/\eta \neq \rho/\sigma . \supset_{\xi, \eta} . \sim R(\xi/\eta)S$

Dem.

$\vdash . *300 \cdot 44 \cdot 45 . *301 \cdot 4 . \supset$

$\vdash : \text{Hp} . P \epsilon (\rho +_0 1)_r . Q \epsilon (\sigma +_0 1)_r . S = P_1 . R = Q_1 .$

$B^\backprime P = B^\backprime Q . B^\backprime \breve{P} = B^\backprime \breve{Q} . C^\backprime P \cap C^\backprime Q = \iota^\backprime B^\backprime P \cup \iota^\backprime B^\backprime \breve{P} . \supset . \dot{\exists} ! R^\sigma \dot{\cap} S^\rho \quad (1)$

$\vdash . *301 \cdot 41 . \supset \vdash : \text{Hp}(1) . \sim (\xi = \rho . \eta = \sigma) . \supset . R^\eta \dot{\cap} S^\xi = \dot{\Lambda} \quad (2)$

$\vdash . (1) . (2) . *303 \cdot 21 . \supset$

$\vdash :. \text{Hp}(1) . \supset : R(\rho/\sigma)S : \xi \,\text{Prm}\, \eta . \sim (\xi = \rho . \eta = \sigma) . \supset_{\xi, \eta} . \sim R(\xi/\eta)S :$

$[*303 \cdot 36] \quad \supset : R(\rho/\sigma)S : \xi \,\text{Prm}\, \eta . \xi/\eta \neq \rho/\sigma . \supset_{\xi, \eta} . \sim R(\xi/\eta)S :$

$[*302 \cdot 22 . *303 \cdot 211]$

$\supset : R(\rho/\sigma)S : \xi, \eta \epsilon C^\backprime U . \sim (\xi = \eta = 0) . \xi/\eta \neq \rho/\sigma . \supset_{\xi, \eta} . \sim R(\xi/\eta)S :$

$[*303 \cdot 182] \supset : R(\rho/\sigma)S : \xi/\eta \neq \rho/\sigma . \supset_{\xi, \eta} . \sim R(\xi/\eta)S \quad (3)$

$\vdash . (3) . *300 \cdot 44 . *303 \cdot 5 . \supset \vdash . \text{Prop}$

***303·52.** $\vdash :. \mu, \nu \epsilon \text{NC ind} - \iota^\backprime 0 . \exists ! (\mu +_0 \nu)_\lambda . \supset :$

$(\exists R, S) : R, S \epsilon t_{00}{}^\backprime\lambda . R(\mu/\nu)S : \xi/\eta \neq \mu/\nu . \supset_{\xi, \eta} . \sim R(\xi/\eta)S$

Dem.

$\vdash . *303 \cdot 24 . *302 \cdot 39 . \supset$

$\vdash : \text{Hp} . \supset . (\exists \rho, \sigma) . \rho \,\text{Prm}\, \sigma . \mu/\nu = \rho/\sigma . \rho \neq 0 . \sigma \neq 0 . \exists ! \rho +_0 \sigma \quad (1)$

$\vdash . (1) . *303 \cdot 51 . \supset \vdash . \text{Prop}$

***303·6.** $\vdash : \nu \epsilon \text{NC ind} - \iota^\backprime 0 . \supset . 0/\nu = 0_q$ $\quad [*303 \cdot 15]$

***303·61.** $\vdash : \nu \epsilon \text{NC ind} - \iota^\backprime 0 . \supset . \nu/0 = \infty_q$ $\quad [*303 \cdot 16]$

***303·62.** $\vdash . 0_q = \text{Cnv}^\backprime \infty_q = \hat{R}\hat{S}(\dot{\exists} ! R \dot{\cap} I \upharpoonright C^\backprime S)$ $\quad [*303 \cdot 6 \cdot 61 \cdot 13 \cdot 15]$

***303·621.** $\vdash . 0_q \upharpoonright \text{Rel num id} = \text{Cnv}^\backprime(\text{Rel num id} \restriction \infty_q)$

$= \hat{R}\hat{S}(R \mathbin{\dot\subset} I . S \epsilon \text{Rel num id} . \exists ! C^\backprime R \cap C^\backprime S)$ $\quad [*303 \cdot 6 \cdot 61 \cdot 13 \cdot 151]$

***303·63.** $\vdash : \exists ! 2_\lambda . \supset . \dot{\exists} ! 0_q \upharpoonright (\text{Rel num} \cap t_{00}`\lambda)$

Dem.

$\vdash . *303·15·6 . \supset \vdash : x \neq y . \supset . I 0_q (x \downarrow y) : \supset \vdash . \text{Prop}$

***303·631.** $\vdash : \exists ! 2_\lambda . \supset . \dot{\exists} ! (\text{Rel num} \cap t_{00}`\lambda) \upharpoonright \infty_q$ [*303·63·62]

***303·65.** $\vdash : \exists ! 2_\lambda . \supset . 0_q \upharpoonright t_{00}`\lambda \neq \infty_q \upharpoonright t_{00}`\lambda$

Dem.

$\vdash . *303·62 . \supset \vdash : x \neq y . \supset . I 0_q (x \downarrow y) . \sim \{I \infty_q (x \downarrow y)\} : \supset \vdash . \text{Prop}$

***303·66.** $\vdash :. \exists ! 2_\lambda . \supset : (\mu/\nu) \upharpoonright t_{00}`\lambda = 0_q . \equiv . \mu = 0 . \nu \in \text{NC ind} - \iota`0$

Dem.

$\vdash . *303·6 . \supset \vdash : \mu = 0 . \nu \in \text{NC ind} - \iota`0 . \supset . \mu/\nu = 0_q$ (1)

$\vdash . *303·6·15 . \supset$

$\vdash : \mu/\nu = 0_q . \supset . \mu/\nu = \hat{R}\hat{S}(R \in \text{Rl}`I . S \in \text{Rel num id} . \exists ! C`R \cap C`S)$ (2)

$\vdash . *300·3 . \supset \vdash : \text{Hp} . \supset . (\exists x, y) . x \neq y . x \downarrow y \in \text{Rel num} \cap t_{00}`\lambda .$

[*10·24] $\supset . \exists ! (\text{Rel num id} - \text{Rl}`I) \cap t_{00}`\lambda$ (3)

$\vdash . (2) . (3) . *303·11·17 . \supset$

$\vdash :: \text{Hp} . \supset :. (\mu/\nu) \upharpoonright t_{00}`\lambda = 0_q . \supset : \mu, \nu \in \text{NC ind} : \mu = 0 . \vee . \nu = 0$ (4)

$\vdash . (2) . (3) . *303·16 . \supset$

$\vdash :. \text{Hp} . \supset : (\mu/\nu) \upharpoonright t_{00}`\lambda = 0_q . \supset . \sim (\mu \neq 0 . \nu = 0)$ (5)

$\vdash . (4) . (5) . \supset \vdash :. \text{Hp} . \supset : (\mu/\nu) \upharpoonright t_{00}`\lambda = 0_q . \supset . \mu = 0 . \nu \in \text{NC ind} - \iota`0$ (6)

$\vdash . (1) . (6) . \supset \vdash . \text{Prop}$

***303·67.** $\vdash :. \exists ! 2_\lambda . \supset : (\mu/\nu) \upharpoonright t_{00}`\lambda = \infty_q . \equiv . \nu = 0 . \mu \in \text{NC ind} - \iota`0$
[*303·66·62]

***303·7.** $\vdash : X \in \text{Rat} . \equiv . (\exists \mu, \nu) . \mu, \nu \in \text{NC ind} . \nu \neq 0 . X = \mu/\nu$
[(*303·04)]

***303·71.** $\vdash : X \in \text{Rat def} . \equiv . (\exists \mu, \nu) . \mu, \nu \in \text{D}`U \cap \text{G}`U . X = (\mu/\nu) \upharpoonright t_{11}`\mu$
[(*303·05)]

***303·72.** $\vdash : X \in \text{Rat} . \supset . (\exists \mu) . \dot{\exists} ! X \upharpoonright t_{11}`\mu$ [*303·26]

***303·721.** $\vdash : X \in \text{Rat} - \iota`0_q . \supset . (\exists \mu) . X \upharpoonright t_{11}`\mu \in \text{Rat def}$
[*300·18 . *303·7·71]

***303·73.** $\vdash : X \in \text{Rat def} . \supset . \dot{\exists} ! X \upharpoonright \text{Rel num}$ [*303·322·324]

***303·731.** $\vdash :. \rho \,\text{Prm}\, \sigma . \supset : (\rho/\sigma) \upharpoonright t_{11}`\rho \in \text{Rat def} . \equiv . \rho, \sigma \in \text{D}`U \cap \text{G}`U$
[*303·71 . *302·39]

***303·74.** $\vdash :. \rho \, \mathrm{Prm} \, \sigma \, . \, X = (\rho/\sigma) \upharpoonright t_{11}{}^{\backprime}\rho \, . \supset : \dot{\exists} ! X \upharpoonright \mathrm{Rel\,num} \, . \equiv . \, \rho, \sigma \, \epsilon \, \mathrm{D}^{\backprime}U \cap \mathrm{G}^{\backprime}U$
[*303·332]

***303·75.** $\vdash : X \, \epsilon \, \mathrm{Rat} \, . \, \dot{\exists} ! X \upharpoonright (t_{11}{}^{\backprime}\mu \cap \mathrm{Rel\,num}) \, . \supset . \, X \upharpoonright t_{11}{}^{\backprime}\mu \, \epsilon \, \mathrm{Rat\,def}$
[*303·74·71]

***303·76.** $\vdash :. \, X, Y \, \epsilon \, \mathrm{Rat} \, . \, X \upharpoonright t_{11}{}^{\backprime}\rho \, \epsilon \, \mathrm{Rat\,def} \, . \supset : X \upharpoonright t_{11}{}^{\backprime}\rho = Y \upharpoonright t_{11}{}^{\backprime}\rho \, . \equiv . \, X = Y$
[*303·391]

***303·77.** $\vdash :. \, \mathrm{Infin\,ax} \, . \supset : \mu, \nu \, \epsilon \, \mathrm{NC\,ind} - \iota^{\backprime}0 \, . \supset . \, \mu/\nu \, \epsilon \, \mathrm{Rat\,def}$
[*300·14 . *303·71]

***303·78.** $\vdash : \mathrm{Infin\,ax} \, . \supset . \, \mathrm{Rat\,def} = \mathrm{Rat} - \iota^{\backprime}0_q$ [*303·7·77]

The above two propositions assume that μ/ν in the first, and "Rat" in the second, have been made typically definite, but they hold however the type may be defined.

$*$304. THE SERIES OF RATIOS.

Summary of $*$304.

In this number we consider the relation of greater and less among ratios, and the series generated by this relation. We need two different notations, one for greater and less between typically indefinite ratios, the other for greater and less between ratios of the same type. The former is more useful where we are dealing merely with inequalities between specified ratios, but the latter is necessary when we wish to consider the *series* of ratios in order of magnitude, since a series must be composed of terms which are all of the same type. We put

$*$304·01. $X <_r Y . = . (\exists \mu, \nu, \rho, \sigma) . \mu, \nu, \rho, \sigma \in \mathrm{NC\,ind} . \sigma \neq 0 . \mu \times_c \sigma < \nu \times_c \rho .$
$X = \mu/\nu . Y = \rho/\sigma$ Df

This definition is so framed as to include 0_q but exclude ∞_q. For the relation "less than" among rationals of given type (excluding 0_q), we use the letter H, to suggest η (defined in $*$273), because, if the axiom of infinity holds, the series of rationals of a given type is an η. The definition is

$*$304·02. $H = \hat{X}\hat{Y}\{X, Y \in \mathrm{Rat\,def} . X <_r Y\}$ Df

When we wish to include 0_q in the series, we use the notation H'; thus

$*$304·03. $H' = \hat{X}\hat{Y}\{X, Y \in \mathrm{Rat\,def} \cup \iota`0_q . X <_r Y\}$ Df

(It will be observed that here $\iota`0_q$ acquires typical definiteness through the fact that it must be of the same type as "Rat def" in order to make "Rat def $\cup\, \iota`0_q$" significant.)

If the axiom of infinity does not hold, H and H' will be finite series: if $\nu +_c 1$ is the greatest integer in a given type ($\nu > 1$), the first term of H is $1/\nu$ and the last is $\nu/1$ ($*$304·281). In a higher type, we shall get a larger series for H, but at no stage shall we get an infinite series. If, on the other hand, the axiom of infinity does hold, H is a compact series ($*$304·3) without beginning or end ($*$304·31) and having $\aleph_0$ terms in its field ($*$304·32), *i.e.* H is an η ($*$304·33). In this case, $C`H = \mathrm{D}`H = \mathrm{Rat} - \iota`0_q$ ($*$304·34), *i.e.* any rational other than $0_q$, as soon as it is made typically definite, belongs to $C`H$.

Under all circumstances, H is a series (*304·23), and H exists in the type $t_{00}{}^{\backprime}\lambda$ if 3 exists in the type $t^{\backprime}\lambda$ (*304·27). In the same case, $C^{\backprime}H = \text{Rat def}$ (*304·28). Similar propositions hold for H'.

$C^{\backprime}H'$ consists of typically definite ratios, and if X is any ratio, there are types in which X belongs to $C^{\backprime}H'$ (*304·52). If the axiom of infinity holds, every ratio is a member of $C^{\backprime}H$ in every type (*304·49).

***304·01.** $X <_r Y . = . (\exists \mu, \nu, \rho, \sigma) . \mu, \nu, \rho, \sigma \in \text{NC ind} . \sigma \neq 0 . \mu \times_c \sigma < \nu \times_c \rho .$
$X = \mu/\nu . Y = \rho/\sigma$ Df

***304·02.** $H = \hat{X}\hat{Y}\{X, Y \in \text{Rat def} . X <_r Y\}$ Df

***304·03.** $H' = \hat{X}\hat{Y}\{X, Y \in \text{Rat def} \cup \iota^{\backprime}0_q . X <_r Y\}$ Df

***304·1.** $\vdash : X <_r Y . \equiv . (\exists \mu, \nu, \rho, \sigma) . \mu, \nu, \rho, \sigma \in \text{NC ind} . \mu \times_c \sigma < \nu \times_c \rho .$
$X = \mu/\nu . Y = \rho/\sigma$ [(*304·01)]

***304·11.** $\vdash : \mu/\nu <_r \rho/\sigma . \equiv . \sigma/\rho <_r \nu/\mu$ [*304·1]

***304·12.** $\vdash : X <_r Y . \equiv . \breve{Y} <_r \breve{X}$ [*304·11 . *303·13]

***304·13.** $\vdash : X <_r Y . \supset . X, Y \in \text{Rat} . Y \neq 0_q$

Dem.

$\vdash . *117·5 . \supset \vdash : \mu \times_c \sigma < \nu \times_c \rho . \supset . \nu \times_c \rho \neq 0 .$
[*113·602] $\supset . \nu \neq 0 . \rho \neq 0$ (1)
$\vdash . (1) . *304·1 . *303·7 . \supset \vdash . \text{Prop}$

***304·14.** $\vdash : XHY . \equiv . X, Y \in \text{Rat def} . X <_r Y$ [(*304·02)]

***304·15.** $\vdash : XHY . \equiv . (\exists \mu, \nu, \rho, \sigma) . \mu, \nu, \rho, \sigma \in \text{D}^{\backprime}U \cap \text{Ǎ}^{\backprime}U .$
$X = (\mu/\nu) \upharpoonright t_{11}{}^{\backprime}\mu . Y = (\rho/\sigma) \upharpoonright t_{11}{}^{\backprime}\mu . \mu \times_c \sigma < \nu \times_c \rho$
[*304·14·1 . *303·71]

***304·151.** $\vdash : XHY . \equiv . (\exists M, N, \mu) . M <_r N . M \upharpoonright t_{11}{}^{\backprime}\mu, N \upharpoonright t_{11}{}^{\backprime}\mu \in \text{Rat def} .$
$X = M \upharpoonright t_{11}{}^{\backprime}\mu . Y = N \upharpoonright t_{11}{}^{\backprime}\mu$ [*304·15]

***304·152.** $\vdash :. \mu \text{ Prm } \nu . \rho \text{ Prm } \sigma . \supset : \{(\mu/\nu) \upharpoonright t_{11}{}^{\backprime}\mu\} H \{(\rho/\sigma) \upharpoonright t_{11}{}^{\backprime}\mu\} . \equiv .$
$\mu/\nu <_r \rho/\sigma . \mu, \nu, \rho, \sigma \in \text{D}^{\backprime}U \cap \text{Ǎ}^{\backprime}U$ [*304·151 . *303·731]

***304·16.** $\vdash : (\mu/\nu) H (\rho/\sigma) . \equiv . (\sigma/\rho) H (\nu/\mu)$ [*304·15]

***304·161.** $\vdash : XHY . \equiv . \breve{Y} H \breve{X}$ [*304·12·151]

***304·2.** $\vdash . H \subseteq J$

Dem.

$\vdash . *303·37 . \supset$
$\vdash : \mu, \nu, \rho, \sigma \in \text{D}^{\backprime}U \cap \text{Ǎ}^{\backprime}U . (\mu/\nu) \upharpoonright t_{11}{}^{\backprime}\mu = (\rho/\sigma) \upharpoonright t_{11}{}^{\backprime}\mu . \supset . \mu \times_c \sigma = \nu \times_c \rho .$
[*304·15] $\supset . \sim \{(\mu/\nu) H (\rho/\sigma)\}$ (1)
$\vdash . (1) . \text{Transp} . \supset \vdash . \text{Prop}$

***304·201.** $\vdash . \sim (X <_r X)$ [Proof as in *304·2]

***304·21.** $\vdash . H \epsilon \text{trans}$

Dem.

$\vdash . *304{\cdot}15 . \supset \vdash : XHY . YHZ . \supset .$

$(\exists \mu, \nu, \rho, \sigma, \xi, \eta) . \mu, \nu, \rho, \sigma, \xi, \eta \epsilon \text{D}`U \cap \text{G}`U . \mu \times_c \sigma < \nu \times_c \rho .$

$\rho \times_c \eta < \sigma \times_c \xi . X = (\mu/\nu) \upharpoonright t_{11}`\mu . Y = (\rho/\sigma) \upharpoonright t_{11}`u . Z = (\xi/\eta) \upharpoonright t_{11}`\mu$ (1)

$\vdash . *117{\cdot}571 . *120{\cdot}51 . \supset$

$\vdash : \mu, \nu, \rho, \sigma, \xi, \eta \epsilon \text{D}`U \cap \text{G}`U . \mu \times_c \sigma < \nu \times_c \rho . \rho \times_c \eta < \sigma \times_c \xi . \supset .$

$\mu \times_c \sigma \times_c \eta < \nu \times_c \rho \times_c \eta < \nu \times_c \sigma \times_c \xi .$

$[*126{\cdot}51] \supset . \mu \times_c \eta < \nu \times_c \xi$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***304·211.** $\vdash : X <_r Y . Y <_r Z . \supset . X <_r Z$ [Proof as in *304·21]

***304·22.** $\vdash . H \epsilon \text{connex}$

Dem.

$\vdash . *126{\cdot}33 . \supset \vdash :. \mu, \nu, \rho, \sigma \epsilon \text{D}`U \cap \text{G}`U . \supset :$

$\mu \times_c \sigma < \nu \times_c \rho . \vee . \mu \times_c \sigma = \nu \times_c \rho . \vee . \mu \times_c \sigma > \nu \times_c \rho$ (1)

$\vdash . (1) . *304{\cdot}15 . \supset \vdash . \text{Prop}$

***304·221.** $\vdash :. X, Y \epsilon \text{Rat} . \supset : X <_r Y . \vee . X = Y . \vee . Y <_r X$ [Proof as in *304·22]

***304·23.** $\vdash . H \epsilon \text{Ser}$ [*304·2·21·22]

***304·24.** $\vdash : \mu, \nu \epsilon \text{D}`U \cap \text{G}`U . \nu \neq 1 . \supset . (\mu/\nu) H \{\mu/(\nu -_c 1)\}$

Dem.

$\vdash . *120{\cdot}414{\cdot}415{\cdot}416 . \supset \vdash : \text{Hp} . \supset . \nu -_c 1 \epsilon \text{D}`U \cap \text{G}`U$ (1)

$\vdash . (1) . *304{\cdot}15 . \supset \vdash . \text{Prop}$

***304·241.** $\vdash : \mu \epsilon \text{D}`U . \mu +_c 1 \epsilon \text{G}`U . \supset . (\mu/1) H \{(\mu +_c 1)/1\}$

Dem.

$\vdash . *300{\cdot}14 . \quad \supset \vdash : \text{Hp} . \supset . \mu, 1 \epsilon \text{G}`U$ (1)

$\vdash . *300{\cdot}14 . *120{\cdot}124 . \supset \vdash : \text{Hp} . \supset . \mu +_c 1 \epsilon \text{D}`U$ (2)

$\vdash . (1) . (2) . *304{\cdot}15 . \supset \vdash . \text{Prop}$

***304·25.** $\vdash : \mu, \nu \epsilon \text{D}`U \cap \text{G}`U . \sim (\mu +_c 1 = B`U . \nu = 1) . \supset . \mu/\nu \epsilon \text{D}`H . \nu/\mu \epsilon \text{G}`H$

[*304·24·241·16]

***304·251.** $\vdash : \mu +_c 1 = B`U . \supset . \mu/1 \sim\epsilon \text{D}`H$

Dem.

$\vdash . *300{\cdot}14 . \supset$

$\vdash : \text{Hp} . \rho, \sigma \epsilon \text{D}`U \cap \text{G}`U . \supset . \rho \leqslant \mu . 1 \leqslant \sigma .$

$[*117{\cdot}571] \quad \supset . \rho \times_c 1 \leqslant \mu \times_c \sigma$ (1)

$\vdash . (1) . *304{\cdot}15 . \supset \vdash . \text{Prop}$

***304·26.** $\vdash :. \mu \text{Prm} \nu . \supset : \mu/\nu \epsilon \text{D}`H . \equiv . \nu/\mu \epsilon \text{G}`H .$

$\equiv . \mu, \nu \epsilon \text{D}`U \cap \text{G}`U . \sim (\mu +_c 1 = B`U . \nu = 1)$

[*302·39 . *304·25·251·15·16]

***304·261.** $\vdash . \text{D}`H = \hat{X}\{(\exists\mu,\nu) . \mu,\nu \in \text{D}`U \cap \text{Q}`U . \sim(\mu +_c 1 = B`U . \nu = 1) . X = (\mu/\nu) \upharpoonright t_{11}`\mu\}$ [*304·25·251·15]

***304·262.** $\vdash . \text{Q}`H = \hat{X}\{(\exists\mu,\nu) . \mu,\nu \in \text{D}`U \cap \text{Q}`U . \sim(\mu +_c 1 = B`U . \nu = 1) . X = (\nu/\mu) \upharpoonright t_{11}`\mu\}$ [*304·261·16]

***304·27.** $\vdash : \dot{\exists} ! H . \equiv . \exists ! 3$

Dem.

$\vdash . *300·14 . \supset$

$\vdash :. \exists ! 3 . \supset : \mu = 1 . \nu = 2 . \supset . \mu,\nu \in \text{D}`U \cap \text{Q}`U . \sim(\mu +_c 1 = B`U . \nu = 1) .$

[*304·25] $\supset . \dot{\exists} ! H$ (1)

$\vdash . *304·261 . \supset$

$\vdash :. \dot{\exists} ! H . \supset : (\exists\mu,\nu) : \mu,\nu \in \text{D}`U \cap \text{Q}`U : \mu +_c 1 \in \text{Q}`U . \vee . \nu \neq 1 :$

[*300·14] $\supset : (\exists\mu) . \mu \geqslant 1 . \exists ! \mu +_c 2 . \vee . (\exists\nu) . \nu > 1 . \exists ! \nu +_c 1 :$

[*117·32] $\supset : \exists ! 3$ (2)

$\vdash . (1) . (2) . \supset \vdash .$ Prop

***304·28.** $\vdash : \exists ! 3 . \supset . C`H = \hat{X}\{(\exists\mu,\nu) . \mu,\nu \in \text{D}`U \cap \text{Q}`U . X = (\mu/\nu) \upharpoonright t_{11}`\mu\}$
$= \text{Rat def}$

Dem.

$\vdash . *300·14 . \supset \vdash :. \text{Hp} . \supset : \mu +_c 1 = B`U . \supset . \mu > 1$ (1)

$\vdash . (1) . \supset \vdash : \text{Hp} . \supset . \sim(\exists\mu,\nu) . \mu +_c 1 = B`U . \nu = 1 . \nu +_c 1 = B`U . \mu = 1$ (2)

$\vdash . (2) . *304·261·262 . *303·71 . \supset \vdash .$ Prop

***304·281.** $\vdash :. \exists ! 3 . \supset : \mu/\nu = B`H . \equiv . \mu = 1 . \nu +_c 1 = B`U . \equiv . \nu/\mu = B`\breve{H}$
[*304·28·261·262]

***304·282.** $\vdash . 0_q \sim\in C`H$ [*304·27·28 . *303·66]

***304·29.** $\vdash : (\mu/\nu) \, H \, (\rho/\sigma) . \mu +_c \rho, \nu +_c \sigma \in \text{Q}`U . \supset .$
$(\mu/\nu) \, H \, \{(\mu +_c \rho)/(\nu +_c \sigma)\} . \{(\mu +_c \rho)/(\nu +_c \sigma)\} \, H \, (\rho/\sigma)$

Dem.

$\vdash . *304·1 . \supset \vdash : \text{Hp} . \supset . \mu \times_c \sigma < \nu \times_c \rho .$

[*126·5] $\supset . \mu \times_c (\nu +_c \sigma) < \nu \times_c (\mu +_c \rho) .$
$(\mu +_c \rho) \times_c \sigma < (\nu +_c \sigma) \times_c \rho .$ (1)

$\vdash . (1) . *304·1 . \supset \vdash .$ Prop

***304·3.** $\vdash : \text{Infin ax} . \supset . H \in \text{Ser} \cap \text{comp}$ [*304·29·23]

***304·31.** $\vdash : \text{Infin ax} . \supset . \sim \text{E} ! B`H . \sim \text{E} ! B`\breve{H}$ [*304·281 . *300·14]

***304·32.** $\vdash : \text{Infin ax} . \supset . C`H \in \aleph_0$

Dem.

$\vdash . *304·15 . *303·211 . *302·22 . \supset$

$\vdash . \text{Nc}`C`H \leqslant \text{Nc}`\hat{X}\{(\exists\rho,\sigma) . \rho \text{ Prm } \sigma . \rho,\sigma \in \text{D}`U \cap \text{Q}`U . X = \rho/\sigma\}$

[*303·36] $\leqslant \text{Nc}`\hat{M}\{(\exists\rho,\sigma) . \rho \text{ Prm } \sigma . \rho,\sigma \in \text{D}`U \cap \text{Q}`U . M = \rho \downarrow \sigma\}$

[*33·161] $\leqslant \text{Nc}`C`U \times_c \text{Nc}`C`U$ (1)

$\vdash . (1) . *123\cdot52 . *300\cdot21 . \supset \vdash : \text{Hp} . \supset . \text{Nc}`C`H \leqslant \aleph_0$ (2)

$\vdash . *304\cdot28 . \supset$

$\vdash : \text{Hp} . \supset . \text{Nc}`C`H \geqslant \text{Nc}`\hat{X} \{(\exists \nu) . \nu \in \text{D}`U \cap \text{Q}`U . X = \nu/1\}$

$[*303\cdot36] \qquad \geqslant \text{Nc}`(\text{D}`U \cap \text{Q}`U)$

$[*300\cdot21] \qquad \geqslant \aleph_0$ (3)

$\vdash . (2) . (3) . *117\cdot23 . \supset \vdash . \text{Prop}$

***304·33.** $\vdash : \text{Infin ax} . \supset . H \in \eta$ $[*304\cdot3\cdot31\cdot32 . *273\cdot1]$

***304·34.** $\vdash : \text{Infin ax} . \supset . C`H = \text{D}`H = \text{Rat} - \iota`0_q$ $[*303\cdot78 . *304\cdot28]$

***304·4.** $\vdash : XH'Y . \equiv . X, Y \in \text{Rat def} \cup \iota`0_q . X <_r Y .$

$\equiv . (\exists \mu, \nu, \rho, \sigma) . \mu, \nu, \rho, \sigma \in \text{Q}`U . \nu \neq 0 . \sigma \neq 0 . \mu \times_c \sigma < \nu \times_c \rho .$

$X = (\mu/\nu) \upharpoonright t_{11}`\mu . Y = (\rho/\sigma) \upharpoonright t_{11}`\mu$ $[*303\cdot71 . (*304\cdot03)]$

***304·401.** $\vdash :. \text{Infin ax} . \supset : X <_r Y . \equiv . XH'Y$ $[*304\cdot4 . *303\cdot78]$

***304·41.** $\vdash . \text{D}`H' = \hat{X} \{(\exists \mu, \nu) . \mu, \nu \in \text{Q}`U . \nu \neq 0 . \sim(\mu +_c 1 = B`U . \nu = 1) .$
$X = (\mu/\nu) \upharpoonright t_{11}`\mu\}$

[Proof as in *304·261]

***304·42.** $\vdash . \text{Q}`H' = \hat{X} \{(\exists \mu, \nu) . \mu, \nu \in \text{Q}`U . \mu \neq 0 . \nu \neq 0 . X = (\mu/\nu) \upharpoonright t_{11}`\mu\}$

***304·43.** $\vdash : \dot{\exists} ! H' . \equiv . \exists ! 2$ $[*304\cdot42]$

***304·44.** $\vdash : \exists ! 2 . \supset . C`H' = \hat{X} \{(\exists \mu, \nu) . \mu, \nu \in \text{Q}`U . \nu \neq 0 . X = (\mu/\nu) \upharpoonright t_{11}`\mu\}$
$[*304\cdot41\cdot42]$

***304·45.** $\vdash : \exists ! 2 . \supset . B`H' = 0_q$ $[*304\cdot41\cdot42 . *303\cdot6]$

***304·46.** $\vdash : \exists ! 3 . \supset . H' = 0_q \leftarrow\!\!\!+ H$ $[*304\cdot45\cdot4\cdot27\cdot1]$

***304·47.** $\vdash : \text{Infin ax} . \supset . H' \in \dot{1} \dotplus \eta$ $[*304\cdot46\cdot33]$

***304·48.** $\vdash . H' \in \text{Ser}$

Dem.

$\vdash . *304\cdot4 . \supset \vdash : \exists ! 2 . \sim \exists ! 3 . \supset . H' = 0_q \downarrow (1/1)$ (1)

$\vdash . (1) . *304\cdot43\cdot46\cdot23 . \supset \vdash . \text{Prop}$

***304·49.** $\vdash : \text{Infin ax} . \supset . C`H' = \text{D}`H' = \text{Rat}$ $[*304\cdot34\cdot46]$

***304·5.** $\vdash : X \in C`H . \supset . \dot{\exists} ! X \upharpoonright \text{Rel num}$ $[*303\cdot73 . *304\cdot14]$

***304·51.** $\vdash : X \in C`H' . \supset . \dot{\exists} ! X \upharpoonright \text{Rel num}$

Dem.

$\vdash . *303\cdot63 . *304\cdot43 . \supset \vdash :. \text{Hp} . \supset . \dot{\exists} ! 0_q \upharpoonright \text{Rel num}$ (1)

$\vdash . (1) . *303\cdot73 . *304\cdot4 . \supset \vdash . \text{Prop}$

***304·52.** $\vdash : X \in \text{Rat} . \supset . (\exists \mu) . X \upharpoonright t_{11}`\mu \in C`H'$ $[*304\cdot44 . *300\cdot18]$

***304·53.** $\vdash : X \in \text{Rat} - \iota`0_q . \supset . (\exists \mu) . X \upharpoonright t_{11}`\mu \in C`H$ $[*304\cdot28 . *300\cdot18]$

*305. MULTIPLICATION OF SIMPLE RATIOS.

Summary of *305.

The ratios hitherto considered are called "simple" ratios in opposition to "generalized" ratios (introduced in *307), which include negative ratios. We deal with multiplication and addition first for simple ratios, and then for generalized ratios. In this number we are only concerned with the multiplication of simple ratios.

In defining multiplication of ratios, we naturally frame our definition so as to secure that the product of μ/ν and ρ/σ shall be $(\mu \times_c \rho)/(\nu \times_c \sigma)$. This is effected by the following definition (where "s" stands for "simple"):

***305·01.** $X \times_s Y = \hat{R}\hat{S}\{(\exists \mu, \nu, \rho, \sigma) . \mu, \nu, \rho, \sigma \in \text{NC ind} . \nu \neq 0 . \sigma \neq 0 .$
$X = \mu/\nu . Y = \rho/\sigma . R \{(\mu \times_c \rho)/(\nu \times_c \sigma)\} S\}$ Df

which gives us

***305·142.** $\vdash : \mu, \rho \in \text{NC ind} . \nu \neq 0 . \sigma \neq 0 . \supset . \mu/\nu \times_s \rho/\sigma = (\mu \times_c \rho)/(\nu \times_c \sigma)$

and

***305·144.** $\vdash : \dot{\exists} ! (\mu/\nu \times_s \rho/\sigma) . \supset . \mu/\nu \times_s \rho/\sigma = (\mu \times_c \rho)/(\nu \times_c \sigma)$

The reason for the hypotheses in these propositions is that, if μ is a cardinal which is not inductive, while $\rho = 0$ and ν, σ are inductive and not 0, $\mu/\nu = \dot{\Lambda}$ and $\mu/\nu \times_s \rho/\sigma = \dot{\Lambda}$, but $(\mu \times_c \rho)/(\nu \times_c \sigma) = 0_q$.

For the applications of the multiplication of ratios, it is essential that we should have, if R, S, T belong to a suitable vector family,

$$R (\mu/\nu) S . S (\rho/\sigma) T . \supset . R (\mu/\nu \times_s \rho/\sigma) T,$$

e.g. we want two-thirds of five-sevenths of T to be $(2/3 \times_s 5/7)$ of T. It will be shown in Section C that our definition satisfies this requirement.

We prove in this number

***305·3.** $\vdash : X, Y \in \text{Rat} . \equiv . X \times_s Y \in \text{Rat}$

***305·22.** $\vdash :. X \times_s Y = 0_q . \equiv : X, Y \in \text{Rat} : X = 0_q . \vee . Y = 0_q$

i.e. a product only vanishes when one of its factors vanishes;

***305·301.** $\vdash : X, Y \in \text{Rat} - \iota`0_q . \equiv . X \times_s Y \in \text{Rat} - \iota`0_q$

***305·25.** $\vdash : \mu, \nu, \rho, \sigma \in \mathrm{D}`U \cap \mathrm{G}`U . \supset . (\mu/\nu \times_s \rho/\sigma) \upharpoonright t_{00}`\mu \in C`H$

Thus a product of two ratios which both exist in a given type exists in the next type, *i.e.*

***305·26.** $\vdash : X, Y \in \mathrm{Rat} . X \upharpoonright t_{11}`\mu, Y \upharpoonright t_{11}`\mu \in \mathrm{Rat\ def} . \supset . (X \times_s Y) \upharpoonright t_{00}`\mu \in C`H$

The formal laws offer no difficulty. We prove the commutative law (*305·11) and the associative law (*305·41); we prove that $X \times_s 1/1 = X$ (*305·51) and that $X \times_s \breve{X} = 1/1$ (*305·52). Division results from

***305·61.** $\vdash :. A \in \mathrm{Rat} - \iota`0_q . A' \in \mathrm{Rat} . \supset : A \times_s X = A' . \equiv . X = A' \times_s \breve{A}$

and the axiom of Archimedes is given by

***305·7.** $\vdash : X, Y \in \mathrm{Rat} - \iota`0_q . \supset . (\exists \alpha) . \alpha \in \mathrm{NC\,ind} . Y <_r (\alpha/1 \times_s X)$

***305·01.** $X \times_s Y = \hat{R}\hat{S}[(\exists \mu, \nu, \rho, \sigma) . \mu, \nu, \rho, \sigma \in \mathrm{NC\,ind} . \nu \neq 0 . \sigma \neq 0 .$
$X = \mu/\nu . Y = \rho/\sigma . R \{(\mu \times_c \rho)/(\nu \times_c \sigma)\} S]$ Df

***305·1.** $\vdash : R (X \times_s Y) S . \equiv . (\exists \mu, \nu, \rho, \sigma) . \mu, \nu, \rho, \sigma \in \mathrm{NC\,ind} . \nu \neq 0 . \sigma \neq 0 .$
$X = \mu/\nu . Y = \rho/\sigma . R \{(\mu \times_c \rho)/(\nu \times_c \sigma)\} S$ [(*305·01)]

***305·11.** $\vdash . X \times_s Y = Y \times_s X$ [*305·1]

***305·12.** $\vdash : X, Y \sim \in \iota`0_q \cup \iota`\infty_q . \mathrm{Cnv}`(X \times_s Y) = \breve{X} \times_s \breve{Y}$ [*305·1 . *303·13]

***305·13.** $\vdash : \mu, \nu, \rho, \sigma \in \mathrm{NC\,ind} - \iota`0 . \mu/\nu = \mu'/\nu' . \rho/\sigma = \rho'/\sigma' . \supset .$
$(\mu \times_c \rho)/(\nu \times_c \sigma) = (\mu' \times_c \rho')/(\nu' \times_c \sigma')$

Dem.

$\vdash . *303·39 . \supset \vdash : \mathrm{Hp} . \supset . \mu \times_c \nu' = \nu \times_c \mu' . \rho \times_c \sigma' = \rho' \times_c \sigma .$
[*120·51] $\supset . \mu \times_c \rho \times_c \nu' \times_c \sigma' = \mu' \times_c \rho' \times_c \nu \times_c \sigma .$
[*303·39] $\supset . (\mu \times_c \rho)/(\nu \times_c \sigma) = (\mu' \times_c \rho')/(\nu' \times_c \sigma') : \supset \vdash . \mathrm{Prop}$

***305·131.** $\vdash : \nu, \rho, \sigma \in \mathrm{NC\,ind} - \iota`0 . 0/\nu = \mu'/\nu' . \rho/\sigma = \rho'/\sigma' . \supset .$
$(0 \times_c \rho)/(\nu \times_c \sigma) = (\mu' \times_c \rho')/(\nu' \times_c \sigma')$

Dem.

$\vdash . *303·66 . \supset \vdash : \mathrm{Hp} . \supset . \mu' = 0 . \nu' \in \mathrm{NC\,ind} - \iota`0$ (1)
$\vdash . (1) . *303·6 . \supset \vdash : \mathrm{Hp} . \supset . (0 \times_c \rho)/(\nu \times_c \sigma) = 0_q = (\mu' \times_c \rho')/(\nu' \times_c \sigma') : \supset \vdash . \mathrm{Prop}$

***305·132.** $\vdash : \mu, \nu, \rho, \sigma \in \mathrm{NC\,ind} . \nu \neq 0 . \sigma \neq 0 . \mu/\nu = \mu'/\nu' . \rho/\sigma = \rho'/\sigma' . \supset .$
$(\mu \times_c \rho)/(\nu \times_c \sigma) = (\mu' \times_c \rho')/(\nu' \times_c \sigma')$

[*305·13·131]

***305·14.** $\vdash : \mu \neq 0 . \rho \neq 0 . \nu \neq 0 . \sigma \neq 0 . \supset . \mu/\nu \times_s \rho/\sigma = (\mu \times_c \rho)/(\nu \times_c \sigma)$

Dem.

$\vdash . *305 \cdot 1 \cdot 132 . \supset$

$\vdash :: \text{Hp} . \supset :. R (\mu/\nu \times_c \rho/\sigma) S . \equiv :$

$(\exists \mu', \nu', \rho', \sigma') . \mu', \nu', \rho', \sigma' \epsilon \text{NC ind} . \mu/\nu = \mu'/\nu' . \rho/\sigma = \rho'/\sigma' . \nu' \neq 0 . \sigma' \neq 0 :$

$R \{(\mu \times_c \rho)/(\nu \times_c \sigma)\} S$ (1)

$\vdash . *303 \cdot 181 . *302 \cdot 36 . *120 \cdot 512 . \supset$

$\vdash : \text{Hp} . R \{(\mu \times_c \rho)/(\nu \times_c \sigma)\} S . \supset . \mu, \nu, \rho, \sigma \epsilon \text{NC ind}$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

The condition $\mu \neq 0 . \rho \neq 0$ is required in the above proposition because if, *e.g.* $\mu = 0 . \rho \epsilon \text{NC infin}$, we shall have (if $\nu, \sigma \epsilon \text{NC ind} - \iota`0$) $\mu/\nu = 0_q . \rho/\sigma = \dot{\Lambda}$, whence $\mu/\nu \times_c \rho/\sigma = \dot{\Lambda}$, but $(\mu \times_c \rho)/(\nu \times_c \sigma) = 0_q$. If we assume $\mu, \rho \epsilon \text{NC ind}$, it is not necessary to assume $\mu \neq 0 . \rho \neq 0$. This is stated in *305·142.

***305·141.** $\vdash :. \nu = 0 . \vee . \sigma = 0 : \supset . \mu/\nu \times_s \rho/\sigma = \dot{\Lambda}$

Dem.

$\vdash . *303 \cdot 67 \cdot 11 . \supset \vdash : \nu = 0 . \mu', \nu' \epsilon \text{NC ind} . \mu/\nu = \mu'/\nu' . \supset . \nu' = 0$ (1)

$\vdash . (1) . *305 \cdot 1 . \supset \vdash . \text{Prop}$

***305·142.** $\vdash : \mu, \rho \epsilon \text{NC ind} . \nu \neq 0 . \sigma \neq 0 . \supset . \mu/\nu \times_s \rho/\sigma = (\mu \times_c \rho)/(\nu \times_c \sigma)$
[Proof as in *305·14]

***305·143.** $\vdash : \dot{\exists} ! (\mu/\nu \times_s \rho/\sigma) . \supset . \mu, \nu, \rho, \sigma \epsilon \text{NC ind} . \nu \neq 0 . \sigma \neq 0$

Dem.

$\vdash . *305 \cdot 1 . \supset \vdash : \dot{\exists} ! (\mu/\nu \times_s \rho/\sigma) . \supset . (\exists \mu', \nu') . \mu', \nu' \epsilon \text{NC ind} . \nu' \neq 0 . \mu/\nu = \mu'/\nu' .$

[*303·182·67] $\supset . \mu, \nu \epsilon \text{NC ind} . \nu \neq 0$ (1)

Similarly $\vdash : \dot{\exists} ! (\mu/\nu \times_s \rho/\sigma) . \supset . \rho, \sigma \epsilon \text{NC ind} . \sigma \neq 0$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***305·144.** $\vdash : \dot{\exists} ! (\mu/\nu \times_s \rho/\sigma) . \supset . \mu/\nu \times_s \rho/\sigma = (\mu \times_c \rho)/(\nu \times_c \sigma)$ [*305·143·142]

***305·15.** $\vdash :. \sim (\mu, \nu, \rho, \sigma \epsilon \text{NC ind}) . \vee . \nu = 0 . \vee . \sigma = 0 : \supset . \mu/\nu \times_s \rho/\sigma = \dot{\Lambda}$
[*305·143 . Transp]

***305·16.** $\vdash :. \mu, \nu, \rho, \sigma \epsilon \text{NC ind} : \mu = 0 . \vee . \rho = 0 : \nu \neq 0 . \sigma \neq 0 : \supset .$
$\mu/\nu \times_s \rho/\sigma = 0_q$ [*305·142 . *303·6]

***305·17.** $\vdash . X \times_s \infty_q = \dot{\Lambda}$ [*305·141 . *303·67]

***305·2.** $\vdash : \dot{\exists} ! X \times_s Y . \supset . X, Y \epsilon \text{Rat}$

Dem.

$\vdash . *305 \cdot 1 . \supset$

$\vdash : \text{Hp} . \supset . (\exists \mu, \nu, \rho, \sigma) . \mu, \nu, \rho, \sigma \epsilon \text{NC ind} . \nu \neq 0 . \sigma \neq 0 . X = \mu/\nu . Y = \rho/\sigma .$

[*303·7] $\supset . X, Y \epsilon \text{Rat} : \supset \vdash . \text{Prop}$

***305·21.** $\vdash : X \times_s Y \in \text{Rat} - \iota`0_q . \supset . X, Y \in \text{Rat} - \iota`0_q$

Dem.

$\vdash . *303·72 . *305·2 . \supset \vdash : \text{Hp} . \supset . X, Y \in \text{Rat}$ (1)

$\vdash . *305·16 . \text{Transp} . \supset \vdash : \text{Hp} . \supset . X \neq 0_q . Y \neq 0_q$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***305·22.** $\vdash :. X \times_s Y = 0_q . \equiv : X, Y \in \text{Rat} : X = 0_q . \vee . Y = 0_q$

Dem.

$\vdash . *305·1·2·142 . *303·66 . \supset$

$\vdash :. X \times_s Y = 0_q . \equiv : (\exists \mu, \nu, \rho, \sigma) . X = \mu/\nu . Y = \rho/\sigma . \mu, \nu, \rho, \sigma \in \text{NC ind} .$
$\mu \times_c \rho = 0 . \nu \times_c \sigma \neq 0 :$

[*303·66] $\equiv : (\exists \mu, \nu, \rho, \sigma) : X = \mu/\nu . Y = \rho/\sigma . \mu, \nu, \rho, \sigma \in \text{NC ind} .$
$\nu \neq 0 . \sigma \neq 0 : \mu/\nu = 0_q . \vee . \rho/\sigma = 0_q :$

[*303·7] $\equiv : X, Y \in \text{Rat} : X = 0_q . \vee . Y = 0_q :. \supset \vdash . \text{Prop}$

***305·222.** $\vdash : X \times_s Y \in \text{Rat} . \supset . X, Y \in \text{Rat}$ [*305·21·22]

The following propositions are lemmas designed to show that if X, Y are ratios which exist in a given type, $X \times_s Y$ exists in the next type.

***305·23.** $\vdash : \mu \in \text{NC ind} . \supset . (2 \times_c \mu) +_c 1 < 2^{\mu +_c 1}$ [*117·652 . *120·429]

***305·231.** $\vdash . (\mu +_c 1)^2 = \mu^2 +_c (2 \times_c \mu) +_c 1$ [*116·34 . *113·43·66]

***305·232.** $\vdash : \mu \in \text{NC ind} . \supset . \mu^2 < 2^{\mu +_c 1}$

Dem.

$\vdash . *116·311·321 . \supset \vdash . 0^2 < 2^{0 +_c 1}$ (1)

$\vdash . *305·231 . \supset \vdash : \text{Hp} . \mu^2 < 2^{\mu +_c 1} . \supset . (\mu +_c 1)^2 < 2^{\mu +_c 1} +_c (2 \times_c \mu) +_c 1$ (2)

$\vdash . (2) . *305·23 . \supset \vdash : \mu \in \text{NC ind} . \mu^2 < 2^{\mu +_c 1} . \supset . (\mu +_c 1)^2 < 2^{\mu +_c 1} +_c 2^{\mu +_c 1} .$

[*113·66 . *116·52] $\supset . (\mu +_c 1)^2 < 2^{\mu +_c 2}$ (3)

$\vdash . (1) . (3) . \text{Induct} . \supset \vdash . \text{Prop}$

***305·24.** $\vdash : \mu, \nu, \rho, \sigma \in \text{D}`U \cap \text{Ɑ}`U . \supset .$
$(\mu \times_c \rho) \cap t`\mu, (\nu \times_c \sigma) \cap t`\mu \in \text{D}`U \cap \text{Ɑ}`U$

Dem.

$\vdash . *116·72 . \supset \vdash : \text{Hp} . \supset . (2^{\mu +_c 1} \cap t`\mu) \in C`U .$

[*305·232] $\supset . \mu^2 \cap t`\mu \in \text{Ɑ}`U$ (1)

$\vdash . *116·35 . \supset \vdash : \text{Hp} . \supset . \mu^2 \cap t`\mu \in \text{D}`U$ (2)

Similarly $\vdash : \text{Hp} . \supset . \nu^2 \cap t`\mu, \rho^2 \cap t`\mu, \sigma^2 \cap t`\mu \in \text{D}`U \cap \text{Ɑ}`U$ (3)

$\vdash . *117·571 . \supset$

$\vdash :. \text{Hp} . \supset : \mu \times_c \rho \leqslant \mu^2 . \vee . \mu \times_c \rho \leqslant \rho^2 : \nu \times_c \sigma \leqslant \nu^2 . \vee . \nu \times_c \sigma \leqslant \sigma^2$ (4)

$\vdash . (1) . (2) . (3) . (4) . \supset \vdash . \text{Prop}$

***305·25.** $\vdash : \mu, \nu, \rho, \sigma \in \mathrm{D}'U \cap \mathrm{G}'U . \supset . (\mu/\nu \times_s \rho/\sigma) \upharpoonright t_{00}{}'\mu \in C'H$

Dem.

$\vdash . *305·14 . \supset \vdash : \mathrm{Hp} . \supset . \mu/\nu \times_s \rho/\sigma = (\mu \times_c \rho)/(\nu \times_c \sigma)$ (1)

$\vdash . (1) . *304·28 . *305·24 . \supset \vdash . \mathrm{Prop}$

***305·26.** $\vdash : X, Y \in \mathrm{Rat} . X \upharpoonright t_{11}{}'\mu, Y \upharpoonright t_{11}{}'\mu \in \mathrm{Rat\,def} . \supset . (X \times_s Y) \upharpoonright t_{00}{}'\mu \in C'H$

[*305·25 . *304·28]

***305·27.** $\vdash : X, Y \in \mathrm{Rat} - \iota'0_q . \supset . (\exists\mu) . (X \times_s Y) \upharpoonright t_{00}{}'\mu \in C'H$

[*305·26 . *303·721]

***305·28.** $\vdash : X, Y \in \mathrm{Rat} . \supset . (\exists\mu) . (X \times_s Y) \upharpoonright t_{00}{}'\mu \in C'H'$ [*305·27·22]

***305·3.** $\vdash : X, Y \in \mathrm{Rat} . \equiv . X \times_s Y \in \mathrm{Rat}$

Dem.

$\vdash . *305·142 . *303·7 . \supset \vdash : X, Y \in \mathrm{Rat} . \supset . X \times_s Y \in \mathrm{Rat}$ (1)

$\vdash . (1) . *305·222 . \supset \vdash . \mathrm{Prop}$

***305·301.** $\vdash : X, Y \in \mathrm{Rat} - \iota'0_q . \equiv . X \times_s Y \in \mathrm{Rat} - \iota'0_q$

[*305·142 . *303·7 . *305·21]

***305·31.** $\vdash : (\exists\mu) . X \upharpoonright t_{11}{}'\mu, Y \upharpoonright t_{11}{}'\mu \in C'H . \equiv . (\exists\nu) . (X \times_s Y) \upharpoonright t_{11}{}'\nu \in C'H$

[*305·301 . *304·53]

***305·32.** $\vdash : (\exists\mu) . X \upharpoonright t_{11}{}'\mu, Y \upharpoonright t_{11}{}'\mu \in C'H' . \equiv . (\exists\nu) . (X \times_s Y) \upharpoonright t_{11}{}'\nu \in C'H'$

[*305·3 . *304·52]

***305·4.** $\vdash : \lambda, \nu, \sigma \in \mathrm{NC\,ind} . \mu \neq 0 . \rho \neq 0 . \tau \neq 0 . \supset .$

$(\lambda/\mu \times_s \nu/\rho) \times_s (\sigma/\tau) = (\lambda \times_c \nu \times_c \sigma)/(\mu \times_c \rho \times_c \tau) = \lambda/\mu \times_s (\nu/\rho \times_s \sigma/\tau)$ [*305·142]

***305·41.** $\vdash . (X \times_s Y) \times_s Z = X \times_s (Y \times_s Z)$ [*305·4·2]

***305·5.** $\vdash : \mu \neq 0 . \supset . (\lambda/\mu) \times_s (1/1) = \lambda/\mu$ [*305·14·142·15]

***305·51.** $\vdash : X \in \mathrm{Rat} . \supset . X \times_s (1/1) = X$ [*305·5]

***305·52.** $\vdash : X \in \mathrm{Rat} - \iota'0_q . \supset . X \times_s \breve{X} = 1/1$

Dem.

$\vdash . *305·14 . *303·13 . \supset$

$\vdash : \mathrm{Hp} . \supset . (\exists\mu, \nu) . \mu, \nu \in \mathrm{NC\,ind} - \iota'0 . X \times_s \breve{X} = (\mu \times_c \nu)/(\nu \times_c \mu) .$

$[*303·23] \supset . X \times_s \breve{X} = 1/1 : \supset \vdash . \mathrm{Prop}$

***305·6.** $\vdash :. A \in \mathrm{Rat} - \iota'0_q . X \in \mathrm{Rat} . \supset : A \times_s X = A' . \equiv . X = A' \times_s \breve{A}$

Dem.

$\vdash . *304·1·4 . *305·32·222 . \supset$

$\vdash : \mathrm{Hp} . \supset . (\exists\mu, \nu, \rho, \sigma, \xi, \eta) . \mu, \nu, \sigma \in \mathrm{NC\,ind} - \iota'0 . \rho, \xi, \eta \in \mathrm{NC\,ind} .$

$A = \mu/\nu . X = \rho/\sigma . A' = \xi/\eta$ (1)

$\vdash . *305\cdot142 . \supset \vdash :. \mu, \nu, \sigma \in \text{NC ind} - \iota'0 . \rho, \xi, \eta \in \text{NC ind} . \supset :$

$\mu/\nu \times_s \rho/\sigma = \xi/\eta . \equiv . (\mu \times_c \rho)/(\nu \times_c \sigma) = \xi/\eta .$

$[*303\cdot38] \qquad \equiv . \mu \times_c \rho \times_c \eta = \nu \times_c \sigma \times_c \xi .$

$[*303\cdot38] \qquad \equiv . \rho/\sigma = (\nu \times_c \xi)/(\mu \times_c \eta)$

$[*305\cdot142 . *303\cdot13] \qquad = \xi/\eta \times_c \text{Cnv}'(\mu/\nu) \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

$*305\cdot61$. $\vdash :. A \in \text{Rat} - \iota'0_q . A' \in \text{Rat} . \supset : A \times_s X = A' . \equiv . X = A' \times_s \breve{A}$

$[*305\cdot6\cdot222\cdot32]$

$*305\cdot7$. $\vdash : X, Y \in \text{Rat} - \iota'0_q . \supset . (\exists \alpha) . \alpha \in \text{NC ind} . Y <_r (\alpha/1 \times_s X)$

Dem.

$\vdash . *117\cdot571 . *120\cdot511 . *117\cdot62 . \supset$

$\vdash : \mu, \nu, \rho, \sigma \in \text{NC ind} - \iota'0 . \xi > \nu . \supset .$

$\mu \times_c \rho \times_c \xi \times_c \sigma > \nu \times_c \rho .$

$[*304\cdot1] \quad \supset . (\rho/\sigma) <_r (\mu \times_c \rho \times_c \xi)/\nu .$

$[*305\cdot14] \supset . (\rho/\sigma) <_r \{\mu/\nu \times_s (\rho \times_c \xi)/1\} \qquad (1)$

$\vdash . (1) . *304\cdot1 . *120\cdot5 . \supset \vdash . \text{Prop}$

$*305\cdot71$. $\vdash :. Z \in \text{Rat} - \iota'0_q . \supset : X <_r Y . \equiv . X \times_s Z <_r Y \times_s Z$

Dem.

$\vdash . *305\cdot142 . \supset \vdash : \text{Hp} . X <_r Y . \supset .$

$(\exists \mu, \nu, \rho, \sigma, \xi, \eta) . \mu, \nu, \rho, \sigma, \xi, \eta \in \text{NC ind} . \nu \neq 0 . \sigma \neq 0 . \xi \neq 0 . \eta \neq 0 .$

$X = \mu/\nu . Y = \rho/\sigma . Z = \xi/\eta . \mu \times_c \sigma < \nu \times_c \rho .$

$X \times_s Z = (\mu \times_c \xi)/(\nu \times_c \eta) . Y \times_s Z = (\rho \times_c \xi)/(\sigma \times_c \eta) .$

$[*304\cdot1 . *126\cdot51] \supset . X \times_s Z <_r Y \times_s Z \qquad (1)$

$\vdash . (1) . \supset \vdash : \text{Hp} . X \times_s Z <_r Y \times_s Z . \supset . X \times_s Z \times_s \breve{Z} <_r Y \times_s Z \times_s \breve{Z} .$

$[*305\cdot51\cdot52] \qquad \supset . X <_r Y \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

Summary of *306.

The addition of simple ratios is treated in a way analogous to that in which their multiplication is treated. We wish to secure that the sum of λ/ν and μ/ν shall be $(\lambda +_c \mu)/\nu$, and that the sum of μ/ν and ρ/σ shall be $\{(\mu \times_c \sigma) +_c (\nu \times_c \rho)\}/(\nu \times_c \sigma)$. This is secured by the definition

***306·01.** $X +_s Y = \hat{R}\hat{S}\,[(\exists \mu, \nu, \rho) \,.\, \mu, \nu, \rho \,\epsilon\, \mathrm{NC\,ind} \,.\, \nu \neq 0 \,.$
$X = \mu/\nu \,.\, Y = \rho/\nu \,.\, R\,\{(\mu +_c \rho)/\nu\}\,S]$ Df

whence we obtain

***306·13.** $\vdash : \nu \neq 0 \,.\supset .\, \mu/\nu +_s \rho/\nu = (\mu +_c \rho)/\nu$

***306·14.** $\vdash : \nu \neq 0 \,.\, \sigma \neq 0 \,.\supset .\, \mu/\nu +_s \rho/\sigma = \{(\mu \times_c \sigma) +_c (\nu \times_c \rho)\}/(\nu \times_c \sigma)$

Our definition is so framed that $\infty_q +_s \infty_q = \Lambda$. This is on the whole convenient, though we could, of course, frame our definition so as to have $\infty_q +_s \infty_q = \infty_q$.

In applications, if R, S, T are members of a suitable vector-family, we want to have

$$R\,(\mu/\nu)\,T \,.\, S\,(\rho/\sigma)\,T \,.\supset .\, (R \,|\, S)\,(\mu/\nu +_s \rho/\sigma)\,T,$$

e.g. if a vector R is $2/3$ of T, and a vector S is $5/7$ of T, we want the vector which consists of first travelling a distance R and then travelling a distance S to be $(2/3 +_s 5/7)$ of T. We shall show in Section C that our definition of addition fulfils this requirement.

As in the case of products, the sum of two ratios is a ratio (*306·22), and the sum of two ratios which exist in a given type exists in the next type (*306·64). A ratio is unchanged by the addition of 0_q (*306·24), and a sum of two ratios is only 0_q if both the summands are 0_q (*306·2). No difficulty is offered by the formal laws: we prove the commutative law (*306·11), the associative law (*306·31), and the distributive law (*306·41).

An important proposition is

***306·52.** $\vdash :.\, X <_r Y \,.\equiv :\, X \,\epsilon\, \mathrm{Rat} : (\exists Z) \,.\, Z \,\epsilon\, \mathrm{Rat} - \iota`0_q \,.\, X +_s Z = Y$

When the axiom of infinity is assumed, this proposition becomes

$$XH'Y \,.\equiv :\, X \,\epsilon\, C`H' : (\exists Z) \,.\, Z \,\epsilon\, C`H \,.\, X +_s Z = Y.$$

We prove also the proposition upon which subtraction depends, namely

***306·54.** $\vdash :. X, Y \epsilon \mathrm{Rat} . \supset : X +_s Y = X +_s Z . \equiv . Y = Z$

***306·01.** $X +_s Y = \hat{R}\hat{S}\{(\exists \mu, \nu, \rho) . \mu, \nu, \rho \epsilon \mathrm{NC\,ind} . \nu \neq 0 .$
$X = \mu/\nu . Y = \rho/\nu . R\{(\mu +_c \rho)/\nu\} S\}$ Df

***306·1.** $\vdash : R(X +_s Y) S . \equiv . (\exists \mu, \nu, \rho) . \mu, \nu, \rho \epsilon \mathrm{NC\,ind} . \nu \neq 0 .$
$X = \mu/\nu . Y = \rho/\nu . R\{(\mu +_c \rho)/\nu\} S$ [(*306·01)]

***306·11.** $\vdash . X +_s Y = Y +_s X$ [*306·1 . *110·51]

***306·12.** $\vdash : \dot{\exists} ! (X +_s Y) . \supset . X, Y \epsilon \mathrm{Rat}$ [*306·1 . *303·7]

***306·121.** $\vdash : \mu/\nu = \mu'/\nu' . \rho/\nu = \rho'/\nu' . \supset . (\mu +_c \rho)/\nu = (\mu' +_c \rho')/\nu'$

Dem.

$\vdash . *303·39 . \supset \vdash : \mathrm{Hp} . \mu, \nu, \rho, \mu', \nu', \rho' \epsilon \mathrm{NC\,ind} . \nu \neq 0 . \nu' \neq 0 . \supset .$
$\mu \times_c \nu' = \mu' \times_c \nu . \rho \times_c \nu' = \rho' \times_c \nu .$

[*113·43] $\supset . (\mu +_c \rho) \times_c \nu' = (\mu' +_c \rho') \times_c \nu .$

[*303·39] $\supset . (\mu +_c \rho)/\nu = (\mu' +_c \rho')/\nu'$ (1)

$\vdash . *303·181 . *302·36 . \supset$

$\vdash : \mathrm{Hp} . \sim(\mu, \nu, \rho, \mu', \nu', \rho' \epsilon \mathrm{NC\,ind}) . \supset . (\mu +_c \rho)/\nu = \dot{\Lambda} . (\mu' +_c \rho')/\nu' = \dot{\Lambda}$ (2)

$\vdash . (1) . (2) . *303·67 . \supset \vdash . \mathrm{Prop}$

***306·13.** $\vdash : \nu \neq 0 . \supset . \mu/\nu +_s \rho/\nu = (\mu +_c \rho)/\nu$

Dem.

$\vdash . *306·1 . \supset \vdash : \mathrm{Hp} . \supset . (\mu +_c \rho)/\nu \subset \mu/\nu +_s \rho/\nu$ (1)

$\vdash . *306·121 . \supset$

$\vdash : \mu/\nu = \mu'/\nu' . \rho/\nu = \rho'/\nu' . X\{(\mu' +_c \rho')/\nu'\} Y . \supset . X\{(\mu +_c \rho)/\nu\} Y$ (2)

$\vdash . (2) . *306·1 . \supset \vdash . \mu/\nu +_s \rho/\nu \subset (\mu +_c \rho)/\nu$ (3)

$\vdash . (1) . (3) . \supset \vdash . \mathrm{Prop}$

***306·14.** $\vdash : \nu \neq 0 . \sigma \neq 0 . \supset . \mu/\nu +_s \rho/\sigma = \{(\mu \times_c \sigma) +_c (\nu \times_c \rho)\}/(\nu \times_c \sigma)$

Dem.

$\vdash . *303·39 . \supset$

$\vdash : \mathrm{Hp} . \mu, \nu, \rho, \sigma \epsilon \mathrm{NC\,ind} . \supset . \mu/\nu = (\mu \times_c \sigma)/(\nu \times_c \sigma) . \rho/\sigma = (\nu \times_c \rho)/(\nu \times_c \sigma) .$

[*306·13] $\supset . \mu/\nu +_s \rho/\sigma = \{(\mu \times_c \sigma) +_c (\nu \times_c \rho)\}/(\nu \times_c \sigma)$ (1)

$\vdash . *306·12 . *303·11 . \supset$

$\vdash : \sim(\mu, \nu, \rho, \sigma \epsilon \mathrm{NC\,ind}) . \supset . \mu/\nu +_s \rho/\sigma = \dot{\Lambda} . \{(\mu \times_c \sigma) +_c (\nu \times_c \rho)\}/(\nu \times_c \sigma) = \dot{\Lambda}$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***306·141.** $\vdash :. \nu = 0 . \vee . \sigma = 0 : \supset . \mu/\nu +_s \rho/\sigma = \dot{\Lambda}$ [*306·12 . Transp . *303·7]

***306·15.** $\vdash : \mu/\nu +_s \rho/\sigma = 0_q . \equiv . \mu = \rho = 0 . \nu, \sigma \in \text{NC ind} - \iota\text{'}0$

Dem.

$\vdash . *306·14 . *303·66 . \supset \vdash : \mu = \rho = 0 . \nu, \sigma \in \text{NC ind} - \iota\text{'}0 . \supset . \mu/\nu +_s \rho/\sigma = 0_q$ (1)

$\vdash . *306·12 . \quad \supset \vdash : \mu/\nu +_s \rho/\sigma = 0_q . \supset . \mu, \nu, \rho, \sigma \in \text{NC ind}$ (2)

$\vdash . *306·141 . \quad \supset \vdash : \mu/\nu +_s \rho/\sigma = 0_q . \supset . \nu \neq 0 . \sigma \neq 0$ (3)

$\vdash . (3) . *306·14 . \supset \vdash : \text{Hp}(3) . \supset . \{(\mu \times_c \sigma) +_c (\nu \times_c \rho)\}/(\nu \times_c \sigma) = 0_q .$

[*303·66] $\quad \supset . (\mu \times_c \sigma) +_c (\nu \times_c \rho) = 0 . \nu \times_c \sigma \neq 0 .$

[*110·62.*113·602] $\quad \supset . \mu = \rho = 0 . \nu \neq 0 . \sigma \neq 0$ (4)

$\vdash . (1) . (2) . (4) . \supset \vdash .$ Prop

***306·16.** $\vdash . X +_s Y = \hat{R}\hat{S}[(\exists \mu, \nu, \rho, \sigma) . \mu, \nu, \rho, \sigma \in \text{NC ind} . \nu \neq 0 . \sigma \neq 0 .$
$X = \mu/\nu . Y = \rho/\sigma . R \{(\overline{\mu \times_c \sigma} +_c \overline{\nu \times_c \rho})/\overline{\nu \times_c \sigma}\} S]$

[*306·14·12]

***306·17.** $\vdash : \mu = 0 . \nu, \rho, \sigma \in \text{NC ind} . \nu \neq 0 . \sigma \neq 0 . \supset . \mu/\nu +_s \rho/\sigma = \rho/\sigma$

Dem.

$\vdash . *303·6 . \supset \vdash : \text{Hp} . \supset . \mu/\nu = 0/\sigma .$

[*306·13] $\quad \supset . \mu/\nu +_s \rho/\sigma = (0 +_c \rho)/\sigma : \supset \vdash .$ Prop

***306·2.** $\vdash : X +_s Y = 0_q . \equiv . X = 0_q . Y = 0_q$ [*306·15·12]

***306·22.** $\vdash : X +_s Y \in \text{Rat} . \equiv . X, Y \in \text{Rat}$

Dem.

$\vdash . *306·16 . *303·7 . \supset \vdash : X +_s Y \in \text{Rat} . \equiv .$

$(\exists \mu, \nu, \rho, \sigma) . \mu, \nu, \rho, \sigma \in \text{NC ind} . X = \mu/\nu . Y = \rho/\sigma . \nu \times_c \sigma \neq 0 .$

[*113·602] $\equiv . (\exists \mu, \nu, \rho, \sigma) . \mu, \nu, \rho, \sigma \in \text{NC ind} . X = \mu/\nu . Y = \rho/\sigma . \nu \neq 0 . \sigma \neq 0 .$

[*303·7] $\quad \equiv . X, Y \in \text{Rat} : \supset \vdash .$ Prop

***306·23** $\vdash : X +_s Y \in \text{Rat} - \iota\text{'}0_q . \equiv . X, Y \in \text{Rat} . \sim(X = Y = 0_q)$
[*306·22 . *303·7 . *306·2]

***306·24.** $\vdash : X \in \text{Rat} . \supset . X +_s 0_q = X$ [*306·17·11]

***306·25.** $\vdash : X +_s Y \in \text{Rat} . \equiv . \dot{\exists} ! (X +_s Y) . \equiv . X, Y \in \text{Rat}$
[*306·12·22 . *303·26 . *306·14]

Here $X +_s Y$ must be taken in a sufficiently high type, otherwise $X +_s Y$ may be null when $X, Y \in \text{Rat}$.

***306·3.** $\vdash . (\lambda/\mu +_s \nu/\rho) +_s \sigma/\tau = \lambda/\mu +_s (\nu/\rho +_s \sigma/\tau)$

Dem.

$\vdash . *306·14 . \supset \vdash : \mu \neq 0 . \rho \neq 0 . \tau \neq 0 . \supset . (\lambda/\mu +_s \nu/\rho) +_s \sigma/\tau$

$= \{(\lambda \times_c \rho) +_c (\mu \times_c \nu)\}/(\mu \times_c \rho) +_s \sigma/\tau$

[*306·14] $= \{(\lambda \times_c \rho \times_c \tau) +_c (\mu \times_c \nu \times_c \tau) +_c (\mu \times_c \rho \times_c \sigma)\}/(\mu \times_c \rho \times_c \tau)$

[*113·43] $= [\{\lambda \times_c (\rho \times_c \tau)\} +_c \{\mu \times_c ((\nu \times_c \tau) +_c (\rho \times_c \sigma))\}]/\{\mu \times_c (\rho \times_c \tau)\}$

[*306·14] $= \lambda/\mu +_s \{(\nu \times_c \tau) +_c (\rho \times_c \sigma)\}/(\rho \times_c \tau)$

[*306·14] $= \lambda/\mu +_s (\nu/\rho +_s \sigma/\tau)$ (1)

$\vdash . (1) . *306·12 . \supset \vdash .$ Prop

∗306·31. $\vdash . (X +_s Y) +_s Z = X +_s (Y +_s Z)$

Dem.

$\vdash . *306·3 . \supset \vdash : X = \lambda/\mu . Y = \nu/\rho . Z = \sigma/\tau . \supset .$

$$(X +_s Y) +_s Z = X +_s (Y +_s Z) \quad (1)$$

$\vdash . *306·25 . \supset \vdash : \sim (\exists \lambda, \mu, \nu, \rho, \sigma, \tau) . X = \lambda/\mu . Y = \nu/\rho . Z = \sigma/\tau . \supset .$

$$(X +_s Y) +_s Z = \dot{\Lambda} . X +_s (Y +_s Z) = \dot{\Lambda} \quad (2)$$

$\vdash . (1) . (2) . \supset \vdash .$ Prop

∗306·4. $\vdash . \lambda/\mu \times_s (\nu/\rho +_s \sigma/\tau) = (\lambda/\mu \times_s \nu/\rho) +_s (\lambda/\mu \times_s \sigma/\tau)$

Dem.

$\vdash . *306·14 . \supset \vdash : \lambda, \mu, \nu, \rho, \sigma, \tau \in \text{NC ind} . \mu \neq 0 . \nu \neq 0 . \sigma \neq 0 . \supset .$

$$\lambda/\mu \times_s (\nu/\rho +_s \sigma/\tau) = \lambda/\mu \times_s \{(\nu \times_c \tau) +_c (\rho \times_c \sigma)\}/(\rho \times_c \tau)$$

[∗305·14] $= [\lambda \times_c \{(\nu \times_c \tau) +_c (\rho \times_c \sigma)\}]/(\mu \times_c \rho \times_c \tau)$

[∗303·23] $= [\lambda \times_c \mu \times_c \{(\nu \times_c \tau) +_c (\rho \times_c \sigma)\}]/(\mu \times_c \rho \times_c \mu \times_c \tau)$

[∗113·43] $= \{(\lambda \times_c \mu \times_c \nu \times_c \tau) +_c (\lambda \times_c \mu \times_c \rho \times_c \sigma)\}/(\mu \times_c \rho \times_c \mu \times_c \tau)$

[∗306·14] $= (\lambda \times_c \nu)/(\mu \times_c \rho) +_s (\lambda \times_c \sigma)/(\mu \times_c \tau)$

[∗305·14] $= (\lambda/\mu \times_s \nu/\rho) +_s (\lambda/\mu \times_s \sigma/\tau)$ (1)

$\vdash . *305·2 . *306·22 . \supset \vdash : \dot{\exists} ! \lambda/\mu \times_s (\nu/\rho +_s \sigma/\tau) . \supset . \lambda/\mu, \nu/\rho, \sigma/\tau \in \text{Rat} .$

[∗303·7] $\supset . \text{Hp}(1)$ (2)

$\vdash . *306·12 . *305·143 . \supset$

$\vdash : \dot{\exists} ! \{(\lambda/\mu \times_s \nu/\rho) +_s (\lambda/\mu \times_s \sigma/\tau)\} . \supset . \lambda/\mu, \nu/\rho, \sigma/\tau \in \text{Rat} .$

[∗303·7] $\supset . \text{Hp}(1)$ (3)

$\vdash . (2) . (3) . \supset$

$\vdash : \sim \text{Hp}(1) . \supset . \lambda/\mu \times_s (\nu/\rho +_s \sigma/\tau) = \dot{\Lambda} = (\lambda/\mu \times_s \nu/\rho) +_s (\lambda/\mu \times_s \sigma/\tau)$ (4)

$\vdash . (1) . (4) . \supset \vdash .$ Prop

∗306·41. $\vdash . X \times_s (Y +_s Z) = (X \times_s Y) +_s (X \times_s Z)$ [∗306·4·25 . ∗305·2]

∗306·51. $\vdash . X +_s (\nu/1 \times_s X) = (\nu +_c 1)/1 \times_s X$

Dem.

$\vdash . *306·12 . \supset \vdash :. \dot{\exists} ! \{X +_s (\nu/1 \times_s X)\} . \supset : X, \nu/1 \times_s X \in \text{Rat} :$

[∗305·3 . ∗303·7] $\supset : \nu \in \text{NC ind} : (\exists \rho, \sigma) . \rho, \sigma \in \text{NC ind} . \sigma \neq 0 . X = \rho/\sigma$ (1)

$\vdash . *305·2 . \supset \vdash :. \dot{\exists} ! \{(\nu +_c 1)/1 \times_s X\} . \supset : (\nu +_c 1)/1, X \in \text{Rat} :$

[∗303·7 . ∗126·31] $\supset : \nu \in \text{NC ind} : (\exists \rho, \sigma) . \rho, \sigma \in \text{NC ind} . \sigma \neq 0 . X = \rho/\sigma$ (2)

$\vdash . *305·142 . \supset \vdash : \nu, \rho, \sigma \in \text{NC ind} . \sigma \neq 0 . \supset . \nu/1 \times_s \rho/\sigma = (\nu \times_c \rho)/\sigma .$

[∗306·13] $\supset . \rho/\sigma +_s (\nu/1 \times_s \rho/\sigma) = \{\rho +_c (\nu \times_c \rho)\}/\sigma$

[∗113·671] $= \{(\nu +_c 1) \times_c \rho\}/\sigma$

[∗305·14] $= (\nu +_c 1)/1 \times_s \rho/\sigma$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash .$ Prop

***306·52.** $\vdash :. X <_r Y . \equiv : X \epsilon \text{Rat} : (\exists Z) . Z \epsilon \text{Rat} - \iota\text{'}0_q . X +_s Z = Y$

Dem.

$\vdash . *306·13 . *119·34 . \supset$

$\vdash : \mu, \nu, \rho, \sigma \epsilon \text{NC ind} . \nu \neq 0 . \sigma \neq 0 . X = \mu/\nu . Y = \rho/\sigma . \mu \times_c \sigma < \nu \times_c \rho .$

$\xi = (\nu \times_c \rho) -_c (\mu \times_c \sigma) . Z = \xi/(\nu \times_c \sigma) . \supset . X +_s Z = (\nu \times_c \rho)/(\nu \times_c \sigma)$

[$*303·23$] $= \rho/\sigma$

[Hp] $= Y$ (1)

$\vdash . (1) . *304·1·13 . \supset$

$\vdash :. X <_r Y . \supset : X \epsilon \text{Rat} : (\exists Z) . Z \epsilon \text{Rat} - \iota\text{'}0_q . X +_s Z = Y$ (2)

$\vdash . *306·14 . \supset$

$\vdash : \mu, \nu, \rho, \sigma \epsilon \text{NC ind} . \nu \neq 0 . \rho \neq 0 . \sigma \neq 0 . X = \mu/\nu . Z = \rho/\sigma . Y = X +_s Z . \supset .$

$Y = \{(\mu \times_c \sigma) +_c (\nu \times_c \rho)\}/(\nu \times_c \sigma) . [\{(\mu \times_c \sigma) +_c (\nu \times_c \rho)\} \times_c \nu] > \mu \times_c (\nu \times_c \sigma) .$

[$*304·1$] $\supset . X <_r Y$ (3)

$\vdash . (3) . *304·1 . \supset \vdash : X \epsilon \text{Rat} . Z \epsilon \text{Rat} - \iota\text{'}0_q . X +_s Z = Y . \supset . X <_r Y$ (4)

$\vdash . (2) . (4) . \supset \vdash .$ Prop

The above proposition requires that X and Y should be taken in a sufficiently high type, namely at least in a type in which, if $X = \mu/\nu$ and $Y = \rho/\sigma$, where $\mu \,\text{Prm}\, \nu$ and $\rho \,\text{Prm}\, \sigma$, $(\nu \times_c \rho) +_c 1$ and $(\mu \times_c \sigma) +_c 1$ are not null. Otherwise there may be no Z such that $X +_s Z = Y$.

***306·53.** $\vdash :. \mu, \nu \epsilon \text{NC ind} . \nu \neq 0 . \sigma \neq 0 . \eta \neq 0 . \supset :$
$\mu/\nu +_s \rho/\sigma = \mu/\nu +_s \xi/\eta . \equiv . \rho/\sigma = \xi/\eta$

Dem.

$\vdash . *306·12 . \supset \vdash : \text{Hp} . \mu/\nu +_s \rho/\sigma = \mu/\nu +_s \xi/\eta . \sim(\rho, \sigma \epsilon \text{NC ind}) . \supset .$

$\mu/\nu +_s \xi/\eta = \dot\Lambda . \rho/\sigma = \dot\Lambda .$ (1)

[$*306·25$] $\supset . \sim \{\mu/\nu, \xi/\eta \epsilon \text{Rat}\} .$

[Hp.$*303·7$] $\supset . \sim(\xi, \eta \epsilon \text{NC ind}) .$

[$*303·11.(1)$] $\supset . \xi/\eta = \rho/\sigma$ (2)

$\vdash . *306·25 . \supset \vdash : \text{Hp} . \mu/\nu +_s \rho/\sigma = \mu/\nu +_s \xi/\eta . \rho, \sigma \epsilon \text{NC ind} . \supset .$

$\xi, \eta \epsilon \text{NC ind}$ (3)

$\vdash . (3) . *306·14 . *303·39 . \supset$

$\vdash : \text{Hp}(3) . \supset . \{(\mu \times_c \sigma) +_c (\nu \times_c \rho)\} \times_c \nu \times_c \eta = \{(\mu \times_c \eta) +_c (\nu \times_c \xi)\} \times_c \nu \times_c \sigma .$

[$*113·43$]

$\supset . (\mu \times_c \sigma \times_c \nu \times_c \eta) +_c (\nu^2 \times_c \rho \times_c \eta) = (\mu \times_c \sigma \times_c \nu \times_c \eta) +_c (\nu^2 \times_c \xi \times_c \sigma) .$

[$*126·4$] $\supset . \nu^2 \times_c (\rho \times_c \eta) = \nu^2 \times_c (\xi \times_c \sigma) .$

[$*303·39$] $\supset . \rho/\sigma = \xi/\eta$ (4)

$\vdash . (2) . (4) . \supset \vdash :. \text{Hp} . \supset : \mu/\nu +_s \rho/\sigma = \mu/\nu +_s \xi/\eta . \supset . \rho/\sigma = \xi/\eta$ (5)

$\vdash . *306·1 . \supset \vdash : \rho/\sigma = \xi/\eta . \supset . \mu/\nu +_s \rho/\sigma = \mu/\nu +_s \xi/\eta$ (6)

$\vdash . (5) . (6) . \supset \vdash .$ Prop

∗306·54. $\vdash :. X, Y \epsilon \text{Rat} . \supset : X +_s Y = X +_s Z . \equiv . Y = Z$

Dem.

$\vdash . ∗306·25 . \supset \vdash :. \text{Hp} . \supset : X +_s Y \epsilon \text{Rat} :$

$[∗306·25] \quad \supset : X +_s Y = X +_s Z . \supset . Z \epsilon \text{Rat} \quad (1)$

$\vdash . (1) . ∗306·53 . ∗303·7 . \supset \vdash . \text{Prop}$

∗306·55. $\vdash : Y <_r X . \supset . \sim (\exists Z) . X +_s Z = Y$

Dem.

$\vdash . ∗117·291 . ∗304·1 . \supset \vdash : \text{Hp} . \supset . \sim (X <_r Y) .$

$[∗306·52] \quad \supset . \sim (\exists Z) . Z \epsilon \text{Rat} - \iota ‘0_q . X +_s Z = Y \quad (1)$

$\vdash . ∗306·24 . ∗304·1 . \supset \vdash : \text{Hp} . \supset . \sim (X +_s 0_q = Y) \quad (2)$

$\vdash . ∗306·25 . \quad \supset \vdash : \text{Hp} . X +_s Z = Y . \supset . Z \epsilon \text{Rat} \quad (3)$

$\vdash . (1) . (2) . (3) . \supset \vdash . \text{Prop}$

The following propositions are concerned with the existence of $X +_s Y$ in definite types. It will be shown that if X, Y exist in a given type, $X +_s Y$ exists in the next type, *i.e.* if $X \upharpoonright t_{11}‘\mu$ and $Y \upharpoonright t_{11}‘\mu$ exist, then $(X +_s Y) \upharpoonright t_{00}‘\mu$ exists, where X, Y are rationals.

∗306·6. $\vdash : \mu, \rho \epsilon \text{D}‘U \cap \text{Ꭰ}‘U . \supset . (\mu +_c \rho) \cap t‘\mu \epsilon \text{D}‘U \cap \text{Ꭰ}‘U$

Dem.

$\vdash . ∗305·23 . \supset \vdash : \text{Hp} . \mu \leqslant \rho . \supset . \mu +_c \rho < 2^{\rho +_c 1} \quad (1)$

Similarly $\quad \vdash : \text{Hp} . \rho \leqslant \mu . \supset . \mu +_c \rho < 2^{\mu +_c 1} \quad (2)$

$\vdash . (1) . (2) . ∗116·72 . \supset \vdash . \text{Prop}$

∗306·61. $\vdash : \mu, \nu, \rho \epsilon \text{D}‘U \cap \text{Ꭰ}‘U . \supset . (\mu/\nu +_s \rho/\nu) \cap t_{00}‘\mu \epsilon \text{Rat def}$

Dem.

$\vdash . ∗306·13·6 . \supset \vdash : \text{Hp} . \supset . \mu/\nu +_s \rho/\nu = (\mu +_c \rho)/\nu . (\mu +_c \rho) \cap t‘\mu, \nu \cap t‘\mu \epsilon \text{D}‘U \cap \text{Ꭰ}‘U .$

$[∗303·71] \quad \supset . (\mu/\nu +_s \rho/\nu) \cap t_{00}‘\mu \epsilon \text{Rat def} : \supset \vdash . \text{Prop}$

∗306·62. $\vdash : \mu, \nu, \rho \epsilon \text{D}‘U \cap \text{Ꭰ}‘U . \supset . (\mu/\nu +_s \rho/\rho) \cap t_{00}‘\mu \epsilon \text{Rat def}$

Dem.

$\vdash . ∗303·39 . \supset \vdash : \text{Hp} . \supset . \mu/\nu +_s \rho/\rho = \mu/\nu +_s \nu/\nu \quad (1)$

$\vdash . (1) . ∗306·61 . \supset \vdash . \text{Prop}$

∗306·621. $\vdash : \sigma \epsilon \text{NC ind} . \supset . \sigma^2 -_c \sigma +_c 1 \leqslant 2^\sigma$

Dem.

$\vdash . ∗116·301·311 . \quad \supset \vdash . 0^2 -_c 0 +_c 1 \leqslant 2^0 \quad (1)$

$\vdash . ∗116·321·331 . \quad \supset \vdash . 1^2 -_c 1 +_c 1 \leqslant 2^1 \quad (2)$

$\vdash . ∗117·55 . ∗126·5 . \supset \vdash . 2^2 -_c 2 +_c 1 \leqslant 2^2 \quad (3)$

$\vdash . ∗305·231 . \supset \vdash : \text{Hp} . \sigma > 1 . \sigma^2 -_c \sigma +_c 1 \leqslant 2^\sigma . \supset .$

$(\sigma +_c 1)^2 -_c (\sigma +_c 1) +_c 1 \leqslant 2^\sigma +_c (2 \times_c \sigma) .$

$[∗117·652 . ∗116·52] \supset . (\sigma +_c 1)^2 -_c (\sigma +_c 1) +_c 1 \leqslant 2^{\sigma +_c 1} \quad (4)$

$\vdash . (1) . (2) . (3) . (4) . \text{Induct} . \supset \vdash . \text{Prop}$

***306·622.** $\vdash : \mu \in \text{NC ind} - \iota`0 . \supset . (\mu -_c 1)^2 = \mu^2 -_c (2 \times_c \mu) +_c 1$

Dem.

$\vdash . *305·231 \frac{\mu -_c 1}{\mu} . \quad \supset \vdash : \text{Hp} . \supset . (\mu -_c 1)^2 +_c \{2 \times_c (\mu -_c 1)\} +_c 1 = \mu^2$ (1)

$\vdash . *113·43 . *120·416 . \supset \vdash : \text{Hp} . \supset . \{2 \times_c (\mu -_c 1)\} +_c 2 = 2 \times_c \mu$ (2)

$\vdash . (1) . (2) . \quad \supset \vdash : \text{Hp} . \supset . (\mu -_c 1)^2 +_c (2 \times_c \mu) = \mu^2 +_c 1$ (3)

$\vdash . (3) . *119·32 . \supset \vdash . \text{Prop}$

***306·623.** $\vdash : \mu, \nu, \rho \in \text{NC ind} . \nu < \mu . \rho \leqslant \mu . \supset . (\mu \times_c \mu) +_c (\nu \times_c \rho) < 2^{\mu +_c 1}$

Dem.

$\vdash . *120·429 . \supset \vdash : \text{Hp} . \supset . (\mu \times_c \mu) +_c (\nu \times_c \rho) \leqslant \mu^2 +_c (\mu -_c 1)^2 .$

$[*120·429 . *306·622] \quad \supset . (\mu \times_c \mu) +_c (\nu \times_c \rho) < (2 \times_c \mu^2) -_c (2 \times_c \mu) +_c 2$

$[*306·621 . *126·51] \quad < 2^{\mu +_c 1} : \supset \vdash . \text{Prop}$

***306·624.** $\vdash : \mu, \nu, \rho, \sigma \in \text{NC ind} . \nu < \mu . \rho \leqslant \mu . \sigma \leqslant \mu . \supset .$

$(\mu \times_c \sigma) +_c (\nu \times_c \rho) < 2^{\mu +_c 1}$ [*306·623]

***306·63.** $\vdash : \mu, \nu, \rho, \sigma \in \text{D}`U \cap \text{Ɑ}`U . \supset . (\mu/\nu +_s \rho/\sigma) \upharpoonright t_{00}`\mu \in \text{Rat def}$

Dem.

$\vdash . *306·62 . \supset \vdash : \text{Hp} . \nu = \mu . \supset . (\mu/\nu +_s \rho/\sigma) \upharpoonright t_{00}`\mu \in \text{Rat def}$ (1)

$\vdash . *306·624 . *305·24 . *303·71 . \supset$

$\vdash : \text{Hp} . \nu < \mu . \rho \leqslant \mu . \sigma \leqslant \mu . \supset . (\mu/\nu +_s \rho/\sigma) \upharpoonright t_{00}`\mu \in \text{Rat def}$ (2)

Similarly

$\vdash : \text{Hp} . \nu < \mu . \mu \leqslant \rho . \sigma \leqslant \mu . \supset . (\mu/\nu +_s \rho/\sigma) \upharpoonright t_{00}`\mu \in \text{Rat def}$ (3)

$\vdash . (2) . (3) . \supset$

$\vdash : \text{Hp} . \nu < \mu . \sigma \leqslant \mu . \supset . (\mu/\nu +_s \rho/\sigma) \upharpoonright t_{00}`\mu \in \text{Rat def}$ (4)

Similarly

$\vdash : \text{Hp} . \mu > \nu . \sigma \leqslant \mu . \supset . (\mu/\nu +_s \rho/\sigma) \upharpoonright t_{00}`\mu \in \text{Rat def}$ (5)

$\vdash . (1) . (4) . (5) . \supset \vdash : \text{Hp} . \sigma \leqslant \mu . \supset . (\mu/\nu +_s \rho/\sigma) \upharpoonright t_{00}`\mu \in \text{Rat def}$ (6)

Similarly $\vdash : \text{Hp} . \mu \leqslant \sigma . \supset . (\mu/\nu +_s \rho/\sigma) \upharpoonright t_{00}`\mu \in \text{Rat def}$ (7)

$\vdash . (6) . (7) . \supset \vdash . \text{Prop}$

The following propositions are immediate consequences of *306·63.

***306·64.** $\vdash : (\mu/\nu) \upharpoonright t_{11}`\mu, (\rho/\sigma) \upharpoonright t_{11}`\mu \in \text{Rat def} . \supset . (\mu/\nu +_s \rho/\sigma) \upharpoonright t_{00}`\mu \in \text{Rat def}$

***306·65.** $\vdash : X, Y \in \text{Rat def} . \supset . (X +_s Y) \upharpoonright t_{00}`C``C`X \in \text{Rat def}$

***306·66.** $\vdash : X, Y \in C`H . \supset . (X +_s Y) \upharpoonright t_{00}`C``C`X \in C`H$

***306·67.** $\vdash : X, Y \in C`H' . \supset . (X +_s Y) \upharpoonright t_{00}`C``C`X \in C`H'$

*307. GENERALIZED RATIOS.

Summary of *307.

In this number we introduce negative ratios. If X is a ratio, what would ordinarily be called $-X$ is $X \mid \text{Cnv}$. This may be seen as follows. Suppose we have RXS. We then have $R(X \mid \text{Cnv})\breve{S}$. Now if R and S are vectors which carry us in the same direction, R and $\breve{S}$ are vectors which carry us in opposite directions, *i.e.* their ratio is negative. Hence calling the class of negative ratios "Rat_n," we may put

***307·01.** $\text{Rat}_n = \mid \text{Cnv}``\text{Rat}$ Df

The sum of "Rat" and "Rat_n" we will call "Rat_g," where "g" stands for "generalized." Thus we put

***307·011.** $\text{Rat}_g = \text{Rat} \cup \text{Rat}_n$ Df

If $\mu/\nu <_r \rho/\sigma$, we have $\{(\mu/\nu) \mid \text{Cnv}\} (\mid \text{Cnv} ; <_r) \{(\rho/\sigma) \mid \text{Cnv}\}$. Hence we put

***307·02.** $<_n = \mid \text{Cnv} ; <_r$ Df

***307·021.** $>_n = \text{Cnv}`<_n$ Df

If X and Y are generalized ratios, we consider X less than Y if either X, Y are both positive and $X <_r Y$, or X, Y are both negative and $X >_n Y$, or X is negative and Y is positive or zero. Hence we put

***307·03.** $<_g = (>_n) \cup (<_r) \cup (\text{Rat}_n - \iota`0_q) \uparrow \text{Rat}$ Df

On the analogy of $<_n$ and $<_g$, we put

***307·04.** $H_n = \mid \text{Cnv} ; H$ Df

***307·05.** $H_g = \breve{H}_n \ddagger H'$ Df

We prove in this number that if X is a ratio, $X \mid \text{Cnv} = \text{Cnv} \mid X$, and $\text{Cnv}`(X \mid \text{Cnv}) = \breve{X} \mid \text{Cnv}$ (*307·21·22). We prove also

***307·25.** $\vdash . C`H \cap C`H_n = \Lambda$

We prove that 0_q and ∞_q are their own negatives, but are not the negatives of anything else (*307·26·27·31). We prove $\text{Nr}`H_n = \text{Nr}`H$ (*307·41) and $\text{Infin ax} . \supset . H_g \in \eta$ (*307·46). None of the propositions of this number offer any difficulty.

***307·01.** $\mathrm{Rat}_n = |\,\mathrm{Cnv}\text{‘‘}\mathrm{Rat}$ Df

***307·011.** $\mathrm{Rat}_g = \mathrm{Rat} \cup \mathrm{Rat}_n$ Df

***307·02.** $<_n = |\,\mathrm{Cnv} ; <_r$ Df

***307·021.** $>_n = \mathrm{Cnv}\text{‘}<_n$ Df

***307·03.** $<_g = (>_n) \dot{\cup} (<_r) \dot{\cup} (\mathrm{Rat}_n - \iota\text{‘}0_q) \uparrow \mathrm{Rat}$ Df

***307·031.** $>_g = \mathrm{Cnv}\text{‘}<_g$ Df

***307·04.** $H_n = |\,\mathrm{Cnv} ; H$ Df

***307·05.** $H_g = \breve{H}_n \mathbin{\updownarrow} H'$ Df

***307·1.** $\vdash : R\,(X\,|\,\mathrm{Cnv})\,S \,.\equiv .\, RX\breve{S}$ [*71·7]

***307·11.** $\vdash : R\,(|\,\mathrm{Cnv};X)\,S \,.\equiv .\, \breve{R}X\breve{S}$ [*307·1]

***307·12.** $\vdash .\, X\,|\,\mathrm{Cnv}\,|\,\mathrm{Cnv} = X$ [*307·1]

***307·13.** $\vdash : X\,|\,\mathrm{Cnv} = Y\,|\,\mathrm{Cnv} \,.\equiv .\, X = Y$ [*307·12]

***307·14.** $\vdash : Y = X\,|\,\mathrm{Cnv} \,.\equiv .\, X = Y\,|\,\mathrm{Cnv}$ [*307·12]

***307·15.** $\vdash : \dot{\exists} !\, X \sqsubset \kappa \,.\equiv .\, \dot{\exists} !\, \kappa \upharpoonleft (X\,|\,\mathrm{Cnv}) \upharpoonright (\mathrm{Cnv}\text{‘‘}\kappa)$ [*307·1]

***307·16.** $\vdash :.\, \kappa = \mathrm{Cnv}\text{‘‘}\kappa \,.\supset : \dot{\exists} !\, X \sqsubset \kappa \,.\equiv .\, \dot{\exists} !\, (X\,|\,\mathrm{Cnv}) \sqsubset \kappa$ [*307·15]

***307·2.** $\vdash .\, (\mu/\nu)\,|\,\mathrm{Cnv} = \mathrm{Cnv}\,|\,(\mu/\nu)$ [*307·1 . *303·19]

***307·21.** $\vdash : X \in \mathrm{Rat} \cup \iota\text{‘}\infty_q \,.\supset .\, X\,|\,\mathrm{Cnv} = \mathrm{Cnv}\,|\,X$ [*307·2 . *303·7·67]

***307·22.** $\vdash : X \in \mathrm{Rat} \cup \iota\text{‘}\infty_q \,.\supset .\, \mathrm{Cnv}\text{‘}(X\,|\,\mathrm{Cnv}) = \breve{X}\,|\,\mathrm{Cnv}$ [*307·21]

***307·23.** $\vdash .\, \mathrm{Cnv}\text{‘‘}C\text{‘}H_n = C\text{‘}H_n$ [*304·28 . *303·13 . *307·22]

***307·24.** $\vdash : \mu, \nu, \rho, \sigma \in \mathrm{NC}\text{‘}U \,.\, \mu \,\mathrm{Prm}\, \nu \,.\, \rho \,\mathrm{Prm}\, \sigma \,.\, \rho \geqslant \sigma \,.\, \sigma \neq 0 \,.\supset .$

$\dot{\exists} !\, (\rho/\sigma) \dot{\wedge} (\mu/\nu)\,|\,\mathrm{Cnv}$

Dem.

$\vdash .\, *303·32 \,.\supset \vdash :.\, \mathrm{Hp} \,.\supset : (\exists P, Q) \,.\, P, Q \in \mathrm{Rel\,num} \,.\, P_{\mathrm{po}} \subseteq Q_{\mathrm{po}} \,.\, P\,(\rho/\sigma)\,Q :$

[*303·21] $\supset : (\exists P, Q) \,.\, P, Q \in \mathrm{Rel\,num} \,.\, P_{\mathrm{po}} \subseteq Q_{\mathrm{po}} \,.\, \dot{\exists} !\, P^{\sigma} \dot{\wedge} Q^{\rho} :$

[*300·3] $\supset : (\exists P, Q) \,.\, P, Q \in \mathrm{Rel\,num} \,.\, \dot{\exists} !\, P^{\sigma} \dot{\wedge} Q^{\rho} \,.\, P^{\nu} \dot{\wedge} \breve{Q}^{\mu} = \dot{\Lambda} :$

[*303·21] $\supset : (\exists P, Q) \,.\, P\,(\rho/\sigma)\,Q \,.\, \sim \{P\,(\mu/\nu)\,\breve{Q}\} :.\supset \vdash .\, \mathrm{Prop}$

***307·25.** $\vdash .\, C\text{‘}H \cap C\text{‘}H_n = \Lambda$

Dem.

$\vdash .\, *307·24 \,.\, *303·13 \,.\supset$

$\vdash : \mu, \nu, \rho, \sigma \in \mathrm{NC}\text{‘}U \,.\, \mu \,\mathrm{Prm}\, \nu \,.\, \rho \,\mathrm{Prm}\, \sigma \,.\supset .\, \mu/\nu \neq (\rho/\sigma)\,|\,\mathrm{Cnv}$ (1)

$\vdash .\, *302·22 \,.\, *303·211 \,.\, *304·27·28 \,.\supset \vdash : X, Y \in C\text{‘}H \,.\supset .$

$(\exists \mu, \nu, \rho, \sigma) \,.\, \mu, \nu, \rho, \sigma \in \mathrm{NC}\text{‘}U \,.\, \mu \,\mathrm{Prm}\, \nu \,.\, \rho \,\mathrm{Prm}\, \sigma \,.\, X = \mu/\nu \,.\, Y = \rho/\sigma$ (2)

$\vdash .\, (1) \,.\, (2) \,.\supset \vdash : X, Y \in C\text{‘}H \,.\supset .\, X \neq Y\,|\,\mathrm{Cnv} : \supset \vdash .\, \mathrm{Prop}$

∗307·26. $\vdash . 0_q \mid \mathrm{Cnv} = 0_q = \mathrm{Cnv} \mid 0_q$

Dem.

$\vdash . ∗307·2 . \supset \vdash . 0_q \mid \mathrm{Cnv} = \mathrm{Cnv} \mid 0_q$ (1)

$\vdash . ∗303·6·15 . ∗307·1 . \supset \vdash : R(0_q \mid \mathrm{Cnv}) S . \equiv . \dot{\exists} ! R \dot{\cap} I \upharpoonright C`\breve{S} .$

[∗33·22] $\equiv . \dot{\exists} ! R \dot{\cap} I \upharpoonright C`S .$

[∗303·15] $\equiv . R 0_q S$ (2)

$\vdash . (1) . (2) . \supset \vdash .$ Prop

∗307·27. $\vdash . \infty_q \mid \mathrm{Cnv} = \infty_q = \mathrm{Cnv} \mid \infty_q$ [∗307·26 . ∗303·62]

∗307·3. $\vdash : X \in C`H . \supset . \dot{\exists} ! (X \mid \mathrm{Cnv}) \mathbin{\dot{\upharpoonright}} \text{Rel num}$ [∗304·5 . ∗307·16 . ∗300·4]

∗307·31. $\vdash : X \in \mathrm{Rat} - \iota`0_q . \supset . X \mid \mathrm{Cnv} \neq 0_q . X \mid \mathrm{Cnv} \neq \infty_q$ [∗307·3 . ∗304·53 . ∗303·62]

∗307·4. $\vdash : X H_n Y . \equiv . (X \mid \mathrm{Cnv}) H (Y \mid \mathrm{Cnv})$ [∗150·41 . (∗307·04)]

∗307·41. $\vdash . \mathrm{Nr}`H_n = \mathrm{Nr}`H$ [∗307·13 . (∗307·04)]

∗307·42. $\vdash :$ Infin ax $. \supset . \mathrm{Nr}`H_n = \mathrm{Nr}`\breve{H}_n = \eta$ [∗307·41 . ∗304·33]

∗307·43. $\vdash : X \in C`H_n . \supset . \dot{\exists} ! X \mathbin{\dot{\upharpoonright}} \text{Rel num}$ [∗307·3]

∗307·44. $\vdash . 0_q, \infty_q \sim \in C`H_n$ [∗307·31]

∗307·45. $\vdash . \mathrm{Nr}`H_g = \mathrm{Nr}`\breve{H} \dot{+} \dot{1} \dot{+} \mathrm{Nr}`H$ [∗307·25·41 . (∗307·05)]

∗307·46. $\vdash :$ Infin ax $. \supset . H_g \in \eta$ [∗307·45 . ∗304·33]

This proposition requires $\eta \dot{+} \dot{1} \dot{+} \eta = \eta$, which is easily proved.

$*308$. ADDITION OF GENERALIZED RATIOS.

Summary of $*308$.

In this number we have to extend addition so as to include negative ratios as addenda, and for this purpose we have to define subtraction of simple ratios. This is defined as follows:

$$*308\cdot01.\quad X -_s Y = \hat{R}\hat{S}\{(\exists Z) : X, Y, Z \in \mathrm{Rat} : Z +_s Y = X \,.\, RZS \,.\, \mathsf{v} \,.\, Z +_s X = Y \,.\, RZ\breve{S}\} \quad \mathrm{Df}$$

That is to say, if $Y <_r X$, $X -_s Y$ is the ratio which must be added to Y to give X, while if $X <_r Y$, $X -_s Y$ is the negative of the ratio which must be added to X to give Y. Thus we have

$$*308\cdot13.\quad \vdash :. Y <_r X \,.\, \mathsf{v} \,.\, Y \in \mathrm{Rat} \,.\, Y = \dot{X} : \supset . X -_s Y = (\imath Z)(Z +_s Y = X)$$

$$*308\cdot14.\quad \vdash :. X <_r Y \,.\, \mathsf{v} \,.\, X \in \mathrm{Rat} \,.\, Y = X : \supset . X -_s Y = \{(\imath Z)(Z +_s X = Y)\} \,|\, \mathrm{Cnv}$$

We have, of course, $X -_s 0_q = X$ ($*308\cdot22$), $0_q -_s X = X \,|\, \mathrm{Cnv}$ ($*308\cdot23$), and $X -_s X = 0_q$ ($*308\cdot12$). Existence-theorems for $X -_s Y$ are closely analogous to those for $X +_s Y$ and $X \times_s Y$. Also we have

$$*308\cdot2.\quad \vdash : X, Y \in \mathrm{Rat} \,.\, \equiv . X -_s Y \in \mathrm{Rat}_g$$

We define the sum of two generalized ratios by means of the sums and differences of simple ratios, as follows:

$$*308\cdot02.\quad X +_g Y = (X +_s Y) \cup (X -_s Y \,|\, \mathrm{Cnv}) \cup (Y -_s X \,|\, \mathrm{Cnv}) \cup (X \,|\, \mathrm{Cnv} +_s Y \,|\, \mathrm{Cnv}) \,|\, \mathrm{Cnv} \quad \mathrm{Df}$$

Of the four relations which occur in the above definition, all but one must be null if neither X nor Y is 0_q. Thus if X and Y are positive, $X -_s Y \,|\, \mathrm{Cnv}$, $Y -_s X \,|\, \mathrm{Cnv}$, and $X \,|\, \mathrm{Cnv} +_s Y \,|\, \mathrm{Cnv}$ are null; if X is positive and Y negative, $X +_s Y$, $Y -_s X \,|\, \mathrm{Cnv}$, and $X \,|\, \mathrm{Cnv} +_s Y \,|\, \mathrm{Cnv}$ are null; if X and Y are both negative, $X +_s Y$, $X -_s Y \,|\, \mathrm{Cnv}$, and $Y -_s X \,|\, \mathrm{Cnv}$ are null.

If X is 0_q and Y is positive,

$$X +_s Y = Y -_s X \,|\, \mathrm{Cnv} \,.\, X -_s Y \,|\, \mathrm{Cnv} = (X \,|\, \mathrm{Cnv} +_s Y \,|\, \mathrm{Cnv}) \,|\, \mathrm{Cnv} = \dot{\Lambda}.$$

If both X and Y are 0_q, all four relations are 0_q.

Hence we find

***308·32.** $\vdash : X, Y \epsilon \text{Rat} . \supset . X +_g Y = X +_s Y$

***308·321.** $\vdash : X \epsilon \text{Rat} . Y \epsilon \text{Rat}_n . \supset . X +_g Y = X -_s Y \,|\, \text{Cnv}$

***308·322.** $\vdash : Y \epsilon \text{Rat} . X \epsilon \text{Rat}_n . \supset . X +_g Y = Y -_s X \,|\, \text{Cnv}$

***308·323.** $\vdash : X, Y \epsilon \text{Rat}_n . \supset . X +_g Y = (X \,|\, \text{Cnv} +_s Y \,|\, \text{Cnv}) \,|\, \text{Cnv}$

The existence-theorems for $X +_g Y$ are closely analogous to those for $X +_s Y$, and the formal laws offer no difficulty. We have

***308·52.** $\vdash :. X, Y \epsilon \text{Rat}_g . \supset : X +_g Y = X +_g Z . \equiv . Y = Z$

***308·54.** $\vdash : X, Y \epsilon \text{Rat}_g . \supset . (\exists Z) . Z \epsilon \text{Rat}_g . X +_g Z = Y$

***308·56.** $\vdash :. X <_g Y . \equiv : X \epsilon \text{Rat}_g : (\exists Z) . Z \epsilon \text{Rat} - \iota\text{'}0_q . X +_g Z = Y$

***308·72.** $\vdash : (X +_g Z) <_g (X +_g Z') . \equiv . X \epsilon \text{Rat}_g . Z <_g Z'$

***308·01.** $X -_s Y = \hat{R}\hat{S} \{(\exists Z) : X, Y, Z \epsilon \text{Rat} : Z +_s Y = X . RZS . \mathbf{v} .$
$Z +_s X = Y . RZ\breve{S}\}$ Df

***308·02.** $X +_g Y = (X +_s Y) \cup (X -_s Y \,|\, \text{Cnv}) \cup$
$(Y -_s X \,|\, \text{Cnv}) \cup (X \,|\, \text{Cnv} +_s Y \,|\, \text{Cnv}) \,|\, \text{Cnv}$ Df

***308·1.** $\vdash : Y <_r X . \supset . X -_s Y = \hat{R}\hat{S} \{(\exists Z) . Z \epsilon \text{Rat} . Z +_s Y = X . RZS\}$

Dem.

$\vdash . *306·55 . \supset \vdash : \text{Hp} . \supset . \sim (\exists Z) . Z +_s X = Y$ (1)

$\vdash . (1) . (*308·01) . \supset \vdash . \text{Prop}$

***308·11.** $\vdash : X <_r Y . \supset . X -_s Y = \hat{R}\hat{S} \{(\exists Z) . Z \epsilon \text{Rat} . Z +_s X = Y . RZ\breve{S}\}$

Dem.

$\vdash . *306·55 . \supset \vdash : \text{Hp} . \supset . \sim (\exists Z) . Z +_s Y = X$ (1)

$\vdash . (1) . (*308·01) . \supset \vdash . \text{Prop}$

***308·12.** $\vdash : X \epsilon \text{Rat} . X = Y . \supset . X -_s Y = 0_q$ [*306·54·24]

***308·13.** $\vdash :. Y <_r X . \mathbf{v} . Y \epsilon \text{Rat} . Y = X : \supset . X -_s Y = (\imath Z)(Z +_s Y = X)$

Dem.

$\vdash . *306·52·24 . \supset \vdash : \text{Hp} . \supset . (\exists Z) . Z +_s Y = X . Z \epsilon \text{Rat}$ (1)

$\vdash . *306·54 . \supset \vdash : \text{Hp} . Z +_s Y = X . Z' +_s Y = X . \supset . Z = Z'$ (2)

$\vdash . (1) . (2) . *308·1·12 . \supset \vdash . \text{Prop}$

***308·14.** $\vdash :. X <_r Y . \mathbf{v} . X \epsilon \text{Rat} . X = Y : \supset . X -_s Y = \{(\imath Z)(Z +_s X = Y)\} \,|\, \text{Cnv}$
[Proof as in *308·13]

***308·15.** $\vdash : \sim (X, Y \epsilon \text{Rat}) . \supset . X -_s Y = \dot{\Lambda}$ [(*308·01)]

***308·16.** $\vdash : X, Y \epsilon \text{Rat} . Y +_s Z = X . \supset . X -_s Y = Z$

Dem.

$\vdash . *306·55 . *304·221 . \supset \vdash :. \text{Hp} . \supset : Y <_r X . \mathbf{v} . Y \epsilon \text{Rat} . Y = X$ (1)

$\vdash . (1) . *308·13 . \supset \vdash . \text{Prop}$

***308·17.** $\vdash : X, Y \epsilon \mathrm{Rat} . X +_s Z = Y . \supset . X -_s Y = Z \mid \mathrm{Cnv}$ [*306·55.*308·14]

***308·18.** $\vdash : Y <_r X . \supset . X -_s Y \epsilon \mathrm{Rat} - \iota`0_q$

Dem.

$\vdash . *306·52 . \supset \vdash : \mathrm{Hp} . \supset . (\exists Z) . Z \epsilon \mathrm{Rat} - \iota`0_q . Y +_s Z = X$ (1)

$\vdash . (1) . *308·13 . \supset \vdash . \mathrm{Prop}$

***308·19.** $\vdash : X <_r Y . \supset . X -_s Y \epsilon \mathrm{Rat}_n - \iota`0_q$

Dem.

$\vdash . *306·52 . \supset \vdash : \mathrm{Hp} . \supset . (\exists Z) . Z \epsilon \mathrm{Rat} - \iota`0_q . X +_s Z = Y$ (1)

$\vdash . (1) . *308·14 . \supset \vdash . \mathrm{Prop}$

***308·2.** $\vdash : X, Y \epsilon \mathrm{Rat} . \equiv . X -_s Y \epsilon \mathrm{Rat}_g$ [*308·12·18·19·15]

***308·21.** $\vdash : X -_s Y = (Y -_s X) \mid \mathrm{Cnv} = \mathrm{Cnv} \mid (Y -_s X)$

Dem.

$\vdash . *308·13·14 . \supset$

$\vdash :. X <_r Y . \mathsf{v} . X \epsilon \mathrm{Rat} - \iota`0_q . X = Y : \supset . X -_s Y = (Y -_s X) \mid \mathrm{Cnv}$ (1)

$\vdash . *308·13·14 . *307·12 . \supset$

$\vdash :. Y <_r X . \mathsf{v} . Y \epsilon \mathrm{Rat} - \iota`0_q . Y = X : \supset . X -_s Y = (Y -_s X) \mid \mathrm{Cnv}$ (2)

$\vdash . (1) . (2) . *304·221 . \supset \vdash : X, Y \epsilon \mathrm{Rat} . \supset . X -_s Y = (Y -_s X) \mid \mathrm{Cnv}$ (3)

[*307·21.*308·2] $= \mathrm{Cnv} \mid (Y -_s X)$ (4)

$\vdash . (3) . (4) . *308·15 . \supset \vdash . \mathrm{Prop}$

***308·22.** $\vdash : X \epsilon \mathrm{Rat} . \supset . X -_s 0_q = X$ [*306·24 . *308·13]

***308·23.** $\vdash : X \epsilon \mathrm{Rat} . \supset . 0_q -_s X = X \mid \mathrm{Cnv}$ [*308·21·22]

***308·24.** $\vdash : (\nu/\rho) <_r (\lambda/\mu) . \supset . \lambda/\mu -_s \nu/\rho = \{(\lambda \times_c \rho) -_c (\mu \times_c \nu)\}/(\mu \times_c \rho)$

Dem.

$\vdash . *304·1 . \supset \vdash : \mathrm{Hp} . \supset . \lambda \times_c \rho > \mu \times_c \nu$ (1)

$\vdash . *303·23 . *306·13 . (1) . \supset$

$\vdash : \mathrm{Hp} . \supset . \{(\lambda \times_c \rho) - (\mu \times_c \nu)\}/(\mu \times_c \rho) +_s \nu/\rho =$

$[\{(\lambda \times_c \rho) - (\mu \times_c \nu)\} +_c (\mu \times_c \nu)]/(\mu \times_c \rho)$

[*303·23.*119·34] $= \lambda/\mu$ (2)

$\vdash . (1) . (2) . *308·16 . \supset \vdash . \mathrm{Prop}$

***308·241.** $\vdash : (\lambda/\mu) <_r (\nu/\rho) . \supset . \lambda/\mu -_s \nu/\rho = [\{(\mu \times_c \nu) -_c (\lambda \times_c \rho)\}/(\mu \times_c \rho)] \mid \mathrm{Cnv}$
[*308·24·21]

***308·25.** $\vdash : \lambda, \mu, \nu, \rho \epsilon \mathrm{D}`U \cap \mathrm{Q}`U . \nu/\rho <_r \lambda/\mu . \supset . (\lambda/\mu -_s \nu/\rho) \upharpoonright t_{00}`\mu \epsilon C`H$

Dem.

$\vdash . *305·24 . \supset$

$\vdash : \mathrm{Hp} . \supset . \{(\lambda \times_c \rho) -_c (\mu \times_c \nu)\} \cap t`\mu, (\mu \times_c \rho) \cap t`\mu \epsilon \mathrm{D}`U \cap \mathrm{Q}`U$ (1)

$\vdash . (1) . *308·24 . *304·28 . \supset \vdash . \mathrm{Prop}$

***308·251.** $\vdash : \lambda, \mu, \nu, \rho \in \mathrm{D}`U \cap \text{ᗡ}`U . \lambda/\mu <_r \nu/\rho . \supset . (\lambda/\mu -_s \nu/\rho) \upharpoonright t_{00}`\mu \in C`H_n$
[*305·24 . *308·241]

***308·252.** $\vdash : \lambda, \mu, \nu, \rho \in \mathrm{D}`U \cap \text{ᗡ}`U . \supset . (\lambda/\mu -_s \nu/\rho) \upharpoonright t_{00}`\mu \in C`H_g$
[*308·25·251·12]

***308·26.** $\vdash : X, Y \in \mathrm{Rat} . X \upharpoonright t_{11}`\mu, Y \upharpoonright t_{11}`\mu \in C`H' . \supset . (X -_s Y) \upharpoonright t_{00}`\mu \in C`H_g$
[*308·252 . *304·28]

***308·261.** $\vdash : X, Y \in C`H' . \supset . (X -_s Y) \upharpoonright t_{00}`C``C`X \in C`H_g$ [*308·26]

***308·3.** $\vdash : \dot{\exists} ! (X -_s Y \mid \mathrm{Cnv}) . \supset . X \in \mathrm{Rat} . Y \in \mathrm{Rat}_n$
[*308·15 . *307·12]

***308·301.** $\vdash : \dot{\exists} ! (X \mid \mathrm{Cnv} +_s Y \mid \mathrm{Cnv}) . \supset . X, Y \in \mathrm{Rat}_n$ [*306·12 . *307·23·12]

***308·31.** $\vdash : \dot{\exists} ! (X +_g Y) . \supset . X, Y \in \mathrm{Rat}_g$ [*306·12 . *308·3·301 . (*308·02)]

***308·32.** $\vdash : X, Y \in \mathrm{Rat} . \supset . X +_g Y = X +_s Y$

Dem.

$\vdash . *308·3·301 . *307·25 . (*308·02) . \supset$
$\vdash : X, Y \in \mathrm{Rat} - \iota`0_q . \supset . X +_g Y = X +_s Y$ (1)
$\vdash . *306·24 . *308·22·3·301 . \supset$
$\vdash : X \in \mathrm{Rat} - \iota`0_q . Y = 0_q . \supset . X +_g Y = X = X +_s Y$ (2)
$\vdash . *306·24 . *308·3·301 . \supset \vdash : X = 0_q . Y = 0_q . \supset . X +_g Y = 0_q = X +_s Y$ (3)
$\vdash . (2) . (3) . \supset$
$\vdash :. X \in \mathrm{Rat} . Y = 0_q . \vee . Y \in \mathrm{Rat} . X = 0_q : \supset . X +_g Y = X +_s Y$ (4)
$\vdash . (1) . (4) . \supset \vdash . \mathrm{Prop}$

***308·321.** $\vdash : X \in \mathrm{Rat} . Y \in \mathrm{Rat}_n . \supset . X +_g Y = X -_s Y \mid \mathrm{Cnv}$
[*306·12 . *308·3·301 . *307·25 . (*308·02)]

***308·322.** $\vdash : Y \in \mathrm{Rat} . X \in \mathrm{Rat}_n . \supset . X +_g Y = Y -_s X \mid \mathrm{Cnv}$
[*306·12 . *308·3·301 . *307·25 . (*308·02)]

***308·323.** $\vdash : X, Y \in \mathrm{Rat}_n . \supset . X +_g Y = (X \mid \mathrm{Cnv} +_s Y \mid \mathrm{Cnv}) \mid \mathrm{Cnv}$
[*306·12 . *308·3·301 . *307·25 . (*308·02)]

***308·33.** $\vdash : X +_g Y \in \mathrm{Rat}_g . \equiv . X, Y \in \mathrm{Rat}_g$
[*306·22 . *308·2·32·31]

***308·4.** $\vdash . X +_g Y = Y +_g X$ [*306·11 . (*308·02)]

***308·41.** $\vdash . X +_g Y = (X \mid \mathrm{Cnv} +_g Y \mid \mathrm{Cnv}) \mid \mathrm{Cnv}$

Dem.

$\vdash . *307·12 . *34·26 . (*308·02) . \supset$
$\vdash . (X \mid \mathrm{Cnv} +_g Y \mid \mathrm{Cnv}) \mid \mathrm{Cnv} = (X \mid \mathrm{Cnv} +_s Y \mid \mathrm{Cnv}) \mid \mathrm{Cnv} \cup (X \mid \mathrm{Cnv} -_s Y) \mid \mathrm{Cnv}$
$\quad \cup (Y \mid \mathrm{Cnv} -_s X) \mid \mathrm{Cnv} \cup (X +_s Y)$
[*308·21] $= (X \mid \mathrm{Cnv} \cup Y \mid \mathrm{Cnv}) \mid \mathrm{Cnv} \cup (Y -_s X \mid \mathrm{Cnv})$
$\quad \cup (X -_s Y \mid \mathrm{Cnv}) \cup (X +_s Y)$
[(*308·02)] $= X +_g Y . \supset \vdash . \mathrm{Prop}$

***308·411.** $\vdash . (X +_g Y) | \text{Cnv} = X | \text{Cnv} +_g Y | \text{Cnv}$ [*308·41 . *307·12]

***308·412.** $\vdash : X | \text{Cnv} +_g Y | \text{Cnv} = Z | \text{Cnv} . \equiv . X +_g Y = Z$
[*308·411 . *307·13]

***308·42.** $\vdash : X, Y \in \text{Rat} . \supset . (X -_s Y) +_g Y = X$

Dem.

$\vdash . *308·12·32 . *306·24 . \supset \vdash : \text{Hp} . X = Y . \supset . (X -_s Y) +_g Y = X$ (1)

$\vdash . *308·18·32 . \supset \vdash : \text{Hp} . Y <_r X . \supset . (X -_s Y) +_g Y = (X -_s Y) +_s Y$

[*308·13] $= X$ (2)

$\vdash . *308·19·322 . \supset \vdash : \text{Hp} . X <_r Y . \supset . (X -_s Y) +_g Y = Y -_s (X -_s Y) | \text{Cnv}$

[*308·21] $= Y -_s (Y -_s X)$ (3)

$\vdash . *308·13 . \supset \vdash : \text{Hp}(3) . \supset . X +_s (Y -_s X) = Y .$

[*308·16·18] $\supset . X = Y -_s (Y -_s X)$ (4)

$\vdash . (3) . (4) . \supset \vdash : \text{Hp} . X <_r Y . \supset . (X -_s Y) +_g Y = X$ (5)

$\vdash . (1) . (2) . (5) . *304·221 . \supset \vdash . \text{Prop}$

***308·43.** $\vdash : X, Y \in \text{Rat} . \supset . (X +_g Y) -_s Y = X$

Dem.

$\vdash . *308·32 . \supset \vdash : \text{Hp} . \supset . X +_g Y = X +_s Y .$

[*308·16.*306·22] $\supset . (X +_g Y) -_s Y = X : \supset \vdash . \text{Prop}$

***308·44.** $\vdash :. X, Z \in \text{Rat} . \supset : X -_s Z = Y -_s Z . \equiv . X = Y$

Dem.

$\vdash . *308·13·14·15 . \supset \vdash : X = Y . \supset . X -_s Z = Y -_s Z$ (1)

$\vdash . *308·2 . \supset \vdash : \text{Hp} . X -_s Z = Y -_s Z . \supset . Y \in \text{Rat} .$

[*308·42] $\supset . (Y -_s Z) +_s Z = Y .$

[Hp] $\supset . (X -_s Z) +_s Z = Y .$

[*308·42] $\supset . X = Y$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***308·45.** $\vdash :. X, Z \in \text{Rat} . \supset : Z -_s X = Z -_s Y . \equiv . X = Y$
[*308·44·21 . *307·13]

***308·46.** $\vdash : X, Y \in \text{Rat} . Y \neq 0_q . \supset . (X -_s Y) <_g X$

Dem.

$\vdash . *308·19 . \supset \vdash : X <_r Y . \supset . (X -_s Y) \in \text{Rat}_n - \iota`0_q . X \in \text{Rat} .$

[(*307·03)] $\supset . (X -_s Y) <_g X$ (1)

$\vdash . *308·12 . \supset \vdash : \text{Hp} . X = Y . \supset . X -_s Y = 0_q .$

[*304·46.(*307·03)] $\supset . (X -_s Y) <_g X$ (2)

$\vdash . *308·13·18 . \supset \vdash : \text{Hp} . Y <_r X . \supset . (X -_s Y) +_s Y = X . X -_s Y \in \text{Rat} - \iota`0_q .$

[*306·52] $\supset . (X -_s Y) <_r X .$

[(*307·03)] $\supset . (X -_s Y) <_g X$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash . \text{Prop}$

***308·47.** $\vdash : X \in \text{Rat} . Y, Z \in \text{Rat} - \iota`0_q . \supset . X -_s Y \neq X +_s Z$

Dem.

$\vdash . *306·52 . *308·46 . \supset \vdash : \text{Hp} . \supset . (X -_s Y) <_r (X +_s Z) .$

$[*304·201] \qquad \supset . X -_s Y \neq X +_s Z : \supset \vdash . \text{Prop}$

***308·51.** $\vdash :. X \in \text{Rat}_g . \supset : X +_g Y = X . \equiv . Y = 0_q$

Dem.

$\vdash . *308·33 . \supset \vdash :. \text{Hp} . \supset : X +_g Y = X . \supset . Y \in \text{Rat}_g \qquad (1)$

$\vdash . *308·32 . \supset \vdash : X \in \text{Rat} . Y = 0_q . \supset . X +_g Y = X +_s Y$

$[*306·24] \qquad = X \qquad (2)$

$\vdash . *308·322 . \supset \vdash : X \in \text{Rat}_n . Y = 0_q . \supset . X +_g Y = Y -_s X \mid \text{Cnv}$

$[*308·23 . *307·12] \qquad = X \qquad (3)$

$\vdash . (2) . (3) . \supset \vdash :. \text{Hp} . \supset : Y = 0_q . \supset . X +_g Y = X \qquad (4)$

$\vdash . *308·32 . \supset \vdash : X, Y \in \text{Rat} . X +_g Y = X . \supset . X +_s Y = X .$

$[*306·24·54] \qquad \supset . Y = 0_q \qquad (5)$

$\vdash . *308·321 . \supset \vdash : X \in \text{Rat} . Y \in \text{Rat}_n . X +_g Y = X . \supset . X -_s Y \mid \text{Cnv} = X .$

$[*308·22·45] \qquad \supset . Y \mid \text{Cnv} = 0_q .$

$[*307·2] \qquad \supset . Y = 0_q \qquad (6)$

$\vdash . *308·322 . \supset \vdash : X \in \text{Rat}_n . Y \in \text{Rat} . X +_g Y = X . \supset . Y -_s X \mid \text{Cnv} = X$

$[*308·23 . *307·12] \qquad = 0_q -_s X \mid \text{Cnv} .$

$[*308·44] \qquad \supset . Y = 0_q \qquad (7)$

$\vdash . *308·323 . *307·14 . \supset$

$\vdash : X, Y \in \text{Rat}_n . X +_g Y = X . \supset . X \mid \text{Cnv} +_s Y \mid \text{Cnv} = X \mid \text{Cnv} .$

$[(5) . *307·26] \qquad \supset . Y = 0_q \qquad (8)$

$\vdash . (1) . (5) . (6) . (7) . (8) . \supset \vdash :. \text{Hp} . \supset : X +_g Y = X . \supset . Y = 0_q \qquad (9)$

$\vdash . (4) . (9) . \supset \vdash . \text{Prop}$

***308·52.** $\vdash :. X, Y \in \text{Rat}_g . \supset : X +_g Y = X +_g Z . \equiv . Y = Z$

Dem.

$\vdash . *308·321·47 . \supset \vdash : X, Y \in \text{Rat} . Y \neq 0_q . X +_g Y = X +_g Z . \supset . Z \sim\in \text{Rat}_n \quad (1)$

$\vdash . *308·51 . \supset \vdash : X \in \text{Rat}_g . Y = 0_q . X +_g Y = X +_g Z . \supset . Z = 0_q \qquad (2)$

$\vdash . (1) . (2) . *308·33 . \supset \vdash : X, Y \in \text{Rat} . X +_g Y = X +_g Z . \supset . Z \in \text{Rat} \qquad (3)$

$\vdash . (3) . *308·32 . \supset \vdash : X, Y \in \text{Rat} . X +_g Y = X +_g Z . \supset . X +_s Y = X +_s Z .$

$[*306·54] \qquad \supset . Y = Z \qquad (4)$

$\vdash . (4) . *308·323 . *307·13 . \supset \vdash : X, Y \in \text{Rat}_n . X +_g Y = X +_g Z . \supset . Y = Z \quad (5)$

$\vdash . *308·321·32·47 . \supset$

$\vdash : X \in \text{Rat} . Y \in \text{Rat}_n . X +_g Y = X +_g Z . \supset . Z \sim\in \text{Rat} - \iota`0_q \qquad (6)$

$\vdash . (2) \dfrac{Z, Y}{Y, Z} . \text{Transp} . \supset$

$\vdash : X \in \text{Rat} . Y \in \text{Rat}_n - \iota`0_q . X +_g Y = X +_g Z . \supset . Z \neq 0_q \qquad (7)$

$\vdash . (6) . (7) . *308\cdot33 . \supset$

$\vdash : X \in \text{Rat} . Y \in \text{Rat}_n - \iota\text{'}0_q . X +_g Y = X +_g Z . \supset . Z \in \text{Rat}_n$ (8)

$\vdash . (8) . *308\cdot321 . \supset \vdash : \text{Hp}(8) . \supset . X -_s Y | \text{Cnv} = X -_s Z | \text{Cnv} .$

$[*308\cdot45 . *307\cdot13] \quad \supset . Y = Z$ (9)

$\vdash . (9) . *308\cdot411 . *307\cdot13 . \supset$

$\vdash : X \in \text{Rat}_n . Y \in \text{Rat} . X +_g Y = X +_g Z . \supset . Y = Z$ (10)

$\vdash . (4) . (5) . (9) . (10) . \supset \vdash : \text{Hp} . X +_g Y = X +_g Z . \supset . Y = Z$ (11)

$\vdash . (11) . (*308\cdot02) . \supset \vdash . \text{Prop}$

***308·53.** $\vdash : X, Y \in \text{Rat}_g . \supset . X +_g (Y +_g X | \text{Cnv}) = Y$

Dem.

$\vdash . *308\cdot321 . *307\cdot12 . \supset \vdash : X, Y \in \text{Rat} . \supset . X +_g (Y +_g X | \text{Cnv}) = X +_g (Y -_s X)$

$[*308\cdot4\cdot42] \quad = Y$ (1)

$\vdash . *308\cdot32 . \supset$

$\vdash : X \in \text{Rat}_n . Y \in \text{Rat} . \supset . X +_g (Y +_g X | \text{Cnv}) = X +_g (Y +_s X | \text{Cnv})$

$[*308\cdot4\cdot321 . *306\cdot22] \quad = (Y +_s X | \text{Cnv}) -_s X | \text{Cnv}$

$[*308\cdot43\cdot32] \quad = Y$ (2)

$\vdash . *308\cdot323 . *307\cdot12 . \supset$

$\vdash : X \in \text{Rat} . Y \in \text{Rat}_n . \supset . X +_g (Y +_g X | \text{Cnv}) = X +_g (Y | \text{Cnv} +_s X) | \text{Cnv}$

$[*308\cdot321 . *306\cdot22] \quad = X -_s (Y | \text{Cnv} +_s X)$

$[*308\cdot17 . *307\cdot12] \quad = Y$ (3)

$\vdash . (1) . \supset \vdash : X, Y \in \text{Rat}_n . \supset . X | \text{Cnv} +_g (Y | \text{Cnv} +_g X | \text{Cnv} | \text{Cnv}) = Y | \text{Cnv} .$

$[*308\cdot411] \quad \supset . X | \text{Cnv} +_g (Y +_g X | \text{Cnv}) | \text{Cnv} = Y | \text{Cnv} .$

$[*308\cdot412] \quad \supset . X +_g (Y +_g X | \text{Cnv}) = Y$ (4)

$\vdash . (1) . (2) . (3) . (4) . \supset \vdash . \text{Prop}$

***308·54.** $\vdash : X, Y \in \text{Rat}_g . \supset . (\exists Z) . Z \in \text{Rat}_g . X +_g Z = Y \quad [*308\cdot53\cdot33]$

***308·55.** $\vdash :. X, Y, Z \in \text{Rat}_g . \supset : X +_g Z = Y . \equiv . X = Y +_g Z | \text{Cnv}$

Dem.

$\vdash . *308\cdot53\cdot52\cdot4 . \supset \vdash : \text{Hp} . X +_g Z = Y . \supset . Y +_g Z | \text{Cnv} = X$ (1)

$\vdash . *308\cdot53\cdot4 . \quad \supset \vdash : \text{Hp} . Y +_g Z | \text{Cnv} = X . \supset . X +_g Z = Y$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***308·56.** $\vdash :. X <_g Y . \equiv : X \in \text{Rat}_g : (\exists Z) . Z \in \text{Rat} - \iota\text{'}0_q . X +_g Z = Y$

Dem.

$\vdash . *306\cdot52 . *308\cdot32 . \supset$

$\vdash :. X <_r Y . \equiv : X \in \text{Rat} : (\exists Z) . Z \in \text{Rat} - \iota\text{'}0_q . X +_g Z = Y :$ (1)

$[*306\cdot52\cdot25] \supset : Y \in \text{Rat} : (\exists Z) . Z \in \text{Rat} - \iota\text{'}0_q . X +_g Z = Y$ (2)

$\vdash . (2) \dfrac{Y | \text{Cnv}, X | \text{Cnv}}{X, Y} . \supset$

$\vdash :. X >_n Y . \supset : X \in \text{Rat}_n : (\exists Z) . Z \in \text{Rat} - \iota\text{'}0_q . Y | \text{Cnv} +_g Z = X | \text{Cnv} :$

$[*308\cdot55\cdot412] \supset : X \in \text{Rat}_n : (\exists Z) . Z \in \text{Rat} - \iota\text{'}0_q . X +_g Z = Y$ (3)

$\vdash . *308{\cdot}32{\cdot}53 . *306{\cdot}23 . \supset \vdash : X \epsilon \mathrm{Rat}_n . Y \epsilon \mathrm{Rat} . \supset .$

$Y +_g X \mid \mathrm{Cnv} \epsilon \mathrm{Rat} - \iota`0_q . X +_g (Y +_g X \mid \mathrm{Cnv}) = Y$ (4)

$\vdash . (1) . (2) . (3) . (4) . (*307{\cdot}03) . \supset$

$\vdash :. X <_g Y . \supset : X \epsilon \mathrm{Rat}_g : (\exists Z) . Z \epsilon \mathrm{Rat} - \iota`0_q . X +_g Z = Y$ (5)

$\vdash . *35{\cdot}103 . (*307{\cdot}03) . \supset \vdash : X \epsilon \mathrm{Rat}_n - \iota`0_q . Y \epsilon \mathrm{Rat} . \supset . X <_g Y$ (6)

$\vdash . *308{\cdot}55{\cdot}412 . \supset$

$\vdash : X, Y \epsilon \mathrm{Rat}_n . Z \epsilon \mathrm{Rat} - \iota`0_q . X +_g Z = Y . \supset . X \mid \mathrm{Cnv} = Y \mid \mathrm{Cnv} +_s Z .$

[$*306{\cdot}52$] $\supset . X >_n Y$ (7)

$\vdash . (6) . (7) . \supset \vdash :. X \epsilon \mathrm{Rat}_n : (\exists Z) . Z \epsilon \mathrm{Rat} - \iota`0_q . X +_g Z = Y : \supset . X <_g Y$ (8)

$\vdash . (1) . (8) . \supset \vdash :. X \epsilon \mathrm{Rat}_g : (\exists Z) . Z \epsilon \mathrm{Rat} - \iota`0_q . X +_g Z = Y : \supset . X <_g Y$ (9)

$\vdash . (5) . (9) . \supset \vdash .$ Prop

$*308{\cdot}561$. $\vdash :. X <_g Y . \equiv : Y \epsilon \mathrm{Rat}_g : (\exists Z) . Z \epsilon \mathrm{Rat} - \iota`0_q . X +_g Z = Y$

[$*308{\cdot}56{\cdot}33$]

$*308{\cdot}57$. $\vdash : X <_g Y . \equiv . X \epsilon \mathrm{Rat}_g . Y +_g X \mid \mathrm{Cnv} \epsilon \mathrm{Rat} - \iota`0_q .$

$\equiv . Y \epsilon \mathrm{Rat}_g . Y +_g X \mid \mathrm{Cnv} \epsilon \mathrm{Rat} - \iota`0_q$

Dem.

$\vdash . *308{\cdot}55{\cdot}56{\cdot}4 . \supset$

$\vdash :. X <_g Y . \equiv : X \epsilon \mathrm{Rat}_g : (\exists Z) . Z \epsilon \mathrm{Rat} - \iota`0_q . Z = Y +_g X \mid \mathrm{Cnv}$ (1)

$\vdash . *308{\cdot}55{\cdot}561{\cdot}4 . \supset$

$\vdash :. X <_g Y . \equiv : Y \epsilon \mathrm{Rat}_g : (\exists Z) . Z \epsilon \mathrm{Rat} - \iota`0_q . Z = Y +_g X \mid \mathrm{Cnv}$ (2)

$\vdash . (1) . (2) . \supset \vdash .$ Prop

$*308{\cdot}6$. $\vdash : X, Y, Z \epsilon \mathrm{Rat} . \supset . (X +_g Y) +_g Z = X +_g (Y +_g Z)$

[$*308{\cdot}32 . *306{\cdot}22{\cdot}31$]

$*308{\cdot}601$. $\vdash : X, Y, Z \epsilon \mathrm{Rat}_n . \supset . (X +_g Y) +_g Z = X +_g (Y +_g Z)$

Dem.

$\vdash . *308{\cdot}323 . *307{\cdot}12$

$\vdash : \mathrm{Hp} . \supset . (X +_g Y) +_g Z = (X \mid \mathrm{Cnv} +_s Y \mid \mathrm{Cnv}) \mid \mathrm{Cnv} +_g (Z \mid \mathrm{Cnv}) \mid \mathrm{Cnv}$

[$*308{\cdot}411$] $= \{(X \mid \mathrm{Cnv} +_s Y \mid \mathrm{Cnv}) +_g Z \mid \mathrm{Cnv}\} \mid \mathrm{Cnv}$

[$*308{\cdot}6 . *306{\cdot}22$] $= \{X \mid \mathrm{Cnv} +_g (Y \mid \mathrm{Cnv} +_g Z \mid \mathrm{Cnv})\} \mid \mathrm{Cnv}$

[$*308{\cdot}411$] $= X +_g (Y \mid \mathrm{Cnv} +_g Z \mid \mathrm{Cnv}) \mid \mathrm{Cnv}$

[$*308{\cdot}323$] $= X +_g (Y +_g Z) : \supset \vdash .$ Prop

$*308{\cdot}602$. $\vdash : \lambda, \mu, \nu, \rho, \sigma, \tau \epsilon \mathrm{NC\,ind} . \mu, \rho, \tau \sim \epsilon \iota`0 . \supset .$

$(\lambda/\mu +_s \nu/\rho) -_s \sigma/\tau = (\lambda/\mu -_s \sigma/\tau) +_g \nu/\rho$

Dem.

$\vdash . *308{\cdot}24 . \supset \vdash : \mathrm{Hp} . \sigma/\tau <_r \lambda/\mu . \supset .$

$(\lambda/\mu +_s \nu/\rho) -_s \sigma/\tau = \{(\lambda \times_c \rho \times_c \tau) +_c (\mu \times_c \nu \times_c \tau) -_c (\mu \times_c \rho \times_c \sigma)\}/(\mu \times_c \rho \times_c \tau) .$

$(\lambda/\mu -_s \sigma/\tau) +_s \nu/\rho = \{(\lambda \times_c \rho \times_c \tau) -_c (\mu \times_c \rho \times_c \sigma) +_c (\mu \times_c \nu \times_c \tau)\}/(\mu \times_c \rho \times_c \tau)$ (1)

$\vdash . *308\cdot241 . \supset \vdash : \text{Hp} . \lambda/\mu +_s \nu/\rho <_r \sigma/\tau . \supset . (\lambda/\mu +_s \nu/\rho) -_s \sigma/\tau$

$= [\{(\mu \times_c \rho \times_c \sigma) - (\lambda \times_c \rho \times_c \tau) - (\mu \times_c \nu \times_c \tau)\}/(\mu \times_c \rho \times_c \tau)] \mid \text{Cnv} .$

$(\lambda/\mu -_s \sigma/\tau) +_g \nu/\rho = [\{(\mu \times_c \tau) -_s (\lambda \times_c \sigma)\}/(\mu \times_c \tau)] \mid \text{Cnv} +_g \nu/\rho$

$[*308\cdot322\cdot21]$

$$= [\{(\mu \times_c \rho \times_c \sigma) - (\lambda \times_c \rho \times_c \tau) - (\mu \times_c \nu \times_c \tau)\}/(\mu \times_c \rho \times_c \tau)] \mid \text{Cnv} \quad (2)$$

$\vdash . *308\cdot24\cdot241 . \supset \vdash : \text{Hp} . \lambda/\mu <_r \sigma/\tau . \sigma/\tau <_r \lambda/\mu +_s \nu/\rho . \supset .$

$(\lambda/\mu +_s \nu/\rho) -_s \sigma/\tau = \{(\lambda \times_c \rho \times_c \tau) +_c (\mu \times_c \nu \times_c \tau) -_c (\mu \times_c \rho \times_c \sigma)\}/(\mu \times_c \rho \times_c \tau)$

$(\lambda/\mu -_s \sigma/\tau) +_g \nu/\rho = [\{(\mu \times_c \sigma) -_c (\lambda \times_c \tau)\}/(\mu \times_c \tau)] \mid \text{Cnv} +_g \nu/\rho$

$$[*308\cdot322\cdot21] = \{(\lambda \times_c \rho \times_c \tau) +_c (\mu \times_c \nu \times_c \tau) -_c (\mu \times_c \rho \times_c \sigma)\}/(\mu \times_c \rho \times_c \tau) \quad (3)$$

$\vdash . *308\cdot16\cdot12 . \supset$

$$\vdash : \text{Hp} . \lambda/\mu = \sigma/\tau . \supset . (\lambda/\mu +_s \nu/\rho) -_s \sigma/\tau = \nu/\rho = (\lambda/\mu -_s \sigma/\tau) +_g \nu/\rho \quad (4)$$

$\vdash . *308\cdot12\cdot53\cdot17 . \supset$

$$\vdash : \text{Hp} . \lambda/\mu +_c \nu/\rho = \sigma/\tau . \supset . (\lambda/\mu +_s \nu/\rho) -_s \sigma/\tau = 0_q = (\lambda/\mu -_s \sigma/\tau) +_g \nu/\rho \quad (5)$$

$\vdash . (1) . (2) . (3) . (4) . (5) . \supset \vdash . \text{Prop}$

$*308\cdot61$. $\vdash : X, Y, Z \in \text{Rat} . \supset . (X +_g Y) -_s Z = (X -_s Z) +_g Y$

$[*308\cdot602\cdot32]$

$*308\cdot62$. $\vdash : X, Y \in \text{Rat} . Z \in \text{Rat}_n . \supset . (X +_g Y) +_g Z = X +_g (Y +_g Z)$

Dem.

$\vdash . *308\cdot33\cdot321 . \supset \vdash : \text{Hp} . \supset . (X +_g Y) +_g Z = (X +_g Y) -_s Z \mid \text{Cnv}$

$[*308\cdot4] \qquad = (Y +_g X) -_s Z \mid \text{Cnv}$

$[*308\cdot61] \qquad = (Y -_s Z \mid \text{Cnv}) +_g X$

$[*308\cdot4] \qquad = X +_g (Y -_s Z \mid \text{Cnv})$

$[*308\cdot321] \qquad = X +_g (Y +_g Z) : \supset \vdash . \text{Prop}$

$*308\cdot621$. $\vdash : X, Y \in \text{Rat}_n . Z \in \text{Rat} . \supset . (X +_g Y) +_g Z = X +_g (Y +_g Z)$

Dem.

$\vdash . *308\cdot62 . \supset$

$\vdash : \text{Hp} . \supset . (X \mid \text{Cnv} +_g Y \mid \text{Cnv}) +_g Z \mid \text{Cnv} = X \mid \text{Cnv} +_g (Y \mid \text{Cnv} +_g Z \mid \text{Cnv}) .$

$[*308\cdot411] \supset . (X +_g Y) \mid \text{Cnv} +_g Z \mid \text{Cnv} = X \mid \text{Cnv} +_g (Y +_g Z) \mid \text{Cnv}$

$[*308\cdot411] \qquad = \{X +_g (Y +_g Z)\} \mid \text{Cnv} .$

$[*308\cdot412] \supset . (X +_g Y) +_g Z = X +_g (Y +_g Z) : \supset \vdash . \text{Prop}$

$*308\cdot63$. $\vdash . (X +_g Y) +_g Z = X +_g (Y +_g Z)$

Dem.

$\vdash . *308\cdot6\cdot601\cdot62\cdot621 . \supset$

$$\vdash : X, Y, Z \in \text{Rat}_g . \supset . (X +_g Y) +_g Z = X +_g (Y +_g Z) \quad (1)$$

$\vdash . *308\cdot31\cdot33 . \supset$

$$\vdash : \sim(X, Y, Z \in \text{Rat}_g) . \supset . (X +_g Y) +_g Z = \dot{\Lambda} . X +_g (Y +_g Z) = \dot{\Lambda} \quad (2)$$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***308·71.** $\vdash : X \epsilon \mathrm{Rat}_g \,.\, Z <_g Z' \,.\, \supset \,.\, (X +_g Z) <_g (X +_g Z')$

Dem.

$\vdash \,.\, *308\cdot57 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, Z' +_g Z \,|\, \mathrm{Cnv} \,\epsilon\, \mathrm{Rat} - \iota\text{‘}0_q \,.$

$[*308\cdot56] \qquad \supset \,.\, (X +_g Z) <_g \{(X +_g Z) +_g (Z' +_g Z \,|\, \mathrm{Cnv})\} \,.$

$[*308\cdot63\cdot53] \qquad \supset \,.\, (X +_g Z) <_g (X +_g Z') : \supset \vdash \,.\, \mathrm{Prop}$

***308·72.** $\vdash : (X +_g Z) <_g (X +_g Z') \,.\, \equiv \,.\, X \epsilon \mathrm{Rat}_g \,.\, Z <_g Z'$

Dem.

$\vdash \,.\, *308\cdot33 \,.\, \supset \vdash : (X +_g Z) <_g (X +_g Z') \,.\, \supset \,.\, X, Z, Z' \epsilon \mathrm{Rat}_g \qquad (1)$

$\vdash \,.\, *308\cdot57 \,.\, \supset$

$\vdash : (X +_g Z) <_g (X +_g Z') \,.\, \supset \,.\, \{(X +_g Z') +_g (X +_g Z) \,|\, \mathrm{Cnv}\} \,\epsilon\, \mathrm{Rat} - \iota\text{‘}0_q \,.$

$[*308\cdot411\cdot63\cdot53] \qquad \supset \,.\, (Z' +_g Z \,|\, \mathrm{Cnv}) \,\epsilon\, \mathrm{Rat} - \iota\text{‘}0_q \qquad (2)$

$\vdash \,.\, (1) \,.\, (2) \,.\, *308\cdot57 \,.\, \supset \vdash : (X +_g Z) <_g (X +_g Z') \,.\, \supset \,.\, Z <_g Z' \qquad (3)$

$\vdash \,.\, (1) \,.\, (3) \,.\, *308\cdot71 \,.\, \supset \vdash \,.\, \mathrm{Prop}$

***308·8.** $\vdash : X, Y \epsilon \mathrm{Rat}_g \,.\, X \upharpoonright t_{11}\text{‘}\mu, Y \upharpoonright t_{11}\text{‘}\mu \,\epsilon\, C\text{‘}H_g \,.\, \supset \,.\, (X +_g Y) \upharpoonright t_{00}\text{‘}\mu \,\epsilon\, C\text{‘}H_g$
$[*308\cdot32\cdot321\cdot322\cdot323 \,.\, *306\cdot64 \,.\, *308\cdot26]$

***308·81.** $\vdash : X, Y \epsilon C\text{‘}H_g \,.\, \supset \,.\, (X +_g Y) \upharpoonright t_{00}\text{‘}C\text{‘‘}C\text{‘}X \,\epsilon\, C\text{‘}H_g \quad [*308\cdot8]$

$*309$. MULTIPLICATION OF GENERALIZED RATIOS.

Summary of $*309$.

The subject of this number is simpler than that of $*308$, because it requires nothing analogous to the consideration of subtraction. The product of two generalized ratios is defined as follows:

$*309\cdot01$. $X \times_g Y = (X \times_s Y) \cup (X \mid \text{Cnv} \times_s Y \mid \text{Cnv})$
$\cup (X \times_s Y \mid \text{Cnv}) \mid \text{Cnv} \cup (X \mid \text{Cnv} \times_s Y) \mid \text{Cnv}$ Df

As in $*308$, three of the four products concerned in this definition will be null in any given case (unless $X = 0_q$ or $Y = 0_q$). Hence

$*309\cdot14$. $\vdash : X, Y \in \text{Rat} . \supset . X \times_g Y = X \times_s Y$

$*309\cdot141$. $\vdash : X \in \text{Rat} . Y \in \text{Rat}_n . \supset . X \times_g Y = (X \times_s Y \mid \text{Cnv}) \mid \text{Cnv}$

$*309\cdot142$. $\vdash : Y \in \text{Rat} . X \in \text{Rat}_n . \supset . X \times_g Y = (X \mid \text{Cnv} \times_s Y) \mid \text{Cnv}$

$*309\cdot143$. $\vdash : X, Y \in \text{Rat}_n . \supset . X \times_g Y = X \mid \text{Cnv} \times_s Y \mid \text{Cnv}$

The propositions of this number are merely generalizations of those of $*305$. The proofs of the formal laws are straightforward, but the proof of the distributive law ($*309\cdot37$) is long, because of the multiplicity of different cases.

$*309\cdot01$. $X \times_g Y = (X \times_s Y) \cup (X \mid \text{Cnv} \times_s Y \mid \text{Cnv})$
$\cup (X \times_s Y \mid \text{Cnv}) \mid \text{Cnv} \cup (X \mid \text{Cnv} \times_s Y) \mid \text{Cnv}$ Df

$*309\cdot1$. $\vdash . X \times_g Y = (X \times_s Y) \cup (X \mid \text{Cnv} \times_s Y \mid \text{Cnv})$
$\cup (X \times_s Y \mid \text{Cnv}) \mid \text{Cnv} \cup (X \mid \text{Cnv} \times_s Y) \mid \text{Cnv}$ [($*309\cdot01$)]

$*309\cdot101$. $\vdash : X \in \text{Rat} - \iota{}^{\backprime}0_q . \supset . X \mid \text{Cnv} \times_s Y = \dot{\Lambda}$ [$*305\cdot2 . *307\cdot25$]

$*309\cdot102$. $\vdash : X \in \text{Rat}_n - \iota{}^{\backprime}0_q . \supset . X \times_s Y = \dot{\Lambda}$ [$*305\cdot2 . *307\cdot25$]

$*309\cdot11$. $\vdash : \dot{\exists} ! X \times_g Y . \supset . X, Y \in \text{Rat}_g$ [$*305\cdot2 . *309\cdot1$]

$*309\cdot12$. $\vdash . X \times_g Y = Y \times_g X$ [$*305\cdot11 . *309\cdot1$]

$*309\cdot121$. $\vdash . X \times_g Y = X \mid \text{Cnv} \times_g Y \mid \text{Cnv}$
$= (X \times_g Y \mid \text{Cnv}) \mid \text{Cnv} = (X \mid \text{Cnv} \times_g Y) \mid \text{Cnv}$ [$*309\cdot1 . *307\cdot12$]

***309·122.** $\vdash . X \times_g Y \,|\, \text{Cnv} = X \,|\, \text{Cnv} \times_g Y = (X \times_g Y) \,|\, \text{Cnv}$
[*309·121 . *307·12]

***309·13.** $\vdash : X, Y \in \text{Rat} - \iota`0_q . \supset . X \times_g Y = X \times_s Y$ [*309·1·101·12]

***309·131.** $\vdash :. X = 0_q . Y \in \text{Rat} - \iota`0_q . \mathbf{v} . Y = 0_q . X \in \text{Rat} - \iota`0_q : \supset .$
$X \times_g Y = X \times_s Y = 0_q$

Dem.

$\vdash . *309·101 . \supset$
$\vdash : X = 0_q . Y \in \text{Rat} - \iota`0_q . \supset . X \times_g Y = (X \times_s Y) \cup (X \,|\, \text{Cnv} \times_s Y) \,|\, \text{Cnv} .$
[*307·26.*305·22] $\supset . X \times_g Y = X \times_s Y = 0_q$ (1)
$\vdash . (1) . *309·12 . \supset \vdash : Y = 0_q . X \in \text{Rat} - \iota`0_q . \supset . X \times_g Y = X \times_s Y = 0_q$ (2)
$\vdash . (1) . (2) . \supset \vdash .$ Prop

***309·133.** $\vdash : X = 0_q . Y = 0_q . \supset . X \times_g Y = X \times_s Y = 0_q$
[*309·1 . *307·26 . *305·22]

***309·14.** $\vdash : X, Y \in \text{Rat} . \supset . X \times_g Y = X \times_s Y$ [*309·13·131·133]

***309·141.** $\vdash : X \in \text{Rat} . Y \in \text{Rat}_n . \supset . X \times_g Y = (X \times_s Y \,|\, \text{Cnv}) \,|\, \text{Cnv}$
[*309·121·14]

***309·142.** $\vdash : Y \in \text{Rat} . X \in \text{Rat}_n . \supset . X \times_g Y = (X \,|\, \text{Cnv} \times_s Y) \,|\, \text{Cnv}$
[*309·141·12]

***309·143.** $\vdash : X, Y \in \text{Rat}_n . \supset . X \times_g Y = X \,|\, \text{Cnv} \times_s Y \,|\, \text{Cnv}$ [*309·14·121]

***309·15.** $\vdash : X, Y \in \text{Rat}_g . \equiv . X \times_g Y \in \text{Rat}_g$

Dem.

$\vdash . *305·3 . *309·14·143 . \supset$
$\vdash :. X, Y \in \text{Rat} . \mathbf{v} . X, Y \in \text{Rat}_n : \supset . X \times_g Y \in \text{Rat}$ (1)
$\vdash . *305·3 . *309·141·142 . \supset$
$\vdash :. X \in \text{Rat} . Y \in \text{Rat}_n . \mathbf{v} . X \in \text{Rat}_n . Y \in \text{Rat} : \supset . X \times_g Y \in \text{Rat}_n$ (2)
$\vdash . (1) . (2) .$ $\supset \vdash : X, Y \in \text{Rat}_g . \supset . X \times_g Y \in \text{Rat}_g$ (3)
$\vdash . *303·72 . (*307·01·011) . \supset \vdash : X \times_g Y \in \text{Rat}_g . \supset . \exists ! X \times_g Y$ (4)
$\vdash . (4) . *309·11 .$ $\supset \vdash : X \times_g Y \in \text{Rat}_g . \supset . X, Y \in \text{Rat}_g$ (5)
$\vdash . (3) . (5) . \supset \vdash .$ Prop

***309·16.** $\vdash . (X \times_g Y) \times_g Z = X \times_g (Y \times_g Z)$ [*305·41 . *309·1]

***309·17.** $\vdash : X, Y \sim \in \iota`0_q \cup \iota`\infty_q . \supset . \breve{X} \times_g \breve{Y} = \text{Cnv}`(X \times_g Y)$

Dem.

$\vdash . *309·1 . \supset \vdash . \breve{X} \times_g \breve{Y} = (\breve{X} \times_s \breve{Y}) \cup (\breve{X} \,|\, \text{Cnv} \times_s \breve{Y} \,|\, \text{Cnv})$
$\cup (\breve{X} \times_s \breve{Y} \,|\, \text{Cnv}) \,|\, \text{Cnv} \cup (\breve{X} \,|\, \text{Cnv} \times_s \breve{Y}) \,|\, \text{Cnv}$ (1)

$\vdash . *305\cdot12 . \supset \vdash : \text{Hp} . \supset . \breve{X} \times_s \breve{Y} = \text{Cnv}`(X \times_s Y)$ (2)

$\vdash . *307\cdot22 . \supset \vdash : X \epsilon \text{Rat} . \supset . \breve{X} | \text{Cnv} = \text{Cnv}`(X | \text{Cnv})$ (3)

$\vdash . (3) . \quad \supset \vdash : Z \epsilon \text{Rat} . X = Z | \text{Cnv} . \supset . \breve{X} | \text{Cnv} = (\breve{Z} | \text{Cnv}) | \text{Cnv}$

[*307·12] $= \breve{Z}$

[*307·14] $= \text{Cnv}`(X | \text{Cnv})$ (4)

$\vdash . (3) . (4) . \supset \vdash : X \epsilon \text{Rat}_g . \supset . \breve{X} | \text{Cnv} = \text{Cnv}`(X | \text{Cnv})$ (5)

$\vdash . (2) . (5) . \supset \vdash : \text{Hp} . X, Y \epsilon \text{Rat}_g . \supset .$

$\breve{X} | \text{Cnv} \times_s \breve{Y} | \text{Cnv} = \text{Cnv}`(X | \text{Cnv} \times_s Y | \text{Cnv}) .$

$\breve{X} \times_s \breve{Y} | \text{Cnv} = \text{Cnv}`(X \times_s Y | \text{Cnv}) .$

$\breve{X} | \text{Cnv} \times_s \breve{Y} = \text{Cnv}`(X | \text{Cnv} \times_s Y)$ (6)

$\vdash . (1) . (2) . (6) . *309\cdot1 . \supset \vdash : \text{Hp} . X, Y \epsilon \text{Rat}_g . \supset . \breve{X} \times_g \breve{Y} = \text{Cnv}`(X \times_g Y)$ (7)

$\vdash . *303\cdot13\cdot7 . \quad \supset \vdash : X, Y \epsilon \text{Rat}_g - \iota`0_q . \equiv . \breve{X}, \breve{Y} \epsilon \text{Rat}_g - \iota`0_q$ (8)

$\vdash . (8) . *309\cdot11 . \supset$

$\vdash : \sim (X, Y \epsilon \text{Rat}_g \cup \iota`\infty_q) . \supset . \breve{X} \times_g \breve{Y} = \dot{\Lambda} . \text{Cnv}`(X \times_g Y) = \dot{\Lambda}$ (9)

$\vdash . (7) . (9) . \supset \vdash . \text{Prop}$

***309·21.** $\vdash :. X, Y \epsilon \text{Rat}_g : X = 0_q . \vee . Y = 0_q : \equiv . X \times_g Y = 0_q$

Dem.

$\vdash . *309\cdot14\cdot141 . *305\cdot22 . *307\cdot26 . \supset \vdash : X \epsilon \text{Rat}_g . Y = 0_q . \supset . X \times_g Y = 0_q$ (1)

$\vdash . *309\cdot15 . \quad \supset \vdash : X \times_g Y = 0_q . \supset . X, Y \epsilon \text{Rat}_g$ (2)

$\vdash . (2) . *309\cdot14\cdot141\cdot142\cdot143 . *307\cdot26 . \supset$

$\vdash :. X \times_g Y = 0_q . \supset : X \times_s Y = 0_q . \vee . X | \text{Cnv} \times_s Y | \text{Cnv} = 0_q .$

$\vee . X \times_s Y | \text{Cnv} = 0_q . \vee . X | \text{Cnv} \times_s Y = 0_q :$

[*305·22.*307·26] $\supset : X = 0_q . \vee . Y = 0_q$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash . \text{Prop}$

***309·22.** $\vdash : X, Y \epsilon \text{Rat}_g - \iota`0_q . \equiv . X \times_g Y \epsilon \text{Rat}_g - \iota`0_q$ [*309·21 . Transp]

***309·23.** $\vdash : X \epsilon \text{Rat}_g - \iota`0_q . \supset . X \times_g \breve{X} = 1/1$

Dem.

$\vdash . *309\cdot13 . \quad \supset \vdash : X \epsilon \text{Rat} - \iota`0_q . \supset . X \times_g \breve{X} = X \times_s \breve{X}$

[*305·52] $= 1/1$ (1)

$\vdash . *309\cdot121 . *307\cdot22 . \supset \vdash : Y \epsilon \text{Rat} - \iota`0_q . X = Y | \text{Cnv} . \supset . X \times_g \breve{X} = Y \times_g \breve{Y}$

[(1)] $= 1/1$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***309·24.** $\vdash : X \epsilon \text{Rat}_g . \supset . X \times_g 1/1 = X$

Dem.

$\vdash . *309\cdot14 . \quad \supset \vdash : X \epsilon \text{Rat} . \supset . X \times_g 1/1 = X \times_s 1/1$

[*305·51] $= X$ (1)

$\vdash . (1) . *309\cdot142 . \supset \vdash : X \epsilon \text{Rat}_n . \supset . X \times_g 1/1 = (X | \text{Cnv}) | \text{Cnv}$

[*307·12] $= X$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***309·25.** $\vdash :. X, A \,\epsilon\, \mathrm{Rat}_g \,.\, A \neq 0_q \,.\, \supset : X \times_g A = A' \,.\, \equiv \,.\, X = A' \times_g \breve{A}$

Dem.

$\vdash . *309{\cdot}23{\cdot}24{\cdot}16 \,.\quad \supset \vdash : \mathrm{Hp} \,.\, \supset .\, X = X \times_g A \times_g \breve{A}$ (1)

$\vdash . (1) \,.\quad \supset \vdash : \mathrm{Hp} \,.\, X \times_g A = A' \,.\, \supset .\, X = A' \times_g \breve{A}$ (2)

$\vdash . (1) \dfrac{A', \breve{A}}{X, A} . *309{\cdot}15 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset .\, A' = A' \times_g \breve{A} \times_g A$ (3)

$\vdash . (3) \,.\quad \supset \vdash : \mathrm{Hp} \,.\, X = A' \times_g \breve{A} \,.\, \supset .\, X \times_g A = A'$ (4)

$\vdash . (2) . (4) . \supset \vdash . \mathrm{Prop}$

***309·251.** $\vdash :. X, A' \,\epsilon\, \mathrm{Rat}_g \,.\, A \neq 0_q \,.\, \supset : X \times_g A = A' \,.\, \equiv \,.\, X = A' \times_g \breve{A}$
[*309·25·15]

***309·26.** $\vdash : X, Y \,\epsilon\, \mathrm{Rat}_g \,.\, X \neq 0_q \,.\, \supset .\, (\exists Z) \,.\, Z \,\epsilon\, \mathrm{Rat}_g \,.\, X \times_g Z = Y$

Dem.

$\vdash . *309{\cdot}25 \,.\, \supset \vdash : \mathrm{Hp} \,.\, Z = Y \times_g \breve{X} \,.\, \supset .\, Z \times_g X = Y$ (1)

$\vdash . (1) . *309{\cdot}15{\cdot}12 \,.\, \supset \vdash . \mathrm{Prop}$

***309·31** $\vdash : X, Y \,\epsilon\, \mathrm{Rat} \,.\, Z \,\epsilon\, \mathrm{Rat}_g \,.\, \supset .\, (X +_g Y) \times_g Z = (X \times_g Z) +_g (Y \times_g Z)$

Dem.

$\vdash . *308{\cdot}32 . *309{\cdot}14 . \supset$

$\vdash : \mathrm{Hp} \,.\, Z \,\epsilon\, \mathrm{Rat} \,.\, \supset .\, (X +_g Y) \times_g Z = (X +_s Y) \times_s Z \,.$
$X \times_g Z = X \times_s Z \,.\, Y \times_g Z = Y \times_s Z \,.$

[*306·41] $\supset .\, (X +_g Y) \times_g Z = (X \times_g Z) +_g (Y \times_g Z)$ (1)

$\vdash . *309{\cdot}122 . \supset$

$\vdash : \mathrm{Hp} \,.\, W \,\epsilon\, \mathrm{Rat} \,.\, Z = W \,|\, \mathrm{Cnv} \,.\, \supset .\, (X +_g Y) \times_g Z = \{(X +_g Y) \times_g W\} \,|\, \mathrm{Cnv}$

[(1)] $= \{(X \times_g W) +_g (Y \times_g W)\} \,|\, \mathrm{Cnv}$

[*308·411.*309·122] $= (X \times_g Z) +_g (Y \times_g Z)$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***309·311.** $\vdash : X, Y \,\epsilon\, \mathrm{Rat}_n \,.\, Z \,\epsilon\, \mathrm{Rat}_g \,.\, \supset .\, (X +_g Y) \times_g Z = (X \times_g Z) +_g (Y \times_g Z)$

Dem.

$\vdash . *308{\cdot}41 . *309{\cdot}122 . \supset$

$\vdash : \mathrm{Hp} \,.\, \supset .\, (X +_g Y) \times_g Z = \{(X \,|\, \mathrm{Cnv} +_g Y \,|\, \mathrm{Cnv}) \times_g Z\} \,|\, \mathrm{Cnv}$

[*309·31] $= \{(X \,|\, \mathrm{Cnv} \times_g Z) +_g (Y \,|\, \mathrm{Cnv} \times_g Z)\} \,|\, \mathrm{Cnv}$

[*309·122.*308·41] $= (X \times_g Z) +_g (Y \times_g Z) : \supset \vdash . \mathrm{Prop}$

***309·32.** $\vdash : (\nu/\rho) <_r (\lambda/\mu) \,.\, \sigma/\tau \,\epsilon\, \mathrm{Rat} \,.\, \supset .$
$(\lambda/\mu -_s \nu/\rho) \times_g \sigma/\tau = \{((\lambda \times_c \rho) -_c (\mu \times_c \nu)) \times_c \sigma\}/(\mu \times_c \rho \times_c \tau)$

Dem.

$\vdash . *308{\cdot}24 . \supset \vdash : \mathrm{Hp} \,.\, \supset .\, \lambda/\mu -_s \nu/\rho = ((\lambda \times_c \rho) -_c (\mu \times_c \nu))/\mu \times_c \rho$ (1)

$\vdash . (1) . *309{\cdot}14 . *305{\cdot}142 . \supset \vdash . \mathrm{Prop}$

***309·33.** $\vdash : \lambda/\mu, \nu/\rho, \sigma/\tau \in \mathrm{Rat} . \supset .$

$$(\lambda/\mu -_s \nu/\rho) \times_g (\sigma/\tau) = (\lambda/\mu \times_g \sigma/\tau) -_s (\nu/\rho \times_g \sigma/\tau)$$

Dem.

$\vdash . *309{\cdot}14 . \supset \vdash : \mathrm{Hp} . \supset . \lambda/\mu \times_g \sigma/\tau = \lambda/\mu \times_s \sigma/\tau . \nu/\rho \times_g \sigma/\tau = \nu/\rho \times_s \sigma/\tau .$

$[*305{\cdot}142] \supset . \lambda/\mu \times_g \sigma/\tau = (\lambda \times_c \sigma)/(\mu \times_c \tau) . \nu/\rho \times_g \sigma/\tau = (\nu \times_c \sigma)/(\rho \times_c \tau)$ (1)

$\vdash . (1) . *308{\cdot}24 . \supset$

$\vdash : \mathrm{Hp} . (\nu/\rho) <_r (\lambda/\mu) . \supset . (\lambda/\mu \times_g \sigma/\tau) -_s (\nu/\rho \times_g \sigma/\tau) =$

$$\{(\lambda \times_c \sigma) \times_c (\rho \times_c \tau) -_c (\mu \times_c \tau) \times_c (\nu \times_c \sigma)\}/(\mu \times_c \rho \times_c \tau^2)$$

$[*303{\cdot}38] \quad = \{(\lambda \times_c \sigma \times_c \rho) - (\mu \times_c \nu \times_c \sigma)\}/(\mu \times_c \rho \times_c \tau)$

$[*309{\cdot}32] \quad = (\lambda/\mu -_s \nu/\rho) \times_g \sigma/\tau$ (2)

$\vdash . (2) . \supset \vdash : \mathrm{Hp} . (\lambda/\mu) <_r (\nu/\rho) . \supset .$

$$(\nu/\rho \times_g \sigma/\tau) -_s (\lambda/\mu \times_g \sigma/\tau) = (\nu/\rho -_s \lambda/\mu) \times_g \sigma/\tau .$$

$[*308{\cdot}21 . *309{\cdot}122] \quad \supset . (\lambda/\mu \times_g \sigma/\tau) -_s (\nu/\rho \times_g \sigma/\tau) = (\lambda/\mu -_s \nu/\rho) \times_g \sigma/\tau$ (3)

$\vdash . *308{\cdot}12 . *309{\cdot}21 . \supset$

$\vdash : \mathrm{Hp} . \lambda/\mu = \nu/\rho . \supset . (\lambda/\mu -_s \nu/\rho) \times_g \sigma/\tau = 0_q .$

$$(\lambda/\mu \times_g \sigma/\tau) -_s (\nu/\rho \times_g \sigma/\tau) = 0_q$$ (4)

$\vdash . (2) . (3) . (4) . \supset \vdash . \mathrm{Prop}$

***309·34.** $\vdash : X, Y, Z \in \mathrm{Rat} . \supset . (X -_s Y) \times_g Z = (X \times_g Z) -_s (Y \times_g Z)$
[*309·33]

***309·35.** $\vdash : X, Z \in \mathrm{Rat} . Y \in \mathrm{Rat}_n . \supset . (X +_g Y) \times_g Z = (X \times_g Z) +_g (Y \times_g Z)$

Dem.

$\vdash . *308{\cdot}321 . \supset \vdash : \mathrm{Hp} . \supset . X +_g Y = X -_s Y \,|\, \mathrm{Cnv} .$

$$(X \times_g Z) +_g (Y \times_g Z) = (X \times_g Z) -_s (Y \,|\, \mathrm{Cnv} \times_g Z)$$ (1)

$\vdash . (1) . *309{\cdot}34 . \supset \vdash . \mathrm{Prop}$

***309·36.** $\vdash : X, Z \in \mathrm{Rat}_n . Y \in \mathrm{Rat} . \supset . (X +_g Y) \times_g Z = (X \times_g Z) +_g (Y \times_g Z)$

Dem.

$\vdash . *308{\cdot}41 . *309{\cdot}121 . \supset$

$\vdash : \mathrm{Hp} . \supset . X +_g Y = (X \,|\, \mathrm{Cnv} +_g Y \,|\, \mathrm{Cnv}) \,|\, \mathrm{Cnv} . X \times_g Z = X \,|\, \mathrm{Cnv} \times_g Z \,|\, \mathrm{Cnv} .$

$$Y \times_g Z = Y \,|\, \mathrm{Cnv} \times_g Z \,|\, \mathrm{Cnv} .$$

$[*309{\cdot}122] \supset . (X +_g Y) \times_g Z = (X \,|\, \mathrm{Cnv} +_g Y \,|\, \mathrm{Cnv}) \times_g Z \,|\, \mathrm{Cnv} .$

$$(X \times_g Z) +_g (Y \times_g Z) = (X \,|\, \mathrm{Cnv} \times_g Z \,|\, \mathrm{Cnv}) +_g (Y \,|\, \mathrm{Cnv} \times_g Z \,|\, \mathrm{Cnv})$$ (1)

$\vdash . (1) . *309{\cdot}35 . \supset \vdash . \mathrm{Prop}$

***309·361.** $\vdash : X \in \mathrm{Rat}_g . Y \in \mathrm{Rat}_n . Z \in \mathrm{Rat} . \supset .$

$$(X +_g Y) \times_g Z = (X \times_g Z) +_g (Y \times_g Z)$$ [*309·311·36]

***309·362.** $\vdash : X, Z \,\epsilon\, \mathrm{Rat}_g \,.\, Y \,\epsilon\, \mathrm{Rat}_n \,.\, \supset . \, (X +_g Y) \times_g Z = (X \times_g Z) +_g (Y \times_g Z)$

Dem.

$\vdash . \, *309{\cdot}122 \,.\, *308{\cdot}41 \,.\, \supset$

$\vdash . \, (X +_g Y) \times_g Z = \{(X +_g Y) \times_g Z \,|\, \mathrm{Cnv}\} \,|\, \mathrm{Cnv} \,.$

$(X \times_g Z) +_g (Y \times_g Z) = \{(X \times_g Z \,|\, \mathrm{Cnv}) +_g (Y \times_g Z \,|\, \mathrm{Cnv})\} \,|\, \mathrm{Cnv}$ (1)

$\vdash . \, *309{\cdot}361 \,.\, \supset$

$\vdash : \mathrm{Hp} \,.\, Z \,\epsilon\, \mathrm{Rat}_n \,.\, \supset . \, (X +_g Y) \times_g Z \,|\, \mathrm{Cnv}$

$= (X \times_g Z \,|\, \mathrm{Cnv}) +_g (Y \times_g Z \,|\, \mathrm{Cnv})$ (2)

$\vdash . \, (1) \,.\, (2) \,.\, \supset \vdash : \mathrm{Hp} \,.\, Z \,\epsilon\, \mathrm{Rat}_n \,.\, \supset . \, (X +_g Y) \times_g Z = (X \times_g Z) +_g (Y \times_g Z)$ (3)

$\vdash . \, (3) \,.\, *309{\cdot}361 \,.\, \supset \vdash . \, \mathrm{Prop}$

***309·363.** $\vdash : X, Y, Z \,\epsilon\, \mathrm{Rat}_g \,.\, \supset . \, (X +_g Y) \times_g Z = (X \times_g Z) +_g (Y \times_g Z)$

Dem.

$\vdash . \, *309{\cdot}35{\cdot}12 \,.\, *308{\cdot}4 \,.\, \supset$

$\vdash : Y, Z \,\epsilon\, \mathrm{Rat} \,.\, X \,\epsilon\, \mathrm{Rat}_n \,.\, \supset . \, (X +_g Y) \times_g Z = (X \times_g Z) +_g (Y \times_g Z)$ (1)

$\vdash . \, *309{\cdot}36 \,.\, \supset$

$\vdash : Y \,\epsilon\, \mathrm{Rat} \,.\, X, Z \,\epsilon\, \mathrm{Rat}_n \,.\, \supset . \, (X +_g Y) \times_g Z = (X \times_g Z) +_g (Y \times_g Z)$ (2)

$\vdash . \, (1) \,.\, (2) \,.\, \supset$

$\vdash : X \,\epsilon\, \mathrm{Rat}_n \,.\, Y \,\epsilon\, \mathrm{Rat} \,.\, Z \,\epsilon\, \mathrm{Rat}_g \,.\, \supset . \, (X +_g Y) \times_g Z = (X \times_g Z) +_g (Y \times_g Z)$ (3)

$\vdash . \, (3) \,.\, *309{\cdot}31 \,.\, \supset$

$\vdash : X \,\epsilon\, \mathrm{Rat}_g \,.\, Y \,\epsilon\, \mathrm{Rat} \,.\, Z \,\epsilon\, \mathrm{Rat}_g \,.\, \supset . \, (X +_g Y) \times_g Z = (X \times_g Z) +_g (Y \times_g Z)$ (4)

$\vdash . \, (4) \,.\, *309{\cdot}362 \,.\, \supset \vdash . \, \mathrm{Prop}$

***309·37.** $\vdash . \, (X +_g Y) \times_g Z = (X \times_g Z) +_g (Y \times_g Z)$
[$*309{\cdot}363{\cdot}11{\cdot}15 \,.\, *308{\cdot}31{\cdot}33$]

***309·41.** $\vdash :. \, A \,\epsilon\, \mathrm{Rat} - \iota`0_q \,.\, \supset : (A \times_g X) <_g Y \,.\, \equiv . \, X <_g (Y \times_g \breve{A})$

Dem.

$\vdash . \, *308{\cdot}56 \,.\, \supset \vdash :. \, (A \times_g X) <_g Y \,.\, \equiv :$

$A \times_g X \,\epsilon\, \mathrm{Rat}_g : (\exists Z) \,.\, Z \,\epsilon\, \mathrm{Rat} - \iota`0_q \,.\, (A \times_g X) +_g Z = Y$ (1)

$\vdash . \, (1) \,.\, *309{\cdot}15 \,.\, \supset \vdash :: \mathrm{Hp} \,.\, \supset :. \, (A \times_g X) <_g Y \,.\, \equiv :$

$X \,\epsilon\, \mathrm{Rat}_g : (\exists Z) \,.\, Z \,\epsilon\, \mathrm{Rat} - \iota`0_q \,.\, (A \times_g X) +_g Z = Y :$

[$*309{\cdot}25{\cdot}37{\cdot}23{\cdot}24$] $\supset : X \,\epsilon\, \mathrm{Rat}_g : (\exists Z) \,.\, Z \,\epsilon\, \mathrm{Rat} - \iota`0_q \,.\, X +_g (Z \times_g \breve{A}) = Y \times_g \breve{A} :$

[$*305{\cdot}31 . *309{\cdot}13$] $\supset : X \,\epsilon\, \mathrm{Rat}_g : (\exists Z') \,.\, Z' \,\epsilon\, \mathrm{Rat} - \iota`0_q \,.\, X +_g Z' = Y \times_g \breve{A} :$

[$*308{\cdot}56$] $\supset : X <_g (Y \times_g \breve{A})$ (2)

Similarly $\vdash :. \, \mathrm{Hp} \,.\, \supset : X <_g (Y \times_g \breve{A}) \,.\, \supset . \, (A \times_g X) <_g Y$ (3)

$\vdash . \, (2) \,.\, (3) \,.\, \supset \vdash . \, \mathrm{Prop}$

***309·42.** $\vdash :. A \epsilon \mathrm{Rat}_n - \iota\text{‘}0_q . \supset : (A \times_g X) <_g Y . \equiv . (Y \times_g \breve{A}) <_g X$

Dem.

$\vdash . *307·4 . *309·122 . \supset$

$\vdash :. \mathrm{Hp} . \supset : (A \times_g X) <_g Y . \equiv . (Y \,|\, \mathrm{Cnv}) <_g (A \,|\, \mathrm{Cnv} \times_g X) .$

[*309·41.*307·22] $\equiv . (Y \,|\, \mathrm{Cnv} \times_g \breve{A} \,|\, \mathrm{Cnv}) <_g X .$

[*309·121] $\equiv . (Y \times_g \breve{A}) <_g X :. \supset \vdash . \mathrm{Prop}$

***309·5.** $\vdash : X, Y \epsilon \mathrm{Rat}_g . X \upharpoonright t_{11}\text{‘}\mu, Y \upharpoonright t_{11}\text{‘}\mu \epsilon C\text{‘}H_g . \supset . (X \times_g Y) \upharpoonright t_{00}\text{‘}\mu \epsilon C\text{‘}H_g$
[*309·14·141·142·143 . *305·26]

***309·51.** $\vdash : X, Y \epsilon C\text{‘}H_g . \supset . (X \times_g Y) \upharpoonright t_{00}\text{‘}C\text{‘‘}C\text{‘}X \epsilon C\text{‘}H_g$ [*309·5]

$*310$. THE SERIES OF REAL NUMBERS.

Summary of $*310$.

Real numbers, as opposed to ratios, are required primarily in order to obtain a Dedekindian series, so as to secure limits to sets of rationals having no *rational* limit. If rationals and irrationals are to form one series, it is necessary to give some definition of "rationals" other than "ratios," since the series of ratios (assuming the axiom of infinity) is not Dedekindian, and is not part of any arithmetically definable Dedekindian series. But in virtue of the propositions of $*212$, the series of segments of the series of ratios, *i.e.* the series $\varsigma`H$, is Dedekindian, and this series contains a series, namely $\overrightarrow{H};H$, which is ordinally similar to H. Thus the properties which we desire real numbers to have will result if we identify them* with segments of H, and give the name "rational real numbers" to segments of the form $\overrightarrow{H}`X$, *i.e.* to segments which have ratios as limits. Thus $\overrightarrow{H}`X$ is the rational real number corresponding to the ratio X, and a real number in general is of the form $H``\lambda$, where $\lambda$ is a class of ratios. $H``\lambda$ will be *irrational* when λ has no limit or maximum in H.

Since real numbers involve *classes* of ratios, the ratios concerned must be of some one type, and cannot be typically indefinite. Thus, as might be expected, hardly any of the properties of real numbers can be proved without assuming the axiom of infinity. In the present number, however, we shall be mainly concerned with just those few simple properties which are independent of the axiom of infinity.

The series $\varsigma`H$, by which real numbers are to be defined, has both a beginning and an end, namely $\Lambda$ and $\mathrm{D}`H$ (which $= C`H$ if the axiom of infinity holds). $\mathrm{D}`H$ will be infinity among real numbers. It is not convenient to include it in the series of real numbers as defined, just as it was not convenient to include ∞_q in the series H or H'. Again Λ is not naturally to be taken as the zero of real numbers, which should rather be taken as being $\iota`0_q$. Thus we are led to the two following definitions, in which Θ is the series of positive real numbers other than zero and infinity,

* On this definition of real numbers, cf. *Principles of Mathematics*, Chap. XXXIII.

while Θ' is the series of zero and the positive real numbers other than infinity:

***310·01.** $\Theta = (\varsigma\text{‘}H) \upharpoonright (-\iota\text{‘}\Lambda - \iota\text{‘}\mathrm{D}\text{‘}H)$ Df

***310·011.** $\Theta' = \iota\text{‘}0_q \mathrel{\leftarrow\!\!\!\!+} \Theta$ Df

These notations are framed on the analogy of H and H', the letter Θ being chosen to suggest θ, the relation-number of the continuum. Although we do not have $\mathrm{Nr}\text{‘}\Theta = \theta$, we have $\mathrm{Nr}\text{‘}\varsigma\text{‘}H = \theta$, and therefore (*310·15) $\dot{1} \dotplus \mathrm{Nr}\text{‘}\Theta \dotplus \dot{1} = \theta$, and $\mathrm{Nr}\text{‘}\Theta' \dotplus \dot{1} = \theta$ (assuming the axiom of infinity). Thus the relation-number of Θ is simply that of a θ with the ends cut off.

We put further, on the analogy of H_n, H_g,

***310·02.** $\Theta_n = (\varsigma\text{‘}H_n) \upharpoonright (-\iota\text{‘}\Lambda - \iota\text{‘}\mathrm{D}\text{‘}H_n)$ Df

***310·021.** $\Theta'_n = \iota\text{‘}0_q \mathrel{\leftarrow\!\!\!\!+} \Theta_n$ Df

***310·03.** $\Theta_g = \breve{\Theta}_n \mathbin{\ddagger} \Theta'$ Df

Thus Θ_n is the series of negative real numbers, Θ'_n the series of zero and the negative real numbers, Θ_g the series of negative and positive real numbers including zero (infinity always excluded). The *class* of positive real numbers is $C\text{‘}\Theta$, of negative real numbers $C\text{‘}\Theta_n$, of all real numbers (excluding infinity) $C\text{‘}\Theta \cup \iota\text{‘}\iota\text{‘}0_q \cup C\text{‘}\Theta_n$. If ν is a positive real number, $|\mathrm{Cnv}\text{‘‘}\nu$ is the corresponding negative real number (*310·16). The properties of Θ, Θ_n, Θ_g in respect of limits, continuity, etc., result from the properties of θ as proved in *275, and from the properties of series of segments as proved in *212.

Instead of taking the series of segments as constituting the real numbers, it is possible to take the series of their relational sums, *i.e.* $\dot{s};\Theta$. This depends on the fact that $\dot{s};\Theta \mathrel{\mathrm{smor}} \Theta$ (*310·33). The chief advantage of $\dot{s};\Theta$ is that it is of the same type as the series of ratios. We shall show in *314 how to construct the arithmetic of real numbers defined as the relational sums of segments; until then, we shall regard real numbers as segments of the series of ratios.

***310·01.** $\Theta = (\varsigma\text{‘}H) \upharpoonright (-\iota\text{‘}\Lambda - \iota\text{‘}\mathrm{D}\text{‘}H)$ Df

***310·011.** $\Theta' = \iota\text{‘}0_q \mathrel{\leftarrow\!\!\!\!+} \Theta$ Df

***310·02.** $\Theta_n = (\varsigma\text{‘}H_n) \upharpoonright (-\iota\text{‘}\Lambda - \iota\text{‘}\mathrm{D}\text{‘}H_n)$ Df

***310·021.** $\Theta'_n = \iota\text{‘}0_q \mathrel{\leftarrow\!\!\!\!+} \Theta_n$ Df

***310·03.** $\Theta_g = \breve{\Theta}_n \mathbin{\ddagger} \Theta'$ Df

***310·1.** $\vdash . \Theta, \Theta', \Theta_n, \Theta'_n, \Theta_g \in \mathrm{Ser}$ [*304·23 . *307·41·25 . *204·5 . *212·31]

***310·11.** $\vdash : \mu \Theta \nu . \equiv . \mu, \nu \in \mathrm{D}\text{‘}H_\epsilon - \iota\text{‘}\Lambda - \iota\text{‘}\mathrm{D}\text{‘}H . \mu \subset \nu . \mu \neq \nu .$

$\equiv . \mu, \nu \in \mathrm{D}\text{‘}H_\epsilon . \exists ! \mu . \exists ! \mathrm{D}\text{‘}H - \nu . \exists ! \nu - \mu .$

$\equiv . \mu, \nu \in \mathrm{D}\text{‘}\varsigma\text{‘}H \cap \mathrm{G}\text{‘}\varsigma\text{‘}H . \mu \subset \nu . \mu \neq \nu$

[*212·23·132 . *211·61 . (*310·01)]

***310·111.** $\vdash : \mu \Theta_n \nu . \equiv . \mu, \nu \in \text{D}`(H_n)_\epsilon - \iota`\Lambda - \iota`\text{D}`H_n . \mu \subset \nu . \mu \neq \nu .$
$\equiv . \mu, \nu \in \text{D}`(H_n)_\epsilon . \exists ! \mu . \exists ! \text{D}`H_n - \nu . \exists ! \nu - \mu .$
$\equiv . \mu, \nu \in \text{D}`\mathfrak{s}`H_n \cap \text{Q}`\mathfrak{s}`H_n . \mu \subset \nu . \mu \neq \nu$ [(*310·02)]

***310·112.** $\vdash :. \mu \Theta_g \nu . \equiv : \mu \breve{\Theta}_n \nu . \vee . \mu \Theta \nu . \vee .$
$\mu \in C`\Theta_n . \nu \in \iota`\iota`0_q \cup C`\Theta . \vee . \mu = \iota`0_q . \nu \in C`\Theta$ [(*310·03)]

***310·113.** $\vdash :. \mu \Theta' \nu . \equiv : \mu = \iota`0_q . \nu \in C`\Theta . \vee . \mu \Theta \nu$ [(*310·011)]

***310·114.** $\vdash :. \mu \Theta'_n \nu . \equiv : \mu = \iota`0_q . \nu \in C`\Theta_n . \vee . \mu \Theta_n \nu$ [(*310·021)]

***310·12.** $\vdash . C`\Theta = \text{D}`\mathfrak{s}`H \cap \text{Q}`\mathfrak{s}`H = \text{D}`H_\epsilon - \iota`\Lambda - \iota`\text{D}`H .$
$C`\Theta_n = \text{D}`\mathfrak{s}`H_n \cap \text{Q}`\mathfrak{s}`H_n = \text{D}`(H_n)_\epsilon - \iota`\Lambda - \iota`\text{D}`H_n$ [*212·132]

***310·121.** $\vdash . C`\Theta \subset \text{Cl ex}`\text{D}`H . C`\Theta_n \subset \text{Cl ex}`\text{D}`H_n$ [*310·12]

***310·122.** $\vdash : \exists ! 3 . \equiv . \dot{\exists} ! \Theta . \equiv . \dot{\exists} ! \Theta' . \equiv . \dot{\exists} ! \Theta_n . \equiv . \dot{\exists} ! \Theta'_n . \equiv . \dot{\exists} ! \Theta_g$
[*212·14 . *161·13 . *304·27]

***310·123.** $\vdash : \exists ! 3 . \supset . C`\Theta' = \iota`\iota`0_q \cup C`\Theta . C`\Theta'_n = \iota`\iota`0_q \cup C`\Theta_n .$
$C`\Theta_g = C`\Theta_n \cup \iota`\iota`0_q \cup C`\Theta$ [*310·122 . *161·14]

***310·13.** $\vdash . C`\Theta \cap C`\Theta_n = \Lambda . s`C`\Theta \cap s`C`\Theta_n = \Lambda$

Dem.

$\vdash . *310{\cdot}11{\cdot}111 . \supset \vdash : \mu \in C`\Theta . \nu \in C`\Theta_n . \supset . \mu \subset \text{D}`H . \nu \subset \text{D}`H_n . \exists ! \mu . \exists ! \nu .$
[*307·25] $\supset . \mu \neq \nu . \mu \cap \nu = \Lambda : \supset \vdash . \text{Prop}$

***310·131.** $\vdash . \iota`0_q \sim\in C`\Theta \cup C`\Theta_n$ [*304·282]

***310·14.** $\vdash . \Theta_n \text{ smor } \Theta$ [*212·72 . *307·41]

***310·15.** $\vdash : \text{Infin ax} . \supset . \Theta' \dashv\!\!\rightarrow C`H, \Theta'_n \dashv\!\!\rightarrow C`H_n, C`H_n \leftarrow\!\!\vdash \Theta_g \dashv\!\!\rightarrow C`H \in \theta$
[*304·33 . *310·14 . *275·21]

***310·151.** $\vdash : \text{Infin ax} . \supset . \Theta', \Theta'_n \in \text{Ser} \cap \text{comp} \cap \text{semi Ded}$
[*310·15 . *275·1 . *271·18 . *214·74]

***310·16.** $\vdash : \nu \in C`\Theta . \equiv . |\text{Cnv}``\nu \in C`\Theta_n$ [*310·12 . (*307·04)]

***310·17.** $\vdash . |\text{Cnv}``|\text{Cnv}``\nu = \nu$ [*307·12]

***310·18.** $\vdash : \mu = |\text{Cnv}``\nu . \equiv . \nu = |\text{Cnv}``\mu$ [*310·17]

***310·19.** $\vdash : \mu = \nu . \equiv . |\text{Cnv}``\mu = |\text{Cnv}``\nu$ [*310·17]

***310·31.** $\vdash : \mu \in C`\Theta \cup C`\Theta_n . \supset . \dot{\exists} ! (\dot{s}`\mu) \upharpoonright \text{Rel num}$ [*304·5 . *310·121]

***310·32.** $\vdash :. \mu, \nu \in C'\Theta_g . \supset : \dot{s}'\mu = \dot{s}'\nu . \equiv . \mu = \nu$

Dem.

$\vdash . *310·31 . *303·62 . \supset$

$\vdash : \mu \in C'\Theta \cup C'\Theta_n . \nu = \iota'0_q . \supset . \dot{\exists} ! (\dot{s}'\mu) \upharpoonright \text{Rel num} . \sim \dot{\exists} ! (\dot{s}'\nu) \upharpoonright \text{Rel num} .$

$\supset . \dot{s}'\mu \neq \dot{s}'\nu$ (1)

$\vdash . *310·12·31 . *307·25 . \supset \vdash : \mu \in C'\Theta . \nu \in C'\Theta_n . \supset . \dot{s}'\mu \neq \dot{s}'\nu$ (2)

$\vdash . *310·11 . \supset \vdash :. \mu \Theta \nu . \supset : \exists ! \nu - \mu :$

[*310·121] $\supset : (\exists \rho, \sigma) : \rho/\sigma \in \nu : \xi/\eta \in \mu . \supset_{\xi, \eta} . \xi/\eta \neq \rho/\sigma :$

[*303·52] $\supset : (\exists \rho, \sigma, R, S) : \rho/\sigma \in \nu . R (\rho/\sigma) S : \xi/\eta \in \mu . \supset_{\xi, \eta} . \sim \{R (\xi/\eta) S\} :$

[*41·11] $\supset : \dot{\exists} ! \dot{s}'\nu \dot{-} \dot{s}'\mu$ (3)

$\vdash . (3) . *310·1 . \supset \vdash : \mu, \nu \in C'\Theta . \mu \neq \nu . \supset . \dot{s}'\mu \neq \dot{s}'\nu$ (4)

Similarly $\vdash : \mu, \nu \in C'\Theta_n . \mu \neq \nu . \supset . \dot{s}'\mu \neq \dot{s}'\nu$ (5)

$\vdash . (1) . (2) . (4) . (5) . \supset \vdash :. \text{Hp} . \supset : \mu \neq \nu . \supset . \dot{s}'\mu \neq \dot{s}'\nu$ (6)

$\vdash . (6) . \text{Transp} . \supset \vdash . \text{Prop}$

***310·33.** $\vdash . \dot{s};\Theta \text{ smor } \Theta . \dot{s};\Theta_n \text{ smor } \Theta_n . \dot{s};\Theta_g \text{ smor } \Theta_g$ [*310·32]

$*311$. ADDITION OF CONCORDANT REAL NUMBERS.

Summary of $*311$.

We define a set of real numbers as *concordant* when all are positive or zero, or all are negative or zero, *i.e.* when all belong to $C'\Theta'$ or all belong to $C'\Theta'_n$. Given two concordant real numbers μ and ν, we define the sum of μ and ν as the class of sums, in the sense of $*308$, of a member of μ and a member of ν, *i.e.* as

$$\hat{W}\{(\exists M, N) \,.\, M \epsilon \mu \,.\, N \epsilon \nu \,.\, W = M +_g N\},$$

i.e. as $s'\mu \underset{,,}{+_g}{}''\nu$, in virtue of $*40\cdot7$. It is easy to prove that, assuming the axiom of infinity, the sum so defined has the properties we require of a sum. We denote the sum so defined by "$\mu +_p \nu$." In order to insure that $\mu +_p \nu$ shall be Λ unless μ, ν are concordant real numbers, we put

$*311\cdot02$. $\mu +_p \nu = \hat{X}\{\text{concord}(\mu, \nu) \,.\, X \epsilon s'\mu \underset{,,}{+_g}{}''\nu\}$ Df

Thus if μ, ν are concordant real numbers, $\mu +_p \nu = s'\mu \underset{,,}{+_g}{}''\nu$ ($*311\cdot11$); if not, $\mu +_p \nu = \Lambda$ ($*311\cdot1$). A definition of addition which applies to real numbers of opposite sign will be given in $*312$.

The commutative and associative laws for $+_p$ ($*311\cdot12\cdot121$) follow at once from the corresponding laws for $+_g$. Assuming the axiom of infinity, we prove without much difficulty that the sum of two positive real numbers is a positive real number ($*311\cdot27$), and the sum of two negative real numbers is a negative real number ($*311\cdot42$). In these proofs, when propositions of previous numbers involving "Rat" are used, "Rat" is replaced by $C'H'$ and "$\text{Rat} - \iota'0_q$" by $C'H$. This is legitimate in virtue of $*304\cdot49\cdot34$. In $*311\cdot511$ we prove (assuming the axiom of infinity) that if ξ is a positive real number, and Y is any positive ratio, however small, there are members X of ξ such that $Y +_g X$ is not a member of ξ, *i.e.* given any positive real number, there are rationals differing from it by less than any assigned positive rational. This proposition is useful, and is used in proving that if ξ, η are positive real numbers, each is less than $\xi +_p \eta$ ($*311\cdot52$). The converse of this proposition, *i.e.* the proposition that, if $\mu\Theta\nu$, there is a positive real number

λ such that $\nu = \mu +_p \lambda$, is proved in $*311{\cdot}621{\cdot}64$, after a considerable amount of work. Thus we have

$*311{\cdot}65$. $\vdash :: \text{Infin ax} . \supset :. \mu \Theta \nu . \equiv : \mu, \nu \in C`\Theta : (\exists \lambda) . \lambda \in C`\Theta . \nu = \mu +_p \lambda$

We have, of course, a corresponding proposition for Θ_n ($*311{\cdot}66$). From $*311{\cdot}65$ we deduce without difficulty that if μ is less than ν (μ, ν being positive real numbers), then $\lambda +_p \mu$ is less than $\lambda +_p \nu$ (λ being a positive real number), *i.e.*

$*311{\cdot}73$. $\vdash : \text{Infin ax} . \lambda \in C`\Theta . \mu \Theta \nu . \supset . (\lambda +_p \mu) \Theta (\lambda +_p \nu)$

whence (with the corresponding proposition for Θ_n) we deduce

$*311{\cdot}75$. $\vdash :. \text{Infin ax} . \text{concord}(\lambda, \mu) . \supset : \lambda +_p \mu = \lambda +_p \nu . \equiv . \mu = \nu$

which secures the uniqueness of subtraction.

$*311{\cdot}01$. $\text{concord}(\mu, \nu, \ldots) . = : \mu, \nu, \ldots \in C`\Theta' . \vee . \mu, \nu, \ldots \in C`\Theta'_n$ Df

$*311{\cdot}02$. $\mu +_p \nu = \hat{X} \{\text{concord}(\mu, \nu) . X \in s`\mu \underset{,,}{+_g}``\nu\}$ Df

$*311{\cdot}1$. $\vdash : \sim \text{concord}(\mu, \nu) . \supset . \mu +_p \nu = \Lambda$ [($*311{\cdot}02$)]

$*311{\cdot}11$. $\vdash : \text{concord}(\mu, \nu) . \supset .$

$$\mu +_p \nu = s`\mu \underset{,,}{+_g}``\nu = \hat{W}\{(\exists M, N) . M \in \mu . N \in \nu . W = M +_g N\}$$

[($*311{\cdot}02$)]

$*311{\cdot}12$. $\vdash . \mu +_p \nu = \nu +_p \mu$ [$*311{\cdot}1{\cdot}11 . *308{\cdot}4$]

$*311{\cdot}121$. $\vdash . (\lambda +_p \mu) +_p \nu = \lambda +_p (\mu +_p \nu)$ [$*311{\cdot}1{\cdot}11 . *308{\cdot}63$]

$*311{\cdot}13$. $\vdash : \text{concord}(\mu, \nu) . \equiv . \text{concord}(| \text{Cnv}``\mu, | \text{Cnv}``\nu)$
[$*310{\cdot}16 . (*311{\cdot}01)$]

$*311{\cdot}14$. $\vdash : \text{concord}(\mu, | \text{Cnv}``\nu) . \equiv . \text{concord}(| \text{Cnv}``\mu, \nu)$ [$*311{\cdot}13 . *310{\cdot}17$]

$*311{\cdot}15$. $\vdash : \text{concord}(\mu, | \text{Cnv}``\nu) . \supset . \sim \text{concord}(\mu, \nu)$ [$*310{\cdot}13{\cdot}16$]

$*311{\cdot}2$. $\vdash : \text{Infin ax} . \xi \subset C`H . X \in C`H . \supset . X +_g``H``\xi = H``X +_g``\xi \cap \overleftarrow{H}`X$

Dem.

$\vdash . *308{\cdot}72 . *304{\cdot}34{\cdot}401 . \supset \vdash :. \text{Hp} . \supset : Y \in X +_g``H``\xi . \equiv .$
$(\exists Z, Z') . Z' \in \xi . Z \in C`H . Y = X +_g Z . (X +_g Z) H (X +_g Z') .$
[$*37{\cdot}6$] $\equiv . (\exists Z, Y') . Z \in C`H . Y = X +_g Z . Y' \in X +_g``\xi . YHY' .$
[$*306{\cdot}52$] $\equiv . Y \in H``X +_g``\xi . XHY :. \supset \vdash . \text{Prop}$

$*311{\cdot}21$. $\vdash : \text{Infin ax} . \xi \subset C`H . \exists ! \xi . X \in C`H . \supset . \overrightarrow{H_*}`X \subset H``X +_g``\xi$

Dem.

$\vdash . *306{\cdot}52 . *304{\cdot}401 . \supset \vdash :. \text{Hp} . \supset : Y \in \xi . \supset . XH(X +_g Y) :$
[$*40{\cdot}51{\cdot}61$] $\supset : X \in H``X +_g``\xi$ (1)
$\vdash . (1) . *304{\cdot}23 . \supset \vdash . \text{Prop}$

***311·22.** $\vdash : \text{Infin ax} \,.\, \xi \subset C\text{‘}H \,.\, \exists ! \,\xi \,.\, X \in C\text{‘}H \,.\, \supset .$

$$H\text{“}X +_g\text{“}\xi = \overrightarrow{H_*}\text{‘}X \cup X +_g\text{“}H\text{“}\xi$$

Dem.

$\vdash . *304\cdot23 \,.\, \supset \vdash .\, H\text{“}X +_g\text{“}\xi = (H\text{“}X +_g\text{“}\xi \cap \overrightarrow{H_*}\text{‘}X) \cup (H\text{“}X +_g\text{“}\xi \cap \overleftarrow{H}\text{‘}X)$ (1)

$\vdash . (1) . *311\cdot2\cdot21 \,.\, \supset \vdash .$ Prop

***311·23.** $\vdash : \text{Infin ax} \,.\, \xi \in C\text{‘}\Theta \,.\, X \in C\text{‘}H \,.\, \supset .\, H\text{“}X +_g\text{“}\xi = \overrightarrow{H_*}\text{‘}X \cup X +_g\text{“}H\text{“}\xi$

[*311·22 . *310·12]

***311·24.** $\vdash :. \text{Infin ax} \,.\, \xi \in C\text{‘}\Theta \,.\, Y \in C\text{‘}H \,.\, \supset :$

$$(\exists Z) \,.\, ZHY \,.\, Y \in Z +_g\text{“}\xi : Y \in s\text{‘}\xi \underset{,,}{+}_g\text{“}\overrightarrow{H}\text{‘}Y$$

Dem.

$\vdash . *304\cdot31 \,.\, \supset \vdash : \text{Hp} \,.\, \supset .\, (\exists W) \,.\, W \in \xi \,.\, WHY \,.$

[*306·52] $\supset .\, (\exists Z, W) \,.\, W \in \xi \,.\, ZHY \,.\, Y = Z +_g W : \supset \vdash .$ Prop

***311·25.** $\vdash : \text{Infin ax} \,.\, \xi, \eta \in C\text{‘}\Theta \,.\, \supset .\, \xi \subset \xi +_p \eta \,.\, \eta \subset \xi +_p \eta$

Dem.

$\vdash . *310\cdot12 \,.\quad \supset \vdash : \text{Hp} \,.\, Y \in \eta \,.\, \supset .\, \overrightarrow{H}\text{‘}Y \subset \eta \,.$

[*311·24] $\supset .\, Y \in s\text{‘}\xi \underset{,,}{+}_g\text{“}\eta$ (1)

$\vdash . (1) . *311\cdot11 \,.\, \supset \vdash : \text{Hp} \,.\, \supset .\, \eta \subset \xi +_p \eta$ (2)

$\vdash . (2) . *311\cdot12 \,.\, \supset \vdash : \text{Hp} \,.\, \supset .\, \xi \subset \xi +_p \eta$ (3)

$\vdash . (2) . (3) \,.\, \supset \vdash .$ Prop

***311·26.** $\vdash : \text{Infin ax} \,.\, \xi, \eta \in C\text{‘}\Theta \,.\, \supset .\, H\text{“}(\xi +_p \eta) = \xi +_p \eta$

Dem.

$\vdash . *311\cdot23 \,.\, \supset \vdash :. \text{Hp} \,.\, \supset : Y \in \eta \,.\, \supset .\, H\text{“}(\xi \underset{,,}{+}_g Y) = \overrightarrow{H_*}\text{‘}Y \cup (H\text{“}\xi) \underset{,,}{+}_g Y :$

[*311·11.*310·12] $\supset : H\text{“}(\xi +_p \eta) = H_*\text{“}\eta \cup (\xi +_p \eta)$

[*311·25.*310·12] $= \xi +_p \eta :. \supset \vdash .$ Prop

***311·27.** $\vdash : \text{Infin ax} \,.\, \xi, \eta \in C\text{‘}\Theta \,.\, \supset .\, \xi +_p \eta \in C\text{‘}\Theta$

Dem.

$\vdash . *311\cdot25 \,.\, *310\cdot12 \,.\, \supset \vdash : \text{Hp} \,.\, \supset .\, \exists ! \,\xi +_p \eta \,.$

[*311·26.*310·12] $\supset .\, \xi +_p \eta \in C\text{‘}\Theta \cup \iota\text{‘}\text{D}\text{‘}H$ (1)

$\vdash . *310\cdot12 \,.\, *211\cdot703 \,.\, \supset$

$\vdash : \text{Hp} \,.\, \supset .\, (\exists M, N) \,.\, M, N \in \text{D}\text{‘}H \,.\, M \in p\text{‘}\overleftarrow{H}\text{“}\xi \,.\, N \in p\text{‘}\overleftarrow{H}\text{“}\eta \,.$

[*308·32·72.*306·23] $\supset .\, (\exists M, N) \,.\, M +_g N \in p\text{‘}\overleftarrow{H}\text{“}(\xi +_p \eta) \cap \text{D}\text{‘}H$ (2)

$\vdash . (2) . *200\cdot5 \,.\, \supset \vdash : \text{Hp} \,.\, \supset .\, \xi +_p \eta \neq \text{D}\text{‘}H$ (3)

$\vdash . (1) . (3) \,.\, \supset \vdash .$ Prop

The axiom of infinity is essential to the truth of the above proposition, for if it fails we have $\text{E} ! \, B\text{‘}H \,.\, B\text{‘}H \sim\in \xi +_p \eta$, while $\mu \in C\text{‘}\Theta \,.\, \supset .\, B\text{‘}H \in \mu$.

***311·31.** $\vdash . \,|\,\mathrm{Cnv}``(\mu +_p \nu) = (|\,\mathrm{Cnv}``\mu) +_p (|\,\mathrm{Cnv}``\nu)$

Dem.

$\vdash . *311·13·1 . \supset$

$\vdash : \sim \mathrm{concord}\,(\mu, \nu) . \supset . \,|\,\mathrm{Cnv}``(\mu +_p \nu) = \Lambda . (|\,\mathrm{Cnv}``\mu) +_p (|\,\mathrm{Cnv}``\nu) = \Lambda$ (1)

$\vdash . *311·13·11 . \supset \vdash : \mathrm{concord}\,(\mu, \nu) . \supset . \,|\,\mathrm{Cnv}``(\mu +_p \nu) = |\,\mathrm{Cnv}``s`\mu \underset{,,}{+_g}``\nu$

$[*308·411] \quad = s`(|\,\mathrm{Cnv}``\mu) \underset{,,}{+_g}``(|\,\mathrm{Cnv}``\nu)$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***311·32.** $\vdash . \,|\,\mathrm{Cnv}``(\mu +_p |\,\mathrm{Cnv}``\nu) = (|\,\mathrm{Cnv}``\mu) +_p \nu$ [*311·31 . *310·17]

***311·33.** $\vdash . \mu +_p \nu = |\,\mathrm{Cnv}``\{(|\,\mathrm{Cnv}``\mu) +_p (|\,\mathrm{Cnv}``\nu)\}$ [*311·31 . *310·18]

***311·41.** $\vdash : \mathrm{Infin\ ax} . \mu, \nu \in C`\Theta_n . \supset . \mu \subset \mu +_p \nu . \nu \subset \mu +_p \nu$

Dem.

$\vdash . *311·25 . *310·16 . \supset \vdash : \mathrm{Hp} . \supset . \,|\,\mathrm{Cnv}``\mu \subset (|\,\mathrm{Cnv}``\mu) +_p (|\,\mathrm{Cnv}``\nu) .$

$[*311·33 . *310·17] \quad \supset . \mu \subset \mu +_p \nu$ (1)

Similarly $\vdash : \mathrm{Hp} . \supset . \nu \subset \mu +_p \nu$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***311·42.** $\vdash : \mathrm{Infin\ ax} . \mu, \nu \in C`\Theta_n . \supset . \mu +_p \nu \in C`\Theta_n$

Dem.

$\vdash . *311·27 . *310·16 . \supset \vdash : \mathrm{Hp} . \supset . (|\,\mathrm{Cnv}``\mu) +_p (|\,\mathrm{Cnv}``\nu) \in C`\Theta .$

$[*311·33 . *310·16] \quad \supset . \mu +_p \nu \in C`\Theta_n : \supset \vdash . \mathrm{Prop}$

***311·43.** $\vdash : \mu \in C`\Theta_g . \supset . \mu +_p \iota`0_q = \mu$

Dem.

$\vdash . *311·11 . \supset \vdash : \mathrm{Hp} . \supset . \mu +_p \iota`0_q = \hat{W}\{(\exists M) . M \in \mu . W = M +_g 0_q\}$

$[*308·51] \quad = \mu : \supset \vdash . \mathrm{Prop}$

***311·44.** $\vdash : \mathrm{Infin\ ax} . \mathrm{concord}\,(\mu, \nu) . \supset . \mu +_p \nu \in C`\Theta_g$ [*311·27·42·43]

***311·45.** $\vdash :. \mathrm{Infin\ ax} . \mathrm{concord}\,(\mu, \nu) : \mu \neq \iota`0_q . \vee . \nu = \iota`0_q : \supset . \mu \subset \mu +_p \nu$ [*311·25·41·43]

***311·51.** $\vdash : \mathrm{Infin\ ax} . \xi \in \mathrm{D}`H_\epsilon - \iota`\Lambda . Y \in C`H . Y +_g``\xi \subset \xi . \supset . \xi = C`H = \mathrm{D}`H$

Dem.

$\vdash . *38·13 . \supset \vdash : \mathrm{Hp} . X \in \xi . \supset . Y +_g X \in \xi .$

$[*306·52] \quad \supset . Y \in \xi$ (1)

$\vdash . *306·51 . \supset$

$\vdash : \mathrm{Hp} . \nu \in \mathrm{NC\,ind} . X \in \xi . Y +_g (\nu/1 \times_s X) \in \xi . \supset . Y +_g \{(\nu +_c 1)/1 \times_s X\} \in \xi$ (2)

$\vdash . (1) . (2) . \mathrm{Induct} . \supset \vdash : \mathrm{Hp} . \nu \in \mathrm{NC\,ind} . X \in \xi . \supset . Y +_g (\nu/1 \times_s X) \in \xi$ (3)

$\vdash . *305·7 . *306·52 . \supset$

$\vdash : \mathrm{Hp} . X \in \xi . Z \in C`H . \supset . (\exists \nu) . \nu \in \mathrm{NC\,ind} . ZH\{Y +_g (\nu/1 \times_s X)\}$ (4)

$\vdash . (3) . (4) . \supset \vdash : \mathrm{Hp} . Z \in C`H . \supset . Z \in \xi : \supset \vdash . \mathrm{Prop}$

$*311\cdot511$. $\vdash : \text{Infin ax} . \xi \in C\text{'}\Theta . Y \in C\text{'}H . \supset . (\exists X) . X \in \xi . Y +_g X \sim\in \xi$
$[*311\cdot51 . \text{Transp}]$

$*311\cdot52$. $\vdash : \text{Infin ax} . \xi, \eta \in C\text{'}\Theta . \supset . \xi \Theta (\xi +_p \eta)$

Dem.

$\vdash . *311\cdot511 . \supset \vdash :. \text{Hp} . \supset : Y \in C\text{'}H . \supset . (\exists X) . X \in \xi . X +_g Y \sim\in \xi :$
$[*311\cdot11] \qquad \supset : (\exists X, Y) . X +_g Y \in (\xi +_p \eta) - \xi :$
$[*310\cdot11 . *311\cdot27] \qquad \supset : \xi \Theta (\xi +_p \eta) :. \supset \vdash . \text{Prop}$

$*311\cdot53$. $\vdash : \text{Infin ax} . \xi, \eta \in C\text{'}\Theta_n . \supset . \xi \Theta_n (\xi +_p \eta)$ $\quad [*311\cdot52\cdot33]$

$*311\cdot56$. $\vdash :. \text{Infin ax} . \xi \in C\text{'}\Theta_g . \supset : \xi = \xi +_p \eta . \equiv . \eta = \iota\text{'}0_q$ $\quad [*311\cdot1\cdot43\cdot52\cdot53]$

$*311\cdot57$. $\vdash :: \text{Infin ax} . \supset :. \xi = \xi +_p \eta . \equiv : \xi = \Lambda . \vee . \xi \in C\text{'}\Theta_g . \eta = \iota\text{'}0_q$
$[*311\cdot56\cdot1]$

$*311\cdot58$. $\vdash : \text{Infin ax} . \mu \in C\text{'}\Theta . \supset . \mu = H\text{''}\mu$ $\quad [*304\cdot3 . *270\cdot31]$

$*311\cdot6$. $\vdash : \text{Infin ax} . \mu \Theta \nu . X, Y \in \nu - \mu . XHY . M \in \mu . \supset . M +_g (Y -_s X) \in \nu$

Dem.

$\vdash . *310\cdot11 . \supset \vdash : \text{Hp} . \supset . MHX .$
$[*308\cdot42\cdot72] \qquad \supset . \{M +_g (Y -_s X)\} HY \qquad (1)$
$\vdash . (1) . *311\cdot58 . \supset \vdash . \text{Prop}$

$*311\cdot61$. $\vdash : \text{Infin ax} . \mu \Theta \nu .$
$\lambda = \hat{L} \{(\exists X, Y) . X, Y \in \nu - \mu . XHY . L = Y -_s X\} . \supset .$
$s\text{'}\mu \underset{,,}{+_g}\text{''}\lambda \subset \nu$ $\quad [*311\cdot6]$

$*311\cdot62$. $\vdash : \text{Infin ax} . \mu \Theta \nu . X \in \nu - \mu . \supset . (\exists Y) . Y \in \nu - \mu . XHY$

Dem.

$\vdash . *311\cdot58 . \supset \vdash : \text{Hp} . \supset . X \in H\text{''}\nu - H\text{''}\mu : \supset \vdash . \text{Prop}$

$*311\cdot621$. $\vdash : \text{Hp} *311\cdot61 . \supset . \lambda \in C\text{'}\Theta$

Dem.

$\vdash . *311\cdot62 . \quad \supset \vdash : \text{Hp} . \supset . \exists ! \lambda \qquad (1)$
$\vdash . *308\cdot46 . \quad \supset \vdash : \text{Hp} . \supset . \lambda \subset H\text{''}\nu \qquad (2)$
$\vdash . *311\cdot62 . \quad \supset \vdash : \text{Hp} . X, Y \in \nu - \mu . XHY . \supset . (\exists Z) . Z \in \nu - \mu . YHZ .$
$[*308\cdot42\cdot72] \qquad \supset . (\exists Z) . Z \in \nu - \mu . (Y -_s X) H (Z -_s X) \qquad (3)$
$\vdash . (3) . *37\cdot1 . \supset \vdash : \text{Hp} . \supset . \lambda \subset H\text{''}\lambda \qquad (4)$
$\vdash . *308\cdot56\cdot42\cdot72 . \supset$
$\vdash : \text{Hp} . X, Y \in \nu - \mu . XHY . LH (Y -_s X) . \supset . XH (X +_g L) . (X +_g L) HY .$
$[*310\cdot11 . *308\cdot43] \qquad \supset . L \in \lambda \qquad (5)$
$\vdash . (5) . *37\cdot1 . \qquad \supset \vdash : \text{Hp} . \supset . H\text{''}\lambda \subset \lambda \qquad (6)$
$\vdash . (1) . (2) . (4) . (6) . \supset \vdash : \text{Hp} . \supset . \lambda \in D\text{'}H_\epsilon - \iota\text{'}\Lambda - \iota\text{'}D\text{'}H .$
$[*310\cdot12] \qquad \supset . \lambda \in C\text{'}\Theta : \supset \vdash . \text{Prop}$

***311·63.** $\vdash :$ Infin ax $. \nu \epsilon C'\Theta . X \epsilon \nu . N \epsilon C'H . \supset . (\exists L) . LHN . X +_g L \epsilon \nu$

Dem.

$\vdash . *311·58 . \supset \vdash : \text{Hp} . \supset . (\exists Y) . Y \epsilon \nu . XHY$ (1)

$\vdash . *308·42 . \supset \vdash : \text{Hp} . Y \epsilon \nu . XHY . Z = Y -_s X . ZHN . \supset . ZHN . X +_g Z \epsilon \nu$ (2)

$\vdash . *308·42·72 . \supset$

$\vdash : \text{Hp} . Y \epsilon \nu . XHY . Z = Y -_s X . NH_* Z . LHN . \supset . LHN . X +_g L \epsilon \nu$ (3)

$\vdash . (3) . *311·58 . \supset$

$\vdash : \text{Hp} . Y \epsilon \nu . XHY . Z = Y -_s X . NH_* Z . \supset . (\exists L) . LHN . X +_g L \epsilon \nu$ (4)

$\vdash . (1) . (2) . (4) . \supset \vdash . \text{Prop}$

***311·631.** $\vdash :$ Infin ax $. \mu \Theta \nu . N \epsilon \mu . \supset .$

$(\exists M, X, Y) . M \epsilon \mu . X, Y \epsilon \nu - \mu . XHY . N = M +_g (Y -_s X)$

Dem.

$\vdash . *311·58 . *308·72 . \supset$

$\vdash : \text{Hp} . X \epsilon \nu - \mu . LHN . X +_g L \epsilon \nu . Y = X +_g L . M = N -_g L . \supset .$

$M \epsilon \mu . X, Y \epsilon \nu - \mu . XHY . N = M +_g (Y -_s X)$ (1)

$\vdash . (1) . *311·63 . \supset \vdash . \text{Prop}$

***311·632.** $\vdash :$ Infin ax $. \mu \Theta \nu . N \epsilon \nu - \mu . \supset .$

$(\exists M, W) . M \epsilon \mu . M +_g W, N +_g W \epsilon \nu - \mu . (M +_g W) H (N +_g W)$

Dem.

$\vdash . *306·52 . *311·63·58 . \supset \vdash : \text{Hp} . \supset . (\exists W) . W \epsilon C'H . N +_g W \epsilon \nu - \mu$ (1)

$\vdash . *311·511 . \supset \vdash : \text{Hp} . W \epsilon C'H . \supset . (\exists M) . M \epsilon \mu . M +_g W \sim \epsilon \mu$ (2)

$\vdash . *311·58 . \supset \vdash : \text{Hp} . M \epsilon \mu . N \epsilon \nu - \mu . W \epsilon C'H . \supset . MHN . W \epsilon C'H .$

[*308·72] $\supset . (M +_g W) H (N +_g W)$ (3)

$\vdash . (3) . *311·58 . \supset \vdash : \text{Hp}(3) . N +_g W \epsilon \nu . \supset . M +_g W \epsilon \nu$ (4)

$\vdash . (2) . (4) . \supset$

$\vdash : \text{Hp} . W \epsilon C'H . N +_g W \epsilon \nu - \mu . \supset . (\exists M) . M \epsilon \mu . M +_g W \epsilon \nu - \mu$ (5)

$\vdash . (1) . (3) . (5) . \supset \vdash . \text{Prop}$

***311·633.** $\vdash :$ Infin ax $. \mu \Theta \nu . N \epsilon \nu . \supset .$

$(\exists M, X, Y) . M \epsilon \mu . X, Y \epsilon \nu - \mu . XHY . N = M +_g (Y -_s X)$

Dem.

$\vdash . *308·61·4·63 . \supset$

$\vdash : \text{Hp} . MHN . X = M +_g W . Y = N +_g W . \supset . N = M +_g (Y -_s X)$ (1)

$\vdash . *311·632 . *308·72 . \supset \vdash : \text{Hp} . N \sim \epsilon \mu . \supset . (\exists M, W, X, Y) .$

$M \epsilon \mu . X = M +_g W . Y = N +_g W . XHY . MHN . X, Y \epsilon \nu - \mu$ (2)

$\vdash . (1) . (2) . \supset \vdash : \text{Hp} . N \sim \epsilon \mu . \supset .$

$(\exists M, X, Y) . M \epsilon \mu . X, Y \epsilon \nu - \mu . XHY . N = M +_g (Y -_s X)$ (3)

$\vdash . (3) . *311·631 . \supset \vdash . \text{Prop}$

***311·64.** $\vdash : \text{Hp} *311\cdot61 . \supset . \nu = \mu +_p \lambda$

Dem.

$\vdash . *311\cdot633 . \supset . \nu \subset s`\mu \underset{,,}{+_g}``\lambda$ (1)

$\vdash . (1) . *311\cdot621\cdot61 . \supset \vdash : \text{Hp} . \supset . \lambda \in C`\Theta . \nu = s`\mu \underset{,,}{+_g}``\lambda .$

$[*311\cdot11] \qquad \supset . \nu = \mu +_p \lambda : \supset \vdash . \text{Prop}$

***311·65.** $\vdash :: \text{Infin ax} . \supset :. \mu \Theta \nu . \equiv : \mu, \nu \in C`\Theta : (\exists \lambda) . \lambda \in C`\Theta . \nu = \mu +_p \lambda$ $[*311\cdot52\cdot64]$

***311·66.** $\vdash :: \text{Infin ax} . \supset :. \mu \Theta_n \nu . \equiv : \mu, \nu \in C`\Theta_n : (\exists \lambda) . \lambda \in C`\Theta_n . \nu = \mu +_p \lambda$

Dem.

$\vdash . *310\cdot11\cdot111 . \supset \vdash : \mu \Theta_n \nu . \equiv . (|\text{Cnv}``\mu) \Theta (|\text{Cnv}``\nu)$ (1)

$\vdash . (1) . *311\cdot65 . \supset \vdash :: \text{Hp} . \supset :.$

$\mu \Theta_n \nu . \equiv : |\text{Cnv}``\mu \in C`\Theta : (\exists \lambda) . \lambda \in C`\Theta . |\text{Cnv}``\nu = |\text{Cnv}``\mu +_p \lambda :$

$[*311\cdot32 . *310\cdot16\cdot19] \equiv : \mu \in C`\Theta_n : (\exists \lambda) . \lambda \in C`\Theta_n . \nu = \mu +_p \lambda :: \supset \vdash . \text{Prop}$

***311·73.** $\vdash : \text{Infin ax} . \lambda \in C`\Theta . \mu \Theta \nu . \supset . (\lambda +_p \mu) \Theta (\lambda +_p \nu)$

Dem.

$\vdash . *311\cdot65 . \supset \vdash : \text{Hp} . \supset . (\exists \rho) . \rho \in C`\Theta . \nu = \mu +_p \rho .$

$[*311\cdot121] \qquad \supset . (\exists \rho) . \rho \in C`\Theta . \lambda +_p \nu = (\lambda +_p \mu) +_p \rho$ (1)

$\vdash . *311\cdot27 . \supset \vdash : \text{Hp} . \supset . \lambda +_p \mu, \lambda +_p \nu \in C`\Theta$ (2)

$\vdash . (1) . (2) . *311\cdot65 . \supset \vdash . \text{Prop}$

***311·731.** $\vdash : \text{Infin ax} . \lambda \in C`\Theta_n . \mu \Theta_n \nu . \supset . (\lambda +_p \mu) \Theta_n (\lambda +_p \nu)$ $[*311\cdot73]$

***311·74.** $\vdash :. \text{Infin ax} : \lambda, \mu \in C`\Theta . \vee . \lambda, \mu \in C`\Theta_n : \supset : \lambda +_p \mu = \lambda +_p \nu . \equiv . \mu = \nu$

Dem.

$\vdash . *311\cdot27\cdot1 . \qquad \supset \vdash : \lambda, \mu \in C`\Theta . \lambda +_p \mu = \lambda +_p \nu . \supset . \nu \in C`\Theta$ (1)

$\vdash . *311\cdot73 . \text{Transp} . \supset \vdash : \text{Hp}(1) . \supset . \sim(\mu \Theta \nu) . \sim(\nu \Theta \mu)$ (2)

$\vdash . (1) . (2) . *310\cdot1 . \supset \vdash : \text{Hp}(1) . \supset . \mu = \nu$ (3)

Similarly $\qquad \vdash : \lambda, \mu \in C`\Theta_n . \lambda +_p \mu = \lambda +_p \nu . \supset . \mu = \nu$ (4)

$\vdash . (3) . (4) . \supset \vdash . \text{Prop}$

***311·75.** $\vdash :. \text{Infin ax} . \text{concord}(\lambda, \mu) . \supset : \lambda +_p \mu = \lambda +_p \nu . \equiv . \mu = \nu$ $[*311\cdot74\cdot43]$

$*312$. ALGEBRAIC ADDITION OF REAL NUMBERS.

Summary of $*312$.

In this number we extend the definition of addition so as to apply to real numbers of opposite sign. As in $*308$, this requires a previous definition of subtraction. We define subtraction as follows: If there is a λ such that $\nu +_p \lambda = \mu$, then $\mu -_p \nu$ is λ; if there is a λ such that $\mu +_p \lambda = \nu$, then $\mu -_p \nu$ is $\mid \mathrm{Cnv}``\lambda$, *i.e.* the negative of λ; in any other case, $\mu -_p \nu = \Lambda$. The formal definition is:

$$*312\cdot01.\quad \mu -_p \nu = \hat{X}\{(\exists\lambda) : \lambda, \mu, \nu \in C`\Theta_g : \nu +_p \lambda = \mu \,.\, X \in \lambda \,.\vee.\, \mu +_p \lambda = \nu \,.\, X \in \mid \mathrm{Cnv}``\lambda\} \quad \mathrm{Df}$$

Hence assuming the axiom of infinity we have

$$\nu\,(\Theta \cup \Theta_n)\,\mu \,.\supset.\, \mu -_p \nu = (\imath\lambda)\,(\nu +_p \lambda = \mu) \quad (*312\cdot18),$$

$$\mu\,(\Theta \cup \Theta_n)\,\nu \,.\supset.\, \mu -_p \nu = (\imath\lambda)\,(\mu +_p \mid \mathrm{Cnv}``\lambda = \nu) \quad (*312\cdot181),$$

$$\lambda \in C`\Theta_g \,.\supset.\, \lambda -_p \lambda = \iota`0_q \quad (*312\cdot191).$$

The algebraic sum of μ and ν is defined as $\mu +_p \nu$ if μ and ν are of the same sign, and as $\mu -_p \mid \mathrm{Cnv}``\nu$ if μ and ν are of opposite signs; *i.e.* we put

$$*312\cdot02.\quad \mu +_a \nu = (\mu +_p \nu) \cup (\mu -_p \mid \mathrm{Cnv}``\nu) \quad \mathrm{Df}$$

This definition is justified because either $\mu +_p \nu$ or $\mu -_p \mid \mathrm{Cnv}``\nu$ must always be Λ. Thus we have

$$*312\cdot32.\quad \vdash : \mathrm{concord}\,(\mu, \nu) \,.\supset.\, \mu +_a \nu = \mu +_p \nu$$

$$*312\cdot33.\quad \vdash : \sim \mathrm{concord}\,(\mu, \nu) \,.\supset.\, \mu +_a \nu = \mu -_p \mid \mathrm{Cnv}``\nu$$

The propositions proved are analogous to those of previous numbers, and offer no difficulty.

$$*312\cdot01.\quad \mu -_p \nu = \hat{X}\{(\exists\lambda) : \lambda, \mu, \nu \in C`\Theta_g : \nu +_p \lambda = \mu \,.\, X \in \lambda \,.\vee.\, \mu +_p \lambda = \nu \,.\, X \in \mid \mathrm{Cnv}``\lambda\} \quad \mathrm{Df}$$

$$*312\cdot02.\quad \mu +_a \nu = (\mu +_p \nu) \cup (\mu -_p \mid \mathrm{Cnv}``\nu) \quad \mathrm{Df}$$

***312·1.** $\vdash :. X \in \mu -_p \nu . \equiv : \mu, \nu \in C`\Theta_g : (\exists\lambda) : \lambda \in C`\Theta_g :$
$\nu +_p \lambda = \mu . X \in \lambda . \vee . \mu +_p \lambda = \nu . X \in | \mathrm{Cnv}``\lambda$ [(*311·01)]

***312·11.** $\vdash : \sim \mathrm{concord}(\mu, \nu) . \supset . \mu -_p \nu = \Lambda$ [*311·1·27·42·43]

***312·12.** $\vdash : \mathrm{Infin\,ax} . \nu\Theta\mu . \supset .$

$$\mu -_p \nu = \hat{X} \{(\exists\lambda) . \lambda \in C`\Theta . \nu +_p \lambda = \mu . X \in \lambda\} = (\imath\lambda)(\nu +_p \lambda = \mu)$$

Dem.

$\vdash . *311·1·65 . \supset \vdash : \mathrm{Hp} . \supset . \sim(\exists\lambda) . \mu +_p \lambda = \nu$ (1)

$\vdash . (1) . *312·1 . \supset \vdash : \mathrm{Hp} . \supset . \mu -_p \nu = \hat{X} \{(\exists\lambda) . \lambda \in C`\Theta . \nu +_p \lambda = \mu . X \in \lambda\}$ (2)

$\vdash . (2) . *311·74 . \supset \vdash . \mathrm{Prop}$

***312·13.** $\vdash : \mathrm{Infin\,ax} . \mu\Theta\nu . \supset .$

$$\mu -_p \nu = \hat{X} \{(\exists\lambda) . \lambda \in C`\Theta . \mu +_p \lambda = \nu . X \in | \mathrm{Cnv}``\lambda\}$$
$$= | \mathrm{Cnv}``(\imath\lambda)(\mu +_p \lambda = \nu)$$ [Proof as in *312·12]

***312·14.** $\vdash : \mathrm{Infin\,ax} . \nu\Theta_n\mu . \supset .$

$$\mu -_p \nu = \hat{X} \{(\exists\lambda) . \lambda \in C`\Theta_n . \nu +_p \lambda = \mu . X \in \lambda\}$$
$$= (\imath\lambda)(\nu +_p \lambda = \mu)$$ [Proof as in *312·12]

***312·15.** $\vdash : \mathrm{Infin\,ax} . \mu\Theta_n\nu . \supset .$

$$\mu -_p \nu = \hat{X} \{(\exists\lambda) . \lambda \in C`\Theta_n . \mu +_p \lambda = \nu . X \in | \mathrm{Cnv}``\lambda\}$$
$$= | \mathrm{Cnv}``(\imath\lambda)(\mu +_p \lambda = \nu)$$ [Proof as in *312·12]

***312·16.** $\vdash : \mu \in C`\Theta_g . \supset . \mu -_p \iota`0_q = \mu$ [*312·1 . *311·43]

***312·17.** $\vdash : \mu \in C`\Theta_g . \supset . \iota`0_q -_p \mu = | \mathrm{Cnv}``\mu$ [*312·1 . *311·43]

***312·18.** $\vdash : \mathrm{Infin\,ax} . \nu(\Theta \cup \Theta_n)\mu . \supset . \mu -_p \nu = (\imath\lambda)(\nu +_p \lambda = \mu)$ [*312·12·14]

***312·181.** $\vdash : \mathrm{Infin\,ax} . \mu(\Theta \cup \Theta_n)\nu . \supset . \mu -_p \nu = | \mathrm{Cnv}``(\imath\lambda)(\mu +_p \lambda = \nu)$
$= (\imath\lambda)(\mu +_p | \mathrm{Cnv}``\lambda = \nu)$ [*312·13·15]

***312·19.** $\vdash : \mathrm{Infin\,ax} . \mathrm{concord}(\lambda, \mu) . \supset . (\lambda +_p \mu) -_p \lambda = \mu$
[*312·18 . *311·65·66·43]

***312·191.** $\vdash : \mathrm{Infin\,ax} . \lambda \in C`\Theta_g . \supset . \lambda -_p \lambda = \iota`0_q$ [*311·52·53·43]

***312·2.** $\vdash . | \mathrm{Cnv}``(\mu -_p \nu) = | \mathrm{Cnv}``\mu -_p | \mathrm{Cnv}``\nu$

Dem.

$\vdash . *312·1 . *310·16 . \supset$

$\vdash :. X \in | \mathrm{Cnv}``\mu -_p | \mathrm{Cnv}``\nu . \equiv : \mu, \nu \in C`\Theta_g :$
$(\exists\lambda) : \lambda \in C`\Theta_g : | \mathrm{Cnv}``\nu +_p \lambda = | \mathrm{Cnv}``\mu . X \in \lambda . \vee .$
$| \mathrm{Cnv}``\mu +_p \lambda = | \mathrm{Cnv}``\nu . X \in | \mathrm{Cnv}``\lambda :$

$[*311·32] \equiv : \mu, \nu \in C`\Theta_g : (\exists\lambda) : \lambda \in C`\Theta_g :$
$\nu +_p | \mathrm{Cnv}``\lambda = \mu . X \in \lambda . \vee . \mu +_p | \mathrm{Cnv}``\lambda = \nu . X \in | \mathrm{Cnv}``\lambda :$

$[*312·1 . *310·16] \equiv : X \in | \mathrm{Cnv}``(\mu -_p \nu) :. \supset \vdash . \mathrm{Prop}$

***312·201.** $\vdash . \mu -_p | \mathrm{Cnv}``\nu = | \mathrm{Cnv}``(| \mathrm{Cnv}``\mu -_p \nu)$ [*312·2]

***312·21.** $\vdash . \mid \mathrm{Cnv}``(\nu -_p \mu) = \mu -_p \nu$

Dem.

$\vdash . *312·1 . \supset \vdash :: X \in \mid \mathrm{Cnv}``(\nu -_p \mu) . \equiv :. (\exists Y) :. \mu, \nu \in C`\Theta_g :.$
$(\exists \lambda) : \lambda \in C`\Theta_g : \mu +_p \lambda = \nu . Y \in \lambda . X = Y \mid \mathrm{Cnv} . \vee .$
$\nu +_p \lambda = \mu . Y \in \mid \mathrm{Cnv}``\lambda . X = Y \mid \mathrm{Cnv} :.$

$[*310·16] \equiv :. \mu, \nu \in C`\Theta_g :. (\exists \lambda) : \lambda \in C`\Theta_g : \mu +_p \lambda = \nu . X \in \mid \mathrm{Cnv}``\lambda . \vee .$
$\nu +_p \lambda = \mu . X \in \lambda :.$

$[*312·1] \equiv :. X \in \mu -_p \nu :: \supset \vdash . \mathrm{Prop}$

***312·211.** $\vdash . \mu -_p \mid \mathrm{Cnv}``\nu = \nu -_p \mid \mathrm{Cnv}``\mu$ [*312·201·21]

***312·22.** $\vdash : \text{Infin ax} . \nu (\Theta \cup \breve{\Theta}_n) \mu . \supset . \mu -_p \nu \in C`\Theta$

Dem.

$\vdash . *311·65 . *312·12 . \supset \vdash : \mathrm{Hp} . \nu \Theta \mu . \supset . \mu -_p \nu \in C`\Theta$ (1)

$\vdash . *311·66 . *312·15 . \supset \vdash : \mathrm{Hp} . \mu \Theta_n \nu . \supset . \mid \mathrm{Cnv}``(\mu -_p \nu) \in C`\Theta_n .$

$[*310·16] \qquad \supset . \mu -_p \nu \in C`\Theta$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***312·23.** $\vdash : \text{Infin ax} . \mu (\Theta \cup \breve{\Theta}_n) \nu . \supset . \mu -_p \nu \in C`\Theta_n$ [*312·21·22 . *310·16]

***312·3.** $\vdash . \mu +_a \nu = (\mu +_p \nu) \cup (\mu -_p \mid \mathrm{Cnv}``\nu)$ [(*312·02)]

***312·31.** $\vdash : \sim (\mu, \nu \in C`\Theta_g) . \supset . \mu +_a \nu = \Lambda$ [*312·3·11 . *311·1]

***312·32.** $\vdash : \text{concord} (\mu, \nu) . \supset . \mu +_a \nu = \mu +_p \nu$ [*312·3·11 . *311·15]

***312·33.** $\vdash : \sim \text{concord} (\mu, \nu) . \supset . \mu +_a \nu = \mu -_p \mid \mathrm{Cnv}``\nu$ [*312·3 . *311·1]

***312·34.** $\vdash : \text{Infin ax} . \mu, \nu \in C`\Theta_g . \supset . \mu +_a \nu \in C`\Theta_g$
[*312·32·33·22·23 . *311·44]

***312·41.** $\vdash . \mu +_a \nu = \nu +_a \mu$

Dem.

$\vdash . *312·32 . *311·12 . \supset \vdash : \text{concord} (\mu, \nu) . \supset . \mu +_a \nu = \nu +_a \mu$ (1)

$\vdash . *312·33·21 . \supset \vdash : \sim \text{concord} (\mu, \nu) . \supset . \mu +_a \nu = \mid \mathrm{Cnv}``(\mid \mathrm{Cnv}``\nu -_p \mu)$

$[*312·201] \qquad = \nu -_p \mid \mathrm{Cnv}``\mu$

$[*312·33] \qquad = \nu +_a \mu$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***312·42.** $\vdash : \text{Infin ax} . \text{concord} (\lambda, \mu, \nu) . \supset . (\lambda +_p \mu) -_p (\lambda +_p \nu) = \mu -_p \nu$

Dem.

$\vdash . *311·27·42·43 . \supset \vdash :. \mathrm{Hp} . \supset : \text{concord} (\lambda +_p \mu, \lambda +_p \nu, \lambda, \mu, \nu) :$

$[*311·75] \qquad \supset : \lambda +_p \rho = \mu . \equiv . (\lambda +_p \rho) +_p \nu = \mu +_p \nu .$

$[*311·12·121] \qquad \equiv . (\lambda +_p \nu) +_p \rho = \mu +_p \nu$ (1)

Similarly $\vdash :. \mathrm{Hp} . \supset : \mu +_p \rho = \lambda . \equiv . (\mu +_p \nu) +_p \rho = \lambda +_p \nu$ (2)

$\vdash . (1) . (2) . *312·1 . \supset \vdash . \mathrm{Prop}$

***312·43.** $\vdash :$ Infin ax . concord $(\lambda, \mu, \nu) . \nu (\Theta \cup \Theta_n) \mu . \supset .$

$$(\lambda +_p \mu) -_p \nu = \lambda +_p (\mu -_p \nu)$$

Dem.

$\vdash . *311·65·66 . \supset \vdash : \text{Hp} . \supset . (\exists \rho) . \rho \in C`\Theta_g . \mu = \nu +_p \rho .$

$[*312·12·13·19] \supset . (\exists \rho) . \rho \in C`\Theta_g . (\lambda +_p \mu) -_p \nu = \lambda +_p \rho . \mu -_p \nu = \rho : \supset \vdash . \text{Prop}$

***312·44.** $\vdash :$ Infin ax . concord $(\lambda, \mu, \nu) . \mu (\Theta \cup \Theta_n) \nu . \supset .$

$$(\lambda +_p \mu) -_p \nu = \lambda -_p (\nu -_p \mu)$$

Dem.

$\vdash . *311·65·66 . \supset \vdash : \text{Hp} . \supset . (\exists \rho) . \rho \in C`\Theta_g . \nu = \mu +_p \rho .$

$[*312·42·19] \supset . (\exists \rho) . \rho \in C`\Theta_g . (\lambda +_p \mu) -_p \nu = \lambda -_p \rho . \rho = \nu -_p \mu : \supset \vdash . \text{Prop}$

***312·45.** $\vdash :$ Infin ax . concord $(\lambda, \mu) . \supset . (\lambda +_p \mu) -_p \mu = \lambda +_p (\mu -_p \mu)$

Dem.

$\vdash . *312·19 . *311·43 . \supset \vdash : \text{Hp} . \supset . \mu -_p \mu = \iota`0_q .$

$[*311·43] \qquad \supset . \lambda +_p (\mu -_p \mu) = \lambda$

$[*312·19] \qquad = (\lambda +_p \mu) -_p \mu : \supset \vdash . \text{Prop}$

***312·451.** $\vdash :$ Infin ax . concord $(\lambda, \mu, \nu) . \supset .$

$$(\lambda +_p \mu) -_p \nu = (\lambda +_a \mu) +_a | \text{Cnv}``\nu = \lambda +_a (\mu +_a | \text{Cnv}``\nu)$$

Dem.

$\vdash . *312·43 . \supset \vdash : \text{Hp} . \nu (\Theta \cup \Theta_n) \mu . \supset . (\lambda +_p \mu) -_p \nu = \lambda +_p (\mu -_p \nu)$

$[*312·33] \qquad = \lambda +_p (\mu +_a | \text{Cnv}``\nu)$

$[*312·32·12·14] \qquad = \lambda +_a (\mu +_a | \text{Cnv}``\nu) \quad (1)$

$\vdash . *312·44 . \supset \vdash : \text{Hp} . \mu (\Theta \cup \Theta_n) \nu . \supset . (\lambda +_p \mu) -_p \nu = \lambda -_p (\nu -_p \mu)$

$[*312·21] \qquad = \lambda -_p | \text{Cnv}``(\mu -_p \nu)$

$[*312·33·12·14] \qquad = \lambda +_a (\mu -_p \nu)$

$[*312·33] \qquad = \lambda +_a (\mu +_a | \text{Cnv}``\nu) \quad (2)$

$\vdash . *312·45 . \supset \vdash : \text{Hp} . \mu = \nu . \supset . (\lambda +_p \mu) -_p \nu = \lambda +_p (\mu -_p \nu)$

$[*312·33·32] \qquad = \lambda +_a (\mu +_a | \text{Cnv}``\nu) \quad (3)$

$\vdash . (1) . (2) . (3) . *312·32 . *311·43 . \supset \vdash . \text{Prop}$

***312·46.** $\vdash :$ Infin ax . concord $(\lambda, \mu) . \supset . (\lambda +_a \mu) +_a \nu = \lambda +_a (\mu +_a \nu)$

Dem.

$\vdash . *312·32 . *311·65·66·43 . \supset \vdash : \text{Hp} . \text{concord} (\lambda, \mu, \nu) . \supset .$

$$(\lambda +_a \mu) +_a \nu = (\lambda +_p \mu) +_p \nu . \lambda +_a (\mu +_a \nu) = \lambda +_p (\mu +_p \nu) \quad (1)$$

$\vdash . *312·451 . \supset$

$\vdash : \text{Hp} . \text{concord} (\lambda, \mu, | \text{Cnv}``\nu) . \supset . (\lambda +_a \mu) +_a \nu = \lambda +_a (\mu +_a \nu) \quad (2)$

$\vdash . *312·31 . \supset \vdash : \nu \sim \in C`\Theta_g . \supset . (\lambda +_a \mu) +_a \nu = \Lambda . \lambda +_a (\mu +_a \nu) = \Lambda \quad (3)$

$\vdash . (1) . (2) . (3) . *311·121 . \supset \vdash . \text{Prop}$

***312·461.** $\vdash$: Infin ax . concord (μ, ν) . $\supset$. $(\lambda +_a \mu) +_a \nu = \lambda +_a (\mu +_a \nu)$

Dem.

$\vdash$. *312·46 . $\supset \vdash$: Hp . $\supset$. $(\nu +_a \mu) +_a \lambda = \nu +_a (\mu +_a \lambda)$ (1)

$\vdash$. (1) . *312·41 . $\supset \vdash$. Prop

***312·47.** $\vdash$: Infin ax . concord (λ, ν) . $\supset$. $(\lambda +_a \mu) +_a \nu = \lambda +_a (\mu +_a \nu)$

Dem.

$\vdash$. *312·461 . $\supset \vdash$: Hp . $\supset$. $(\mu +_a \lambda) +_a \nu = \mu +_a (\lambda +_a \nu)$.

[*312·41] $\supset$. $(\lambda +_a \mu) +_a \nu = \mu +_a (\lambda +_a \nu)$

[*312·41] $= (\lambda +_a \nu) +_a \mu$

[*312·46] $= \lambda +_a (\nu +_a \mu)$

[*312·41] $= \lambda +_a (\mu +_a \nu)$: $\supset \vdash$. Prop

***312·48.** $\vdash$: Infin ax . $\supset$. $(\lambda +_a \mu) +_a \nu = \lambda +_a (\mu +_a \nu)$

Dem.

$\vdash$. *312·31 . $\supset$

$\vdash$: $\sim \{\lambda, \mu, \nu \in C\text{‘}\Theta_g\}$. $\supset$. $(\lambda +_a \mu) +_a \nu = \Lambda$. $\lambda +_a (\mu +_a \nu) = \Lambda$ (1)

$\vdash$. *310·12 . $\supset \vdash$:. $\lambda, \mu, \nu \in C\text{‘}\Theta_g$. $\supset$: concord (λ, μ) . v . concord (λ, ν) :

[*312·46·47] $\supset$: $(\lambda +_a \mu) +_a \nu = \lambda +_a (\mu +_a \nu)$ (2)

$\vdash$. (1) . (2) . $\supset \vdash$. Prop

***312·51.** $\vdash$: $\lambda \in C\text{‘}\Theta_g$. $\supset$. $\lambda +_a \iota\text{‘}0_q = \lambda$ [*312·32 . *311·43]

***312·52.** $\vdash$: Infin ax . $\lambda \in C\text{‘}\Theta_g$. $\supset$. $\lambda +_a |\,\mathrm{Cnv}\text{“}\lambda = \iota\text{‘}0_q$

Dem.

$\vdash$. *312·33 . $\supset \vdash$: Hp . $\supset$. $\lambda +_a |\,\mathrm{Cnv}\text{“}\lambda = \lambda -_p \lambda$

[*312·191] $= \iota\text{‘}0_q$: $\supset \vdash$. Prop

***312·53.** $\vdash$:. Infin ax . $\lambda, \mu, \nu \in C\text{‘}\Theta_g$. $\supset$: $\lambda +_a \mu = \nu$. $\equiv$. $\lambda = \nu +_a |\,\mathrm{Cnv}\text{“}\mu$
[*312·48·51·52]

***312·54.** $\vdash$: Infin ax . $\lambda, \mu \in C\text{‘}\Theta_g$. $\supset$. $(\exists \sigma)$. $\sigma \in C\text{‘}\Theta_g$. $\lambda +_a \sigma = \mu$

Dem.

$\vdash$. *312·48·51·52 . $\supset \vdash$: Hp . $\supset$. $\lambda +_a (|\,\mathrm{Cnv}\text{“}\lambda +_a \mu) = \mu$ (1)

$\vdash$. *312·34 . $\supset \vdash$: Hp . $\supset$. $|\,\mathrm{Cnv}\text{“}\lambda +_a \mu \in C\text{‘}\Theta_g$ (2)

$\vdash$. (1) . (2) . $\supset \vdash$. Prop

***312·55.** $\vdash$:. Infin ax . $\lambda, \mu, \nu \in C\text{‘}\Theta_g$. $\supset$: $\lambda +_a \mu = \lambda +_a \nu$. $\equiv$. $\mu = \nu$

Dem.

$\vdash$. *312·41·53 . $\supset \vdash$:. Hp . $\supset$: $\lambda +_a \mu = \lambda +_a \nu$. $\equiv$. $\mu = (\lambda +_a \nu) +_a |\,\mathrm{Cnv}\text{“}\lambda$.

[*312·41·48] $\equiv$. $\mu = \nu +_a (\lambda +_a |\,\mathrm{Cnv}\text{“}\lambda)$.

[*312·51·52] $\equiv$. $\mu = \nu$:. $\supset \vdash$. Prop

***312·56.** $\vdash :. \text{Infin ax} . \text{concord}(\lambda, \mu) . \supset : \lambda \Theta_g \mu . \equiv . (\exists \sigma) . \sigma \in C\text{'}\Theta . \lambda +_a \sigma = \mu$

Dem.

$\vdash . *311{\cdot}65 . *312{\cdot}32 . \supset$

$\vdash :. \text{Hp} . \lambda, \mu \in C\text{'}\Theta . \supset : \lambda \Theta_g \mu . \equiv . (\exists \sigma) . \sigma \in C\text{'}\Theta . \lambda +_a \sigma = \mu$ (1)

$\vdash . *311{\cdot}66 . *310{\cdot}16 . \supset$

$\vdash :. \text{Hp} . \lambda, \mu \in C\text{'}\Theta_n . \supset : \lambda \Theta_g \mu . \equiv . (\exists \sigma) . \sigma \in C\text{'}\Theta . \mu +_p \mid \text{Cnv}\text{''}\sigma = \lambda .$

$[*312{\cdot}53{\cdot}32] \qquad \equiv . (\exists \sigma) . \sigma \in C\text{'}\Theta . \lambda +_a \sigma = \mu$ (2)

$\vdash . *312{\cdot}51 . \quad \supset \vdash :. \text{Hp} . \lambda = \iota\text{'}0_q . \supset : \lambda \Theta_g \mu . \equiv . (\exists \sigma) . \sigma \in C\text{'}\Theta . \lambda +_a \sigma = \mu$ (3)

$\vdash . *312{\cdot}53{\cdot}51 . \supset \vdash :. \text{Hp} . \mu = \iota\text{'}0_q . \supset : \lambda \Theta_g \mu . \equiv . (\exists \sigma) . \sigma \in C\text{'}\Theta . \lambda +_a \sigma = \mu$ (4)

$\vdash . (1) . (2) . (3) . (4) . \supset \vdash . \text{Prop}$

***312·57.** $\vdash :. \text{Infin ax} . \lambda, \mu \in C\text{'}\Theta_g . \sim \text{concord}(\lambda, \mu) . \supset :$

$\lambda \Theta_g \mu . \equiv . (\exists \sigma) . \sigma \in C\text{'}\Theta . \lambda +_a \sigma = \mu$

Dem.

$\vdash . *312{\cdot}48{\cdot}51{\cdot}52 . \quad \supset \vdash : \lambda \in C\text{'}\Theta_n . \mu \in C\text{'}\Theta . \supset . \mu = \lambda +_a (\mid \text{Cnv}\text{''}\lambda +_a \mu)$ (1)

$\vdash . *312{\cdot}32 . *311{\cdot}27 . \supset \vdash : \text{Hp}(1) . \supset . (\mid \text{Cnv}\text{''}\lambda +_a \mu) \in C\text{'}\Theta$ (2)

$\vdash . (1) . (2) . \quad \supset \vdash : \lambda \in C\text{'}\Theta_n . \mu \in C\text{'}\Theta . \supset . (\exists \sigma) . \sigma \in C\text{'}\Theta . \lambda +_a \sigma = \mu$ (3)

$\vdash . *312{\cdot}32 . *311{\cdot}27 . *310{\cdot}13 . \supset$

$\vdash : \lambda \in C\text{'}\Theta . \mu \in C\text{'}\Theta_n . \supset . \sim (\exists \sigma) . \sigma \in C\text{'}\Theta . \lambda +_a \sigma = \mu$ (4)

$\vdash . (3) . (4) . \supset$

$\vdash :. \text{Hp} . \supset : \lambda \in C\text{'}\Theta_n . \mu \in C\text{'}\Theta . \equiv . (\exists \sigma) . \sigma \in C\text{'}\Theta . \lambda +_a \sigma = \mu :. \supset \vdash . \text{Prop}$

***312·58.** $\vdash :. \text{Infin ax} . \lambda, \mu \in C\text{'}\Theta_g . \supset :$

$\lambda \Theta_g \mu . \equiv . (\exists \sigma) . \sigma \in C\text{'}\Theta . \lambda +_a \sigma = \mu \quad [*312{\cdot}56{\cdot}57]$

$*313$. MULTIPLICATION OF REAL NUMBERS.

Summary of $*313$.

Multiplication of real numbers is simpler than addition, because it is not necessary to distinguish between factors of the same sign and factors of opposite signs. Thus we put

$*313{\cdot}01$. $\mu \times_a \nu = \hat{X}\{\mu, \nu \in C`\Theta_g \,.\, X \in s`\mu \underset{\prime\prime}{\times_g}{}``\nu\}$ Df

Thus if μ, ν are real numbers, their product is the class of products (in the sense of $*309$) of members of μ and members of ν; otherwise their product is Λ. The propositions of this number are analogous to those of previous numbers, and the proofs are as a rule analogous to those of $*311$, except in the case of the distributive law ($*313{\cdot}55$).

$*313{\cdot}01$. $\mu \times_a \nu = \hat{X}\{\mu, \nu \in C`\Theta_g \,.\, X \in s`\mu \underset{\prime\prime}{\times_g}{}``\nu\}$ Df

Proofs in this number are mostly analogous to those for addition, and are therefore often omitted.

$*313{\cdot}11$. $\vdash : \sim(\mu, \nu \in C`\Theta_g) \,.\, \supset \,.\, \mu \times_a \nu = \Lambda$

$*313{\cdot}12$. $\vdash : \mu, \nu \in C`\Theta_g \,.\, \supset \,.\, \mu \times_a \nu = s`\mu \underset{\prime\prime}{\times_g}{}``\nu$

$*313{\cdot}21$. $\vdash : \mu, \nu \in C`\Theta \cup \iota`\iota`0_q \,.\, \supset \,.\, \mu \times_a \nu = s`\mu \underset{\prime\prime}{\times_s}{}``\nu$

$*313{\cdot}22$. $\vdash : \mu, \nu \in C`\Theta_n \cup \iota`\iota`0_q \,.\, \supset \,.\, \mu \times_a \nu = s`(|\,\mathrm{Cnv}``\mu) \underset{\prime\prime}{\times_s}{}``(|\,\mathrm{Cnv}``\nu)$

$*313{\cdot}23$. $\vdash : \mu \in C`\Theta_n \,.\, \nu \in C`\Theta \,.\, \supset \,.\, \mu \times_a \nu = |\,\mathrm{Cnv}``s`(|\,\mathrm{Cnv}``\mu) \underset{\prime\prime}{\times_s}{}``\nu$

$*313{\cdot}24$. $\vdash : \mu \in C`\Theta \,.\, \nu \in C`\Theta_n \,.\, \supset \,.\, \mu \times_a \nu = |\,\mathrm{Cnv}``s`(\mu \underset{\prime\prime}{\times_s})``|\,\mathrm{Cnv}``\nu$

$*313{\cdot}25$. $\vdash \,.\, \mu \times_a \nu = |\,\mathrm{Cnv}``(|\,\mathrm{Cnv}``\mu \times_a \nu) = |\,\mathrm{Cnv}``\mu \times_a |\,\mathrm{Cnv}``\nu$

$*313{\cdot}26$. $\vdash \,.\, \mu \times_a |\,\mathrm{Cnv}``\nu = |\,\mathrm{Cnv}``\mu \times_a \nu = |\,\mathrm{Cnv}``(\mu \times_a \nu)$

$*313{\cdot}31$. $\vdash : \mathrm{Infin\ ax} \,.\, \xi \in C`\Theta \,.\, X \in C`H \,.\, \supset \,.\, X \times_g{}``\xi \subset H``X \times_g{}``\xi$

$*313{\cdot}32$. $\vdash : \mathrm{Infin\ ax} \,.\, \xi \in C`\Theta \,.\, X \in C`H \,.\, \supset \,.\, X \times_g{}``\xi = H``X \times_g{}``\xi$

$*313{\cdot}33$. $\vdash : \mathrm{Infin\ ax} \,.\, \xi \in C`\Theta \,.\, X \in C`H \,.\, \supset \,.\, X \times_g{}``\xi \in C`\Theta$

$*313{\cdot}34$. $\vdash : \mathrm{Infin\ ax} \,.\, \xi \in C`\Theta_n \,.\, X \in C`H_n \,.\, \supset \,.\, X \times_g{}``\xi \in C`\Theta$

***313·35.** $\vdash :$ Infin ax $. \xi \in C`\Theta . X \in C`H_n . \supset . X \times_g `` \xi \in C`\Theta_n$

***313·351.** $\vdash :$ Infin ax $. \xi \in C`\Theta_n . X \in C`H . \supset . X \times_g `` \xi \in C`\Theta_n$

***313·36.** $\vdash : \xi \in C`\Theta_g . \supset . 0_q \times_g `` \xi = \iota`0_q$

***313·37.** $\vdash : X \in C`H_g . \supset . X \times_g `` \iota`0_q = \iota`0_q$

***313·38.** $\vdash :$ Infin ax $. \xi \in C`\Theta_g . X \in C`H_g . \supset . X \times_g `` \xi \in C`\Theta_g$

***313·41.** $\vdash :$ Infin ax . concord $(\mu, \nu) . \mu \neq \iota`0_q . \nu \neq \iota`0_q . \supset . \mu \times_a \nu \in C`\Theta$

***313·42.** $\vdash :$ Infin ax $. \sim$ concord $(\mu, \nu) . \mu, \nu \in C`\Theta_g . \supset . \mu \times_a \nu \in C`\Theta_n$

***313·43.** $\vdash :. \mu = \iota`0_q . \vee . \nu = \iota`0_q : \mu, \nu \in C`\Theta_g : \supset . \mu \times_a \nu = \iota`0_q$

***313·44.** $\vdash :$ Infin ax $. \mu, \nu \in C`\Theta_g . \supset . \mu \times_a \nu \in C`\Theta_g$

***313·45.** $\vdash . \mu \times_a \nu = \nu \times_a \mu$

***313·46.** $\vdash :$ Infin ax $. \supset . (\lambda \times_a \mu) \times_a \nu = \lambda \times_a (\mu \times_a \nu)$

The following propositions are concerned with the proof of the distributive law.

***313·51.** $\vdash :$ Infin ax . concord $(\lambda, \mu, \nu) . \supset . (\nu \times_a \lambda) +_a (\nu \times_a \mu) =$

$\hat{M}[(\exists X, Y, Z, Z') . X \in \lambda . Y \in \mu . Z, Z' \in \nu . M = (Z \times_g X) +_g (Z' \times_g Y)]$

[*313·12 . *312·32 . *311·11 . *313·41]

***313·511.** $\vdash :$ Infin ax $. \lambda, \mu \in C`\Theta . Z, Z' \in \mu . Z H Z' . X \in \lambda . \supset . Z \times_g \breve{Z}' \times_g X \in \lambda$

Dem.

$\vdash .$ *304·1·401 . *305·14 $. \supset \vdash :$ Hp $. \supset . (Z \times_g X) H (Z' \times_g X) .$

[*309·41] $\supset . (Z \times_g \breve{Z}' \times_g X) H X .$

[*311·58] $\supset . Z \times_g \breve{Z}' \times_g X \in \lambda : \supset \vdash .$ Prop

***313·52.** $\vdash :$ Infin ax . concord $(\lambda, \mu, \nu) . \supset . (\nu \times_a \lambda) +_a (\nu \times_a \mu) = \nu \times_a (\lambda +_a \mu)$

Dem.

$\vdash .$ *313·51·511 $. \supset \vdash :$ Hp $. \supset .$

$(\nu \times_a \lambda) +_a (\nu \times_a \mu) = \hat{M}[(\exists X, Y, Z) . X \in \lambda . Y \in \mu . Z \in \nu . M = (Z \times_g X) +_g (Z \times_g Y)]$

[*309·37] $= \hat{M}[(\exists X, Y, Z) . X \in \lambda . Y \in \mu . Z \in \nu . M = Z \times_g (X +_g Y)]$

[*313·12.*312·32.*311·11] $= \nu \times_a (\lambda +_a \mu) : \supset \vdash .$ Prop

***313·53.** $\vdash :$ Infin ax . concord $(\lambda, \mu) . \sim$ concord $(\lambda, \nu) . \nu \in C`\Theta_g . \supset .$

$(\nu \times_a \lambda) +_a (\nu \times_a \mu) = \nu \times_a (\lambda +_a \mu)$

Dem.

$\vdash .$ *313·25 $. \supset \vdash . (\lambda +_a \mu) \times_a \nu = | \text{Cnv}``\{(\lambda +_a \mu) \times_a | \text{Cnv}``\nu\}$ (1)

$\vdash .$ *313·52 $. \supset \vdash :$ Hp $. \supset . (\lambda +_a \mu) \times_a | \text{Cnv}``\nu = (\lambda \times_a | \text{Cnv}``\nu) +_a (\mu \times_a | \text{Cnv}``\nu)$

[*313·26.*311·31] $= \text{Cnv}``\{(\lambda \times_a \nu) +_a (\mu \times_a \nu)\}$ (2)

$\vdash . (1) . (2) . \supset \vdash .$ Prop

***313·54.** $\vdash :$ Infin ax . concord (λ, ν) . $\sim$ concord (λ, μ) . $\mu \,\varepsilon\, C\text{‘}\Theta_g$. $\supset$.

$$\nu \times_a (\lambda +_a \mu) = (\nu \times_a \lambda) +_a (\nu \times_a \mu)$$

Dem.

$\vdash$. *312·33·34 . $\supset \vdash$:. Hp . $\lambda +_a \mu = \rho$. $\supset$: concord (λ, ρ) . v . concord (μ, ρ) (1)

$\vdash$. *313·52 . $\supset \vdash$: Hp (1) . concord (λ, ρ) . $\supset$.

$$(\rho \times_a \nu) +_a (\mid \mathrm{Cnv}\text{“}\mu \times_a \nu) = (\rho +_a \mid \mathrm{Cnv}\text{“}\mu) \times_a \nu$$

[*312·53] $= \lambda \times_a \nu$.

[*312·53.*313·26] $\supset$. $\rho \times_a \nu = (\lambda \times_a \nu) +_a (\mu \times_a \nu)$ (2)

Similarly $\vdash$: Hp (1) . concord (μ, ρ) . $\supset$. $\rho \times_a \nu = (\lambda \times_a \nu) +_a (\mu \times_a \nu)$ (3)

$\vdash$. (1) . (2) . (3) . $\supset \vdash$. Prop

***313·55.** $\vdash$: Infin ax . $\supset$. $(\nu \times_a \lambda) +_a (\nu \times_a \mu) = \nu \times_a (\lambda +_a \mu)$
[*313·52·53·54·11 . *312·31]

$*314$. REAL NUMBERS AS RELATIONS.

Summary of $*314$.

In this number we take up the definition of real numbers suggested in $*310$, namely $\dot{s}``C`\Theta_g$ instead of $C`\Theta_g$. The series of real numbers is now $\dot{s};\Theta_g$ instead of Θ_g. Everything in this number depends upon

$*310\cdot32$. $\vdash :. \mu, \nu \in C`\Theta_g . \supset : \dot{s}`\mu = \dot{s}`\nu . \equiv . \mu = \nu$

In consequence of this proposition, $\dot{s} \upharpoonright C`\Theta_g$ is a correlation of the two sorts of real numbers, and the properties of the relational sort can be immediately deduced from the propositions of previous numbers. We define addition and multiplication of relational real numbers so as to secure that, if μ, ν are real numbers of our previous sort, the arithmetical sum of $\dot{s}`\mu$ and $\dot{s}`\nu$ is $\dot{s}`(\mu +_a \nu)$ and their product is $\dot{s}`(\mu \times_a \nu)$. This is effected by putting

$*314\cdot01$. $X +_r Y = \hat{R}\hat{S}[(\exists\mu, \nu) . X = \dot{s}`\mu . Y = \dot{s}`\nu . R\{\dot{s}`(\mu +_a \nu)\} S]$ Df

with a similar definition for $X \times_r Y$. The zero of real numbers is now 0_q instead of $\iota`0_q$, and the negative of a real number X is $X \mid \mathrm{Cnv}$. The fundamental propositions are

$*314\cdot13$. $\vdash : \mu, \nu \in C`\Theta_g . \supset . \dot{s}`\mu +_r \dot{s}`\nu = \dot{s}`(\mu +_a \nu)$

$*314\cdot14$. $\vdash : \mu, \nu \in C`\Theta_g . \supset . \dot{s}`\mu \times_r \dot{s}`\nu = \dot{s}`(\mu \times_a \nu)$

in virtue of which the arithmetical properties of relational real numbers follow at once from those of real numbers as segments.

Relational real numbers are useful in applying measurement by means of real numbers to vector-families, since it is convenient to have real numbers of the same type as ratios.

For some purposes, a somewhat different definition of real numbers as relations is more convenient. Instead of deriving our relations from Θ_g, we may derive them from $\varsigma`H_g$, *i.e.* we may consider the relations $\dot{s}``C`\varsigma`H_g$ instead of the relations $\dot{s}``C`\Theta_g$. In virtue of $*217\cdot43$, $(\varsigma`H_g) \upharpoonright (-\iota`\Lambda - \iota`C`H_g)$ is ordinally similar to Θ_g; hence the requisite properties of $\dot{s}``C`\varsigma`H_g$ follow at once.

***314·01.** $X +_r Y = \hat{R}\hat{S}\,[(\exists\mu, \nu) \,.\, X = \dot{s}`\mu \,.\, Y = \dot{s}`\nu \,.\, R\,\{\dot{s}`(\mu +_a \nu)\}\,S]$ Df

***314·02.** $X \times_r Y = \hat{R}\hat{S}\,[(\exists\mu, \nu) \,.\, X = \dot{s}`\mu \,.\, Y = \dot{s}`\nu \,.\, R\,\{\dot{s}`(\mu \times_a \nu)\}\,S]$ Df

***314·03.** $♂ = (\breve{H}_n)_\epsilon \,|\, (C`H_n -) \upharpoonright \{D`(H_n)_\epsilon - \iota`\Lambda - \iota`C`H_n\}$
$\cup\, (C`H_n) \downarrow (\iota`0_q) \cup (C`H_n \cup) \upharpoonright (D`H_\epsilon - \iota`\Lambda - \iota`C`H)$ Df

***314·04.** $M +_\sigma N = \hat{R}\hat{S}\,[(\exists\mu, \nu) \,.\, M = \dot{s}`♂`\mu \,.\, N = \dot{s}`♂`\nu \,.\, R\,\{\dot{s}`♂`(\mu +_a \nu)\}\,S]$ Df

***314·05.** $M \times_\sigma N = \hat{R}\hat{S}\,[(\exists\mu, \nu) \,.\, M = \dot{s}`♂`\mu \,.\, N = \dot{s}`♂`\nu \,.\, R\,\{\dot{s}`♂`(\mu \times_a \nu)\}\,S]$ Df

***314·1.** $\vdash :\, \dot{\exists}\,!\, X +_r Y \,.\, \supset .\, X, Y \in \dot{s}``C`\Theta_g$ [*312·31 . (*314·01)]

***314·11.** $\vdash :.\, \text{Infin ax} \,.\, \supset :\, \dot{\exists}\,!\, X +_r Y \,.\, \equiv .\, X, Y \in \dot{s}``C`\Theta_g$ [*314·1 . *312·34]

***314·12.** $\vdash :.\, \text{Infin ax} \,.\, \supset :\, \dot{\exists}\,!\, X \times_r Y \,.\, \equiv .\, X, Y \in \dot{s}``C`\Theta_g$

***314·13.** $\vdash :\, \mu, \nu \in C`\Theta_g \,.\, \supset .\, \dot{s}`\mu +_r \dot{s}`\nu = \dot{s}`(\mu +_a \nu)$

Dem.

$\vdash .\, *314·1 \,.\, (*314·01) \,.\, \supset \vdash :\, R\,\{\dot{s}`\mu +_r \dot{s}`\nu\}\,S \,.\, \equiv .$
$(\exists\rho, \sigma) \,.\, \rho, \sigma \in C`\Theta_g \,.\, \dot{s}`\mu = \dot{s}`\rho \,.\, \dot{s}`\nu = \dot{s}`\sigma \,.\, R\,\{\dot{s}`(\rho +_a \sigma)\}\,S$ (1)

$\vdash .\, (1) \,.\, *310·32 \,.\, \supset \vdash .\, \text{Prop}$

***314·14.** $\vdash :\, \mu, \nu \in C`\Theta_g \,.\, \supset .\, \dot{s}`\mu \times_r \dot{s}`\nu = \dot{s}`(\mu \times_a \nu)$

***314·2.** $\vdash :\, R \in \dot{s}``(C`\Theta_g - \iota`\iota`0_q) \,.\, \supset .\, \dot{\exists}\,!\, R \upharpoonright \text{Rel num}$ [*310·31]

***314·21.** $\vdash :.\, \text{Infin ax} \,.\, \supset :\, R, S \in \dot{s}``C`\Theta_g \,.\, \equiv .\, R +_r S \in \dot{s}``C`\Theta_g \,.$
$\equiv .\, R \times_r S \in \dot{s}``C`\Theta_g$

Dem.

$\vdash .\, *314·13·14 \,.\, *312·34 \,.\, *313·44 \,.\, \supset$
$\vdash :\, \text{Hp} \,.\, R, S \in \dot{s}``C`\Theta_g \,.\, \supset .\, R +_r S, R \times_r S \in \dot{s}``C`\Theta_g$ (1)
$\vdash .\, (1) \,.\, *314·11·12 \,.\, \supset \vdash .\, \text{Prop}$

***314·22.** $\vdash :\, R \in \dot{s}``C`\Theta_g \,.\, \supset .\, R +_r 0_q = R \,.\, R \times_r 0_q = 0_q$

Dem.

$\vdash .\, *314·13·14 \,.\, \supset$
$\vdash :\, \mu \in C`\Theta_g \,.\, \supset .\, \dot{s}`\mu +_r 0_q = \dot{s}`(\mu +_a \iota`0_q) \,.\, \dot{s}`\mu \times_r 0_q = \dot{s}`(\mu \times_a \iota`0_q) \,.$
$[*312·51 . *313·43]\, \supset .\, \dot{s}`\mu +_r 0_q = \dot{s}`\mu \,.\, \dot{s}`\mu \times_r 0_q = 0_q :\supset \vdash .\, \text{Prop}$

***314·23.** $\vdash :\, \text{Infin ax} \,.\, R \in \dot{s}``C`\Theta_g \,.\, \supset .\, R +_r R \,|\, \text{Cnv} = 0_q$

Dem.

$\vdash .\, *314·13 \,.\, \supset \vdash :\, \mu \in C`\Theta_g \,.\, \supset .\, \dot{s}`\mu +_r \dot{s}`|\,\text{Cnv}``\mu = \dot{s}`(\mu +_a |\,\text{Cnv}``\mu) \,.$
$[*43·421]$ $\supset .\, \dot{s}`\mu +_r (\dot{s}`\mu)|\,\text{Cnv} = \dot{s}`(\mu +_a |\,\text{Cnv}``\mu)$
$[*312·52]$ $= 0_q :\supset \vdash .\, \text{Prop}$

***314·24.** $\vdash . R +_r S = S +_r R$ [*312·41 . (*314·01)]

***314·25.** $\vdash . R \times_r S = S \times_r R$ [*313·45 . (*314·02)]

***314·26.** $\vdash :$ Infin ax $. \supset . (R +_r S) +_r T = R +_r (S +_r T)$

Dem.

$\vdash . *314{\cdot}13 . \supset \vdash : \text{Hp} . \rho, \sigma, \tau \in C`\Theta_g . R = \dot{s}`\rho . S = \dot{s}`\sigma . T = \dot{s}`\tau . \supset .$

$(R +_r S) +_r T = \dot{s}`\{(\rho +_a \sigma) +_a \tau\}$

[*312·48] $= \dot{s}`\{\rho +_a (\sigma +_a \tau)\}$

[*314·13] $= R +_r (S +_r T)$ (1)

$\vdash . *314{\cdot}11{\cdot}21 . \supset$

$\vdash : \sim(\exists \rho, \sigma, \tau) . \rho, \sigma, \tau \in C`\Theta_g . R = \dot{s}`\rho . S = \dot{s}`\sigma . T = \dot{s}`\tau . \supset .$

$(R +_r S) +_r T = \dot{\Lambda} . R +_r (S +_r T) = \dot{\Lambda}$ (2)

$\vdash . (1) . (2) . \supset \vdash .$ Prop

***314·27.** $\vdash :$ Infin ax $. \supset . (R \times_r S) \times_r T = R \times_r (S \times_r T)$
[*314·14 . *313·46 . *314·12·21]

***314·28.** $\vdash :$ Infin ax $. \supset . (R \times_r S) +_r (R \times_r T) = R \times_r (S +_r T)$

Dem.

$\vdash . *314{\cdot}13{\cdot}14 . \supset \vdash : \text{Hp} . \rho, \sigma, \tau \in C`\Theta_g . R = \dot{s}`\rho . S = \dot{s}`\sigma . T = \dot{s}`\tau . \supset .$

$(R \times_r S) +_r (R \times_r T) = \dot{s}`(\rho \times_a \sigma) +_r \dot{s}`(\rho \times_a \tau)$

[*314·21·13] $= \dot{s}`\{(\rho \times_a \sigma) +_a (\rho \times_a \tau)\}$

[*313·55] $= \dot{s}`\{\rho \times_a (\sigma +_a \tau)\}$

[*314·21·14] $= \dot{s}`\rho \times_r \dot{s}`(\sigma +_a \tau)$

[*314·13] $= \dot{s}`\rho \times_r (\dot{s}`\sigma +_r \dot{s}`\tau)$

[Hp] $= R \times_r (S +_r T)$ (1)

$\vdash . *314{\cdot}21{\cdot}11{\cdot}12 . \supset$

$\vdash . \sim(\exists \rho, \sigma, \tau) . \rho, \sigma, \tau \in C`\Theta_g . R = \dot{s}`\rho . S = \dot{s}`\sigma . T = \dot{s}`\tau . \supset .$

$(R \times_r S) +_r (R \times_r T) = \dot{\Lambda} . R \times_r (S +_r T) = \dot{\Lambda}$ (2)

$\vdash . (1) . (2) . \supset \vdash .$ Prop

***314·4.** $\vdash :$ Infin ax $. \supset . ♂ \in \{(\varsigma`H_g) \upharpoonleft (-\iota`\Lambda - \iota`C`H_g)\} \overline{\text{smor}}\ \Theta_g$
[*217·43 . *304·31·282·23 . *307·41·44·46·25 . (*310·01·011·02·03)]

***314·41.** $\vdash . \dot{s} \upharpoonright (C`\varsigma`H_g) \in 1 \to 1$ [The proof is analogous to that of *310·32]

***314·42.** $\vdash :$ Infin ax $. \supset . \dot{s}\,;♂\,;\Theta_g$ smor Θ_g [*314·4·41]

***314·5.** $\vdash :.$ Infin ax $. \supset :$

$\dot{\exists} ! M +_\sigma N . \equiv . \dot{\exists} ! M \times_\sigma N . \equiv . M, N \in \dot{s}``(\text{D}`\varsigma`H_g - \iota`\Lambda)$
[*312·34 . *313·44 . *314·42 . (*314·04·05)]

***314·51.** $\vdash :$ Infin ax $. \mu, \nu \in C`\Theta_g . \supset .$

$\dot{s}`♂`\mu +_\sigma \dot{s}`♂`\nu = \dot{s}`♂`(\mu +_a \nu) . \dot{s}`♂`\mu \times_\sigma \dot{s}`♂`\nu = \dot{s}`♂`(\mu \times_a \nu)$
[*314·42 . (*314·04·05)]

The properties of $M +_\sigma N$ and $M \times_\sigma N$ result from this proposition exactly as those of $X +_r Y$ and $X \times_r Y$ result from *314·13·14.

SECTION B.

VECTOR-FAMILIES.

Summary of Section B.

The present Section is concerned with the theory of magnitude, so far as this can be developed without measurement. Measurement—*i.e.* the application of ratios and real numbers to magnitudes—will be dealt with in Section C; for the present, we shall confine ourselves to those properties of magnitude which are presupposed in measurement. But throughout this Section, measurement is the goal: the hypotheses introduced and the propositions proved will be such as are relevant to the possibility of measurement.

We conceive a magnitude as a vector, *i.e.* as an operation, *i.e.* as a descriptive function in the sense of $*30$. Thus for example, we shall so define our terms that 1 gramme would not be a magnitude, but the difference between 2 grammes and 1 gramme would be a magnitude, *i.e.* the relation "$+1$ gramme" would be a magnitude. On the other hand a centimetre and a second will both be magnitudes according to our definition, because distances in space and time are vectors. It will be remembered that we defined ratios as relations between relations; hence if ratios are to hold between magnitudes, magnitudes must be taken as relations.

We demand of a vector (1) that it shall be a one-one relation, (2) that it shall be capable of indefinite repetition, *i.e.* that if the vector takes us from a to b, there shall always be a point c such that the vector takes us from b to c. If R is the vector, the point to which it takes us from a is $R'a$; thus the above requisite is expressed by "$\mathrm{E}\,!\,R'a \,.\supset_a.\, \mathrm{E}\,!\,R'R'a$," *i.e.* by "$\mathrm{D}'R \subset \text{ꓷ}'R$." It will be observed that the points which are starting-points of the vector form the class $\text{ꓷ}'R$, *i.e.* the class of possible arguments to R considered as a descriptive function, while the points which are the end-points of the vector form the class $\mathrm{D}'R$, *i.e.* the class of values of R considered as a descriptive function. Since $\mathrm{D}'R \subset \text{ꓷ}'R$, we have $\text{ꓷ}'R = C'R$; thus the field of the vector consists of all points from which the vector can start. By

assuming $D`R \subset \text{ɑ}`R$, we exclude magnitudes of kinds which have a definite maximum, unless they are circular, like the angles at a point or the distances on an elliptic straight line; but, except when they are circular, such magnitudes are of little importance.

According to what has just been said, if R is a vector whose field is α, we have

$$R \epsilon 1 \to 1 \,.\, \text{ɑ}`R = \alpha \,.\, D`R \subset \alpha.$$

A relation which fulfils this hypothesis is called a "correspondence" of α, because it makes a part of α correspond with α. The class of correspondences of α we denote by "cr$`\alpha$," which is the cardinal correlative of "cror$`P$," defined in *208. Thus we put

$$\text{cr}`\alpha = (1 \to 1) \cap \overleftarrow{\text{ɑ}}`\alpha \cap \breve{D}``\text{Cl}`\alpha \quad \text{Df.}$$

We proceed next to define a vector-family of α." This we define as an existent sub-class of cr$`\alpha$ such that, if R and S are any two members of it, $R \mid S = S \mid R$. We define a class of relations as "Abelian" when the relative product of any two members of the class is commutative, *i.e.* we put

$$\text{Abel} = \hat{\kappa}\,(R, S \epsilon \kappa \,.\, \supset_{R,S} \,.\, R \mid S = S \mid R) \quad \text{Df.}$$

Thus a vector-family of α is an existent Abelian sub-class of cr$`\alpha$, *i.e.* writing "fm$`\alpha$" for "vector-family of α," we put

$$\text{fm}`\alpha = \text{Abel} \cap \text{Cl ex}`\text{cr}`\alpha \quad \text{Df.}$$

The class of vector-families is then defined as everything which is a vector-family of some α, *i.e.* we put

$$FM = s`D`\text{fm} \quad \text{Df.}$$

Thus a vector-family is an existent Abelian class of one-one relations which all have the same converse domain, and all have their domains contained in this common converse domain. If κ is a vector-family, the common converse domain is $\breve{\iota}`\text{ɑ}``\kappa$, which is identical with $s`\text{ɑ}``\kappa$, and will be called the "field" of the family. Thus we have

$$\vdash : \kappa \epsilon FM \,.\, \equiv \,.\, \kappa \epsilon \text{Abel} \,.\, \kappa \subset 1 \to 1 \,.\, \text{ɑ}``\kappa \epsilon 1 \,.\, s`D``\kappa \subset s`\text{ɑ}``\kappa.$$

A vector-family may be regarded as a kind of magnitude. In order to render measurement possible, we require various hypotheses as to the nature of the family. Measurement within a given family κ is obtained by limiting the fields of ratios to κ, *i.e.* by considering $X \upharpoonright \kappa$ where X is a ratio, or $Z \upharpoonright \kappa$ where Z is a relational real number of the kind defined in *314. In order to make measurement possible, we wish κ to be such that, if X is a ratio, $X \upharpoonright \kappa$ shall be one-one; again, if R, S, T are members of κ, and R has the ratio X to S, while S has the ratio Y to T, we wish R to have the ratio $X \times_s Y$ to T, *i.e.* we wish to have

$$X \upharpoonright \kappa \mid Y \upharpoonright \kappa \subseteq (X \times_s Y) \upharpoonright \kappa\,;$$

again, if R has the ratio X to T, and S has the ratio Y to T, we wish $R \mid S$ (which represents the "sum" of R and S) to have the ratio $X +_s Y$ to T, *i.e.* we wish to have

$$(X \lceil \kappa`T) \mid (Y \lceil \kappa`T) \subset (X +_s Y) \lceil \kappa`T.$$

The above and other similar properties will be proved, with suitable hypotheses, in Section C; for the present, we shall proceed with the theory of vector-families without explicit regard to measurement.

The first and most important hypothesis as to a family which we consider is the hypothesis that it is "connected," *i.e.* that there is at least one member of its field from which we can reach any member of the field by a vector belonging to the family or by the converse of a vector belonging to the family. Such a member of the field of κ we shall call a "connected point" of κ; the class of such points will be denoted by "conx$`\kappa$"; the definition is

$$\mathrm{conx}`\kappa = s`\mathrm{C}``\kappa \cap \hat{a}\,(\overrightarrow{\dot{s}`\kappa}`a \cup \overleftarrow{\dot{s}`\kappa}`a = s`\mathrm{C}``\kappa) \quad \mathrm{Df.}$$

It will be observed that $\overrightarrow{\dot{s}`\kappa}`a$ are the points to which there is a vector from a, while $\overleftarrow{\dot{s}`\kappa}`a$ are the points from which there is a vector to a. The definition states that these two classes together make up the whole field of the family. We define a connected family as one which has at least one connected point, *i.e.* we put

$$FM\,\mathrm{conx} = FM \cap \hat{\kappa}\,(\exists\,!\,\mathrm{conx}`\kappa) \quad \mathrm{Df.}$$

The properties of connected families are many and important. Among these may be mentioned the following: If κ is a connected family, the logical product of any two different members of κ is null, *i.e.* if $P, Q \in \kappa \,.\, P \neq Q$, then $P \dot{\cap} Q = \dot{\Lambda}$, or, what comes to the same thing, if $P, Q \in \kappa$, and if we ever have $P`x = Q`x$, then $P = Q$; if $P \in \kappa$, all the powers of P are either members of κ or the converses of members; if $P, Q \in \kappa$, then $P \mid Q$ is either a member of κ or the converse of a member. A connected family may not form a group, *i.e.* we do not necessarily have

$$P, Q \in \kappa \,.\, \supset_{P,Q} .\, P \mid Q \in \kappa,$$

but we shall show at a later stage (*354) that a group can be derived from a connected family κ by merely adding to it the converses of those members of κ (if any) whose domains are equal to their converse domains. The result of this addition is to give us a connected family which is a group.

Another important property of a connected family κ is that $I \upharpoonright s`\mathrm{C}``\kappa$ is always a member of it. $I \upharpoonright s`\mathrm{C}``\kappa$ is the zero vector. In a connected family, every vector except $I \upharpoonright s`\mathrm{C}``\kappa$ is contained in diversity. For many purposes, the class of vectors excluding $I \upharpoonright s`\mathrm{C}``\kappa$ is important. We therefore put

$$\kappa_\partial = \kappa - \mathrm{Rl}`I \quad \mathrm{Df.}$$

In the study of a vector-family κ, an important derived class of relations is the class of all relations of the form $\breve{R} | S$, where $R, S \epsilon \kappa$. The operation $\breve{R} | S$ consists of an S-step forward, followed by an R-step backward; that is to say, if $\breve{R}‘S‘a$ exists, it is obtained by moving a distance S forward from a to $S‘a$, and then a distance R backward from $S‘a$ to $\breve{R}‘S‘a$. The class of such relations as $\breve{R} | S$, where $R, S \epsilon \kappa$, we call κ_ι; *i.e.* we put

$$\kappa_\iota = s‘(\mathrm{Cnv}‘‘\kappa) |_{,,}‘‘\kappa \quad \mathrm{Df.}$$

The class κ_ι will have different properties according to the nature of κ. We may distinguish three cases:

(1) The field of κ may have a first term, *i.e.* there may be a member of $s‘\mathrm{Cl}‘‘\kappa$ which is not a member of $s‘\mathrm{D}‘‘\kappa_\partial$. This case is illustrated, *e.g.* by a family of distances from left to right on the portion of a given line not lying to the left of a given point. This given point will then belong to $s‘\mathrm{Cl}‘‘\kappa$, since there are vectors which start from it, but it will not belong to $s‘\mathrm{D}‘‘\kappa_\partial$, since there are no vectors which end at it except the zero vector. A connected point a which belongs to $s‘\mathrm{Cl}‘‘\kappa$ but not to $s‘\mathrm{D}‘‘\kappa_\partial$ is called the "initial" point, and a family which has an initial point is called an "initial" family. A family cannot have more than one initial point. Thus we put

$$\mathrm{init}‘\kappa = \breve{\iota}‘(\mathrm{conx}‘\kappa - s‘\mathrm{D}‘‘\kappa_\partial) \quad \mathrm{Df,}$$

$$FM\,\mathrm{init} = FM \cap \mathrm{Cl}‘\mathrm{init} \quad \mathrm{Df.}$$

(2) It may happen that, even if κ is not an initial family, none of the converses of members of κ_∂ are members of κ. (If κ is an initial family, this must happen.) This case is illustrated by the case of all distances towards the right on a straight line. It is also illustrated by the family of vectors of the form $(+_s X) \upharpoonright C‘H$, where $X \epsilon C‘H'$. In this case, as in (1), it is possible, by adding suitable hypotheses, to secure that $\dot{s}‘\kappa_\partial$ shall be a series. This case divides into two, which are illustrated by the above two instances: it may happen, as in our first instance, that the domain of a vector is always equal to its converse domain, *i.e.* $\mathrm{D}‘‘\kappa = \mathrm{Cl}‘‘\kappa$; or it may happen, as in our second instance, that the domain is only part of the converse domain. (The domain of $(+_s X) \upharpoonright C‘H$ consists of all ratios greater than X.)

(3) It may happen that κ_∂ contains pairs of vectors which are each other's converses. In this case, it is obvious that $\dot{s}‘\kappa_\partial$ cannot be serial, since $R, \breve{R} \epsilon \kappa_\partial . \supset . \breve{R} | R = I \upharpoonright s‘\mathrm{Cl}‘‘\kappa . \breve{R} | R \subset (\dot{s}‘\kappa_\partial)^2$, so that $(\dot{s}‘\kappa_\partial)^2$ is not contained in diversity (except in the trivial case $\kappa = \iota‘\dot{\Lambda}$).

In considering κ_ι, we do not at first explicitly introduce any of the above possibilities, but it is necessary to bear them in mind in order to realize the

purpose of the propositions proved concerning κ_ι. If L is a member of κ_ι, and $L = \breve{R} | S$, where $R, S \epsilon \kappa$, then if a is a connected point, and $L'a$ exists, it follows that there is a member T of $\kappa \cup \text{Cnv}``\kappa$ such that $L'a = T'a$. It is easy to deduce from this that $L = T$, hence $L \epsilon \kappa \cup \text{Cnv}``\kappa$. The same holds if $\breve{L}'a$ exists. Hence if $\text{E} ! L'a . \vee . \text{E} ! \breve{L}'a$, *i.e.* if $a \epsilon C'L$, L is a member of $\kappa \cup \text{Cnv}``\kappa$. Thus if $a$ belongs to the field of every member of $\kappa_\iota$, we shall have $\kappa_\iota = \kappa \cup \text{Cnv}``\kappa$. We say that a family "has connexity" (not to be confounded with "being connected") if $\exists ! \text{conx}'\kappa \cap p'C``\kappa_\iota$; thus we put

$$FM\,\text{connex} = FM \cap \hat{\kappa}\,(\exists ! \text{conx}'\kappa \cap p'C``\kappa_\iota) \quad \text{Df}$$

and by what has just been said we have

$$\vdash : \kappa \epsilon FM\,\text{connex} . \supset . \kappa_\iota = \kappa \cup \text{Cnv}``\kappa.$$

We also have $\vdash : \kappa \epsilon FM\,\text{connex} . \supset . \dot{s}'\kappa_\partial \epsilon \text{connex}$

and $\vdash :. \kappa \epsilon FM\,\text{conx} . \supset : \kappa \epsilon FM\,\text{connex} . \equiv . \dot{s}'\kappa_\partial \epsilon \text{connex}.$

It is these propositions that justify the notation "*FM* connex."

It is obvious that we shall have $\exists ! p'C``\kappa_\iota$ if $\text{D}``\kappa = \text{Ɑ}``\kappa$, unless $\kappa = \iota'\dot{\Lambda}$.

Some illustrations will serve to make clearer the nature of the hypothesis $\exists ! p'C``\kappa_\iota$. This hypothesis states that there is at least one term a in the field of κ such that, if R, S are any two members of κ, we can either take an R-step forward from a, followed by an S-step backward, or we can take an S-step forward followed by an R-step backward. Suppose, for example, that our family consists of all vectors of the form $(+_0 \mu) \upharpoonright \text{NC induct}$, where $\mu \epsilon \text{NC induct}$. Then if R is the operation of adding μ, and S is the operation of adding ν, $\breve{R} | S$ is the operation of adding $\nu -_0 \mu$ if $\nu > \mu$, and is the operation of subtracting $\mu -_0 \nu$ if $\mu > \nu$. In the former case $\breve{R} | S \epsilon \kappa$, while in the latter case $\breve{S} | R \epsilon \kappa$. In the former case, if ϖ is any inductive cardinal, $(\breve{R} | S)'\varpi = \nu -_0 \mu +_0 \varpi$; in the latter case, $(\breve{S} | R)'\varpi = \mu -_0 \nu +_0 \varpi$. Thus in either case $\varpi \epsilon C'(\breve{R} | S)$. Thus the family in question has connexity, and $\kappa_\iota = \kappa \cup \text{Cnv}``\kappa$.

But now consider the family consisting of all vectors of the form $(\times_0 \mu) \upharpoonright (\text{NC induct} - \iota'0)$, where $\mu \epsilon \text{NC induct} - \iota'0$. This is an initial family, its initial point being 1. But it does not have connexity. If $R = (\times_0 \mu) \upharpoonright (\text{NC induct} - \iota'0)$ and $S = (\times_0 \nu) \upharpoonright (\text{NC induct} - \iota'\Lambda)$, $\breve{R} | S$ is the operation of multiplying by ν and dividing by μ, with its field confined to inductive cardinals other than 0. If ν is prime to μ, this relation has only multiples of μ in its converse domain and only multiples of ν in its domain. Hence its field consists of multiples of μ together with multiples of ν. Thus no member of κ_ι except $I \upharpoonright s'\text{Ɑ}``\kappa$, *i.e.* $(\times_0 1) \upharpoonright (\text{NC induct} - \iota'0)$, has the whole of $s'\text{Ɑ}``\kappa$ for its field, and there is no number which belongs to the

field of every member of κ_ι. The above family may be usefully borne in mind in considering κ_ι, since it affords good illustrations of most of the general theorems concerning κ_ι.

If κ is any family except $\iota`\Lambda$, any finite number of members of $\kappa_\iota$ have an existent relative product, and their converse domains have an existent logical product. If $\kappa$ is a connected family, any two members $L, M$ of $\kappa_\iota$ whose logical product exists, *i.e.* for which $(\exists y) . L`y = M`y$, are identical, and if $x, y$ are any two members of $s`\mathrm{C}``\kappa$, there is just one member of κ_ι such that $x = L`y$. If $M \epsilon \kappa_\iota$ and P is a power of M, there is some member L of κ_ι such that $P \subset L$. But P is not in general itself a member of κ_ι. For the application of ratio, the member of κ_ι which contains P is important. We call it the "representative" of P. The general definition of a representative is

$$\mathrm{rep}_\kappa`P = \dot{s}`(\kappa_\iota \cap \overleftarrow{\subset}`P) \quad \mathrm{Df.}$$

In a connected family, $\kappa_\iota \cap \overleftarrow{\subset}`P$ cannot have more than one member; hence if there is any member of $\kappa_\iota$ which contains $P$, that member is $\mathrm{rep}_\kappa`P$, and if there is no member of κ_ι which contains P, $\mathrm{rep}_\kappa`P = \dot{\Lambda}$.

If $\breve{P} \mid Q$ is any member of κ_ι (where $P, Q \epsilon \kappa$), we shall have

$$\mathrm{rep}_\kappa`(\breve{P} \mid Q)^\rho = \breve{P}^\rho \mid Q^\rho;$$

and if $L, M \epsilon \kappa_\iota$, we shall have

$$\mathrm{rep}_\kappa`(L \mid M)^\rho = \mathrm{rep}_\kappa`(L^\rho \mid M^\rho) = \mathrm{rep}_\kappa`\{(\mathrm{rep}_\kappa`L^\rho) \mid (\mathrm{rep}_\kappa`M^\rho)\}.$$

These two formulae are the most useful in determining representatives.

In order to apply the above theory to the measurement of vectors, it is necessary to distinguish between open and cyclic families. An open family is one in which, if $M \epsilon \kappa_\iota - \mathrm{Rl}`I$, $M_{\mathrm{po}} \subset J$, *i.e.* one in which no number of repetitions of a non-zero member of κ_ι will bring us back to our starting-point. If this condition fails, as in the case of angles, or distances on the elliptic straight line, the problem of measurement is more complicated, since, if θ is a measure of an angle, so is $2\nu\pi + \theta$ for any integral ν. The case of cyclic families will be considered in Section D; for the present, we proceed to consider open families, and we shall still be concerned almost exclusively with open families in Section C. It should be observed that in cyclic families, as we shall define them, members of κ_∂ return into themselves, whereas in open families, not merely no member of κ_∂, but no member of $\kappa_\iota - \mathrm{Rl}`I$, returns into itself. In most of the families that naturally occur, it happens either that no member of $\kappa_\iota - \mathrm{Rl}`I$ returns into itself, or that there are members of κ_∂ which do so. But there is no logical necessity in this, as the following instance shows: Consider the family consisting of positive and negative integral multipliers other than -1, with their fields confined to positive and negative integers other than -1. Then 1 is a

connected point of this family, in fact the initial point. Multiplication by -1 is a member of κ_ι, since it can be obtained by multiplying by any integer μ and then dividing by $-\mu$. Also the square of multiplication by -1 is contained in identity, and is the zero vector of our family. Hence there is a member of $\kappa_\iota - \mathrm{Rl}`I$ whose square is contained in identity, although no power of any member of κ_∂ is contained in identity.

In order to avoid brackets, we put

$$\kappa_{\iota\partial} = (\kappa_\iota)_\partial \quad \mathrm{Df},$$

i.e.

$$\kappa_{\iota\partial} = \kappa_\iota - \mathrm{Rl}`I.$$

Then the definition of open families is

$$FM\,\mathrm{ap} = FM \cap \hat{\kappa}\,(s`\mathrm{Pot}``\kappa_{\iota\partial} \subset \mathrm{Rl}`J) \quad \mathrm{Df}.$$

Hence $\quad \vdash :. \kappa \in FM\,\mathrm{ap} . \equiv : \kappa \in FM : M \in \kappa_{\iota\partial} . \supset_M . M_{\mathrm{po}} \mathrel{\mathsf{G}} J.$

It will be observed that if κ is an open family, κ_ι is contained in Rel num id (cf. ∗300), and $\kappa_{\iota\partial} \subset$ Rel num. Hence if $M \in \kappa_{\iota\partial}$, $M^\nu = M_\nu$ (cf. ∗121), and the propositions on intervals in ∗121 become available. Also if $M \in \kappa_{\iota\partial}$, and $a \in s`\mathrm{C}``\kappa$, we have

$$\breve{M} \upharpoonright \overrightarrow{M_*}`a \in \mathrm{Prog} . \breve{M}_{\mathrm{po}} \upharpoonright \overrightarrow{M_*}`a \in \omega.$$

The chief use of these facts is to show that the existence of open families implies the axiom of infinity and the existence of $\aleph_0$. Hence as applied to open families, the theory of ratio undergoes the very great simplification which results from the axiom of infinity.

If κ is open and connected, and $L, M \in \kappa_\iota$, and σ is any inductive cardinal other than 0, we shall have $L = M$ if $L^\sigma = M^\sigma$ or $\mathrm{rep}_\kappa`L^\sigma = \mathrm{rep}_\kappa`M^\sigma$ or $\dot{\exists} ! L^\sigma \dot{\cap} M^\sigma$. If ρ, τ are also inductive cardinals other than 0, we shall have $\mathrm{rep}_\kappa`L^\rho = \mathrm{rep}_\kappa`M^\sigma$ if $L^{\rho \times_0 \tau} = M^{\sigma \times_0 \tau}$, or if $\mathrm{rep}_\kappa`L^{\rho \times_0 \tau} = \mathrm{rep}_\kappa`M^{\sigma \times_0 \tau}$, or if $\dot{\exists} ! L^{\rho \times_0 \tau} \dot{\cap} M^{\sigma \times_0 \tau}$. We have in fact

$$\mathrm{rep}_\kappa`L^\rho = \mathrm{rep}_\kappa`M^\sigma . \equiv . \dot{\exists} ! L^\rho \dot{\cap} M^\sigma$$
$$\equiv . \dot{\exists} ! L^{\rho \times_0 \tau} \dot{\cap} M^{\sigma \times_0 \tau}$$

and

$$\mathrm{rep}_\kappa`M^\rho = \mathrm{rep}_\kappa`M^\sigma . \equiv . M^\rho = M^\sigma . \equiv . \rho = \sigma.$$

On applying the definition of ratio (∗303·01), we see from the above propositions that, with the above hypothesis,

$$M\,(\rho/\sigma)\,N . \equiv . \dot{\exists} ! M^\sigma \dot{\cap} N^\rho . \equiv . \mathrm{rep}_\kappa`M^\sigma = \mathrm{rep}_\kappa`N^\rho,$$

while if R, T are members of κ,

$$R\,(\rho/\sigma)\,S . \equiv . R^\sigma = S^\rho.$$

Further, we have, in virtue of the above propositions,

$$\dot{\exists} ! L^\sigma \dot{\cap} M^\rho . \dot{\exists} ! L^\nu \dot{\cap} M^\mu . \supset . \mu \times_0 \sigma = \nu \times_0 \rho,$$

whence

$$X, Y \in C`H' . \dot{\exists} ! X \upharpoonright \kappa_{\iota\partial} \dot{\cap} Y \upharpoonright \kappa_{\iota\partial} . \supset . X = Y.$$

These propositions, together with

$$X \in C\text{`}H \,.\, \supset \,.\, X \upharpoonright \kappa_\iota \in 1 \rightarrow 1,$$

belong to Section C. They are mentioned here as showing why the propositions of this Section are useful in connection with measurement.

We next proceed to consider serial families, which are those in which $\dot{s}\text{`}\kappa_\partial$ is an existent serial relation. For this purpose we require the definition of "*FM* connex" already given, and also the definition of "transitive" families. We define a as a "transitive point" of κ if

$$(\dot{s}\text{`}\kappa_\partial)\text{``}\overrightarrow{\dot{s}\text{`}\kappa_\partial}\text{`}a \subset \overrightarrow{\dot{s}\text{`}\kappa_\partial}\text{`}a,$$

i.e. if any point which can be reached from a in two non-null steps can also be reached in one non-null step. We define a family as transitive when it has at least one transitive point. If $\kappa \in FM\,\text{conx}$, the hypothesis that κ is transitive is equivalent to the hypothesis that κ_∂ forms a group, and implies that κ forms a group. We define a serial family as one which is transitive and has connexity, *i.e.* we put

$$FM\,\text{sr} = FM\,\text{trs} \cap FM\,\text{connex} \quad \text{Df.}$$

Then if $\kappa \in FM\,\text{sr}$, $\dot{s}\text{`}\kappa_\partial$ is a serial relation, so that the points of the field of κ are arranged in a series by means of relations of distance.

When a family is serial, the vectors also can be arranged in a series, by means of a relation which may be regarded as that of greater and less. After a short number on initial families (explained above), we proceed to the consideration of greater and less (as it may be called) among vectors. We may call a point y "earlier" than a point z when there is a non-null vector which goes from y to z, *i.e.* when $z\,(\dot{s}\text{`}\kappa_\partial)\,y$. If $M, N \in \kappa_\iota$, we then say that N is "less" than M if the N-step from some point x takes us to an earlier point than the M-step. Writing V_κ for "greater than" among members of κ_ι, our definition is

$$V_\kappa = \hat{M}\hat{N}\{M, N \in \kappa_\iota : (\exists x) \,.\, (M\text{`}x)\,(\dot{s}\text{`}\kappa_\partial)\,(N\text{`}x)\} \quad \text{Df.}$$

For the same relation confined to members of κ, we use the notation U_κ; thus

$$U_\kappa = V_\kappa \upharpoonright \kappa \quad \text{Df.}$$

If $\kappa \in FM\,\text{conx}$, we have

$$U_\kappa = \hat{P}\hat{Q}\{P, Q \in \kappa : (\exists T) \,.\, T \in \kappa_\partial \,.\, P = T \,|\, Q\};$$

this is generally the most serviceable formula for U_κ.

If κ is a serial family, U_κ and V_κ are series; and if κ is an initial family, U_κ is similar to $\dot{s}\text{`}\kappa_\partial$.

The last number in this Section is concerned with the axiom of Archimedes and with the existence of sub-multiples of vectors. The axiom of Archimedes will be expressed by saying that if a is any member of the

field of κ, and R is any vector, then $R^{\nu}{}^{\backprime}a$, for a sufficiently great finite ν, will be later than any assigned member of the field of κ. In other words, putting $P = \mathrm{Cnv}{}^{\backprime}\dot{s}{}^{\backprime}\kappa_\partial$, we wish to have

$$x \in C{}^{\backprime}P \,.\, \supset_x .\, (\exists \nu) \,.\, \nu \in \mathrm{NC\,ind} - \iota{}^{\backprime}0 \,.\, xP(R^{\nu}{}^{\backprime}a),$$

or, what comes to the same thing,

$$P{}^{\backprime\backprime}\overrightarrow{R_*}{}^{\backprime}a = C{}^{\backprime}P.$$

This will hold if κ is a serial family and P is semi-Dedekindian (cf. $*214$). If, further, P is compact (*i.e.* $P^2 = P$), then all finite sub-multiples of a given vector exist, *i.e.*

$$S \in \kappa \,.\, \nu \in \mathrm{NC\,ind} - \iota{}^{\backprime}0 \,.\, \supset .\, (\exists L) \,.\, L \in \kappa \,.\, S = L^{\nu}.$$

It will be observed that, according to our definition of ratio, if $S = L^{\nu}$ and $S \neq \dot{\Lambda}$, L has to S the ratio $1/\nu$, so that L is the νth sub-multiple of S.

Instead of treating vector-families by the method we have adopted, we might have started from a double descriptive function, which we may denote by $x + y$, and concerning which we should make various hypotheses. By the general notation of $*38$, we obtain various relations of the form $+\,y$ or $x\,+$. These relations may replace the κ employed in our method. For convenience of notation, we may put

$$\overrightarrow{+}{}^{\backprime}y = +\,y \quad \mathrm{Df},$$

$$\overleftarrow{+}{}^{\backprime}x = x\,+ \quad \mathrm{Df}.$$

Then if $+$ has suitable properties, and γ is a suitable class, $\overrightarrow{+}{}^{\backprime\backprime}\gamma$ will be a vector family.

Let us assume that $x + y$ exists when, and only when, x and y both belong to the class γ, and that when x and y both belong to the class γ, $x + y$ also belongs to this class. Then if $x + y$ exists, so does $x + y + y$; hence $\mathrm{D}{}^{\backprime}+y \subset \mathrm{Cl}{}^{\backprime}+y$. Further, by our assumptions, if $x, y \in \gamma$, $x + y$ exists, and therefore $x \in \mathrm{Cl}{}^{\backprime}+y$. Hence $y \in \gamma \,.\, \supset .\, \mathrm{Cl}{}^{\backprime}+y = \gamma$. Hence if γ exists,

$$\mathrm{Cl}{}^{\backprime\backprime}\overrightarrow{+}{}^{\backprime\backprime}\gamma \in 1 \,.\, s{}^{\backprime}\mathrm{D}{}^{\backprime\backprime}\overrightarrow{+}{}^{\backprime\backprime}\gamma \subset s{}^{\backprime}\mathrm{Cl}{}^{\backprime\backprime}\overrightarrow{+}{}^{\backprime\backprime}\gamma.$$

If we now assume $\quad y + x = z + x \,.\, \supset_{x,y,z} .\, y = z,$

then $\overrightarrow{+}{}^{\backprime\backprime}\gamma \subset 1 \rightarrow 1$. Hence we now have

$$\overrightarrow{+}{}^{\backprime\backprime}\gamma \in \mathrm{Cl\ ex}{}^{\backprime}\mathrm{cr}{}^{\backprime}\gamma.$$

In order to obtain the Abelian property, we require

$$(x + y) + z = (x + z) + y,$$

which holds if $+$ obeys the permutative and associative laws. Thus in this case,

$$\overrightarrow{+}{}^{\backprime\backprime}\gamma \in \mathrm{fm}{}^{\backprime}\gamma.$$

In order that $\overrightarrow{+}{}^{\text{``}}\gamma$ may be a *connected* family, we require

$$(\exists a) :. z \,\epsilon\, \gamma \,.\, \supset_z : (\exists y) : a = z + y \,.\, \vee \,.\, z = a + y.$$

A sufficient, though not a necessary, condition for this is that there should be a zero, *i.e.*

$$(\exists a) : z \,\epsilon\, \gamma \,.\, \supset_z \,.\, z = a + z.$$

In this case, $+a$ is the zero vector, and if a is not the sum of two terms other than itself, a is the initial point of the family.

The condition that if x, y are members of γ so is $x + y$ secures that $\overrightarrow{+}{}^{\text{``}}\gamma$ is a group. Families which are groups we denote by "FM grp."

Thus collecting what has been said, we find that

$$\overrightarrow{+}{}^{\text{``}}\gamma \,\epsilon\, FM \,\text{conx grp}$$

if $+$ fulfils the following conditions:

(1) $x + y$ exists when, and only when, $x, y \,\epsilon\, \gamma$;

(2) $x, y \,\epsilon\, \gamma \,.\, \supset_{x,y} \,.\, x + y \,\epsilon\, \gamma$;

(3) $x + y = x + z \,.\, \supset_{x,y,z} \,.\, y = z$;

(4) $x + y = y + x$;

(5) $(x + y) + z = x + (y + z)$;

(6) $(\exists a) : z \,\epsilon\, \gamma \,.\, \supset_z \,.\, z = a + z$.

From (3) and (4) it follows that the a of (6) is unique, *i.e.* there cannot be more than one zero.

In order to insure that our family shall have connexity, we require further

(7) $x, y \,\epsilon\, \gamma \,.\, \supset_{x,y} : (\exists z) : z \,\epsilon\, \gamma : x + z = y \,.\, \vee \,.\, y + z = x$;

(8) in order that our family may be an initial family we require that $x + y$ shall only be zero when x and y are zero.

With this further condition, our family becomes serial.

The above is only a sketch of one of the simplest ways of generating families by means of double descriptive functions. Other ways are possible, and by greater complication greater generality can be obtained.

There are some advantages in the above manner of treatment. First, it is possible to take our magnitudes as being the x and y which appear in "$x + y$," instead of having to take them as the vectors $+y$ or $x+$. Secondly, our vector-family derives unity from the fact of being generated by the single operation $+$. Thirdly, the method is more in agreement with current conceptions of quantity than the method we have adopted. The choice

between the two methods is a matter of taste; but it would seem that the method we have adopted is capable of somewhat greater generality than the other, and that it requires less new technical apparatus than the other. We have not elsewhere had occasion to treat of double descriptive functions which only exist when their arguments belong to assigned classes, though it is to be observed that our definitions of various kinds of addition and multiplication might quite easily have been so framed as to give this result. For instance, we might have put

$$\mu +_0 \nu = (\imath \varpi) \{(\exists \alpha, \beta) \,.\, \mu = N_0 c\text{'}\alpha \,.\, \nu = N_0 c\text{'}\beta \,.\, \varpi = Nc\text{'}(\alpha + \beta)\} \quad \text{Df.}$$

In that case, $E ! (\mu +_0 \nu)$ would have implied $\mu, \nu \in N_0C$, whereas with our definition it is only $\exists ! (\mu +_0 \nu)$ that implies $\mu, \nu \in N_0C$. The general treatment of double functions which only exist in certain cases would require a considerable logical apparatus not required elsewhere in our work, and this is, for us, a reason against adopting the method of treating vector-families which derives them, as in the above sketch, from a single function $x + y$.

*330. ELEMENTARY PROPERTIES OF VECTOR-FAMILIES.

Summary of *330

In this number, we begin by defining the class of "correspondences" of α. A "correspondence" of α is a one-one relation R which makes every member of α correspond to an α, *i.e.* which is such that, if $x \epsilon \alpha$, $R‘x$ always exists and is a member of α. Thus, for example, if μ is an inductive number, $+_0 \mu$, with its field limited to inductive numbers, is a correspondence of the class of inductive numbers, provided the axiom of infinity holds. (Otherwise, $(+_0 \mu) \upharpoonright$ NC induct is not one-one.) The definition of correspondences of α is

*330·01. $\mathrm{cr}‘\alpha = (1 \rightarrow 1) \cap \overleftarrow{\mathrm{G}}‘\alpha \cap \breve{\mathrm{D}}‘‘\mathrm{Cl}‘\alpha$ Df

I.e. a correspondence of α is a one-one relation whose converse domain is α and whose domain is contained in α. The definition should be compared with the definition of "cror‘P" in *208.

It will be seen that if $R \epsilon \mathrm{cr}‘\alpha$ and $x \epsilon \alpha$, $R‘x$ exists and is an α, and therefore $R‘R‘x$ exists and is an α, and so on. Hence all the powers of R exist (*330·23). Similarly if $R, S, T, \ldots$ are any finite number of correspondences of α, $R \mid S \mid T \mid \ldots$ exists. This is proved for two and three factors in *330·21·22.

We define a "vector-family of α" as an existent Abelian class of correspondences of α, where an Abelian class of relations is defined as one such that the relative product of any two of its members is commutative. Thus we put

*330·02. $\mathrm{Abel} = \hat{\kappa}(R, S \epsilon \kappa \,.\, \supset_{R,S} \,.\, R \mid S = S \mid R)$ Df

*330·03. $\mathrm{fm}‘\alpha = \mathrm{Abel} \cap \mathrm{Cl\,ex}‘\mathrm{cr}‘\alpha$ Df

*330·04. $FM = s‘\mathrm{D}‘\mathrm{fm}$ Df

It will be remembered that Potid‘P and (for certain kinds of relations) finid‘P are Abelian classes of relations (*91·34 and *121·352). If $P \epsilon 1 \rightarrow 1$, Potid‘$P$ will be a vector-family of $C‘P$, and if further $P_{\mathrm{po}} \subseteq J$, finid‘$P$ will be the same vector-family.

One other definition belongs to this number, namely

*330·05. $\kappa_\iota = s‘(\mathrm{Cnv}‘‘\kappa) \,\big|_{,,}‘‘\kappa$ Df

This definition has been sufficiently discussed in the summary of the present Section.

After some preliminary propositions on Cl ex‘cr‘α (*330·1—·32) and on κ_ι (*330·4—·43), we proceed to such properties of families as do not require any further hypothesis as to the nature of the family concerned. These properties are mainly such as assert the existence of relative products, and of logical products of converse domains, or such as assert commutativeness of the relative product under certain circumstances. The earlier propositions deal with members of κ, the later propositions mainly with members of κ_ι. The most useful propositions are:

***330·54.** $\vdash : \kappa \in FM \,.\, Q, R \in \kappa \,.\, E\,!\,\breve{R}\text{‘}x \,.\, \supset \,.\, E\,!\,\breve{R}\text{‘}Q\text{‘}x$

***330·56.** $\vdash : \kappa \in FM \,.\, Q, R \in \kappa \,.\, E\,!\,\breve{R}\text{‘}a \,.\, \supset \,.\, \breve{R}\text{‘}Q\text{‘}a = Q\text{‘}\breve{R}\text{‘}a$

***330·61.** $\vdash : \kappa \in FM - \iota\text{‘}\iota\text{‘}\dot{\Lambda} \,.\, L, M \in \kappa_\iota \,.\, \supset \,.$

$\exists\,!\,\text{Ɑ}\text{‘}L \cap \text{Ɑ}\text{‘}M \,.\, \exists\,!\,\text{D}\text{‘}L \cap \text{Ɑ}\text{‘}M \,.\, \exists\,!\,\text{Ɑ}\text{‘}L \cap \text{D}\text{‘}M \,.\, \exists\,!\,\text{D}\text{‘}L \cap \text{D}\text{‘}M$

***330·611.** $\vdash : \kappa \in FM - \iota\text{‘}\iota\text{‘}\dot{\Lambda} \,.\, L, M \in \kappa_\iota \,.\, \supset \,.\, \dot{\exists}\,!\,L \mid M$

***330·624.** $\vdash : \kappa \in FM - \iota\text{‘}\iota\text{‘}\dot{\Lambda} \,.\, L \in \kappa_\iota \,.\, \supset \,.\, \dot{\Lambda} \sim\in \text{Pot}\text{‘}L$

***330·63.** $\vdash : \kappa \in FM \,.\, L, M \in \kappa_\iota \,.\, E\,!\,L\text{‘}x \,.\, E\,!\,L\text{‘}M\text{‘}x \,.\, \supset \,.\, L\text{‘}M\text{‘}x = M\text{‘}L\text{‘}x$

***330·642.** $\vdash : \kappa \in FM - \iota\text{‘}\iota\text{‘}\dot{\Lambda} \,.\, L, M \in \kappa_\iota \,.\, \supset \,.\, (\exists x) \,.\, E\,!\,L\text{‘}x \,.\, E\,!\,L\text{‘}M\text{‘}x$

***330·71.** $\vdash : \kappa \in FM \,.\, P, Q \in \kappa \,.\, \rho \in \text{NC ind} - \iota\text{‘}0 \,.\, E\,!\,\breve{P}^\rho\text{‘}x \,.\, \supset \,.\, E\,!\,(\breve{P} \mid Q)^\rho\text{‘}x$

***330·72.** $\vdash : \kappa \in FM - \iota\text{‘}\iota\text{‘}\dot{\Lambda} \,.\, L, M \in \kappa_\iota \,.\, \rho, \sigma \in \text{NC induct} \,.\, \supset \,.\, \exists\,!\,\text{Ɑ}\text{‘}L^\rho \cap \text{Ɑ}\text{‘}M^\sigma$

***330·73.** $\vdash : \kappa \in FM \,.\, P, Q \in \kappa \,.\, \rho \in \text{NC ind} \,.\, E\,!\,(\breve{P} \mid Q)^\rho\text{‘}x \,.\, \supset \,.\, (\breve{P} \mid Q)^\rho\text{‘}x = \breve{P}^\rho\text{‘}Q^\rho\text{‘}x$

***330·01.** $\text{cr}\text{‘}\alpha = (1 \to 1) \cap \overleftarrow{\text{Ɑ}}\text{‘}\alpha \cap \breve{\text{D}}\text{‘‘}\text{Cl}\text{‘}\alpha$ Df

***330·02.** $\text{Abel} = \hat{\kappa}\,(R, S \in \kappa \,.\, \supset_{R,S} \,.\, R \mid S = S \mid R)$ Df

***330·03.** $\text{fm}\text{‘}\alpha = \text{Abel} \cap \text{Cl ex}\text{‘}\text{cr}\text{‘}\alpha$ Df

***330·04.** $FM = s\text{‘}\text{D}\text{‘}\text{fm}$ Df

***330·05.** $\kappa_\iota = s\text{‘}(\text{Cnv}\text{‘‘}\kappa) \underset{\text{„}}{\mid}\text{‘‘}\kappa$ Df

***330·1.** $\vdash : \kappa \in \text{Cl ex}\text{‘}\text{cr}\text{‘}\alpha \,.\, \equiv \,.\, \kappa \subset 1 \to 1 \,.\, \text{Ɑ}\text{‘‘}\kappa = \iota\text{‘}\alpha \,.\, \text{D}\text{‘‘}\kappa \subset \text{Cl}\text{‘}\alpha$ [(*330·01)]

***330·11.** $\vdash :.\, (\exists\alpha) \,.\, \kappa \in \text{Cl ex}\text{‘}\text{cr}\text{‘}\alpha \,.\, \equiv : \kappa \subset 1 \to 1 : (\exists\alpha) \,.\, \text{Ɑ}\text{‘‘}\kappa = \iota\text{‘}\alpha \,.\, s\text{‘}\text{D}\text{‘‘}\kappa \subset \alpha$
[*330·1]

***330·12.** $\vdash : \kappa \in \text{Cl ex}\text{‘}\text{cr}\text{‘}\alpha \,.\, \supset \,.\, s\text{‘}\text{Ɑ}\text{‘‘}\kappa = \alpha$ [*330·1 . *53·02]

***330·13.** $\vdash : \kappa \in \text{Cl ex}\text{‘}\text{cr}\text{‘}\alpha \,.\, \supset \,.\, \text{D}\text{‘‘}\kappa \subset \text{Cl}\text{‘}s\text{‘}\text{Ɑ}\text{‘‘}\kappa \,.\, s\text{‘}\text{D}\text{‘‘}\kappa \subset s\text{‘}\text{Ɑ}\text{‘‘}\kappa$ [*330·1·12]

***330·131.** $\vdash : (\exists\alpha) \,.\, \kappa \in \text{Cl ex}\text{‘}\text{cr}\text{‘}\alpha \,.\, \equiv \,.\, \kappa \subset 1 \to 1 \,.\, \text{Ɑ}\text{‘‘}\kappa \in 1 \,.\, s\text{‘}\text{D}\text{‘‘}\kappa \subset s\text{‘}\text{Ɑ}\text{‘‘}\kappa$
[*330·11·12]

***330·14.** $\vdash : \kappa \in \text{Cl ex}\text{‘}\text{cr}\text{‘}\alpha \,.\, \supset \,.\, \text{D}\text{‘‘}\kappa \subset \text{Nc}\text{‘}\alpha$ [*330·1]

*330·15. $\vdash . \text{Cl ex}`\text{cr}`\Lambda = \iota`\iota`\Lambda$ [*330·1]

*330·151. $\vdash : \exists ! \alpha . \kappa \in \text{Cl ex}`\text{cr}`\alpha . \supset . \Lambda \sim\in \kappa$ [*330·14]

*330·16. $\vdash :. (\exists \alpha) . \kappa \in \text{Cl ex}`\text{cr}`\alpha : \kappa \neq \iota`\dot{\Lambda} : \supset . \Lambda \sim\in \kappa$ [*330·15·151]

*330·17. $\vdash : \exists ! \alpha . \kappa \in \text{Cl ex}`\text{cr}`\alpha . \supset . \text{D}``\kappa \subset \text{Cl ex}`s`\text{Cl}``\kappa$ [*330·13·151]

*330·18. $\vdash :. (\exists \alpha) . \kappa \in \text{Cl ex}`\text{cr}`\alpha : \kappa \neq \iota`\dot{\Lambda} : \supset . \text{D}``\kappa \subset \text{Cl ex}`s`\text{Cl}``\kappa$ [*330·15·17]

*330·19. $\vdash . \iota`(I \upharpoonright \alpha) \in \text{Cl ex}`\text{cr}`\alpha$ [*330·1]

*330·2. $\vdash : \kappa \in \text{Cl ex}`\text{cr}`\alpha . R \in \kappa . \exists ! \text{D}`M \cap s`\text{Cl}``\kappa . \supset . \dot{\exists} ! R | M$

Dem.

$\vdash . *330·1·12 . \supset \vdash : \text{Hp} . \supset . \exists ! \text{D}`M \cap \text{Cl}`R : \supset \vdash . \text{Prop}$

*330·21. $\vdash : \kappa \in \text{Cl ex}`\text{cr}`\alpha . \kappa \neq \iota`\dot{\Lambda} . R, S \in \kappa . \supset . \dot{\exists} ! R | S$

Dem.

$\vdash . *330·18 . \supset \vdash : \text{Hp} . \supset . \exists ! \text{D}`S \cap s`\text{Cl}``\kappa$ (1)

$\vdash . (1) . *330·2 . \supset \vdash . \text{Prop}$

*330·22. $\vdash : \kappa \in \text{Cl ex}`\text{cr}`\alpha . \kappa \neq \iota`\dot{\Lambda} . R, S, T \in \kappa . \supset . \dot{\exists} ! R | S | T$

Dem.

$\vdash . *330·21·18 . \supset \vdash : \text{Hp} . \supset . \dot{\exists} ! \text{D}`(S | T) \cap s`\text{Cl}``\kappa$ (1)

$\vdash . (1) . *330·2 . \supset \vdash . \text{Prop}$

*330·23. $\vdash : \kappa \in \text{Cl ex}`\text{cr}`\alpha . \kappa \neq \iota`\dot{\Lambda} . R \in \kappa . \supset . \dot{\Lambda} \sim\in \text{Potid}`R$

Dem.

$\vdash . *330·16 . \supset \vdash : \text{Hp} . \supset . \dot{\exists} ! I \upharpoonright C`R$ (1)

$\vdash . *330·18 . \supset \vdash : \text{Hp} . P \in \text{Potid}`R . \dot{\exists} ! P . \supset . \exists ! \text{D}`P \cap s`\text{Cl}``\kappa .$

[*330·2] $\supset . \dot{\exists} ! R | P$ (2)

$\vdash . (1) . (2) . \text{Induct} . \supset \vdash . \text{Prop}$

*330·3. $\vdash : \kappa \in \text{Cl ex}`\text{cr}`\alpha . I \upharpoonright \alpha \in \kappa . \supset . \kappa \subset s`\kappa |``\kappa$

Dem.

$\vdash . *330·1 . \supset \vdash :. \text{Hp} . \supset : R \in \kappa . \supset . R = R | I \upharpoonright \alpha :. \supset \vdash . \text{Prop}$

*330·31. $\vdash : \kappa \in \text{Cl ex}`\text{cr}`\alpha . R \in \kappa . \supset . \breve{R} | R = I \upharpoonright s`\text{Cl}``\kappa$ [*330·1]

*330·32. $\vdash :. \kappa \in \text{Cl ex}`\text{cr}`\alpha . R, S \in \kappa . \supset : \breve{R} | S = I \upharpoonright s`\text{Cl}``\kappa . \equiv . R = S$

Dem.

$\vdash . *330·31 . \supset \vdash :. \text{Hp} . \supset : R = S . \supset . \breve{R} | S = I \upharpoonright s`\text{Cl}``\kappa$ (1)

$\vdash . *330·1 . \supset \vdash : \text{Hp} . \supset . R | \breve{R} | S = (\text{D}`R) \upharpoonleft S$ (2)

$\vdash . (2) . \supset \vdash :. \text{Hp} . \supset : \breve{R} | S = I \upharpoonright s`\text{Cl}``\kappa . \supset . R = (\text{D}`R) \upharpoonleft S .$

[*72·92] $\supset . R = S \upharpoonright \text{Cl}`R .$

[*330·1] $\supset . R = S$ (3)

$\vdash . (1) . (3) . \supset \vdash . \text{Prop}$

***330·4.** $\vdash : M \in \kappa_\iota . \equiv . (\exists R, S) . R, S \in \kappa . M = \breve{R} \,|\, S$ [(*330·05)]

***330·41.** $\vdash . \mathrm{Cnv}``\kappa_\iota = \kappa_\iota$ [*330·4]

***330·42.** $\vdash : \kappa \in \mathrm{Cl\,ex}`\mathrm{cr}`\alpha . I \upharpoonright \alpha \in \kappa . \supset . \kappa \cup \mathrm{Cnv}``\kappa \subset \kappa_\iota$

Dem.

$\vdash . *330·1 . *50·5·51 . \supset \vdash : \mathrm{Hp} . R \in \kappa . \supset . R = (I \upharpoonright \alpha) \,|\, R . I \upharpoonright \alpha \in \mathrm{Cnv}``\kappa$ (1)

$\vdash . (1) . *330·4·41 . \supset \vdash . \mathrm{Prop}$

***330·43.** $\vdash : \kappa \in \mathrm{Cl\,ex}`\mathrm{cr}`\alpha . \supset . I \upharpoonright s`\mathrm{C}``\kappa \in \kappa_\iota$ [*330·31·4]

***330·5.** $\vdash :. \kappa \in \mathrm{Abel} . \equiv : R, S \in \kappa . \supset_{R,S} . R \,|\, S = S \,|\, R$ [(*330·02)]

***330·51.** $\vdash : \kappa \in \mathrm{fm}`\alpha . \equiv . \kappa \in \mathrm{Abel} \cap \mathrm{Cl\,ex}`\mathrm{cr}`\alpha$ [(*330·03)]

***330·52.** $\vdash : \kappa \in FM . \equiv . (\exists \alpha) . \kappa \in \mathrm{Abel} \cap \mathrm{Cl\,ex}`\mathrm{cr}`\alpha .$
$\equiv . \kappa \in \mathrm{Abel} . \kappa \subset 1 \rightarrow 1 . \mathrm{C}``\kappa \in 1 . s`\mathrm{D}``\kappa \subset s`\mathrm{C}``\kappa$
[*330·51·131 . (*330·04)]

***330·53.** $\vdash : \kappa \in FM . Q, R \in \kappa . \mathrm{E} ! \breve{R}`\breve{Q}`x . \supset . \mathrm{E} ! \breve{Q}`x . \mathrm{E} ! \breve{R}`x$

Dem.

$\vdash . *330·5 . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{E} ! \breve{Q}`\breve{R}`x$ (1)

$\vdash . (1) . *30·5 . \supset \vdash . \mathrm{Prop}$

***330·54.** $\vdash : \kappa \in FM . Q, R \in \kappa . \mathrm{E} ! \breve{R}`x . \supset . \mathrm{E} ! \breve{R}`Q`x$

Dem.

$\vdash . *330·31·52 . \supset \vdash : \mathrm{Hp} . \supset . \breve{R}`x = \breve{R}`\breve{Q}`Q`x$ (1)

$\vdash . (1) . *330·53 . \supset \vdash . \mathrm{Prop}$

***330·541.** $\vdash : \kappa \in FM . Q, R \in \kappa . \supset . Q``\mathrm{D}`R \subset \mathrm{D}`R$ [*330·54]

***330·542.** $\vdash : \kappa \in FM . R \in \kappa . \supset . \mathrm{D}`R \in \mathrm{sect}`\dot{s}`\kappa$ [*330·541 . *211·1]

***330·55.** $\vdash : \kappa \in FM - \iota`\iota`\Lambda . Q, R \in \kappa . \supset . \exists ! \mathrm{D}`Q \cap \mathrm{D}`R . \exists ! Q``\mathrm{D}`R$

Dem.

$\vdash . *330·54 . \quad \supset \vdash :. \mathrm{Hp} . \supset : x \in \mathrm{D}`R . \supset . Q`x \in \mathrm{D}`R :$

[*33·43] $\supset : \exists ! \mathrm{D}`R . \supset . \exists ! \mathrm{D}`Q \cap \mathrm{D}`R$ (1)

$\vdash . (1) . *330·16 . \supset \vdash : \mathrm{Hp} . \supset . \exists ! \mathrm{D}`Q \cap \mathrm{D}`R$ (2)

$\vdash . *330·11·16 . \quad \supset \vdash : \mathrm{Hp} . \supset . \mathrm{D}`R \subset \mathrm{C}`Q . \exists ! \mathrm{D}`R .$

[*37·43] $\supset . \exists ! Q``\mathrm{D}`R$ (3)

$\vdash . (2) . (3) . \supset \vdash . \mathrm{Prop}$

***330·551.** $\vdash : \mathrm{Hp}\, *330·55 . \supset . \dot{\exists} ! Q \,|\, R$ [*330·55 . *37·32]

$*330{\cdot}56.\quad \vdash : \kappa \in FM \,.\, Q, R \in \kappa \,.\, \mathrm{E}\,!\,\breve{R}`a \,.\, \supset .\, \breve{R}`Q`a = Q`\breve{R}`a$

Dem.

$\vdash .\, *330{\cdot}5{\cdot}11 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset .\, Q`R`\breve{R}`a = R`Q`\breve{R}`a \,.$

$[*72{\cdot}24] \quad \supset .\, Q`a = R`Q`\breve{R}`a \,.$

$[*330{\cdot}31{\cdot}54] \quad \supset .\, \breve{R}`Q`a = Q`\breve{R}`a : \supset \vdash .\, \mathrm{Prop}$

$*330{\cdot}561.\quad \vdash : \kappa \in FM \,.\, Q, R \in \dot{\kappa} \,.\, \supset .\, \breve{R} \,|\, Q \upharpoonright \mathrm{D}`R = Q \,|\, \breve{R} \quad [*330{\cdot}56]$

$*330{\cdot}562.\quad \vdash : \kappa \in FM \,.\, Q, R \in \kappa \,.\, \supset .\, R \,\dot{;}\, Q \in Q \quad [*330{\cdot}561]$

$*330{\cdot}563.\quad \vdash : \kappa \in FM \,.\, R \in \kappa \,.\, \lambda \subset \kappa \,.\, \supset .\, R \,\dot{;}\, \dot{s}`\lambda \in \dot{s}`\lambda \quad [*330{\cdot}562]$

$*330{\cdot}57.\quad \vdash : \kappa \in \mathrm{Abel} \,.\, R, S \in \kappa \,.\, \nu \in \mathrm{NC\,induct} \,.\, \supset .\, R^\nu | S^\nu = (R|S)^\nu \,.\, R|S^\nu = S^\nu|R$

Dem.

$\vdash .\, *301{\cdot}2 \,.\quad \supset \vdash .\, R^0 \,|\, S^0 = (R \,|\, S)^0 \,.\, R \,|\, S^0 = S^0 \,|\, R \quad (1)$

$\vdash .\, *330{\cdot}5 \,.\, *301{\cdot}21 \,.\, \supset \vdash : \mathrm{Hp} \,.\, R \,|\, S^\nu = S^\nu \,|\, R \,.\, \supset .\, R \,|\, S^{\nu +_c 1} = S^{\nu +_c 1} \,|\, R \quad (2)$

$\vdash .\, (1) \,.\, (2) \,.\, \mathrm{Induct} \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset .\, R \,|\, S^\nu = S^\nu \,|\, R \quad (3)$

$\vdash .\, (3) \,.\, *301{\cdot}21 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset .\, R^{\nu +_c 1} \,|\, S^{\nu +_c 1} = R^\nu \,|\, S^\nu \,|\, R \,|\, S \quad (4)$

$\vdash .\, (4) \,.\, *301{\cdot}21 \,.\, \supset \vdash : \mathrm{Hp} \,.\, R^\nu \,|\, S^\nu = (R \,|\, S)^\nu \,.\, \supset .\, R^{\nu +_c 1} \,|\, S^{\nu +_c 1} = (R \,|\, S)^{\nu +_c 1} \quad (5)$

$\vdash .\, (1) \,.\, (5) \,.\, \mathrm{Induct} \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset .\, R^\nu \,|\, S^\nu = (R \,|\, S)^\nu \quad (6)$

$\vdash .\, (3) \,.\, (6) \,.\, \supset \vdash .\, \mathrm{Prop}$

$*330{\cdot}6.\quad \vdash : \kappa \in FM - \iota`\iota`\dot{\Lambda} \,.\, L \in \kappa_\iota \,.\, \supset .\, \dot{\exists}\,!\,L$

Dem.

$\vdash .\, *330{\cdot}16{\cdot}4 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset .\, (\exists Q, R) \,.\, Q, R \in \kappa \,.\, \dot{\exists}\,!\,R \,.\, L = \breve{R} \,|\, Q \,.$

$[*330{\cdot}54] \quad \supset .\, (\exists Q, R, x) \,.\, Q, R \in \kappa \,.\, \mathrm{E}\,!\,\breve{R}`Q`x \,.\, L = \breve{R} \,|\, Q \,.$

$[*34{\cdot}41] \quad \supset .\, \dot{\exists}\,!\,L : \supset \vdash .\, \mathrm{Prop}$

$*330{\cdot}61.\quad \vdash : \kappa \in FM - \iota`\iota`\dot{\Lambda} \,.\, L, M \in \kappa_\iota \,.\, \supset .$

$\exists\,!\,\mathrm{C}`L \cap \mathrm{C}`M \,.\, \exists\,!\,\mathrm{D}`L \cap \mathrm{C}`M \,.\, \exists\,!\,\mathrm{C}`L \cap \mathrm{D}`M \,.\, \exists\,!\,\mathrm{D}`L \cap \mathrm{D}`M$

Dem.

$\vdash .\, *330{\cdot}55{\cdot}4 \,.\, \supset$

$\vdash : \mathrm{Hp} \,.\, \supset .\, (\exists Q, R, S, T) \,.\, Q, R, S, T \in \kappa \,.\, L = \breve{R} \,|\, Q \,.\, M = \breve{T} \,|\, S \,.\, \exists\,!\,\mathrm{D}`R \cap \mathrm{D}`T \,.$

$[*330{\cdot}54]$

$\supset .\, (\exists Q, R, S, T, x) \,.\, Q, R, S, T \in \kappa \,.\, L = \breve{R} \,|\, Q \,.\, M = \breve{T} \,|\, S \,.\, \mathrm{E}\,!\,\breve{R}`Q`x \,.\, \mathrm{E}\,!\,\breve{T}`S`x \,.$

$[*34{\cdot}41] \supset .\, (\exists x) \,.\, \mathrm{E}\,!\,L`x \,.\, \mathrm{E}\,!\,M`x \,.$

$[*33{\cdot}43] \supset .\, \exists\,!\,\mathrm{C}`L \cap \mathrm{C}`M \quad (1)$

$\vdash .\, (1) \,.\, *330{\cdot}41 \,.\, \supset \vdash .\, \mathrm{Prop}$

***330·611.** $\vdash : \kappa \in FM - \iota`\iota`\Lambda \,.\, L, M \in \kappa_\iota \,.\, \supset \,.\, \dot{\exists} \,!\, L \,|\, M$ [*330·61 . *34·3]

***330·612.** $\vdash : \kappa \in FM - \iota`\iota`\Lambda \,.\, L, M, N \in \kappa_\iota \,.\, \supset \,.\, \exists \,!\, \text{Ɑ}`L \cap \text{Ɑ}`M \cap \text{Ɑ}`N$

Dem.

$\vdash \,.\, *330{\cdot}22{\cdot}4 \,.\, \supset$

$\vdash : \mathrm{Hp} \,.\, \supset \,.\, (\exists P, Q, R, S, T, W) \,.\, P, Q, R, S, T, W \in \kappa \,.$

$L = \breve{P} \,|\, Q \,.\, M = \breve{R} \,|\, S \,.\, N = \breve{T} \,|\, W \,.\, \dot{\exists} \,!\, P \,|\, R \,|\, T \,.$

$[*330{\cdot}53] \supset \,.\, (\exists P, Q, R, S, T, W, x) \,.\, P, Q, R, S, T, W \in \kappa \,.$

$L = \breve{P} \,|\, Q \,.\, M = \breve{R} \,|\, S \,.\, N = \breve{T} \,|\, W \,.\, E \,!\, \breve{P}`x \,.\, E \,!\, \breve{R}`x \,.\, E \,!\, \breve{T}`x \,.$

$[*330{\cdot}54] \supset \,.\, (\exists x) \,.\, E \,!\, L`x \,.\, E \,!\, M`x \,.\, E \,!\, N`x : \supset \vdash \,.\,$ Prop

***330·613.** $\vdash : \kappa \in FM - \iota`\iota`\Lambda \,.\, L, M, N \in \kappa_\iota \,.\, \supset \,.\, \dot{\exists} \,!\, L \,|\, M \,|\, N$

Dem.

$\vdash \,.\, *330{\cdot}22{\cdot}4 \,.\, \supset$

$\vdash : \mathrm{Hp} \,.\, \supset \,.\, (\exists P, Q, R, S, T, W, x) \,.\, P, Q, R, S, T, W \in \kappa \,.$

$L = \breve{P} \,|\, Q \,.\, M = \breve{R} \,|\, S \,.\, N = \breve{T} \,|\, W \,.\, E \,!\, \breve{P}`\breve{R}`\breve{T}`x \,.$

$[*330{\cdot}54] \supset \,.\, (\exists P, Q, R, S, x) \,.\, P, Q, R, S \in \kappa \,.$

$L = \breve{P} \,|\, Q \,.\, M = \breve{R} \,|\, S \,.\, E \,!\, \breve{P}`\breve{R}`(N`x) \,.$

$[*330{\cdot}54] \supset \,.\, (\exists P, Q) \,.\, P, Q \in \kappa \,.\, L = \breve{P} \,|\, Q \,.\, E \,!\, \breve{P}`(M`N`x) \,.$

$[*330{\cdot}54] \supset \,.\, (\exists x) \,.\, E \,!\, L`M`N`x : \supset \vdash \,.\,$ Prop

***330·62.** $\vdash : \kappa \in FM \,.\, L \in \kappa_\iota \,.\, S \in \kappa \,.\, \supset \,.\, S \,|\, L \Subset L \,|\, S$

Dem.

$\vdash \,.\, *330{\cdot}561 \,.\, \supset \vdash : \mathrm{Hp} \,.\, P, Q \in \kappa \,.\, L = \breve{P} \,|\, Q \,.\, \supset \,.\, S \,|\, \breve{P} \Subset \breve{P} \,|\, S \,.$

$[*330{\cdot}5] \qquad \supset \,.\, S \,|\, \breve{P} \,|\, Q \Subset \breve{P} \,|\, Q \,|\, S : \supset \vdash \,.\,$ Prop

***330·621.** $\vdash :. \kappa \in FM - \iota`\iota`\Lambda \,.\, L \in \kappa_\iota \,.\, C`P \subset s`\text{Ɑ}``\kappa \,.\, \dot{\exists} \,!\, P :$

$S \in \kappa \,.\, \supset_S \,.\, S \,|\, P \Subset P \,|\, S : \supset \,.\, \dot{\exists} \,!\, P \,|\, L$

Dem.

$\vdash \,.\, *330{\cdot}11 \,.\, \supset \vdash :. \mathrm{Hp} \,.\, Q, R \in \kappa \,.\, L = \breve{Q} \,|\, R \,.\, \supset :$

$xPy \,.\, \supset \,.\, (\exists u, z) \,.\, uRx \,.\, zQy \,.\, xPy \,.$

$[*34{\cdot}1] \qquad \supset \,.\, \dot{\exists} \,!\, R \,|\, P \,|\, \breve{Q} \,.$

$[*330{\cdot}5] \qquad \supset \,.\, \dot{\exists} \,!\, P \,|\, R \,|\, \breve{Q} \,.$

$[*330{\cdot}561] \qquad \supset \,.\, \dot{\exists} \,!\, P \,|\, \breve{Q} \,|\, R \,.$

$[\mathrm{Hp}] \qquad \supset \,.\, \dot{\exists} \,!\, P \,|\, L :. \supset \vdash \,.\,$ Prop

***330·622.** $\vdash : \mathrm{Hp}\ *330{\cdot}621 \,.\supset .\, \dot{\exists} \,!\, L \,|\, P$

Dem.

$\vdash . *330{\cdot}11 . *72{\cdot}59 . \supset \vdash : \mathrm{Hp} \,.\, Q, R \in \kappa \,.\, L = \breve{Q} \,|\, R \,.\supset .\, P \mathrel{\dot{\subset}} \breve{Q} \,|\, P \,|\, Q \,.$

$[*72{\cdot}59] \qquad \supset .\, P \,|\, \breve{Q} \mathrel{\dot{\subset}} \breve{Q} \,|\, P \,.$

$[*330{\cdot}621] \qquad \supset .\, \dot{\exists} \,!\, \breve{Q} \,|\, P \,|\, R \,.$

$[*330{\cdot}5] \qquad \supset .\, \dot{\exists} \,!\, \breve{Q} \,|\, R \,|\, P \,.$

$[\mathrm{Hp}] \qquad \supset .\, \dot{\exists} \,!\, L \,|\, P : \supset \vdash . \mathrm{Prop}$

***330·623.** $\vdash : \kappa \in FM \,.\, S \in \kappa \,.\, L \in \kappa_\iota \,.\, M \in \mathrm{Pot}`L \,.\supset .\, S \,|\, M \mathrel{\dot{\subset}} M \,|\, S$

Dem.

$\vdash . *34{\cdot}34 . \supset \vdash : \mathrm{Hp} \,.\, S \,|\, M \mathrel{\dot{\subset}} M \,|\, S \,.\supset .\, S \,|\, M \,|\, L \mathrel{\dot{\subset}} M \,|\, S \,|\, L \,.$

$[*330{\cdot}62] \qquad \supset .\, S \,|\, M \,|\, L \mathrel{\dot{\subset}} M \,|\, L \,|\, S \qquad (1)$

$\vdash . (1) . *330{\cdot}62 . \mathrm{Induct} . \supset \vdash . \mathrm{Prop}$

***330·624.** $\vdash : \kappa \in FM - \iota`\iota`\dot{\Lambda} \,.\, L \in \kappa_\iota \,.\supset .\, \dot{\Lambda} \sim\in \mathrm{Pot}`L$

Dem.

$\vdash . *330{\cdot}6 . \qquad \supset \vdash : \mathrm{Hp} \,.\supset .\, \dot{\exists} \,!\, L \qquad (1)$

$\vdash . *330{\cdot}622{\cdot}623 . \qquad \supset \vdash : \mathrm{Hp} \,.\, M \in \mathrm{Pot}`L \,.\, \dot{\exists} \,!\, M \,.\supset .\, \dot{\exists} \,!\, M \,|\, L \qquad (2)$

$\vdash . (1) . (2) . \mathrm{Induct} . \supset \vdash :. \mathrm{Hp} . \supset : M \in \mathrm{Pot}`L \,.\supset_M .\, \dot{\exists} \,!\, M :. \supset \vdash . \mathrm{Prop}$

***330·625.** $\vdash : \kappa \in FM \,.\, L, M \in \kappa_\iota \,.\, Q \in \mathrm{Pot}`(L \,|\, M) \,.\, S \in \kappa \,.\supset .\, S \,|\, Q \mathrel{\dot{\subset}} Q \,|\, S$

Dem.

$\vdash . *330{\cdot}62 . \supset \vdash : \mathrm{Hp} \,.\supset .\, S \,|\, L \,|\, M \mathrel{\dot{\subset}} L \,|\, M \,|\, S \qquad (1)$

$\vdash . *34{\cdot}34 . \supset$

$\vdash : \mathrm{Hp} \,.\, R \in \mathrm{Pot}`(L \,|\, M) \,.\, S \,|\, R \mathrel{\dot{\subset}} R \,|\, S \,.\supset .\, S \,|\, R \,|\, L \,|\, M \mathrel{\dot{\subset}} R \,|\, S \,|\, L \,|\, M$

$[(1)] \qquad \mathrel{\dot{\subset}} R \,|\, L \,|\, M \,|\, S \qquad (2)$

$\vdash . (1) . (2) . \mathrm{Induct} . \supset \vdash . \mathrm{Prop}$

***330·626.** $\vdash : \kappa \in FM - \iota`\iota`\dot{\Lambda} \,.\, L, M \in \kappa_\iota \,.\supset .\, \dot{\Lambda} \sim\in \mathrm{Pot}`(L \,|\, M)$

Dem.

$\vdash . *330{\cdot}611 . \qquad \supset \vdash : \mathrm{Hp} \,.\supset .\, \dot{\exists} \,!\, L \,|\, M \qquad (1)$

$\vdash . *330{\cdot}621{\cdot}625 . \supset \vdash : \mathrm{Hp} \,.\, Q \in \mathrm{Pot}`(L \,|\, M) \,.\, \dot{\exists} \,!\, Q \,.\supset .\, \dot{\exists} \,!\, Q \,|\, L \qquad (2)$

$\vdash . *330{\cdot}625 . \qquad \supset \vdash : \mathrm{Hp} \,.\, Q \in \mathrm{Pot}`(L \,|\, M) \,.\, S \in \kappa \,.\supset .\, S \,|\, Q \,|\, L \mathrel{\dot{\subset}} Q \,|\, S \,|\, L$

$[*330{\cdot}62] \qquad \mathrel{\dot{\subset}} Q \,|\, L \,|\, S \qquad (3)$

$\vdash . (2) . (3) . *330{\cdot}621 . \supset \vdash : \mathrm{Hp} \,.\, Q \in \mathrm{Pot}`(L \,|\, M) \,.\, \dot{\exists} \,!\, Q \,.\supset .\, \dot{\exists} \,!\, Q \,|\, L \,|\, M \qquad (4)$

$\vdash . (1) . (4) . \mathrm{Induct} . \supset \vdash . \mathrm{Prop}$

***330·627.** $\vdash : \kappa \epsilon FM - \iota`\iota`\Lambda \,.\, L, M \epsilon \kappa_\iota \,.\, P \epsilon \text{Pot}`M \,.\supset .\, \dot{\exists} ! P | L \,.\, \dot{\exists} ! L | P$

Dem.

$$\begin{array}{lll}
\vdash . *330·611 . & \supset \vdash : \text{Hp} . \supset . \dot{\exists} ! M | L . \dot{\exists} ! L | M & (1) \\
\vdash . *330·623 . & \supset \vdash : \text{Hp} . S \epsilon \kappa . \supset . S | P | L \subset P | S | L . & \\
[*330·62] & \supset . S | P | L \subset P | L | S & (2) \\
\vdash . (2) . *330·622 . & \supset \vdash : \text{Hp} . \dot{\exists} ! P | L . \supset . \dot{\exists} ! M | P | L & (3) \\
\vdash . (1) . (3) . \text{Induct} . & \supset \vdash : \text{Hp} . \supset . \dot{\exists} ! P | L & (4) \\
\vdash . (2) . *330·621 . & \supset \vdash : \text{Hp} . \dot{\exists} ! L | P . \supset . \dot{\exists} ! L | P | M & (5) \\
\vdash . (1) . (5) . \text{Induct} . & \supset \vdash : \text{Hp} . \supset . \dot{\exists} ! L | P & (6) \\
\vdash . (4) . (6) . \supset \vdash . \text{Prop} & &
\end{array}$$

***330·63.** $\vdash : \kappa \epsilon FM \,.\, L, M \epsilon \kappa_\iota \,.\, E ! L`x \,.\, E ! L`M`x \,.\supset .\, L`M`x = M`L`x$

Dem.

$$\begin{array}{ll}
\vdash . *330·56 . \supset \vdash : \text{Hp} . Q, R, S, T \epsilon \kappa . L = \breve{Q} | R . M = \breve{S} | T . \supset . & \\
& \breve{Q}`R`\breve{S}`T`x = \breve{Q}`\breve{S}`R`T`x \\
[*330·5] & = \breve{S}`\breve{Q}`T`R`x \\
[*330·56.\text{Hp}] & = \breve{S}`T`\breve{Q}`R`x : \supset \vdash . \text{Prop}
\end{array}$$

***330·64.** $\vdash :. \kappa \epsilon FM \,.\, L, M \epsilon \kappa_\iota \,.\supset :$

$$E ! L`x \,.\, E ! L`M`x \,.\equiv .\, E ! M`x \,.\, E ! M`L`x \quad [*330·63]$$

***330·641.** $\vdash :. \kappa \epsilon FM \,.\, L, M \epsilon \kappa_\iota \,.\, E ! L`x \,.\, E ! M`x \,.\supset :$

$$E ! L`M`x \,.\equiv .\, E ! M`L`x \,.\equiv .\, L`M`x = M`L`x \quad [*330·63·64]$$

***330·642.** $\vdash : \kappa \epsilon FM - \iota`\iota`\Lambda \,.\, L, M \epsilon \kappa_\iota \,.\supset .\, (\exists x) \,.\, E ! L`x \,.\, E ! L`M`x$

Dem.

$$\begin{array}{l}
\vdash . *330·21 . \supset \\
\vdash : \text{Hp} . \supset . (\exists P, Q, R, S, x) . P, Q, R, S \epsilon \kappa . L = \breve{P} | Q . M = \breve{R} | S . E ! \breve{P}`\breve{R}`x . \\
[*330·53·54] \supset . (\exists P, Q, R, S, x) . P, Q, R, S \epsilon \kappa . L = \breve{P} | Q . M = \breve{R} | S . \\
\qquad E ! \breve{P}`Q`x . E ! \breve{P}`Q`\breve{R}`S`x : \supset \vdash . \text{Prop}
\end{array}$$

***330·643.** $\vdash : \kappa \epsilon FM \,.\, P \epsilon \kappa \,.\, L \epsilon \kappa_\iota \,.\, E ! L`x \,.\supset .\, P`L`x = L`P`x \quad [*330·56·5]$

***330·65.** $\vdash : \kappa \epsilon FM \,.\, Q, R, S, T \epsilon \kappa \,.\, \breve{R}`Q`x = \breve{T}`S`x \,.\supset .\, T`Q`x = R`S`x$

Dem.

$$\begin{array}{ll}
\vdash . *72·24 . \supset \vdash : \text{Hp} . \supset . Q`x = R`\breve{T}`S`x & \\
[*330·56] & = \breve{T}`R`S`x . \\
[*72·24] & \supset . T`Q`x = R`S`x : \supset \vdash . \text{Prop}
\end{array}$$

***330·66.** $\vdash :. \kappa \in FM \,.\, Q, R, S, T \in \kappa \,.\, \mathrm{E}\,!\, \breve{R}\text{‘}Q\text{‘}x \,.\, \mathrm{E}\,!\, \breve{T}\text{‘}S\text{‘}x \,.\, \supset :$

$$\breve{R}\text{‘}Q\text{‘}x = \breve{T}\text{‘}S\text{‘}x \,.\, \equiv \,.\, T\text{‘}Q\text{‘}x = R\text{‘}S\text{‘}x$$

Dem.

$\vdash \,.\, *330{\cdot}56 \,.\, \supset \vdash : \mathrm{Hp} \,.\, T\text{‘}Q\text{‘}x = R\text{‘}S\text{‘}x \,.\, \supset \,.\, T\text{‘}\breve{R}\text{‘}Q\text{‘}x = \breve{R}\text{‘}R\text{‘}S\text{‘}x$

[*72·241] $\qquad = S\text{‘}x \,.$

[*72·241] $\qquad \supset \,.\, \breve{R}\text{‘}Q\text{‘}x = \breve{T}\text{‘}S\text{‘}x \qquad (1)$

$\vdash \,.\, (1) \,.\, *330{\cdot}65 \,.\, \supset \vdash \,.\, \mathrm{Prop}$

***330·7.** $\vdash : \kappa \in FM \,.\, P, Q \in \kappa \,.\, \rho \in \mathrm{NC\,ind} - \iota\text{‘}0 \,.\, \mathrm{E}\,!\, Q\text{‘}(\breve{P}\,|\,Q)^{\rho -_c 1}\text{‘}\breve{P}\text{‘}x \,.\, \supset \,.$

$$Q\text{‘}(\breve{P}\,|\,Q)^{\rho -_c 1}\text{‘}\breve{P}\text{‘}x = (\breve{P}\,|\,Q)^{\rho}\text{‘}x$$

Dem.

$\vdash \,.\, *330{\cdot}56 \,.\, *301{\cdot}2 \,.\, \supset$

$\vdash : \mathrm{Hp} \,.\, \mathrm{E}\,!\, Q\text{‘}(\breve{P}\,|\,Q)^{0}\text{‘}P\text{‘}x \,.\, \supset \,.\, Q\text{‘}(\breve{P}\,|\,Q)^{0}\text{‘}\breve{P}\text{‘}x = (\breve{P}\,|\,Q)^{1}\text{‘}x \qquad (1)$

$\vdash \,.\, *330{\cdot}56 \,.\, *301{\cdot}21 \,.\, \supset$

$\vdash :. \mathrm{Hp} : \mathrm{E}\,!\, Q\text{‘}(\breve{P}\,|\,Q)^{\rho -_c 1}\text{‘}\breve{P}\text{‘}x \,.\, \supset_x \,.\, Q\text{‘}(\breve{P}\,|\,Q)^{\rho -_c 1}\text{‘}\breve{P}\text{‘}x = (\breve{P}\,|\,Q)^{\rho}\text{‘}x : \supset :$

$\qquad \mathrm{E}\,!\, Q\text{‘}(\breve{P}\,|\,Q)^{\rho}\text{‘}\breve{P}\text{‘}x \,.\, \supset \,.\, Q\text{‘}(\breve{P}\,|\,Q)^{\rho}\text{‘}\breve{P}\text{‘}x = (\breve{P}\,|\,Q)^{\rho}\text{‘}Q\text{‘}\breve{P}\text{‘}x$

[*330·56.*301·21] $\qquad = (\breve{P}\,|\,Q)^{\rho + 1}\text{‘}x \qquad (2)$

$\vdash \,.\, (1) \,.\, (2) \,.\, \mathrm{Induct} \,.\, \supset \vdash \,.\, \mathrm{Prop}$

***330·71.** $\vdash : \kappa \in FM \,.\, P, Q \in \kappa \,.\, \rho \in \mathrm{NC\,ind} - \iota\text{‘}0 \,.\, \mathrm{E}\,!\, \breve{P}^{\rho}\text{‘}x \,.\, \supset \,.\, \mathrm{E}\,!\, (\breve{P}\,|\,Q)^{\rho}\text{‘}x$

Dem.

$\vdash \,.\, *330{\cdot}54 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \mathrm{E}\,!\, \breve{P}^{1}\text{‘}x \,.\, \supset \,.\, \mathrm{E}\,!\, (\breve{P}\,|\,Q)^{1}\text{‘}x \qquad (1)$

$\vdash \,.\, *301{\cdot}21 \,.\, \supset \vdash :. \mathrm{Hp} : \mathrm{E}\,!\, \breve{P}^{\rho}\text{‘}x \,.\, \supset_x \,.\, \mathrm{E}\,!\, (\breve{P}\,|\,Q)^{\rho}\text{‘}x : \supset :$

$\qquad \mathrm{E}\,!\, \breve{P}^{\rho +_c 1}\text{‘}x \,.\, \supset \,.\, \mathrm{E}\,!\, (\breve{P}\,|\,Q)^{\rho}\text{‘}\breve{P}\text{‘}x \,.$

[*330·52] $\qquad \supset \,.\, \mathrm{E}\,!\, Q\text{‘}(\breve{P}\,|\,Q)^{\rho}\text{‘}\breve{P}\text{‘}x \,.$

[*330·7] $\qquad \supset \,.\, \mathrm{E}\,!\, (\breve{P}\,|\,Q)^{\rho +_c 1}\text{‘}x \qquad (2)$

$\vdash \,.\, (1) \,.\, (2) \,.\, \mathrm{Induct} \,.\, \supset \vdash \,.\, \mathrm{Prop}$

***330·711.** $\vdash : \kappa \in FM \,.\, Q \in s\text{‘}\mathrm{Pot}\text{“}\kappa \,.\, \supset \,.\, \mathrm{C}\text{‘}Q = s\text{‘}\mathrm{C}\text{“}\kappa$

Dem.

$\vdash \,.\, *330{\cdot}52 \,.\, \supset \vdash : \mathrm{Hp} \,.\, P \in \kappa \,.\, \supset \,.\, \mathrm{C}\text{‘}P = s\text{‘}\mathrm{C}\text{“}\kappa \qquad (1)$

$\vdash \,.\, *37{\cdot}322 \,.\, \supset$

$\vdash : \mathrm{Hp} \,.\, P \in \kappa \,.\, Q \in \mathrm{Pot}\text{‘}P \,.\, \mathrm{C}\text{‘}Q = s\text{‘}\mathrm{C}\text{“}\kappa \,.\, \supset \,.\, \mathrm{C}\text{‘}(Q\,|\,P) = s\text{‘}\mathrm{C}\text{“}\kappa \qquad (2)$

$\vdash \,.\, (1) \,.\, (2) \,.\, \mathrm{Induct} \,.\, \supset \vdash \,.\, \mathrm{Prop}$

***330·72.** $\vdash : \kappa \in FM - \iota`\iota`\Lambda \,.\, L, M \in \kappa_\iota \,.\, \rho, \sigma \in \mathrm{NC\ induct} \,.\supset .\, \exists ! \text{Ɑ}`L^\rho \cap \text{Ɑ}`M^\sigma$

Dem.

$\vdash .\, *330·711·23 \,.\supset$

$\vdash : \mathrm{Hp} \,.\, P, R \in \kappa \,.\supset .\, (\exists a) \,.\, E ! R^{\sigma}`a \,.\, R^{\sigma}`a \in \text{Ɑ}`P^\sigma \,.$

$[*330·52] \qquad \supset .\, (\exists a) \,.\, E ! P^{\sigma}`R^{\sigma}`a \qquad (1)$

$\vdash .\, *330·57 \,.\supset \vdash : \mathrm{Hp}(1) \,.\, x = P^{\sigma}`R^{\sigma}`a \,.\supset .\, E ! \breve{P}^{\sigma}`x \,.\, E ! \breve{R}^{\sigma}`x \qquad (2)$

$\vdash .\, (2) \,.\, *330·71 \,.\supset$

$\vdash : \mathrm{Hp}(2) \,.\, Q, S \in \kappa \,.\, L = \breve{P} \,|\, Q \,.\, M = \breve{R} \,|\, S \,.\supset .\, E ! L^{\rho}`x \,.\, E ! M^{\sigma}`x \,.$

$[*33·43] \qquad \supset .\, x \in \text{Ɑ}`L^\rho \cap \text{Ɑ}`M^\sigma \qquad (3)$

$\vdash .\, (1) \,.\, (3) \,.\supset \vdash .\, \mathrm{Prop}$

We have "NC induct" in the above proposition, not "NC ind," because it is necessary to have $E ! L^\rho \,.\, E ! M^\sigma$, and by *301·16 this may fail if either ρ or σ is null in the type of L and M. The existence of a family does not imply the axiom of infinity, since the family may be cyclic.

***330·73.** $\vdash : \kappa \in FM \,.\, P, Q \in \kappa \,.\, \rho \in \mathrm{NC\ ind} \,.\, E ! (\breve{P} \,|\, Q)^{\rho}`x \,.\supset .$

$$(\breve{P} \,|\, Q)^{\rho}`x = \breve{P}^{\rho}`Q^{\rho}`x$$

Dem.

$\vdash .\, *330·56 \,.\supset \vdash : \mathrm{Hp} \,.\, E ! \breve{P}`y \,.\supset .\, Q`\breve{P}`y = \breve{P}`Q`y \qquad (1)$

$\vdash .\, (1) \,.\supset \vdash : \mathrm{Hp} \,.\, Q`\breve{P}^{\rho -_c 1}`x = \breve{P}^{\rho -_c 1}`Q`x \,.\, E ! \breve{P}^{\rho}`y \,.\supset .\, Q`\breve{P}^{\rho}`y = \breve{P}`Q`\breve{P}^{\rho -_c 1}`y$

$[\mathrm{Hp}] \qquad = \breve{P}`\breve{P}^{\rho -_c 1}`Q`y$

$[*301·23] \qquad = \breve{P}^{\rho}`Q`y \qquad (2)$

$\vdash .\, (1) \,.\, (2) \,.\, \mathrm{Induct} \,.\supset \vdash : \mathrm{Hp} \,.\, E ! \breve{P}^{\rho}`y \,.\supset .\, Q`\breve{P}^{\rho}`y = \breve{P}^{\rho}`Q`y \qquad (3)$

$\vdash .\, *301·23 \,.\supset \vdash : \mathrm{Hp} \,.\, (\breve{P} \,|\, Q)^{\rho}`x = \breve{P}^{\rho}`Q^{\rho}`x \,.\, E ! (\breve{P} \,|\, Q)^{\rho +_c 1}`x \,.\supset .$

$\qquad (\breve{P} \,|\, Q)^{\rho +_c 1}`x = \breve{P}`Q`\breve{P}^{\rho}`Q^{\rho}`x$

$[(3)] \qquad = \breve{P}`\breve{P}^{\rho}`Q`Q^{\rho}`x$

$[*301·23] \qquad = \breve{P}^{\rho +_c 1}`Q^{\rho +_c 1}`x \qquad (4)$

$\vdash .\, (4) \,.\, \mathrm{Induct} \,.\supset \vdash .\, \mathrm{Prop}$

*331. CONNECTED FAMILIES.

Summary of *331.

A "connected point" of a family κ is a point of the field of κ from which every member of the field can be reached by a member of κ or the converse of a member. That is, if a is a connected point, we are to have

$$x \in s`\text{Ɑ}``\kappa \,.\supset_x .\, (\exists R) \,.\, R \in \kappa \,.\, x(R \cup \breve{R})\, a$$

as well as $a \in s`\text{Ɑ}``\kappa$. This amounts to saying that every member of $s`\text{Ɑ}``\kappa$ is of the form $R`a$ or $\breve{R}`a$, where $R \in \kappa$. The definition is

***331·01.** $\mathrm{conx}`\kappa = s`\text{Ɑ}``\kappa \cap \hat{a}\,(\overrightarrow{\dot{s}`\kappa}`a \cup \overleftarrow{\dot{s}`\kappa}`a = s`\text{Ɑ}``\kappa)$ Df

Here we include the factor $s`\text{Ɑ}``\kappa$ in the definition, in order to exclude the case when $\kappa = \iota`\dot{\Lambda}$. If $s`\text{Ɑ}``\kappa$ were not included, we should have $\mathrm{conx}`\iota`\dot{\Lambda} = \mathrm{V}$, whereas with the above definition $\mathrm{conx}`\iota`\dot{\Lambda} = \Lambda$.

In the case of any other family, the factor $s`\text{Ɑ}``\kappa$ makes no difference, since if $s`\text{Ɑ}``\kappa$ exists,

$$\overrightarrow{\dot{s}`\kappa}`a \cup \overleftarrow{\dot{s}`\kappa}`a = s`\text{Ɑ}``\kappa \,.\supset .\, a \in C`\dot{s}`\kappa,$$

and if κ is a family, $C`\dot{s}`\kappa = s`\text{Ɑ}``\kappa$. But in the case of $\iota`\dot{\Lambda}$, the factor $s`\text{Ɑ}``\kappa$ insures that no connected point exists, thus securing, conversely, that a family which has a connected point is not $\iota`\dot{\Lambda}$. This is convenient since the case of $\iota`\dot{\Lambda}$, which is trivial, would often otherwise have to be explicitly excluded.

The definition would be more analogous to the definition of a connected relation in *202 if we put

$$\mathrm{conx}`\kappa = s`\text{Ɑ}``\kappa \cap \hat{a}\,(\overrightarrow{\dot{s}`\kappa_\partial}`a \cup \overleftarrow{\dot{s}`\kappa_\partial}`a \cup \iota`a = s`\text{Ɑ}``\kappa) \quad \text{Df.}$$

But this definition fails to give us the information that there is a member of κ which relates a to itself, whereas our definition does give this information, and hence leads to the proof that $I \upharpoonright s`\text{Ɑ}``\kappa \in \kappa$, *i.e.* that there is a zero vector.

We say that a family "is connected" when it has at least one connected point, *i.e.* we put

***331·02.** $FM\,\mathrm{conx} = FM \cap \hat{\kappa}\,(\exists\,!\,\mathrm{conx}`\kappa)$ Df

When *all* points of the field are connected points, the family "has connexity" (cf. $*334\cdot27$), provided $\kappa \neq \iota`\Lambda$. For the present, we only assume that at least one of the points of the field is a connected point. To take an illustration: the family whose members are of the form $(\times_0 \mu) \upharpoonright (\text{NC induct} - \iota`0)$, where $\mu \in \text{NC induct} - \iota`0$, has only one connected point, namely 1. If we had taken positive and negative integers, both as multipliers and as constituting the field, we should have had two connected points, namely 1 and -1.

Almost all our future propositions on vector-families will be confined to *connected* families. In the present number, we prove first that in a connected family κ, the vector which relates a connected point to itself also relates any other member of the field to itself ($*331\cdot2$), whence it follows that $I \upharpoonright s`\text{C}``\kappa$ is a member of $\kappa$ ($*331\cdot22$), and that every other member of $\kappa$ is wholly contained in diversity ($*331\cdot23$), and that $\kappa \cup \text{Cnv}``\kappa \subset \kappa_\iota$ ($*331\cdot24$). We next prove that the product of two members of κ is a member of κ or of $\text{Cnv}``\kappa$ ($*331\cdot33$). We then proceed to consider $\kappa_\iota$, and prove at once the two fundamental properties of $\kappa_\iota$ in a connected family, namely (1) that between any two members of $s`\text{C}``\kappa$ there is a relation which is a member of κ_ι ($*331\cdot4$), and (2) that two members of κ_ι whose logical product exists are identical ($*331\cdot42$). From these two propositions it follows that there is just one member of κ_ι which relates any two members of $s`\text{C}``\kappa$ ($*331\cdot43$). Finally we prove that any power of a member of $\kappa$ is a member of $\kappa \cup \text{Cnv}``\kappa$ ($*331\cdot54$), and that any power of a member of κ_ι is contained in some member of κ_ι ($*331\cdot56$).

Stated symbolically, the above propositions are as follows:

$*331\cdot2$. $\vdash :. \kappa \in FM . a \in \text{conx}`\kappa . x \in s`\text{C}``\kappa . R \in \kappa . \supset : R`a = a . \equiv . R`x = x$

$*331\cdot22$. $\vdash : \kappa \in FM\,\text{conx} . \supset . I \upharpoonright s`\text{C}``\kappa \in \kappa$

$*331\cdot23$. $\vdash : \kappa \in FM\,\text{conx} . \supset . \kappa \subset \text{Rl}`I \cup \text{Rl}`J$

$*331\cdot24$. $\vdash : \kappa \in FM\,\text{conx} . \supset . \kappa \cup \text{Cnv}``\kappa \subset \kappa_\iota$

$*331\cdot33$. $\vdash : \kappa \in FM\,\text{conx} . \supset . s`\kappa \,|_{,,}``\kappa \subset \kappa \cup \text{Cnv}``\kappa$

$*331\cdot4$. $\vdash : \kappa \in FM\,\text{conx} . x, y \in s`\text{C}``\kappa . \supset . (\exists L) . L \in \kappa_\iota . x = L`y$

$*331\cdot42$. $\vdash :. \kappa \in FM\,\text{conx} . L, M \in \kappa_\iota . \supset : \dot{\exists} ! L \dot{\cap} M . \equiv . L = M$

$*331\cdot43$. $\vdash : \kappa \in FM\,\text{conx} . x, y \in s`\text{C}``\kappa . \supset . \hat{M}(M \in \kappa_\iota . xMy) \in 1$

$*331\cdot54$. $\vdash : \kappa \in FM\,\text{conx} . P \in \kappa . \supset . \text{Pot}`P \subset \kappa \cup \text{Cnv}``\kappa$

$*331\cdot56$. $\vdash : \kappa \in FM\,\text{conx} . L \in \kappa_\iota . M \in \text{Pot}`L . \supset . (\exists N) . N \in \kappa_\iota . M \,\dot{\subset}\, N$

$*331\cdot01$. $\text{conx}`\kappa = s`\text{C}``\kappa \cap \hat{a}(\overrightarrow{\dot{s}`\kappa`a} \cup \overleftarrow{\dot{s}`\kappa`a} = s`\text{C}``\kappa)$ Df

$*331\cdot02$. $FM\,\text{conx} = FM \cap \hat{\kappa}(\exists ! \text{conx}`\kappa)$ Df

***331·1.** $\vdash : a \,\epsilon\, \mathrm{conx}'\kappa \,.\equiv .\, a \,\epsilon\, s'\mathrm{C}''\kappa \,.\, \overrightarrow{\dot{s}'\kappa}'a \cup \overleftarrow{\dot{s}'\kappa}'a = s'\mathrm{C}''\kappa$ [(*331·01)]

***331·11.** $\vdash :. a \,\epsilon\, \mathrm{conx}'\kappa \,.\equiv : a \,\epsilon\, s'\mathrm{C}''\kappa : x \,\epsilon\, s'\mathrm{C}''\kappa \,.\supset_x .\, (\exists R) \,.\, R \,\epsilon\, \kappa \,.\, x (R \,\dot{\cup}\, \breve{R})\, a$
[*331·1]

***331·12.** $\vdash : \exists \,!\, \mathrm{conx}'\kappa \,.\supset .\, \kappa \neq \iota'\dot{\Lambda}$ [*331·1]

***331·13.** $\vdash :. \kappa \,\epsilon\, \mathrm{Cl\,ex}'\mathrm{cr}'\alpha \,.\supset : a \,\epsilon\, \mathrm{conx}'\kappa \,.\equiv .\, \kappa \neq \iota'\dot{\Lambda} \,.\, \overrightarrow{\dot{s}'\kappa}'a \cup \overleftarrow{\dot{s}'\kappa}'a = s'\mathrm{C}''\kappa$

Dem.

$\vdash .\, \text{*}53\cdot24 \,.\supset \vdash : \mathrm{Hp} \,.\, \kappa \neq \iota'\dot{\Lambda} \,.\, \overrightarrow{\dot{s}'\kappa}'a \cup \overleftarrow{\dot{s}'\kappa}'a = s'\mathrm{C}''\kappa \,.\supset .\, \exists \,!\, \overrightarrow{\dot{s}'\kappa}'a \cup \overleftarrow{\dot{s}'\kappa}'a \,.$
[*330·13] $\supset .\, a \,\epsilon\, s'\mathrm{C}''\kappa$ (1)

$\vdash .\, (1) \,.\, \text{*}331\cdot1\cdot12 \,.\supset \vdash .$ Prop

***331·131.** $\vdash :: \kappa \,\epsilon\, \mathrm{Cl\,ex}'\mathrm{cr}'\alpha \,.\supset :. a \,\epsilon\, \mathrm{conx}'\kappa \,.\equiv : \kappa \neq \iota'\dot{\Lambda} : x \,\epsilon\, s'\mathrm{C}''\kappa \,.\supset_x .$
$(\exists R) \,.\, R \,\epsilon\, \kappa \,.\, x (R \,\dot{\cup}\, \breve{R})\, a$ [*331·13]

***331·14.** $\vdash :. \lambda = \kappa \cup \mathrm{Cnv}''\kappa \,.\supset : a \,\epsilon\, \mathrm{conx}'\kappa \,.\equiv .\, a \,\epsilon\, s'\mathrm{C}''\kappa \,.\, \overrightarrow{\dot{s}'\lambda}'a = s'\mathrm{C}''\kappa$
[*331·1]

***331·2.** $\vdash :. \kappa \,\epsilon\, FM \,.\, a \,\epsilon\, \mathrm{conx}'\kappa \,.\, x \,\epsilon\, s'\mathrm{C}''\kappa \,.\, R \,\epsilon\, \kappa \,.\supset : R'a = a \,.\equiv .\, R'x = x$

Dem.

$\vdash .\, \text{*}331\cdot11 \,.\quad \supset \vdash : \mathrm{Hp} \,.\supset .\, (\exists S) \,.\, S \,\epsilon\, \kappa \,.\, x (S \,\dot{\cup}\, \breve{S})\, a$ (1)

$\vdash .\, \text{*}330\cdot5 \,.\quad \supset \vdash : \mathrm{Hp} \,.\, S \,\epsilon\, \kappa \,.\, x = S'a \,.\, R'a = a \,.\supset .\, R'x = S'R'a$

[Hp] $= S'a$

[Hp] $= x$ (2)

$\vdash .\, \text{*}330\cdot56 \,.\quad \supset \vdash : \mathrm{Hp} \,.\, S \,\epsilon\, \kappa \,.\, x = \breve{S}'a \,.\, R'a = a \,.\supset .\, R'x = \breve{S}'R'a$

[Hp] $= \breve{S}'a$

[Hp] $= x$ (3)

$\vdash .\, (1) \,.\, (2) \,.\, (3) \,.\supset \vdash :. \mathrm{Hp} \,.\supset : R'a = a \,.\supset .\, R'x = x$ (4)

Similarly $\vdash :. \mathrm{Hp} \,.\supset : R'x = x \,.\supset .\, R'a = a$ (5)

$\vdash .\, (4) \,.\, (5) \,.\supset \vdash .$ Prop

***331·21.** $\vdash :. \kappa \,\epsilon\, FM \,.\, a \,\epsilon\, \mathrm{conx}'\kappa \,.\, R \,\epsilon\, \kappa \,.\supset : R'a = a \,.\equiv .\, I \upharpoonright s'\mathrm{C}''\kappa = R$

Dem.

$\vdash .\, \text{*}331\cdot2 \,.\quad \supset \vdash : \mathrm{Hp} \,.\, R'a = a \,.\supset .\, I \upharpoonright s'\mathrm{C}''\kappa = R$ (1)

$\vdash .\, \text{*}331\cdot1 \,.\quad \supset \vdash : \mathrm{Hp} \,.\, I \upharpoonright s'\mathrm{C}''\kappa = R \,.\supset .\, R'a = a$ (2)

$\vdash .\, (1) \,.\, (2) \,.\supset \vdash .$ Prop

***331·22.** $\vdash : \kappa \,\epsilon\, FM\,\mathrm{conx} \,.\supset .\, I \upharpoonright s'\mathrm{C}''\kappa \,\epsilon\, \kappa$

Dem.

$\vdash .\, \text{*}331\cdot11 \,.\supset \vdash : \mathrm{Hp} \,.\, a \,\epsilon\, \mathrm{conx}'\kappa \,.\supset .\, (\exists R) \,.\, R \,\epsilon\, \kappa \,.\, R'a = a$ (1)

$\vdash .\, (1) \,.\, \text{*}331\cdot21 \,.\supset \vdash .$ Prop

***331·23.** $\vdash : \kappa \epsilon FM \text{ conx} . \supset . \kappa \subset \text{Rl}`I \cup \text{Rl}`J$

Dem.

$\vdash . *331\cdot2\cdot21 . \supset \vdash : \text{Hp} . R \epsilon \kappa . \dot{\exists} ! R \dot{\wedge} I . \supset . R \mathrel{\mathsf{G}} I : \supset \vdash . \text{Prop}$

***331·24.** $\vdash : \kappa \epsilon FM \text{ conx} . \supset . \kappa \cup \text{Cnv}``\kappa \subset \kappa_\iota$ [*330·42 . *331·22]

***331·25.** $\vdash : \kappa \epsilon FM \text{ conx} - 1 . \supset . \exists ! \kappa \cap \text{Rl}`J$ [*331·22·23]

***331·26.** $\vdash : \kappa \epsilon FM \text{ conx} - 1 . \supset . \dot{s}`\kappa, \dot{s}`\kappa_\iota \sim \epsilon \kappa_\iota$

Dem.

$\vdash . *331\cdot22\cdot25 . \supset \vdash : \text{Hp} . \supset . (\exists a, R, S, x) . R, S \epsilon \kappa . aRa . aSx . a \neq x .$

[*71·172.*41·11] $\supset . \dot{s}`\kappa \sim \epsilon 1 \rightarrow 1 .$ (1)

[*331·24] $\supset . \dot{s}`\kappa_\iota \sim \epsilon 1 \rightarrow 1$ (2)

$\vdash . (1) . (2) . *330\cdot52 . \supset \vdash . \text{Prop}$

***331·31.** $\vdash :. \kappa \epsilon FM . a \epsilon \text{conx}`\kappa . x \epsilon s`\text{C}``\kappa . P \epsilon \kappa . N \epsilon \kappa_\iota . \supset :$
$P`a = N`a . \equiv . P`x = N`x$

Dem.

$\vdash . *331\cdot11 . *330\cdot4 . \supset$

$\vdash : \text{Hp} . \supset . (\exists Q, R, S) . Q, R, S \epsilon \kappa . x (Q \cup \breve{Q}) a . N = \breve{R} | S$ (1)

$\vdash . *330\cdot5 . \supset$

$\vdash : \text{Hp} . Q, R, S \epsilon \kappa . x = Q`a . N = \breve{R} | S . P`a = N`a . \supset . P`x = Q`\breve{R}`S`a$

[*330·56] $= \breve{R}`Q`S`a$

[*330·5] $= \breve{R}`S`Q`a$

[Hp] $= N`x$ (2)

$\vdash . *330\cdot56 . \supset$

$\vdash : \text{Hp} . Q, R, S \epsilon \kappa . x = \breve{Q}`a . N = \breve{R} | S . P`a = N`a . \supset . P`x = \breve{Q}`\breve{R}`S`a$

[*330·5] $= \breve{R}`\breve{Q}`S`a$

[*330·56.Hp] $= \breve{R}`S`\breve{Q}`a$

[Hp] $= N`x$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash : \text{Hp} . P`a = N`a . \supset . P`x = N`x$ (4)

Similarly $\vdash : \text{Hp} . P`x = N`x . \supset . P`a = N`a$ (5)

$\vdash . (4) . (5) . \supset \vdash . \text{Prop}$

***331·32.** $\vdash :. \kappa \epsilon FM \text{ conx} . P \epsilon \kappa . N \epsilon \kappa_\iota . \supset : \dot{\exists} ! P \dot{\wedge} N . \equiv . P = N$

Dem.

$\vdash . *331\cdot31 . \supset \vdash :: \text{Hp} . a \epsilon \text{conx}`\kappa . \supset :. x, y \epsilon s`\text{C}``\kappa . \supset :$
$P`x = N`x . \equiv . P`a = N`a . \equiv . P`y = N`y$ (1)

$\vdash . (1) . (*331\cdot02) . \supset \vdash :. \text{Hp} . \supset : x, y \epsilon s`\text{C}``\kappa . P`x = N`x . \supset . P`y = N`y :$

[*33·45.*72·94] $\supset : \dot{\exists} ! P \dot{\wedge} N . \supset . P = N$ (2)

$\vdash . *331\cdot12 . *330\cdot16 . \supset \vdash :. \text{Hp} . \supset : P = N . \supset . \dot{\exists} ! P \dot{\wedge} N$ (3)

$\vdash . (2) . (3) . \supset \vdash . \text{Prop}$

$*331\cdot321$. $\vdash :. \kappa \in FM \text{ conx} . P, Q \in \kappa . \supset : \dot{\exists} ! P \mathbin{\dot{\cap}} Q . \equiv . P = Q$ $[*331\cdot32\cdot24]$

$*331\cdot33$. $\vdash : \kappa \in FM \text{ conx} . \supset . s'\kappa \underset{,,}{|}{}''\kappa \subset \kappa \cup \text{Cnv}''\kappa$

Dem.

$\vdash . *331\cdot11 . \supset \vdash :. \text{Hp} . \supset : (\exists a) : P, Q \in \kappa . \supset_{P,Q} . (\exists R) . (P'Q'a)(R \cup \breve{R})\, a$ (1)

$\vdash . *330\cdot5 . \supset$

$\vdash : \text{Hp} . P, Q, R \in \kappa . P'Q'a = R'a . S \in \kappa . y = S'a . \supset . P'Q'y = S'P'Q'a$

$[\text{Hp}]$ $= S'R'a$

$[*330\cdot5 . \text{Hp}]$ $= R'y$ (2)

$\vdash . *330\cdot56 . \supset$

$\vdash : \text{Hp} . P, Q, R \in \kappa . P'Q'a = R'a . S \in \kappa . y = \breve{S}'a . \supset . P'Q'y = \breve{S}'P'Q'a$

$[\text{Hp}]$ $= \breve{S}'R'a$

$[*330\cdot56 . \text{Hp}]$ $= R'y$ (3)

$\vdash . (2) . (3) . *331\cdot11 . \supset \vdash : \text{Hp} . P, Q, R \in \kappa . P'Q'a = R'a . \supset . P \,|\, Q = R$ (4)

Similarly $\vdash : \text{Hp} . P, Q, R \in \kappa . P'Q'a = \breve{R}'a . \supset . P \,|\, Q = \breve{R}$ (5)

$\vdash . (1) . (4) . (5) . \supset$

$\vdash :. \text{Hp} . P, Q \in \kappa . \supset : (\exists R) : R \in \kappa : P \,|\, Q = R . \vee . P \,|\, Q = \breve{R} :. \supset \vdash . \text{Prop}$

$*331\cdot4$. $\vdash : \kappa \in FM \text{ conx} . x, y \in s'\text{C}''\kappa . \supset . (\exists L) . L \in \kappa_\iota . x = L'y$

Dem.

$\vdash . *331\cdot11 . \supset \vdash : \text{Hp} . \supset . (\exists a, R, S) . R, S \in \kappa . x (R \cup \breve{R})\, a . y (S \cup \breve{S})\, a$ (1)

$\vdash . *330\cdot56 . \supset \vdash : \text{Hp} . R, S \in \kappa . x = R'a . y = S'a . \supset . x = \breve{S}'R'y .$

$[*330\cdot4]$ $\supset . (\exists L) . L \in \kappa_\iota . x = L'y$ (2)

$\vdash . *331\cdot24\cdot33 . \supset$

$\vdash : \text{Hp} . R, S \in \kappa . x = R'a . y = \breve{S}'a . \supset . R \,|\, S \in \kappa_\iota . x = (R \,|\, S)'y$ (3)

$\vdash . *331\cdot24\cdot33 . \supset$

$\vdash : \text{Hp} . R, S \in \kappa . x = \breve{R}'a . y = S'a . \supset . \breve{R} \,|\, \breve{S} \in \kappa_\iota . x = (\breve{R} \,|\, \breve{S})'y$ (4)

$\vdash . *330\cdot4 . \supset$

$\vdash : \text{Hp} . R, S \in \kappa . x = \breve{R}'a . y = \breve{S}'a . \supset . \breve{R} \,|\, S \in \kappa_\iota . x = (\breve{R} \,|\, S)'y$ (5)

$\vdash . (1) . (2) . (3) . (4) . (5) . \supset \vdash . \text{Prop}$

$*331\cdot41$. $\vdash : \kappa \in FM \text{ conx} . \supset . \dot{s}'\kappa_\iota = (s'\text{C}''\kappa) \uparrow (s'\text{C}''\kappa)$ $[*331\cdot4]$

$*331\cdot42$. $\vdash :. \kappa \in FM \text{ conx} . L, M \in \kappa_\iota . \supset : \dot{\exists} ! L \mathbin{\dot{\cap}} M . \equiv . L = M$

Dem.

$\vdash . *330\cdot6 . *331\cdot12 . \supset \vdash : \text{Hp} . L = M . \supset . \dot{\exists} ! L \mathbin{\dot{\cap}} M$ (1)

$\vdash . *331\cdot4 . \supset$

$\vdash : \text{Hp} . L'x = M'x . \text{E} ! L'y . \supset . (\exists N) . N \in \kappa_\iota . N'x = y . \text{E} ! L'y .$

[$*330{\cdot}63$] $\supset . (\exists N) . N \in \kappa_\iota . N`x = y . L`y = N`L`x$

[Hp] $= N`M`x$

[$*330{\cdot}63$] $= M`N`x$

[$*13{\cdot}12$] $\supset . L`y = M`y$ (2)

Similarly $\vdash : \text{Hp} . L`x = M`x . E ! M`y . \supset . L`y = M`y$ (3)

$\vdash . (2) . (3) . *71{\cdot}35 . \supset \vdash : \text{Hp} . \dot{\exists} ! L \dot{\wedge} M . \supset . L = M$ (4)

$\vdash . (1) . (4) . \supset \vdash . \text{Prop}$

$*331{\cdot}43$. $\vdash : \kappa \in FM \text{ conx} . x, y \in s`\text{C}``\kappa . \supset . \hat{M}(M \in \kappa_\iota . xMy) \in 1$

Dem.

$\vdash . *331{\cdot}4 . \supset \vdash : \text{Hp} . \supset . (\exists M) . (M \in \kappa_\iota . xMy)$ (1)

$\vdash . *331{\cdot}42 . \supset \vdash : \text{Hp} . L, M \in \kappa_\iota . xMy . xLy . \supset . L = M$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

$*331{\cdot}44$. $\vdash :. \kappa \in FM \text{ conx} . P, Q \in \kappa . \supset : \dot{\exists} ! P \dot{\wedge} Q . \equiv . P = Q$ [$*331{\cdot}42{\cdot}24$]

$*331{\cdot}45$. $\vdash :. \kappa \in FM \text{ conx} . L, M, N \in \kappa_\iota . \supset :$

$\dot{\exists} ! L | M \dot{\wedge} N . \equiv . L | M = N \upharpoonright \text{C}`(L | M)$

Dem.

$\vdash . *330{\cdot}611 . \supset \vdash : \text{Hp} . L | M = N \upharpoonright \text{C}`(L | M) . \supset . \dot{\exists} ! L | M \dot{\wedge} N$ (1)

$\vdash . *330{\cdot}63 . \supset \vdash : \text{Hp} . L`M`x = N`x . E ! L`M`y . X \in \kappa_\iota . y = X`x . \supset .$

$L`M`y = L`X`M`x . E ! L`M`x . E ! L`X`M`x . E ! X`x .$

[$*330{\cdot}63$] $\supset . L`M`y = X`L`M`x . E ! X`x .$

[Hp] $\supset . L`M`y = X`N`x . E ! X`x .$

[$*330{\cdot}63$] $\supset . L`M`y = N`X`x$

[Hp] $= N`y$ (2)

$\vdash . (2) . *331{\cdot}4 . \supset \vdash : \text{Hp} . L`M`x = N`x . y \in \text{C}`(L | M) . \supset . L`M`y = N`y$ (3)

$\vdash . (1) . (3) . \supset \vdash . \text{Prop}$

$*331{\cdot}46$. $\vdash :. \text{Hp} *331{\cdot}45 . \supset : M | L = N \upharpoonright \text{C}`(M | L) . \equiv . L | M = N \upharpoonright \text{C}`(L | M)$

Dem.

$\vdash . *330{\cdot}642{\cdot}63 . \supset \vdash : \text{Hp} . L | M = N \upharpoonright \text{C}`(L | M) . \supset . (\exists x) . M`L`x = N`x .$

[$*331{\cdot}45$] $\supset . M | L = N \upharpoonright \text{C}`(M | L)$ (1)

Similarly $\vdash : \text{Hp} . M | L = N \upharpoonright \text{C}`(M | L) . \supset . L | M = N \upharpoonright \text{C}`(L | M)$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

$*331{\cdot}47$. $\vdash : \kappa \in FM \text{ conx} . L, M \in \kappa_\iota . \supset . (\exists N) . N \in \kappa_\iota . L | M \mathrel{\dot{\subset}} N . M | L \mathrel{\dot{\subset}} N$

[$*331{\cdot}46{\cdot}45{\cdot}4$]

$*331{\cdot}48$. $\vdash : \kappa \in FM . L \in \kappa_\iota . \exists ! \text{conx}`\kappa \cap C`L . \supset . L \in \kappa \cup \text{Cnv}``\kappa$

Dem.

$\vdash . *330{\cdot}41 . \supset \vdash :. \text{Hp} . a \in \text{conx}`\kappa \cap C`L . \supset : L, \breve{L} \in \kappa_\iota : E ! L`a . \vee . E ! \breve{L}`a :$

[$*331{\cdot}11$] $\supset : L, \breve{L} \in \kappa_\iota : (\exists R) : R \in \kappa \cup \text{Cnv}``\kappa : L`a = R`a . \vee . \breve{L}`a = R`a :$

[$*331{\cdot}24{\cdot}42$] $\supset : (\exists R) : R \in \kappa \cup \text{Cnv}``\kappa : L = R . \vee . \breve{L} = R :. \supset \vdash . \text{Prop}$

***331·5.** $\vdash : \kappa \epsilon FM\,\mathrm{conx} \,.\, P \epsilon \kappa \,.\, L \epsilon \kappa_\iota \,.\, \supset \,.\, L | P, \breve{P} | L \epsilon \kappa_\iota$

Dem.

$\vdash \,.\, *331{\cdot}33 \,.\, \supset$

$\vdash : \mathrm{Hp} \,.\, Q, R \epsilon \kappa \,.\, L = \breve{Q} | R \,.\, \supset \,.\, (\exists S) \,.\, S \epsilon \kappa \cup \mathrm{Cnv}``\kappa \,.\, L | P = \breve{Q} | S$ (1)

$\vdash \,.\, *330{\cdot}4 \,.\, \supset \vdash : \mathrm{Hp}\,(1) \,.\, S \epsilon \kappa \,.\, L | P = \breve{Q} | S \,.\, \supset \,.\, L | P \epsilon \kappa_\iota$ (2)

$\vdash \,.\, *34{\cdot}2 \,.\, \supset$

$\vdash : \mathrm{Hp}\,(1) \,.\, S \epsilon \mathrm{Cnv}``\kappa \,.\, L | P = \breve{Q} | S \,.\, \supset \,.\, (\exists T) \,.\, T \epsilon \kappa \,.\, L | P = \mathrm{Cnv}`(T | Q) \,.$

[*331·33] $\supset \,.\, L | P \epsilon \kappa \cup \mathrm{Cnv}``\kappa \,.$

[*331·24] $\supset \,.\, L | P \epsilon \kappa_\iota$ (3)

$\vdash \,.\, (1) \,.\, (2) \,.\, (3) \,.\, *330{\cdot}41 \,.\, \supset \vdash \,.\,$ Prop

***331·51.** $\vdash : \kappa \epsilon FM\,\mathrm{conx} \,.\, P \epsilon \kappa \,.\, \supset \,.\, \mathrm{Pot}`P \subset \kappa_\iota$ [*331·5 . Induct]

***331·52.** $\vdash : \kappa \epsilon FM\,\mathrm{conx} \,.\, P, Q \epsilon \kappa \,.\, L \epsilon \kappa_\iota \,.\, \supset \,.\, \breve{P} | L | Q \epsilon \kappa_\iota$ [*331·5]

***331·53.** $\vdash : \kappa \epsilon FM\,\mathrm{conx} \,.\, P, Q \epsilon \kappa \,.\, \rho, \sigma \epsilon \mathrm{NC\,induct} \,.\, \supset \,.\, \breve{P}^\rho | Q^\sigma \epsilon \kappa_\iota$
[*331·5 . Induct . *331·51 . *330·43]

***331·54.** $\vdash : \kappa \epsilon FM\,\mathrm{conx} \,.\, P \epsilon \kappa \,.\, \supset \,.\, \mathrm{Pot}`P \subset \kappa \cup \mathrm{Cnv}``\kappa$

Dem.

$\vdash \,.\, *330{\cdot}711 \,.\, \supset \vdash : \mathrm{Hp} \,.\, a \epsilon \mathrm{conx}`\kappa \,.\, Q \epsilon \mathrm{Pot}`P \,.\, \supset \,.\, E\,!\, Q`a \,.$

[*331·11] $\supset \,.\, (\exists T) \,.\, T \epsilon \kappa \cup \mathrm{Cnv}``\kappa \,.\, Q`a = T`a \,.$

[*331·51·42·24] $\supset \,.\, Q \epsilon \kappa \cup \mathrm{Cnv}``\kappa : \supset \vdash \,.\,$ Prop

***331·55.** $\vdash : \kappa \epsilon FM\,\mathrm{conx} \,.\, P, Q \epsilon \kappa_\iota \,.\, \rho \epsilon \mathrm{NC\,induct} \,.\, \supset \,.$
$(\breve{P} | Q)^\rho \subseteq \breve{P}^\rho | Q^\rho \,.\, \breve{P}^\rho | Q^\rho \epsilon \kappa_\iota$ [*330·73 . *331·53]

***331·56.** $\vdash : \kappa \epsilon FM\,\mathrm{conx} \,.\, L \epsilon \kappa_\iota \,.\, M \epsilon \mathrm{Pot}`L \,.\, \supset \,.\, (\exists N) \,.\, N \epsilon \kappa_\iota \,.\, M \subseteq N$
[*331·55 . *330·4]

$*332$. ON THE REPRESENTATIVE OF A RELATION IN A FAMILY.

Summary of $*332$.

We saw at the end of the last number ($*331\cdot56$) that any power of a member of κ_ι is contained in a member of κ_ι. When a relation is contained in a member of κ_ι, we call this member the "representative" of the relation in the family. For purposes connected with the application of ratio, the "representative" is an important function of a relation, especially when the relation concerned is a power of a member of κ_ι. By the definition of ratio ($*303\cdot01$), we shall have $L(\rho/\sigma)M$ if $\dot{\exists} ! L^\sigma \dot{\land} M^\rho$ and $\rho \operatorname{Prm} \sigma$. Now if L^σ and M^ρ each have a representative, then they must have the *same* representative if $\dot{\exists} ! L^\sigma \dot{\land} M^\rho$ (by $*331\cdot42$). Hence we are enabled to substitute an equality for $\dot{\exists} ! L^\sigma \dot{\land} M^\rho$ in dealing with ratios of members of κ_ι. The elementary properties of representatives are dealt with in the present number.

We denote the representative of P in the family κ by "$\operatorname{rep}_\kappa{}^\backprime P$." In order to insure $\mathrm{E} ! \operatorname{rep}_\kappa{}^\backprime P$ under all circumstances, we do not define $\operatorname{rep}_\kappa{}^\backprime P$ as the only member of κ_ι which contains P, but as the logical sum of the class of members of κ_ι which contain P, *i.e.* we put

$*332\cdot01$. $\operatorname{rep}_\kappa{}^\backprime P = \dot{s}{}^\backprime(\kappa_\iota \cap \overleftarrow{\mathrm{G}}{}^\backprime P)$ Df

In a connected family, if P is not null, $\kappa_\iota \cap \overleftarrow{\mathrm{G}}{}^\backprime P$ cannot have more than one member ($*332\cdot21$), and therefore the representative of P, if it is not null, must be a member of κ_ι ($*332\cdot22$). If P is a member of κ_ι, it is its own representative ($*332\cdot241$).

We prove in this number that, if $P, Q, R, \ldots$ have existent representatives, the representative of their relative product (unless this product is null) is the representative of the relative product of their representatives ($*332\cdot37$). Among other important propositions in this number are the following:

$*332\cdot32$. $\vdash : \kappa \in FM \operatorname{conx} . L, M \in \kappa_\iota . \supset . \operatorname{rep}_\kappa{}^\backprime(L \mid M) = \operatorname{rep}_\kappa{}^\backprime(M \mid L)$

$*332\cdot51$. $\vdash : \kappa \in FM \operatorname{conx} . P, Q \in \kappa . \supset . \operatorname{rep}_\kappa{}^\backprime(P \mid \breve{Q}) = \breve{Q} \mid P$

$*332\cdot53$. $\vdash : \kappa \in FM \operatorname{conx} . P, Q \in \kappa . \rho \in \mathrm{NC} \operatorname{induct} . \supset . \operatorname{rep}_\kappa{}^\backprime(\breve{P} \mid Q)^\rho = \breve{P}^\rho \mid Q^\rho$

$*332\cdot61$. $\vdash : \kappa \in FM \operatorname{conx} . L \in \kappa_\iota . \supset . \operatorname{rep}_\kappa{}^{\backprime\backprime}\operatorname{Potid}{}^\backprime L \subset \kappa_\iota$

***332·8.** $\vdash : \kappa \in FM \text{ conx} \,.\, L, M \in \kappa_\iota \,.\, \xi \in \text{NC ind} \,.\, \supset .$
$$\text{rep}_\kappa\text{‘}(L \mid M)^\xi = \text{rep}_\kappa\text{‘}(L^\xi \mid M^\xi)$$

***332·81.** $\vdash : \kappa \in FM \text{ conx} \,.\, \nu, \sigma \in \text{NC ind} - \iota\text{‘}0 \,.\, L \in \kappa_\iota \,.\, \supset .$
$$\text{rep}_\kappa\text{‘}L^{\nu \times_c \sigma} = \text{rep}_\kappa\text{‘}(\text{rep}_\kappa\text{‘}L^\nu)^\sigma$$

***332·01.** $\text{rep}_\kappa\text{‘}P = \dot{s}\text{‘}(\kappa_\iota \cap \overleftarrow{\text{G}}\text{‘}P)$ Df

***332·1.** $\vdash . \text{rep}_\kappa\text{‘}P = \dot{s}\text{‘}(\kappa_\iota \cap \overleftarrow{\text{G}}\text{‘}P) = \hat{x}\hat{y}\,\{(\exists L) \,.\, L \in \kappa_\iota \,.\, P \mathrel{\text{G}} L \,.\, xLy\}$
[(*332·01)]

***332·11.** $\vdash : \dot{\exists} ! \text{rep}_\kappa\text{‘}P \,.\, \supset .\, P \mathrel{\text{G}} \text{rep}_\kappa\text{‘}P$ [*332·1]

***332·12.** $\vdash : \dot{\exists} ! \text{rep}_\kappa\text{‘}P \,.\, \supset .\, \exists ! (\kappa_\iota \cap \overleftarrow{\text{G}}\text{‘}P)$ [*332·1]

***332·13.** $\vdash . \text{rep}_\kappa\text{‘}\dot{\Lambda} = \dot{s}\text{‘}\kappa_\iota$ [*332·1]

***332·14.** $\vdash : P \mathrel{\text{G}} Q \,.\, \supset .\, \text{rep}_\kappa\text{‘}Q \mathrel{\text{G}} \text{rep}_\kappa\text{‘}P$ [*332·1]

***332·15.** $\vdash . \text{rep}_\kappa\text{‘}\breve{P} = \text{Cnv}\text{‘}\text{rep}_\kappa\text{‘}P$

Dem.

$\vdash . *330{\cdot}41 \,.\, \supset \vdash .\, \kappa_\iota \cap \overleftarrow{\text{G}}\text{‘}\breve{P} = \text{Cnv}\text{“}(\kappa_\iota \cap \overleftarrow{\text{G}}\text{‘}P)$ (1)

$\vdash . (1) \,.\, *332{\cdot}1 \,.\, \supset \vdash .$ Prop

***332·16.** $\vdash : \kappa = \iota\text{‘}\dot{\Lambda} \,.\, \supset .\, \text{rep}_\kappa\text{‘}P = \dot{\Lambda}$ [*332·1]

***332·2.** $\vdash :. \kappa \in FM - \iota\text{‘}\iota\text{‘}\dot{\Lambda} \,.\, \supset : \exists ! (\kappa_\iota \cap \overleftarrow{\text{G}}\text{‘}P) \,.\, \equiv .\, \dot{\exists} ! \text{rep}_\kappa\text{‘}P$

Dem.

$\vdash . *330{\cdot}6 \,.\, \supset \vdash : \text{Hp} \,.\, \exists ! (\kappa_\iota \cap \overleftarrow{\text{G}}\text{‘}P) \,.\, \supset .\, \exists ! (\kappa_\iota \cap \overleftarrow{\text{G}}\text{‘}P) - \iota\text{‘}\dot{\Lambda} \,.$

[*332·1] $\supset .\, \dot{\exists} ! \text{rep}_\kappa\text{‘}P$ (1)

$\vdash . (1) \,.\, *332{\cdot}12 \,.\, \supset \vdash .$ Prop

***332·21.** $\vdash : \kappa \in FM \text{ conx} \,.\, \dot{\exists} ! P \,.\, \supset .\, (\kappa_\iota \cap \overleftarrow{\text{G}}\text{‘}P) \in 0 \cup 1$

Dem.

$\vdash . *331{\cdot}42 \,.\, \supset \vdash : \text{Hp} \,.\, L, M \in \kappa_\iota \,.\, P \mathrel{\text{G}} L \,.\, P \mathrel{\text{G}} M \,.\, \supset .\, L = M : \supset \vdash .$ Prop

***332·22.** $\vdash :. \kappa \in FM \text{ conx} \,.\, \dot{\exists} ! P \,.\, \supset : \text{rep}_\kappa\text{‘}P \in \kappa_\iota \,.\, \vee .\, \text{rep}_\kappa\text{‘}P = \dot{\Lambda}$

Dem.

$\vdash . *332{\cdot}21{\cdot}12 \,.\, \supset \vdash : \text{Hp} \,.\, \dot{\exists} ! \text{rep}_\kappa\text{‘}P \,.\, \supset .\, (\kappa_\iota \cap \overleftarrow{\text{G}}\text{‘}P) \in 1 \,.$

[*332·1] $\supset .\, \text{rep}_\kappa\text{‘}P \in \kappa_\iota : \supset \vdash .$ Prop

***332·23.** $\vdash :. \kappa \in FM \text{ conx} \,.\, \dot{\exists} ! P \,.\, \supset : \text{rep}_\kappa\text{‘}P \in \kappa_\iota \,.\, \equiv .\, \exists ! (\kappa_\iota \cap \overleftarrow{\text{G}}\text{‘}P)$

Dem.

$\vdash . *332{\cdot}22{\cdot}2 \,.\, \supset \vdash : \text{Hp} \,.\, \text{rep}_\kappa\text{‘}P \sim\in \kappa_\iota \,.\, \supset .\, (\kappa_\iota \cap \overleftarrow{\text{G}}\text{‘}P) = \Lambda$ (1)

$\vdash . *330{\cdot}6 \,.\, \supset \vdash : \text{Hp} \,.\, \text{rep}_\kappa\text{‘}P \in \kappa_\iota \,.\, \supset .\, \dot{\exists} ! \text{rep}_\kappa\text{‘}P \,.$

[*332·2] $\supset .\, \exists ! (\kappa_\iota \cap \overleftarrow{\text{G}}\text{‘}P)$ (2)

$\vdash . (1) \,.\, (2) \,.\, \supset \vdash .$ Prop

***332·231.** $\vdash :. \kappa \in FM \operatorname{conx} - 1 . \supset : \operatorname{rep}_\kappa{}'P \in \kappa_\iota . \equiv . \dot{\exists} ! P . \exists ! (\kappa_\iota \cap \overleftarrow{\mathrm{G}}{}'P)$

Dem.

$\vdash . *331{\cdot}26 . \supset \vdash :. \mathrm{Hp} . \supset : \operatorname{rep}_\kappa{}'P \in \kappa_\iota . \supset . \operatorname{rep}_\kappa{}'P \neq \dot{s}'\kappa_\iota .$

$[*332{\cdot}13] \qquad \supset . P \neq \dot{\Lambda} \qquad (1)$

$\vdash . (1) . *332{\cdot}23 . \supset \vdash . \mathrm{Prop}$

***332·232.** $\vdash :. \kappa \in FM \operatorname{conx} - 1 . \supset : \operatorname{rep}_\kappa{}'P \in \kappa_\iota . \equiv . \dot{\exists} ! P . \dot{\exists} ! \operatorname{rep}_\kappa{}'P$
$[*332{\cdot}231{\cdot}2]$

***332·24.** $\vdash :. \kappa \in FM \operatorname{conx} . \dot{\exists} ! P . \supset : L \in (\kappa_\iota \cap \overleftarrow{\mathrm{G}}{}'P) . \equiv . \dot{\exists} ! \operatorname{rep}_\kappa{}'P . \operatorname{rep}_\kappa{}'P = L$

Dem.

$\vdash . *332{\cdot}21{\cdot}1 . \quad \supset \vdash :. \mathrm{Hp} . \supset : L \in \kappa_\iota \cap \overleftarrow{\mathrm{G}}{}'P . \supset . \operatorname{rep}_\kappa{}'P = L \qquad (1)$

$\vdash . *332{\cdot}2 . \quad \supset \vdash :. \mathrm{Hp} . \supset : L \in \kappa_\iota \cap \overleftarrow{\mathrm{G}}{}'P . \supset . \dot{\exists} ! \operatorname{rep}_\kappa{}'P \qquad (2)$

$\vdash . *332{\cdot}22 . \quad \supset \vdash :. \mathrm{Hp} . \supset : \dot{\exists} ! \operatorname{rep}_\kappa{}'P . \supset . \operatorname{rep}_\kappa{}'P \in \kappa_\iota :$

$[*13{\cdot}12] \qquad \supset : \dot{\exists} ! \operatorname{rep}_\kappa{}'P . \operatorname{rep}_\kappa{}'P = L . \supset . L \in \kappa_\iota \qquad (3)$

$\vdash . (3) . *332{\cdot}11 . \supset \vdash :. \mathrm{Hp} . \supset : \dot{\exists} ! \operatorname{rep}_\kappa{}'P . \operatorname{rep}_\kappa{}'P = L . \supset . L \in (\kappa_\iota \cap \overleftarrow{\mathrm{G}}{}'P) \qquad (4)$

$\vdash . (1) . (2) . (4) . \supset \vdash . \mathrm{Prop}$

***332·241.** $\vdash : \kappa \in FM \operatorname{conx} . P \in \kappa_\iota . \supset . P = \operatorname{rep}_\kappa{}'P$

Dem.

$\vdash . *332{\cdot}24 . \supset \vdash :. \mathrm{Hp} . \dot{\exists} ! P . \supset : P \in \kappa_\iota \cap \overleftarrow{\mathrm{G}}{}'P . \equiv . \dot{\exists} ! \operatorname{rep}_\kappa{}'P . \operatorname{rep}_\kappa{}'P = P :$

$[\mathrm{Hp}] \qquad \supset : \operatorname{rep}_\kappa{}'P = P \qquad (1)$

$\vdash . *330{\cdot}6 . \quad \supset \vdash : \mathrm{Hp} . \sim \dot{\exists} ! P . \supset . \kappa = \iota'\dot{\Lambda} .$

$[*332{\cdot}13] \qquad \supset . \operatorname{rep}_\kappa{}'P = \dot{\Lambda} \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***332·242.** $\vdash : \kappa \in FM \operatorname{conx} . \dot{\exists} ! P . \dot{\exists} ! \operatorname{rep}_\kappa{}'P . \supset . \operatorname{rep}_\kappa{}'P = \operatorname{rep}_\kappa{}'\operatorname{rep}_\kappa{}'P$

Dem.

$\vdash . *332{\cdot}22 . \supset \vdash : \mathrm{Hp} . \supset . \operatorname{rep}_\kappa{}'P \in \kappa_\iota \qquad (1)$

$\vdash . (1) . *332{\cdot}241 . \supset \vdash . \mathrm{Prop}$

***332·243.** $\vdash : \kappa \in FM \operatorname{conx} . \dot{\exists} ! P . P \mathrel{\mathrm{G}} I \restriction s'\mathrm{C\!l}''\kappa . \supset . \operatorname{rep}_\kappa{}'P = I \restriction s'\mathrm{C\!l}''\kappa$
$[*332{\cdot}24 . *330{\cdot}43]$

***332·244.** $\vdash :. \kappa \in FM \operatorname{conx} - 1 . \supset :$

$\dot{\exists} ! P . P \mathrel{\mathrm{G}} I \restriction s'\mathrm{C\!l}''\kappa . \equiv . \operatorname{rep}_\kappa{}'P = I \restriction s'\mathrm{C\!l}''\kappa$

Dem.

$\vdash . *331{\cdot}26 . *330{\cdot}43 . \supset \vdash :. \mathrm{Hp} . \supset : \dot{s}'\kappa_\iota \neq I \restriction s'\mathrm{C\!l}''\kappa :$

$[*332{\cdot}13] \qquad \supset : \operatorname{rep}_\kappa{}'P = I \restriction s'\mathrm{C\!l}''\kappa . \supset . \dot{\exists} ! P \qquad (1)$

$\vdash . *332{\cdot}11 . \quad \supset \vdash :. \mathrm{Hp} . \supset : \operatorname{rep}_\kappa{}'P = I \restriction s'\mathrm{C\!l}''\kappa . \supset . P \mathrel{\mathrm{G}} I \restriction s'\mathrm{C\!l}''\kappa \qquad (2)$

$\vdash . (1) . (2) . *332{\cdot}243 . \supset \vdash . \mathrm{Prop}$

***332·25.** $\vdash : \kappa \in FM \text{ conx} . \dot{\exists} ! P . \dot{\exists} ! \text{rep}_\kappa{}^{\backprime}Q . P \mathrel{G} Q . \supset . \text{rep}_\kappa{}^{\backprime}P = \text{rep}_\kappa{}^{\backprime}Q$

Dem.

$\vdash . *332·11 . \supset \vdash : \text{Hp} . \supset . P \mathrel{G} \text{rep}_\kappa{}^{\backprime}Q$ (1)

$\vdash . *332·22 . \supset \vdash : \text{Hp} . \supset . \text{rep}_\kappa{}^{\backprime}Q \in \kappa_\iota$ (2)

$\vdash . (1) . (2) . *332·24 . \supset \vdash . \text{Prop}$

***332·26.** $\vdash : \kappa \in FM \text{ conx} . \dot{\exists} ! P \dot{\frown} Q . \dot{\exists} ! \text{rep}_\kappa{}^{\backprime}P . \dot{\exists} ! \text{rep}_\kappa{}^{\backprime}Q . \supset .$
$\text{rep}_\kappa{}^{\backprime}P = \text{rep}_\kappa{}^{\backprime}Q = \text{rep}_\kappa{}^{\backprime}(P \dot{\frown} Q)$ [*332·25]

***332·27.** $\vdash : \kappa \in FM \text{ conx} . \dot{\exists} ! P . \dot{\exists} ! \text{rep}_\kappa{}^{\backprime}Q . \dot{\exists} ! Q \dot{\frown} \text{rep}_\kappa{}^{\backprime}P . \supset . \text{rep}_\kappa{}^{\backprime}P = \text{rep}_\kappa{}^{\backprime}Q$

Dem.

$\vdash . *332·11 . \supset \vdash : \text{Hp} . \supset . Q \mathrel{G} \text{rep}_\kappa{}^{\backprime}Q .$

[Hp] $\supset . \dot{\exists} ! \text{rep}_\kappa{}^{\backprime}P \dot{\frown} \text{rep}_\kappa{}^{\backprime}Q$ (1)

$\vdash . *332·22 . \supset \vdash : \text{Hp} . \supset . \text{rep}_\kappa{}^{\backprime}P, \text{rep}_\kappa{}^{\backprime}Q \in \kappa_\iota$ (2)

$\vdash . (1) . (2) . *331·42 . \supset \vdash . \text{Prop}$

***332·31.** $\vdash : \kappa \in FM \text{ conx} . L, M \in \kappa_\iota . \supset . \text{rep}_\kappa{}^{\backprime}(L \mid M) \in \kappa_\iota$
[*330·611 . *331·47·12 . *332·23]

***332·32.** $\vdash : \kappa \in FM \text{ conx} . L, M \in \kappa_\iota . \supset . \text{rep}_\kappa{}^{\backprime}(L \mid M) = \text{rep}_\kappa{}^{\backprime}(M \mid L)$
[*330·611 . *331·47·12 . *332·24]

***332·33.** $\vdash : \kappa \in FM \text{ conx} . \text{rep}_\kappa{}^{\backprime}P, \text{rep}_\kappa{}^{\backprime}Q \in \kappa_\iota . \dot{\exists} ! P \mid Q . \supset . \text{rep}_\kappa{}^{\backprime}(P \mid Q)$
$= \text{rep}_\kappa{}^{\backprime}\{(\text{rep}_\kappa{}^{\backprime}P) \mid (\text{rep}_\kappa{}^{\backprime}Q)\} = \text{rep}_\kappa{}^{\backprime}\{(\text{rep}_\kappa{}^{\backprime}P) \mid Q\} = \text{rep}_\kappa{}^{\backprime}\{P \mid \text{rep}_\kappa{}^{\backprime}Q\}$

Dem.

$\vdash . *330·6 . *331·12 . \supset \vdash : \text{Hp} . \supset . \dot{\exists} ! \text{rep}_\kappa{}^{\backprime}P . \dot{\exists} ! \text{rep}_\kappa{}^{\backprime}Q .$

[*332·11] $\supset . P \mathrel{G} \text{rep}_\kappa{}^{\backprime}P . Q \mathrel{G} \text{rep}_\kappa{}^{\backprime}Q .$ (1)

[Hp] $\supset . \dot{\exists} ! P \mid \text{rep}_\kappa{}^{\backprime}Q$ (2)

$\vdash . *330·6 . *332·31 . (1) . \supset$

$\vdash : \text{Hp} . \supset . P \mid \text{rep}_\kappa{}^{\backprime}Q \mathrel{G} \text{rep}_\kappa{}^{\backprime}P \mid \text{rep}_\kappa{}^{\backprime}Q . \dot{\exists} ! \text{rep}_\kappa{}^{\backprime}\{\text{rep}_\kappa{}^{\backprime}P \mid \text{rep}_\kappa{}^{\backprime}Q\} .$

[(2).*332·25]

$\supset . \text{rep}_\kappa{}^{\backprime}(P \mid \text{rep}_\kappa{}^{\backprime}Q) = \text{rep}_\kappa{}^{\backprime}\{\text{rep}_\kappa{}^{\backprime}P \mid \text{rep}_\kappa{}^{\backprime}Q\} . \dot{\exists} ! \text{rep}_\kappa{}^{\backprime}(P \mid \text{rep}_\kappa{}^{\backprime}Q)$ (3)

Similarly $\vdash : \text{Hp} . \supset . \text{rep}_\kappa{}^{\backprime}\{(\text{rep}_\kappa{}^{\backprime}P) \mid Q\} = \text{rep}_\kappa{}^{\backprime}\{(\text{rep}_\kappa{}^{\backprime}P) \mid (\text{rep}_\kappa{}^{\backprime}Q)\}$ (4)

$\vdash . (1) . \supset \vdash : \text{Hp} . \supset . P \mid Q \mathrel{G} P \mid \text{rep}_\kappa{}^{\backprime}Q .$

[Hp.(3).*332·25] $\supset . \text{rep}_\kappa{}^{\backprime}(P \mid Q) = \text{rep}_\kappa{}^{\backprime}(P \mid \text{rep}_\kappa{}^{\backprime}Q)$ (5)

$\vdash . (3) . (4) . (5) . \supset \vdash . \text{Prop}$

***332·34.** $\vdash : \text{Hp } *332·33 . \supset . \text{rep}_\kappa{}^{\backprime}(P \mid Q) \in \kappa_\iota$ [*332·31·33]

***332·35.** $\vdash : \kappa \in FM \text{ conx} . L, M, N \in \kappa_\iota . \supset .$
$\text{rep}_\kappa{}^{\backprime}(L \mid M \mid N) = \text{rep}_\kappa{}^{\backprime}\{L \mid \text{rep}_\kappa{}^{\backprime}(M \mid N)\} = \text{rep}_\kappa{}^{\backprime}[\{\text{rep}_\kappa{}^{\backprime}(L \mid M)\} \mid N]$
[*330·613 . *332·31·33]

***332·36.** $\vdash : \text{Hp } *332·35 . \supset . \text{rep}_\kappa{}^{\backprime}(L \mid M \mid N) \in \kappa_\iota$ [*332·35·31]

***332·37.** $\vdash : \kappa \in FM \,\mathrm{conx} \,.\, \mathrm{rep}_\kappa{}‘P, \mathrm{rep}_\kappa{}‘Q, \mathrm{rep}_\kappa{}‘R \in \kappa_\iota \,.\, \dot{\exists}\,!\, P \,|\, Q \,|\, R \,.\, \supset .$

$$\begin{aligned} \mathrm{rep}_\kappa{}‘(P \,|\, Q \,|\, R) &= \mathrm{rep}_\kappa{}‘\{\mathrm{rep}_\kappa{}‘P \,|\, \mathrm{rep}_\kappa{}‘Q \,|\, \mathrm{rep}_\kappa{}‘R\} \\ &= \mathrm{rep}_\kappa{}‘\{\mathrm{rep}_\kappa{}‘P \,|\, \mathrm{rep}_\kappa{}‘R \,|\, \mathrm{rep}_\kappa{}‘Q\} \\ &= \mathrm{rep}_\kappa{}‘\{\mathrm{rep}_\kappa{}‘Q \,|\, \mathrm{rep}_\kappa{}‘R \,|\, \mathrm{rep}_\kappa{}‘P\} \end{aligned}$$

Dem.

$\vdash . *332·33 . \supset$

$\vdash : \mathrm{Hp} . \supset . \mathrm{rep}_\kappa{}‘(P \,|\, Q \,|\, R) = \mathrm{rep}_\kappa{}‘\{\mathrm{rep}_\kappa{}‘P \,|\, \mathrm{rep}_\kappa{}‘(Q \,|\, R)\}$

[*332·33] $\qquad = \mathrm{rep}_\kappa{}‘\{\mathrm{rep}_\kappa{}‘P \,|\, \mathrm{rep}_\kappa{}‘(\mathrm{rep}_\kappa{}‘Q \,|\, \mathrm{rep}_\kappa{}‘R)\}$ (1)

[*332·35] $\qquad = \mathrm{rep}_\kappa{}‘\{\mathrm{rep}_\kappa{}‘P \,|\, \mathrm{rep}_\kappa{}‘Q \,|\, \mathrm{rep}_\kappa{}‘R\}$ (2)

$\vdash . (1) . *332·32 . \supset$

$\vdash : \mathrm{Hp} . \supset . \mathrm{rep}_\kappa{}‘(P \,|\, Q \,|\, R) = \mathrm{rep}_\kappa{}‘\{\mathrm{rep}_\kappa{}‘P \,|\, \mathrm{rep}_\kappa{}‘(\mathrm{rep}_\kappa{}‘R \,|\, \mathrm{rep}_\kappa{}‘Q)\}$

[*332·35] $\qquad = \mathrm{rep}_\kappa{}‘\{\mathrm{rep}_\kappa{}‘P \,|\, \mathrm{rep}_\kappa{}‘R \,|\, \mathrm{rep}_\kappa{}‘Q\}$ (3)

$\vdash . (1) . *332·33·32 . \supset$

$\vdash : \mathrm{Hp} . \supset . \mathrm{rep}_\kappa{}‘(P \,|\, Q \,|\, R) = \mathrm{rep}_\kappa{}‘[\{\mathrm{rep}_\kappa{}‘(\mathrm{rep}_\kappa{}‘Q \,|\, \mathrm{rep}_\kappa{}‘R)\} \,|\, \mathrm{rep}_\kappa{}‘P]$

[*332·35] $\qquad = \mathrm{rep}_\kappa{}‘\{\mathrm{rep}_\kappa{}‘Q \,|\, \mathrm{rep}_\kappa{}‘R \,|\, \mathrm{rep}_\kappa{}‘P\}$ (4)

$\vdash . (2) . (3) . (4) . \supset \vdash . \mathrm{Prop}$

***332·41.** $\vdash :. \kappa \in FM \,\mathrm{conx} \,.\, L, M, N \in \kappa_\iota \,.\, \supset :$

$$\mathrm{rep}_\kappa{}‘(L \,|\, M) = \mathrm{rep}_\kappa{}‘(L \,|\, N) \,.\, \equiv .\, M = N$$

Dem.

$\vdash . *34·34 . \supset \vdash : \mathrm{Hp} . \mathrm{rep}_\kappa{}‘(L \,|\, M) = \mathrm{rep}_\kappa{}‘(L \,|\, N) . \supset .$

$\qquad \breve{L} \,|\, \mathrm{rep}_\kappa{}‘(L \,|\, M) = \breve{L} \,|\, \mathrm{rep}_\kappa{}‘(L \,|\, N) .$

[*332·35] $\qquad \supset . \mathrm{rep}_\kappa{}‘(\breve{L} \,|\, L \,|\, M) = \mathrm{rep}_\kappa{}‘(\breve{L} \,|\, L \,|\, N) .$

[*330·31] $\qquad \supset . \mathrm{rep}_\kappa{}‘M = \mathrm{rep}_\kappa{}‘N .$

[*332·241] $\qquad \supset . M = N : \supset \vdash . \mathrm{Prop}$

***332·411.** $\vdash :. \kappa \in FM \,\mathrm{conx} . L, M, N \in \kappa_\iota . \supset : \mathrm{rep}_\kappa{}‘(M \,|\, L) = \mathrm{rep}_\kappa{}‘(N \,|\, L) . \equiv . M = N$
[*332·32·41]

***332·42.** $\vdash : \kappa \in FM \,\mathrm{conx} \,.\, L, M \in \kappa_\iota \,.\, \supset .\, \mathrm{Cnv}‘\mathrm{rep}_\kappa{}‘(L \,|\, M) = \mathrm{rep}_\kappa{}‘(\breve{L} \,|\, \breve{M})$
[*332·32·15]

***332·43.** $\vdash :. \kappa \in FM \,\mathrm{conx} \,.\, L, M, N \in \kappa_\iota \,.\, \supset :$

$$\begin{aligned} N = \mathrm{rep}_\kappa{}‘(L \,|\, M) \,.\, \equiv .\, L = \mathrm{rep}_\kappa{}‘(N \,|\, \breve{M}) \,.\, \equiv .\, L = \mathrm{rep}_\kappa{}‘(\breve{M} \,|\, N) \,. \\ \equiv .\, M = \mathrm{rep}_\kappa{}‘(N \,|\, \breve{L}) \,.\, \equiv .\, M = \mathrm{rep}_\kappa{}‘(\breve{L} \,|\, N) \end{aligned}$$

Dem.

$\vdash . *332·35 . *330·41 . \supset$

$\vdash : \mathrm{Hp} . N = \mathrm{rep}_\kappa{}‘(L \,|\, M) . \supset . \mathrm{rep}_\kappa{}‘(L \,|\, M \,|\, \breve{M}) = \mathrm{rep}_\kappa{}‘(N \,|\, \breve{M}) .$

[*330·31] $\qquad \supset . \mathrm{rep}_\kappa{}‘L = \mathrm{rep}_\kappa{}‘(N \,|\, \breve{M}) .$

[*332·241] $\qquad \supset . L = \mathrm{rep}_\kappa{}‘(N \,|\, \breve{M}) .$ (1)

[*332·32.*330·41] $\qquad \supset . L = \mathrm{rep}_\kappa{}‘(\breve{M} \,|\, N)$ (2)

$\vdash . (1) . *330·41 . \supset \vdash : \mathrm{Hp} . L = \mathrm{rep}_\kappa{}‘(N \,|\, \breve{M}) . \supset . N = \mathrm{rep}_\kappa{}‘(L \,|\, M)$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash . \mathrm{Prop}$

***332·44.** $\vdash :. \kappa \epsilon FM \text{ conx} . L, M, N \epsilon \kappa_\iota . \supset : \text{rep}_\kappa\text{‘}(L \mid M) = N . \equiv . L \mid M \subset N$
[*330·6 . *332·24·31]

***332·45.** $\vdash :. \text{Hp} *332\cdot44 . \supset : \text{rep}_\kappa\text{‘}(L \mid M) = N . \equiv . \text{rep}_\kappa\text{‘}(L \mid M \mid \breve{N}) = I \upharpoonright s\text{‘}\text{G}\text{“}\kappa$

Dem.

$\vdash$ *332·35 . $\supset \vdash :. \text{Hp} . \supset : \text{rep}_\kappa\text{‘}(L \mid M) = N . \supset . \text{rep}_\kappa\text{‘}(L \mid M \mid \breve{N}) = \text{rep}_\kappa\text{‘}(N \mid \breve{N})$
[*332·24.*330·31] $= I \upharpoonright s\text{‘}\text{G}\text{“}\kappa$ (1)
$\vdash$. *332·35 . $\supset \vdash :. \text{Hp} . \supset : \text{rep}_\kappa\text{‘}(L \mid M \mid \breve{N}) = I \upharpoonright s\text{‘}\text{G}\text{“}\kappa . \supset .$
$\text{rep}_\kappa\text{‘}[\{\text{rep}_\kappa\text{‘}(L \mid M)\} \mid \breve{N}] = I \upharpoonright s\text{‘}\text{G}\text{“}\kappa .$
[*332·31·43] $\supset . \text{rep}_\kappa\text{‘}(L \mid M) = \text{rep}_\kappa\text{‘}N$
[*332·241] $= N$ (2)
$\vdash . (1) . (2) . \supset \vdash$. Prop

***332·46.** $\vdash :. \kappa \epsilon FM \text{ conx} . L, M \epsilon \kappa_\iota . \supset : L \mid M \subset I . \equiv . L = \breve{M}$

Dem.

$\vdash$. *330·43·611 . *332·243 . $\supset$
$\vdash : \text{Hp} . L \mid M \subset I . \supset . \text{rep}_\kappa\text{‘}(L \mid M) = I \upharpoonright s\text{‘}\text{G}\text{“}\kappa .$
[*332·43.*330·43] $\supset . L = \text{rep}_\kappa\text{‘}\breve{M}$
[*332·241.*330·41] $= \breve{M}$ (1)
$\vdash$. *71·191 . $\supset \vdash : \text{Hp} . L = \breve{M} . \supset . L \mid M \subset I$ (2)
$\vdash . (1) . (2) . \supset \vdash$. Prop

***332·51.** $\vdash : \kappa \epsilon FM \text{ conx} . P, Q \epsilon \kappa . \supset . \text{rep}_\kappa\text{‘}(P \mid \breve{Q}) = \breve{Q} \mid P$

Dem.

$\vdash$. *331·24 . *332·32 . $\supset \vdash : \text{Hp} . \supset . \text{rep}_\kappa\text{‘}(P \mid \breve{Q}) = \text{rep}_\kappa\text{‘}(\breve{Q} \mid P)$
[*332·241] $= \breve{Q} \mid P : \supset \vdash$. Prop

***332·52.** $\vdash : \kappa \epsilon FM \text{ conx} . P, Q, R, S \epsilon \kappa . \supset . \text{rep}_\kappa\text{‘}(P \mid \breve{Q} \mid R \mid \breve{S}) = \breve{Q} \mid \breve{S} \mid P \mid R$

Dem.

$\vdash$. *330·613 . *331·12·124 . $\supset \vdash : \text{Hp} . \supset . \dot{\exists} ! (P \mid \breve{Q}) \mid (R \mid \breve{S}) .$
[*332·33·51] $\supset . \text{rep}_\kappa\text{‘}(P \mid \breve{Q} \mid R \mid \breve{S}) = \text{rep}_\kappa\text{‘}(\breve{Q} \mid P \mid \breve{S} \mid R)$ (1)
$\vdash$. *330·561·611 . $\supset \vdash : \text{Hp} . \supset . \breve{Q} \mid P \mid \breve{S} \mid R \subset \breve{Q} \mid \breve{S} \mid P \mid R . \dot{\exists} ! \breve{Q} \mid P \mid \breve{S} \mid R$ (2)
$\vdash$. *331·52 . $\supset \vdash : \text{Hp} . \supset . \breve{Q} \mid \breve{S} \mid P \mid R \epsilon \kappa_\iota$ (3)
$\vdash . (1) . (2) . (3)$. *332·24 . $\supset \vdash$. Prop

***332·53.** $\vdash : \kappa \in FM \text{ conx} . P, Q \in \kappa . \rho \in \text{NC induct} . \supset . \text{rep}_\kappa{}^\prime(\breve{P} \mid Q)^\rho = \breve{P}^\rho \mid Q^\rho$

Dem.

$\vdash . *330·624 . \supset \vdash : \text{Hp} . \supset . \dot{\exists} ! (\breve{P} \mid Q)^\rho$ (1)

$\vdash . *330·73 . \supset \vdash : \text{Hp} . \supset . (\breve{P} \mid Q)^\rho \mathrel{\mathsf{G}} \breve{P}^\rho \mid Q^\rho$ (2)

$\vdash . *331·53 . \supset \vdash : \text{Hp} . \supset . \breve{P}^\rho \mid Q^\rho \in \kappa_\iota$ (3)

$\vdash . (1) . (2) . (3) . *332·24 . \supset \vdash . \text{Prop}$

***332·61.** $\vdash : \kappa \in FM \text{ conx} . L \in \kappa_\iota . \supset . \text{rep}_\kappa{}^{\prime\prime}\text{Potid}^\prime L \subset \kappa_\iota$

Dem.

$\vdash . *332·243 . *330·43 . \supset \vdash : \text{Hp} . \supset . \text{rep}_\kappa{}^\prime(I \upharpoonright C^\prime L) \in \kappa_\iota$ (1)

$\vdash . *332·31 . \supset \vdash : \text{Hp} . M \in \text{Pot}^\prime L . \text{rep}_\kappa{}^\prime M \in \kappa_\iota . \supset . \text{rep}_\kappa{}^\prime\{L \mid \text{rep}_\kappa{}^\prime M\} \in \kappa_\iota$ (2)

$\vdash . *330·624 . \supset \vdash : \text{Hp} . M \in \text{Pot}^\prime L . \supset . \dot{\exists} ! L \mid M$ (3)

$\vdash . (2) . (3) . *332·33 . \supset \vdash : \text{Hp}(2) . \supset . \text{rep}_\kappa{}^\prime(L \mid M) \in \kappa_\iota$ (4)

$\vdash . (1) . (4) . \text{Induct} . \supset \vdash . \text{Prop}$

***332·62.** $\vdash : \kappa \in FM \text{ conx} . \dot{\Lambda} \sim \in \text{Pot}^\prime P . \dot{\exists} ! \text{rep}_\kappa{}^\prime P . \supset .$

$\text{rep}_\kappa{}^{\prime\prime}\text{Pot}^\prime P \subset \text{rep}_\kappa{}^{\prime\prime}\text{Pot}^\prime\text{rep}_\kappa{}^\prime P$

Dem.

$\vdash . *332·242 . \supset \vdash : \text{Hp} . \supset . \text{rep}_\kappa{}^\prime P = \text{rep}_\kappa{}^\prime\text{rep}_\kappa{}^\prime P$ (1)

$\vdash . *332·22 . \supset \vdash : \text{Hp} . \supset . \text{rep}_\kappa{}^\prime P \in \kappa_\iota$ (2)

$\vdash . (2) . *332·61 . \supset$

$\vdash : \text{Hp} . Q \in \text{Pot}^\prime P . \text{rep}_\kappa{}^\prime Q \in \text{rep}_\kappa{}^{\prime\prime}\text{Pot}^\prime\text{rep}_\kappa{}^\prime P . \supset . \text{rep}_\kappa{}^\prime Q \in \kappa_\iota$ (3)

$\vdash . *91·36 . \supset \vdash : \text{Hp} . Q \in \text{Pot}^\prime P . \supset . \dot{\exists} ! P \mid Q$ (4)

$\vdash . (2) . (3) . (4) . *332·33 . \supset \vdash : \text{Hp}(3) . \supset . \text{rep}_\kappa{}^\prime(P \mid Q) = \text{rep}_\kappa{}^\prime\{\text{rep}_\kappa{}^\prime P \mid \text{rep}_\kappa{}^\prime Q\} .$

[Hp.*91·36] $\supset . \text{rep}_\kappa{}^\prime(P \mid Q) \in \text{rep}_\kappa{}^{\prime\prime}\text{Pot}^\prime\text{rep}_\kappa{}^\prime P$ (5)

$\vdash . (1) . (5) . \text{Induct} . \supset \vdash . \text{Prop}$

***332·63.** $\vdash : \text{Hp } *332·62 . \supset . \text{rep}_\kappa{}^{\prime\prime}\text{Pot}^\prime P \subset \kappa_\iota$

Dem.

$\vdash . *332·22 . \supset \vdash : \text{Hp} . \supset . \text{rep}_\kappa{}^\prime P \in \kappa_\iota$ (1)

$\vdash . (1) . *332·62·61 . \supset \vdash . \text{Prop}$

***332·64.** $\vdash : \kappa \in FM \text{ conx} . \text{rep}_\kappa{}^{\prime\prime}\text{Pot}^\prime P \subset \kappa_\iota . \supset . \text{rep}_\kappa{}^{\prime\prime}\text{Pot}^\prime P \subset \text{rep}_\kappa{}^{\prime\prime}\text{Pot}^\prime\text{rep}_\kappa{}^\prime P$

Dem.

$\vdash . *331·26 . *332·13 . \supset \vdash : \text{Hp} . \kappa \sim \in 1 . \supset . \dot{\Lambda} \sim \in \text{Pot}^\prime P$ (1)

$\vdash . *330·6 . *331·12 . \supset \vdash : \text{Hp} . \supset . \dot{\Lambda} \sim \in \text{rep}_\kappa{}^{\prime\prime}\text{Pot}^\prime P$ (2)

$\vdash . (1) . (2) . *332·62 . \supset \vdash : \text{Hp} . \kappa \sim \in 1 . \supset . \text{rep}_\kappa{}^{\prime\prime}\text{Pot}^\prime P \subset \text{rep}_\kappa{}^{\prime\prime}\text{Pot}^\prime\text{rep}_\kappa{}^\prime P$ (3)

$\vdash . *330·45 . *331·22 . \supset \vdash : \text{Hp} . \kappa \in 1 . \supset . \kappa_\iota = \iota^\prime(I \upharpoonright s^\prime \text{G}^{\prime\prime}\kappa) = \kappa$ (4)

$\vdash . (2) . (4) . *332{\cdot}12 . \supset \vdash : \text{Hp}(4) . \supset . P \mathrel{G} I \upharpoonright s`\text{Cl}``\kappa .$ (5)

$[*332{\cdot}243{\cdot}13.(4)] \qquad \supset . \text{rep}_\kappa`P = I \upharpoonright s`\text{Cl}``\kappa$ (6)

$\vdash . (5) . *301{\cdot}3 . \qquad \supset \vdash : \text{Hp}(4) . \supset . \text{Pot}`P = \iota`P .$

$[(6).*332{\cdot}241] \qquad \supset . \text{rep}_\kappa``\text{Pot}`P = \iota`\text{rep}_\kappa`\text{rep}_\kappa`P$ (7)

$\vdash . (3) . (7) . \supset \vdash . \text{Prop}$

$*332{\cdot}65$. $\vdash : \dot{\Lambda} \sim \epsilon \text{Pot}`P . \dot{\exists} ! \text{rep}_\kappa`P . \supset . \text{Pot}`P \subset s`\text{Rl}``\text{Pot}`\text{rep}_\kappa`P$

Dem.

$\vdash . *332{\cdot}11 . \supset \vdash : \text{Hp} . \supset . P \mathrel{G} \text{rep}_\kappa`P$ (1)

$\vdash . (1) . \supset \vdash : \text{Hp} . Q \epsilon \text{Pot}`P . R \epsilon \text{Pot}`\text{rep}_\kappa`P . Q \mathrel{G} R . \supset . Q | P \mathrel{G} R | \text{rep}_\kappa`P$ (2)

$\vdash . (1) . (2) . \text{Induct} . \supset \vdash . \text{Prop}$

$*332{\cdot}66$. $\vdash : \dot{\exists} ! \text{rep}_\kappa`P . R \epsilon \text{Pot}`\text{rep}_\kappa`P . \supset . (\exists Q) . Q \epsilon \text{Pot}`P . Q \mathrel{G} R$
[Proof as in $*332{\cdot}65$]

$*332{\cdot}67$. $\vdash : \kappa \epsilon FM \text{ conx} . \dot{\Lambda} \sim \epsilon \text{Pot}`P . \dot{\exists} ! \text{rep}_\kappa`P . \supset .$
$\text{rep}_\kappa``\text{Pot}`\text{rep}_\kappa`P = \text{rep}_\kappa``\text{Pot}`P$

Dem.

$\vdash . *332{\cdot}242 . \qquad \supset \vdash : \text{Hp} . \supset . \text{rep}_\kappa`\text{rep}_\kappa`P = \text{rep}_\kappa`P$ (1)

$\vdash . *332{\cdot}66 . \qquad \supset \vdash :. \text{Hp} . \supset : R \epsilon \text{Pot}`\text{rep}_\kappa`P . \supset . \dot{\exists} ! R | P$ (2)

$\vdash . *332{\cdot}22 . \qquad \supset \vdash : \text{Hp} . \supset . \text{rep}_\kappa`P \epsilon \kappa_\iota$ (3)

$\vdash . (3) . *332{\cdot}61 . \supset \vdash :. \text{Hp} . \supset : R \epsilon \text{Pot}`\text{rep}_\kappa`P . \supset . \text{rep}_\kappa`R \epsilon \kappa_\iota$ (4)

$\vdash . (2) . (3) . (4) . *332{\cdot}33 . \supset$

$\vdash :. \text{Hp} . \supset : R \epsilon \text{Pot}`\text{rep}_\kappa`P . \supset . \text{rep}_\kappa`(\text{rep}_\kappa`R | \text{rep}_\kappa`P) = \text{rep}_\kappa`(R | \text{rep}_\kappa`P)$ (5)

$\vdash . *332{\cdot}33 . \supset \vdash : \text{Hp} . R \epsilon \text{Pot}`\text{rep}_\kappa`P . Q \epsilon \text{Pot}`P . \text{rep}_\kappa`R = \text{rep}_\kappa`Q . \supset .$
$\text{rep}_\kappa`(Q | P) = \text{rep}_\kappa`(\text{rep}_\kappa`R | \text{rep}_\kappa`P)$

$[(5)] \qquad = \text{rep}_\kappa`(R | \text{rep}_\kappa`P)$ (6)

$\vdash . (6) . \supset \vdash : \text{Hp} . R \epsilon \text{Pot}`\text{rep}_\kappa`P . \text{rep}_\kappa`R \epsilon \text{rep}_\kappa``\text{Pot}`P . \supset .$
$\text{rep}_\kappa`(R | \text{rep}_\kappa`P) \epsilon \text{rep}_\kappa``\text{Pot}`P$ (7)

$\vdash . (1) . (7) . \text{Induct} . \supset \vdash : \text{Hp} . \supset . \text{rep}_\kappa``\text{Pot}`\text{rep}_\kappa`P \subset \text{rep}_\kappa``\text{Pot}`P$ (8)

$\vdash . (8) . *332{\cdot}62 . \supset \vdash . \text{Prop}$

$*332{\cdot}71$. $\vdash : \kappa \epsilon FM \text{ conx} . L, M \epsilon \kappa_\iota . \supset .$
$\text{rep}_\kappa``\text{Pot}`(L | M) = \text{rep}_\kappa``\text{Pot}`\text{rep}_\kappa`(L | M)$

Dem.

$\vdash . *330{\cdot}626 . \qquad \supset \vdash : \text{Hp} . \supset . \dot{\Lambda} \sim \epsilon \text{Pot}`(L | M)$ (1)

$\vdash . *332{\cdot}31 . *330{\cdot}6 . \supset \vdash : \text{Hp} . \supset . \dot{\exists} ! \text{rep}_\kappa`(L | M)$ (2)

$\vdash . (1) . (2) . *332{\cdot}67 . \supset \vdash . \text{Prop}$

***332·72.** $\vdash : \mathrm{Hp} *332{\cdot}71 . \supset . \mathrm{rep}_\kappa``\mathrm{Pot}`(L \mid M) \subset \kappa_\iota$ [*332·31·61·71]

***332·73.** $\vdash : \kappa \in FM \,\mathrm{conx} . L, M \in \kappa_\iota . \supset . \mathrm{Pot}`(L \mid M) \subset s`\mathrm{Rl}``\mathrm{Pot}`\mathrm{rep}_\kappa`(L \mid M)$
[*332·65·31 . *330·626]

***332·74.** $\vdash : \kappa \in FM \,\mathrm{conx} . L, M \in \kappa_\iota . P \in \mathrm{Pot}`M . \supset .$
$$\mathrm{rep}_\kappa`(L \mid P) = \mathrm{rep}_\kappa`(P \mid L) = \mathrm{rep}_\kappa`(L \mid \mathrm{rep}_\kappa`P)$$

Dem.

$\vdash . *330{\cdot}627 . *332{\cdot}61{\cdot}33 . \supset$

$\vdash : \mathrm{Hp} . \supset . \mathrm{rep}_\kappa`(L \mid P) = \mathrm{rep}_\kappa`\{L \mid \mathrm{rep}_\kappa`P\}$ (1)

[*332·61·32] $= \mathrm{rep}_\kappa`\{\mathrm{rep}_\kappa`P \mid L\}$

[*330·627.*332·61·33] $= \mathrm{rep}_\kappa`(P \mid L)$ (2)

$\vdash . (1) . (2) . \supset \vdash .$ Prop

***332·75.** $\vdash : \mathrm{Hp} *332{\cdot}74 . \supset . \exists ! \,\mathrm{rep}_\kappa`(L \mid P)$ [*332·74·61·31 . *330·6]

***332·8.** $\vdash : \kappa \in FM \,\mathrm{conx} . L, M \in \kappa_\iota . \xi \in \mathrm{NC\,ind} . \supset .$
$$\mathrm{rep}_\kappa`(L \mid M)^\xi = \mathrm{rep}_\kappa`(L^\xi \mid M^\xi)$$

Dem.

$\vdash . *332{\cdot}243 . \supset$

$\vdash : \mathrm{Hp} . \xi = 0 . \supset . \mathrm{rep}_\kappa`(L \mid M)^\xi = I \upharpoonright s`\mathrm{C}``\kappa = \mathrm{rep}_\kappa`(L^\xi \mid M^\xi)$ (1)

$\vdash . *301{\cdot}21 . *332{\cdot}33 . *330{\cdot}626 . \supset$

$\vdash : \mathrm{Hp} . \mathrm{rep}_\kappa`(L \mid M)^\xi = \mathrm{rep}_\kappa`(L^\xi \mid M^\xi) . \supset .$

$\mathrm{rep}_\kappa`(L \mid M)^{\xi +_c 1} = \mathrm{rep}_\kappa`\{L^\xi \mid M^\xi \mid L \mid M\}$

[*332·37] $= \mathrm{rep}_\kappa`\{L^\xi \mid \mathrm{rep}_\kappa`(M^\xi \mid L) \mid M\}$

[*332·32·33] $= \mathrm{rep}_\kappa`\{L^\xi \mid \mathrm{rep}_\kappa`(L \mid M^\xi) \mid M\}$

[*332·37] $= \mathrm{rep}_\kappa`\{L^{\xi +_c 1} \mid M^{\xi +_c 1}\}$ (2)

$\vdash . (1) . (2) . \mathrm{Induct} . \supset \vdash .$ Prop

***332·81.** $\vdash : \kappa \in FM \,\mathrm{conx} . \nu, \sigma \in \mathrm{NC\,ind} - \iota`0 . L \in \kappa_\iota . \supset .$
$$\mathrm{rep}_\kappa`L^{\nu \times_c \sigma} = \mathrm{rep}_\kappa`(\mathrm{rep}_\kappa`L^\nu)^\sigma$$

Dem.

$\vdash . *301{\cdot}23 . \supset \vdash : \mathrm{Hp} . \mathrm{rep}_\kappa`L^{\nu \times_c \sigma} = \mathrm{rep}_\kappa`(\mathrm{rep}_\kappa`L^\nu)^\sigma . \supset .$

$\mathrm{rep}_\kappa`L^{\nu \times_c (\sigma +_c 1)} = \mathrm{rep}_\kappa`(L^{\nu \times_c \sigma} \mid L^\nu)$

[*332·33] $= \mathrm{rep}_\kappa`\{(\mathrm{rep}_\kappa`L^\nu)^\sigma \mid \mathrm{rep}_\kappa`L^\nu\}$

[*301·23] $= \mathrm{rep}_\kappa`(\mathrm{rep}_\kappa`L^\nu)^{\sigma +_c 1}$ (1)

$\vdash . (1) . \mathrm{Induct} . \supset \vdash .$ Prop

***332·82.** $\vdash : \kappa \in FM \,\mathrm{conx} . \nu \in \mathrm{NC\,ind} - \iota`0 . L, M \in \kappa_\iota . \supset .$
$$\mathrm{rep}_\kappa`(L \mid M)^\nu = \mathrm{rep}_\kappa`\{\mathrm{rep}_\kappa`(L \mid M)\}^\nu$$

Dem.

$\vdash . *332{\cdot}33 . \supset \vdash : \mathrm{Hp} . \mathrm{rep}_\kappa`(L \mid M)^\nu = \{\mathrm{rep}_\kappa`(L \mid M)\}^\nu . \supset .$

$\mathrm{rep}_\kappa`(L \mid M)^{\nu +_c 1} = \mathrm{rep}_\kappa`[\{\mathrm{rep}_\kappa`(L \mid M)\}^\nu \mid \mathrm{rep}_\kappa`(L \mid M)]$

[*301·23] $= \mathrm{rep}_\kappa`\{\mathrm{rep}_\kappa`(L \mid M)\}^{\nu +_c 1}$ (1)

$\vdash . (1) . *113{\cdot}621 . *301{\cdot}2 . \mathrm{Induct} . \supset \vdash .$ Prop

*333. OPEN FAMILIES.

Summary of *333.

An "open" family is defined as one such that, if L is any member of κ_ι which is not contained in identity, then every power of L is contained in diversity, *i.e.* $L_{\text{po}} \subset J$. We shall often have occasion, both in this number and later, to consider the class $\kappa_\iota - \text{Rl}`I$, and in later numbers we shall often have occasion to consider the class $\kappa - \text{Rl}`I$. We therefore put

***333·01.** $\kappa_\partial = \kappa - \text{Rl}`I$ Df

***333·011.** $\kappa_{\iota\partial} = (\kappa_\iota)_\partial$ Df

Thus $\kappa_{\iota\partial}$ consists of all members of κ_ι which are not contained in identity, *i.e.* (if κ is a connected family) all members of κ_ι except $I \upharpoonright s`\text{C}``\kappa$. The definition of an "open" family is

***333·02.** $FM\,\text{ap} = FM \cap \hat{\kappa}\,\{s`\text{Pot}``\kappa_{\iota\partial} \subset \text{Rl}`J\}$ Df

From the point of view of the application of ratio, the hypothesis that a family is open is very important. To begin with, it insures (*333·18) that $\kappa_{\iota\partial}$ consists of "numerical" relations (cf. *300), so that if $L \in \kappa_{\iota\partial}$, we have $\text{Pot}`L = \text{fin}`L$ (*333·15), and in virtue of *300·491, the existence of open families implies the axiom of infinity (*333·19).

Again, in an open connected family, if L, M are two different members of κ_ι, all the powers of $L \,|\, \breve{M}$ are contained in diversity, and therefore the representatives of these powers are members of $\kappa_{\iota\partial}$; that is, we have

***333·22.** $\vdash : \kappa \in FM\,\text{ap}\,\text{conx} \,.\, L, M \in \kappa_\iota \,.\, L \neq M \,.\, \supset .\, \text{rep}_\kappa``\text{Pot}`(L \,|\, \breve{M}) \subset \kappa_{\iota\partial}$

It follows from this proposition that, with the above hypothesis, if σ is any inductive cardinal other than 0, $L^\sigma \,|\, \breve{M}^\sigma$ is not contained in identity, and therefore $L^\sigma \neq M^\sigma$ and $\text{rep}_\kappa`L^\sigma \neq \text{rep}_\kappa`M^\sigma$. Hence by transposition we obtain the two propositions:

***333·41.** $\vdash :.\, \kappa \in FM\,\text{ap}\,\text{conx} \,.\, L, M \in \kappa_\iota \,.\, \sigma \in \text{NC ind} - \iota`0 \,.\, \supset :$
$\text{rep}_\kappa`L^\sigma = \text{rep}_\kappa`M^\sigma \,.\, \equiv .\, L = M$

***333·42.** $\vdash :. \text{Hp} *333\cdot41 . \supset : L^\sigma = M^\sigma . \equiv . L = M$

Hence we obtain

***333·43.** $\vdash :. \text{Hp} *333\cdot41 . \supset : \dot{\exists} ! L^\sigma \dot{\frown} M^\sigma . \equiv . L = M$

This proposition shows that in an open connected family, no two members of κ_ι have the ratio 1/1 unless they are identical. Again it follows from *333·41 that if $L^{\rho \times_c \tau}$ and $M^{\sigma \times_c \tau}$ have the same representative, then L^ρ and M^σ have the same representative, and vice versa, *i.e.*

***333·44.** $\vdash :. \kappa \in FM \text{ ap conx} . L, M \in \kappa_\iota . \rho, \sigma, \tau \in \text{NC ind} - \iota'0 . \supset :$

$$\text{rep}_\kappa{}'L^{\rho \times_c \tau} = \text{rep}_\kappa{}'M^{\sigma \times_c \tau} . \equiv . \text{rep}_\kappa{}'L^\rho = \text{rep}_\kappa{}'M^\sigma$$

Hence we obtain two propositions which are vital for the application of ratio, namely:

***333·47.** $\vdash :. \kappa \in FM \text{ ap conx} . L, M \in \kappa_\iota . \rho, \sigma \in \text{NC ind} - \iota'0 . \supset :$

$$\text{rep}_\kappa{}'L^\rho = \text{rep}_\kappa{}'M^\sigma . \equiv . \dot{\exists} ! L^\rho \dot{\frown} M^\sigma$$

***333·48.** $\vdash :. \kappa \in FM \text{ ap conx} . L, M \in \kappa_\iota . \rho, \sigma, \tau \in \text{NC ind} - \iota'0 . \supset :$

$$\dot{\exists} ! L^\rho \dot{\frown} M^\sigma . \equiv . \dot{\exists} ! L^{\rho \times_c \tau} \dot{\frown} M^{\sigma \times_c \tau}$$

On comparing this last proposition with the definition of ratio (*303·01), it will be seen that, whether ρ is prime to σ or not, L has to M the ratio σ/ρ when, and only when, $\dot{\exists} ! L^\rho \dot{\frown} M^\sigma$, *i.e.* (by *333·47) when, and only when, $\text{rep}_\kappa{}'L^\rho = \text{rep}_\kappa{}'M^\sigma$.

From *333·47 it follows also that, if $M \in \kappa_{\iota\partial}$, M^ρ and M^σ will not have the same representative unless $\rho = \sigma$ (*333·51), *i.e.*

***333·51.** $\vdash :. \kappa \in FM \text{ ap conx} . M \in \kappa_{\iota\partial} . \rho, \sigma \in \text{NC ind} . \supset :$

$$\text{rep}_\kappa{}'M^\rho = \text{rep}_\kappa{}'M^\sigma . \equiv . \rho = \sigma$$

From this it follows that no member of $\kappa_{\iota\partial}$ has any other ratio to itself than 1/1. Again, by *333·47·48·51, we have

***333·53.** $\vdash : \kappa \in FM \text{ ap conx} . L, M \in \kappa_{\iota\partial} . \dot{\exists} ! L^\sigma \dot{\frown} M^\rho . \dot{\exists} ! L^\nu \dot{\frown} M^\mu . \supset .$

$$\mu \times_0 \sigma = \nu \times_0 \rho$$

Hence if L and M have the two ratios ρ/σ, μ/ν, we have $\rho/\sigma = \mu/\nu$; that is, no two members of $\kappa_{\iota\partial}$ have more than one ratio.

The applications of ratio indicated in this summary will not be made till the following Section; they are here mentioned in order to show the utility of the propositions of the present number.

***333·01.** $\kappa_\partial = \kappa - \text{Rl}'I$ Df

***333·011.** $\kappa_{\iota\partial} = (\kappa_\iota)_\partial$ Df

***333·02.** $FM \text{ ap} = FM \cap \hat{\kappa} \{s'\text{Pot}``\kappa_{\iota\partial} \subset \text{Rl}'J\}$ Df

***333·03.** $FM \text{ ap conx} = FM \text{ ap} \cap FM \text{ conx}$ Df

$*333\cdot1.\quad \vdash : M \epsilon \kappa_{\iota\partial} . \equiv . (\exists P, Q) . P, Q \epsilon \kappa . M = \breve{P} | Q . \dot{\exists} ! M \dot{\wedge} J .$

$\equiv . M \epsilon \kappa_\iota . \dot{\exists} ! M \dot{\wedge} J \qquad [(*333\cdot01\cdot011)]$

$*333\cdot101.\quad \vdash :. \kappa \epsilon FM \,\mathrm{ap} . \equiv : \kappa \epsilon FM : M \epsilon \kappa_{\iota\partial} . P \epsilon \mathrm{Pot}`M . \supset_{M,P} . P \mathrel{\dot{\subset}} J :$

$\equiv : \kappa \epsilon FM : M \epsilon \kappa_{\iota\partial} . \supset_M . M_{\mathrm{po}} \mathrel{\dot{\subset}} J \qquad [(*333\cdot02)]$

$*333\cdot11.\quad \vdash : \kappa \epsilon FM \,\mathrm{ap} . L \epsilon \kappa_{\iota\partial} . \supset . L \mathrel{\dot{\subset}} J . L^2 \mathrel{\dot{\subset}} J . L \dot{\wedge} \breve{L} = \dot{\Lambda} . L \neq \breve{L} . \dot{\exists} ! L$
$[*333\cdot1\cdot101]$

$*333\cdot12.\quad \vdash : \kappa \epsilon FM \,\mathrm{ap\,conx} . \dot{\exists} ! \mathrm{rep}_\kappa`P . \dot{\exists} ! P \dot{\wedge} J . \supset .$

$\mathrm{rep}_\kappa`P \epsilon \kappa_{\iota\partial} . (\mathrm{rep}_\kappa`P)_{\mathrm{po}} \mathrel{\dot{\subset}} J$

Dem.

$\vdash . *332\cdot11 . \supset \vdash : \mathrm{Hp} . \supset . \dot{\exists} ! \mathrm{rep}_\kappa`P \dot{\wedge} J .$

$[*332\cdot22 . *333\cdot1] \qquad \supset . \mathrm{rep}_\kappa`P \epsilon \kappa_{\iota\partial} \qquad (1)$

$\vdash . (1) . *333\cdot101 . \supset \vdash . \mathrm{Prop}$

$*333\cdot13.\quad \vdash : \kappa \epsilon FM \,\mathrm{ap\,conx} . \dot{\exists} ! \mathrm{rep}_\kappa`P . \dot{\exists} ! P \dot{\wedge} J . \supset . P_{\mathrm{po}} \mathrel{\dot{\subset}} J$

Dem.

$\vdash . *332\cdot11 . \qquad \supset \vdash : \mathrm{Hp} . \supset . P \mathrel{\dot{\subset}} \mathrm{rep}_\kappa`P \qquad (1)$

$\vdash . (1) . *332\cdot22 . \supset \vdash : \mathrm{Hp} . \supset . \dot{\exists} ! (\mathrm{rep}_\kappa`P) \dot{\wedge} J . P_{\mathrm{po}} \mathrel{\dot{\subset}} (\mathrm{rep}_\kappa`P)_{\mathrm{po}} . \mathrm{rep}_\kappa`P \epsilon \kappa_\iota .$

$[*333\cdot1] \qquad \supset . P_{\mathrm{po}} \mathrel{\dot{\subset}} (\mathrm{rep}_\kappa`P)_{\mathrm{po}} . \mathrm{rep}_\kappa`P \epsilon \kappa_{\iota\partial} .$

$[*333\cdot101] \qquad \supset . P_{\mathrm{po}} \mathrel{\dot{\subset}} J : \supset \vdash . \mathrm{Prop}$

$*333\cdot14.\quad \vdash : \kappa \epsilon FM \,\mathrm{ap\,conx} . L, M \epsilon \kappa_\iota . L \neq \breve{M} . \supset . (L | M)_{\mathrm{po}} \mathrel{\dot{\subset}} J$

Dem.

$\vdash . *330\cdot626 . \qquad \supset \vdash : \mathrm{Hp} . \supset . \dot{\Lambda} \sim \epsilon \mathrm{Pot}`(L | M) \qquad (1)$

$\vdash . *332\cdot31 . *330\cdot6 . \supset \vdash : \mathrm{Hp} . \supset . \dot{\exists} ! \mathrm{rep}_\kappa`(L | M) \qquad (2)$

$\vdash . *332\cdot46 . \mathrm{Transp} . \supset \vdash : \mathrm{Hp} . \supset . \dot{\exists} ! (L | M) \dot{\wedge} J \qquad (3)$

$\vdash . (1) . (2) . (3) . *333\cdot13 . \supset \vdash . \mathrm{Prop}$

$*333\cdot15.\quad \vdash : \kappa \epsilon FM \,\mathrm{ap} . L \epsilon \kappa_{\iota\partial} . \supset . \mathrm{Pot}`L = \mathrm{fin}`L = \mathrm{finid}`L - \iota`L_0$
$[*121\cdot501 . *333\cdot11\cdot101]$

$*333\cdot16.\quad \vdash : \kappa \epsilon FM \,\mathrm{ap\,conx} . L, M \epsilon \kappa_\iota . L \neq \breve{M} . \supset .$

$\mathrm{Pot}`(L | M) = \mathrm{fin}`(L | M) = \mathrm{finid}`(L | M) - \iota`(L | M)_0$

$[*121\cdot501 . *333\cdot14]$

$*333\cdot17.\quad \vdash : \kappa \epsilon FM \,\mathrm{ap\,conx} . \dot{\exists} ! \mathrm{rep}_\kappa`P . \dot{\exists} ! P \dot{\wedge} J . \supset .$

$\mathrm{Pot}`P = \mathrm{fin}`P = \mathrm{finid}`P - \iota`P_0 \quad [*121\cdot501 . *333\cdot13]$

$*333\cdot18.\quad \vdash : \kappa \epsilon FM \,\mathrm{ap} . \supset . \kappa_{\iota\partial} \subset \mathrm{Rel\,num} \qquad [*333\cdot101 . *300\cdot3]$

$*333\cdot19.\quad \vdash : \kappa \epsilon FM \,\mathrm{ap} - \iota`\iota`\dot{\Lambda} . \supset . \mathrm{Infin\,ax} \qquad [*333\cdot18 . *330\cdot624 . *300\cdot491]$

$*333\cdot2.\quad \vdash : \exists ! FM \,\mathrm{ap\,conx} . \supset . \mathrm{Infin\,ax} \qquad [*333\cdot19 . *331\cdot12]$

***333·21.** $\vdash : \kappa \epsilon FM \text{ ap conx} . L \epsilon \kappa_{\iota\partial} . \supset . \text{rep}_\kappa``\text{Pot}`L \subset \kappa_{\iota\partial}$

Dem.

$\vdash . *332·61 . \quad \supset \vdash : \text{Hp} . \supset . \text{rep}_\kappa``\text{Pot}`L \subset \kappa_\iota \quad (1)$

$\vdash . *333·101 . *330·624 . \supset \vdash :. \text{Hp} . \supset : \dot\Lambda \sim \epsilon \text{Pot}`L . \text{Pot}`L \subset \text{Rl}`J :$

$[*332·11.(1)] \quad \supset : M \epsilon \text{rep}_\kappa``\text{Pot}`L . \supset . \dot\exists ! M \dot\cap J \quad (2)$

$\vdash . (1) . (2) . *333·1 . \supset \vdash . \text{Prop}$

***333·22.** $\vdash : \kappa \epsilon FM \text{ ap conx} . L, M \epsilon \kappa_\iota . L \neq M . \supset . \text{rep}_\kappa``\text{Pot}`(L \mid \breve{M}) \subset \kappa_{\iota\partial}$

Dem.

$\vdash . *332·71 . \supset \vdash : \text{Hp} . \supset . \text{rep}_\kappa``\text{Pot}`(L \mid \breve{M}) = \text{rep}_\kappa``\text{Pot}`\text{rep}_\kappa`(L \mid \breve{M}) \quad (1)$

$\vdash . *332·46·11·232·31 . \supset \vdash : \text{Hp} . \supset . \text{rep}_\kappa`(L \mid \breve{M}) \epsilon \kappa_{\iota\partial} \quad (2)$

$\vdash . (1) . (2) . *333·21 . \supset \vdash . \text{Prop}$

***333·23.** $\vdash : \kappa \epsilon FM \text{ ap conx} . \dot\Lambda \sim \epsilon \text{Pot}`P . \dot\exists ! \text{rep}_\kappa`P . \dot\exists ! P \dot\cap J . \supset .$
$\text{rep}_\kappa``\text{Pot}`P \subset \kappa_{\iota\partial}$

Dem.

$\vdash . *332·62 . \quad \supset \vdash : \text{Hp} . \supset . \text{rep}_\kappa``\text{Pot}`P \subset \text{rep}_\kappa``\text{Pot}`\text{rep}_\kappa`P \quad (1)$

$\vdash . *332·11·22 . *333·1 . \supset \vdash : \text{Hp} . \supset . \text{rep}_\kappa`P \epsilon \kappa_{\iota\partial} \quad (2)$

$\vdash . (1) . (2) . *333·21 . \supset \vdash . \text{Prop}$

***333·24.** $\vdash : \kappa \epsilon FM \text{ conx} . \dot\Lambda \sim \epsilon \text{Pot}`P . \dot\exists ! \text{rep}_\kappa`P . \nu \epsilon \text{NC ind} . \exists !$
$(\nu +_c 1) \cap t^2`P . \supset . \text{rep}_\kappa`P^\nu = \text{rep}_\kappa`(\text{rep}_\kappa`P)^\nu$

Dem.

$\vdash . *301·2 . *332·243 . \supset \vdash : \text{Hp} . \supset . \text{rep}_\kappa`P^0 = I \upharpoonright s`\text{C}``\kappa = \text{rep}_\kappa`(\text{rep}_\kappa`P)^0 \quad (1)$

$\vdash . *332·63 . *330·6 . *301·16·22 . \supset$

$\vdash : \text{Hp} . \supset . \text{rep}_\kappa`P^\nu, \text{rep}_\kappa`P \epsilon \kappa_\iota . \dot\exists ! P^{\nu +_c 1} . \quad (2)$

$[*301·21.*332·33] \supset . \text{rep}_\kappa`P^{\nu +_c 1} = \text{rep}_\kappa`\{(\text{rep}_\kappa`P^\nu) \mid \text{rep}_\kappa`P\} \quad (3)$

$\vdash . (2) . (3) . \supset \vdash : \text{Hp} . \text{rep}_\kappa`P^\nu = \text{rep}_\kappa`(\text{rep}_\kappa`P)^\nu . \supset .$

$\text{rep}_\kappa`P^{\nu +_c 1} = \text{rep}_\kappa`\{\text{rep}_\kappa`(\text{rep}_\kappa`P)^\nu \mid \text{rep}_\kappa`P\} .$

$\text{rep}_\kappa`(\text{rep}_\kappa`P)^\nu, \text{rep}_\kappa`P \epsilon \kappa_\iota \quad (4)$

$\vdash . (2) . *330·624 . *301·21 . \supset \vdash : \text{Hp} . \supset . \dot\exists ! (\text{rep}_\kappa`P)^\nu \mid \text{rep}_\kappa`P \quad (5)$

$\vdash . (4) . (5) . *332·33 . \supset \vdash : \text{Hp} (4) . \supset . \text{rep}_\kappa`P^{\nu +_c 1} = \text{rep}_\kappa`\{(\text{rep}_\kappa`P)^\nu \mid \text{rep}_\kappa`P\}$

$[*301·21] \quad = \text{rep}_\kappa`(\text{rep}_\kappa`P)^{\nu +_c 1} \quad (6)$

$\vdash . (1) . (6) . \text{Induct} . \supset \vdash . \text{Prop}$

A hypothesis equivalent to $\nu \epsilon \text{NC ind} . \exists ! (\nu +_c 1) \cap t^2`P$ is $\nu \epsilon \text{C}`U \upharpoonright t^3`P$. It is sometimes convenient to substitute this for the other.

***333·25.** $\vdash : \kappa \epsilon FM \text{ conx} . L, M \epsilon \kappa_\iota . \nu \epsilon \text{NC ind} . \exists ! (\nu +_c 1) \cap t^2`L . \supset .$
$\text{rep}_\kappa`(L \mid M)^\nu = \text{rep}_\kappa`\{\text{rep}_\kappa`(L \mid M)\}^\nu$

Dem.

$\vdash . *330·626 . *331·12 . \supset \vdash : \text{Hp} . \supset . \dot\Lambda \sim \epsilon \text{Pot}`(L \mid M) \quad (1)$

$\vdash . *332·31 . *330·6 . \quad \supset \vdash : \text{Hp} . \supset . \dot\exists ! \text{rep}_\kappa`(L \mid M) \quad (2)$

$\vdash . (1) . (2) . *333·24 . \quad \supset \vdash . \text{Prop}$

***333·32.** $\vdash : \kappa \in FM \,\mathrm{conx} \,.\, L, M \in \kappa_\iota \,.\, \rho, \sigma \in \mathrm{C}'(U \upharpoonright t^3{}'L) \,.\, \supset .\, \dot{\exists} \,!\, L^\rho \,|\, M^\sigma$

Dem.

$\vdash . *330{\cdot}61 . *301{\cdot}2 . \supset \vdash : \mathrm{Hp} . \supset . \dot{\exists} \,!\, L^0 \,|\, M^0$ (1)

$\vdash . *330{\cdot}623 . \quad \supset \vdash :. \mathrm{Hp} . \supset : S \in \kappa . \supset_S . S \,|\, L^\rho \,|\, M^\sigma \mathrel{\dot{\subset}} L^\rho \,|\, M^\sigma \,|\, S :$ (2)

$[*330{\cdot}622] \quad \supset : \dot{\exists} \,!\, L^\rho \,|\, M^\sigma . \supset . \dot{\exists} \,!\, L^{\rho +_c 1} \,|\, M^\sigma$ (3)

$\vdash . (2) . *330{\cdot}621 . \quad \supset \vdash :. \mathrm{Hp} . \supset : \dot{\exists} \,!\, L^\rho \,|\, M^\sigma . \supset . \dot{\exists} \,!\, L^\rho \,|\, M^{\sigma +_c 1}$ (4)

$\vdash . (1) . (3) . (4) . \mathrm{Induct} . \supset \vdash . \mathrm{Prop}$

***333·33.** $\vdash : \kappa \in FM \,\mathrm{conx} \,.\, L, M \in \kappa_\iota \,.\, \sigma \in \mathrm{C}'(U \upharpoonright t^3{}'L) \,.\, \supset .$

$$\mathrm{rep}_\kappa{}'(L^\sigma \,|\, M^\sigma) = \mathrm{rep}_\kappa{}'(L \,|\, M)^\sigma$$

Dem.

$\vdash . *333{\cdot}32 . *332{\cdot}243 . \supset$

$\vdash : \mathrm{Hp} . \supset . \mathrm{rep}_\kappa{}'(L^0 \,|\, M^0) = I \upharpoonright s'\mathrm{C}''\kappa = \mathrm{rep}_\kappa{}'(L \,|\, M)^0$ (1)

$\vdash . *332{\cdot}37 . *301{\cdot}21 . \supset$

$\vdash : \mathrm{Hp} . \supset . \mathrm{rep}_\kappa{}'(L^{\sigma +_c 1} \,|\, M^{\sigma +_c 1}) = \mathrm{rep}_\kappa{}'\{\mathrm{rep}_\kappa{}'(L^\sigma \,|\, M^\sigma) \,|\, \mathrm{rep}_\kappa{}'L \,|\, \mathrm{rep}_\kappa{}'M\}$ (2)

$\vdash . (2) . \supset \vdash : \mathrm{Hp} . \mathrm{rep}_\kappa{}'(L^\sigma \,|\, M^\sigma) = \mathrm{rep}_\kappa{}'(L \,|\, M)^\sigma . \supset .$

$\mathrm{rep}_\kappa{}'(L^{\sigma +_c 1} \,|\, M^{\sigma +_c 1}) = \mathrm{rep}_\kappa{}'\{\mathrm{rep}_\kappa{}'(L \,|\, M)^\sigma \,|\, \mathrm{rep}_\kappa{}'L \,|\, \mathrm{rep}_\kappa{}'M\}$ (3)

$\vdash . (3) . *333{\cdot}32 . *332{\cdot}37 . \supset$

$\vdash : \mathrm{Hp}\,(3) . \supset . \mathrm{rep}_\kappa{}'(L^{\sigma +_c 1} \,|\, M^{\sigma +_c 1}) = \mathrm{rep}_\kappa{}'\{(L \,|\, M)^\sigma \,|\, L \,|\, M\}$

$[*301{\cdot}21] \quad = \mathrm{rep}_\kappa{}'(L \,|\, M)^{\sigma +_c 1}$ (4)

$\vdash . (1) . (4) . \mathrm{Induct} . \supset \vdash . \mathrm{Prop}$

***333·34.** $\vdash : \mathrm{Hp}\, *333{\cdot}33 . \supset . \mathrm{rep}_\kappa{}'(L^\sigma \,|\, M^\sigma) = \mathrm{rep}_\kappa{}'\{\mathrm{rep}_\kappa{}'(L|M)\}^\sigma = \mathrm{rep}_\kappa{}'(L|M)^\sigma$

Dem.

$\vdash . *330{\cdot}626{\cdot}6 . *332{\cdot}31 . \supset$

$\vdash : \mathrm{Hp} . \supset . \dot{\Lambda} \sim\in \mathrm{Pot}'(L \,|\, M) . \dot{\exists} \,!\, \mathrm{rep}_\kappa{}'(L \,|\, M)$ (1)

$\vdash . (1) . *333{\cdot}24 . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{rep}_\kappa{}'\{\mathrm{rep}_\kappa{}'(L \,|\, M)\}^\sigma = \mathrm{rep}_\kappa{}'(L \,|\, M)^\sigma$ (2)

$\vdash . (2) . *333{\cdot}33 . \supset \vdash . \mathrm{Prop}$

***333·41.** $\vdash :. \kappa \in FM \,\mathrm{ap\,conx} \,.\, L, M \in \kappa_\iota \,.\, \sigma \in \mathrm{NC\,ind} - \iota'0 \,.\, \supset :$

$$\mathrm{rep}_\kappa{}'L^\sigma = \mathrm{rep}_\kappa{}'M^\sigma . \equiv . L = M$$

Dem.

$\vdash . *333{\cdot}34{\cdot}22{\cdot}2 . \supset \vdash : \mathrm{Hp} . L \neq M . \supset . \mathrm{rep}_\kappa{}'(L^\sigma \,|\, \breve{M}^\sigma) \in \kappa_{\iota\partial} .$

$[*333{\cdot}21{\cdot}32 . *332{\cdot}33] \quad \supset . \mathrm{rep}_\kappa{}'\{\mathrm{rep}_\kappa{}'L^\sigma \,|\, \mathrm{rep}_\kappa{}'\breve{M}^\sigma\} \in \kappa_{\iota\partial} .$

$[*332{\cdot}44 . \mathrm{Transp}] \quad \supset . \sim\{\mathrm{rep}_\kappa{}'L^\sigma \,|\, \mathrm{rep}_\kappa{}'\breve{M}^\sigma \mathrel{\dot{\subset}} I \upharpoonright s'\mathrm{C}''\kappa\} .$

$[*332{\cdot}15{\cdot}46 . \mathrm{Transp}] \quad \supset . \mathrm{rep}_\kappa{}'L^\sigma \neq \mathrm{rep}_\kappa{}'M^\sigma$ (1)

$\vdash . (1) . \mathrm{Transp} . \supset \vdash . \mathrm{Prop}$

***333·42.** $\vdash :. \mathrm{Hp}\, *333{\cdot}41 . \supset : L^\sigma = M^\sigma . \equiv . L = M \quad [*333{\cdot}41]$

***333·43.** $\vdash :. \mathrm{Hp} *333\cdot41 . \supset : \dot{\exists} ! L^\sigma \dot{\wedge} M^\sigma . \equiv . L = M$

Dem.

$\vdash . *333\cdot21 . *332\cdot26 . \supset \vdash : \mathrm{Hp} . \dot{\exists} ! L^\sigma \dot{\wedge} M^\sigma . \supset . \mathrm{rep}_\kappa{}^{\backprime}L^\sigma = \mathrm{rep}_\kappa{}^{\backprime}M^\sigma .$

$[*333\cdot41] \qquad \supset . L = M \qquad (1)$

$\vdash . (1) . *330\cdot624 . \supset \vdash . \mathrm{Prop}$

***333·44.** $\vdash :. \kappa \in FM \,\mathrm{ap\,conx} . L, M \in \kappa_\iota . \rho, \sigma, \tau \in \mathrm{NC\,ind} - \iota{}^{\backprime}0 . \supset :$

$\mathrm{rep}_\kappa{}^{\backprime}L^{\rho \times_c \tau} = \mathrm{rep}_\kappa{}^{\backprime}M^{\sigma \times_c \tau} . \equiv . \mathrm{rep}_\kappa{}^{\backprime}L^\rho = \mathrm{rep}_\kappa{}^{\backprime}M^\sigma$

Dem.

$\vdash . *301\cdot5 . *333\cdot24 . \supset$

$\vdash :. \mathrm{Hp} . \supset : \mathrm{rep}_\kappa{}^{\backprime}L^{\rho \times_c \tau} = \mathrm{rep}_\kappa{}^{\backprime}M^{\sigma \times_c \tau} . \equiv . \mathrm{rep}_\kappa{}^{\backprime}(\mathrm{rep}_\kappa{}^{\backprime}L^\rho)^\tau = \mathrm{rep}_\kappa{}^{\backprime}(\mathrm{rep}_\kappa{}^{\backprime}M^\sigma)^\tau$

$[*333\cdot41\cdot21] \qquad \equiv . \mathrm{rep}_\kappa{}^{\backprime}L^\rho = \mathrm{rep}_\kappa{}^{\backprime}M^\sigma :. \supset \vdash . \mathrm{Prop}$

***333·45.** $\vdash :. \mathrm{Hp} *333\cdot44 . \supset : L^{\rho \times_c \tau} = M^{\sigma \times_c \tau} . \supset . \mathrm{rep}_\kappa{}^{\backprime}L^\rho = \mathrm{rep}_\kappa{}^{\backprime}M^\sigma \quad [*333\cdot44]$

***333·46.** $\vdash :. \mathrm{Hp} *333\cdot44 . \supset : \dot{\exists} ! L^{\rho \times_c \tau} \dot{\wedge} M^{\sigma \times_c \tau} . \supset . \mathrm{rep}_\kappa{}^{\backprime}L^\rho = \mathrm{rep}_\kappa{}^{\backprime}M^\sigma$

Dem.

$\vdash . *332\cdot26 . *333\cdot21 . \supset$

$\vdash : \mathrm{Hp} . \dot{\exists} ! L^{\rho \times_c \tau} \dot{\wedge} M^{\sigma \times_c \tau} . \supset . \mathrm{rep}_\kappa{}^{\backprime}L^{\rho \times_c \tau} = \mathrm{rep}_\kappa{}^{\backprime}M^{\sigma \times_c \tau} \qquad (1)$

$\vdash . (1) . *333\cdot44 . \supset \vdash . \mathrm{Prop}$

***333·47.** $\vdash :. \kappa \in FM \,\mathrm{ap\,conx} . L, M \in \kappa_\iota . \rho, \sigma \in \mathrm{NC\,ind} - \iota{}^{\backprime}0 . \supset :$

$\mathrm{rep}_\kappa{}^{\backprime}L^\rho = \mathrm{rep}_\kappa{}^{\backprime}M^\sigma . \equiv . \dot{\exists} ! L^\rho \dot{\wedge} M^\sigma$

Dem.

$\vdash . *333\cdot46 . \supset \vdash : \mathrm{Hp} . \dot{\exists} ! L^\rho \dot{\wedge} M^\sigma . \supset . \mathrm{rep}_\kappa{}^{\backprime}L^\rho = \mathrm{rep}_\kappa{}^{\backprime}M^\sigma \qquad (1)$

$\vdash . *332\cdot53 . *72\cdot92 . \supset$

$\vdash : \mathrm{Hp} . P, Q, R, S \in \kappa . L = \breve{P} \,|\, Q . M = \breve{R} \,|\, S . \supset . L^\rho = (\breve{P}^\rho \,|\, Q^\rho) \upharpoonright C{}^{\backprime}L^\rho .$

$M^\sigma = (\breve{R}^\sigma \,|\, S^\sigma) \upharpoonright C{}^{\backprime}M^\sigma . \mathrm{rep}_\kappa{}^{\backprime}L^\rho = \breve{P}^\rho \,|\, Q^\rho . \mathrm{rep}_\kappa{}^{\backprime}M^\sigma = \breve{R}^\sigma \,|\, S^\sigma \qquad (2)$

$\vdash . (2) . *35\cdot14 . \supset$

$\vdash : \mathrm{Hp}\,(2) . \mathrm{rep}_\kappa{}^{\backprime}L^\rho = \mathrm{rep}_\kappa{}^{\backprime}M^\sigma . \supset . L^\rho \dot{\wedge} M^\sigma = (\breve{P}^\rho \,|\, Q^\rho) \upharpoonright (C{}^{\backprime}L^\rho \cap C{}^{\backprime}M^\sigma) .$

$[*330\cdot72] \qquad \supset . \dot{\exists} ! L^\rho \dot{\wedge} M^\sigma \qquad (3)$

$\vdash . (1) . (3) . \supset \vdash . \mathrm{Prop}$

***333·48.** $\vdash :. \kappa \in FM \,\mathrm{ap\,conx} . L, M \in \kappa_\iota . \rho, \sigma, \tau \in \mathrm{NC\,ind} - \iota{}^{\backprime}0 . \supset :$

$\dot{\exists} ! L^\rho \dot{\wedge} M^\sigma . \equiv . \dot{\exists} ! L^{\rho \times_c \tau} \dot{\wedge} M^{\sigma \times_c \tau}$

Dem.

$\vdash . *333\cdot46 . \qquad \supset \vdash : \mathrm{Hp} . \dot{\exists} ! L^\rho \dot{\wedge} M^\sigma . \supset . \mathrm{rep}_\kappa{}^{\backprime}L^\rho = \mathrm{rep}_\kappa{}^{\backprime}M^\sigma \qquad (1)$

$\vdash . *330\cdot624 . *332\cdot61 . \supset \vdash : \mathrm{Hp} . \supset . \dot{\Lambda} \sim\in \mathrm{Pot}{}^{\backprime}L^\rho . \dot{\exists} ! \mathrm{rep}_\kappa{}^{\backprime}L^\rho .$

$[*333\cdot24] \qquad \supset . \mathrm{rep}_\kappa{}^{\backprime}L^{\rho \times_c \tau} = \mathrm{rep}_\kappa{}^{\backprime}(\mathrm{rep}_\kappa{}^{\backprime}L^\rho)^\tau \qquad (2)$

Similarly $\qquad \vdash : \mathrm{Hp} . \supset . \mathrm{rep}_\kappa{}^{\backprime}M^{\sigma \times_c \tau} = \mathrm{rep}_\kappa{}^{\backprime}(\mathrm{rep}_\kappa{}^{\backprime}M^\sigma)^\tau \qquad (3)$

$\vdash . (1) . (2) . (3) . \qquad \supset \vdash : \mathrm{Hp} . \dot{\exists} ! L^\rho \dot{\wedge} M^\sigma . \supset . \mathrm{rep}_\kappa{}^{\backprime}L^{\rho \times_c \tau} = \mathrm{rep}_\kappa{}^{\backprime}M^{\sigma \times_c \tau} .$

$[*333\cdot47] \qquad \supset . \dot{\exists} ! L^{\rho \times_c \tau} \dot{\wedge} M^{\sigma \times_c \tau} \qquad (4)$

$\vdash . *333\cdot46\cdot47 . \qquad \supset \vdash : \mathrm{Hp} . \dot{\exists} ! L^{\rho \times_c \tau} \dot{\wedge} M^{\sigma \times_c \tau} . \supset . \dot{\exists} ! L^\rho \dot{\wedge} M^\sigma \qquad (5)$

$\vdash . (4) . (5) . \supset \vdash . \mathrm{Prop}$

∗333·49. $\vdash : \kappa \in FM \text{ ap conx} . L, M \in \kappa_\iota . \rho, \sigma \in \text{NC ind} - \iota`0 . \text{rep}_\kappa`L^\rho = \text{rep}_\kappa`M^\sigma .$
$\supset . L^\rho \upharpoonright \text{C}`M^\sigma = M^\sigma \upharpoonright \text{C}`L^\rho . (\text{D}`M^\sigma) \upharpoonright L^\rho = (\text{D}`L^\rho) \upharpoonright M^\sigma$

Dem.

$\vdash . ∗333·21 . ∗330·6 . \supset \vdash : \text{Hp} . \supset . \dot{\exists} ! \text{rep}_\kappa`L^\rho .$

[∗332·11] $\supset . L^\rho \mathrel{G} \text{rep}_\kappa`L^\rho .$

[∗72·92] $\supset . L^\rho = (\text{rep}_\kappa`L^\rho) \upharpoonright \text{C}`L^\rho$ (1)

Similarly $\vdash : \text{Hp} . \supset . M^\sigma = (\text{rep}_\kappa`M^\sigma) \upharpoonright \text{C}`M^\sigma .$

[Hp] $\supset . M^\sigma = (\text{rep}_\kappa`L^\rho) \upharpoonright \text{C}`M^\sigma$ (2)

$\vdash . (1) . (2) . \supset \vdash : \text{Hp} . \supset . L^\rho \upharpoonright \text{C}`M^\sigma = (\text{rep}_\kappa`L^\rho) \upharpoonright (\text{C}`L^\rho \cap \text{C}`M^\sigma) = M^\sigma \upharpoonright \text{C}`L^\rho$ (3)

Similarly $\vdash : \text{Hp} . \supset . (\text{D}`M^\sigma) \upharpoonright L^\rho = (\text{D}`L^\rho) \upharpoonright M^\sigma$ (4)

$\vdash . (3) . (4) . \supset \vdash . \text{Prop}$

∗333·5. $\vdash :. \kappa \in FM \text{ ap conx} . P, Q \in \kappa . \sigma \in \text{NC ind} - \iota`0 . \supset :$
$P^\sigma = Q^\sigma . \equiv . \dot{\exists} ! P^\sigma \dot{\cap} Q^\sigma . \equiv . P = Q$ [∗333·42·43 . ∗331·24]

∗333·51. $\vdash :. \kappa \in FM \text{ ap conx} . M \in \kappa_{\iota\partial} . \rho, \sigma \in \text{NC ind} . \supset :$
$\text{rep}_\kappa`M^\rho = \text{rep}_\kappa`M^\sigma . \equiv . \rho = \sigma$

Dem.

$\vdash . ∗333·47 . \supset \vdash :. \text{Hp} . \text{rep}_\kappa`M^\rho = \text{rep}_\kappa`M^\sigma . \supset : \dot{\exists} ! M^\rho \dot{\cap} M^\sigma :$

[∗301·23 . ∗120·412·416] $\supset : \rho \geqslant \sigma . \supset . \dot{\exists} ! M^{\rho -_c \sigma} \dot{\cap} I .$

[∗333·101] $\supset . \rho = \sigma$ (1)

Similarly $\vdash :. \text{Hp}(1) . \supset : \sigma \geqslant \rho . \supset . \rho = \sigma$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

∗333·52. $\vdash :. \text{Hp } ∗333·51 . \supset : M^\rho = M^\sigma . \equiv . \rho = \sigma$ [∗333·51]

∗333·53. $\vdash : \kappa \in FM \text{ ap conx} . L, M \in \kappa_{\iota\partial} . \dot{\exists} ! L^\sigma \dot{\cap} M^\rho . \dot{\exists} ! L^\nu \dot{\cap} M^\mu . \supset .$
$\mu \times_c \sigma = \nu \times_c \rho$

Dem.

$\vdash . ∗333·48 . ∗301·16 . \supset \vdash : \text{Hp} . \supset . \dot{\exists} ! L^{\mu \times_c \sigma} \dot{\cap} M^{\mu \times_c \rho} . \dot{\exists} ! L^{\nu \times_c \rho} \dot{\cap} M^{\mu \times_c \rho} .$

[∗333·47] $\supset . \text{rep}_\kappa`L^{\mu \times_c \sigma} = \text{rep}_\kappa`M^{\mu \times_c \rho} = \text{rep}_\kappa`L^{\nu \times_c \rho} .$

[∗333·51] $\supset . \mu \times_c \sigma = \nu \times_c \rho : \supset \vdash . \text{Prop}$

*334. SERIAL FAMILIES.

Summary of *334.

The purpose of the present number is to consider what properties of a family κ will insure that $\dot{s}`\kappa_\partial$ is serial, or has one or more of the properties characteristic of serial relations. Suppose, for example, that κ consists of distances on a line. Then κ_∂ consists of those distances which are members of κ and are not zero. Any selection of distances on the line may constitute κ; thus *e.g.* κ may consist of all distances which are integral multiples of a given distance, or of all which are rational multiples of a given distance, or of all distances from left to right, or of all distances on the line in either direction. It is plain to begin with that if $\dot{s}`\kappa_\partial$ is to be serial, $\kappa$ must not contain equal distances in opposite directions, since if it does, $(\dot{s}`\kappa_\partial)^2$ will not be contained in diversity, *i.e.* $\dot{s}`\kappa_\partial$ will not be asymmetrical We call a family κ asymmetrical when no member of κ_∂ has a converse which is also a member of κ_∂. The definition is

***334·05.** $FM\,\mathrm{asym} = FM \cap \hat{\kappa}(\kappa \cap \mathrm{Cnv}``\kappa \subset \mathrm{Rl}`I)$ Df

It will be observed that $\dot{s}`\kappa_\partial \in J$ in any connected family, by *331·23. If $\kappa \in FM\,\mathrm{asym}$, we have also $(\dot{s}`\kappa_\partial)^2 \in J$.

In order to secure that $\dot{s}`\kappa_\partial$ shall be *transitive*, we require that the field of κ should contain at least one "transitive point," where a "transitive point" means a point a such that any point which can be reached from a by two successive non-zero steps can also be reached by one non-zero step, *i.e.* such that

$$(\dot{s}`\kappa_\partial)``\overrightarrow{\dot{s}`\kappa_\partial}`a \subset \overrightarrow{\dot{s}`\kappa_\partial}`a.$$

The definition of transitive points is

***334·01.** $\mathrm{trs}`\kappa = s`\mathrm{C}``\kappa \cap \hat{a}\{(\dot{s}`\kappa_\partial)``\overrightarrow{\dot{s}`\kappa_\partial}`a \subset \overrightarrow{\dot{s}`\kappa_\partial}`a\}$ Df

Thus if a is a transitive point, and $R, S \in \kappa_\partial$, there is always a member of κ_∂, say T, such that $R`S`a = T`a$. It will be seen that if κ is a connected family, the existence of a transitive point implies that the family is asymmetrical. Again, if there is a transitive point in a connected family, then $R, S \in \kappa_\partial . \supset . R | S \in \kappa_\partial$, by *331·32; hence κ_∂ is a group. The converse also

holds, *i.e.* if κ_∂ is a group, any member $s'\text{Ɑ}''\kappa$ is a transitive point (*334·11). Hence if there is any transitive point, every point of $s'\text{Ɑ}''\kappa$ is a transitive point.

The definition of a transitive family is

***334·02.** $FM\,\mathrm{trs} = FM \cap \hat{\kappa}\,(\exists !\,\mathrm{trs}'\kappa)$ Df

By what has just been said, a connected transitive family is one in which κ_∂ is a group, *i.e.*

***334·13.** $\vdash :.\ \kappa \in FM\,\mathrm{conx}\,.\supset :\kappa \in FM\,\mathrm{trs}\,.\equiv .\, s'\kappa_\partial \underset{\prime\prime}{|}{}''\kappa_\partial \subset \kappa_\partial$

A connected family is transitive when, and only when, $\dot{s}'\kappa_\partial$ is a transitive relation, *i.e.*

***334·14.** $\vdash :.\ \kappa \in FM\,\mathrm{conx}\,.\supset :\kappa \in FM\,\mathrm{trs}\,.\equiv .\,\dot{s}'\kappa_\partial \in \mathrm{trans}$

In order to secure that $\dot{s}'\kappa_\partial$ shall be a *connected* relation, it is not enough that κ should be an FM conx, *i.e.* that $s'\text{Ɑ}''\kappa$ should have at least one connected point. We require that *every* point of $s'\text{Ɑ}''\kappa$ should be a connected point. This will be secured if there is a connected point which belongs to the field of every member of κ_ι, *i.e.* if

$$\exists !\,\mathrm{conx}'\kappa \cap p'C''\kappa_\iota .$$

For suppose $a \in \mathrm{conx}'\kappa \cap p'C''\kappa_\iota$. Then if $L \in \kappa_\iota$, either $L'a$ or $\breve{L}'a$ exists, and is of the form $R'a$ or $\breve{R}'a$, where $R \in \kappa$. Hence, by *331·32, L is identical with R or with $\breve{R}$; hence $\kappa_\iota = \kappa \cup \mathrm{Cnv}''\kappa$. Hence by *331·4, $\dot{s}'\kappa_\partial \in$ connex. Conversely, if $\kappa \in FM\,\mathrm{conx}$ and $\dot{s}'\kappa_\partial \in \mathrm{connex}$, it follows from *331·32 that $\kappa_\iota = \kappa \cup \mathrm{Cnv}''\kappa$; hence $p'C''\kappa_\iota = s'\text{Ɑ}''\kappa$, and therefore we have $\exists !\,\mathrm{conx}'\kappa \cap p'C''\kappa_\iota$. Hence putting

***334·03.** $FM\,\mathrm{connex} = FM \cap \hat{\kappa}\,(\exists !\,\mathrm{conx}'\kappa \cap p'C''\kappa_\iota)$ Df

where "FM connex" means "families having connexity," we have

***334·26.** $\vdash :.\ \kappa \in FM\,\mathrm{conx}\,.\supset :\kappa \in FM\,\mathrm{connex}\,.\equiv .\,\dot{s}'\kappa_\partial \in \mathrm{connex}\,.$
$$\equiv .\,\kappa_\iota = \kappa \cup \mathrm{Cnv}''\kappa\,.\equiv .\,C''\kappa_\iota = \text{Ɑ}''\kappa$$

and

***334·27.** $\vdash .\ FM\,\mathrm{connex} = FM \cap \hat{\kappa}\,(s'\text{Ɑ}''\kappa = \mathrm{conx}'\kappa\,.\,\kappa \neq \iota'\dot{\Lambda})$

I.e. a family having connexity is one whose field consists wholly of connected points and is not null.

We thus secure (1) $\dot{s}'\kappa_\partial \mathrel{\dot{\subset}} J$ by the hypothesis $\kappa \in FM\,\mathrm{conx}$, (2) $\dot{s}'\kappa_\partial \in \mathrm{trans}$ by the hypothesis $\kappa \in FM\,\mathrm{conx} \cap FM\,\mathrm{trs}$, (3) $\dot{s}'\kappa_\partial \in \mathrm{connex}$ by the hypothesis $\kappa \in FM\,\mathrm{connex}$ (which implies $\kappa \in FM\,\mathrm{conx}$). Hence we secure $\dot{s}'\kappa_\partial \in \mathrm{Ser}$ by the hypothesis $\kappa \in FM\,\mathrm{trs} \cap FM\,\mathrm{connex}$. When this hypothesis is fulfilled, we call κ a "serial" family; thus we put

***334·04.** $FM \,\mathrm{sr} = FM \,\mathrm{trs} \cap FM \,\mathrm{connex}$ Df

and we have

***334·3.** $\vdash : \kappa \in FM \,\mathrm{sr} . \supset . \dot{s}`\kappa_\partial \in \mathrm{Ser}$

***334·31.** $\vdash :. \kappa \in FM . I \upharpoonright s`\mathrm{G}``\kappa \in \kappa . \supset : \kappa \in FM \,\mathrm{sr} . \equiv . \dot{s}`\kappa_\partial \in \mathrm{Ser} - \iota`\dot{\Lambda}$

An important special case, which is briefly considered in this number, is the case when the domains of members of κ are the same as their converse domains, *i.e.* when

$$\mathrm{D}``\kappa = \mathrm{G}``\kappa.$$

This case is illustrated, *e.g.* by the family whose members are all relations of the form $(+_g X) \upharpoonright C`H_g$, where $X \in C`H'$. It is also illustrated by cyclic families, which are considered in the next Section but one. When $\mathrm{D}``\kappa = \mathrm{G}``\kappa$, if κ is a family, so is $\kappa \cup \mathrm{Cnv}``\kappa$ (*334·4), and if $\kappa$ is a connected family, so is $\kappa \cup \mathrm{Cnv}``\kappa$ (*334·41). In the case of the above family, whose members are $(+_g X) \upharpoonright C`H_g$ where $X \in C`H'$, $\kappa \cup \mathrm{Cnv}``\kappa$ will consist of all relations $(+_g X) \upharpoonright C`H_g$ where $X \in C`H_g$, *i.e.* it will consist of all additions of positive or negative ratios to positive or negative ratios.

A connected family in which $\mathrm{D}``\kappa = \mathrm{G}``\kappa$ is a family having connexity, *i.e.*

***334·42.** $\vdash : \kappa \in FM \,\mathrm{conx} . \mathrm{D}``\kappa = \mathrm{G}``\kappa . \supset . \kappa \in FM \,\mathrm{connex}$

The definitions and propositions of this number are much used throughout the remainder of Part VI.

***334·01.** $\mathrm{trs}`\kappa = s`\mathrm{G}``\kappa \cap \hat{a} \{(\dot{s}`\kappa_\partial)``\overrightarrow{\dot{s}`\kappa_\partial}`a \subset \overrightarrow{\dot{s}`\kappa_\partial}`a\}$ Df

***334·02.** $FM \,\mathrm{trs} = FM \cap \hat{\kappa} (\exists ! \,\mathrm{trs}`\kappa)$ Df

***334·03.** $FM \,\mathrm{connex} = FM \cap \hat{\kappa} (\exists ! \,\mathrm{conx}`\kappa \cap p`C``\kappa_\iota)$ Df

***334·04.** $FM \,\mathrm{sr} = FM \,\mathrm{trs} \cap FM \,\mathrm{connex}$ Df

***334·05.** $FM \,\mathrm{asym} = FM \cap \hat{\kappa} (\kappa \cap \mathrm{Cnv}``\kappa \subset \mathrm{Rl}`I)$ Df

***334·09.** $\vdash : \kappa \in FM \,\mathrm{conx} . \supset . \dot{s}`\kappa_\partial \in J$ [*331·23]

***334·1.** $\vdash :: \kappa \in FM . \supset :. a \in \mathrm{trs}`\kappa . \equiv :$
$a \in s`\mathrm{G}``\kappa : R, S \in \kappa_\partial . \supset_{R,S} . (\exists T) . T \in \kappa_\partial . R`S`a = T`a$ [(*334·01)]

***334·11.** $\vdash :. \kappa \in FM \,\mathrm{conx} . \supset : a \in \mathrm{trs}`\kappa . \equiv . a \in s`\mathrm{G}``\kappa . s`\kappa_\partial |``\kappa_\partial \subset \kappa_\partial$
,,

Dem.

$\vdash . *331·33·24 . \supset \vdash : \mathrm{Hp} . R, S \in \kappa_\partial . \supset . R | S \in \kappa_\iota$ (1)

$\vdash . (1) . *331·32 . \supset \vdash : \mathrm{Hp} . T \in \kappa_\partial . R`S`a = T`a . \supset . R | S = T$ (2)

$\vdash . (2) . *334·1 . \supset \vdash :: \mathrm{Hp} . \supset :.$

$a \in \mathrm{trs}`\kappa . \equiv : a \in s`\mathrm{G}``\kappa : R, S \in \kappa_\partial . \supset_{R,S} . (\exists T) . T \in \kappa_\partial . R | S = T :$

[*13·195] $\equiv : a \in s`\mathrm{G}``\kappa : R, S \in \kappa_\partial . \supset_{R,S} . R | S \in \kappa_\partial :: \supset \vdash . \mathrm{Prop}$

***334·12.** $\vdash :. \kappa \in FM \text{ conx} . \alpha, x \in s'\text{Ɑ}''\kappa . \supset :$
$\alpha \in \text{trs}'\kappa . \equiv . x \in \text{trs}'\kappa . \equiv . s'\kappa_\partial \underset{,,}{|} ''\kappa_\partial \subset \kappa_\partial$ [*334·11]

***334·13.** $\vdash :. \kappa \in FM \text{ conx} . \supset : \kappa \in FM \text{ trs} . \equiv . s'\kappa_\partial \underset{,,}{|} ''\kappa_\partial \subset \kappa_\partial$
[*334·12 . *331·12 . (*334·02)]

***334·131.** $\vdash : \kappa \in FM \text{ conx} \cap FM \text{ trs} . R \in \kappa_\partial . \supset . \text{Pot}'R \subset \kappa_\partial$ [*334·13 . Induct]

***334·132.** $\vdash : \kappa \in FM \text{ conx} \cap FM \text{ trs} . \supset . s'\text{Pot}''\kappa \subset \kappa$ [*334·131]

***334·14.** $\vdash :. \kappa \in FM \text{ conx} . \supset : \kappa \in FM \text{ trs} . \equiv . \dot{s}'\kappa_\partial \in \text{trans}$

Dem.

$\vdash . *41·51 . *334·13 . \supset \vdash :. \text{Hp} . \supset : \kappa \in FM \text{ trs} . \supset . (\dot{s}'\kappa_\partial)^2 \mathrel{\dot{\subset}} \dot{s}'\kappa_\partial$ (1)

$\vdash . *330·52 . \supset \vdash :: \text{Hp} . \supset :. \dot{s}'\kappa_\partial \in \text{trans} . \supset :$
$R, S \in \kappa_\partial . x \in s'\text{Ɑ}''\kappa . \supset_{R,S,x} . (\exists T) . T \in \kappa_\partial . R'S'x = T'x .$

[*331·31·33·24] $\supset_{R,S,x} . (\exists T) . T \in \kappa_\partial . R \,|\, S = T .$

[*13·195] $\supset_{R,S,x} . R \,|\, S \in \kappa_\partial$ (2)

$\vdash . (2) . *331·12 . \supset \vdash :: \text{Hp} . \supset :. \dot{s}'\kappa_\partial \in \text{trans} . \supset : R, S \in \kappa_\partial . \supset_{R,S} . R \,|\, S \in \kappa_\partial :$

[*334·13] $\supset : \kappa \in FM \text{ trs}$ (3)

$\vdash . (1) . (3) . \supset \vdash . \text{Prop}$

***334·15.** $\vdash : \kappa \in FM \text{ conx} \cap FM \text{ trs} . \supset . s'\kappa \underset{,,}{|} ''\kappa = \kappa$

Dem.

$\vdash . *331·321·22 . \supset \vdash :. \text{Hp} . R \in \kappa - \kappa_\partial . \supset : R = I \upharpoonright s'\text{Ɑ}''\kappa :$

[*50·62·63] $\supset : S \in \kappa . \supset . R \,|\, S, S \,|\, R \in \kappa$ (1)

$\vdash . (1) . *334·13 . \supset \vdash : \text{Hp} . \supset . s'\kappa \underset{,,}{|} ''\kappa \subset \kappa$ (2)

$\vdash . *331·22 . *50·62·63 . \supset \vdash : \text{Hp} . \supset . \kappa \subset s'\kappa \underset{,,}{|} ''\kappa$ (3)

$\vdash . (2) . (3) . \supset \vdash . \text{Prop}$

***334·16.** $\vdash : \kappa \in FM \text{ conx} \cap FM \text{ trs} . R \in \kappa_\partial . \supset . R_{po} \mathrel{\dot{\subset}} J$ [*334·131·09]

***334·161.** $\vdash : \kappa \in FM \text{ conx} \cap FM \text{ trs} . R \in \kappa_\partial . a \in s'\text{Ɑ}''\kappa . \supset . \overrightarrow{R_*}'a \in \aleph_0$
[*334·16 . *123·191]

***334·162.** $\vdash : \exists ! FM \text{ conx} \cap FM \text{ trs} - 1 . \supset . \text{Infin ax}$ [*334·161]

***334·17.** $\vdash : \kappa \in FM \text{ conx} \cap 1 . \supset . \kappa_\partial = \Lambda$ [*331·22]

***334·18.** $\vdash : \kappa \in FM \text{ conx} - 1 . \supset . C'\dot{s}'\kappa_\partial = s'\text{Ɑ}''\kappa = s'\text{Ɑ}''\kappa_\partial . \dot{\exists} ! \dot{s}'\kappa_\partial . \exists ! \kappa_\partial$

Dem.

$\vdash . *331·22·321 . \supset \vdash :. \text{Hp} . \supset : \exists ! \kappa_\partial :$

[*330·52] $\supset : a \in s'\text{Ɑ}''\kappa . \supset . (\exists R) . R \in \kappa_\partial . a \in \text{Ɑ}'R .$

[*40·4] $\supset . a \in s'\text{Ɑ}''\kappa_\partial .$ (1)

[*41·45] $\supset . a \in C'\dot{s}'\kappa_\partial$ (2)

$\vdash . (1) . (2) . *331·12 . \supset \vdash . \text{Prop}$

∗334·19. $\vdash : \kappa \epsilon FM . \supset . C`\dot{s}`\kappa_\partial \subset s`\text{Ⴀ}``\kappa$ [∗41·45 . ∗330·52]

∗334·2. $\vdash :: . \kappa \epsilon FM . \supset :: a \epsilon p`C``\kappa_\iota . \equiv :. L \epsilon \kappa_\iota . \supset_L : E ! L`a . \vee . E ! \breve{L}`a$ [∗330·52]

∗334·21. $\vdash : \kappa \epsilon FM \text{ connex} . \supset . \kappa_\iota = \kappa \cup \text{Cnv}``\kappa$

Dem.

$\vdash . ∗334·2 . ∗331·11 . \supset \vdash :. \text{Hp} . a \epsilon \text{conx}`\kappa \cap p`C``\kappa_\iota . L \epsilon \kappa_\iota . \supset :$

$(\exists R) : R \epsilon \kappa \cup \text{Cnv}``\kappa : L`a = R`a . \vee . \breve{L}`a = R`a :$

[∗331·42·24] $\supset : (\exists R) : R \epsilon \kappa \cup \text{Cnv}``\kappa : L = R . \vee . \breve{L} = R$ (1)

$\vdash . (1) . ∗331·24 . \supset \vdash . \text{Prop}$

∗334·22. $\vdash : \kappa \epsilon FM \text{ connex} . \supset . p`C``\kappa_\iota = s`\text{Ⴀ}``\kappa$ [∗334·21 . ∗330·52]

∗334·23. $\vdash : \kappa \epsilon FM \text{ connex} . \supset . \text{conx}`\kappa = s`\text{Ⴀ}``\kappa$ [∗334·21 . ∗331·4]

∗334·24. $\vdash : \kappa \epsilon FM \text{ connex} . \supset . \dot{s}`\kappa_\partial \epsilon \text{ connex}$

Dem.

$\vdash . ∗334·21 . ∗331·4 . \supset$

$\vdash :. \text{Hp} . x, y \epsilon s`\text{Ⴀ}``\kappa . x \neq y . \supset : (\exists R) : R \epsilon \kappa_\partial : xRy . \vee . yRx :. \supset \vdash . \text{Prop}$

∗334·25. $\vdash : \kappa \epsilon FM \text{ connex} . \supset . C``\kappa_\iota = \text{Ⴀ}``\kappa$ [∗334·21 . ∗330·52]

∗334·251. $\vdash : \kappa \epsilon FM . \kappa_\iota = \kappa \cup \text{Cnv}``\kappa . \supset . p`C``\kappa_\iota = s`\text{Ⴀ}``\kappa$

Dem.

$\vdash . ∗40·18 . ∗33·22 . \supset \vdash : \text{Hp} . \supset . p`C``\kappa_\iota = p`C``\kappa$ (1)

$\vdash . (1) . ∗330·52 . \supset \vdash . \text{Prop}$

∗334·252. $\vdash : \kappa \epsilon FM \text{ conx} . \dot{s}`\kappa_\partial \epsilon \text{ connex} . \supset . \kappa_\iota = \kappa \cup \text{Cnv}``\kappa$

Dem.

$\vdash . ∗41·11 . \supset \vdash : \text{Hp} . L \epsilon \kappa_\iota . x = L`y . \supset . (\exists R) . R \epsilon \kappa \cup \text{Cnv}``\kappa . xRy .$

[∗331·42·24] $\supset . L \epsilon \kappa \cup \text{Cnv}``\kappa$ (1)

$\vdash . (1) . ∗330·6 . ∗331·12 . \supset \vdash . \text{Prop}$

∗334·253. $\vdash : \kappa \epsilon FM \text{ conx} . C``\kappa_\iota = \text{Ⴀ}``\kappa . \supset . \kappa \epsilon FM \text{ connex}$

Dem.

$\vdash . ∗330·52 . \supset \vdash : \text{Hp} . \supset . p`C``\kappa = s`\text{Ⴀ}``\kappa .$

[∗331·1] $\supset . \exists ! p`C``\kappa_\iota \cap \text{conx}`\kappa : \supset \vdash . \text{Prop}$

∗334·26. $\vdash :. \kappa \epsilon FM \text{ conx} . \supset : \kappa \epsilon FM \text{ connex} . \equiv . \dot{s}`\kappa_\partial \epsilon \text{ connex} .$

$\equiv . \kappa_\iota = \kappa \cup \text{Cnv}``\kappa . \equiv . C``\kappa_\iota = \text{Ⴀ}``\kappa$ [∗334·21·24·25·251·252·253]

∗334·27. $\vdash . FM \text{ connex} = FM \cap \hat{\kappa}\, (s`\text{Ⴀ}``\kappa = \text{conx}`\kappa . \kappa \neq \iota`\dot{\Lambda})$

Dem.

$\vdash . ∗331·1 . \supset \vdash : \kappa \epsilon FM . \kappa \neq \iota`\dot{\Lambda} . s`\text{Ⴀ}``\kappa = \text{conx}`\kappa . \supset . \dot{s}`\kappa_\partial \epsilon \text{ connex} .$

[∗334·26.(∗331·02)] $\supset . \kappa \epsilon FM \text{ connex}$ (1)

$\vdash . ∗334·23 . (∗334·03) . \supset \vdash : \kappa \epsilon FM \text{ connex} . \supset . s`\text{Ⴀ}``\kappa = \text{conx}`\kappa . \kappa \neq \iota`\dot{\Lambda}$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***334·3.** $\vdash : \kappa \epsilon FM \,\mathrm{sr} \,.\, \supset . \, \dot{s}'\kappa_\partial \epsilon \mathrm{Ser}$

Dem.

$\vdash . *334·09 . \supset \vdash : \mathrm{Hp} . \supset . \dot{s}'\kappa_\partial \,\mathsf{C}\, J$ (1)

$\vdash . *334·14 . \supset \vdash : \mathrm{Hp} . \supset . \dot{s}'\kappa_\partial \epsilon \mathrm{trans}$ (2)

$\vdash . *334·24 . \supset \vdash : \mathrm{Hp} . \supset . \dot{s}'\kappa_\partial \epsilon \mathrm{connex}$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash . \mathrm{Prop}$

***334·31.** $\vdash :. \kappa \epsilon FM . I \restriction s'\text{Ɑ}``\kappa \epsilon \kappa . \supset : \kappa \epsilon FM \,\mathrm{sr} . \equiv . \dot{s}'\kappa_\partial \epsilon \mathrm{Ser} - \iota'\dot{\Lambda}$

Dem.

$\vdash . *41·11 . \supset \vdash :. \mathrm{Hp} . \dot{s}'\kappa_\partial \epsilon \mathrm{Ser} - \iota'\dot{\Lambda} . \supset :$

$x, y \epsilon s'\text{Ɑ}``\kappa . \supset_{x,y} . (\exists R) . R \epsilon \kappa . x (R \,\dot{\cup}\, \breve{R}) y :$

[*331·11] $\supset : s'\text{Ɑ}``\kappa = \mathrm{conx}'\kappa$ (1)

$\vdash . (1) . *334·14·26 . \supset \vdash : \mathrm{Hp}(1) . \supset . \kappa \epsilon FM \,\mathrm{trs} . \kappa \epsilon FM \,\mathrm{connex}$ (2)

$\vdash . (2) . *334·3 . *331·12 . \supset \vdash . \mathrm{Prop}$

***334·32.** $\vdash . FM \,\mathrm{sr} \subset FM \,\mathrm{ap}$ [*334·16·21 . *333·101]

***334·4.** $\vdash : \kappa \epsilon FM . \mathrm{D}``\kappa = \text{Ɑ}``\kappa . \supset . \kappa \cup \mathrm{Cnv}``\kappa \epsilon FM$

Dem.

$\vdash . *33·2·21 . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{D}``(\kappa \cup \mathrm{Cnv}``\kappa) = \text{Ɑ}``(\kappa \cup \mathrm{Cnv}``\kappa) = \text{Ɑ}``\kappa$ (1)

$\vdash . *330·561 . \supset \vdash :. \mathrm{Hp} . \supset : R, S \epsilon \kappa . \supset . \breve{R} | S = S | \breve{R}$ (2)

$\vdash . (1) . (2) . *330·52 . \supset \vdash . \mathrm{Prop}$

***334·41.** $\vdash : \kappa \epsilon FM \,\mathrm{conx} . \mathrm{D}``\kappa = \text{Ɑ}``\kappa . \supset . \kappa \cup \mathrm{Cnv}``\kappa \epsilon FM \,\mathrm{conx}$
[*334·4 . *331·11]

***334·42.** $\vdash : \kappa \epsilon FM \,\mathrm{conx} . \mathrm{D}``\kappa = \text{Ɑ}``\kappa . \supset . \kappa \epsilon FM \,\mathrm{connex}$

Dem.

$\vdash . *37·323 . \supset \vdash :. \mathrm{Hp} . \supset : R, S \epsilon \kappa . \supset . \text{Ɑ}'(\breve{R} | S) = \text{Ɑ}'S :$

[*330·4] $\supset : C``\kappa_\iota = \text{Ɑ}``\kappa$ (1)

$\vdash . (1) . *334·26 . \supset \vdash . \mathrm{Prop}$

***334·43.** $\vdash : \kappa \epsilon FM \,\mathrm{conx} \cap FM \,\mathrm{trs} . \mathrm{D}``\kappa = \text{Ɑ}``\kappa . \supset . \kappa \epsilon FM \,\mathrm{sr}$
[*334·42 . (*334·04)]

***334·44.** $\vdash : \kappa \epsilon FM \,\mathrm{conx} . \mathrm{D}``\kappa = \text{Ɑ}``\kappa . L \epsilon \kappa_\iota . \supset . \mathrm{D}'L = \text{Ɑ}'L = C'L = s'\text{Ɑ}``\kappa$

Dem.

$\vdash . *37·323 . \supset \vdash : \mathrm{Hp} . R, S \epsilon \kappa . \supset . \text{Ɑ}'(\breve{R} | S) = \text{Ɑ}'S : \supset \vdash . \mathrm{Prop}$

***334·45.** $\vdash : \kappa \in FM \text{ conx} . \text{D}``\kappa = \text{G}``\kappa . L, M \in \kappa_\iota . \supset . \text{G}`(L \mid M) = s`\text{G}``\kappa$
[*334·44]

***334·451.** $\vdash : \text{Hp} *334\cdot44 . S \in \text{Pot}`L . \supset . \text{D}`S = \text{G}`S = C`S = s`\text{G}``\kappa$ [*334·44]

***334·46.** $\vdash :. \text{Hp} *334\cdot44 . M, N \in \kappa_\iota . \supset : \dot{\exists} ! L \mid M \dot{\frown} N . \equiv . L \mid M = N$
[*334·45 . *331·45]

***334·5.** $\vdash : \kappa \in FM \text{ conx} \frown FM \text{ asym} . \supset . (\dot{s}`\kappa_\partial)^2 \subseteq J$

Dem.

$\vdash . *332\cdot46 . \quad \supset \vdash : \text{Hp} . R, S \in \kappa . R \mid S \subseteq I . \supset . R = \breve{S} .$

$[(*334\cdot05)] \qquad \supset . R = I \upharpoonright s`\text{G}``\kappa \quad (1)$

$\vdash . (1) . \text{Transp} . \supset \vdash :. \text{Hp} . \supset : R, S \in \kappa_\partial . \supset . \sim(R \mid S \subseteq I) .$

$[*331\cdot33\cdot23] \qquad \supset . R \mid S \subseteq J :. \supset \vdash . \text{Prop}$

*335. INITIAL FAMILIES.

Summary of *335.

A family of vectors may or may not have a point in its field which is a starting-point but not an end-point of non-zero vectors. For example, the family of which a member is $(+_s X) \upharpoonright C'H'$, where $X \in C'H'$, has such a point in its field, namely 0_q; but the family of which a member is $(+_s X) \upharpoonright C'H$, where $X \in C'H'$, has no such point in its field, and no more has the family of which a member is $(+_g X) \upharpoonright C'H_g$, where $X \in C'H'$. If such a point exists, it is a member of $s'\text{Ꭰ}''\kappa$ but not of $s'\text{D}''\kappa_\partial$. Such a point, if it is also a connected point, must be unique, *i.e.* we have

***335·12.** $\vdash : \kappa \in FM . \supset . \text{conx}'\kappa - s'\text{D}''\kappa_\partial \in 0 \cup 1$

When $\text{conx}'\kappa - s'\text{D}''\kappa_\partial$ exists, we call its only member "the initial point of κ," putting

***335·01.** $\text{init}'\kappa = \breve{\iota}'(\text{conx}'\kappa - s'\text{D}''\kappa_\partial)$ Df

If the initial point of κ exists, we call κ an "initial" family; thus we put

***335·02.** $FM\,\text{init} = FM \cap \text{Ꭰ}'\text{init}$ Df

An initial family is asymmetrical (*335·16) and transitive (*335·18), and forms a group (*335·17); and if its initial point is a member of $p'C''\kappa_\iota$, it is a serial family (*335·3).

***335·01.** $\text{init}'\kappa = \breve{\iota}'(\text{conx}'\kappa - s'\text{D}''\kappa_\partial)$ Df

***335·02.** $FM\,\text{init} = FM \cap \text{Ꭰ}'\text{init}$ Df

***335·11.** $\vdash : \kappa \in FM . a \in \text{conx}'\kappa - s'\text{D}''\kappa_\partial . \supset . s'\text{Ꭰ}''\kappa = \overrightarrow{s'\kappa}'a . \iota'a = \overleftarrow{s'\kappa}'a$

Dem.

$\vdash . *41\cdot43 . *33\cdot4 . \supset \vdash : \text{Hp} . \supset . \overleftarrow{s'\kappa_\partial}'a = \Lambda$ (1)

$\vdash . *331\cdot23\cdot22 . \quad \supset \vdash : \text{Hp} . \supset . \overleftarrow{s'\kappa}'a = \overleftarrow{s'\kappa_\partial}'a \cup \iota'a$ (2)

$\vdash . *331\cdot1\cdot23\cdot22 . \quad \supset \vdash : \text{Hp} . \supset . s'\text{Ꭰ}''\kappa = \overrightarrow{s'\kappa}'a \cup \overleftarrow{s'\kappa\partial}'a$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash . \text{Prop}$

***335·12.** $\vdash : \kappa \in FM . \supset . \text{conx}`\kappa - s`\text{D}``\kappa_\partial \in 0 \cup 1$

Dem.

$\vdash . *335·11 . \supset \vdash : \text{Hp} . a, b \in \text{conx}`\kappa - s`\text{D}``\kappa_\partial . \supset . b \in \overrightarrow{\dot{s}`\kappa}`a .$

$[*32·182] \quad \supset . a \in \overleftarrow{\dot{s}`\kappa}`b .$

$[*335·11] \quad \supset . a = b : \supset \vdash . \text{Prop}$

***335·13.** $\vdash :. \kappa \in FM . \supset : \text{E} ! \text{init}`\kappa . \equiv . \exists ! \text{conx}`\kappa - s`\text{D}``\kappa_\partial$
$[*335·12 . (*335·01)]$

***335·14.** $\vdash : \kappa \in FM \text{init} . \equiv . \kappa \in FM . \exists ! \text{conx}`\kappa - s`\text{D}``\kappa_\partial$ $[*335·13 . (*335·02)]$

***335·15.** $\vdash : \kappa \in FM \text{init} . \supset . s`\text{G}``\kappa = \overrightarrow{\dot{s}`\kappa}`\text{init}`\kappa$ $[*335·11 . (*335·01)]$

***335·16.** $\vdash . FM \text{init} \subset FM \text{asym}$

Dem.

$\vdash . *335·14 . \supset \vdash :. \kappa \in FM \text{init} . \supset :$

$(\exists a) : a \in s`\text{G}``\kappa : R \in \kappa . a \in \text{D}`R . \supset_R . R \in \text{Rl}`I$ (1)

$\vdash . *330·52 . \supset \vdash : \kappa \in FM . a \in s`\text{G}``\kappa . R \in \kappa \cap \text{Cnv}``\kappa . \supset . a \in \text{D}`R$ (2)

$\vdash . (1) . (2) . \supset \vdash :. \kappa \in FM \text{init} . \supset : R \in \kappa \cap \text{Cnv}``\kappa . \supset_R . R \in \text{Rl}`I :$

$[(*334·05)] \quad \supset : \kappa \in FM \text{asym} :. \supset \vdash . \text{Prop}$

***335·17.** $\vdash : \kappa \in FM \text{init} . \supset . s`\kappa_{,,}|``\kappa = \kappa$

Dem.

$\vdash . *335·15 . \supset \vdash :. \text{Hp} . \supset : R, S \in \kappa . \supset . (\exists T) . T \in \kappa . R`S`\text{init}`\kappa = T`\text{init}`\kappa .$

$[*331·24·33·32] \quad \supset . (\exists T) . T \in \kappa . R | S = T .$

$[*13·195] \quad \supset . R | S \in \kappa$ (1)

$\vdash . *331·22 . \supset \vdash : \text{Hp} . \supset . \kappa \subset s`\kappa_{,,}|``\kappa$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***335·18.** $\vdash . FM \text{init} \subset FM \text{trs}$

Dem.

$\vdash . *335·17 . \quad \supset \vdash :. \kappa \in FM \text{init} . \supset : R, S \in \kappa_\partial . \supset . R | S \in \kappa$ (1)

$\vdash . *334·5 . *335·16 . \supset \vdash :. \kappa \in FM \text{init} . \supset : R, S \in \kappa_\partial . \supset . R | S \mathrel{\mathsf{G}} J$ (2)

$\vdash . (1) . (2) . *330·551 . \supset \vdash :. \kappa \in FM \text{init} . \supset : R, S \in \kappa_\partial . \supset . R | S \in \kappa_\partial$ (3)

$\vdash . (3) . *334·13 . \supset \vdash . \text{Prop}$

***335·19.** $\vdash :. \kappa \in FM \text{init} . \supset : \kappa \in FM \text{connex} . \equiv . \text{init}`\kappa \in p`C``\kappa_\iota$
$[*334·23 . (*334·03 . *335·02·01)]$

***335·21.** $\vdash : \kappa \in FM \text{init} . \supset . \dot{s}`\kappa_\partial \in \text{trans} . (\dot{s}`\kappa_\partial)^2 \mathrel{\mathsf{G}} J$ $[*335·18·16 . *334·14·5]$

***335·22.** $\vdash :. \kappa \in FM \text{init} . \supset : \dot{s}`\kappa_\partial \in \text{connex} . \equiv . C``\kappa_\iota = \text{G}``\kappa . \equiv . \text{init}`\kappa \in p`C``\kappa_\iota$
$[*334·26 . *335·19]$

***335·23.** $\vdash :. \kappa \in FM \text{ init} \cap FM \text{ connex} . L \in \kappa_{\iota\partial} . \supset :$
$$\text{init}`\kappa \in \text{D}`L . \equiv . \text{init}`\kappa \sim\in \text{Ɑ}`L$$

Dem.

$\vdash . *335·19 . \supset \vdash :. \text{Hp} . \supset : \text{init}`\kappa \in \text{D}`L . \text{v} . \text{init}`\kappa \in \text{Ɑ}`L$ (1)

$\vdash . *334·21 . \supset \vdash : \text{Hp} . \supset . L \in \kappa_\partial \cup \text{Cnv}``\kappa_\partial$ (2)

$\vdash . *335·11 . \supset \vdash :. \text{Hp} . \supset : L \in \kappa_\partial . \supset . \text{init}`\kappa \sim\in \text{D}`L :$
$L \in \text{Cnv}``\kappa_\partial . \supset . \text{init}`\kappa \sim\in \text{Ɑ}`L$ (3)

$\vdash . (2) . (3) . \supset \vdash :. \text{Hp} . \supset : \text{init}`\kappa \sim\in \text{D}`L . \text{v} . \text{init}`\kappa \sim\in \text{Ɑ}`L$ (4)

$\vdash . (1) . (4) . *5·17 . \supset \vdash . \text{Prop}$

***335·24.** $\vdash :. \kappa \in FM \text{ init} \cap FM \text{ connex} . R, S \in \kappa . R \neq S . \supset :$
$$R`\text{init}`\kappa \in \text{D}`S . \equiv . S`\text{init}`\kappa \sim\in \text{D}`R$$

Dem.

$\vdash . *71·162 . \supset \vdash :. \text{Hp} . \supset : R`\text{init}`\kappa \in \text{D}`S . \equiv . \text{init}`\kappa \in \text{Ɑ}`(\breve{S} \mid R) .$

[*333·1.*335·23] $\equiv . \text{init}`\kappa \sim\in \text{D}`(\breve{S} \mid R) .$

[*71·162] $\equiv . S`\text{init}`\kappa \sim\in \text{D}`R :. \supset \vdash . \text{Prop}$

***335·25.** $\vdash ::. \kappa \in FM \text{ init} . \supset :: \dot{s}`\kappa_\partial \in \text{connex} . \equiv :.$
$$R, S \in \kappa . \supset_{R,S} : \text{D}`R \subset \text{D}`S . \text{v} . \text{D}`S \subset \text{D}`R :.$$
$$\equiv :. \alpha, \beta \in \text{D}``\kappa . \supset_{\alpha,\beta} : \alpha \subset \beta . \text{v} . \beta \subset \alpha$$

Dem.

$\vdash . *202·135 . \supset \vdash :: \text{Hp} . \dot{s}`\kappa_\partial \in \text{connex} . \supset :. \dot{s}`\kappa \in \text{connex} :.$

[*211·6.*330·542] $\supset :. R, S \in \kappa . \supset : \text{D}`R \subset \text{D}`S . \text{v} . \text{D}`S \subset \text{D}`R$ (1)

$\vdash . *71·162 . \supset \vdash : \text{Hp} . R`\text{init}`\kappa \in \text{D}`S . \supset . \text{init}`\kappa \in \text{Ɑ}`(\breve{R} \mid S)$ (2)

$\vdash . *71·162 . \supset \vdash : \text{Hp} . S`\text{init}`\kappa \in \text{D}`R . \supset . \text{init}`\kappa \in \text{D}`(\breve{R} \mid S)$ (3)

$\vdash . (2) . (3) . \supset \vdash :. \text{Hp} . R, S \in \kappa : \text{D}`R \subset \text{D}`S . \text{v} . \text{D}`S \subset \text{D}`R : \supset .$
$\text{init}`\kappa \in C`(\breve{R} \mid S)$ (4)

$\vdash . (4) . *330·4 . \supset \vdash :: \text{Hp} :. R, S \in \kappa . \supset_{R,S} : \text{D}`R \subset \text{D}`S . \text{v} . \text{D}`S \subset \text{D}`R :. \supset .$
$\text{init}`\kappa \in p`C``\kappa_\iota .$

[*335·22] $\supset . \dot{s}`\kappa_\partial \in \text{connex}$ (5)

$\vdash . (1) . (5) . *37·63 . \supset \vdash . \text{Prop}$

***335·26.** $\vdash : \kappa \in FM \text{ init} \cap FM \text{ connex} . \supset . \text{D} \restriction \kappa \in 1 \to 1$

Dem.

$\vdash . *33·43 . \supset \vdash : \text{Hp} . R, S \in \kappa . R`\text{init}`\kappa \sim\in \text{D}`S . \supset . \text{D}`R \neq \text{D}`S$ (1)

$\vdash . *335·24 . \supset \vdash : \text{Hp} . R, S \in \kappa . R \neq S . R`\text{init}`\kappa \in \text{D}`S . \supset . S`\text{init}`\kappa \sim\in \text{D}`R .$

[*33·43] $\supset . \text{D}`R \neq \text{D}`S$ (2)

$\vdash . (1) . (2) . \supset \vdash : \text{Hp} . R, S \in \kappa . R \neq S . \supset . \text{D}`R \neq \text{D}`S : \supset \vdash . \text{Prop}$

***335·3.** $\vdash : \kappa \in FM . \text{init}`\kappa \in p`C``\kappa_\iota . \supset . \dot{s}`\kappa_\partial \in \text{Ser}$ [*335·21·22]

Summary of *336.

In this number we consider a relation between members of κ or of κ_ι which, with suitable limitations as to the nature of the family, may be identified with the relation of greater and less. If there is a member of κ which takes us from a point z to a point y, *i.e.* if $y (\dot{s}\text{‘}\kappa_\partial) z$, we say that z is an earlier point than y; thus we regard $\dot{s}\text{‘}\kappa_\partial$ as the relation of later to earlier. If now M and N are two members of κ_ι, and if, for some x, $M\text{‘}x$ is later than $N\text{‘}x$, we shall say that M is "greater" than N with respect to κ. This relation we denote by V_κ, where "V" is intended to suggest that the relation holds between *vectors*. The definition is:

***336·01.** $V_\kappa = \hat{M}\hat{N} \{M, N \epsilon \kappa_\iota : (\exists x) . (M\text{‘}x) (\dot{s}\text{‘}\kappa_\partial) (N\text{‘}x)\}$ Df

For the same relation when confined to members of κ, we use the notation U_κ; thus we put

***336·011.** $U_\kappa = V_\kappa \upharpoonright\!\!\upharpoonleft \kappa$ Df

In dealing with V_κ and U_κ it is desirable to be able to express $M\text{‘}x$ as a function of M. We wish to consider (say) a fixed origin a, and the various points $R\text{‘}a$, $S\text{‘}a$, $T\text{‘}a$, ... to which the various vectors which are members of κ carry us from a. For this purpose we put

$$R\text{‘}a = A_a\text{‘}R,$$

where "A" stands for "argument," and "$A_a\text{‘}R$" may be read "the value, for the argument a, of R." The definition is

$$A_a = \hat{x}\hat{R} (xRa) \quad \text{Df},$$

whence we obtain

***336·101.** $\vdash : E ! R\text{‘}a . \supset . R\text{‘}a = A_a\text{‘}R$

Then the points $R\text{‘}a, S\text{‘}a, T\text{‘}a, \ldots$, where $R, S, T, \ldots$ are the various members of κ, form the class $A_a\text{“}\kappa$, which is thus the same class as $\overrightarrow{\dot{s}\text{‘}\kappa}\text{‘}a$. The relation $A_a \upharpoonright \kappa$ correlates the point $R\text{‘}a$ with the vector R. The vector R is analogous to the coordinate of $R\text{‘}a$ when a is the origin; thus $A_a \upharpoonright \kappa$ is analogous to the relation of a point to its coordinate. A relation which is more exactly that of a point to its coordinate will be explained in Section C, where, in

addition to the above correlator $A_a \upharpoonright \kappa$, we shall also correlate a vector with its numerical measure in terms of an assigned unit.

If κ is a connected family, and a is any point of its field, $A_a \upharpoonright \kappa_\iota$ is a one-one relation (*336·2). If κ is an initial family, and a is its initial point, $A_a \upharpoonright \kappa$ is a correlator of $s`\mathrm{C}``\kappa$ and κ (*336·21), so that in an initial family the class of vectors is similar to the field (*336·22). If κ is a connected family, and a is any point of the field, and λ is those members L of κ_ι for which $L`a$ exists, then $A_a \upharpoonright \lambda$ correlates the field with λ, so that λ is similar to the field (*336·24).

By the definition of A_a, if $M \epsilon \kappa_\iota$ and $M`a$ exists, we have

$$M`a = A_a`M = A_a \upharpoonright \kappa_\iota`M.$$

Hence by the definition of V_κ,

$$\vdash : MV_\kappa N . \equiv . (\exists a) . (A_a \upharpoonright \kappa_\iota`M)(\dot{s}`\kappa_\partial)(A_a \upharpoonright \kappa_\iota`N) .$$

$$\equiv . (\exists a) . M(\kappa_\iota \upharpoonright \breve{A}_a ; \dot{s}`\kappa_\partial) N, \text{ by } *150{\cdot}41.$$

Similarly $\quad \vdash : PU_\kappa Q . \equiv . (\exists a) . P(\kappa \upharpoonright \breve{A}_a ; \dot{s}`\kappa_\partial) Q.$

Now in a connected family, if a and b are any two members of the field, and $P, Q \epsilon \kappa$,

$$(P`a)(\dot{s}`\kappa_\partial)(Q`a) . \equiv . (P`b)(\dot{s}`\kappa_\partial)(Q`b) \quad (*336{\cdot}38);$$

hence $$\kappa \upharpoonright \breve{A}_a ; \dot{s}`\kappa_\partial = \kappa \upharpoonright \breve{A}_b ; \dot{s}`\kappa_\partial,$$

and hence $$U_\kappa = \kappa \upharpoonright \breve{A}_a ; \dot{s}`\kappa_\partial \quad (*336{\cdot}43).$$

Since $\kappa \upharpoonright \breve{A}_a$ is one-one (by *336·2), the above gives an ordinal correlation of U_κ with $(\dot{s}`\kappa_\partial) \upharpoonright A_a``\kappa$ (*336·461), *i.e.* U_κ is ordinally similar to $\dot{s}`\kappa_\partial$ with its field confined to those points which can be reached from $a$ by vectors which are members of $\kappa$. If $\kappa$ is an initial family, it follows that $U_\kappa$ is similar to $\dot{s}`\kappa_\partial$ (*336·44); if not, U_κ is in general only similar to a segment of $\dot{s}`\kappa_\partial$ (in the sense of *213).

It should be observed that $\kappa_\iota \upharpoonright \breve{A}_a`x$ is the member of $\kappa_\iota$ which takes us from $a$ to $x$, and $\kappa \upharpoonright \breve{A}_a`x$ (if it exists) is the member of κ which takes us from a to x. Thus $\kappa \upharpoonright \breve{A}_a ; \dot{s}`\kappa_\partial$ is the series of vectors which take us from a to all the various points which can be reached from a by members of κ, the order of the series being that of the points to which the various vectors take us from a.

If κ is a connected family, U_κ is the relation which holds between two members of κ when one of them is the relative product of the other and a third (other than the zero vector), *i.e.*

***336·41.** $\vdash : \kappa \epsilon FM \,\mathrm{conx} . \supset . U_\kappa = \hat{P}\hat{Q}\{P, Q \epsilon \kappa : (\exists T) . T \epsilon \kappa_\partial . P = T \mid Q\}$

This is for many purposes the most convenient formula for U_κ. If, in addition, we have $\mathrm{D}``\kappa = \mathrm{Q}``\kappa$, a similar formula holds for V_κ, *i.e.*

***336·54.** $\vdash : \kappa \epsilon FM \,\mathrm{conx} \,.\, \mathrm{D}``\kappa = \mathrm{Q}``\kappa \,.\, \supset .$

$$V_\kappa = \hat{M}\hat{N}\{M, N \epsilon \kappa_\iota : (\exists T) \,.\, T \epsilon \kappa_\partial \,.\, M = T \mid N\}$$

If $\kappa \epsilon FM$ conx, V_κ is contained in diversity (*336·6); if κ is also transitive, V_κ is transitive (336·61); and if κ has connexity, so has V_κ (*336·62). Hence if κ is a serial family, V_κ and U_κ are serial (*336·63·64).

In addition to the above-mentioned propositions, the following propositions in this number are important:

***336·411.** $\vdash :. \kappa \epsilon FM \,\mathrm{conx} \,.\, s`\kappa \mid``\kappa \subset \kappa \,.\, \supset : PU_\kappa Q \,.\, R \epsilon \kappa \,.\, \supset . (P \mid R)\, U_\kappa (Q \mid R)$

***336·511.** $\vdash :. \kappa \epsilon FM \,\mathrm{sr} \,.\, \nu \epsilon \mathrm{NC\,ind} - \iota`0 \,.\, \supset : RU_\kappa S \,.\, \equiv .\, R^\nu U_\kappa S^\nu$

***336·53.** $\vdash :. \kappa \epsilon FM \,\mathrm{conx} \,.\, M, N \epsilon \kappa_\iota \,.\, \supset : MV_\kappa N \,.\, \equiv .\, \breve{N} V_\kappa \breve{M}$

The present number is important, since V_κ and U_κ are the general relations from which greater and less among magnitudes are derived, and the subject of magnitude is therefore intimately dependent upon them.

***336·01.** $V_\kappa = \hat{M}\hat{N}\{M, N \epsilon \kappa_\iota : (\exists x) \,.\, (M`x)\,(\dot{s}`\kappa_\partial)\,(N`x)\}$ Df

***336·011.** $U_\kappa = V_\kappa \upharpoonright \kappa$ Df

***336·02.** $A_a = \hat{x}\hat{R}\,(xRa)$ Df

***336·1.** $\vdash : xA_a R \,.\, \equiv .\, xRa$ [(*336·02)]

***336·101.** $\vdash : \mathrm{E}\,!\, R`a \,.\, \supset .\, R`a = A_a`R$ [*336·1]

***336·11.** $\vdash : x\,(A_a \upharpoonright \kappa)\, R \,.\, \equiv .\, R \epsilon \kappa \,.\, xRa$ [*336·1]

***336·12.** $\vdash .\, \overrightarrow{\dot{s}`\kappa}`a = A_a``\kappa = \mathrm{D}`(A_a \upharpoonright \kappa)$

Dem.

$\vdash .\, *41·11 \,.\, \supset \vdash .\, \overrightarrow{\dot{s}`\kappa}`a = \hat{x}\{(\exists R) \,.\, R \epsilon \kappa \,.\, xRa\}$

[*336·1] $= \hat{x}\{(\exists R) \,.\, R \epsilon \kappa \,.\, xA_a R\} \,.\, \supset \vdash .\, \mathrm{Prop}$

***336·13.** $\vdash .\, \mathrm{D}`A_a \upharpoonright \kappa \subset s`\mathrm{D}``\kappa$

Dem.

$\vdash .\, *336·12 \,.\, *33·15 \,.\, \supset \vdash .\, \mathrm{D}`A_a \upharpoonright \kappa \subset \mathrm{D}`\dot{s}`\kappa \,.\, \supset \vdash .\, \mathrm{Prop}$

***336·14.** $\vdash : \kappa \subset 1 \to \mathrm{Cls} \,.\, \supset .\, A_a \upharpoonright \kappa \epsilon 1 \to \mathrm{Cls}$

Dem.

$\vdash .\, *336·11 \,.\, \supset \vdash : x\,(A_a \upharpoonright \kappa)\, R \,.\, y\,(A_a \upharpoonright \kappa)\, R \,.\, \supset .\, R \epsilon \kappa \,.\, xRa \,.\, yRa$ (1)

$\vdash .\, (1) \,.\, *71·17 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \mathrm{Hp}(1) \,.\, \supset .\, x = y$ (2)

$\vdash .\, (2) \,.\, *71·17 \,.\, \supset \vdash .\, \mathrm{Prop}$

***336·15.** $\vdash : \kappa \subset \mathrm{cr}`\alpha \,.\, a \in \alpha \,.\, \supset \,.\, \mathrm{G}`(A_a \upharpoonright \kappa) = \kappa$

Dem.

$\vdash \,.\, *336\cdot11 \,.\, \supset \vdash : R \in \mathrm{G}`(A_a \upharpoonright \kappa) \,.\, \equiv \,.\, (\exists x) \,.\, R \in \kappa \,.\, xRa$ (1)

$\vdash \,.\, (1) \,.\, (*330\cdot01) \,.\, \supset \vdash \,.\, \mathrm{Prop}$

***336·16.** $\vdash : a \in \mathrm{conx}`\kappa \,.\, \equiv \,.\, a \in s`\mathrm{G}``\kappa \,.\, A_a``(\kappa \cup \mathrm{Cnv}``\kappa) = s`\mathrm{G}``\kappa$

Dem.

$\vdash \,.\, *331\cdot1 \,.\, *336\cdot12 \,.\, \supset$

$\vdash : a \in \mathrm{conx}`\kappa \,.\, \equiv \,.\, a \in s`\mathrm{G}``\kappa \,.\, A_a``\kappa \cup A_a``\mathrm{Cnv}``\kappa = s`\mathrm{G}``\kappa$ (1)

$\vdash \,.\, (1) \,.\, *37\cdot22 \,.\, \supset \vdash \,.\, \mathrm{Prop}$

***336·17.** $\vdash : \kappa \in FM\,\mathrm{conx} \cap FM\,\mathrm{trs} - 1 \,.\, \breve{P} = \dot{s}`\kappa_\partial \,.\, \supset \,.\, A_a``\kappa = \overleftarrow{P_*}`a$

Dem.

$\vdash \,.\, *334\cdot14\cdot18 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, \overleftarrow{P_*}`a = \overleftarrow{P}`a \cup \overrightarrow{I \upharpoonright s`\mathrm{G}``\kappa}`a$

$[*331\cdot22\cdot23] \qquad = \overrightarrow{\dot{s}`\kappa}`a$

$[*336\cdot12] \qquad = A_a``\kappa : \supset \vdash \,.\, \mathrm{Prop}$

***336·2.** $\vdash : \kappa \in FM\,\mathrm{conx} \,.\, a \in s`\mathrm{G}``\kappa \,.\, \supset \,.\, A_a \upharpoonright \kappa_\iota \in 1 \rightarrow 1$

Dem.

$\vdash \,.\, *336\cdot14 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, A_a \upharpoonright \kappa_\iota \in 1 \rightarrow \mathrm{Cls}$ (1)

$\vdash \,.\, *336\cdot11 \,.\, \supset \vdash : \mathrm{Hp} \,.\, x(A_a \upharpoonright \kappa_\iota)L \,.\, x(A_a \upharpoonright \kappa_\iota)M \,.\, \supset \,.\, L, M \in \kappa_\iota \,.\, xLa \,.\, xMa \,.$

$[*331\cdot42] \qquad \supset \,.\, L = M$ (2)

$\vdash \,.\, (1) \,.\, (2) \,.\, \supset \vdash \,.\, \mathrm{Prop}$

***336·21.** $\vdash : \kappa \in FM \,.\, a = \mathrm{init}`\kappa \,.\, \supset \,.\, A_a \upharpoonright \kappa \in (s`\mathrm{G}``\kappa)\,\overline{\mathrm{sm}}\,\kappa$

Dem.

$\vdash \,.\, *336\cdot2 \,. \qquad \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, A_a \upharpoonright \kappa \in 1 \rightarrow 1$ (1)

$\vdash \,.\, *335\cdot15 \,.\, *336\cdot12 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, \mathrm{D}`A_a \upharpoonright \kappa = s`\mathrm{G}``\kappa$ (2)

$\vdash \,.\, *336\cdot15 \,. \qquad \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, \mathrm{G}`A_a \upharpoonright \kappa = \kappa$ (3)

$\vdash \,.\, (1) \,.\, (2) \,.\, (3) \,.\, \supset \vdash \,.\, \mathrm{Prop}$

***336·22.** $\vdash : \kappa \in FM\,\mathrm{init} \,.\, \supset \,.\, (s`\mathrm{G}``\kappa)\,\mathrm{sm}\,\kappa \quad [*336\cdot21]$

***336·23.** $\vdash : \kappa \in FM\,\mathrm{conx} \,.\, a \in s`\mathrm{G}``\kappa \,.\, \lambda = \kappa_\iota \cap \hat{L}(a \in \mathrm{G}`L) \,.\, \supset \,.$

$A_a \upharpoonright \lambda \in (s`\mathrm{G}``\kappa)\,\overline{\mathrm{sm}}\,\lambda$

Dem.

$\vdash \,.\, *336\cdot2 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, A_a \upharpoonright \lambda \in 1 \rightarrow 1$ (1)

$\vdash \,.\, *336\cdot11 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, \mathrm{D}`(A_a \upharpoonright \lambda) = \hat{x}\{(\exists L) \,.\, L \in \lambda \,.\, xLa\}$

$[\mathrm{Hp}] \qquad = \hat{x}\{(\exists L) \,.\, L \in \kappa_\iota \,.\, xLa\}$

$[*331\cdot4] \qquad = s`\mathrm{G}``\kappa$ (2)

$\vdash \,.\, *336\cdot11 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, \mathrm{G}`(A_a \upharpoonright \lambda) = \hat{L}\{(\exists x) \,.\, L \in \lambda \,.\, xLa\}$

$[\mathrm{Hp}] \qquad = \lambda$ (3)

$\vdash \,.\, (1) \,.\, (2) \,.\, (3) \,.\, \supset \vdash \,.\, \mathrm{Prop}$

***336·24.** $\vdash : \mathrm{Hp} * 336 \cdot 23 . \supset . (s\text{‘}\text{Ɑ}\text{“}\kappa) \,\mathrm{sm}\, \lambda \quad [*336 \cdot 23]$

***336·25.** $\vdash : \kappa \in FM \,\mathrm{conx} . a, b \in s\text{‘}\text{Ɑ}\text{“}\kappa . \lambda = \kappa_\iota \cap \hat{L}(a \in \text{Ɑ}\text{‘}L) .$

$$\mu = \kappa_\iota \cap \hat{M}(b \in \text{Ɑ}\text{‘}M) . \supset . \lambda \,\mathrm{sm}\, \mu \quad [*336 \cdot 24]$$

***336·26.** $\vdash : \kappa \in FM . a \in \mathrm{conx}\text{‘}\kappa . \lambda = \kappa \cup \mathrm{Cnv}\text{“}\hat{R}(R \in \kappa . a \in \mathrm{D}\text{‘}R) . \supset .$

$$A_a \upharpoonright \lambda \in (s\text{‘}\text{Ɑ}\text{“}\kappa) \,\overline{\mathrm{sm}}\, \lambda \quad [*336 \cdot 23 . *331 \cdot 48]$$

***336·3.** $\vdash :. \kappa \subset 1 \rightarrow \mathrm{Cls} . \supset : R(\kappa \rceil \breve{A}_a ; P) S . \equiv . R, S \in \kappa . (R\text{‘}a) P (S\text{‘}a)$

Dem.

$\vdash . *150 \cdot 11 . \supset \vdash : R(\kappa \rceil \breve{A}_a ; P) S . \equiv . (\exists x, y) . R, S \in \kappa . x A_a R . y A_a S . x P y .$

$[*336 \cdot 1] \qquad \equiv . (\exists x, y) . R, S \in \kappa . x R a . y S a . x P y \qquad (1)$

$\vdash . (1) . *71 \cdot 36 . \supset \vdash . \mathrm{Prop}$

***336·31.** $\vdash : \kappa \in FM \,\mathrm{conx} . a \in s\text{‘}\text{Ɑ}\text{“}\kappa . \supset . \kappa_\partial \subset \mathrm{D}\text{‘}(\kappa \rceil \breve{A}_a ; \dot{s}\text{‘}\kappa_\partial)$

Dem.

$\vdash . *336 \cdot 3 . \supset$

$\vdash :. \mathrm{Hp} . \supset : R \in \dot{\mathrm{D}}\text{‘}(\kappa \rceil \breve{A}_a ; \dot{s}\text{‘}\kappa_\partial) . \equiv . (\exists S, T) . R, S \in \kappa . T \in \kappa_\partial . R\text{‘}a = T\text{‘}S\text{‘}a \quad (1)$

$\vdash . *331 \cdot 22 . \supset \vdash : \mathrm{Hp} . R \in \kappa_\partial . \supset . R \in \kappa_\partial . I \upharpoonright s\text{‘}\text{Ɑ}\text{“}\kappa \in \kappa . R\text{‘}a = R\text{‘}(I \upharpoonright s\text{‘}\text{Ɑ}\text{“}\kappa)\text{‘}a .$

$[(1)] \qquad \supset . R \in \mathrm{D}\text{‘}(\kappa \rceil \breve{A}_a ; \dot{s}\text{‘}\kappa_\partial) : \supset \vdash . \mathrm{Prop}$

***336·311.** $\vdash : \kappa \in FM \,\mathrm{conx} - 1 . a \in s\text{‘}\text{Ɑ}\text{“}\kappa . \supset . I \upharpoonright s\text{‘}\text{Ɑ}\text{“}\kappa \in \text{Ɑ}\text{‘}(\kappa \rceil \breve{A}_a ; \dot{s}\text{‘}\kappa_\partial)$

Dem.

$\vdash . *336 \cdot 3 . \supset$

$\vdash :. \mathrm{Hp} . \supset : S \in \text{Ɑ}\text{‘}(\kappa \rceil \breve{A}_a ; \dot{s}\text{‘}\kappa_\partial) . \equiv . (\exists R, T) . R, S \in \kappa . T \in \kappa_\partial . R\text{‘}a = T\text{‘}S\text{‘}a :$

$[*331 \cdot 22] \supset : I \upharpoonright s\text{‘}\text{Ɑ}\text{“}\kappa \in \text{Ɑ}\text{‘}(\kappa \rceil \breve{A}_a ; \dot{s}\text{‘}\kappa_\partial) . \equiv . (\exists R, T) . R \in \kappa . T \in \kappa_\partial . R\text{‘}a = T\text{‘}a .$

$[*330 \cdot 52] \qquad \equiv . \exists ! \kappa_\partial \qquad (1)$

$\vdash . (1) . *334 \cdot 18 . \supset \vdash . \mathrm{Prop}$

***336·312.** $\vdash : \kappa \in FM \,\mathrm{conx} - 1 . \supset . C\text{‘}(\kappa \rceil \breve{A}_a ; \dot{s}\text{‘}\kappa_\partial) = \kappa \quad [*336 \cdot 31 \cdot 311]$

***336·313.** $\vdash : \kappa \in FM \,\mathrm{conx} \cap FM \,\mathrm{asym} . a \in s\text{‘}\text{Ɑ}\text{“}\kappa . \supset . \mathrm{D}\text{‘}(\kappa \rceil \breve{A}_a ; \dot{s}\text{‘}\kappa_\partial) = \kappa_\partial$

Dem.

$\vdash . *336 \cdot 3 . \supset$

$\vdash :. \mathrm{Hp} . \supset : I \upharpoonright s\text{‘}\text{Ɑ}\text{“}\kappa \in \mathrm{D}\text{‘}(\kappa \rceil \breve{A}_a ; \dot{s}\text{‘}\kappa_\partial) . \equiv . (\exists S, T) . S \in \kappa . T \in \kappa_\partial . a = T\text{‘}S\text{‘}a \quad (1)$

$\vdash . (1) . *334 \cdot 5 . \supset \vdash : \mathrm{Hp} . \supset . I \upharpoonright s\text{‘}\text{Ɑ}\text{“}\kappa \sim\in \mathrm{D}\text{‘}(\kappa \rceil \breve{A}_a ; \dot{s}\text{‘}\kappa_\partial) \qquad (2)$

$\vdash . (2) . *336 \cdot 31 . \supset \vdash . \mathrm{Prop}$

***336·32.** $\vdash : \kappa \in FM \,.\, a \in \text{conx}\text{‘}\kappa \,.\, \lambda = \kappa \cap \hat{R}(a \in \text{D}\text{‘}R) \,.\, \supset .$
$$C\text{‘}\{(\kappa \cup \text{Cnv}\text{“}\kappa) \upharpoonright \breve{A}_a \,\dot{;}\, \dot{s}\text{‘}\kappa_\partial\} = \kappa \cup \text{Cnv}\text{“}\lambda$$

Dem.

$\vdash .\, *336·16 \,.\, *334·18 \,.\, \supset \vdash : \text{Hp} \,.\, \supset .\, C\text{‘}\dot{s}\text{‘}\kappa_\partial = \text{C\!\!\!I}\text{‘}(\kappa \cup \text{Cnv}\text{“}\kappa) \upharpoonright \breve{A}_a \,.$

$[*150·23] \quad \supset .\, C\text{‘}\{(\kappa \cup \text{Cnv}\text{“}\kappa) \upharpoonright \breve{A}_a \,\dot{;}\, \dot{s}\text{‘}\kappa_\partial\} = \text{D}\text{‘}(\kappa \cup \text{Cnv}\text{“}\kappa) \upharpoonright \breve{A}_a$

$[*336·15·11] \quad = \kappa \cup \hat{R}\{(\exists x) \,.\, R \in \text{Cnv}\text{“}\kappa \,.\, xRa\}$

$[\text{Hp}] \quad = \kappa \cup \text{Cnv}\text{“}\lambda : \supset \vdash .\, \text{Prop}$

***336·34.** $\vdash : \kappa \in FM \,.\, a = \text{init}\text{‘}\kappa \,.\, \supset .\, (\kappa \upharpoonright \breve{A}_a \,\dot{;}\, \dot{s}\text{‘}\kappa_\partial) \text{ smor } (\dot{s}\text{‘}\kappa_\partial)$

Dem.

$\vdash .\, *336·21 \,.\, \supset \vdash : \text{Hp} \,.\, \supset .\, \kappa \upharpoonright \breve{A}_a \in 1 \to 1 \,.\, \text{C\!\!\!I}\text{‘}\kappa \upharpoonright \breve{A}_a = C\text{‘}\dot{s}\text{‘}\kappa_\partial : \supset \vdash .\, \text{Prop}$

***336·35.** $\vdash : \kappa \in FM \,.\, a \in \text{conx}\text{‘}\kappa \,.\, \supset .\, \{(\kappa \cup \text{Cnv}\text{“}\kappa) \upharpoonright \breve{A}_a \,\dot{;}\, \dot{s}\text{‘}\kappa_\partial\} \text{ smor } (\dot{s}\text{‘}\kappa_\partial)$ $[*336·2·16]$

***336·351.** $\vdash : \kappa \in FM \text{ conx} \,.\, a \in s\text{‘}\text{C\!\!\!I}\text{“}\kappa \,.\, \supset .\, (\kappa \upharpoonright \breve{A}_a \,\dot{;}\, \dot{s}\text{‘}\kappa_\partial) \text{ smor } (\dot{s}\text{‘}\kappa_\partial) \upharpoonleft A_a\text{“}\kappa$

Dem.

$\vdash .\, *336·2 \,.\, \supset \vdash : \text{Hp} \,.\, \supset .\, \kappa \upharpoonright \breve{A}_a \in 1 \to 1 \quad (1)$

$\vdash .\, *150·37 \,.\, \supset \vdash : \text{Hp} \,.\, \supset .\, \kappa \upharpoonright \breve{A}_a \,\dot{;}\, \dot{s}\text{‘}\kappa_\partial = \kappa \upharpoonright \breve{A}_a \,\dot{;}\, (\dot{s}\text{‘}\kappa_\partial) \upharpoonleft A_a\text{“}\kappa \quad (2)$

$\vdash .\, (1) \,.\, (2) \,.\, \supset \vdash .\, \text{Prop}$

***336·36.** $\vdash :. \kappa \in FM \text{ conx} \,.\, L, M \in \kappa_\iota \,.\, a, b \in \text{C\!\!\!I}\text{‘}L \cap \text{C\!\!\!I}\text{‘}M \,.\, T \in \kappa_\iota \,.\, \supset :$
$$L\text{‘}a = T\text{‘}M\text{‘}a \,.\, \equiv .\, L\text{‘}b = T\text{‘}M\text{‘}b : L\text{‘}a = \breve{T}\text{‘}M\text{‘}a \,.\, \equiv .\, L\text{‘}b = \breve{T}\text{‘}M\text{‘}b$$

Dem.

$\vdash .\, *13·12 \,.\, \supset \vdash :. \text{Hp} \,.\, N \in \kappa_\iota \,.\, a = N\text{‘}b \,.\, \supset : L\text{‘}a = T\text{‘}M\text{‘}a \,.\, \equiv .\, L\text{‘}N\text{‘}b = T\text{‘}M\text{‘}N\text{‘}b \,.$

$[*330·63] \quad \equiv .\, N\text{‘}L\text{‘}b = N\text{‘}T\text{‘}M\text{‘}b \,.$

$[*71·56] \quad \equiv .\, L\text{‘}b = T\text{‘}M\text{‘}b \quad (1)$

$\vdash .\, (1) \,.\, *331·4 \,.\, \supset \vdash :. \text{Hp} \,.\, \supset : L\text{‘}a = T\text{‘}M\text{‘}a \,.\, \equiv .\, L\text{‘}b = T\text{‘}M\text{‘}b \quad (2)$

$\vdash .\, *71·362 \,.\, \supset \vdash :. \text{Hp} \,.\, \supset : L\text{‘}a = \breve{T}\text{‘}M\text{‘}a \,.\, \equiv .\, M\text{‘}a = T\text{‘}L\text{‘}a \,.$

$\left[(2)\dfrac{M, L}{L, M}\right] \quad \equiv .\, M\text{‘}b = T\text{‘}L\text{‘}b \,.$

$[*71·362] \quad \equiv .\, L\text{‘}b = \breve{T}\text{‘}M\text{‘}b \quad (3)$

$\vdash .\, (2) \,.\, (3) \,.\, \supset \vdash .\, \text{Prop}$

***336·37.** $\vdash :. \kappa \in FM \text{ conx} \,.\, L, M \in \kappa_\iota \,.\, a, b \in \text{C\!\!\!I}\text{‘}L \cap \text{C\!\!\!I}\text{‘}M \,.\, \supset :$
$$(L\text{‘}a)\,(\dot{s}\text{‘}\kappa_\partial)\,(M\text{‘}a) \,.\, \equiv .\, (L\text{‘}b)\,(\dot{s}\text{‘}\kappa_\partial)\,(M\text{‘}b)$$

Dem.

$\vdash .\, *336·36 \,.\, \supset$

$\vdash :. \text{Hp} \,.\, \supset : (\exists T) \,.\, T \in \kappa_\partial \,.\, L\text{‘}a = T\text{‘}M\text{‘}a \,.\, \equiv .\, (\exists T) \,.\, T \in \kappa_\partial \,.\, L\text{‘}b = T\text{‘}M\text{‘}b :. \supset \vdash .\, \text{Prop}$

***336·371.** $\vdash :. \kappa \in FM \text{ conx} . L, M \in \kappa_\iota . a \in \text{Cl}'L \cap \text{Cl}'M . \supset :$
$LV_\kappa M . \equiv . (L'a)(\dot{s}'\kappa_\partial)(M'a)$ [*336·37 . (*336·01)]

***336·38.** $\vdash :. \kappa \in FM \text{ conx} . P, Q \in \kappa . a, b \in s'\text{Cl}''\kappa . \supset :$
$(P'a)(\dot{s}'\kappa_\partial)(Q'a) . \equiv . (P'b)(\dot{s}'\kappa_\partial)(Q'b)$ [*336·37 . *331·24]

***336·4.** $\vdash : \kappa \in FM \text{ conx} . a \in s'\text{Cl}''\kappa . \supset . U_\kappa = \hat{P}\hat{Q}\{P, Q \in \kappa . (P'a)(\dot{s}'\kappa_\partial)(Q'a)\}$

Dem.

$\vdash . *336·38 . \supset$
$\vdash :. \text{Hp} . \supset : b \in s'\text{Cl}''\kappa . (P'b)(\dot{s}'\kappa_\partial)(Q'b) . \equiv . b \in s'\text{Cl}''\kappa . (P'a)(\dot{s}'\kappa_\partial)(Q'a) :$
$[*10·11·281.\text{Hp}] \supset : (\exists b) . b \in s'\text{Cl}''\kappa . (P'b)(\dot{s}'\kappa_\partial)(Q'b) . \equiv . (P'a)(\dot{s}'\kappa_\partial)(Q'a)$ (1)
$\vdash . (1) . (*336·011) . \supset \vdash . \text{Prop}$

***336·41.** $\vdash : \kappa \in FM \text{ conx} . \supset . U_\kappa = \hat{P}\hat{Q}\{P, Q \in \kappa : (\exists T) . T \in \kappa_\partial . P = T | Q\}$

Dem.

$\vdash . *41·11 . \supset \vdash : \text{Hp} . a \in s'\text{Cl}''\kappa . P, Q \in \kappa . T \in \kappa_\partial . P = T | Q . \supset . (P'a)(\dot{s}'\kappa_\partial)(Q'a)$ (1)
$\vdash . *41·11 . \supset \vdash : \text{Hp} . a \in s'\text{Cl}''\kappa . (P'a)(\dot{s}'\kappa_\partial)(Q'a) . \supset . (\exists T) . T \in \kappa_\partial . P'a = T'Q'a .$
$[*331·32·33·24]$ $\supset . (\exists T) . T \in \kappa_\partial . P = T | Q$ (2)
$\vdash . (1) . (2) . *336·4 . \supset \vdash . \text{Prop}$

***336·411.** $\vdash :. \kappa \in FM \text{ conx} . s'\kappa_{,}|''\kappa \subset \kappa . \supset : PU_\kappa Q . R \in \kappa . \supset . (P | R) U_\kappa (Q | R)$
[*336·41]

***336·412.** $\vdash : \text{Hp} *336·411 . P, Q, R \in \kappa . (P | R) U_\kappa (Q | R) . \supset . PU_\kappa Q$

Dem.

$\vdash . *336·41 . \supset \vdash : \text{Hp} . \supset . (\exists T) . T \in \kappa_\partial . P | R = T | Q | R .$
$[*330·5]$ $\supset . (\exists T) . T \in \kappa_\partial . \breve{R} | R | P = \breve{R} | R | T | Q$
$[*330·31]$ $\supset . (\exists T) . T \in \kappa_\partial . P = T | Q .$
$[*336·41]$ $\supset . PU_\kappa Q : \supset \vdash . \text{Prop}$

***336·413.** $\vdash :. \text{Hp} *336·411 . P, Q, R \in \kappa . \supset : PU_\kappa Q . \equiv . (P | R) U_\kappa (Q | R)$
[*336·411·412]

***336·42.** $\vdash : \kappa \in FM \text{ conx} . a \in p'\text{D}''\kappa . \supset . V_\kappa = \hat{L}\hat{M}\{L, M \in \kappa_\iota . (L'a)(\dot{s}'\kappa_\partial)(M'a)\}$

Dem.

$\vdash . *330·54 . \supset \vdash :. \text{Hp} . L, M \in \kappa_\iota . \supset : a \in \text{Cl}'L \cap \text{Cl}'M :$
$[*336·37]$ $\supset : (L'b)(\dot{s}'\kappa_\partial)(M'b) . \supset . (L'a)(\dot{s}'\kappa_\partial)(M'a) :$
$[(*336·01)]$ $\supset : LV_\kappa M . \supset . (L'a)(\dot{s}'\kappa_\partial)(M'a)$ (1)
$\vdash . (1) . (*336·01) . \supset \vdash . \text{Prop}$

***336·43.** $\vdash : \kappa \in FM \text{ conx} . a \in s'\text{Cl}''\kappa . \supset . U_\kappa = \kappa \upharpoonleft \breve{A}_a ; \dot{s}'\kappa_\partial$

Dem.

$\vdash . *336·4·101 . \supset \vdash : \text{Hp} . \supset . U_\kappa = \hat{P}\hat{Q}\{P, Q \in \kappa . (A_a'P)(\dot{s}'\kappa_\partial)(A_a'Q)\}$
$[*35·7]$ $= \hat{P}\hat{Q}\{(A_a \upharpoonright \kappa'P)(\dot{s}'\kappa_\partial)(A_a \upharpoonright \kappa'Q)\}$
$[*150·41 . *336·2]$ $= \kappa \upharpoonleft \breve{A}_a ; \dot{s}'\kappa_\partial : \supset \vdash . \text{Prop}$

***336·44.** $\vdash : \kappa \in FM \text{ init} . \supset . U_\kappa \text{ smor } (\dot{s}`\kappa_\partial)$

Dem.

$\vdash . *336·41 . \supset \vdash : \text{Hp} . a = \text{init}`\kappa . \supset . U_\kappa = \kappa \upharpoonright \breve{A}_a \dot{;} \dot{s}`\kappa_\partial$ (1)

$\vdash . *336·21 . \supset \vdash : \text{Hp} . a = \text{init}`\kappa . \supset . \kappa \upharpoonright \breve{A}_a \in 1 \rightarrow 1 . \text{C}`(\kappa \upharpoonright \breve{A}_a) = s`\text{C}``\kappa$ (2)

$\vdash . (1) . (2) . *334·19 . \supset \vdash . \text{Prop}$

***336·45.** $\vdash : \kappa \in FM . a \in \text{conx}`\kappa . \lambda = \kappa \cap \hat{R}(a \in \text{D}`R_\prime) . \supset .$

$V_\kappa \upharpoonright (\kappa \cup \text{Cnv}``\lambda) = (\kappa \cup \text{Cnv}``\kappa) \upharpoonright \breve{A}_a \dot{;} \dot{s}`\kappa_\partial$

Dem.

$\vdash . *41·11 . (*336·01) . \supset$

$\vdash :. P \{V_\kappa \upharpoonright (\kappa \cup \text{Cnv}``\lambda)\} Q . \equiv : P, Q \in \kappa \cup \text{Cnv}``\lambda : (\exists x, T) . T \in \kappa_\partial . P`x = T`Q`x$ (1)

$\vdash . (1) . *336·36 . \supset \vdash :: \text{Hp} . \supset :.$

$P \{V_\kappa \upharpoonright (\kappa \cup \text{Cnv}``\lambda)\} Q . \equiv : P, Q \in \kappa \cup \text{Cnv}``\lambda : (\exists T) . T \in \kappa_\partial . P`a = T`Q`a :$

[*14·21.Hp] $\equiv : P, Q \in \kappa \cup \text{Cnv}``\kappa : (\exists T) . T \in \kappa_\partial . P`a = T`Q`a :$

[*41·11] $\equiv : P, Q \in \kappa \cup \text{Cnv}``\kappa . (P`a)(\dot{s}`\kappa_\partial)(Q`a) :$

[*336·3] $\equiv : P \{(\kappa \cup \text{Cnv}``\kappa) \upharpoonright \breve{A}_a \dot{;} \dot{s}`\kappa_\partial\} Q :: \supset \vdash . \text{Prop}$

***336·46.** $\vdash : \text{Hp } *336·45 . \supset . V_\kappa \upharpoonright (\kappa \cup \text{Cnv}``\lambda) \text{ smor } (\dot{s}`\kappa_\partial)$ [*336·45·2·16]

***336·461.** $\vdash : \kappa \in FM \text{ conx} . a \in s`\text{C}``\kappa . \supset . U_\kappa \text{ smor } (\dot{s}`\kappa_\partial) \upharpoonright (A_a``\kappa)$
[*336·351·43]

***336·462.** $\vdash : \kappa \in FM \text{ conx} \cap FM \text{ trs} . a \in s`\text{C}``\kappa . \breve{P} = \dot{s}`\kappa_\partial . \supset . \breve{U}_\kappa \text{ smor } (P \upharpoonright \overleftarrow{P_*}`a)$
[*336·461·17 . *334·17]

***336·47.** $\vdash : \kappa \in FM \text{ conx} . \supset . \kappa_\partial \subset \text{D}`U_\kappa$ [*336·31·43]

***336·471.** $\vdash : \kappa \in FM \text{ conx} - 1 . \supset . \kappa = C`U_\kappa$ [*336·312·43]

***336·472.** $\vdash : \kappa \in FM \text{ conx} \cap FM \text{ asym} . \supset . \kappa_\partial = \text{D}`U_\kappa$ [*336·313·43]

***336·51.** $\vdash :. \kappa \in FM \text{ sr} . R, S \in \kappa . \nu \in \text{NC ind} - \iota`0 . \supset :$

$(R`a)(\dot{s}`\kappa_\partial)(S`a) . \equiv . (R^\nu`a)(\dot{s}`\kappa_\partial)(S^\nu`a)$

Dem.

$\vdash . *333·42 . *334·32 . *330·57 . *331·42 . \supset$

$\vdash :. \text{Hp} . \supset : T \in \kappa_\partial . R`a = T`S`a . \supset . R^\nu`a = T^\nu`S^\nu`a .$

[*334·131] $\supset . (R^\nu`a)(\dot{s}`\kappa_\partial)(S^\nu`a)$ (1)

$\vdash . (1) . *41·11 . \supset \vdash :. \text{Hp} . \supset : (R`a)(\dot{s}`\kappa_\partial)(S`a) . \supset . (R^\nu`a)(\dot{s}`\kappa_\partial)(S^\nu`a)$ (2)

$\vdash . (2) \frac{S, R}{R, S} . \supset \vdash :. \text{Hp} . \supset : (S`a)(\dot{s}`\kappa_\partial)(R`a) . \supset . (S^\nu`a)(\dot{s}`\kappa_\partial)(R^\nu`a)$ (3)

$\vdash . *331·42 . \supset \vdash :. \text{Hp} . \supset : R`a = S`a . \supset . R^\nu`a = S^\nu`a$ (4)

$\vdash . (3) . (4) . *334·3 . \supset$

$\vdash :. \text{Hp} . \supset : \sim \{(R`a)(\dot{s}`\kappa_\partial)(S`a)\} . \supset . \sim \{(R^\nu`a)(\dot{s}`\kappa_\partial)(S^\nu`a)\}$ (5)

$\vdash . (2) . (5) . \supset \vdash . \text{Prop}$

***336·511.** $\vdash :. \kappa \in FM \,\mathrm{sr} \,.\, \nu \in \mathrm{NC\,ind} - \iota`0 \,.\, \supset : RU_\kappa S \,.\, \equiv \,.\, R^\nu U_\kappa S^\nu$ [*336·51·4]

***336·52.** $\vdash :. \kappa \in FM \,\mathrm{conx} \,.\, Q, R, S, T \in \kappa \,.\, x \in \text{Ɑ}`(\breve{Q} \,|\, R) \cap \text{Ɑ}`(\breve{S} \,|\, T) \,.\, \supset :$

$$(\breve{Q} \,|\, R)\, V_\kappa\, (\breve{S} \,|\, T) \,.\, \equiv \,.\, (S`R`x)\,(\dot{s}`\kappa_\partial)\,(Q`T`x)$$

Dem.

$\vdash . *336·371 . \supset$

$\vdash :. \mathrm{Hp} \,.\, \supset : (\breve{Q} \,|\, R)\, V_\kappa\, (\breve{S} \,|\, T) \,.\, \equiv \,.\, (\exists P) \,.\, P \in \kappa_\partial \,.\, \breve{Q}`R`x = P`\breve{S}`T`x$ (1)

$\vdash . *330·56 . \supset \vdash :. \mathrm{Hp} \,.\, P \in \kappa_\partial \,.\, \supset : \breve{Q}`R`x = P`\breve{S}`T`x \,.\, \equiv \,.\, \breve{Q}`R`x = \breve{S}`P`T`x \,.$

[*71·362] $\equiv \,.\, R`x = Q`\breve{S}`P`T`x \,.$

[*330·54·56] $\equiv \,.\, R`x = \breve{S}`Q`P`T`x \,.$

[*71·362.*330·5] $\equiv \,.\, S`R`x = P`Q`T`x$ (2)

$\vdash . (1) . (2) . \supset \vdash :. \mathrm{Hp} \,.\, \supset : (\breve{Q} \,|\, R)\, V_\kappa (\breve{S} \,|\, T) \,.\, \equiv \,.\, (\exists P) \,.\, P \in \kappa_\partial \,.\, S`R`x = P`Q`T`x \,.$

[*41·11] $\equiv \,.\, (S`R`x)(\dot{s}`\kappa_\partial)(Q`T`x) :. \supset \vdash . \mathrm{Prop}$

***336·53.** $\vdash :. \kappa \in FM \,\mathrm{conx} \,.\, M, N \in \kappa_\iota \,.\, \supset : M V_\kappa N \,.\, \equiv \,.\, \breve{N} V_\kappa \breve{M}$

Dem.

$\vdash . *330·5·54 . \supset$

$\vdash : \mathrm{Hp} \,.\, Q, R, S, T \in \kappa \,.\, M = \breve{Q} \,|\, R \,.\, N = \breve{S} \,|\, T \,.\, a \in s`\text{Ɑ}``\kappa \,.\, x = Q`R`S`T`a \,.\, \supset \,.$

$$\mathrm{E} \,!\, M`x \,.\, \mathrm{E} \,!\, N`x \,.\, \mathrm{E} \,!\, \breve{M}`x \,.\, \mathrm{E} \,!\, \breve{N}`x \quad (1)$$

$\vdash . (1) . *336·52 . \supset \vdash :. \mathrm{Hp}\,(1) \,.\, \supset : M V_\kappa N \,.\, \equiv \,.\, (S`R`x)\,(\dot{s}`\kappa_\partial)\,(Q`T`x) \,.$

[*330·5] $\equiv \,.\, (R`S`x)\,(\dot{s}`\kappa_\partial)\,(T`Q`x) \,.$

[*336·52] $\equiv \,.\, (\breve{T} \,|\, S)\, V_\kappa\, (\breve{R} \,|\, Q) \,.$

[Hp] $\equiv \,.\, \breve{N} V_\kappa \breve{M}$ (2)

$\vdash . (2) . *331·12 . \supset \vdash . \mathrm{Prop}$

***336·54.** $\vdash : \kappa \in FM \,\mathrm{conx} \,.\, \mathrm{D}``\kappa = \text{Ɑ}``\kappa \,.\, \supset \,.$

$$V_\kappa = \hat{M}\hat{N}\{M, N \in \kappa_\iota : (\exists T) \,.\, T \in \kappa_\partial \,.\, M = T \,|\, N\}$$

Dem.

$\vdash . *334·46 . \supset \vdash :. \mathrm{Hp} \,.\, M, N \in \kappa_\iota \,.\, \supset :$

$$(\exists T, x) \,.\, T \in \kappa_\partial \,.\, M`x = T`N`x \,.\, \equiv \,.\, (\exists T) \,.\, T \in \kappa_\partial \,.\, M = T \,|\, N \quad (1)$$

$\vdash . (1) . (*336·01) . \supset \vdash . \mathrm{Prop}$

***336·6.** $\vdash : \kappa \in FM \,\mathrm{conx} \,.\, \supset \,.\, V_\kappa \subseteq J$

Dem.

$\vdash . *331·23 . \supset \vdash :. \mathrm{Hp} \,.\, \supset : M V_\kappa N \,.\, \supset \,.\, (\exists x) \,.\, M`x \neq N`x :. \supset \vdash . \mathrm{Prop}$

Observe that, by the conventions explained in *14, "$M`x \neq N`x$" implies $\mathrm{E} \,!\, M`x \,.\, \mathrm{E} \,!\, N`x$. From "$(\exists x) \,.\, \sim (M`x = N`x)$" we cannot infer $M \neq N$.

***336·61.** $\vdash : \kappa \in FM \text{ conx trs} . \supset . V_\kappa \in \text{trs}$

Dem.

$\vdash . *330·612 . \supset \vdash : \text{Hp} . L, M, N \in \kappa_\iota . \supset . \exists ! \text{C}`L \cap \text{C}`M \cap \text{C}`N$ (1)

$\vdash . *336·371 . \supset \vdash : \text{Hp} . L V_\kappa M . M V_\kappa N . a \in \text{C}`L \cap \text{C}`M \cap \text{C}`N . \supset .$

$(L`a)(\dot{s}`\kappa_\partial)(M`a) . (M`a)(\dot{s}`\kappa_\partial)(N`a) .$

[*334·14] $\supset . (L`a)(\dot{s}`\kappa_\partial)(N`a) .$

[(*336·01)] $\supset . L V_\kappa N$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***336·62.** $\vdash : \kappa \in FM \text{ connex} . \supset . V_\kappa \in \text{connex}$

Dem.

$\vdash . *330·61 . \supset \vdash : \text{Hp} . L, M \in \kappa_\iota . \supset . \exists ! \text{C}`L \cap \text{C}`M$ (1)

$\vdash . *334·24 . \supset \vdash :. \text{Hp} . L, M \in \kappa_\iota . a \in \text{C}`L \cap \text{C}`M . \supset :$

$L`a = M`a . \vee . (L`a)(\dot{s}`\kappa_\partial)(M`a) . \vee . (M`a)(\dot{s}`\kappa_\partial)(L`a) :$

[*331·42.(*336·01)] $\supset : L = M . \vee . L V_\kappa M . \vee . M V_\kappa L$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***336·63.** $\vdash : \kappa \in FM \text{ sr} . \supset . V_\kappa \in \text{Ser}$ [*336·6·61·62]

***336·64.** $\vdash : \kappa \in FM \text{ sr} . \supset . U_\kappa \in \text{Ser}$ [*336·63]

$*337$. MULTIPLES AND SUB-MULTIPLES OF VECTORS.

Summary of $*337$.

In this number, we are concerned with the axiom of Archimedes and the axiom of divisibility. If κ is a family of vectors, κ obeys the axiom of Archimedes if, given any two points x, a in the field of κ, and any vector R which is a member of κ, there is some power R^ν of R such that $R^{\nu}{}^{\text{\textquoteleft}}a$ is later than x. That is, κ obeys the axiom of Archimedes if, starting from any given point in the field, a sufficient finite number of repetitions of any given vector will take us beyond any other assigned point. A sufficient hypothesis for this is that κ should be serial and $\text{Cnv}{}^{\text{\textquoteleft}}\dot{s}{}^{\text{\textquoteleft}}\kappa_\partial$ should be semi-Dedekindian (cf. $*214$), *i.e.* we have

$$*337{\cdot}13. \quad \vdash :. \kappa \in FM\,\text{sr} . \breve{P} = \dot{s}{}^{\text{\textquoteleft}}\kappa_\partial . P \in \text{semi Ded} . R \in \kappa_\partial . a \in C{}^{\text{\textquoteleft}}P . \supset :$$
$$x \in C{}^{\text{\textquoteleft}}P . \supset . (\exists \nu) . \nu \in \text{NC ind} - \iota{}^{\text{\textquoteleft}}0 . xP(R^{\nu}{}^{\text{\textquoteleft}}a)$$

The hypothesis $\breve{P} = \dot{s}{}^{\text{\textquoteleft}}\kappa_\partial$, which appears in the above proposition, is often notationally convenient. It will be observed that $\dot{s}{}^{\text{\textquoteleft}}\kappa_\partial$ gives us the series in the opposite order to that in which it is usually wanted; hence the introduction of the above relation P tends to avoid confusions.

A family κ is said to obey the axiom of divisibility when, given any member R of κ, and any inductive cardinal ν other than 0, there is a member L of κ such that $L^\nu = R$. When this axiom holds, every vector can be divided into any assigned finite number of equal parts. We shall in the next Section ($*351$) define a family for which this holds as a "sub-multipliable family," denoted by "FM subm." For the present we are concerned to find a hypothesis as to $\dot{s}{}^{\text{\textquoteleft}}\kappa_\partial$ from which this property can be deduced. The hypothesis in question is that $\text{Cnv}{}^{\text{\textquoteleft}}\dot{s}{}^{\text{\textquoteleft}}\kappa_\partial$ is serial, compact, and semi-Dedekindian; *i.e.* we have

$$*337{\cdot}27. \quad \vdash :. \kappa \in FM\,\text{sr} . \text{Cnv}{}^{\text{\textquoteleft}}\dot{s}{}^{\text{\textquoteleft}}\kappa_\partial \in \text{comp} \cap \text{semi Ded} . \supset :$$
$$S \in \kappa . \nu \in \text{NC ind} - \iota{}^{\text{\textquoteleft}}0 . \supset . (\exists L) . L \in \kappa . S = L^\nu$$

The proof proceeds by taking two points a, x in the field of κ, of which a is earlier than x, and considering the class

$$\lambda = \kappa_\partial \cap \breve{R}\{(R^{\nu}{}^{\text{\textquoteleft}}a)\,Px\},$$

i.e. the class of vectors such that ν repetitions of them, starting from a, do not take us as far as x. It is easy to show that, when P is compact, this class has no maximum (*337·23), and therefore, when P is also semi-Dedekindian, has a limit, whose νth power is the vector which takes us from a to x (*337·26). Hence our result follows.

***337·1.** $\vdash : \kappa \in FM \,.\, \breve{P} = \dot{s}'\kappa_\partial \,.\, R \in \kappa_\partial \,.\, a \in C'P \,.\supset .\, \overrightarrow{R_*}'a \subset P``\overrightarrow{R_*}'a$

Dem.

$\vdash .\, {*}90{\cdot}16 \,.\, {*}41{\cdot}141 \,.\supset \vdash : \mathrm{Hp} \,.\, xR_* a \,.\, y = R'x \,.\supset .\, y \in \overrightarrow{R_*}'a \,.\, xPy \,.$

$[{*}37{\cdot}1] \qquad \supset .\, x \in P``\overrightarrow{R_*}'a : \supset \vdash .\, \mathrm{Prop}$

***337·11.** $\vdash : \kappa \in FM \text{ connex asym} \,.\, \breve{P} = \dot{s}'\kappa_\partial \,.\, R \in \kappa_\partial \,.\, a \in C'P \,.\supset .\, \overrightarrow{\mathrm{seq}_P}'\overrightarrow{R_*}'a = \Lambda$

Dem.

$\vdash .\, {*}206{\cdot}15 \,. \qquad \supset \vdash : \mathrm{Hp} \,.\supset .\, \overrightarrow{\mathrm{seq}_P}'\overrightarrow{R_*}'a = p'\overleftarrow{P}``\overrightarrow{R_*}'a - \breve{P}``p'\overleftarrow{P}``\overrightarrow{R_*}'a \quad (1)$

$\vdash .\, {*}330{\cdot}542 \,.\, {*}40{\cdot}61 \,.\supset \vdash : \mathrm{Hp} \,.\, x \in p'\overleftarrow{P}``\overrightarrow{R_*}'a \,.\supset .\, x \in \mathrm{D}'R \,.$

$[\mathrm{Hp}] \qquad \supset .\, (\exists c) \,.\, x = R'c \,.\, cPx \quad (2)$

$\vdash .\, {*}90{\cdot}172 \,. \qquad \supset \vdash : c \in \overrightarrow{R_*}'a \,.\supset .\, R'c \in \overrightarrow{R_*}'a \quad (3)$

$\vdash .\, (3) \,.\, \mathrm{Transp} \,.\, {*}200{\cdot}5 \,.\, {*}334{\cdot}5 \,.\supset \vdash : \mathrm{Hp}\,(2) \,.\, x = R'c \,.\supset .\, c \sim \in \overrightarrow{R_*}'a \quad (4)$

$\vdash .\, {*}37{\cdot}1 \,. \qquad \supset \vdash : c \in P``\overrightarrow{R_*}'a \,.\supset .\, (\exists b) \,.\, b \in \overrightarrow{R_*}'a \,.\, cPb \quad (5)$

$\vdash .\, (5) \,.\, {*}208{\cdot}2 \,.\supset \vdash : \mathrm{Hp} \,.\, c \in P``\overrightarrow{R_*}'a \,.\, x = R'c \,.\supset .\, (\exists b) \,.\, b \in \overrightarrow{R_*}'a \,.\, xP(R'b) \,.$

$[{*}90{\cdot}172] \qquad \supset .\, x \in P``\overrightarrow{R_*}'a \quad (6)$

$\vdash .\, (6) \,.\, \mathrm{Transp} \,.\, {*}200{\cdot}53 \,. \qquad \supset \vdash : \mathrm{Hp}\,(2) \,.\, x = R'c \,.\supset .\, c \sim \in P``\overrightarrow{R_*}'a \quad (7)$

$\vdash .\, (4) \,.\, (7) \,.\, {*}202{\cdot}502 \,.\, {*}334{\cdot}24 \,.\supset \vdash : \mathrm{Hp}\,(2) \,.\, x = R'c \,.\supset .\, c \in p'\overleftarrow{P}``\overrightarrow{R_*}'a \quad (8)$

$\vdash .\, (2) \,.\, (8) \,. \qquad \supset \vdash : \mathrm{Hp}\,(2) \,.\supset .\, x \in \breve{P}``p'\overleftarrow{P}``\overrightarrow{R_*}'a \quad (9)$

$\vdash .\, (1) \,.\, (9) \,.\supset \vdash .\, \mathrm{Prop}$

***337·12.** $\vdash : \kappa \in FM \text{ sr} \,.\, \breve{P} = \dot{s}'\kappa_\partial \,.\, P \in \text{semi Ded} \,.\, R \in \kappa_\partial \,.\, a \in C'P \,.\supset .\, P``\overrightarrow{R_*}'a = C'P$

Dem.

$\vdash .\, {*}337{\cdot}1 \,. \qquad \supset \vdash : \mathrm{Hp} \,.\supset .\, \sim \exists \,!\, \overrightarrow{\max_P}'\overrightarrow{R_*}'a \,.$

$[{*}205{\cdot}7] \qquad \supset .\, \sim \exists \,!\, \overrightarrow{\max_P}'P``\overrightarrow{R_*}'a \quad (1)$

$\vdash .\, (1) \,.\, {*}206{\cdot}33 \,.\, {*}337{\cdot}11 \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, \sim \exists \,!\, \mathrm{seq}_P'P``\overrightarrow{R_*}'a \quad (2)$

$\vdash .\, (1) \,.\, (2) \,.\, {*}214{\cdot}7 \,.\supset \vdash .\, \mathrm{Prop}$

***337·13.** $\vdash :.\, \kappa \in FM \text{ sr} \,.\, \breve{P} = \dot{s}'\kappa_\partial \,.\, P \in \text{semi Ded} \,.\, R \in \kappa_\partial \,.\, a \in C'P \,.\supset :$
$x \in C'P \,.\supset .\, (\exists \nu) \,.\, \nu \in \mathrm{NC\ ind} - \iota'0 \,.\, xP(R^\nu{}'a) \quad [{*}337{\cdot}12 \,.\, {*}301{\cdot}26]$

***337·14.** $\vdash : \kappa \in FM \,\mathrm{sr} \,.\, \breve{P} = \dot{s}`\kappa_\partial \,.\, P \in \text{semi Ded} \,.\, \supset .\, \breve{U}_\kappa \in \text{semi Ded}$ [*336·462 . *214·74·75]

***337·2.** $\vdash : \kappa \in FM \,\mathrm{conx} \,.\, LU_\kappa R \,.\, R \neq I \upharpoonright s`\text{C}``\kappa \,.\, \supset .\, LU_\kappa(\breve{R} \mid L)$

Dem.

$\vdash . *336\cdot41 . \supset \vdash : \mathrm{Hp} . \supset . (\exists T) . L, R \in \kappa . T \in \kappa_\partial . L = T \mid R .$

[*330·31] $\supset . (\exists T) . T \in \kappa_\partial . \breve{R} \mid L = T . L = T \mid R .$

[*13·195] $\supset . \breve{R} \mid L \in \kappa_\partial . L = (\breve{R} \mid L) \mid R .$

[*330·5.*336·41] $\supset . LU_\kappa(\breve{R} \mid L) : \supset \vdash . \mathrm{Prop}$

***337·21.** $\vdash : \kappa \in FM \,\mathrm{conx} \cap FM \,\mathrm{trs} \,.\, R \in \kappa_\partial \,.\, \nu \in \mathrm{NC\,ind} - \iota`0 - \iota`1 \,.\, \supset .\, R^\nu U_\kappa R$

Dem.

$\vdash . *334\cdot162 . *301\cdot23 . \supset \vdash : \mathrm{Hp} . \supset . R^\nu = R^{\nu \dot{-} 1} \mid R \quad (1)$

$\vdash . *334\cdot131 . \quad \supset \vdash : \mathrm{Hp} . \supset . R, R^\nu, R^{\nu \dot{-} 1} \in \kappa_\partial \quad (2)$

$\vdash . (1) . (2) . *336\cdot41 . \supset \vdash . \mathrm{Prop}$

***337·22.** $\vdash : \kappa \in FM \,\mathrm{sr} \,.\, \breve{P} = \dot{s}`\kappa_\partial \,.\, P \in \mathrm{comp} \,.\, aPx \,.\, \nu \in \mathrm{NC\,ind} - \iota`0 \,.\, \supset .$
$(\exists R) . R \in \kappa \,\, . (R^\nu`a) Px$

Dem.

$\vdash . *270\cdot11 . \supset \vdash : \mathrm{Hp} . \supset . (\exists y) . aPy . yPx .$

[*41·11] $\supset . (\exists R, y) . R \in \kappa_\partial . y = R`a . (R`a) Px \quad (1)$

$\vdash . (1) \dfrac{R^\nu`a}{a} . \supset \vdash : \mathrm{Hp} . R \in \kappa_\partial . (R^\nu`a) Px . \supset . (\exists S) . S \in \kappa_\partial . (S`R^\nu`a) Px \quad (2)$

$\vdash . *336\cdot64 . \supset \vdash :. \mathrm{Hp}(2) . S \in \kappa_\partial . (S`R^\nu`a) Px . \supset : R = S . \vee . RU_\kappa S . \vee . SU_\kappa R :$

[*336·511·4] $\supset : R = S . \vee . (R^{\nu \dot{+} 1}`a) P (S`R^\nu`a) . \vee . (S^{\nu \dot{+} 1}`a) P (S`R^\nu`a) \quad (3)$

$\vdash . (2) . (3) . *334\cdot3 . \supset \vdash : \mathrm{Hp}(2) . \supset . (\exists S) . S \in \kappa_\partial . (S^{\nu \dot{+} 1}`a) Px \quad (4)$

$\vdash . (1) . (4) . \mathrm{Induct} . \supset \vdash . \mathrm{Prop}$

***337·23.** $\vdash : \mathrm{Hp}\, *337\cdot22 \,.\, \lambda = \kappa_\partial \cap \hat{R}\{(R^\nu`a) Px\} \,.\, \supset .\, \lambda = \breve{U}_\kappa``\lambda$

Dem.

$\vdash . *336\cdot511 . \supset \vdash : \mathrm{Hp} . R \in \lambda . S\breve{U}_\kappa R . \supset . (S^\nu`a) P (R^\nu`a) . (R^\nu`a) Px .$

[*334·3.Hp] $\supset . S \in \lambda \quad (1)$

$\vdash . *337\cdot22 . \supset \vdash : \mathrm{Hp} . R \in \lambda . \supset . (\exists S) . S \in \kappa_\partial . (S^\nu`R^\nu`a) Px .$

[*330·57·5.*334·13] $\supset . (\exists S) . R \mid S \in \kappa_\partial . \{(R \mid S)^\nu`a\} Px .$

[*336·41] $\supset . (\exists S) . R \mid S \in \kappa_\partial . \{(R \mid S)^\nu`a\} Px . R\breve{U}_\kappa(R \mid S) .$

[*37·1] $\supset . R \in \breve{U}_\kappa``\lambda \quad (2)$

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***337·24.** $\vdash : \text{Hp} *337\cdot23 . L = \text{tl}(U_\kappa)`\lambda . \supset . \sim \{(L^\nu`a) Px\}$

Dem.

$\vdash . *206\cdot2 . \supset \vdash : \text{Hp} . \supset . L \sim\epsilon \lambda .$

$[\text{Hp}] \qquad \supset . \sim \{(L^\nu`a) Px\} : \supset \vdash . \text{Prop}$

***337·241.** $\vdash : \text{Hp} *337\cdot24 . \supset . \sim \{xP(L^\nu`a)\}$

Dem.

$\vdash . *337\cdot2\cdot23 . \supset \vdash : \text{Hp} . R \epsilon \lambda . \supset . \breve{R} | L \epsilon \lambda .$

$[*332\cdot53\cdot241 . *334\cdot131] \qquad \supset . \breve{R} | L \epsilon \lambda . (\breve{R} | L)^\nu = \breve{R}^\nu | L^\nu .$

$[\text{Hp}] \qquad \supset . (\breve{R}^\nu`L^\nu`a) Px .$

$[*71\cdot362 . *41\cdot11] \qquad \supset . (L^\nu`a) P (R^\nu`x) \qquad (1)$

$\vdash . *337\cdot23 . \supset \vdash : \text{Hp} . R \epsilon \kappa_\partial - \lambda . \supset . \sim \{LU_\kappa R\} .$

$[*336\cdot511] \qquad \supset . \sim \{(R^\nu`a) P (L^\nu`a)\} .$

$[*330\cdot5 . \text{Hp} . *334\cdot14] \qquad \supset . \sim \{(R^\nu`x) P (L^\nu`a)\} \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash : \text{Hp} . \supset . \sim (\exists R) . R \epsilon \kappa_\partial . (R^\nu`x) P (L^\nu`a) .$

$[*337\cdot22 . \text{Transp}] \qquad \supset . \sim \{xP(L^\nu`a)\} : \supset \vdash . \text{Prop}$

***337·25.** $\vdash : \text{Hp} *337\cdot24 . \supset . L^\nu = \kappa \upharpoonright \breve{A}_a`x$

Dem.

$\vdash . *337\cdot24\cdot241 . \supset \vdash : \text{Hp} . \supset . L^\nu`a = x : \supset \vdash . \text{Prop}$

***337·26.** $\vdash : \text{Hp} *337\cdot23 . P \epsilon \text{semi Ded} . \supset . \{\text{tl}(U_\kappa)`\lambda\}^\nu = \kappa \upharpoonright \breve{A}_a`x$

Dem.

$\vdash . *337\cdot21 . \qquad \supset \vdash :. \text{Hp} . \supset : R \epsilon \lambda . \supset_R . (R`a) Px :$

$[*336\cdot4] \qquad \supset : \kappa \upharpoonright \breve{A}_a`x \epsilon p`\overrightarrow{U_\kappa}``\lambda \qquad (1)$

$\vdash . (1) . *337\cdot23\cdot14 . \supset \vdash : \text{Hp} . \supset . \text{E} ! \, \text{tl}(U_\kappa)`\lambda \qquad (2)$

$\vdash . (2) . *337\cdot25 . \supset \vdash . \text{Prop}$

***337·27.** $\vdash :. \kappa \epsilon FM \text{sr} . \text{Cnv}`\dot{s}`\kappa_\partial \epsilon \text{comp} \cap \text{semi Ded} . \supset :$

$S \epsilon \kappa . \nu \epsilon \text{NC ind} - \iota`0 . \supset . (\exists L) . L \epsilon \kappa . S = L^\nu \quad [*337\cdot26]$

SECTION C.

MEASUREMENT.

Summary of Section C.

In this Section, the "pure" theory of ratios and real numbers developed in Section A is applied to vector-families. A vector-family, if it has suitable properties, may be regarded as a kind of magnitude. In order to derive from the "pure" theory of ratio a theory of measurement having the properties which we should expect, it is necessary to confine ourselves to some one vector-family; that is, instead of considering the general relation X, where X is a ratio, we consider the relation $X \upharpoonright \kappa$, where κ is the vector-family in question; or sometimes we consider $X \upharpoonright \kappa_\iota$, or sometimes $X \upharpoonright (\kappa \cup \mathrm{Cnv}``\kappa)$.

Concerning ratios with their fields thus limited, which are what we may call "applied" ratios, we have to prove various propositions.

(1) No two members of a family must have two different ratios. This is proved, for an open and connected family, in $*350{\cdot}44$.

(2) All ratios except 0_q and ∞_q must be one-one relations when limited to a single family. This is proved, for an open and connected family, in $*350{\cdot}5$; with the same hypothesis, 0_q is one-many ($*350{\cdot}51$).

(3) The relative product of two applied ratios ought to be equal to the arithmetical product of the corresponding pure ratios with its field limited, *i.e.* if X, Y are ratios, we ought to have

$$X \upharpoonright \kappa \mid Y \upharpoonright \kappa = (X \times_s Y) \upharpoonright \kappa$$

or

$$X \upharpoonright \kappa_\iota \mid Y \upharpoonright \kappa_\iota = (X \times_s Y) \upharpoonright \kappa_\iota .$$

That is to say, two-thirds of half a pound of cheese ought to be $(2/3 \times_s 1/2)$ of a pound of cheese; and similarly in any other case. For any open connected family, we have ($*350{\cdot}6$)

$$X \upharpoonright \kappa_\iota \mid Y \upharpoonright \kappa_\iota \subset (X \times_s Y) \upharpoonright \kappa_\iota ,$$

but in order to obtain an equation instead of an inclusion, it is necessary ($*351{\cdot}31$) that κ should be "submultipliable," *i.e.* that if R is any member of κ, and ν any inductive cardinal other than zero, there should be a member of κ whose νth power is R. The class of such families is denoted by "FM subm," and considered in $*351$.

(4) If X, Y are ratios and T is a member of the family κ, we ought to have

$$(X \underset{\cdot}{[} \kappa{}^{\backprime}T) \mid (Y \underset{\cdot}{[} \kappa{}^{\backprime}T) = (X +_s Y) \underset{\cdot}{[} \kappa{}^{\backprime}T,$$

that is, two-thirds of a pound of cheese together with half a pound of cheese ought to be $(2/3 +_s 1/2)$ of a pound of cheese, and similarly in any other instance. This property is shown, in $*351{\cdot}43$, to hold for any open connected submultipliable family in which all powers of members are members. In any open connected family, if $R, S, T \in \kappa$, we have

$$RXT \,.\, SYT \,.\, \supset . \,(R \mid S)(X +_s Y)\, T \quad (*350{\cdot}62).$$

The remainder of the hypothesis of $*351{\cdot}43$ is required in order to prove (a) that $X \underset{\cdot}{[} \kappa{}^{\backprime}T$, $Y \underset{\cdot}{[} \kappa{}^{\backprime}T$ and $(X +_s Y) \underset{\cdot}{[} \kappa{}^{\backprime}T$ exist, (b) that $(X \underset{\cdot}{[} \kappa{}^{\backprime}T) \mid (Y \underset{\cdot}{[} \kappa{}^{\backprime}T)$, which is the $R \mid S$ of $*350{\cdot}62$, is a member of κ. As applied to κ_ι, we have to take the representative (cf. $*332$) of the relative product; if $L \in \kappa_\iota$, we have ($*351{\cdot}42$)

$$\text{rep}_\kappa{}^{\backprime}\{(X \underset{\cdot}{[} \kappa_\iota{}^{\backprime}L) \mid (Y \underset{\cdot}{[} \kappa_\iota{}^{\backprime}L)\} = (X +_s Y) \underset{\cdot}{[} \kappa_\iota{}^{\backprime}L,$$

provided κ is open and connected and submultipliable.

The fact that the above propositions can be proved for suitable vector-families constitutes the reason for studying such families, as we did in Section B. The proof of the above propositions, together with other elementary properties of applied ratios, occupies the first two numbers of this Section.

We proceed next ($*352$) to consider all the rational multiples of a given vector in a given family, *i.e.* all the members of a given family κ which have, to a given vector T, a ratio which is a member of $C{}^{\backprime}H'$, or, alternatively, all the members of κ_ι which have to T a ratio which is a member of $C{}^{\backprime}H_g$. It will be observed that, in virtue of $*307$, if R and S have a ratio X which is a member of $C{}^{\backprime}H'$, R and $\breve{S}$ have the corresponding negative ratio $X \mid \text{Cnv}$. The members of κ which have to T a ratio which is a member of $C{}^{\backprime}H'$ are those vectors R for which we have

$$(\exists X) \,.\, X \in C{}^{\backprime}H' \,.\, RXT,$$

i.e. using the notation of $*336$, those for which we have

$$(\exists X) \,.\, X \in C{}^{\backprime}H' \,.\, RA_TX.$$

Thus they constitute the class

$$\kappa \cap A_T{}^{\backprime\backprime}C{}^{\backprime}H'.$$

Assuming that $T \in \kappa$, the vector which has the ratio X to T is $\kappa \rceil A_T{}^{\backprime}X$. This is the vector whose measure is X when T is the unit. Thus $\kappa \rceil A_T \upharpoonright C{}^{\backprime}H'$ is the correlator of a vector with its measure. It is easy to prove ($*352{\cdot}12$) that $\kappa \rceil A_T \upharpoonright C{}^{\backprime}H'$ is one-one.

We can arrange the vectors which are rational multiples of T in a series by correlation with their measures, putting vectors with smaller measures before those with larger measures. The ordering relation is T_κ, where

$$T_\kappa = \kappa \upharpoonright A_T ; H' \quad \text{Df.}$$

Similarly the members of κ_ι which are positive or negative rational multiples of T may be ordered by the relation $T_{\kappa\iota}$, where

$$T_{\kappa\iota} = \kappa_\iota \upharpoonright A_T ; H_g \quad \text{Df.}$$

We prove that change of units makes no difference to T_κ, *i.e.* if S is any member of κ which is a rational multiple of T, then $S_\kappa = T_\kappa$ (∗352·45). The corresponding proposition holds for $T_{\kappa\iota}$ if S has a positive ratio to T, but if S has a negative ratio, $S_{\kappa\iota} = \breve{T}_{\kappa\iota}$ (∗352·56·57).

If κ is a serial family, T_κ is the converse of U_κ (cf. ∗336) with its field limited to rational multiples of T (∗352·72). This proposition connects the generalized form of greater and less represented by U_κ with the form of greater and less derived from greater and less among the measures of vectors, since it shows that, in a serial family, the vectors which have greater measures come later in the series $\breve{U}_\kappa$, and those with smaller measures come earlier.

We next proceed (∗353) to consider "rational" families. These are families in which every member is a rational multiple of some one unit T, *i.e.* in which

$$(\exists T) . T \in \kappa_\partial . \kappa \subset A_T{}^{\prime\prime}C^{\prime}H'.$$

It is obvious that, given any family, the rational multiples of one of its members constitute a rational sub-family. In a rational family, rationals are sufficient for measurement, and irrationals are not required. If the family has connexity, it will be serial; in fact, if T is one of its vectors and a is a member of its field, we have (cf. ∗353·32·33)

$$U_\kappa = \kappa \upharpoonright A_T ; \breve{H}' . \dot{s}^{\prime}\kappa_\partial = A_a ; \kappa \upharpoonright A_T ; \breve{H}'.$$

Thus both U_κ and $\dot{s}^{\prime}\kappa_\partial$ are ordinally similar to $\breve{H}' \upharpoonright \breve{A}_T{}^{\prime\prime}\kappa$. If κ is submultipliable, U_κ is ordinally similar to $\breve{H}'$ (∗353·44).

We proceed next (∗354) to consider "rational nets," which are important in connection with the introduction of coordinates in geometry. A rational net is obtained from a given family, roughly speaking, by selecting those vectors which are rational multiples of a given vector, and then limiting their fields to the points which can be reached by means of them from a given point. In order to make this more precise, we proceed as follows: Let us define as the "connection" of a with respect to κ the class $A_a{}^{\prime\prime}\kappa_\iota$, *i.e.* all the points which can be reached from a by a member of κ_ι. We will now define as the "a-connected derivative of κ" the class of relations obtained by limiting

the field of every member of κ to the connection of a with respect to κ. This class of relations we denote by $\text{cx}_a\text{‘}\kappa$, putting

$$\text{cx}_a\text{‘}\kappa = \complement\, (A_a\text{‘‘}\kappa_\iota)\text{‘‘}\kappa \quad \text{Df.}$$

Instead of κ, we take, in order to obtain a rational net, all the rational multiples (in κ) of a given member T of κ, *i.e.* $C\text{‘}T_\kappa$. Then $\text{cx}_a\text{‘}C\text{‘}T_\kappa$ is a rational net, namely the rational net associated with the origin a and the unit vector T.

In proving propositions concerning the rational net $\text{cx}_a\text{‘}C\text{‘}T_\kappa$, we often require the hypothesis that κ is a group. In order to avoid having to make this hypothesis concerning our original family, we construct a closely allied family, which is always a group when κ is connected. This family, which we call κ_g, is obtained from κ by including the converses of those members of κ, if any, whose domains are equal to their converse domains, *i.e.* we put

$$\kappa_g = \kappa \cup \text{Cnv}\text{‘‘}(\kappa \cap \overleftarrow{\text{D}}\text{‘}s\text{‘}\text{Cl}\text{‘‘}\kappa) \quad \text{Df.}$$

Then if κ is a connected family, κ_g is a connected family which is a group (*354·14·16), and $(\kappa_g)_\iota = \kappa_\iota$ (*354·15). Then putting $\lambda = \kappa_g$, we take $\text{cx}_a\text{‘}C\text{‘}T_\lambda$ rather than $\text{cx}_a\text{‘}C\text{‘}T_\kappa$ as the rational net to be considered. If κ is an open and connected family, this rational net is a family which is open, connected, rational, transitive and asymmetrical (*354·41).

We proceed next (*356) to the application of real numbers to vector-families. For the application of real numbers, it is essential that our family should be serial. Given a serial family in which a given vector S is the limit (in the series U_κ) of a set of vectors which are rational multiples of another vector R, it is natural to take as the measure of S, with the unit R, the limit of the measures of the vectors whose limit is S. It is convenient to take our real numbers in the relational form given in *314, *i.e.* if ξ is a segment of H, we take $\dot{s}\text{‘}\xi$ as the corresponding real number. Thus positive real numbers are the class $\dot{s}\text{‘‘}C\text{‘}\Theta$, while positive and negative real numbers together with zero are the class $\dot{s}\text{‘‘}C\text{‘}\Theta_g$. If $\xi \in C\text{‘}\Theta$, a vector which has to R a ratio which is a member of ξ has a measure which is less than $\dot{s}\text{‘}\xi$. The class of all such vectors is $\overrightarrow{\dot{s}\text{‘}\xi}\text{‘}R$, *i.e.* if $X = \dot{s}\text{‘}\xi$, it is $\overrightarrow{X}\text{‘}R$. The limit of such vectors in the series U_κ, if it exists, will naturally be taken as the vector whose measure is X. Remembering that U_κ proceeds from greater to smaller vectors, we see that the first vector which is greater than every member of $\overrightarrow{X}\text{‘}R$ will be the *lower* limit of $\overrightarrow{X}\text{‘}R$ with respect to U_κ. Hence, if we write $X_\kappa\text{‘}R$ for the vector whose measure with the unit R is X, we have

$$X_\kappa\text{‘}R = \text{prec}\,(U_\kappa)\text{‘}\overrightarrow{X}\text{‘}R.$$

Hence we may take as our definition of X_κ

$$X_\kappa = \text{prec}\,(U_\kappa) \,|\, \overrightarrow{X} \restriction \kappa \quad \text{Df.}$$

Then X_κ is an "applied" real number.

The properties to be proved concerning applied real numbers almost all require that the family to which they are applied should be serial and submultipliable, and most of them also require that $\mathrm{Cnv}`\dot{s}`\kappa_\partial$ should be semi-Dedekindian. Assuming this, we can prove that, if $X, Y \epsilon \dot{s}``C`\Theta$, $X_\kappa \upharpoonright \kappa$ is one-one, and, with various hypotheses,

$$(X \upharpoonright \kappa) | (Y \upharpoonright \kappa) = (X \times_r Y) \upharpoonright \kappa \quad (*356{\cdot}31),$$

$$X_\kappa | Y_\kappa = (X \times_r Y)_\kappa \quad (*356{\cdot}33),$$

$$(X_\kappa`R) | (Y_\kappa`R) = (X +_r Y)_\kappa`R \quad (*356{\cdot}54).$$

These are the essential properties required of measurement, as in the analogous case of ratios.

We might proceed to consider "real" multiples of a given vector, and "real" nets. But these subjects have less importance than in the analogous case of rationals, and are therefore not discussed.

The Section ends (*359) with a number on existence-theorems for vector-families. The most important of these are derived from rationals and real numbers. The family whose members are of the form $(+_s X) \upharpoonright C`H'$, where $X \epsilon C`H'$, is initial, serial, and submultipliable (*359·21). The family whose members are of the form $(+_p \mu) \upharpoonright C`\Theta'$, where $\mu \epsilon C`\Theta'$, is initial, serial, and submultipliable, and has $\mathrm{Cnv}`\dot{s}`\kappa_\partial = \Theta'$, so that $\mathrm{Cnv}`\dot{s}`\kappa_\partial \epsilon \text{semi Ded}$ (*359·31). Finally we prove that the properties of families are unaffected by the application of correlators, whence it follows that, given any series P whose relation-number is $\dot{1} \dotplus \eta$, or is θ' where $\theta' \dotplus \dot{1} = \theta$, there is an initial serial submultipliable family κ such that $\mathrm{Cnv}`\dot{s}`\kappa_\partial = P$. Such a family may be used for the measurement of distances in P.

It is of some interest to observe that, given a suitable family κ, ratios with their field limited to κ_∂ form a family whose field is κ_∂. In this family, the zero vector is $(1/1) \upharpoonright \kappa_\partial$, and the family is connected if κ is a rational family. If we wish to obtain a serial family, we must limit ourselves to ratios not less than 1/1, *i.e.* to

$$\upharpoonright \kappa_\partial `` \overleftarrow{H_*}`(1/1).$$

This family is serial, and if we call it λ, we have (with a suitable hypothesis)

$$\dot{s}`\lambda_\partial = U_\kappa \upharpoonright \kappa_\partial.$$

It is necessary, however, if we are to obtain a family, that our original family should be submultipliable, since otherwise we do not necessarily have $\mathrm{G}`X \upharpoonright \kappa_\partial = \kappa_\partial$. For this reason, we cannot use the family of ratios without a frequent loss of generality in the resulting theorems.

The theory of measurement developed in this Section is only applicable to open families. The application of ratio to cyclic families is more complicated and is considered separately in Section D.

$*350$. RATIOS OF MEMBERS OF A FAMILY.

Summary of $*350$.

In this number we introduce no new definitions, but merely bring together the propositions of $*303$ on the pure theory of ratio, and the propositions of $*333$ on powers of vectors in open connected families, especially $*333{\cdot}47{\cdot}48$. We thus find that, if κ is an open connected family, and μ, ν are inductive cardinals which are not both zero,

$$M\{(\mu/\nu)\upharpoonright\kappa_\iota\}N \,.\equiv.\, M, N \in \kappa_\iota \,.\, \exists ! M^\nu \dot\cap N^\mu \,. \qquad (*350{\cdot}4)$$

$$\equiv.\, M, N \in \kappa_\iota \,.\, \text{rep}_\kappa{}^{\backprime}M^\nu = \text{rep}_\kappa{}^{\backprime}N^\mu \qquad (*350{\cdot}41),$$

while if R, T are members of κ,

$$R\,(\mu/\nu)\,T \,.\equiv.\, R^\nu = T^\mu \qquad (*350{\cdot}43).$$

We prove also, by means of $*333{\cdot}53$, that if L and M are members of κ_ι other than $I \upharpoonright s{}^{\backprime}\text{C}{}^{\backprime\backprime}\kappa$, they cannot have more than one ratio, *i.e.*

$*350{\cdot}44$. $\vdash : \kappa \in FM\,\text{ap}\,\text{conx} \,.\, X, Y \in C{}^{\backprime}H' \,.\, \exists ! X \upharpoonright \kappa_{\iota\partial} \dot\cap Y \upharpoonright \kappa_{\iota\partial} \,.\supset.\, X = Y$

We next prove that any ratio other than 0_q and ∞_q becomes one-one when its field is limited to κ_ι ($*350{\cdot}5$), while 0_q becomes one-many ($*350{\cdot}51$) and ∞_q becomes many-one ($*350{\cdot}511$), 0_q being in fact the ratio of the zero vector $I \upharpoonright s{}^{\backprime}\text{C}{}^{\backprime\backprime}\kappa$ to any member of κ_ι, and ∞_q being the converse of 0_q.

We consider next the multiplication and addition of ratios, but in this subject we cannot obtain some of the main theorems without the hypothesis that our family is submultipliable (introduced in $*351$). In the present number, we prove that, if κ is an open connected family, and μ, ν are inductive cardinals other than 0,

$$(\mu/1)\upharpoonright\kappa_\iota \,|\, (1/\nu)\upharpoonright\kappa_\iota \subset (\mu/\nu)\upharpoonright\kappa_\iota \qquad (*350{\cdot}53),$$

$$(1/\nu)\upharpoonright\kappa_\iota \,|\, (\mu/1)\upharpoonright\kappa_\iota = (\mu/\nu)\upharpoonright\kappa_\iota \qquad (*350{\cdot}54),$$

$$(\mu/1)\upharpoonright\kappa_\iota \,|\, (\nu/1)\upharpoonright\kappa_\iota = \{(\mu \times_c \nu)/1\}\upharpoonright\kappa_\iota \qquad (*350{\cdot}55),$$

and

$$(1/\mu)\upharpoonright\kappa_\iota \,|\, (1/\nu)\upharpoonright\kappa_\iota = \{1/(\mu \times_c \nu)\}\upharpoonright\kappa_\iota \qquad (*350{\cdot}56).$$

Hence we find that, if X, Y are ratios other than 0_q and ∞_q,

$$X\upharpoonright\kappa_\iota \quad Y\upharpoonright\kappa_\iota \subset (X \times_s Y)\upharpoonright\kappa_\iota \qquad (*350{\cdot}6),$$

while if R, S, T are members of κ,

$$RXT \,.\, SYT \,.\supset.\, (R \,|\, S)(X +_s Y)\,T \qquad (*350{\cdot}62),$$

and if L, M, N are members of κ_ι,

$$LXN \,.\, MYN \,.\supset.\, \{\text{rep}_\kappa{}^{\backprime}(L \,|\, M)\}(X +_s Y)\,N \qquad (*350{\cdot}63).$$

We then prove similar results for subtraction, and thus arrive at the following proposition concerning generalized addition of positive or negative ratios:

***350·66.** $\vdash : \kappa \epsilon FM \text{ ap conx} . L, M, N \epsilon \kappa_\iota . X, Y \epsilon C\text{‘}H_g . LXN . MYN . \supset .$
$\text{rep}_\kappa\text{‘}(L \mid M) = (X +_g Y) \upharpoonright \kappa_\iota\text{‘}N$

***350·1.** $\vdash : \kappa \epsilon FM \text{ ap} . \supset . \kappa_\iota \subset \text{Rel num id} . \kappa_{\iota\partial} \subset \text{Rel num}$

Dem.

$\vdash . *333\cdot101 . \quad \supset \vdash : \text{Hp} . L \epsilon \kappa_{\iota\partial} . \supset . L \epsilon 1 \to 1 . L_{\text{po}} \mathrel{G} J$ (1)
$\vdash . (1) . *300\cdot3 . \supset \vdash : \text{Hp} . \supset . \kappa_{\iota\partial} \subset \text{Rel num}$ (2)
$\vdash . *333\cdot1\cdot101 . \supset \vdash : \text{Hp} . L \epsilon \kappa_\iota - \kappa_{\iota\partial} . \supset . L \mathrel{G} I .$
$[*300\cdot325] \qquad \supset . L \epsilon \text{Rel num id}$ (3)
$\vdash . (2) . (3) . \supset \vdash . \text{Prop}$

***350·2.** $\vdash : \kappa \epsilon FM \text{ ap conx} . \exists ! \kappa_{\iota\partial} . \supset . \text{Infin ax}$

Dem.

$\vdash . *330\cdot624 . *333\cdot15 . \supset \vdash :. \text{Hp} . L \epsilon \kappa_{\iota\partial} . \supset : \dot\Lambda \sim\epsilon \text{ finid‘}L :$
$[*121\cdot11\cdot12] \qquad \supset : \nu \epsilon \text{NC induct} . \supset_\nu . (\exists x, y) . L(x \mapsto y) \epsilon \nu +_c 1 :$
$[*120\cdot3] \qquad \supset : \text{Infin ax} :. \supset \vdash . \text{Prop}$

***350·21.** $\vdash : \exists ! FM \text{ ap conx} - 1 . \supset . \text{Infin ax} \quad [*334\cdot18 . *350\cdot2]$

***350·31.** $\vdash :. \kappa \epsilon FM \text{ ap conx} . \mu, \nu \epsilon \text{NC ind} - \iota\text{‘}0 . M, N \epsilon \kappa_{\iota\partial} . \supset :$
$M(\mu/\nu) N . \equiv . \dot\exists ! M^\nu \dot\frown N^\mu$

Dem.

$\vdash . *303\cdot1 . (*302\cdot02\cdot03) . *113\cdot602 . \supset$
$\vdash :: \text{Hp} . \supset :. M(\mu/\nu) N . \equiv : (\exists \rho, \sigma, \tau) . \rho \text{ Prm } \sigma . \tau \epsilon \text{NC ind} - \iota\text{‘}0 .$
$\mu = \rho \times_c \tau . \nu = \sigma \times_c \tau . \dot\exists ! M^\sigma \dot\frown N^\rho . \rho \neq 0 . \sigma \neq 0 :$
$[*333\cdot48] \equiv : (\exists \rho, \sigma, \tau) . \rho \text{ Prm } \sigma . \tau \epsilon \text{NC ind} - \iota\text{‘}0 . \rho \neq 0 . \sigma \neq 0 .$
$\mu = \rho \times_c \tau . \nu = \sigma \times_c \tau : \dot\exists ! M^\nu \dot\frown N^\mu :$
$[*113\cdot602 . (*302\cdot02\cdot03)] \equiv : (\exists \rho, \sigma) . (\rho, \sigma) \text{ Prm } (\mu, \nu) : \dot\exists ! M^\nu \dot\frown N^\mu :$
$[*302\cdot36] \qquad \equiv : \dot\exists ! M^\nu \dot\frown N^\mu :: \supset \vdash . \text{Prop}$

***350·32.** $\vdash :. \text{Hp} *350\cdot31 . \supset : M(\mu/\nu) N . \equiv . \text{rep}_\kappa\text{‘}M^\nu = \text{rep}_\kappa\text{‘}N^\mu$
$[*350\cdot31 . *333\cdot47]$

***350·33.** $\vdash :. \kappa \epsilon FM \text{ ap conx} . \mu, \nu \epsilon \text{NC ind} - \iota\text{‘}0 . M = I \upharpoonright s\text{‘}\text{Œ}\text{‘‘}\kappa . N \epsilon \kappa_\iota . \supset :$
$M(\mu/\nu) N . \equiv . M = N . \equiv . \dot\exists ! M^\nu \dot\frown N^\mu$

Dem.

$\vdash . *301\cdot3 . *333\cdot2 . \supset \vdash :. \text{Hp} . \supset : \sigma \epsilon \text{NC ind} - \iota\text{‘}0 . \supset . M^\sigma = M$ (1)
$\vdash . (1) . *303\cdot1 . \supset$
$\vdash :. \text{Hp} . \supset : M(\mu/\nu) N . \equiv . (\exists \rho, \sigma) . (\rho, \sigma) \text{ Prm } (\mu, \nu) . \dot\exists ! M \dot\frown N^\rho .$
$[*333\cdot101] \qquad \equiv . (\exists \rho, \sigma) . (\rho, \sigma) \text{ Prm } (\mu, \nu) . M = N .$
$[*302\cdot36] \qquad \equiv . M = N .$ (2)
$[(1) . *331\cdot42] \qquad \equiv . \dot\exists ! M^\nu \dot\frown N^\mu$ (3)
$\vdash . (2) . (3) . \supset \vdash . \text{Prop}$

***350·331.** $\vdash :. \kappa \epsilon FM \text{ ap conx} . \mu, \nu \epsilon \text{NC ind} - \iota\text{‘}0 . M \epsilon \kappa_\iota . N = I \upharpoonright s\text{‘}\text{Ɑ}\text{“}\kappa . \supset :$

$M (\mu/\nu) N . \equiv . M = N . \equiv . \dot{\exists} ! M^\nu \dot{\wedge} N^\mu$ [*350·33 . *303·13]

***350·34.** $\vdash :. \kappa \epsilon FM \text{ ap conx} . \nu \epsilon \text{NC ind} - \iota\text{‘}0 . M, N \epsilon \kappa_\iota . \supset :$

$M (0/\nu) N . \equiv . M = I \upharpoonright s\text{‘}\text{Ɑ}\text{“}\kappa$

Dem.

$\vdash . *303\cdot151 . \supset \vdash :. \text{Hp} . \supset : M (0/\nu) N . \equiv . M \subseteq I . \exists ! C\text{‘}M \cap C\text{‘}N .$

[*330·43·61] $\equiv . M = I \upharpoonright s\text{‘}\text{Ɑ}\text{“}\kappa :. \supset \vdash . \text{Prop}$

***350·35.** $\vdash :. \kappa \epsilon FM \text{ ap conx} . \nu \epsilon \text{NC ind} - \iota\text{‘}0 . M, N \epsilon \kappa_\iota . \supset :$

$M (0/\nu) N . \equiv . \dot{\exists} ! M^\nu \dot{\wedge} N^0$

Dem.

$\vdash . *301\cdot2 . \supset \vdash :. \text{Hp} . \supset : \dot{\exists} ! M^\nu \dot{\wedge} N^0 . \equiv . \dot{\exists} ! M^\nu \dot{\wedge} I \upharpoonright s\text{‘}\text{Ɑ}\text{“}\kappa .$

[*333·101.*331·12] $\equiv . M = I \upharpoonright s\text{‘}\text{Ɑ}\text{“}\kappa$ (1)

$\vdash . (1) . *350\cdot34 . \supset \vdash . \text{Prop}$

***350·351.** $\vdash :. \kappa \epsilon FM \text{ ap conx} . \mu \epsilon \text{NC ind} - \iota\text{‘}0 . \supset :$

$M (\mu/0) N . \equiv . N = I \upharpoonright s\text{‘}\text{Ɑ}\text{“}\kappa$ [*350·35 . *303·13]

***350·4.** $\vdash :. \kappa \epsilon FM \text{ ap conx} . \mu, \nu \epsilon \text{NC ind} . \sim(\mu = \nu = 0) . \supset :$

$M \{(\mu/\nu) \upharpoonright \kappa_\iota\} N . \equiv . M, N \epsilon \kappa_\iota . \dot{\exists} ! M^\nu \dot{\wedge} N^\mu$ [*350·31·33·331·35·351]

***350·41.** $\vdash :. \text{Hp} *350\cdot4 . \supset : M \{(\mu/\nu) \upharpoonright \kappa_\iota\} N . \equiv . M, N \epsilon \kappa_\iota . \text{rep}_\kappa\text{‘}M^\nu = \text{rep}_\kappa\text{‘}N^\mu$

Dem.

$\vdash . *332\cdot243 . *301\cdot3 . \supset \vdash : \text{Hp} . M = I \upharpoonright s\text{‘}\text{Ɑ}\text{“}\kappa . \supset . \text{rep}_\kappa\text{‘}M^\nu = M$ (1)

$\vdash . (1) . *350\cdot33\cdot331\cdot32 . \supset \vdash . \text{Prop}$

***350·42.** $\vdash :. \text{Hp} *350\cdot4 . Q, R, S, T \epsilon \kappa . \supset :$

$(\breve{Q} \mid R) (\mu/\nu) (\breve{S} \mid T) . \equiv . \breve{Q}^\nu \mid R^\nu = \breve{S}^\mu \mid T^\mu$ [*350·41 . *332·53]

***350·43.** $\vdash :. \text{Hp} *350\cdot4 . R, T \epsilon \kappa . \supset : R (\mu/\nu) T . \equiv . R^\nu = T^\mu$

$\left[*350\cdot42 \dfrac{I \upharpoonright s\text{‘}\text{Ɑ}\text{“}\kappa, \; I \upharpoonright s\text{‘}\text{Ɑ}\text{“}\kappa}{Q, \qquad S}\right]$

***350·44.** $\vdash : \kappa \epsilon FM \text{ ap conx} . X, Y \epsilon C\text{‘}H' . \dot{\exists} ! X \upharpoonright \kappa_{\iota\partial} \dot{\wedge} Y \upharpoonright \kappa_{\iota\partial} . \supset . X = Y$

Dem.

$\vdash . *350\cdot4 . \supset \vdash : \text{Hp} . \supset . (\exists L, M, \mu, \nu, \rho, \sigma) . L, M \epsilon \kappa_{\iota\partial} .$

$\dot{\exists} ! L^\sigma \dot{\wedge} M^\rho . \dot{\exists} ! L^\nu \dot{\wedge} M^\mu . X = \mu/\nu . Y = \rho/\sigma .$

[*333·53] $\supset . \mu \times_0 \sigma = \nu \times_0 \rho . X = \mu/\nu . Y = \rho/\sigma .$

[*303·39] $\supset . X = Y : \supset \vdash . \text{Prop}$

***350·5.** $\vdash : \kappa \in FM \text{ ap conx} . \mu, \nu \in \text{NC ind} - \iota\text{'}0 . \supset . (\mu/\nu) \upharpoonright \kappa_\iota \in 1 \rightarrow 1$

Dem.

$\vdash . *350·41 . \supset \vdash :. \text{Hp} . \supset :$

$L, M, N \in \kappa_\iota . L (\mu/\nu) N . M (\mu/\nu) N . \supset . \text{rep}_\kappa\text{'}L^\nu = \text{rep}_\kappa\text{'}N^\mu = \text{rep}_\kappa\text{'}M^\nu .$

[*333·41] $\supset . L = M$ (1)

$\vdash . (1) . \supset \vdash : \text{Hp} . \supset . (\mu/\nu) \upharpoonright \kappa_\iota \in 1 \rightarrow \text{Cls}$ (2)

Similarly $\vdash : \text{Hp} . \supset . (\mu/\nu) \upharpoonright \kappa_\iota \in \text{Cls} \rightarrow 1$ (3)

$\vdash . (2) . (3) . \supset \vdash . \text{Prop}$

***350·51.** $\vdash : \kappa \in FM \text{ ap conx} . \nu \in \text{NC ind} - \iota\text{'}0 . \supset .$

$(0/\nu) \upharpoonright \kappa_\iota \in 1 \rightarrow \text{Cls} . \text{Q}\text{'}(0/\nu) \upharpoonright \kappa_\iota = \kappa_\iota . \text{D}\text{'}(0/\nu) \upharpoonright \kappa_\iota = \iota\text{'}I \upharpoonright s\text{'}\text{Q}\text{''}\kappa$ [*350·34]

***350·511.** $\vdash : \text{Hp} *350·51 . \supset .$

$(\nu/0) \upharpoonright \kappa_\iota \in \text{Cls} \rightarrow 1 . \text{D}\text{'}(\nu/0) \upharpoonright \kappa_\iota = \kappa_\iota . \text{Q}\text{'}(\nu/0) \upharpoonright \kappa_\iota = \iota\text{'}I \upharpoonright s\text{'}\text{Q}\text{''}\kappa$

[*350·51 . *303·13]

***350·52.** $\vdash : \kappa \in FM \text{ ap conx} . X \in C\text{'}H . \supset . X \upharpoonright \kappa_\iota \in 1 \rightarrow 1$

[*350·5 . *304·34 . *333·2]

***350·521.** $\vdash : \kappa \in FM \text{ ap conx} . X \in C\text{'}H' . \supset . X \upharpoonright \kappa_\iota \in 1 \rightarrow \text{Cls}$

[*350·52·51 . *303·1]

***350·53.** $\vdash : \text{Hp} *350·5 . \supset . \{(\mu/1) \upharpoonright \kappa_\iota\} \mid \{(1/\nu) \upharpoonright \kappa_\iota\} \subseteq (\mu/\nu) \upharpoonright \kappa_\iota$

Dem.

$\vdash . *350·4 . \supset \vdash : \text{Hp} . L \{(\mu/1) \upharpoonright \kappa_\iota\} M . M \{(1/\nu) \upharpoonright \kappa_\iota\} N . \supset .$

$L, M, N \in \kappa_\iota . \dot{\exists} ! L \dot{\frown} M^\mu . \dot{\exists} ! N \dot{\frown} M^\nu .$

[*333·48] $\supset . L, M, N \in \kappa_\iota . \dot{\exists} ! L^\nu \dot{\frown} M^{\mu \times_c \nu} . \dot{\exists} ! N^\mu \dot{\frown} M^{\mu \times_c \nu} .$

[*333·47] $\supset . L, M, N \in \kappa_\iota . \text{rep}_\kappa\text{'}L^\nu = \text{rep}_\kappa\text{'}M^{\mu \times_c \nu} = \text{rep}_\kappa\text{'}N^\mu .$

[*350·41] $\supset . L \{(\mu/\nu) \upharpoonright \kappa_\iota\} N : \supset \vdash . \text{Prop}$

***350·54.** $\vdash : \text{Hp} *350·5 . \supset . \{(1/\nu) \upharpoonright \kappa_\iota\} \mid \{(\mu/1) \upharpoonright \kappa_\iota\} = (\mu/\nu) \upharpoonright \kappa_\iota$

Dem.

$\vdash . *350·41 . *332·241 . \supset$

$\vdash :. \text{Hp} . \supset : L [\{(1/\nu) \upharpoonright \kappa_\iota\} \mid \{(\mu/1) \upharpoonright \kappa_\iota\}] N . \equiv .$

$(\exists M) . L, M, N \in \kappa_\iota . \text{rep}_\kappa\text{'}L^\nu = M = \text{rep}_\kappa\text{'}N^\mu .$

[*332·22] $\equiv . L, N \in \kappa_\iota . \text{rep}_\kappa\text{'}L^\nu = \text{rep}_\kappa\text{'}N^\mu .$

[*350·41] $\equiv . L (\mu/\nu) N :. \supset \vdash . \text{Prop}$

***350·55.** $\vdash : \mathrm{Hp} *350\cdot5 \,.\supset.\, \{(\mu/1) \upharpoonright \kappa_\iota\} \mid \{(\nu/1) \upharpoonright \kappa_\iota\} = \{(\mu \times_c \nu)/1\} \upharpoonright \kappa_\iota$

$= \{(\nu/1) \upharpoonright \kappa_\iota\} \mid \{(\mu/1) \upharpoonright \kappa_\iota\}$

Dem.

$\vdash . *350\cdot4 . \supset \vdash :. \mathrm{Hp} . \supset : L [\{(\mu/1) \upharpoonright \kappa_\iota\} \mid \{(\nu/1) \upharpoonright \kappa_\iota\}] N . \equiv .$

$(\exists M) . L, M, N \in \kappa_\iota . \dot{\exists} ! L \dot{\cap} M^\mu . \dot{\exists} ! M \dot{\cap} N^\nu .$

[*333·47] $\equiv . (\exists M) . L, M, N \in \kappa_\iota . \dot{\exists} ! L \cap M^\mu . M = \mathrm{rep}_\kappa{}^\prime N^\nu .$

[*333·21] $\equiv . L, N \in \kappa_\iota . \dot{\exists} ! L \cap (\mathrm{rep}_\kappa{}^\prime N^\nu)^\mu .$

[*333·47] $\equiv . L, N \in \kappa_\iota . L = \mathrm{rep}_\kappa{}^\prime \{(\mathrm{rep}_\kappa{}^\prime N^\nu)^\mu\} .$

[*333·24] $\equiv . L, N \in \kappa_\iota . L = \mathrm{rep}_\kappa{}^\prime (N^\nu)^\mu .$

[*350·41.*301·5] $\equiv . L [\{(\nu \times_c \mu)/1\} \upharpoonright \kappa_\iota] N$ (1)

$\vdash . (1) . *113\cdot27 . \supset \vdash . \mathrm{Prop}$

***350·56.** $\vdash : \mathrm{Hp} *350\cdot5 \,.\supset.\, \{(1/\mu) \upharpoonright \kappa_\iota\} \mid \{(1/\nu) \upharpoonright \kappa_\iota\} = \{1/(\mu \times_c \nu)\} \upharpoonright \kappa_\iota$

$= \{(1/\nu) \upharpoonright \kappa_\iota\} \mid \{(1/\mu) \upharpoonright \kappa_\iota\}$ [*350·55 . *303·13]

***350·6.** $\vdash : \kappa \in FM \,\mathrm{ap\ conx} . X, Y \in C^\prime H . \supset . (X \upharpoonright \kappa_\iota) \mid (Y \upharpoonright \kappa_\iota) \subset (X \times_s Y) \upharpoonright \kappa_\iota$

Dem.

$\vdash . *304\cdot34 . \supset$

$\vdash : \mathrm{Hp} . \supset . (\exists \mu, \nu, \rho, \sigma) . \mu, \nu, \rho, \sigma \in \mathrm{NC\ induct} - \iota^\prime 0 . X = \mu/\nu . Y = \rho/\sigma$ (1)

$\vdash . *350\cdot54 . \supset \vdash : \kappa \in FM \,\mathrm{ap\ conx} . \mu, \nu, \rho, \sigma \in \mathrm{NC\ induct} - \iota^\prime 0 . \supset .$

$\{(\mu/\nu) \upharpoonright \kappa_\iota\} \mid \{(\rho/\sigma) \upharpoonright \kappa_\iota\} = \{(1/\nu) \upharpoonright \kappa_\iota\} \mid \{(\mu/1) \upharpoonright \kappa_\iota\} \mid \{(1/\sigma) \upharpoonright \kappa_\iota\} \mid \{(\rho/1) \upharpoonright \kappa_\iota\}$

[*350·53·54] $\subset \{(1/\nu) \upharpoonright \kappa_\iota\} \mid \{(1/\sigma) \upharpoonright \kappa_\iota\} \mid \{(\mu/1) \upharpoonright \kappa_\iota\} \mid \{(\rho/1) \upharpoonright \kappa_\iota\}$

[*350·56·55] $\subset \{1/(\nu \times_c \sigma)\} \upharpoonright \kappa_\iota \mid \{(\mu \times_c \rho)/1\} \upharpoonright \kappa_\iota$

[*350·54] $\subset \{(\mu \times_c \rho)/(\nu \times_c \sigma)\} \upharpoonright \kappa_\iota$

[*305·14] $\subset \{\mu/\nu \times_s \rho/\sigma\} \upharpoonright \kappa_\iota$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***350·61.** $\vdash :. \kappa \in FM \,\mathrm{ap\ conx} . X \in C^\prime H . \supset : M = (X \upharpoonright \kappa_\iota)^\prime N . \equiv . N = (\breve{X} \upharpoonright \kappa_\iota)^\prime M$

[*350·52]

***350·62.** $\vdash : \kappa \in FM \,\mathrm{ap\ conx} . X, Y \in C^\prime H^\prime . R, S, T \in \kappa . RXT . SYT . \supset .$

$(R \mid S)(X +_s Y) T$

Dem.

$\vdash . *350\cdot43 . \supset \vdash : \mathrm{Hp} . X = \mu/\nu . Y = \rho/\sigma . \supset .$

$R^\nu = T^\mu . S^\sigma = T^\rho$

[*301·5] $\supset . R^{\nu \times_c \sigma} = T^{\mu \times_c \sigma} . S^{\nu \times_c \sigma} = T^{\nu \times_c \rho} .$

[*330·57] $\supset . (R \mid S)^{\nu \times_c \sigma} = T^{(\mu \times_c \sigma) +_c (\nu \times_c \rho)} .$

[*350·41.*306·14] $\supset . (R \mid S)(X +_s Y) T : \supset \vdash . \mathrm{Prop}$

***350·63.** $\vdash : \kappa \epsilon FM \text{ ap conx} . X, Y \epsilon C\text{‘}H . L, M, N \epsilon \kappa_\iota . LXN . MYN . \supset .$
$\{\text{rep}_\kappa\text{‘}(L \mid M)\} (X +_s Y) N$

Dem.

$\vdash . *350\cdot41 . \supset$

$\vdash : \text{Hp} . X = \mu/\nu . Y = \rho/\sigma . \supset . \text{rep}_\kappa\text{‘}L^\nu = \text{rep}_\kappa\text{‘}N^\mu . \text{rep}_\kappa\text{‘}M^\sigma = \text{rep}_\kappa\text{‘}N^\rho .$

$[*332\cdot81] \supset . \text{rep}_\kappa\text{‘}L^{\nu \times_c \sigma} = \text{rep}_\kappa\text{‘}N^{\mu \times_c \sigma} . \text{rep}_\kappa\text{‘}M^{\nu \times_c \sigma} = \text{rep}_\kappa\text{‘}N^{\nu \times_c \rho} .$

$[*332\cdot33] \supset . \text{rep}_\kappa\text{‘}(L^{\nu \times_c \sigma} \mid M^{\nu \times_c \sigma}) = \text{rep}_\kappa\text{‘}N^{(\mu \times_c \sigma) +_c (\nu \times_c \rho)} .$

$[*332\cdot8] \quad \supset . \text{rep}_\kappa\text{‘}(L \mid M)^{\nu \times_c \sigma} = \text{rep}_\kappa\text{‘}N^{(\mu \times_c \sigma) +_c (\nu \times_c \rho)} .$

$[*332\cdot82] \supset . \text{rep}_\kappa\text{‘}\{\text{rep}_\kappa\text{‘}(L \mid M)\}^{\nu \times_c \sigma} = \text{rep}_\kappa\text{‘}N^{(\mu \times_c \sigma) +_c (\nu \times_c \rho)} .$

$[*350\cdot41] \supset . \{\text{rep}_\kappa\text{‘}(L \mid M)\} [\{(\mu \times_c \sigma) +_c (\nu \times_c \rho)\}/(\nu \times_c \sigma)] N .$

$[*306\cdot14] \supset . \{\text{rep}_\kappa\text{‘}(L \mid M)\} (X +_s Y) N : \supset \vdash . \text{Prop}$

***350·64.** $\vdash : \text{Hp} *350\cdot63 . XHY . \supset . \{\text{rep}_\kappa\text{‘}(\breve{L} \mid M)\} (Y -_s X) N$

Dem.

$\vdash . *332\cdot15\cdot81 . \supset \vdash : \text{Hp} . \supset . \text{rep}_\kappa\text{‘}\breve{L}^{\nu \times_c \sigma} = \text{Cnv}\text{‘}(\text{rep}_\kappa\text{‘}L)^{\nu \times_c \sigma}$ (1)

Thence the proof proceeds as in *350·63.

***350·65.** $\vdash : \text{Hp} *350\cdot62 . \supset . (\breve{R} \mid S)(Y -_s X) T$ [*350·64 . *308·21]

***350·66.** $\vdash : \kappa \epsilon FM \text{ ap conx} . L, M, N \epsilon \kappa_\iota . X, Y \epsilon C\text{‘}H_g . LXN . MYN . \supset .$
$\text{rep}_\kappa\text{‘}(L \mid M) = (X +_g Y) \upharpoonright \kappa_\iota\text{‘}N$

Dem.

$\vdash . *350\cdot63 . \supset$

$\vdash :. \text{Hp} . W = \text{rep}_\kappa\text{‘}(L \mid M) . \supset : X, Y \epsilon C\text{‘}H . \supset . W = (X +_g Y) \upharpoonright \kappa_\iota\text{‘}N$ (1)

$\vdash . *350\cdot64 . \supset \vdash : \text{Hp}(1) . X \epsilon C\text{‘}H_n . Y \epsilon C\text{‘}H . \supset . W = (X +_g Y) \upharpoonright \kappa_\iota\text{‘}N$ (2)

$\vdash . *350\cdot63 . *307\cdot1 . \supset \vdash : \text{Hp}(1) . X, Y \epsilon C\text{‘}H_n . \supset . W = (X +_g Y) \upharpoonright \kappa_\iota\text{‘}N$ (3)

$\vdash . *350\cdot34 . \supset \vdash : \text{Hp} . X = 0_q . \supset . \text{rep}_\kappa\text{‘}(L \mid M) = M$

$[*308\cdot51] \qquad = (X +_g Y) \upharpoonright \kappa_\iota\text{‘}N$ (4)

Similarly $\vdash : \text{Hp} . Y = 0_q . \supset . \text{rep}_\kappa\text{‘}(L \mid M) = (X +_g Y) \upharpoonright \kappa_\iota\text{‘}N$ (5)

$\vdash . (1) . (2) . (3) . (4) . (5) . \supset \vdash . \text{Prop}$

$*351$. SUBMULTIPLIABLE FAMILIES.

Summary of $*351$.

A "submultipliable" family is one in which any vector can be divided into ν equal parts (where ν is any inductive cardinal other than 0), *i.e.* in which, if $R \epsilon \kappa$, there is a vector S which is a member of κ and is such that $S^\nu = R$. The definition is

$*351 \cdot 01$. FM subm $=$

$$FM \cap \hat{\kappa} \{R \epsilon \kappa \,.\, \nu \epsilon \mathrm{NC\,ind} - \iota`0 \,.\, \supset_{R,\nu} \,.\, (\exists S) \,.\, S \epsilon \kappa \,.\, R = S^\nu\} \quad \mathrm{Df}$$

In open families, such as we are considering in this Section, S will be unique when R and ν are given. But in cyclic families, as we shall show in Section D, there will be ν values of S. For example, let κ be a family of angles. Then the vector-angle $2\mu\pi/\nu$ has its νth power equal to 2π for any integral value of μ, since $2\mu\pi$ is the same vector as 2π; and $2\mu\pi/\nu$ has ν different values, since, considered as a vector, any angle θ is identical with $\theta + 2\pi$. In the present Section, however, these complications are excluded, owing to the fact that we confine our attention to open families.

In virtue of $*337 \cdot 27$, a family is submultipliable if it is serial and $\mathrm{Cnv}`\dot{s}`\kappa_\partial$ is compact and semi-Dedekindian ($*351 \cdot 11$).

When κ is a family which is open, connected, and submultipliable, if $L \epsilon \kappa_\iota$ and $\mu \epsilon \mathrm{NC\,ind} - \iota`0$, we have

$$(\exists M) \,.\, M \epsilon \kappa_\iota \,.\, \mathrm{rep}_\kappa`M^\mu = L \quad (*351 \cdot 2).$$

Hence if X is any ratio (excluding ∞_q, now and always henceforth), we have

$$\mathrm{E}\,!\, X \upharpoonright \kappa_\iota`L \quad (*351 \cdot 21).$$

In order to obtain the same result for κ, we have to assume that all powers of members of κ are members of κ ($*351 \cdot 22$), but we can obtain the same result for $\kappa \cup \mathrm{Cnv}``\kappa$ without this assumption ($*351 \cdot 221$), because of $*331 \cdot 54$, which shows that in any connected family all powers of members of $\kappa \cup \mathrm{Cnv}``\kappa$ are members of $\kappa \cup \mathrm{Cnv}``\kappa$.

In virtue of the above propositions, the propositions on products and sums of ratios, which in $*350$ only stated inclusions, now state identities. Thus if $X, Y \epsilon C`H'$, we have

$$(X \upharpoonright \kappa_\iota) \,|\, (Y \upharpoonright \kappa_\iota) = (X \times_s Y) \upharpoonright \kappa_\iota \quad (*351 \cdot 31),$$

$$\mathrm{rep}_\kappa`\{(X \upharpoonright \kappa_\iota`L) \,|\, (Y \upharpoonright \kappa_\iota`L)\} = (X +_s Y) \upharpoonright \kappa_\iota`L \quad (*351 \cdot 42),$$

where $L \epsilon \kappa_\iota$; also

$$\text{rep}_\kappa\text{‘}\{(X \upharpoonleft \kappa_\iota\text{‘}L) \mid (Y \upharpoonleft \kappa_\iota\text{‘}\breve{L})\} = (X -_s Y) \upharpoonleft \kappa_\iota\text{‘}L \quad (*351\cdot45).$$

The corresponding propositions for ratios confined to κ instead of to κ_ι require the additional hypothesis $s\text{‘Pot“}\kappa \subset \kappa$, because this hypothesis is required in $*351\cdot22$; on the other hand, in the analogue of $*351\cdot42$ "rep_κ" does not appear, and we have (with the above hypothesis)

$$(X \upharpoonleft \kappa\text{‘}R) \mid (Y \upharpoonleft \kappa\text{‘}R) = (X +_s Y) \upharpoonleft \kappa\text{‘}R \quad (*351\cdot43),$$

where $R \epsilon \kappa$. For ratios confined to $\kappa \cup \text{Cnv“}\kappa$ instead of to κ, the corresponding result can be proved without the hypothesis $s\text{‘Pot“}\kappa \subset \kappa$ ($*351\cdot431$). It will be observed that the hypothesis $s\text{‘Pot“}\kappa \subset \kappa$ is satisfied if κ is a group, though it may also be satisfied when κ is not a group. Since a transitive connected family is a group, a transitive connected family always satisfies $s\text{‘Pot“}\kappa \subset \kappa$, as has been proved already ($*334\cdot132$).

$*351\cdot01$. $FM\,\text{subm} =$

$$FM \cap \hat{\kappa}\{R \epsilon \kappa \,.\, \nu \epsilon \text{NC ind} - \iota\text{‘}0 \,.\, \supset_{R,\nu} .\, (\exists S) \,.\, S \epsilon \kappa \,.\, R = S^\nu\} \quad \text{Df}$$

$*351\cdot1$. $\vdash :.\ \kappa \epsilon FM\,\text{subm} \,.\equiv:\ \kappa \epsilon FM : R \epsilon \kappa \,.\, \nu \epsilon \text{NC ind} - \iota\text{‘}0 \,.\supset_{R,\nu} .$
$$(\exists S) \,.\, S \epsilon \kappa \,.\, R = S^\nu \quad [(*351\cdot01)]$$

$*351\cdot101$. $\vdash : \exists !\, FM\,\text{subm} \,.\supset .\, \text{Infin ax} \quad [*351\cdot1 \,.\, *301\cdot16 \,.\, *300\cdot14]$

$*351\cdot11$. $\vdash : \kappa \epsilon FM\,\text{sr} \,.\, \text{Cnv‘}\dot{s}\text{‘}\kappa_\partial \epsilon \text{comp} \cap \text{semi Ded} \,.\supset .\, \kappa \epsilon FM\,\text{subm}$
$[*337\cdot27]$

$*351\cdot2$. $\vdash :.\ \kappa \epsilon FM\,\text{ap subm conx} \,.\supset : \mu \epsilon \text{NC ind} - \iota\text{‘}0 \,.\, L \epsilon \kappa_\iota \,.\supset .$
$$(\exists M) \,.\, M \epsilon \kappa_\iota \,.\, \text{rep}_\kappa\text{‘}M^\mu = L$$

Dem.

$$\vdash .\, *351\cdot1 \,.\supset \vdash : \text{Hp} \,.\, \mu \epsilon \text{NC ind} - \iota\text{‘}0 \,.\, Q, R \epsilon \kappa \,.\, L = \breve{Q} \mid R \,.\supset .$$
$$(\exists S, T) \,.\, S, T \epsilon \kappa \,.\, Q = S^\mu \,.\, R = T^\mu \,.$$
$$[*332\cdot53] \quad \supset .\, (\exists S, T) \,.\, S, T \epsilon \kappa \,.\, L = \text{rep}_\kappa\text{‘}(\breve{S} \mid T)^\mu : \supset \vdash .\, \text{Prop}$$

$*351\cdot21$. $\vdash : \text{Hp}\,*351\cdot2 \,.\, X \epsilon C\text{‘}H' \,.\, L \epsilon \kappa_\iota \,.\supset .\, \text{E}!\, X \upharpoonleft \kappa_\iota\text{‘}L$

Dem.

$\vdash .\, *351\cdot2 \,.\, *332\cdot61 \,.\supset$

$\vdash : \text{Hp} \,.\, \mu, \nu \epsilon \text{NC ind} - \iota\text{‘}0 \,.\, X = \mu/\nu \,.\supset .\, (\exists M) \,.\, M \epsilon \kappa_\iota \,.\, \text{rep}_\kappa\text{‘}M^\mu = \text{rep}_\kappa\text{‘}L^\nu \,.$

$[*350\cdot41\cdot5] \qquad \supset .\, \text{E}!\, X \upharpoonleft \kappa_\iota\text{‘}L \qquad (1)$

$\vdash .\, *350\cdot34 \,.\supset$

$\vdash : \text{Hp} \,.\, \mu = 0 \,.\, \nu \epsilon \text{NC ind} - \iota\text{‘}0 \,.\, X = \mu/\nu \,.\supset .\, X \upharpoonleft \kappa_\iota\text{‘}L = I \upharpoonright s\text{‘}\text{G“}\kappa \qquad (2)$

$\vdash .\, (1) \,.\, (2) \,.\supset \vdash .\, \text{Prop}$

***351·22.** $\vdash : \mathrm{Hp} \ast 351{\cdot}2 \,.\, s`\mathrm{Pot}``\kappa \subset \kappa \,.\, X \in C`H' \,.\, R \in \kappa \,.\, \supset \,.\, \mathrm{E}\,!\, X \upharpoonleft \kappa`R$

Dem.

$\vdash \,.\, \ast 301{\cdot}22 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \mu, \nu \in \mathrm{NC\,ind} \,.\, \nu \neq 0 \,.\, \supset \,.\, R^\mu \in \kappa \,.$

$[\ast 351{\cdot}1] \qquad \supset \,.\, (\exists S) \,.\, S \in \kappa \,.\, R^\mu = S^\nu \,.$

$[\ast 350{\cdot}4 \,.\, \ast 331{\cdot}12] \qquad \supset \,.\, (\exists S) \,.\, S \in \kappa \,.\, S(\mu/\nu) R \quad (1)$

$\vdash \,.\, (1) \,.\, \ast 350{\cdot}521 \,.\, \supset \vdash \,.\, \mathrm{Prop}$

***351·221.** $\vdash : \mathrm{Hp} \ast 351{\cdot}2 \,.\, X \in C`H' \,.\, \lambda = \kappa \cup \mathrm{Cnv}``\kappa \,.\, R \in \lambda \,.\, \supset \,.\, \mathrm{E}\,!\, X \upharpoonleft \lambda`R$

[Proof as in *351·22, using *331·54]

***351·3.** $\vdash : \mathrm{Hp} \ast 351{\cdot}2 \,.\, \mu, \nu \in \mathrm{NC\,ind} \,.\, \nu \neq 0 \,.\, \supset \,.$

$$\{(\mu/1) \upharpoonleft \kappa_\iota\} \,|\, \{(1/\nu) \upharpoonleft \kappa_\iota\} = (\mu/\nu) \upharpoonleft \kappa_\iota$$

Dem.

$\vdash \,.\, \ast 350{\cdot}41 \,.\, \supset \vdash :. \mathrm{Hp} \,.\, \mu \neq 0 \,.\, \supset :$

$$L \{(\mu/\nu) \upharpoonleft \kappa_\iota\} N \,.\, \equiv \,.\, L, N \in \kappa_\iota \,.\, \mathrm{rep}_\kappa`L^\nu = \mathrm{rep}_\kappa`N^\mu \,.$$

$[\ast 351{\cdot}2] \equiv \,.\, (\exists M) \,.\, L, M, N \in \kappa_\iota \,.\, L = \mathrm{rep}_\kappa`M^\mu \,.\, \mathrm{rep}_\kappa`L^\nu = \mathrm{rep}_\kappa`N^\mu \,.$

$[\ast 333{\cdot}24] \equiv \,.\, (\exists M) \,.\, L, M, N \in \kappa_\iota \,.\, L = \mathrm{rep}_\kappa`M^\mu \,.\, \mathrm{rep}_\kappa`M^{\mu \times_c \nu} = \mathrm{rep}_\kappa`N^\mu \,.$

$[\ast 333{\cdot}44] \equiv \,.\, (\exists M) \,.\, L, M, N \in \kappa_\iota \,.\, L = \mathrm{rep}_\kappa`M^\mu \,.\, \mathrm{rep}_\kappa`M^\nu = \mathrm{rep}_\kappa`N \,.$

$[\ast 350{\cdot}41] \equiv \,.\, (\exists M) \,.\, L \{(\mu/1) \upharpoonleft \kappa_\iota\} M \,.\, M \{(1/\nu) \upharpoonleft \kappa_\iota\} N \quad (1)$

$\vdash \,.\, \ast 350{\cdot}34 \,.\, \supset \vdash :. \mathrm{Hp} \,.\, \mu = 0 \,.\, \supset :$

$$L \{(\mu/\nu) \upharpoonleft \kappa_\iota\} N \,.\, \equiv \,.\, L = I \upharpoonleft s`\mathrm{G}``\kappa \,.\, N \in \kappa_\iota \quad (2)$$

$\vdash \,.\, \ast 350{\cdot}34 \,.\, \ast 351{\cdot}21 \,.\, \supset \vdash :. \mathrm{Hp} \,.\, \mu = 0 \,.\, \supset :$

$$L \{(\mu/1) \upharpoonleft \kappa_\iota\} \,|\, \{(1/\nu) \upharpoonleft \kappa_\iota\} N \,.\, \equiv \,.\, L = I \upharpoonleft s`\mathrm{G}``\kappa \,.\, N \in \kappa_\iota \quad (3)$$

$\vdash \,.\, (1) \,.\, (2) \,.\, (3) \,.\, \supset \vdash \,.\, \mathrm{Prop}$

***351·31.** $\vdash : \mathrm{Hp} \ast 351{\cdot}2 \,.\, X, Y \in C`H' \,.\, \supset \,.\, (X \upharpoonleft \kappa_\iota) \,|\, (Y \upharpoonleft \kappa_\iota) = (X \times_s Y) \upharpoonleft \kappa_\iota$

[Proof as in *350·6, using *351·3 instead of *350·53]

***351·4.** $\vdash : \kappa \in FM \,\mathrm{ap\,subm\,conx} \,.\, \mu, \nu, \rho, \sigma \in \mathrm{NC\,ind} \,.\, \nu \neq 0 \,.\, \sigma \neq 0 \,.\, L \in \kappa_\iota \,.\, \supset \,.$

$$\mathrm{rep}_\kappa`[\{(\mu/\nu) \upharpoonleft \kappa_\iota`L\} \,|\, \{(\rho/\sigma) \upharpoonleft \kappa_\iota`L\}] = (\mu/\nu +_s \rho/\sigma) \upharpoonleft \kappa_\iota`L$$

Dem.

$\vdash \,.\, \ast 350{\cdot}41 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \mu \neq 0 \,.\, \rho \neq 0 \,.\, M = (\mu/\nu) \upharpoonleft \kappa_\iota`L \,.\, \supset \,.\, \mathrm{rep}_\kappa`M^\nu = \mathrm{rep}_\kappa`L^\mu \,.$

$[\ast 333{\cdot}44] \qquad \supset \,.\, \mathrm{rep}_\kappa`M^{\nu \times_c \sigma} = \mathrm{rep}_\kappa`L^{\mu \times_c \sigma} \quad (1)$

Similarly

$\vdash : \mathrm{Hp} \,.\, \mu \neq 0 \,.\, \rho \neq 0 \,.\, N = (\rho/\sigma) \upharpoonleft \kappa_\iota`L \,.\, \supset \,.\, \mathrm{rep}_\kappa`N^{\nu \times_c \sigma} = \mathrm{rep}_\kappa`L^{\nu \times_c \rho} \quad (2)$

$\vdash \,.\, (1) \,.\, (2) \,.\, \ast 333{\cdot}34 \,.\, \ast 332{\cdot}33 \,.\, \supset$

$\vdash : \mathrm{Hp}(1) \,.\, \mathrm{Hp}(2) \,.\, \supset \,.\, \mathrm{rep}_\kappa`(M \,|\, N)^{\nu \times_c \sigma} = \mathrm{rep}_\kappa`\{L^{(\mu \times_c \sigma)} \,|\, L^{(\nu \times_c \rho)}\} \,.$

$[\ast 301{\cdot}23 \,.\, \ast 333{\cdot}24] \quad \supset \,.\, \{\mathrm{rep}_\kappa`(M \,|\, N)\}^{\nu \times_c \sigma} = \mathrm{rep}_\kappa`L^{(\mu \times_c \sigma) +_c (\nu \times_c \rho)} \,.$

$[\ast 306{\cdot}14 \,.\, \ast 350{\cdot}41] \quad \supset \,.\, \mathrm{rep}_\kappa`(M \,|\, N) = (\mu/\nu +_s \rho/\sigma) \upharpoonleft \kappa_\iota`L \quad (3)$

$\vdash \,.\, (3) \,.\, \ast 351{\cdot}21 \,.\, \ast 350{\cdot}34 \,.\, \supset \vdash \,.\, \mathrm{Prop}$

***351·41.** $\vdash : \kappa \in FM \text{ ap subm conx} \,.\, s\text{‘}\mathrm{Pot}\text{“}\kappa \subset \kappa \,.$

$$\mu, \nu, \rho, \sigma \in \mathrm{NC\ ind} \,.\, \nu \neq 0 \,.\, \sigma \neq 0 \,.\, R \in \kappa \,.\, \supset .$$
$$\{(\mu/\nu) \upharpoonright \kappa\text{‘}R\} \mid \{(\rho/\sigma) \upharpoonright \kappa\text{‘}R\} = (\mu/\nu +_s \rho/\sigma) \upharpoonright \kappa\text{‘}R$$

Dem.

$\vdash . *351{\cdot}21{\cdot}22 \,.\, \supset$

$\vdash : \mathrm{Hp} \,.\, \supset . \, (\mu/\nu) \upharpoonright \kappa\text{‘}R = (\mu/\nu) \upharpoonright \kappa_\iota\text{‘}R \,.\, (\rho/\sigma) \upharpoonright \kappa\text{‘}R = (\rho/\sigma) \upharpoonright \kappa_\iota\text{‘}R$ (1)

$\vdash . (1) . *332{\cdot}241 \,.\, *331{\cdot}24{\cdot}33 \,.\, \supset$

$\vdash : \mathrm{Hp} \,.\, \supset . \, \{(\mu/\nu) \upharpoonright \kappa\text{‘}R\} \mid \{(\rho/\sigma) \upharpoonright \kappa\text{‘}R\} = \mathrm{rep}_\kappa\text{‘}[\{(\mu/\nu) \upharpoonright \kappa_\iota\text{‘}R\} \mid \{(\rho/\sigma) \upharpoonright \kappa_\iota\text{‘}R\}]$

$[*351{\cdot}4.(1)]$ $\qquad = (\mu/\nu +_s \rho/\sigma) \upharpoonright \kappa\text{‘}R : \supset \vdash . \mathrm{Prop}$

***351·411.** $\vdash : \mathrm{Hp} *351{\cdot}4 \,.\, \lambda = \kappa \cup \mathrm{Cnv}\text{“}\kappa \,.\, S \in \lambda \,.\, \supset .$

$$\{(\mu/\nu) \upharpoonright \lambda\text{‘}S\} \mid \{(\rho/\sigma) \upharpoonright \lambda\text{‘}S\} = (\mu/\nu +_s \rho/\sigma) \upharpoonright \lambda\text{‘}S$$

[Proof as in *351·41, using *331·54]

***351·42.** $\vdash : \kappa \in FM \text{ ap subm conx} \,.\, X, Y \in C\text{‘}H' \,.\, L \in \kappa_\iota \,.\, \supset .$

$$\mathrm{rep}_\kappa\text{‘}\{(X \upharpoonright \kappa_\iota\text{‘}L) \mid (Y \upharpoonright \kappa_\iota\text{‘}L)\} = (X +_s Y) \upharpoonright \kappa_\iota\text{‘}L \quad [*351{\cdot}4]$$

***351·43.** $\vdash : \kappa \in FM \text{ ap subm conx} \,.\, s\text{‘}\mathrm{Pot}\text{“}\kappa \subset \kappa \,.\, X, Y \in C\text{‘}H' \,.\, R \in \kappa \,.\, \supset .$

$$(X \upharpoonright \kappa\text{‘}R) \mid (Y \upharpoonright \kappa\text{‘}R) = (X +_s Y) \upharpoonright \kappa\text{‘}R \quad [*351{\cdot}41]$$

***351·431.** $\vdash : \mathrm{Hp} *351{\cdot}42 \,.\, \lambda = \kappa \cup \mathrm{Cnv}\text{“}\kappa \,.\, S \in \lambda \,.\, \supset .$

$$(X \upharpoonright \lambda\text{‘}S) \mid (Y \upharpoonright \lambda\text{‘}S) = (X +_s Y) \upharpoonright \lambda\text{‘}S \quad [*351{\cdot}411]$$

***351·44.** $\vdash : \kappa \in FM \text{ ap subm conx} \,.$

$$\mu, \nu, \rho, \sigma \in \mathrm{NC\ ind} \,.\, \nu \neq 0 \,.\, \sigma \neq 0 \,.\, (\rho/\sigma)\, H' \,(\mu/\nu) \,.\, L \in \kappa_\iota \,.\, \supset .$$
$$\mathrm{rep}_\kappa\text{‘}[\{(\mu/\nu) \upharpoonright \kappa_\iota\text{‘}L\} \mid \{(\rho/\sigma) \upharpoonright \kappa_\iota\text{‘}\breve{L}\}] = (\mu/\nu -_s \rho/\sigma) \upharpoonright \kappa_\iota\text{‘}L$$

Dem.

As in *351·4,

$\vdash : \mathrm{Hp} \,.\, M = (\mu/\nu) \upharpoonright \kappa_\iota\text{‘}L \,.\, N = (\rho/\sigma) \upharpoonright \kappa_\iota\text{‘}\breve{L} \,.\, \supset .$

$$\{\mathrm{rep}_\kappa\text{‘}(M \mid N)\}^{\nu \times_c \sigma} = \mathrm{rep}_\kappa\text{‘}\{L^{\mu \times_c \sigma} \mid \breve{L}^{\nu \times_c \rho}\} \quad (1)$$

$\vdash . *301{\cdot}23 \,.\, *308{\cdot}13 \,.\, \supset \vdash : \mathrm{Hp} \,.\, \tau = (\mu \times_c \sigma) -_c (\nu \times_c \rho) \,.\, \supset .$

$$\mathrm{rep}_\kappa\text{‘}\{L^{\mu \times_c \sigma} \mid \breve{L}^{\nu \times_c \rho}\} = \mathrm{rep}_\kappa\text{‘}\{L^\tau \mid L^{\nu \times_c \rho} \mid \breve{L}^{\nu \times_c \rho}\}$$

$[*72{\cdot}59.*332{\cdot}25]$ $\qquad = \mathrm{rep}_\kappa\text{‘}L^\tau$ (2)

$\vdash . (1) . (2) . *350{\cdot}41 \,.\, \supset$

$\vdash : \mathrm{Hp}(1) \,.\, \mathrm{Hp}(2) \,.\, \supset . \, \mathrm{rep}_\kappa\text{‘}(M \mid N) = \{\tau/(\nu \times_c \sigma)\} \upharpoonright \kappa_\iota\text{‘}L$ (3)

$\vdash . (3) . *308{\cdot}24 \,.\, \supset \vdash . \mathrm{Prop}$

***351·441.** $\vdash : \kappa \in FM \text{ ap subm conx} .$

$$\mu, \nu, \rho, \sigma \in \text{NC ind} . \nu \neq 0 . \sigma \neq 0 . (\mu/\nu) H' (\rho/\sigma) . L \in \kappa_\iota . \supset .$$
$$\text{rep}_\kappa `[\{(\mu/\nu) \upharpoonright \kappa_\iota `L\} \mid \{(\rho/\sigma) \upharpoonright \kappa_\iota `\breve{L}\}] = (\mu/\nu -_s \rho/\sigma) \upharpoonright \kappa_\iota `L$$

Dem.

$$\vdash . *332\cdot15 . *303\cdot19 . \supset$$
$$\vdash : \text{Hp} . \supset . \text{rep}_\kappa `[\{(\mu/\nu) \upharpoonright \kappa_\iota `L\} \mid \{(\rho/\sigma) \upharpoonright \kappa_\iota `\breve{L}\}] =$$
$$\text{Cnv}`\text{rep}_\kappa `[\{(\rho/\sigma) \upharpoonright \kappa_\iota `L\} \mid \{(\mu/\nu) \upharpoonright \kappa_\iota `\breve{L}\}]$$
$$[*351\cdot44] = \text{Cnv}`(\rho/\sigma -_s \mu/\nu) \upharpoonright \kappa_\iota `L$$
$$[*303\cdot19] = (\rho/\sigma -_s \mu/\nu) \upharpoonright \kappa_\iota `\breve{L}$$
$$[*308\cdot21] = (\mu/\nu -_s \rho/\sigma) \upharpoonright \kappa_\iota `L : \supset \vdash . \text{Prop}$$

***351·45.** $\vdash : \kappa \in FM \text{ ap subm conx} . X, Y \in C`H' . L \in \kappa_\iota . \supset .$

$$\text{rep}_\kappa `\{(X \upharpoonright \kappa_\iota `L) \mid (Y \upharpoonright \kappa_\iota `\breve{L})\} = (X -_s Y) \upharpoonright \kappa_\iota `L$$

Dem.

$\vdash . *351\cdot21 . *350\cdot34 . *308\cdot12 . \supset \vdash : \text{Hp} . X = Y . \supset .$

$$\text{rep}_\kappa `\{(X \upharpoonright \kappa_\iota `L) \mid (Y \upharpoonright \kappa_\iota `\breve{L})\} = I \upharpoonright s`\text{Cl}``\kappa = (X -_s Y) \upharpoonright \kappa_\iota `L \quad (1)$$

$\vdash . (1) . *351\cdot44\cdot441 . \supset \vdash . \text{Prop}$

***351·46.** $\vdash : \kappa \in FM \text{ ap subm conx} . s`\text{Pot}``\kappa \subset \kappa . X, Y \in C`H' . R \in \kappa . \supset .$

$$(\text{Cnv}`Y \upharpoonright \kappa`R) \mid (X \upharpoonright \kappa`R) \in \kappa_\iota$$

Dem.

$\vdash . *351\cdot22 . \supset \vdash : \text{Hp} . \supset . X \upharpoonright \kappa`R \in \kappa . Y \upharpoonright \kappa`R \in \kappa .$

$[*37\cdot62] \qquad \supset . X \upharpoonright \kappa`R \in \kappa . \text{Cnv}`Y \upharpoonright \kappa`R \in \text{Cnv}``\kappa : \supset \vdash . \text{Prop}$

***351·47.** $\vdash : \text{Hp} *351\cdot46 . \supset . (\text{Cnv}`Y \upharpoonright \kappa`R) \mid (X \upharpoonright \kappa`R) = (X -_s Y) \upharpoonright \kappa_\iota `R$

$[*351\cdot45\cdot46]$

*352. RATIONAL MULTIPLES OF A GIVEN VECTOR.

Summary of *352.

By a "rational multiple" of a given vector in a family κ we mean, if we are dealing with κ, any vector in the family which has to the given vector a relation which is a member of $C'H'$, and if we are dealing with κ_ι, we mean any member of κ_ι which has to the given member of κ_ι a relation which is a member of $C'H_g$. We will call the former "rational κ-multiples" and the latter "*generalized* rational multiples." It will be observed that if κ contains pairs of members which are each other's converses, only one member of such a pair can be contained among the rational κ-multiples of a given member of κ, provided κ is an open family. Hence the rational κ-multiples of a given vector all have one "sense," even if this was not the case with the original family.

Rational multiples of a given vector T can be arranged in a series by correlation with their measures with T as unit. These measures are ordered, in the case of rational κ-multiples, by the relation H', and in the case of generalized rational multiples, by the relation H_g. Moreover if X is the measure of a given member of κ with T as unit, the given member of κ is $\kappa \rceil A_T{}^{\backprime}X$; while if X is the measure of a given member of κ_ι, the given member of κ_ι is $\kappa_\iota \rceil A_T{}^{\backprime}X$. Hence the rational κ-multiples of T are ordered by the relation $\kappa \rceil A_T ; H'$, and the generalized rational multiples are ordered by the relation $\kappa_\iota \rceil A_T ; H_g$. These two relations, therefore, are the relations we shall consider in this number. We put

***352·01.** $T_\kappa = \kappa \rceil A_T ; H'$ Df

***352·02.** $T_{\kappa\iota} = \kappa_\iota \rceil A_T ; H_g$ Df

We assume throughout this number that κ is open and connected. In dealing with T_κ, we assume $T \in \kappa_\partial$, and in dealing with $T_{\kappa\iota}$, we assume $T \in \kappa_{\iota\partial}$. We then prove the following propositions among others:

$$\kappa \rceil A_T \upharpoonright C'H' \in 1 \to 1 \quad (*352{\cdot}12),$$

$$\kappa_\iota \rceil A_T \upharpoonright C'H_g \in 1 \to 1 \quad (*352{\cdot}15),$$

i.e. the relation of a rational multiple of T to its measure is one-one.

$$T_\kappa, T_{\kappa\iota} \epsilon \mathrm{Ser} \quad (*352\cdot16\cdot17).$$

Observe that this requires only that κ should be open and connected. The serial property results from the correlation with H' or H_g.

$$C`T_\kappa = \kappa \cap A_T``C`H' \,.\, C`T_{\kappa\iota} = \kappa_\iota \cap A_T``C`H_g \quad (*352\cdot3\cdot31).$$

If S is any non-zero member of $C`T_\kappa$, $C`S_\kappa = C`T_\kappa$ ($*352\cdot41$), *i.e.* the rational $\kappa$-multiples of $T$ are the same as those of any rational $\kappa$-multiple of $T$; with a similar proposition for $C`T_{\kappa\iota}$ ($*352\cdot42$).

$$RT_\kappa S \,.\equiv: R, S \epsilon \kappa \cap A_T``C`H' : (\exists \mu, \nu) \,.\, \mu, \nu \epsilon \mathrm{NC\,ind} \,.\, \mu < \nu \,.\, R^\nu = S^\mu \quad (*352\cdot43).$$

This is a convenient formula for T_κ, and leads immediately to

$$T_\kappa = \{\dot{s}`\overrightarrow{H'}`(1/1)\} \upharpoonright (\kappa \cap A_T``C`H') \quad (*352\cdot44).$$

Observe that $\overrightarrow{H'}`(1/1)$ is the class of rational proper fractions, including $0_q$. By $*352\cdot44$ and $*352\cdot41\cdot3$, we see that, if $S \neq I \upharpoonright s`\mathrm{C}``\kappa$,

$$S \epsilon C`T_\kappa \,.\supset.\, S_\kappa = T_\kappa \quad (*352\cdot45),$$

i.e. the order of magnitude of a set of vectors which are rational κ-multiples of a given unit is independent of the choice of the unit.

In order to establish the analogous property for $T_{\kappa\iota}$, we first prove a formula analogous to $*352\cdot44$, namely

$$T_{\kappa\iota} = \mathrm{Cnv}\,;\{\dot{s}`\overleftarrow{H}`(1/1)\} \upharpoonright (\kappa_\iota \cap A_T``C`H) \,\dot{+}\, \{\dot{s}`\overrightarrow{H'}`(1/1)\} \upharpoonright (\kappa_\iota \cap A_T``C`H') \quad (*352\cdot54).$$

Here the first term gives the series of negative multiples of T, while the second gives the series of positive multiples of T (including $I \upharpoonright s`\mathrm{C}``\kappa$).

From the above formula it follows, as in the case of T_κ, that if S is a positive multiple of T (not including $I \upharpoonright s`\mathrm{C}``\kappa$), $S_{\kappa\iota} = T_{\kappa\iota}$, while if S is a negative multiple of T, $S_{\kappa\iota} = \breve{T}_{\kappa\iota}$ ($*352\cdot56\cdot57$).

Finally we deal with the relation of U_κ to T_κ. Here we have to assume that κ is a *serial* family. We then find that U_κ with its field confined to rational κ-multiples of T is the converse of T_κ, *i.e.* we have

$*352\cdot72$. $\vdash : \kappa \epsilon FM\mathrm{sr} \,.\, T \epsilon \kappa_\partial \,.\supset.\, U_\kappa \upharpoonright C`T_\kappa = \kappa \,\rceil\, A_T\,;\breve{H}' = \breve{T}_\kappa$

$*352\cdot01$. $T_\kappa = \kappa \,\rceil\, A_T\,;H'$ Df

$*352\cdot02$. $T_{\kappa\iota} = \kappa_\iota \,\rceil\, A_T\,;H_g$ Df

$*352\cdot1$. $\vdash :. RT_\kappa S \,.\equiv: R, S \epsilon \kappa : (\exists X, Y) \,.\, XH'Y \,.\, RXT \,.\, SYT$ [($*352\cdot01$)]

$*352\cdot11$. $\vdash :. RT_{\kappa\iota} S \,.\equiv: R, S \epsilon \kappa_\iota : (\exists X, Y) \,.\, XH_gY \,.\, RXT \,.\, SYT$ [($*352\cdot02$)]

***352·12.** $\vdash : \kappa \epsilon FM \text{ ap conx} . T \epsilon \kappa_\partial . \supset . \kappa \rceil A_T \upharpoonright C`H' \epsilon 1 \rightarrow 1$

Dem.

$\vdash . *336·1 . \quad \supset \vdash : R(\kappa \rceil A_T \upharpoonright C`H')X . \equiv . R \epsilon \kappa . X \epsilon C`H' . RXT$ (1)

$\vdash . *350·521 . \supset \vdash : \text{Hp} . R, S \epsilon \kappa . X \epsilon C`H' . RXT . SXT . \supset . R = S$ (2)

$\vdash . *350·44 . \quad \supset \vdash : \text{Hp} . R \epsilon \kappa_\partial . X, Y \epsilon C`H' . RXT . RYT . \supset . X = Y$ (3)

$\vdash . *350·34·4 . \supset$

$\vdash : \text{Hp} . R = I \upharpoonright s`\text{Ɑ}``\kappa . X, Y \epsilon C`H' . RXT . SYT . \supset . X = 0_q . Y = 0_q$ (4)

$\vdash . (3) . (4) . \supset \vdash : \text{Hp} . R \epsilon \kappa . X, Y \epsilon C`H' . RXT . SYT . \supset . X = Y$ (5)

$\vdash . (1) . (2) . (5) . \supset \vdash . \text{Prop}$

***352·13.** $\vdash : \kappa \epsilon FM \text{ ap conx} . T \epsilon \kappa_{\iota\partial} . \supset . \kappa_\iota \cap A_T``C`H \subset \kappa_{\iota\partial}$

Dem.

$\vdash . *350·4 . \supset \vdash : \text{Hp} . R \epsilon \kappa_\iota \cap A_T``C`H . \supset .$

$(\exists \mu, \nu) . \mu, \nu \epsilon \text{NC ind} - \iota`0 . \dot{\exists} ! R^\nu \dot{\cap} T^\mu .$

$[*333·101] \quad \supset . R \epsilon \kappa_{\iota\partial} : \supset \vdash . \text{Prop}$

***352·131.** $\vdash : \text{Hp } *352·13 . \supset . \kappa_\iota \cap A_T``C`H_n = \text{Cnv}``(\kappa_\iota \cap A_T``C`H)$ [*307·1]

***352·132.** $\vdash : \text{Hp } *352·13 . \supset . \kappa_\iota \cap A_T``C`H_n \subset \kappa_{\iota\partial}$ [*352·13·131]

***352·14.** $\vdash : \kappa \epsilon FM \text{ ap conx} . T \epsilon \kappa_{\iota\partial} . \supset . \kappa_\iota \cap A_T``C`H' \cap A_T``C`H_n = \Lambda$

Dem.

$\vdash . *307·1 . *350·4 . *352·132 . \supset \vdash : \text{Hp} . R, S \epsilon \kappa_\iota . R \epsilon A_T``C`H_n . S \epsilon A_T``C`H' . \supset .$

$(\exists \mu, \nu, \rho, \sigma) . \mu, \nu, \rho, \sigma \epsilon \text{NC ind} . \nu \neq 0 . \rho \neq 0 . \sigma \neq 0 . R \epsilon \kappa_{\iota\partial} .$

$\text{rep}_\kappa`\breve{R}^\nu = \text{rep}_\kappa`T^\mu . \text{rep}_\kappa`S^\sigma = \text{rep}_\kappa`T^\rho .$

$[*333·44] \quad \supset . (\exists \mu, \nu, \rho, \sigma) . \mu, \nu, \rho, \sigma \epsilon \text{NC ind} . \nu \neq 0 . \rho \neq 0 . \sigma \neq 0 . R \epsilon \kappa_{\iota\partial} .$

$\text{rep}_\kappa`\breve{R}^{\nu \times_c \rho} = \text{rep}_\kappa`T^{\mu \times_c \rho} = \text{rep}_\kappa`S^{\sigma \times_c \mu} .$

$[*333·47] \quad \supset . (\exists \xi, \eta) . \xi, \eta \epsilon \text{NC ind} . \xi \neq 0 . \dot{\exists} ! \breve{R}^\xi \dot{\cap} S^\eta . R \epsilon \kappa_{\iota\partial} .$

$[*71·192] \quad \supset . (\exists \xi, \eta) . \xi, \eta \epsilon \text{NC ind} . \xi \neq 0 . \dot{\exists} ! I \dot{\cap} R^\xi | S^\eta . R \epsilon \kappa_{\iota\partial} .$

$[*333·101 . \text{Transp}] \supset . R \neq S : \supset \vdash . \text{Prop}$

***352·15.** $\vdash : \kappa \epsilon FM \text{ ap conx} . T \epsilon \kappa_{\iota\partial} . \supset . \kappa_\iota \rceil A_T \upharpoonright C`H_g \epsilon 1 \rightarrow 1$

Dem.

$\vdash . *336·1 . \supset \vdash : \text{Hp} . R(\kappa_\iota \rceil A_T \upharpoonright C`H_g) X . R(\kappa_\iota \rceil A_T \upharpoonright C`H_g) Y . \supset .$

$R \epsilon \kappa_\iota . X, Y \epsilon C`H_g . RXT . RYT$ (1)

$\vdash . (1) . *352·14 . \supset \vdash :. \text{Hp}(1) . \supset :$

$R \epsilon \kappa_\iota . X, Y \epsilon C`H' . RXT . RYT . \vee . R \epsilon \kappa_\iota . X, Y \epsilon C`H_n . RXT . RYT :$

$[*307·1 . *350·44 . *352·13·132] \supset : X = Y$ (2)

$\vdash . *336·1 . \supset \vdash : \text{Hp} . R(\kappa_\iota \rceil A_T \upharpoonright C`H_g) X . S(\kappa_\iota \rceil A_T \upharpoonright C`H_g) X . \supset .$

$R, S \epsilon \kappa_\iota . X \epsilon C`H_g . RXT . SXT .$

$[*350·521 . *307·1] \supset . R = S$ (3)

$\vdash . (2) . (3) . \supset \vdash . \text{Prop}$

***352·16.** $\vdash : \kappa \in FM \text{ ap conx} \,.\, T \in \kappa_\partial \,.\, \supset \,.\, T_\kappa \in \text{Ser}$ [*352·12 . *304·48]

***352·17.** $\vdash : \kappa \in FM \text{ ap conx} \,.\, T \in \kappa_{\iota\partial} \,.\, \supset \,.\, T_{\kappa\iota} \in \text{Ser}$ [*352·15 . *307·45 . *304·23]

***352·18.** $\vdash : \kappa \in FM \text{ ap conx} \,.\, s\text{`Pot``}\kappa_\partial \subset \kappa_\partial \,.\, \kappa_\partial \cap \text{Cnv``}\kappa_\partial = \Lambda \,.\, T \in \kappa_\partial \,.\, \supset \,.$
$\kappa \cap A_T\text{``}C\text{`}H_n = \Lambda$

Dem.

$\vdash \,.\, *350\cdot43 \,.\, \supset$

$\vdash :. \text{Hp} \,.\, \mu, \nu \in \text{NC ind} - \iota\text{`}0 \,.\, X = (\mu/\nu) \,|\, \text{Cnv} \,.\, S \in \kappa \,.\, \supset : SXT \,.\, \equiv \,.\, S^\nu = \breve{T}^\mu \,.$

$[\text{Hp}] \qquad \supset \,.\, S^\nu \in \kappa_\partial \cap \text{Cnv``}\kappa_\partial \qquad (1)$

$\vdash \,.\, (1) \,.\, \text{Transp} \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, \sim (\exists X, S) \,.\, X \in C\text{`}H_n \,.\, S \in \kappa \,.\, SXT : \supset \vdash \,.\, \text{Prop}$

***352·181.** $\vdash : \kappa \in FM \text{ init} \,.\, T \in \kappa_\partial \,.\, \supset \,.\, \kappa \cap A_T\text{``}C\text{`}H_n = \Lambda$ [*352·18 . *335·21]

***352·2.** $\vdash : \kappa \in FM \text{ ap conx} \,.\, T \in \kappa_\partial \,.\, \supset \,.\, (I \upharpoonright s\text{`}\text{Cl``}\kappa)\, T_\kappa\, T$

Dem.

$\vdash \,.\, *350\cdot34 \,.\, *331\cdot22 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, (I \upharpoonright s\text{`}\text{Cl``}\kappa)\, 0_q\, T \qquad (1)$

$\vdash \,.\, *350\cdot31 \,. \qquad \supset \vdash : \text{Hp} \,.\, \supset \,.\, T\, (1/1)\, T \qquad (2)$

$\vdash \,.\, *304\cdot45\cdot48 \,. \qquad \supset \vdash : \text{Hp} \,.\, \supset \,.\, 0_q\, H'\, (1/1) \qquad (3)$

$\vdash \,.\, (1) \,.\, (2) \,.\, (3) \,.\, *352\cdot1 \,.\, \supset \vdash \,.\, \text{Prop}$

***352·21.** $\vdash : \kappa \in FM \text{ ap conx} \,.\, T \in \kappa_{\iota\partial} \,.\, \supset \,.\, (I \upharpoonright s\text{`}\text{Cl``}\kappa)\, T_{\kappa\iota}\, T$ [Proof as in *352·2]

***352·22.** $\vdash : \kappa \in FM \text{ ap conx} \,.\, T \in \kappa_\partial \,.\, \supset \,.\, \dot{\exists} \,!\, T_\kappa$ [*352·2]

***352·23.** $\vdash : \kappa \in FM \text{ ap conx} \,.\, T \in \kappa_{\iota\partial} \,.\, \supset \,.\, \dot{\exists} \,!\, T_{\kappa\iota}$ [*352·21]

***352·3.** $\vdash : \kappa \in FM \text{ ap conx} \,.\, T \in \kappa_\partial \,.\, \supset \,.\, C\text{`}T_\kappa = \kappa \cap A_T\text{``}C\text{`}H'$

Dem.

$\vdash \,.\, *350\cdot31 \,.\, *304\cdot48 \,.\, \supset$

$\vdash : \text{Hp} \,.\, X \in C\text{`}H' \,.\, X \neq 1/1 \,.\, \supset \,.\, X\, (H' \cup \breve{H}')\, (1/1) \,.\, T\, (1/1)\, T \,.$

$[*306\cdot1] \qquad \supset \,.\, X \in (H' \cup \breve{H}')\text{``}\breve{A}_T\text{``}\kappa \qquad (1)$

$\vdash \,.\, *350\cdot34 \,.\, *331\cdot22 \,.\, *304\cdot45\cdot48 \,.\, \supset$

$\vdash : \text{Hp} \,.\, X = 1/1 \,.\, \supset \,.\, X \breve{H}'\, 0_q \,.\, (I \upharpoonright s\text{`}\text{Cl``}\kappa)\, 0_q\, T \,.\, I \upharpoonright s\text{`}\text{Cl``}\kappa \in \kappa \,.$

$[*306\cdot1] \qquad \supset \,.\, X \in \breve{H}'\text{``}\breve{A}_T\text{``}\kappa \qquad (2)$

$\vdash \,.\, (1) \,.\, (2) \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, C\text{`}H' \subset (H' \cup \breve{H}')\text{``}\breve{A}_T\text{``}\kappa \qquad (3)$

$\vdash \,.\, *150\cdot201 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, C\text{`}T_\kappa = \kappa \,⫞\, A_T\text{``}(H' \cup \breve{H}')\text{``}\breve{A}_T\text{``}\kappa \,.$

$[(3)] \qquad \supset \,.\, \kappa \,⫞\, A_T\text{``}C\text{`}H' \subset C\text{`}T_\kappa \qquad (4)$

$\vdash \,.\, (4) \,.\, *150\cdot202 \,.\, \supset \vdash \,.\, \text{Prop}$

***352·31.** $\vdash : \kappa \in FM \text{ ap conx} \,.\, T \in \kappa_{\iota\partial} \,.\, \supset \,.\, C\text{‘}T_{\kappa\iota} = \kappa_\iota \cap A_T\text{‘‘}C\text{‘}H_g$

Dem.

As in *352·3, $\vdash : \text{Hp} \,.\, \supset \,.\, C\text{‘}H' \subset (H_g \cup \breve{H}_g)\text{‘‘}\breve{A}_T\text{‘‘}\kappa$ (1)

$\vdash \,.\, $*350·31 . (*307·05) . $\supset \vdash : \text{Hp} \,.\, X \in C\text{‘}H_n \,.\, \supset \,.\, X H_g (1/1) \,.\, T (1/1) T \,.$

[*336·1] $\supset \,.\, X \in \breve{H}_g\text{‘‘}\breve{A}_T\text{‘‘}\kappa$ (2)

$\vdash \,.\, (1) \,.\, (2) \,.$ $\supset \vdash : \text{Hp} \,.\, \supset \,.\, C\text{‘}H_g \subset (H_g \cup \breve{H}_g)\text{‘‘}\breve{A}_T\text{‘‘}\kappa$ (3)

$\vdash \,.\, (3) \,.\, $*150·201·202 . $\supset \vdash \,.\,$ Prop

***352·32.** $\vdash :. \text{Hp}$ *352·3 . $X, Y \in C\text{‘}H' \,.\, R = X \upharpoonright \kappa\text{‘}T \,.\, S = Y \upharpoonright \kappa\text{‘}T \,.\, \supset :$

$RT_\kappa S \,.\, \equiv \,.\, XH'Y$ [*352·1 . *350·521]

***352·33.** $\vdash :. \text{Hp}$ *352·31 . $X, Y \in C\text{‘}H_g \,.\, R = X \upharpoonright \kappa_\iota\text{‘}T \,.\, S = Y \upharpoonright \kappa_\iota\text{‘}T \,.\, \supset :$

$RT_{\kappa\iota} S \,.\, \equiv \,.\, XH_g Y$ [*352·11·15]

***352·34.** $\vdash :. \text{Hp}$ *352·3 . $\supset : RT_\kappa T \,.\, \equiv \,.\, (\exists X) \,.\, XH'(1/1) \,.\, R = X \upharpoonright \kappa\text{‘}T$ [*352·1 . *350·521·31]

***352·341.** $\vdash :. \text{Hp}$ *352·3 . $\supset : TT_\kappa R \,.\, \equiv \,.\, (\exists X) \,.\, (1/1) H'X \,.\, R = X \upharpoonright \kappa\text{‘}T$

***352·35.** $\vdash :. \text{Hp}$ *352·31 . $\supset : RT_{\kappa\iota} T \,.\, \equiv \,.\, (\exists X) \,.\, XH_g (1/1) \,.\, R = X \upharpoonright \kappa_\iota\text{‘}T$ [*352·11·15]

***352·351.** $\vdash :. \text{Hp}$ *352·31 . $\supset : TT_{\kappa\iota} R \,.\, \equiv \,.\, (\exists X) \,.\, (1/1) H_g X \,.\, R = X \upharpoonright \kappa_\iota\text{‘}T$

***352·36.** $\vdash : \text{Hp}$ *352·3 . $s\text{‘Pot‘‘}\kappa \subset \kappa \,.\, \supset \,.\, \text{Pot‘}T - \iota\text{‘}T \subset \overleftarrow{T_\kappa}\text{‘}T$

Dem.

$\vdash \,.\, $*350·43 . $\supset \vdash : \text{Hp} \,.\, \nu \in \text{NC ind} - \iota\text{‘}0 - \iota\text{‘}1 \,.\, \supset \,.\, T^\nu (\nu/1) T \,.$

[*304·4.*352·341] $\supset \,.\, TT_\kappa T^\nu : \supset \vdash \,.\,$ Prop

***352·37.** $\vdash : \text{Hp}$ *352·31 . $T \in \kappa \cup \text{Cnv‘‘}\kappa \,.\, \supset \,.\, \text{Pot‘}T - \iota\text{‘}T \subset \overleftarrow{T_{\kappa\iota}}\text{‘}T$

Dem.

$\vdash \,.\, $*331·24·54 . $\supset \vdash : \text{Hp} \,.\, \supset \,.\, \text{Pot‘}T \subset \kappa_\iota$

Hence as in *352·36.

***352·38.** $\vdash : \text{Hp}$ *352·31 . $\supset \,.\, \text{rep}_\kappa\text{‘‘}(\text{Pot‘}T - \iota\text{‘}T) \subset \overleftarrow{T_{\kappa\iota}}\text{‘}T$

Dem.

$\vdash \,.\, $*332·61 . $\supset \vdash : \text{Hp} \,.\, \supset \,.\, \text{rep}_\kappa\text{‘‘}(\text{Pot‘}T - \iota\text{‘}T) \subset \kappa_\iota$

Hence as in *352·36.

***352·41.** $\vdash : \kappa \in FM \text{ ap conx} \,.\, S, T \in \kappa_\partial \,.\, S \in C\text{‘}T_\kappa \,.\, \supset \,.$

$C\text{‘}S_\kappa = C\text{‘}T_\kappa = \kappa \cap A_T\text{‘‘}C\text{‘}H' = \kappa \cap A_S\text{‘‘}C\text{‘}H$

Dem.

$\vdash \,.\, $*352·3 . *350·43 . $\supset \vdash : \text{Hp} \,.\, \supset \,.\, (\exists \mu, \nu) \,.\, \mu, \nu \in \text{NC ind} - \iota\text{‘}0 \,.\, S^\mu = T^\nu \,.$ (1)

[*352·3] $\supset \,.\, T \in C\text{‘}S_\kappa$ (2)

$\vdash . (1) . *352\cdot3 . *350\cdot43 . \supset \vdash : \text{Hp} . R \in C`S_\kappa . \supset .$

$(\exists \mu, \nu, \rho, \sigma) . \mu, \nu, \sigma \in \text{NC ind} - \iota`0 . \rho \in \text{NC ind} . S^\mu = T^\nu . R^\sigma = S^\rho .$

[*301·504] $\supset . (\exists \mu, \nu, \rho, \sigma) . \mu, \nu, \sigma \in \text{NC ind} - \iota`0 . \rho \in \text{NC ind} . R^{\sigma \times_c \mu} = T^{\nu \times_c \rho} .$

[*352·3.*350·43] $\supset . R \in C`T_\kappa$ (3)

$\vdash . (2) . (3) \frac{T, S}{S, T} . \supset \vdash : \text{Hp} . R \in C`T_\kappa . \supset . R \in C`S_\kappa$ (4)

$\vdash . (3) . (4) . *352\cdot3 . \supset \vdash . \text{Prop}$

***352·42.** $\vdash : \kappa \in FM \text{ ap conx} . S, T \in \kappa_{\iota \partial} . S \in C`T_{\kappa\iota} . \supset . C`S_{\kappa\iota} = C`T_{\kappa\iota}$

Dem.

$\vdash . *352\cdot3 . *350\cdot4 . *307\cdot1 . \supset$

$\vdash :. \text{Hp} . \supset : (\exists \mu, \nu) : \mu, \nu \in \text{NC ind} - \iota`0 : \dot{\exists} ! S^\nu \dot{\wedge} T^\mu . \vee . \dot{\exists} ! \breve{S}^\nu \dot{\wedge} T^\mu :$ (1)

[*352·31] $\supset : T \in C`S_{\kappa\iota}$ (2)

$\vdash . (1) . *352\cdot3 . *350\cdot4 . *307\cdot1 . \supset$

$\vdash :. \text{Hp} . R \in C`S_{\kappa\iota} . \supset : (\exists \mu, \nu, \rho, \sigma) : \mu, \nu, \sigma \in \text{NC ind} - \iota`0 . \rho \in \text{NC ind} :$

$\dot{\exists} ! S^\nu \dot{\wedge} T^\mu . \vee . \dot{\exists} ! \breve{S}^\nu \dot{\wedge} T^\mu : \dot{\exists} ! R^\sigma \dot{\wedge} S^\rho . \vee . \dot{\exists} ! \breve{R}^\sigma \dot{\wedge} S^\rho :$

[*333·48] $\supset : (\exists \mu, \nu, \rho, \sigma) : \mu, \nu, \sigma \in \text{NC ind} - \iota`0 . \rho \in \text{NC ind} :$

$\dot{\exists} ! R^{\sigma \times_c \mu} \dot{\wedge} T^{\nu \times_c \rho} . \vee . \dot{\exists} ! \breve{R}^{\sigma \times_c \mu} \dot{\wedge} T^{\nu \times_c \rho} :$

[*352·31] $\supset : R \in C`T_{\kappa\iota}$ (3)

$\vdash . (2) . (3) \frac{T, S}{S, T} . \supset \vdash : \text{Hp} . R \in C`T_{\kappa\iota} . \supset . R \in C`S_{\kappa\iota}$ (4)

$\vdash . (3) . (4) . \supset \vdash . \text{Prop}$

***352·43.** $\vdash :: \kappa \in FM \text{ ap conx} . T \in \kappa_\partial . \supset :.$

$RT_\kappa S . \equiv : R, S \in \kappa \cap A_T``C`H' : (\exists \mu, \nu) . \mu, \nu \in \text{NC ind} . \mu < \nu . R^\nu = S^\mu$

Dem.

$\vdash . *33\cdot17 . \supset \vdash : RT_\kappa S . \equiv . R, S \in C`T_\kappa . RT_\kappa S$ (1)

$\vdash . (1) . *352\cdot31 . *350\cdot43 . \supset \vdash :: \text{Hp} . \supset :.$

$RT_\kappa S . \equiv : R, S \in \kappa \cap A_T``C`H' : (\exists \rho, \sigma, \xi, \eta) . \sigma, \xi, \eta \in \text{NC ind} - \iota`0 . \rho \in \text{NC ind} .$

$\rho \times_c \eta < \sigma \times_c \xi . R^\sigma = T^\rho . S^\eta = T^\xi :$

[*333·5] $\equiv : R, S \in \kappa \cap A_T``C`H' : (\exists \rho, \sigma, \xi, \eta) . \sigma, \xi, \eta \in \text{NC ind} - \iota`0 . \rho \in \text{NC ind} .$

$\rho \times_c \eta < \sigma \times_c \xi . R^{\sigma \times_c \xi} = T^{\rho \times_c \xi} = S^{\rho \times_c \eta} :$

[*126·14] $\supset : R, S \in \kappa \cap A_T``C`H' : (\exists \mu, \nu) . \mu, \nu \in \text{NC ind} . \mu < \nu . R^\nu = S^\mu$ (2)

$\vdash . *350\cdot43 . *304\cdot4 . \supset$

$\vdash :. R, S \in \kappa \cap A_T``C`H' : (\exists \mu, \nu) . \mu, \nu \in \text{NC ind} . \mu < \nu . R^\nu = S^\mu : \supset :$

$R, S \in \kappa \cap A_T``C`H' : (\exists X) . XH' (1/1) . RXS :$

[*336·1] $\supset : R, S \in \kappa : (\exists X, Y, Z) . XH' (1/1) . Y, Z \in C`H' . RXS . RYT . SZT :$

[*350·6.*305·71·51] $\supset : R, S \in \kappa : (\exists X, Z) . (X \times_s Z) H'Z . R (X \times_s Z) T . SZT :$

[*352·1] $\supset : RT_\kappa S$ (3)

$\vdash . (2) . (3) . \supset \vdash . \text{Prop}$

***352·44.** $\vdash : \kappa \epsilon FM \text{ ap conx} . T \epsilon \kappa_\partial . \supset . T_\kappa = \{\dot{s}`\overrightarrow{H'}`(1/1)\} \upharpoonright (\kappa \cap A_T``C`H')$

Dem.

$\vdash . *352·43 . *304·4 . \supset \vdash :: \text{Hp} . \supset :.$

$RT_\kappa S . \equiv : R, S \epsilon \kappa \cap A_T``C`H' : (\exists X) . XH'(1/1) . RXS :: \supset \vdash . \text{Prop}$

***352·45.** $\vdash : \kappa \epsilon FM \text{ ap conx} . S, T \epsilon \kappa_\partial . S \epsilon C`T_\kappa . \supset . S_\kappa = T_\kappa$ [*352·44·41]

***352·5.** $\vdash : \kappa \epsilon FM \text{ ap conx} . T \epsilon \kappa_{\iota\partial} . \supset . C`\kappa_\iota \upharpoonright A_T ; H' = \kappa_\iota \cap A_T``C`H'$

[Proof as in *352·3]

***352·51.** $\vdash : \kappa \epsilon FM \text{ ap conx} . T \epsilon \kappa_{\iota\partial} . \supset . C`\kappa_\iota \upharpoonright A_T ; H_n = \kappa_\iota \cap A_T``C`H_n$

Dem.

$\vdash . *150·202 . \supset \vdash : \text{Hp} . \supset . C`\kappa_\iota \upharpoonright A_T ; H_n \subset \kappa_\iota \cap A_T``C`H_n$ (1)

$\vdash . *352·131 . \supset \vdash : \text{Hp} . R \epsilon \kappa_\iota \cap A_T``C`H_n . \supset . (\exists X) . X \epsilon C`H . \breve{R} \epsilon \kappa_\iota . \breve{R}XT$ (2)

$\vdash . *304·23 . \supset \vdash : \text{Hp} . X \epsilon C`H - \iota`(1/1) . \breve{R} \epsilon \kappa_\iota . \breve{R}XT . \supset .$

$X (H \cup \breve{H})(1/1) . \breve{R} \epsilon \kappa_\iota . \breve{R}XT . T(1/1)T .$

[*307·1.*336·1] $\supset . R \epsilon C`\kappa_\iota \upharpoonright A_T ; H_n$ (3)

$\vdash . *352·38 . \supset \vdash : \text{Hp} . X = 1/1 . \breve{R} \epsilon \kappa_\iota . \breve{R}XT . \supset . \breve{R}(\kappa_\iota \upharpoonright A_T ; H)(\text{rep}_\kappa`T^2) .$

[*307·1] $\supset . R \epsilon C`\kappa_\iota \upharpoonright A_T ; H_n$ (4)

$\vdash . (3) . (4) . \supset \vdash : \text{Hp} . X \epsilon C`H . \breve{R} \epsilon \kappa_\iota . \breve{R}XT . \supset . R \epsilon C`\kappa_\iota \upharpoonright A_T ; H_n$ (5)

$\vdash . (2) . (5) . \supset \vdash : \text{Hp} . \supset . \kappa_\iota \cap A_T``C`H_n \subset C`\kappa_\iota \upharpoonright A_T ; H_n$ (6)

$\vdash . (1) . (6) . \supset \vdash . \text{Prop}$

***352·52.** $\vdash : \kappa \epsilon FM \text{ ap conx} . T \epsilon \kappa_{\iota\partial} . \supset . T_{\kappa\iota} = \kappa_\iota \upharpoonright A_T ; \breve{H}_n \dagger \kappa_\iota \upharpoonright A_T ; H'$

$= \text{Cnv} ; \kappa_\iota \upharpoonright A_T ; \breve{H} \dagger \kappa_\iota \upharpoonright A_T ; H'$

Dem.

$\vdash . *160·43 . (*307·05) . \supset$

$\vdash . T_{\kappa\iota} = \kappa_\iota \upharpoonright A_T ; \breve{H}_n \cup \kappa_\iota \upharpoonright A_T ; \dot{H}' \cup (\kappa_\iota \upharpoonright A_T``C`H_n) \uparrow (\kappa_\iota \upharpoonright A_T``C`H')$ (1)

$\vdash . (1) . *352·5·51 . *307·1 . \supset \vdash . \text{Prop}$

***352·53.** $\vdash : \kappa \epsilon FM \text{ ap conx} . T \epsilon \kappa_{\iota\partial} . \supset .$

$\kappa_\iota \upharpoonright A_T ; H' = \{\dot{s}`\overrightarrow{H'}`(1/1)\} \upharpoonright (\kappa_\iota \cap A_T``C`H')$ [Proof as in *352·44]

***352·531.** $\vdash : \text{Hp} *352·53 . \supset . \kappa_\iota \upharpoonright A_T ; \breve{H} = \{\dot{s}`\overleftarrow{H}`(1/1)\} \upharpoonright (\kappa_\iota \cap A_T``C`H)$

[Proof as in *352·44]

***352·54.** $\vdash : \text{Hp} *352·53 . \supset . T_{\kappa\iota} = \text{Cnv} ; \{\dot{s}`\overleftarrow{H}`(1/1)\} \upharpoonright (\kappa_\iota \cap A_T``C`H) \dagger$

$\{\dot{s}`\overrightarrow{H'}`(1/1)\} \upharpoonright (\kappa_\iota \cap A_T``C`H')$ [*352·52·53·531]

***352·55.** $\vdash : \kappa \epsilon FM \text{ ap conx} . S, T \epsilon \kappa_{\iota\partial} . S \epsilon \kappa_\iota \cap A_T``C`H . \supset .$

$\kappa_\iota \cap A_S``C`H' = \kappa_\iota \cap A_T``C`H' . \kappa_\iota \cap A_S``C`H = \kappa_\iota \cap A_T``C`H$

[Proof as in *352·41]

***352·56.** $\vdash : \kappa \epsilon FM \text{ ap conx} . S, T \epsilon \kappa_{\iota\partial} . S \epsilon \kappa_\iota \cap A_T``C`H . \supset . S_{\kappa\iota} = T_{\kappa\iota}$
[*352·54·55]

***352·57.** $\vdash : \kappa \epsilon FM \text{ ap conx} . S, T \epsilon \kappa_{\iota\partial} . S \epsilon \kappa_\iota \cap A_T``C`H_n . \supset . S_{\kappa\iota} = \breve{T}_{\kappa\iota}$
[*352·54·55 . *307·1]

***352·7.** $\vdash :. \kappa \epsilon FM \text{ sr} . X, Y \epsilon C`H' . T \epsilon \kappa_\partial . P, Q \epsilon \kappa . PXT . QYT . \supset :$

$$PU_\kappa Q . \equiv . X\breve{H}'Y$$

Dem.

$\vdash . *352\cdot18 . \quad \supset \vdash : \text{Hp} . \breve{Q} | P \epsilon \kappa_\partial . \supset . \breve{Q} | P \sim \epsilon A_T``C`H_n .$

$[*350\cdot65] \quad \supset . X -_s Y \epsilon C`H' \quad (1)$

$\vdash . *350\cdot52 . \quad \supset \vdash : \text{Hp}(1) . \supset . X \neq Y \quad (2)$

$\vdash . (1) . (2) . *336\cdot41 . \supset \vdash : \text{Hp} . PU_\kappa Q . \supset . X -_s Y \epsilon C`H .$

$[*308\cdot12\cdot19.\text{Transp}] \quad \supset . X\breve{H}'Y \quad (3)$

$\vdash . *336\cdot64 . \quad \supset \vdash :. \text{Hp} . \sim(PU_\kappa Q) . \supset : P = Q . \text{v} . QU_\kappa P :$

$[*350\cdot44.(3)] \quad \supset : X = Y . \text{v} . Y\breve{H}'X :$

$[*304\cdot48] \quad \supset : \sim(X\breve{H}'Y) \quad (4)$

$\vdash . (3) . (4) . \supset \vdash .$ Prop

***352·71.** $\vdash :. \kappa \epsilon FM \text{ sr} . T \epsilon \kappa_\partial . P, Q \epsilon C`T_\kappa . \supset : PU_\kappa Q . \equiv . P(A_T \mathbin{;} \breve{H}')Q$
[*352·7·3]

***352·72.** $\vdash : \kappa \epsilon FM \text{ sr} . T \epsilon \kappa_\partial . \supset . U_\kappa \upharpoonright C`T_\kappa = \kappa \upharpoonleft A_T \mathbin{;} \breve{H}' = \breve{T}_\kappa$ [*352·71]

***352·73.** $\vdash :. \kappa \epsilon FM \text{ sr subm} . X, Y \epsilon C`H' . T \epsilon \kappa_\partial . \supset :$

$$(X \upharpoonright \kappa`T) U_\kappa (Y \upharpoonright \kappa`T) . \equiv . X\breve{H}'Y \quad [*352\cdot7 . *351\cdot22]$$

$*353$. RATIONAL FAMILIES.

Summary of $*353$.

A "rational family" is one which consists entirely of positive rational multiples of one of its members. We denote rational families by "$FM\,\mathrm{rt}$"; the definition is

$*353{\cdot}01$. $FM\,\mathrm{rt} = FM \cap \hat{\kappa}\{(\exists T) . T \epsilon \kappa_\partial . \kappa \subset A_T``C`H'\}$ Df

It is obvious that, if κ is any family, $\kappa \cap A_T``C`H'$, which we considered in the last number, is a rational family. If $\kappa$ is a connected family, it does not follow that $\kappa \cap A_T``C`H'$ is a connected family, but the proofs of its properties, as we saw in $*352$, make use of the fact that it is contained in a connected family. Many of the most important properties of connected families hold equally of sub-classes of connected families, notably the property that two members of κ or κ_ι whose logical product exists are identical ($*331{\cdot}42{\cdot}24$). In dealing with rational families, a good many propositions can be proved by merely assuming that they are contained in connected families. We put

$*353{\cdot}02$. $FM\,\mathrm{cx} = FM \cap \hat{\lambda}\{(\exists\kappa) . \kappa \epsilon FM\,\mathrm{conx} . \lambda \subset \kappa\}$ Df

$*353{\cdot}03$. $FM\,\mathrm{rt}\,\mathrm{cx} = FM\,\mathrm{rt} \cap FM\,\mathrm{cx}$ Df

We will call a family "sub-connected" when it is contained in a connected family. When a family κ is open, rational, and sub-connected, any member of κ_∂ may be taken as the T of the definition $*353{\cdot}01$ (this is proved in $*353{\cdot}13$); and if S, T are any two members of κ_∂, some power of S will be identical with some power of T ($*353{\cdot}12$). An open rational sub-connected family is asymmetrical ($*353{\cdot}2$); no power of a member, and no product of two members, is the converse of a non-zero member ($*353{\cdot}22{\cdot}23$). Hence by $*331{\cdot}54{\cdot}33$, if the family is connected, and not merely sub-connected, it is a group and transitive ($*353{\cdot}25{\cdot}27$).

If λ is a family which, besides being open and rational, has connexity, then if a is a member of the field and $T \epsilon \kappa_\partial$ we shall have

$$\dot{s}`\lambda_\partial = A_a ; \lambda \upharpoonright A_T ; \breve{H}' . U_\lambda = \lambda \upharpoonright A_T ; \breve{H}' \quad (*353{\cdot}32{\cdot}33).$$

That is, the series of points in the field and the series of vectors are both

ordinally similar to part or the whole of the series of ratios; they will be similar to the whole if λ is submultipliable (*353·44). But when λ is submultipliable, a smaller hypothesis suffices, for in that case we can prove that if λ is connected, then $\lambda_\iota = \lambda \cup \mathrm{Cnv}``\lambda$ (*353·41), so that λ has connexity, and is serial (*353·42). Thus we have

***353·44.** $\vdash : \lambda \in FM \,\mathrm{ap\, conx\, rt\, subm} \,.\supset.\, s`\lambda_\partial \,\mathrm{smor}\, \breve{H}'$

***353·45.** $\vdash .\, FM \,\mathrm{ap\, conx\, rt\, subm} \subset FM\,\mathrm{sr}$

***353·01.** $FM\,\mathrm{rt} = FM \cap \hat{\kappa}\{(\exists T) \,.\, T \in \kappa_\partial \,.\, \kappa \subset A_T``C`H'\}$ Df

***353·02.** $FM\,\mathrm{cx} = FM \cap \hat{\lambda}\{(\exists \kappa) \,.\, \kappa \in FM\,\mathrm{conx} \,.\, \lambda \subset \kappa\}$ Df

***353·03.** $FM\,\mathrm{rt\,cx} = FM\,\mathrm{rt} \cap FM\,\mathrm{cx}$ Df

***353·1.** $\vdash :.\, \kappa \in FM\,\mathrm{rt} \,.\equiv:\, \kappa \in FM : (\exists T) \,.\, T \in \kappa_\partial \,.\, \kappa \subset A_T``C`H'$ [(*353·01)]

***353·12.** $\vdash : \lambda \in FM\,\mathrm{ap\,rt\,cx} \,.\, S, T \in \lambda_\partial \,.\, \lambda \subset A_T``C`H' \,.\supset.$
$(\exists \mu, \nu) \,.\, \mu, \nu \in \mathrm{NC\,ind} \,.\, \nu \neq 0 \,.\, S^\nu = T^\mu$ [*350·43]

***353·13.** $\vdash : \lambda \in FM\,\mathrm{ap\,rt\,cx} \,.\, T \in \lambda_\partial \,.\supset.\, \lambda \subset A_T``C`H'$

Dem.

$\vdash .\, *353·12 \,.\supset\vdash : \mathrm{Hp} \,.\, S \in \lambda_\partial \,.\, \lambda \subset A_S``C`H' \,.\, R \in \lambda \,.\supset.$
$(\exists \mu, \nu, \rho, \sigma) \,.\, \mu, \nu, \rho, \sigma \in \mathrm{NC\,ind} \,.\, \rho \neq 0 \,.\, \nu \neq 0 \,.\, \sigma \neq 0 \,.\, R^\nu = S^\mu \,.\, T^\sigma = S^\rho \,.$
[*333·5]
$\supset .\, (\exists \mu, \nu, \rho, \sigma) \,.\, \mu, \nu, \rho, \sigma \in \mathrm{NC\,ind} \,.\, \rho \neq 0 \,.\, \nu \neq 0 \,.\, \sigma \neq 0 \,.\, R^{\nu \times_c \rho} = S^{\mu \times_c \rho} = T^{\mu \times_c \sigma} \,.$
[*350·43] $\supset .\, R \in A_T``C`H' : \supset \vdash .\, \mathrm{Prop}$

***353·14.** $\vdash : \mathrm{Hp}\, *353·13 \,.\supset.\, \lambda_\iota \subset A_T``C`H_g$

Dem.

$\vdash .\, *353·13 \,.\supset\vdash : \mathrm{Hp} \,.\, R, S \in \lambda \,.\supset.\, (\exists X, Y) \,.\, X, Y \in C`H' \,.\, RXT \,.\, SYT \,.$
[*350·65] $\supset .\, (\breve{R} \,|\, S)(Y -_s X)\, T \,.$
[*308·2] $\supset .\, \breve{R} \,|\, S \in A_T``C`H_g : \supset \vdash .\, \mathrm{Prop}$

***353·15.** $\vdash : \kappa \in FM\,\mathrm{conx} \,.\, T \in \kappa_\partial \,.\supset.\, \kappa \cap A_T``C`H' \in FM\,\mathrm{rt\,cx}$
[*353·1 . (*353·02)]

***353·2.** $\vdash : \lambda \in FM\,\mathrm{ap\,rt\,cx} \,.\supset.\, \lambda_\partial \cap \mathrm{Cnv}``\lambda_\partial = \Lambda \,.\, \lambda \in FM\,\mathrm{asym}$

Dem.

$\vdash .\, *353·12·13 \,.\supset$

$\vdash : \mathrm{Hp} \,.\, R, \breve{R} \in \lambda_\partial \,.\supset.\, (\exists \mu, \nu) \,.\, \mu, \nu \in \mathrm{NC\,ind} - \iota`0 \,.\, R^\mu = \breve{R}^\nu$ (1)

$\vdash .\, (1) \,.\, *301·23 \,.\supset\vdash : \mathrm{Hp}(1) \,.\supset.\, (\exists \mu, \nu) \,.\, \mu, \nu \in \mathrm{NC\,ind} - \iota`0 \,.\, R^{\mu +_c \nu} \in I$ (2)

$\vdash .\, *333·101 \,. \quad \supset\vdash : \mathrm{Hp}(1) \,.\supset.\, \mathrm{Pot}`R \subset \mathrm{Rl}`J$ (3)

$\vdash .\, (3) \,.\, (2) \,.\, \mathrm{Transp} \,.\, (*334·05) \,.\supset\vdash .\, \mathrm{Prop}$

$*353{\cdot}22$. $\vdash : \text{Hp} *353{\cdot}2 . \supset . s`\text{Pot}``\lambda_\partial \cap \text{Cnv}``\lambda_\partial = \Lambda$

Dem.

$\vdash . *353{\cdot}12{\cdot}13 . *301{\cdot}5 . \supset \vdash : \text{Hp} . \sigma \in \text{NC ind} - \iota`0 . R, \breve{R}^\sigma \in \lambda_\partial . \supset .$

$(\exists \mu, \nu) . \mu, \nu \in \text{NC ind} - \iota`0 . \breve{R}^{\sigma \times_c \nu} = R^\mu .$

$[*301{\cdot}23] \supset . (\exists \mu, \nu) . \mu, \nu \in \text{NC ind} - \iota`0 . R^{\mu +_c (\sigma \times_c \nu)} \in I$ (1)

$\vdash . *333{\cdot}101 . *330{\cdot}23 . \supset \vdash : \text{Hp} . R \in \lambda_\partial . \supset . \text{Pot}`R \in J . \dot{\Lambda} \sim \in \text{Pot}`R$ (2)

$\vdash . (2) . (1) . \text{Transp} . \supset \vdash : \text{Hp} . R \in \lambda_\partial . \supset . \sim \exists ! \text{Pot}`R \cap \text{Cnv}``\lambda : \supset \vdash . \text{Prop}$

$*353{\cdot}23$. $\vdash : \text{Hp} *353{\cdot}2 . \supset . (s`\lambda \underset{\prime\prime}{|} ``\lambda) \cap \text{Cnv}``\lambda_\partial = \Lambda$ [Proof as in $*353{\cdot}22$]

$*353{\cdot}24$. $\vdash : \text{Hp} *353{\cdot}2 . \lambda \in FM \text{ conx} . \supset . s`\text{Pot}``\lambda \subset \lambda$ [$*353{\cdot}22 . *331{\cdot}54$]

$*353{\cdot}25$. $\vdash : \text{Hp} *353{\cdot}24 . \supset . s`\lambda \underset{\prime\prime}{|} ``\lambda \subset \lambda$ [$*353{\cdot}23 . *331{\cdot}33$]

$*353{\cdot}26$. $\vdash : \text{Hp} *353{\cdot}24 . \supset . s`\lambda_\partial \underset{\prime\prime}{|} ``\lambda_\partial \subset \lambda_\partial$

Dem.

$\vdash . *353{\cdot}12{\cdot}13 . \supset \vdash : \text{Hp} . R, S \in \lambda_\partial . \supset . (\exists \mu, \nu) . \mu, \nu \in \text{NC ind} - \iota`0 . R^\nu = S^\mu .$

$[*330{\cdot}57] \quad \supset . (\exists \mu, \nu) . \mu, \nu \in \text{NC ind} - \iota`0 . (R | S)^\nu = S^{\mu +_c \nu} .$

$[*333{\cdot}101] \quad \supset . \exists ! \text{Pot}`(R | S)^\nu \cap \text{Rl}`J .$

$[*301{\cdot}3 . \text{Transp} . *331{\cdot}23] \quad \supset . R | S \in \text{Rl}`J$ (1)

$\vdash . (1) . *353{\cdot}25 . \supset \vdash . \text{Prop}$

$*353{\cdot}27$. $\vdash : \text{Hp} *353{\cdot}24 . \supset . \lambda \in FM \text{ trs asym}$ [$*353{\cdot}26{\cdot}2 . *334{\cdot}13$]

$*353{\cdot}3$. $\vdash :. \text{Hp} *353{\cdot}2 . \nu \in \text{NC ind} - \iota`0 . s`\text{Pot}``\lambda \subset \lambda . \supset : R U_\lambda S . \supset . R^\nu U_\lambda S^\nu$

Dem.

$\vdash . *336{\cdot}41 . \supset \vdash : \text{Hp} . \supset . (\exists T) . T \in \lambda_\partial . R = T | S .$

$[*330{\cdot}57] \quad \supset . (\exists T) . T \in \lambda_\partial . R^\nu = T^\nu | S^\nu .$

$[*336{\cdot}41 . \text{Hp}] \quad \supset . R^\nu U_\lambda S^\nu : \supset \vdash . \text{Prop}$

$*353{\cdot}31$. $\vdash :. \lambda \in FM \text{ ap rt connex} . R, S \in \lambda . \nu \in \text{NC ind} - \iota`0 . \supset :$

$R U_\lambda S . \equiv . R^\nu U_\lambda S^\nu$

Dem.

$\vdash . *336{\cdot}62 . \supset \vdash : \text{Hp} . R \neq S . \sim (R U_\lambda S) . \supset . S U_\lambda R .$

$[*353{\cdot}3{\cdot}24] \quad \supset . S^\nu U_\lambda R^\nu .$

$[*336{\cdot}6{\cdot}61 . *353{\cdot}27] \quad \supset . \sim (R^\nu U_\lambda S^\nu)$ (1)

$\vdash . *336{\cdot}6 . \supset \vdash : \text{Hp} . R = S . \supset . \sim (R^\nu U_\lambda S^\nu)$ (2)

$\vdash . (1) . (2) . \supset \vdash : \text{Hp} . \sim (R U_\lambda S) . \supset . \sim (R^\nu U_\lambda S^\nu)$ (3)

$\vdash . (3) . *353{\cdot}3 . \supset \vdash . \text{Prop}$

***353·32.** $\vdash : \lambda \epsilon FM \text{ ap rt connex} . T \epsilon \lambda_\partial . \supset . U_\lambda = \lambda \upharpoonright A_T ; \breve{H}'$

Dem.

$\vdash . *353{\cdot}12{\cdot}13 . *350{\cdot}5 . \supset \vdash : \text{Hp} . R, S \epsilon \lambda . R \neq S . \supset .$

$(\exists \mu, \nu, \rho, \sigma) . \mu, \nu, \rho, \sigma \epsilon \text{NC ind} . \nu \neq 0 . \sigma \neq 0 . R^\nu = T^\mu . S^\sigma = T^\rho . \mu/\nu \neq \rho/\sigma$ (1)

$\vdash . (1) . *350{\cdot}43 . \supset \vdash :. \text{Hp}(1) . \supset : R(\lambda \upharpoonright A_T ; H') S . \vee . S(\lambda \upharpoonright A_T ; H') R$ (2)

$\vdash . *301{\cdot}5 . \supset$

$\vdash : \text{Hp}(1) . \mu, \nu, \rho, \sigma \epsilon \text{NC ind} . \nu \neq 0 . \sigma \neq 0 . R^\nu = T^\mu . S^\sigma = T^\rho . \mu \times_c \sigma < \nu \times_c \rho . \supset .$

$\breve{R}^{\nu \times_c \sigma} | S^{\nu \times_c \sigma} = T^{(\nu \times_c \rho) -_c (\mu \times_c \sigma)}$ (3)

$\vdash . *334{\cdot}21 . \supset \vdash : \text{Hp}(3) . \supset . \breve{R} | S \epsilon \lambda \cup \text{Cnv}``\lambda .$

$[*331{\cdot}54 . *332{\cdot}241] \quad \supset . (\breve{R} | S)^{\nu \times_c \sigma} = \text{rep}_\kappa`(\breve{R} | S)^{\nu \times_c \sigma}$

$[*332{\cdot}53 . (3)] \quad = T^{(\nu \times_c \rho) -_c (\mu \times_c \sigma)}$ (4)

$\vdash . (4) . *353{\cdot}24{\cdot}2 . \supset \vdash : \text{Hp}(3) . \supset . \breve{R} | S \epsilon \lambda .$

$[*336{\cdot}41] \quad \supset . SU_\kappa R$ (5)

$\vdash . (1) . (5) . *304{\cdot}4 . \supset \vdash : \text{Hp} . R(\lambda \upharpoonright A_T ; H') S . \supset . SU_\kappa R$ (6)

$\vdash . (2) . *304{\cdot}4 . \supset \vdash : \text{Hp}(1) . \sim \{R(\lambda \upharpoonright A_T ; H') S\} . \supset . S(\lambda \upharpoonright A_T ; H') R$

$[(6)] \quad \supset . RU_\kappa S .$

$[*336{\cdot}6{\cdot}61 . *353{\cdot}27] \quad \supset . \sim (SU_\kappa R)$ (7)

$\vdash . *336{\cdot}6 . \supset \vdash : \text{Hp} . R = S . \supset . \sim (SU_\kappa R)$ (8)

$\vdash . (6) . (7) . (8) . \supset \vdash . \text{Prop}$

***353·33.** $\vdash : \text{Hp } *353{\cdot}32 . a \epsilon s`\text{C}``\lambda . \supset . \dot{s}`\lambda_\partial = A_a ; \lambda \upharpoonright A_T ; \breve{H}'$

Dem.

$\vdash . *336{\cdot}43 . \quad \supset \vdash : \text{Hp} . \supset . U_\lambda = \lambda \upharpoonright \breve{A}_a ; \dot{s}`\lambda_\partial$ (1)

$\vdash . (1) . *336{\cdot}2 . \quad \supset \vdash : \text{Hp} . \supset . \dot{s}`\lambda_\partial = A_a ; U_\lambda$ (2)

$\vdash . (2) . *353{\cdot}32 . \supset \vdash . \text{Prop}$

***353·34.** $\vdash . FM \text{ ap rt connex} \subset FM \text{ sr} \quad [*353{\cdot}27]$

***353·4.** $\vdash : \lambda \epsilon FM \text{ ap rt cx} . s`\text{Pot}``\lambda \subset \lambda . L \epsilon \lambda_{\iota\partial} . \supset .$

$(\exists \sigma) . \sigma \epsilon \text{NC ind} - \iota`0 . \text{rep}_\lambda`L^\sigma \epsilon \lambda \cup \text{Cnv}``\lambda$

Dem.

$\vdash . *353{\cdot}12{\cdot}13 . \supset$

$\vdash : \text{Hp} . \supset . (\exists \mu, \nu, R, S) . \mu, \nu \epsilon \text{NC ind} . R, S \epsilon \lambda . L = \breve{R} | S . \mu \neq \nu . R^\nu = S^\mu$ (1)

$\vdash . *301{\cdot}23 . \supset$

$\vdash :. \text{Hp} . \mu, \nu \epsilon \text{NC ind} . R, S \epsilon \lambda . R^\nu = S^\mu . \supset : \mu < \nu . \supset . \breve{R}^\nu | S^\nu = S^{\nu -_c \mu} .$

$[*332{\cdot}53] \quad \supset . \text{rep}_\kappa`(\breve{R} | S)^\nu \epsilon \lambda$ (2)

Similarly $\vdash :. \text{Hp}(2) . \supset : \mu > \nu . \supset . \text{rep}_\kappa`(\breve{R} | S)^\mu \epsilon \text{Cnv}``\kappa$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash . \text{Prop}$

***353·41.** $\vdash : \lambda \in FM$ ap conx rt subm $. \supset . \lambda_\iota = \lambda \cup \mathrm{Cnv}``\lambda$

Dem.

$\vdash . *353·4 . \supset$

$\vdash : \mathrm{Hp} . L \in \lambda_{\iota\partial} . \supset . (\exists R, \sigma) . R \in \lambda \cup \mathrm{Cnv}``\lambda . \sigma \in \mathrm{NC\,ind} - \iota`0 . \mathrm{rep}_\kappa`L^\sigma = R^\sigma .$

[*333·41] $\supset . L \in \lambda \cup \mathrm{Cnv}``\lambda : \supset \vdash .$ Prop

***353·42.** $\vdash : \mathrm{Hp}\, *353·41 . \supset . \lambda \in FM$ sr [*353·41 . *334·26 . *353·27]

***353·43.** $\vdash : \lambda \in FM$ ap cx rt subm $. T \in \lambda_\partial .$ Potid$`T \subset \lambda . \supset . C`H' \subset \breve{A}_T``\lambda$

Dem.

$\vdash . *351·1 . \supset \vdash : \mathrm{Hp} . \mu, \nu \in \mathrm{NC\,ind} . \nu \neq 0 . \supset . (\exists S) . S \in \lambda . S^\nu = T^\mu .$

[*350·43] $\supset . (\exists S) . S \in \lambda . S(\mu/\nu) T$ (1)

$\vdash . (1) . *336·1 . \supset \vdash : \mathrm{Hp} . X \in C`H' . \supset . (\exists S) . S \in \lambda . SA_T X : \supset \vdash .$ Prop

***353·44.** $\vdash : \lambda \in FM$ ap conx rt subm $. \supset . \dot{s}`\lambda_\partial$ smor $\breve{H}'$

Dem.

$\vdash . *353·42·33 . \supset \vdash : \mathrm{Hp} . a \in s`\mathrm{G}``\lambda . \supset . \dot{s}`\lambda_\partial = A_a ; \lambda \upharpoonright A_T ; \breve{H}'$ (1)

$\vdash . *353·43 . \supset \vdash : \mathrm{Hp}\,(1) . \supset . C`H' \subset \mathrm{G}`(A_a | \lambda \upharpoonright A_T)$ (2)

$\vdash . *336·2 . *352·15 . \supset \vdash : \mathrm{Hp}\,(1) . \supset . A_a | \lambda \upharpoonright A_T \upharpoonright C`H' \in 1 \to 1$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash .$ Prop

***353·45.** $\vdash . FM$ ap conx rt subm $\subset FM$ sr [*353·42]

*354. RATIONAL NETS.

Summary of *354.

The subject of "rational nets," which is to be considered in this number, is of importance for the introduction of coordinates in geometry. We have three stages in the construction of a rational net. First, taking any vector T in a family κ, we construct $C'T_\kappa$, *i.e.* the positive rational multiples of T, as in *352. The result is, as a rule, a family which is not connected, even when the family κ is connected. For if there are in κ any vectors other than $C'T_\kappa$, any point of the field which is reached from a given point a by one of these "irrational" vectors cannot be reached from a by a member of $C'T_\kappa$, though it will be in the field of $C'T_\kappa$. Thus in order to obtain from $C'T_\kappa$ a connected family, we shall have to limit the fields of its members to the points which can be reached from a given point a by one or more rational steps backwards or forwards, *i.e.* to the points $A_a``(C'T_\kappa)_\iota$. It will be observed that whereas, in the construction of $C'T_\kappa$, only positive vectors are used, negative vectors, *i.e.* the converses of positive vectors, are also admitted in constructing what we may call the "rational points" with respect to a and T. Having constructed these points, *i.e.* the class $A_a``(C'T_\kappa)_\iota$ we then proceed to the third and last stage in constructing a rational net, by limiting the field of every member of $C'T_\kappa$ to $A_a``(C'T_\kappa)_\iota$.

Many of the propositions concerning rational nets require the hypothesis that the family concerned is a group. If this is not the case with the family κ from which we start, we replace κ by κ_g, where κ_g is formed by adding to κ the converses of those members of κ (if any) whose domains are identical with the common converse domain of members of κ. The definition is

***354·01.** $\kappa_g = \kappa \cup \mathrm{Cnv}``(\kappa \cap \overleftarrow{\mathrm{D}}`s`\mathrm{G}``\kappa)$ Df

We put also

***354·03.** $FM\,\mathrm{grp} = FM \cap \hat{\kappa}(s`\kappa \,|\, ``\kappa \subset \kappa)$ Df

We then easily prove that if κ is connected, κ_g is a group (*354·14), and if κ is open and connected, κ_g is open and connected and a group (*354·17). If κ is connected, $(\kappa_g)_\iota = \kappa_\iota$ (*354·15), so that properties only dependent on κ_ι, like that of openness, always hold for κ_g when they hold for κ.

Next, we prove that if κ is open, connected, and a group, $C'T_\kappa$ is open, rational, sub-connected and a group (*354·22). Hence if κ is open and connected, and $\lambda = \kappa_g$, $C'T_\lambda$ is open, rational, sub-connected and a group (*354·24).

The "rational points" with respect to a and T are $A_a``(C'T_\kappa)_\iota$. In order to study them, we consider $A_a``\lambda_\iota$, where λ is a family concerning which we make hypotheses which will be fulfilled in the case of $C'T_\kappa$. We prove that if λ is a family which is a group, and $S \in \lambda \,.\, a \in s`\text{Cl}``\lambda$, then

$$A_a``\lambda_\iota \subset \breve{S}``A_a``\lambda_\iota \quad (*354·31),$$

whence $\quad S \upharpoonright (A_a``\lambda_\iota) = (A_a``\lambda_\iota) \upharpoonright S = S \upharpoonright (A_a``\lambda_\iota) \quad (*354·312).$

Next we prove that, with the same hypothesis, if b is any other member of $A_a``\lambda_\iota$, then

$$A_a``\lambda_\iota = A_b``\lambda_\iota \quad (*354·33).$$

Thus the rational points with respect to a and T are the same as the rational points with respect to b and T, if b is one of these rational points.

The "rational net" is the family $\upharpoonright \{A_a``(C'T_\kappa)_\iota\}``C'T_\kappa$. Writing λ for $C'T_\kappa$, this becomes $\upharpoonright (A_a``\lambda_\iota)``\lambda$. In order to obtain the properties of the rational net, we therefore continue to consider a family λ, concerning which we make hypotheses which are verified in the case of $C'T_\kappa$, and we put

***354·02.** $\text{cx}_a`\lambda = \upharpoonright (A_a``\lambda_\iota)``\lambda \quad \text{Df}$

Thus $\text{cx}_a`C'T_\kappa$ is the rational net defined by $\kappa$, $T$, and $a$. We prove (*354·4) that if $\lambda$ is a group, $\text{cx}_a`\lambda$ is a family whose field is $A_a``\lambda_\iota$. We prove that if $\lambda$ is a family, and $a$ a member of its field such that any member $L$ of $\lambda_\iota$ for which $L`a$ exists is a member of $\lambda \cup \text{Cnv}``\lambda$, then a is a connected point of $\text{cx}_a`\lambda$, *i.e.*

***354·32** $\vdash : \lambda \in FM \,.\, a \in s`\text{Cl}``\lambda \,.\, \lambda_\iota \cap \text{Cl}`A_a \subset \lambda \cup \text{Cnv}``\lambda \,.\, \supset .\, a \in \text{conx}`\text{cx}_a`\lambda$

The hypothesis $\lambda_\iota \cap \text{Cl}`A_a \subset \lambda \cup \text{Cnv}``\lambda$ would be verified if λ were a connected family and a were a connected point of λ. But we want to be able to replace λ by $C'T_\kappa$, which is in general not connected. The above hypothesis, unlike $\lambda \in FM\,\text{conx}$, is satisfied by $C'T_\kappa$, provided κ is open and a group and a is a connected point of κ (*354·34). Hence it follows that if κ is a family which is open, connected, and a group, and a is a connected point of κ, $\text{cx}_a`C'T_\kappa$ is open and connected, and $a$ is a connected point of $\text{cx}_a`C'T_\kappa$ (*354·401). Again, in virtue of *354·312, if λ is a family which is a group, and a is any member of its field, $\text{cx}_a`\lambda$ is a group (*354·313); hence when $\kappa$ is a family which is open, connected, and a group, $\text{cx}_a`C'T_\kappa$ is a group (*354·402); and it is easy to prove that it is also a rational family (*354·403). Hence, by *353·27, $\text{cx}_a`C'T_\kappa$ is a family which is open, connected, rational, a group, transitive, and asymmetrical (*354·404). If our original family is open and connected but not a group, we only have to

substitute κ_g for κ, *i.e.* putting $\lambda = \kappa_g$, we only have to take $\text{cx}_a{}'C'T_\lambda$, in order to obtain a rational net with all the above properties. This is stated in the proposition

***354·41.** $\vdash : \kappa \in FM \text{ ap conx} \,.\, T \in \kappa_\partial \,.\, a \in \text{conx}'\kappa \,.\, \lambda = \kappa_g \,.\, \supset .$
$\text{cx}_a{}'C'T_\lambda \in FM \text{ ap conx rt trs asym}$

***354·01.** $\kappa_g = \kappa \cup \text{Cnv}``(\kappa \cap \overleftarrow{D}`s`\text{Ɑ}``\kappa)$ Df

***354·02.** $\text{cx}_a{}'\lambda = \dot{[}\,(A_a``\lambda_\iota)``\lambda$ Df

***354·03.** $FM \text{ grp} = FM \cap \hat{\kappa}\,(s'\kappa\,|_{,,}``\kappa \subset \kappa)$ Df

***354·1.** $\vdash :. R \in \kappa_g \,.\, \equiv : R \in \kappa \,.\, \vee \,.\, \breve{R} \in \kappa \,.\, \text{Ɑ}'R = s'\text{Ɑ}``\kappa$ [(*354·01)]

***354·11.** $\vdash : \kappa \in FM \text{ conx} \,.\, R, S \in \kappa \,.\, \supset .\, R \,|\, S \in \kappa_g$ [*331·33 . *354·1]

***354·12.** $\vdash : \text{Hp} \ast 354{\cdot}11 \,.\, \text{D}'R = s'\text{Ɑ}``\kappa \,.\, \supset .\, \breve{R} \,|\, S = S \,|\, \breve{R} \,.\, \breve{R} \,|\, S \in \kappa_g$

Dem.

$\vdash . \ast 330{\cdot}52 \,.\, \supset \vdash : \text{Hp} \,.\, a \in \text{conx}'\kappa \,.\, \supset .\, \text{E}!\, \breve{R}'S'a \,.\, \text{Ɑ}'(\breve{R} \,|\, S) = s'\text{Ɑ}``\kappa \,.$

$[\ast 331{\cdot}11{\cdot}42] \qquad \supset .\, \breve{R} \,|\, S \in \kappa \cup \text{Cnv}``\kappa \,.\, \text{Ɑ}'(\breve{R} \,|\, S) = s'\text{Ɑ}``\kappa \,.$

$[\ast 354{\cdot}1 . \ast 330{\cdot}561] \qquad \supset .\, \breve{R} \,|\, S \in \kappa_g \,.\, S \,|\, \breve{R} = \breve{R} \,|\, S : \supset \vdash . \text{Prop}$

***354·13.** $\vdash : \text{Hp} \ast 354{\cdot}11 \,.\, \text{D}'R = \text{D}'S = s'\text{Ɑ}``\kappa \,.\, \supset .\, \breve{R} \,|\, \breve{S} \in \kappa_g$

Dem.

$\vdash . \ast 331{\cdot}33 \,.\, \supset \vdash : \text{Hp} \,.\, \supset .\, \breve{R} \,|\, \breve{S} \in \kappa \cup \text{Cnv}``\kappa$ (1)

$\vdash . \ast 37{\cdot}323 \,.\, \supset \vdash : \text{Hp} \,.\, \supset .\, \text{Ɑ}'(\breve{R} \,|\, \breve{S}) = s'\text{Ɑ}``\kappa$ (2)

$\vdash . (1) . (2) . \ast 354{\cdot}1 \,.\, \supset \vdash . \text{Prop}$

***354·14.** $\vdash : \kappa \in FM \text{ conx} \,.\, \supset .\, s'\kappa_g\,|_{,,}``\kappa_g \subset \kappa_g$ [*354·11·12·13·1]

***354·15.** $\vdash : \kappa \in FM \text{ conx} \,.\, \supset .\, (\kappa_g)_\iota = \kappa_\iota$

Dem.

$\vdash . \ast 354{\cdot}1 \,.\, \supset \vdash :. \text{Hp} \,.\, R, S \in \kappa_g \,.\, \supset :$

$R, S \in \kappa \,.\, \vee \,.\, \breve{R}, S \in \kappa \,.\, \vee \,.\, R, \breve{S} \in \kappa \,.\, \vee \,.\, \breve{R}, \breve{S} \in \kappa \,.\, \text{Ɑ}'R = \text{Ɑ}'S = s'\text{Ɑ}``\kappa$ (1)

$\vdash . \ast 330{\cdot}4 \,.\, \supset \vdash : \text{Hp} \,.\, R, S \in \kappa \,.\, \supset .\, \breve{R} \,|\, S \in \kappa_\iota$ (2)

$\vdash . \ast 331{\cdot}33{\cdot}24 \,.\, \supset \vdash :. \text{Hp} : \breve{R}, S \in \kappa \,.\, \vee \,.\, R, \breve{S} \in \kappa : \supset .\, \breve{R} \,|\, S \in \kappa_\iota$ (3)

$\vdash . \ast 354{\cdot}12 \,.\, \supset \vdash : \text{Hp} \,.\, \breve{R}, \breve{S} \in \kappa \,.\, \text{Ɑ}'R = \text{Ɑ}'S = s'\text{Ɑ}``\kappa \,.\, \supset .\, \breve{R} \,|\, S \in \kappa_\iota$ (4)

$\vdash . (1) . (2) . (3) . (4) \,.\, \supset \vdash . \text{Prop}$

***354·16.** $\vdash : \kappa \in FM \text{ conx} \,.\, \supset .\, \kappa_g \in FM \text{ conx}$ [*354·1·12]

***354·17.** $\vdash : \kappa \in FM \text{ ap conx} \,.\, \supset .\, \kappa_g \in FM \text{ ap conx grp}$
[*354·16·15·14 . *333·101]

***354·18.** $\vdash :. \kappa \in FM \,\mathrm{grp} \,.\equiv : \kappa \in FM : R, S \in \kappa \,.\supset_{R,S} .\, R \mid S \in \kappa$ [(*354·03)]

***354·19.** $\vdash : \kappa \in FM \,\mathrm{grp} \,.\supset .\, s`\mathrm{Pot}``\kappa \subset \kappa$ [*354·18 . Induct]

***354·2.** $\vdash : \kappa \in FM \,\mathrm{ap\ conx} \,.\, T \in \kappa_\partial \,.\supset .\, C`T_\kappa \in FM \,\mathrm{ap\ rt\ cx}$
[*353·15 . *352·3]

***354·22.** $\vdash : \kappa \in FM \,\mathrm{ap\ conx\ grp} \,.\, T \in \kappa_\partial \,.\supset .\, C`T_\kappa \in FM \,\mathrm{ap\ rt\ cx\ grp}$

Dem.

$\vdash .\,$*350·62 . *354·18 $.\supset \vdash : \mathrm{Hp} \,.\, R, S, T \in \kappa \,.\, X, Y \in C`H' \,.\, RXT \,.\, SYT \,.\supset .$
$(R \mid S)(X +_s Y) T \,.\, R \mid S \in \kappa \,.$

[*306·67.*352·3] $\supset .\, R \mid S \in C`T_\kappa$ (1)

$\vdash .\,(1)\,.\,$*352·3 $.\supset \vdash : \mathrm{Hp} \,.\, R, S \in C`T_\kappa \,.\supset .\, R \mid S \in C`T_\kappa$ (2)

$\vdash .\,(2)\,.\,$*354·2 $.\supset \vdash .$ Prop

***354·23** $\vdash : \kappa \in FM \,\mathrm{rt\ conx} \,.\, T \in \kappa_\partial \,.\supset .\, C`T_\kappa = \kappa$ [*353·13 . *352·3]

***354·24.** $\vdash : \kappa \in FM \,\mathrm{ap\ conx} \,.\, T \in \kappa_\partial \,.\, \lambda = \kappa_g \,.\supset .\, C`T_\lambda \in FM \,\mathrm{ap\ rt\ cx\ grp}$
[*354·22·17]

***354·31.** $\vdash : \lambda \in FM \,\mathrm{grp} \,.\, a \in s`\mathrm{G}``\lambda \,.\, S \in \lambda \,.\supset .\, A_a``\lambda_\iota \subset \breve{S}``A_a``\lambda_\iota$

Dem.

$\vdash .\,$*336·1 $.\supset \vdash :. \mathrm{Hp} \,.\supset : x \in A_a``\lambda_\iota \,.\supset .\, (\exists P, Q) \,.\, P, Q \in \kappa \,.\, x = \breve{P}`Q`a \,.$

[*330·56] $\supset .\, (\exists P, Q) \,.\, P, Q \in \kappa \,.\, S`x = \breve{P}`S`Q`a \,.$

[*354·18] $\supset .\, (\exists P, R) \,.\, P, R \in \kappa \,.\, S`x = \breve{P}`R`a \,.$

[*336·1] $\supset .\, S`x \in A_a``\lambda_\iota \,.$

[*37·106] $\supset .\, x \in \breve{S}``A_a``\lambda_\iota :. \supset \vdash .$ Prop

***354·311.** $\vdash : \mathrm{Hp}\,$*354·31 $.\supset .\, S``A_a``\lambda_\iota \subset A_a``\lambda_\iota$ [*354·31]

***354·312.** $\vdash : \mathrm{Hp}\,$*354·31 $.\supset .\, S \upharpoonleft (A_a``\lambda_\iota) = (A_a``\lambda_\iota) \upharpoonleft S = S \upharpoonright (A_a``\lambda_\iota)$
[*354·31·311]

***354·313.** $\vdash : \lambda \in FM \,\mathrm{grp} \,.\, a \in s`\mathrm{G}``\lambda \,.\, \mu = \mathrm{cx}_a`\lambda \,.\supset .\, s`\mu_{\mid\,,,}``\mu \subset \mu$

Dem.

$\vdash .\,$*354·312 $.\supset$

$\vdash : \mathrm{Hp} \,.\, R, S \in \lambda \,.\supset .\, \{R \upharpoonleft (A_a``\lambda_\iota)\} \mid \{S \upharpoonleft (A_a``\lambda_\iota)\} = (R \mid S) \upharpoonleft (A_a``\lambda_\iota)$ (1)

$\vdash .\,(1)\,.\,$*354·18 $.\supset$

$\vdash : \mathrm{Hp} \,.\, R, S \in \lambda \,.\supset .\, \{R \upharpoonleft (A_a``\lambda_\iota)\} \mid \{S \upharpoonleft (A_a``\lambda_\iota)\} \in \mathrm{cx}_a`\lambda : \supset \vdash .$ Prop

***354·32.** $\vdash : \lambda \in FM \,.\, a \in s`\mathrm{G}``\lambda \,.\, \lambda_\iota \cap \mathrm{G}`A_a \subset \lambda \cup \mathrm{Cnv}``\lambda \,.\supset .\, a \in \mathrm{conx}`\mathrm{cx}_a`\lambda$

Dem.

$\vdash .\,$*336·1 $.\supset \vdash :. \mathrm{Hp} \,.\supset : x \in A_a``\lambda_\iota \,.\supset .\, (\exists L) \,.\, L \in \lambda_\iota \,.\, x = L`a \,.\, L \in \mathrm{G}`A_a \,.$

[Hp] $\supset .\, (\exists L) \,.\, L \in \lambda \cup \mathrm{Cnv}``\lambda \,.\, x = L`a \,.$

[*330·43] $\supset .\, (\exists M) \,.\, M \in \mathrm{cx}_a`\lambda \cup \mathrm{Cnv}``\mathrm{cx}_a`\lambda \,.\, x = M`a :$

[*331·11] $\supset : a \in \mathrm{conx}`\mathrm{cx}_a`\lambda :. \supset \vdash .$ Prop

***354·33.** $\vdash : \lambda \in FM \operatorname{grp} . a \in s`\text{C}``\lambda . b \in A_a``\lambda_\iota . \supset . A_a``\lambda_\iota = A_b``\lambda_\iota$

Dem.

$\vdash . *336·1 . \supset$

$\vdash : \text{Hp} . c \in A_b``\lambda_\iota . \supset . (\exists P, Q, R, S) . P, Q, R, S \in \kappa . c = \breve{R}`S`\breve{P}`Q`a .$

$[*330·56] \quad \supset . (\exists P, Q, R, S) . P, Q, R, S \in \kappa . c = \breve{R}`\breve{P}`S`Q`a .$

$[*354·18] \quad \supset . (\exists M, N) . M, N \in \kappa . c = \breve{M}`N`a .$

$[*336·1] \quad \supset . c \in A_a``\lambda_\iota \qquad (1)$

Similarly $\vdash : \text{Hp} . c \in A_a``\lambda_\iota . \supset . c \in A_b``\lambda_\iota \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***354·34.** $\vdash : \kappa \in FM \text{ ap conx grp} . T \in \kappa_{\partial} . \lambda = C`T_\kappa . a \in \text{conx}`\kappa . \supset .$
$\lambda_\iota \cap \text{C}`A_a \subset \lambda \cup \text{Cnv}``\lambda$

Dem.

$\vdash . *354·22 . \quad \supset \vdash : \text{Hp} . \supset . \lambda \in FM \text{ ap rt cx} .$

$[*353·14] \quad \supset . \lambda_\iota \cap (\kappa \cup \text{Cnv}``\kappa) \subset \lambda \cup \text{Cnv}``\lambda \qquad (1)$

$\vdash . *331·11·32 . \supset \vdash : \text{Hp} . L \in \lambda_\iota \cap \text{C}`A_a . \supset . L \in \kappa \cup \text{Cnv}``\kappa .$

$[(1)] \quad \supset . L \in \lambda \cup \text{Cnv}``\lambda : \supset \vdash . \text{Prop}$

***354·35.** $\vdash : \kappa \in FM \text{ ap conx} . T \in \kappa_{\partial} . \mu = \kappa_g . \lambda = C`T_\mu . a \in \text{conx}`\kappa . \supset .$
$\lambda_\iota \cap \text{C}`A_a \subset \lambda \cup \text{Cnv}``\lambda \quad [*354·34·17]$

***354·4.** $\vdash : \lambda \in FM \operatorname{grp} . a \in s`\text{C}``\lambda . \supset . \text{cx}_a`\lambda \in FM . s`\text{C}``\text{cx}_a`\lambda = A_a``\lambda_\iota$

Dem.

$\vdash . *330·52 . \quad \supset \vdash : \text{Hp} . \supset . \text{cx}_a`\lambda \subset 1 \rightarrow 1 \qquad (1)$

$\vdash . *354·311 . \supset \vdash :. \text{Hp} . \supset : R \in \lambda . \supset . \text{C}`R = A_a``\lambda_\iota . \text{D}`R \subset \text{C}`R \qquad (2)$

$\vdash . *354·312 . \supset \vdash : \text{Hp} . R, S \in \lambda . \supset . \{R \upharpoonright (A_a``\lambda_\iota)\} | \{S \upharpoonright (A_a``\lambda_\iota)\} = (R | S) \upharpoonright (A_a``\lambda_\iota)$

$[*330·5·52] \quad = (S | R) \upharpoonright (A_a``\lambda_\iota)$

$[*354·312] \quad = \{S \upharpoonright (A_a``\lambda_\iota)\} | \{R \upharpoonright (A_a``\lambda_\iota)\} \qquad (3)$

$\vdash . (3) . *330·5 . \supset \vdash : \text{Hp} . \supset . \text{cx}_a`\lambda \in \text{Abel} \qquad (4)$

$\vdash . (1) . (2) . (4) . *330·52 . \supset \vdash . \text{Prop}$

***354·401.** $\vdash : \kappa \in FM \text{ ap conx grp} . a \in \text{conx}`\kappa . T \in \kappa_{\partial} . \supset .$
$\text{cx}_a`C`T_\kappa \in FM \text{ ap conx} . a \in \text{conx}`\text{cx}_a`C`T_\kappa$

Dem.

$\vdash . *354·4·22 . \quad \supset \vdash : \text{Hp} . \supset . \text{cx}_a`C`T_\kappa \in FM \qquad (1)$

$\vdash . *354·34·32·2 . \supset \vdash : \text{Hp} . \supset . a \in \text{conx}`\text{cx}_a`C`T_\kappa \qquad (2)$

$\vdash . (1) . (2) . *333·101 . \supset \vdash . \text{Prop}$

***354·402.** $\vdash : \text{Hp} *354·401 . \supset . \text{cx}_a`C`T_\kappa \in FM \operatorname{grp} \quad [*354·313·22·401]$

***354·403.** $\vdash : \mathrm{Hp} *354\cdot401 . \supset . \mathrm{cx}_a{}^{\backprime}C{}^{\backprime}T_\kappa \in FM\,\mathrm{rt}$

Dem.

$\vdash . *353\cdot12 . *354\cdot2 . \supset$

$\vdash : \mathrm{Hp} . S \in C{}^{\backprime}T_\kappa . \lambda = C{}^{\backprime}T_\kappa . \supset . (\exists\mu, \nu) . \mu, \nu \in \mathrm{NC\,ind} . \nu \neq 0 . S^\nu = T^\mu .$

$[*354\cdot312.\mathrm{Induct}] \quad \supset . (\exists\mu, \nu) . \mu, \nu \in \mathrm{NC\,ind} . \nu \neq 0 .$

$\{S \upharpoonright (A_a{}^{\backprime\backprime}\lambda_\iota)\}^\nu = S^\nu \upharpoonright (A_a{}^{\backprime\backprime}\lambda_\iota) = T^\mu \upharpoonright (A_a{}^{\backprime\backprime}\lambda_\iota) = \{T \upharpoonright (A_a{}^{\backprime\backprime}\lambda_\iota)\}^\mu .$

$[*350\cdot43.*354\cdot401]$

$\supset . (\exists\mu, \nu) . \mu, \nu \in \mathrm{NC\,ind} . \nu \neq 0 . \{S \upharpoonright (A_a{}^{\backprime\backprime}\lambda_\iota)\} (\mu/\nu) \{T \upharpoonright (A_a{}^{\backprime\backprime}\lambda_\iota)\} \quad (1)$

$\vdash . (1) . *353\cdot1 . \supset \vdash . \mathrm{Prop}$

***354·404.** $\vdash : \kappa \in FM\,\mathrm{ap\,conx\,grp} . a \in \mathrm{conx}{}^{\backprime}\kappa . T \in \kappa_\partial . \supset .$
$\mathrm{cx}_a{}^{\backprime}C{}^{\backprime}T_\kappa \in FM\,\mathrm{ap\,conx\,rt\,grp\,trs\,asym} \quad [*354\cdot401\cdot402\cdot403 . *353\cdot27]$

***354·41.** $\vdash : \kappa \in FM\,\mathrm{ap\,conx} . T \in \kappa_\partial . a \in \mathrm{conx}{}^{\backprime}\kappa . \lambda = \kappa_g . \supset .$
$\mathrm{cx}_a{}^{\backprime}C{}^{\backprime}T_\lambda \in FM\,\mathrm{ap\,conx\,rt\,trs\,asym} \quad [*354\cdot17\cdot404]$

*356. MEASUREMENT BY REAL NUMBERS.

Summary of *356.

In this number we consider the application of real numbers to the measurement of vectors in a family. The principle of this application is as follows: If a given set of vectors, all of which are rational multiples of a given vector R, have a limit with respect to U_κ, and if their measures determine a segment of H, then we take the real number represented by this segment as the measure of the limit of the given set of vectors. For the sake of homogeneity with rational measures, it is well to take our real numbers in the relational form given in *314; *i.e.* if $\xi \in C`\Theta$, we take $\dot{s}`\xi$ as the corresponding real number. With a suitable hypothesis, the result of the above principle for applying real numbers is, where rational multiples of the unit R are concerned, to replace the ratio X by the rational real number $\dot{s}`\overrightarrow{H}`X$, as the measure of the vector $X \upharpoonright \kappa`R$ (cf. *356·63). Then the measure of the limit of a set of rational vectors will be, by our principle, the limit of their measures. Thus our principle is conformable to what is required for an application of real numbers.

It should be observed that, if any application of irrationals is to be possible, it is necessary that the vectors of the family concerned should have a serial or quasi-serial order, independently of the order generated by their measures. The order generated, among rational multiples of T, by the ratios which are measures of these multiples, is T_κ (cf. *352). A vector which is not a member of $C`T_\kappa$ cannot be the limit of any set of vectors with respect to T_κ. But we saw (*352·72) that if κ is a serial family,

$$T_\kappa = \breve{U}_\kappa \upharpoonright C`T_\kappa.$$

Hence when κ is a serial family, a vector which is not a member of $C`T_\kappa$ may be the limit of a set of members of $C`T_\kappa$ with respect to U_κ. It is the existence of an independent series U_κ, not generated by measurement, which makes the application of irrationals as measures possible.

The following phraseology may be found convenient. Taking a unit T in a family κ, and an origin a in its field, if $X \in C`H'$ and $S = X \upharpoonright \kappa`T$ and $x = S`a = (X \upharpoonright \kappa`T)`a$, we call X the "rational measure" of S and the "rational coordinate" of x. We have, in the same circumstances,

$$S = \kappa \urcorner A_T`X \,.\, x = A_a`S = A_a`\kappa \urcorner A_T`X.$$

We will call S the vector of X, and x the point of X; and the same phraseology will be employed for the vectors and points obtained by measures which are real numbers. We may now state the principle according to which we apply real numbers as measures as follows. Given a segment ξ of H, take all the vectors of ξ's: these form the class $\kappa \cap A_T``\xi$. Then the real number $\dot{s}`\xi$ is to be the measure of the limit (with respect to $U_\kappa$) of the class $\kappa \cap A_T``\xi$. Since U_κ has the opposite sense to that of T_κ, *i.e.* U_κ proceeds from the vectors with bigger measures to those with smaller ones, the limit we shall have to take will be the *lower* lim. with respect to U_κ. Thus the vector whose measure is $\dot{s}`\xi$ will be

$$\text{prec}\,(U_\kappa)`(\kappa \cap A_T``\xi).$$

Now if we put $X = \dot{s}`\xi$, $A_T``\xi = \overrightarrow{X}`T$, and $X$ is a relational real number. Hence using *206·131, the vector whose measure is $X$ is $\text{prec}\,(U_\kappa)`\overrightarrow{X}`T$. Hence if "$X_\kappa`T$" represents the vector whose measure is X (unit T), we put

***356·01.** $X_\kappa = \text{prec}\,(U_\kappa) \,|\, \overrightarrow{X} \upharpoonright \kappa$ Df

Assuming now that κ is a serial submultipliable family, in which we take R as the unit and a as the origin, and putting, for notational convenience,

$$\breve{P} = U_\kappa \,.\, \breve{Q} = \dot{s}`\kappa_\partial,$$

we have first a set of preliminary propositions (*356·1—·191), of which the most important are

$$H' = (C`H') \rceil \breve{A}_R \,;\, P = (C`H') \rceil \breve{A}_R \,;\, \breve{A}_a \,;\, Q \quad (*356·13),$$

$$P \upharpoonright C`R_\kappa = \kappa \rceil A_R \,;\, H' \quad (*356·14),$$

giving the relations between the series of ratios, the series of their vectors, and the series of their points.

We proceed next (*356·2—·26) to the proof that $X_\kappa \upharpoonright \kappa \in 1 \to 1$. This requires, in addition to our previous hypothesis, that Q should be semi-Dedekindian. With this hypothesis, we first prove that if X, Y are relational real numbers,

$$\text{Ɑ}`X_\kappa = \text{Ɑ}`Y_\kappa = \kappa_\partial : X_\kappa = Y_\kappa \,.\equiv.\, X = Y \quad (*356·21).$$

We then prove, by the help of some arithmetical lemmas, that the lower limit of the submultiples of a given vector is the zero vector, *i.e.*

$$\text{tl}_P`\hat{S}\{S \in \kappa : (\exists \nu) \,.\, R = S^\nu\} = I \upharpoonright C`Q \quad (*356·22).$$

Hence we easily prove that, if R is any non-zero vector, and λ is a class of vectors having a lower limit L, the lower limit of the relative products of R and members of λ is the relative product of R and L, *i.e.*

$$\lambda \subset \kappa \,.\, L = \text{tl}_P`\lambda \,.\, R \in \kappa_\partial \,.\supset.\, R \,|\, L = \text{tl}_P`R \,|``\lambda \quad (*356·221).$$

Remembering that the relative product is represented arithmetically by the sum, we may express the above proposition by saying that the limit of the sums of a given vector and a set of vectors is the sum of the given vector and the limit of the set. From this proposition we easily deduce that if RPS, $X_\kappa{}^{\backprime}R \nmid X_\kappa{}^{\backprime}S$, whence it follows that

$$X_\kappa \upharpoonright \kappa \epsilon 1 \rightarrow 1 \quad (*356\cdot26).$$

Our next set of propositions ($*356\cdot3$—$\cdot33$) is concerned in connecting the relative product of X_κ and Y_κ with the arithmetical product $X \times_r Y$, where "$\times_r$" has the meaning defined in $*314$. Here we only require that κ should be serial and submultipliable, and we obtain

$$X_\kappa \mid Y_\kappa = (X \times_r Y)_\kappa \quad (*356\cdot33).$$

This proposition is the analogue of $*351\cdot31$ (except that κ_ι is replaced by κ); it has a similar importance, and calls for similar remarks.

Our next set of propositions ($*356\cdot4$—$\cdot43$) is concerned in proving that the limit of the points of a segment of ratios is the point of their limit, in other words, that the limit of a set of points whose coordinates are a segment of rationals is the point whose coordinate is the limit of the segment. Here we again require that our family should be semi-Dedekindian; then if ξ is a segment of ratios, and $X = \dot{s}{}^{\backprime}\xi$, the above proposition is

$$(X_\kappa{}^{\backprime}R){}^{\backprime}a = \text{seq}_Q{}^{\backprime}A_a{}^{\backprime\backprime}A_R{}^{\backprime\backprime}\xi = \text{seq}_Q{}^{\backprime}A_a{}^{\backprime\backprime}\overrightarrow{X}{}^{\backprime}R \quad (*356\cdot43).$$

Here $X_\kappa{}^{\backprime}R$ is the vector of X, $(X_\kappa{}^{\backprime}R){}^{\backprime}a$ is the point of X; $A_R{}^{\backprime\backprime}\xi = \overrightarrow{X}{}^{\backprime}R$, and each is the class of vectors of members of ξ; and $A_a{}^{\backprime\backprime}A_R{}^{\backprime\backprime}\xi$ or $A_a{}^{\backprime\backprime}\overrightarrow{X}{}^{\backprime}R$ is the class of points of members of ξ. Moreover X is a relational real number. Thus the above proposition states that the point of X is the segment (*i.e.* the limit) of the points of the ratios contained in X; *i.e.* of the ratios which may be considered less than X.

We next proceed ($*356\cdot5$—$\cdot54$) to connect the relative multiplication of vectors with the addition of their measures. Here we require that κ should be semi-Dedekindian as well as serial and submultipliable. We then find that if X, Y are relational real numbers, and R is a non-zero vector,

$$(X_\kappa{}^{\backprime}R) \mid (Y_\kappa{}^{\backprime}R) = (X +_r Y)_\kappa{}^{\backprime}R \quad (*356\cdot54).$$

This proposition is the analogue of $*351\cdot43$, and calls for similar remarks. The proof proceeds without much difficulty by means of $*356\cdot43$.

Finally we have a set of propositions ($*356\cdot6$—$\cdot63$) to prove that the real number which measures a rational vector is the real number corresponding to the ratio which is its measure; *i.e.* if X is a ratio, the vector which has the ratio X to the unit has the real number $\dot{s}{}^{\backprime}\overrightarrow{H}{}^{\backprime}X$ for its measure. It is to be remembered that rational real numbers must not be identified with ratios,

any more than integral ratios (*i.e.* ratios of the form $\nu/1$) must be identified with cardinals. The real number corresponding to a ratio X is $\dot{s}`\overrightarrow{H}`X$; this is what we call a "rational real number." In measurement, when we are measuring by ratios, if R is our unit, X will be the measure of $X \upharpoonright \kappa`R$; but when we are measuring by real numbers, the measure of $X \upharpoonright \kappa`R$ must be a real number. The real number which is the measure of $X \upharpoonright \kappa`R$ will, by our definition, be a real number Z such that

$$X \upharpoonright \kappa`R = \text{prec}(U_\kappa)`\overrightarrow{Z}`R.$$

Thus we have to prove that, if X is a ratio, the above equation is satisfied if we put $Z = \dot{s}`\overrightarrow{H}`X$. This requires that κ should be serial, submultipliable and semi-Dedekindian; we then have

$$X \in C`H . \supset . (\dot{s}`\overrightarrow{H}`X)_\kappa = X \upharpoonright \kappa \quad (*356\cdot63).$$

Thus although the "pure" real number $\dot{s}`\overrightarrow{H}`X$ is not identical with the "pure" ratio X, yet the "applied" real number $(\dot{s}`\overrightarrow{H}`X)_\kappa$ *is* identical with the "applied" ratio $X \upharpoonright \kappa$. This fact explains why the results of the habitual confusion between a ratio and a rational real number have not been even more disastrous.

$*356\cdot01$. $X_\kappa = \text{prec}(U_\kappa) \,|\, \overrightarrow{X} \upharpoonright \kappa$ Df

$*356\cdot1$. $\vdash :. R \in \kappa . \supset : S = X_\kappa`R . \equiv . S = \text{prec}(U_\kappa)`\overrightarrow{X}`R$ $[(*356\cdot01)]$

$*356\cdot11$. $\vdash :. R \in \kappa . \supset : S = (\dot{s}`\xi)_\kappa`R . \equiv . S = \text{prec}(U_\kappa)`A_R``\xi$
$[*356\cdot1 . *336\cdot12]$

$*356\cdot12$. $\vdash :. \kappa \in FM \,\text{sr}\,\text{subm}\,.$
$X, Y \in C`H' . R \in \kappa_\partial . a \in s`\mathrm{Cl}``\kappa . \breve{Q} = \dot{s}`\kappa_\partial . \breve{P} = U_\kappa . \supset :$
$XH'Y . \equiv . (X \upharpoonright \kappa`R) P (Y \upharpoonright \kappa`R) . \equiv . \{(X \upharpoonright \kappa`R)`a\} Q \{(Y \upharpoonright \kappa`R)`a\}$
$[*352\cdot73 . *336\cdot4]$

$*356\cdot13$. $\vdash : \kappa \in FM \,\text{sr}\,\text{subm} . R \in \kappa_\partial . a \in s`\mathrm{Cl}``\kappa . \breve{Q} = \dot{s}`\kappa_\partial . \breve{P} = U_\kappa . \supset .$
$$H' = (C`H') \rceil \breve{A}_R ; P = (C`H') \rceil \breve{A}_R ; \breve{A}_a ; Q \quad [*356\cdot12]$$

$*356\cdot14$. $\vdash : \text{Hp}\, *356\cdot13 . \supset . P \upharpoonright C`R_\kappa = \kappa \rceil A_R ; H'$ $[*352\cdot72]$

$*356\cdot15$. $\vdash : \text{Hp}\, *356\cdot13 . \lambda \subset C`H . X = \dot{s}`\lambda . \supset . \overrightarrow{\max}_P`\overrightarrow{X}`R = \kappa \rceil A_R``\overrightarrow{\max}_H`\lambda$

Dem.

$\vdash . *352\cdot41 . \quad \supset \vdash : \text{Hp} . \supset . \kappa \cap \overrightarrow{X}`R \subset C`R_\kappa . \overrightarrow{X}`R = A_R``\lambda$ (1)

$\vdash . (1) . *356\cdot14 . \supset \vdash : \text{Hp} . \supset . \overrightarrow{\max}_P`\overrightarrow{X}`R = \max(P \upharpoonright C`R_\kappa)`\overrightarrow{X}`R$

$[*356\cdot14]$ $\qquad = \kappa \rceil A_R``\overrightarrow{\max}_H`\lambda : \supset \vdash . \text{Prop}$

***356·16.** $\vdash : \text{Hp} \ast 356 \cdot 13 \,.\, \lambda \in C'\Theta \,.\, X = \dot{s}'\lambda \,.\, \supset .\, \max_P{}'\overrightarrow{X}{}'R = \Lambda$ [*356·15]

***356·17.** $\vdash : \text{Hp} \ast 356 \cdot 16 \,.\, \supset .\, X_\kappa = \text{lt}_P \,|\, \overrightarrow{X} \upharpoonright C'P$ [*356·16]

***356·18.** $\vdash : \kappa \in FM \text{ connex} \,.\, \supset .\, X_\kappa \in 1 \rightarrow \text{Cls}$
[*206·161 . *336·62 . (*353·01)]

***356·19.** $\vdash :.\, \kappa \in FM \text{ sr} \,.\, P = \breve{U}_\kappa \,.\, \supset : Z \in C'H \,.\, \supset .\, Z \lceil \kappa \,;\, P \Subset P$

Dem.

$\vdash . \ast 336 \cdot 511 \,.\, \supset \vdash :.\, \text{Hp} \,.\, R, S \in \kappa \,.\, \mu, \nu \in \text{NC ind} - \iota'0 \,.\, Z = \mu/\nu \,.\, \supset :$
$RPS \,.\, \equiv .\, R^\mu P S^\mu \,.$

[*350·43] $\supset : RPS \,.\, M = (\mu/\nu) \lceil \kappa'R \,.\, N = (\mu/\nu) \lceil \kappa'S \,.\, \supset .\, M^\nu P N^\nu \,.$

[*336·511] $\supset .\, MPN :.\, \supset \vdash .\, \text{Prop}$

***356·191.** $\vdash : \text{Hp} \ast 356 \cdot 19 \,.\, X \in \dot{s}``C'\Theta \,.\, \supset .\, X \lceil \kappa \,|\, P \Subset P \,|\, X \lceil \kappa$

Dem.

$\vdash . \ast 356 \cdot 19 \,.\, \supset$

$\vdash :.\, \text{Hp} \,.\, \supset : \lambda \in C'\Theta \,.\, X = \dot{s}'\lambda \,.\, Z \in \lambda \,.\, \supset .\, Z \lceil \kappa \,|\, P \Subset P \,|\, Z \lceil \kappa :.\, \supset \vdash .\, \text{Prop}$

***356·2.** $\vdash : \text{Hp} \ast 356 \cdot 16 \,.\, \mu \in C'\Theta \,.\, L \in \lambda - \mu \,.\, \supset .\, \kappa \rceil A_R{}'L \in p'\overleftarrow{P}``A_R``\mu$

Dem.

$\vdash . \ast 310 \cdot 11 \,.\, \supset \vdash : \text{Hp} \,.\, \supset .\, L \in p'\overleftarrow{H}``\mu \,.$

[*206·6.*352·12] $\supset .\, \kappa \rceil A_R{}'L \in p'\overleftarrow{\kappa \rceil A_R ; H}``A_R``\mu \,.$

[*356·141] $\supset .\, \kappa \rceil A_R{}'L \in p'\overleftarrow{P}``A_R``\mu : \supset \vdash .\, \text{Prop}$

***356·21.** $\vdash :.\, \kappa \in FM \text{ sr subm} \,.\, \text{Cnv}'\dot{s}'\kappa_\partial \in \text{semi Ded} \,.\, X, Y \in \dot{s}``C'\Theta \,.\, \supset :$
$\text{Ɑ}'X_\kappa = \text{Ɑ}'Y_\kappa = \kappa_\partial : X_\kappa = Y_\kappa \,.\, \equiv .\, X = Y$

Dem.

$\vdash . \ast 356 \cdot 16 \,.\, \ast 214 \cdot 7 \,.\, \supset$

$\vdash : \text{Hp} \,.\, \lambda, \mu \in C'\Theta \,.\, X = \dot{s}'\lambda \,.\, Y = \dot{s}'\mu \,.\, R \in \kappa_\partial \,.\, \supset .\, E!\, X_\kappa{}'R \,.\, E!\, Y_\kappa{}'R$ (1)

$\vdash . (1) . \ast 356 \cdot 2 \,.\, \supset \vdash : \text{Hp}(1) \,.\, P = \breve{U}_\kappa \,.\, \exists !\, \lambda - \mu \,.\, \supset .\, (Y_\kappa{}'R)\, P\, (X_\kappa{}'R)$ (2)

Similarly $\vdash : \text{Hp}(1) \,.\, P = \breve{U}_\kappa \,.\, \exists !\, \mu - \lambda \,.\, \supset .\, (X_\kappa{}'R)\, P\, (Y_\kappa{}'R)$ (3)

$\vdash . (1) . (2) . (3) \,.\, \supset \vdash : \text{Hp}(1) \,.\, X_\kappa{}'R = Y_\kappa{}'R \,.\, \supset .\, \lambda = \mu .$

[Hp] $\supset .\, X = Y$ (4)

$\vdash . (1) . (4) \,.\, \supset \vdash .\, \text{Prop}$

***356·211.** $\vdash : \sigma, \tau \in \text{NC ind} - \iota'0 \,.\, \nu \in \text{NC ind} - \iota'0 - \iota'1 \,.\, \supset .$
$(\sigma +_c \tau)^\nu > \sigma^\nu +_c (\nu \times_c \sigma^{\nu -_c 1} \times_c \tau)$

Dem.

$\vdash . \ast 113 \cdot 43 \cdot 66 \,.\, \ast 116 \cdot 34 \,.\, \supset \vdash .\, (\sigma \times_c \tau)^2 = \sigma^2 +_c (2 \times_c \sigma \times_c \tau) +_c \tau^2$ (1)

$\vdash . \ast 126 \cdot 5 \,.\, \supset \vdash :.\, \text{Hp} \,.\, \supset : (\sigma +_c \tau)^\nu > \sigma^\nu +_c (\nu \times_c \sigma^{\nu -_c 1} \times_c \tau) \,.\, \supset .$
$(\sigma +_c \tau)^{\nu +_c 1} > \sigma^{\nu +_c 1} +_c (\nu \times_c \sigma^\nu \times_c \tau) +_c (\sigma^\nu \times_c \tau)$ (2)

$\vdash . (1) . (2) . \text{Induct} \,.\, \supset \vdash .\, \text{Prop}$

***356·212.** $\vdash : \rho > \sigma . \rho, \sigma, \zeta \in \mathrm{NC\,ind} . \supset . (\exists \nu) . \nu \in \mathrm{NC\,ind} . \rho^\nu > \sigma^\nu \times_c \zeta$

Dem.

$\vdash . *356·211 . \supset$

$\vdash : \mathrm{Hp} . \nu \in \mathrm{NC\,ind} . \rho = \sigma +_c \tau . \supset . \rho^\nu > \sigma^{\nu -_c 1} \times_c \{\sigma +_c (\nu \times_c \tau)\}$ (1)

$\vdash . (1) . *126·51 . \supset \vdash : \mathrm{Hp}(1) . \sigma +_c (\nu \times_c \tau) > \sigma \times_c \zeta . \supset . \rho^\nu > \sigma^\nu \times_c \zeta$ (2)

$\vdash . (2) . *113·43 . *120·416 . *126·5 . \supset$

$\vdash : \mathrm{Hp}(1) . \nu \times_c \tau > \sigma \times_c (\zeta -_c 1) . \supset . \rho^\nu > \sigma^\nu \times_c \zeta : \supset \vdash . \mathrm{Prop}$

***356·213.** $\vdash : \rho > \sigma . \rho, \sigma, \xi, \eta \in \mathrm{NC\,ind} . \eta \neq 0 . \supset .$
$(\exists \nu) . \nu \in \mathrm{NC\,ind} . \rho^\nu \times_c \eta > \sigma^\nu \times_c \xi$

Dem.

$\vdash . *356·212 . \supset \vdash : \mathrm{Hp} . \supset . (\exists \nu) . \nu \in \mathrm{NC\,ind} . \rho^\nu > \sigma^\nu \times_c \xi : \supset \vdash . \mathrm{Prop}$

***356·214.** $\vdash : \rho, \sigma \in \mathrm{NC\,ind} - \iota`0 . \rho > \sigma . X \in C`H . \supset .$
$(\exists \nu) . \nu \in \mathrm{NC\,ind} . (\rho/\sigma)^\nu \breve{H} X$ [*356·213]

***356·215.** $\vdash : \lambda \in C`\Theta . \rho, \sigma \in \mathrm{NC\,ind} - \iota`0 . \rho > \sigma . \supset .$
$(\exists X) . X \in \lambda . X \times_s \rho/\sigma \sim\in \lambda$

Dem.

$\vdash . *305·142 . \mathrm{Induct} . \supset \vdash :. \lambda \subset C`H . \exists ! \lambda . \nu \in \mathrm{NC\,ind} - \iota`0 :$
$X \in \lambda . \supset_X . X \times_s \rho/\sigma \in \lambda : \supset : X \in \lambda . \supset_X . X \times_s \rho^\nu/\sigma^\nu \in \lambda :$

[*356·214] $\supset : H``\lambda = C`H$ (1)

$\vdash . (1) . \mathrm{Transp} . \supset \vdash . \mathrm{Prop}$

***356·22.** $\vdash : \mathrm{Hp}\, *356·13 . Q \in \mathrm{semi\,Ded} . \supset .$
$\mathrm{tl}_P`\hat{S} \{S \in \kappa : (\exists \nu) . R = S^\nu\} = I \upharpoonright C`Q$

Dem.

$\vdash . *336·511 . \supset \vdash :. \mathrm{Hp} . L = \mathrm{tl}_P`\hat{S} \{(\exists \nu) . R = S^\nu\} . \mu, \nu \in \mathrm{NC\,ind} - \iota`0 . \supset :$
$S \in \kappa . S^{\mu \times_c \nu} = R . \supset . L^\nu P S^\nu :$

[*301·5] $\supset : T \in \kappa . T^\mu = R . \supset . L^\nu P T :$

[Hp] $\supset : L^\nu P_* L$ (1)

$\vdash . *337·21 . \supset \vdash : \mathrm{Hp} . \nu \in \mathrm{NC\,ind} - \iota`0 - \iota`1 . L \in \kappa_\partial . \supset . L P L^\nu$ (2)

$\vdash . (1) . (2) . \supset \vdash : \mathrm{Hp} . \supset . L \sim\in \kappa_\partial : \supset \vdash . \mathrm{Prop}$

***356·221.** $\vdash : \mathrm{Hp}\, *356·19 . \breve{Q} = \dot{s}`\kappa_\partial . \lambda \subset \kappa . L = \mathrm{tl}_P`\lambda . R \in \kappa_\partial . \supset .$
$R \mid L = \mathrm{tl}_P`R \mid ``\lambda$

Dem.

$\vdash . *334·15 . *336·411 . \supset \vdash :. \mathrm{Hp} . \supset : LPM . \supset . (R \mid L) P (R \mid M) :$

[Hp] $\supset : M \in \lambda . \supset . (R \mid L) P (R \mid M) :$

[*37·61] $\supset : R \mid ``\lambda \subset \overleftarrow{P}`(R \mid L)$ (1)

$\vdash . *336\cdot41 . \supset \vdash : \text{Hp} . (R \mid L) PM . \supset . (\exists N) . N \epsilon \kappa_\partial . M = R \mid L \mid N .$

[*330·31] $\supset . (\exists N) . N \epsilon \kappa_\partial . \breve{R} \mid M = L \mid N .$

[*336·41.*334·13] $\supset . LP(\breve{R} \mid M) . \breve{R} \mid M \epsilon \kappa_\partial .$ (2)

[Hp] $\supset . (\exists N) . N \epsilon \lambda . NP(\breve{R} \mid M) .$

[*336·411.(2)] $\supset . (\exists N) . N \epsilon \lambda . (R \mid N) PM .$

[*37·1] $\supset . M \epsilon \breve{P}``R \mid ``\lambda$ (3)

$\vdash . (1) . (3) . *207\cdot21 . \supset \vdash .$ Prop

***356·23.** $\vdash : \text{Hp} *356\cdot22 . RPS . \supset . (\exists \nu) . \nu \epsilon \text{NC ind} - \iota`0 . [\{(\nu +_c 1)/\nu\} \upharpoonright \kappa`R] PS$

Dem.

$\vdash . *356\cdot22\cdot221 . \supset \vdash : \text{Hp} . \lambda = \hat{T}\{T \epsilon \kappa : (\exists \nu) . R = T^\nu\} . \supset . \text{tl}_P`R \mid ``\lambda = R .$

[Hp] $\supset . (\exists T) . T \epsilon \lambda . (R \mid T) PS .$

[Hp] $\supset . (\exists \nu) . \nu \epsilon \text{NC ind} - \iota`0 . \{R \mid (1/\nu) \upharpoonright \kappa`R\} PS .$

[*350·62.*334·32] $\supset . (\exists \nu) . \nu \epsilon \text{NC ind} - \iota`0 . [\{(\nu +_c 1)/\nu\} \upharpoonright \kappa`R] PS : \supset \vdash .$ Prop

***356·231.** $\vdash : \text{Hp} *356\cdot23 . \supset . (\exists \nu) . \nu \epsilon \text{NC ind} - \iota`0 . SP[\{(\nu -_c 1)/\nu\} \upharpoonright \kappa`R]$
[Proof as in *356·23]

***356·24** $\vdash : \text{Hp} *356\cdot23 . X \epsilon \dot{s}``C`\Theta . \supset . X_\kappa`R \neq X_\kappa`S$

Dem.

$\vdash . *356\cdot23 . \supset \vdash : \text{Hp} . \lambda \epsilon C`\Theta . X = \dot{s}`\lambda . \supset .$
$(\exists \rho, \sigma) . \rho, \sigma \epsilon \text{NC ind} - \iota`0 . \rho > \sigma . \{(\rho/\sigma) \upharpoonright \kappa`R\} PS .$

[*356·215] $\supset . (\exists \rho, \sigma, Y) . \rho, \sigma \epsilon \text{NC induct} - \iota`0 . \rho > \sigma . Y \epsilon \lambda . Y \times_s \rho/\sigma \sim \epsilon \lambda .$
$\{(\rho/\sigma) \upharpoonright \kappa`R\} PS .$

[*336·511] $\supset . (\exists \rho, \sigma, Y) . \rho, \sigma \epsilon \text{NC ind} - \iota`0 . \rho > \sigma . Y \epsilon \lambda . Y \times_s \rho/\sigma \epsilon p`\overleftarrow{H}``\lambda .$
$\{Y \upharpoonright \kappa`(\rho/\sigma) \upharpoonright \kappa`R\} P \{Y \upharpoonright \kappa`S\} .$

[*351·31.*356·13] $\supset . (\exists \rho, \sigma, Y) . Y \upharpoonright \kappa`(\rho/\sigma) \upharpoonright \kappa`R \epsilon p`\overleftarrow{P}``\overrightarrow{X}`R \cap P``\overrightarrow{X}`S .$

[*356·1] $\supset . X_\kappa`R \neq Y_\kappa`R : \supset \vdash .$ Prop

***356·25.** $\vdash : \text{Hp} *356\cdot22 . X \epsilon \dot{s}``C`\Theta . \supset . X_\kappa`R \subset \breve{Q}$

Dem.

$\vdash . *356\cdot1\cdot21 . \supset \vdash : \text{Hp} . \supset . X_\kappa`R \epsilon \kappa_\partial$ (1)

$\vdash . (1) . *41\cdot13 . \supset \vdash .$ Prop

***356·26.** $\vdash : \text{Hp} *356\cdot25 . \supset . X_\kappa \upharpoonright \kappa \epsilon 1 \to 1$

Dem.

$\vdash . *356\cdot24 . \text{Transp} . \supset \vdash : \text{Hp} . R, S \epsilon \kappa_\partial . X_\kappa`R = X_\kappa`S . \supset . R = S$ (1)

$\vdash . (1) . *356\cdot18\cdot21 . \supset \vdash .$ Prop

***356·3.** $\vdash :. \kappa \epsilon FM \text{ ap conx subm} . \dot{s}`\text{Pot}``\kappa \subset \kappa . \mu, \nu \epsilon C`\Theta . R, S \epsilon \kappa . \supset :$
$R(\dot{s}`\mu \times_r \dot{s}`\nu) S . \equiv . R \{(\dot{s}`\mu) \restriction \kappa \ (\dot{s}`\nu)\} S$

Dem.

$\vdash . *314·14 . *313·21 . \supset \vdash : \text{Hp} . \supset . \dot{s}`\mu \times_r \dot{s}`\nu = \dot{s}`s`\mu \underset{,,}{\times_s} ``\nu$ (1)

$\vdash . (1) . \supset \vdash :. \text{Hp} . \supset : R(\dot{s}`\mu \times_r \dot{s}`\nu) S . \equiv . (\exists M, N) . M \epsilon \mu . N \epsilon \nu . R(M \times_s N) S .$

[*351·31·22] $\equiv . (\exists M, N) . M \epsilon \mu . N \epsilon \nu . R(M \restriction \kappa \mid N) S :. \supset \vdash . \text{Prop}$

***356·31.** $\vdash : \kappa \epsilon FM \text{ ap conx subm} . \dot{s}`\text{Pot}``\kappa \subset \kappa . X, Y \epsilon \dot{s}``C`\Theta . \supset .$
$(X \times_r Y) \upharpoonright \kappa = (X \upharpoonright \kappa) \mid (Y \upharpoonright \kappa)$ [*356·3]

***356·32.** $\vdash : \kappa \epsilon FM \text{ sr subm} . X, Y \epsilon \dot{s}``C`\Theta . R \epsilon \kappa_\partial . \supset . X_\kappa`Y_\kappa`R = (X \mid Y)_\kappa`R$

Dem.

$\vdash . *356·191 . \supset \vdash :. \text{Hp} . \supset : S \epsilon \kappa \cap \overrightarrow{Y}`R . \supset . \kappa \cap \overrightarrow{X}`S \subset P``\overrightarrow{X}`Y_r`R :$

[*37·63] $\supset : X``(\kappa \cap \overrightarrow{Y}`R) \subset P``\overrightarrow{X}`Y_\kappa`R$ (1)

$\vdash . *305·6 . \supset \vdash : \text{Hp} . \lambda \epsilon C`\Theta . X = \dot{s}`\lambda . Z, Z' \epsilon \lambda . ZHZ' . \supset .$
$Z \upharpoonright \kappa`Y_\kappa`R = Z' \upharpoonright \kappa`(Z \mid \breve{Z}') \upharpoonright \kappa`Y_\kappa`R .$

[*356·12] $\supset . Z \upharpoonright \kappa`Y_\kappa`R \epsilon Z' \upharpoonright \kappa``\overrightarrow{P}`Y_\kappa`R .$

[*356·17] $\supset . Z \upharpoonright \kappa`Y_\kappa`R \epsilon Z' \upharpoonright \kappa``P``\overrightarrow{Y}`R .$

[*356·19] $\supset . Z \upharpoonright \kappa`Y_\kappa`R \epsilon P``Z' \upharpoonright \kappa``\overrightarrow{Y}`R .$

[Hp] $\supset . Z \upharpoonright \kappa`Y_\kappa`R \epsilon P``X``\overrightarrow{Y}`R$ (2)

$\vdash . (1) . (2) . \supset \vdash : \text{Hp} . \supset . P``X``\overrightarrow{Y}`R = P``\overrightarrow{X}`Y_\kappa`R .$

[*356·1] $\supset . (X \mid Y)_\kappa`R = X_\kappa`Y_\kappa`R : \supset \vdash . \text{Prop}$

***356·33.** $\vdash : \text{Hp} *356·32 . \supset . X_\kappa \mid Y_\kappa = (X \times_r Y)_\kappa$ [*356·31·32]

***356·4.** $\vdash : \kappa \epsilon FM \text{ conx} . Q = \text{Cnv}`\dot{s}`\kappa_\partial . S \epsilon \kappa . \alpha \subset C`Q . \exists ! \alpha . E ! \text{seq}_Q`\alpha . \supset .$
$S`\text{seq}_Q`\alpha = \text{seq}_Q`S``\alpha$

Dem.

$\vdash . *330·563 . \supset \vdash : \text{Hp} . \supset . S`\text{seq}_Q`\alpha \epsilon p`\overleftarrow{Q}``S``\alpha$ (1)

$\vdash . *37·1 . \supset \vdash :: \text{Hp} . \supset :. S`z \epsilon \breve{Q}``p`\overleftarrow{Q}``S``\alpha . \equiv :$
$(\exists y) : x \epsilon \alpha . \supset_x . S`xQy : yQS`z :$

[*330·542] $\equiv : (\exists w) : x \epsilon \alpha . \supset_x . S`x \, Q \, S`w : S`w \, Q \, S`z :$

[*208·2] $\equiv : (\exists w) : x \epsilon \alpha . \supset_x . xQw : wQz :$

[*37·1] $\equiv : z \epsilon \breve{Q}``p`\overleftarrow{Q}``\alpha$ (2)

$\vdash . (2) . \text{Transp} . \supset \vdash :. \text{Hp} . \supset : z \sim\epsilon \breve{Q}``p`\overleftarrow{Q}``\alpha . \equiv . S`z \sim\epsilon \breve{Q}``p`\overleftarrow{Q}``S``\alpha$ (3)

$\vdash . (1) . (3) . *330·542 . \supset \vdash . \text{Prop}$

***356·41.** $\vdash :. \kappa \in FM \operatorname{conx} \operatorname{trs} . \breve{P} = U_\kappa . \breve{Q} = \dot{s}`\kappa_\partial . a \in C`Q . \lambda \subset \kappa . \exists ! \lambda . \supset :$
$N = \operatorname{seq}_P`\lambda . \equiv . N \in \kappa . \operatorname{seq}_Q`A_a``\lambda = N`a$

Dem.

$\vdash . *336·43·2 . *206·61 . \supset$

$\vdash :. \mathrm{Hp} . \supset : N = \operatorname{seq}_P`\lambda . \equiv . N \in \kappa . A_a`N = \operatorname{seq}(Q \upharpoonright A_a``\kappa)`A_a``\lambda$ (1)

$\vdash . *206·211 . \supset \vdash : \mathrm{Hp} . b = \operatorname{seq}_Q`A_a``\lambda . \supset . (\exists R) . R \in \lambda . R`aQb .$

[Hp] $\supset . (\exists S) . S \in \kappa . bSa .$

[*336·11] $\supset . b \in A_a``\kappa$ (2)

$\vdash . (1) . (2) . \supset \vdash :. \mathrm{Hp} . \supset : N = \operatorname{seq}_P`\lambda . \equiv . N \in \kappa . A_a`N = \operatorname{seq}_Q`A_a``\lambda$ (3)

$\vdash . (3) . *336·11 . \supset \vdash . \mathrm{Prop}$

***356·42.** $\vdash : \mathrm{Hp}\, *356·41 . \mathrm{E} ! \operatorname{seq}_P`\lambda . \supset . (\operatorname{seq}_P`\lambda)`a = \operatorname{seq}_Q`A_a``\lambda$ [*356·41]

***356·43.** $\vdash : \mathrm{Hp}\, *356·22 . \xi \in C`\Theta . X = \dot{s}`\xi . a \in C`Q . \supset .$
$(X_\kappa`R)`a = \operatorname{seq}_Q`A_a``A_R``\xi = \operatorname{seq}_Q`A_a``\overrightarrow{X}`R$
[*356·42·11·21 . *336·12]

***356·5.** $\vdash : \mathrm{Hp}\, *356·22 .$
$X, Y \in \dot{s}``C`\Theta . a \in C`Q . R \in \kappa . \lambda = \kappa \cap \overrightarrow{X}`R . \mu = \kappa \cap \overrightarrow{Y}`R . \supset .$
$(X_\kappa`R)`(Y_\kappa`R)`a = \operatorname{seq}_Q`\dot{s}`\overrightarrow{\lambda}`\operatorname{seq}_Q`\dot{s}`\overrightarrow{\mu}`a$

Dem.

$\vdash . *356·43 . *336·12 . \supset \vdash : \mathrm{Hp} . \supset . (X_\kappa`R)`(Y_\kappa`R)`a = \operatorname{seq}_Q`\dot{s}`\overrightarrow{\lambda}`(Y_\kappa`R)`a$

[*356·43.*336·12] $= \operatorname{seq}_Q`\dot{s}`\overrightarrow{\lambda}`\operatorname{seq}_Q`\dot{s}`\overrightarrow{\mu}`a : \supset \vdash . \mathrm{Prop}$

***356·51.** $\vdash : \mathrm{Hp}\, *356·5 . \supset . (X +_r Y)_\kappa`R = \operatorname{seq}_P`s`\lambda \,|_{,,}``\mu$

Dem.

$\vdash . *356·11 . *314·13 . \supset \vdash : \mathrm{Hp} . \xi, \eta \in C`\Theta . X = \dot{s}`\xi . Y = \dot{s}`\eta . \supset .$
$(X +_r Y)_\kappa`R = \operatorname{seq}_P`A_R``(\xi +_a \eta)$

[*312·32.*311·11.*308·32] $= \operatorname{seq}_P`A_R``s`\xi \underset{,,}{+_s}``\eta$

[*336·11] $= \operatorname{seq}_P`\hat{N}\{(\exists L, M) . L \in \xi . M \in \eta . N = (L +_s M) \upharpoonright \kappa`R\}$

[*351·43] $= \operatorname{seq}_P`\hat{N}\{(\exists L, M) . L \in \xi . M \in \eta . N = (L \upharpoonright \kappa`R) | (M \upharpoonright \kappa`R)\}$

[Hp] $= \operatorname{seq}_P`\hat{N}\{(\exists U, W) . U \in \lambda . W \in \mu . N = U | W\} : \supset \vdash . \mathrm{Prop}$

***356·52.** $\vdash : \mathrm{Hp}\, *356·5 . \supset . \{(X +_r Y)_\kappa`R\}`a = \operatorname{seq}_Q`(\dot{s}`\lambda)``\dot{s}`\overrightarrow{\mu}`a$

Dem.

$\vdash . *356·51 . \supset \vdash : \mathrm{Hp} . \supset . \{(X +_r Y)_\kappa`R\}`a = (\operatorname{seq}_P`s`\lambda \,|_{,,}``\mu)`a$

[*356·42] $= \operatorname{seq}_Q`A_a``s`\lambda \,|_{,,}``\mu$

[*336·11] $= \operatorname{seq}_Q`\hat{x}\{(\exists X, Y) . X \in \lambda . Y \in \mu . x = (X | Y)`a\}$

[*41·11] $= \operatorname{seq}_Q`\hat{x}\{(\exists X) . X \in \lambda . x \in X``\dot{s}`\overrightarrow{\mu}`a\}$

[*41·11] $= \operatorname{seq}_Q`(\dot{s}`\lambda)``\dot{s}`\overrightarrow{\mu}`a : \supset \vdash . \mathrm{Prop}$

***356·53.** $\vdash : \text{Hp} *356\cdot5 \,.\supset .\, \text{seq}_Q{}^{\backprime}\overrightarrow{\dot{s}{}^{\backprime}\lambda}{}^{\backprime}\text{seq}_Q{}^{\backprime}\overrightarrow{\dot{s}{}^{\backprime}\mu}{}^{\backprime}a = \text{seq}_Q{}^{\backprime}(\dot{s}{}^{\backprime}\lambda){}^{\backprime\backprime}\overrightarrow{\dot{s}{}^{\backprime}\mu}{}^{\backprime}a$

Dem.

$\vdash . *356\cdot16 . \supset \vdash : \text{Hp} . \supset . \text{seq}_Q{}^{\backprime}\overrightarrow{\dot{s}{}^{\backprime}\lambda}{}^{\backprime}\text{seq}_Q{}^{\backprime}\overrightarrow{\dot{s}{}^{\backprime}\mu}{}^{\backprime}a = \text{lt}_Q{}^{\backprime}\overrightarrow{\dot{s}{}^{\backprime}\lambda}{}^{\backprime}\text{seq}_Q{}^{\backprime}\overrightarrow{\dot{s}{}^{\backprime}\mu}{}^{\backprime}a$

$[*41\cdot11] \quad = \text{lt}_Q{}^{\backprime}\hat{x}\,\{(\exists L) . L \in \lambda . x = L{}^{\backprime}\text{seq}_Q{}^{\backprime}\overrightarrow{\dot{s}{}^{\backprime}\mu}{}^{\backprime}a\}$

$[*356\cdot4] \quad = \text{lt}_Q{}^{\backprime}\hat{x}\,\{(\exists L) . L \in \lambda . x = \text{seq}_Q{}^{\backprime}L{}^{\backprime\backprime}\overrightarrow{\dot{s}{}^{\backprime}\mu}{}^{\backprime}a\}$

$[*356\cdot16.\text{Hp}] = \text{lt}_Q{}^{\backprime}\hat{x}\,\{(\exists L) . L \in \lambda . x = \text{lt}_Q{}^{\backprime}L{}^{\backprime\backprime}\overrightarrow{\dot{s}{}^{\backprime}\mu}{}^{\backprime}a\}$

$[*207\cdot55] \quad = \text{lt}_Q{}^{\backprime}s{}^{\backprime}\hat{\alpha}\,\{(\exists L) . L \in \lambda . \alpha = L{}^{\backprime\backprime}\overrightarrow{\dot{s}{}^{\backprime}\mu}{}^{\backprime}a\}$

$[*41\cdot11] \quad = \text{lt}_Q{}^{\backprime}(\dot{s}{}^{\backprime}\lambda){}^{\backprime\backprime}\overrightarrow{\dot{s}{}^{\backprime}\mu}{}^{\backprime}a$

$[*356\cdot16] \quad = \text{seq}_Q{}^{\backprime}(\dot{s}{}^{\backprime}\lambda){}^{\backprime\backprime}\overrightarrow{\dot{s}{}^{\backprime}\mu}{}^{\backprime}a : \supset \vdash . \text{Prop}$

***356·54.** $\vdash : \kappa \in FM \text{ sr subm} . \text{Cnv}{}^{\backprime}\dot{s}{}^{\backprime}\kappa_\partial \in \text{semi Ded} . X, Y \in \dot{s}{}^{\backprime\backprime}C{}^{\backprime}\Theta . R \in \kappa_\partial . \supset .$
$(X_\kappa{}^{\backprime}R) \,|\, (Y_\kappa{}^{\backprime}R) = (X +_r Y)_\kappa{}^{\backprime}R \quad [*356\cdot5\cdot53\cdot52]$

***356·6.** $\vdash : \kappa \in FM \text{ sr} . R \in \kappa_\partial . \breve{P} = U_\kappa . \breve{Q} = \dot{s}{}^{\backprime}\kappa_\partial . X \in C{}^{\backprime}H . \supset .$
$\kappa \cap A_R{}^{\backprime\backprime}\overrightarrow{H}{}^{\backprime}X \subset \overrightarrow{P}{}^{\backprime}X \upharpoonright \kappa{}^{\backprime}R$

Dem.

$\vdash . *37\cdot6 . \supset \vdash :. \text{Hp} . \supset : M \in A_R{}^{\backprime\backprime}\overrightarrow{H}{}^{\backprime}X . \equiv . (\exists Y) . YHX . MYR .$

$[*352\cdot7] \quad \supset . MP(X \upharpoonright \kappa{}^{\backprime}R) :. \supset \vdash . \text{Prop}$

***356·61.** $\vdash : \text{Hp} *356\cdot6 . \kappa \in FM \text{ subm} . Q \in \text{semi Ded} . SP(X \upharpoonright \kappa{}^{\backprime}R) . \supset .$
$(\exists Y) . YHX . SP(Y \upharpoonright \kappa{}^{\backprime}R)$

Dem.

$\vdash . *356\cdot231 . \supset \vdash : \text{Hp} . \supset . (\exists \nu) . \nu \in \text{NC ind} - \iota{}^{\backprime}0 . SP[\{(\nu -_c 1)/\nu\} \upharpoonright \kappa{}^{\backprime}X \upharpoonright \kappa{}^{\backprime}R]$

$[*351\cdot31] \quad \supset . (\exists \nu) . \nu \in \text{NC ind} - \iota{}^{\backprime}0 . SP[\{(\nu -_c 1)/\nu \times_s X\} \upharpoonright \kappa{}^{\backprime}R]$

$[*305\cdot71\cdot51] \quad \supset . (\exists Y) . YHX . SP(Y \upharpoonright \kappa{}^{\backprime}R) : \supset \vdash . \text{Prop}$

***356·62.** $\vdash : \text{Hp} *356\cdot6 . \kappa \in FM \text{ subm} . Q \in \text{semi Ded} . \supset .$
$\overrightarrow{P}{}^{\backprime}X \upharpoonright \kappa{}^{\backprime}R \subset P{}^{\backprime\backprime}A_R{}^{\backprime\backprime}\overrightarrow{H}{}^{\backprime}X \quad [*356\cdot61]$

***356·63.** $\vdash : \text{Hp} *356\cdot62 . \supset . (\dot{s}{}^{\backprime}\overrightarrow{H}{}^{\backprime}X)_\kappa = X \upharpoonright \kappa$

Dem.

$\vdash . *356\cdot6\cdot62 . \supset \vdash : \text{Hp} . \supset . X \upharpoonright \kappa{}^{\backprime}R = \text{lt}_P{}^{\backprime}A_R{}^{\backprime\backprime}\overrightarrow{H}{}^{\backprime}X .$

$[*356\cdot11] \quad \supset . X \upharpoonright \kappa{}^{\backprime}R = (\dot{s}{}^{\backprime}\overrightarrow{H}{}^{\backprime}X)_\kappa{}^{\backprime}R \quad (1)$

$\vdash . (1) . *356\cdot21 . \supset \vdash . \text{Prop}$

*359. EXISTENCE-THEOREMS FOR VECTOR-FAMILIES.

Summary of *359.

In this number we prove that, assuming the axiom of infinity, there are vector-families of the various kinds considered in previous numbers.

If P is any well-ordered series having no last term, the converses of the interval-relations, *i.e.* the class finid‘$\breve{P}$, form an open family of $C\text{‘}P$ (*359·11). If P is a progression, this family is serial and initial (*359·12).

The family consisting of additions of positive ratios to positive ratios (including 0_q), *i.e.* consisting of all terms of the form $(+_s X) \upharpoonright C\text{‘}H'$, where $X \epsilon C\text{‘}H'$, is initial, serial, open, and submultipliable (*359·21), assuming the axiom of infinity. The family consisting of generalized additions of positive ratios to generalized ratios is serial, open, and submultipliable, but not initial (*359·25).

The family consisting of multiplications of positive ratios not 0_q by positive ratios not 0_q is open and connected, but not serial or submultipliable (*359·22); if we confine the multipliers to ratios not less than 1/1, the family becomes serial (*359·25).

The family consisting of additions of positive real numbers to positive real numbers (including $\iota\text{‘}0_q$) is serial, initial, and submultipliable (*359·31); the family consisting of generalized additions of positive real numbers (including $\iota\text{‘}0_q$) to generalized real numbers is serial and submultipliable, but not initial (*359·32). Similar propositions hold for multiplication, provided $\iota\text{‘}0_q$ is omitted; but the resulting families will not be serial. In the case where the field is confined to positive real numbers, however, the family becomes serial if the multipliers are confined to such as are not less than $\overrightarrow{H}\text{‘}(1/1)$, which is the real number 1.

The last set of propositions in this number (*359·4—·44) are concerned in proving that, given a family κ whose field is β, if S is a correlator of α and β, $S\dagger\text{“}\kappa$ is a family whose field is α, and which has the same properties of being connected, open, etc. as the original family κ. Hence if κ is a family whose field is the real numbers, and we are given any class α similar to the real numbers (in other words the field of any continuous series), if S is the correlator

of this class with the real numbers, $S\dagger``\kappa$ gives a family whose field is α. Hence from our previous existence-theorems we derive the existence, for α, of an initial serial family, giving us a system of measurement for α. Similarly if α is similar to the rationals.

***359·1.** $\vdash : P \epsilon \Omega . \sim E ! B`\breve{P} . \supset . \text{finid}`\breve{P} \epsilon \text{Cl ex}`\text{cr}`C`P$

Dem.

$\vdash . *260·23·28 . \supset \vdash : \text{Hp} . \supset . \text{finid}`\breve{P} \subset 1 \rightarrow 1$ (1)

$\vdash . *121·302 . \supset \vdash : \text{Hp} . \supset . D`P_0 = C`P$ (2)

$\vdash . (2) . *121·302·35 . *260·28 . \supset$

$\vdash : \text{Hp} . \nu \epsilon \text{NC ind} . D`P_\nu = C`P . \supset . D`P_{\nu +_c 1} = C`P$ (3)

$\vdash . (2) . (3) . \text{Induct} . \supset \vdash : \text{Hp} . R \epsilon \text{finid}`P . \supset . D`R = C`P$ (4)

$\vdash . *121·322 . \supset \vdash : R \epsilon \text{finid}`P . \supset . \text{Cl}`R \subset C`P$ (5)

$\vdash . (1) . (4) . (5) . *330·1 . \supset \vdash . \text{Prop}$

***359·11.** $\vdash : P \epsilon \Omega . \sim E ! B`\breve{P} . \supset . \text{finid}`\breve{P} \epsilon \text{fm ap}`C`P$

Dem

$\vdash . *260·28 . *121·352 . \supset \vdash : \text{Hp} . \supset . \text{finid}`\breve{P} \epsilon \text{Abel}$ (1)

$\vdash . *71·19 . \supset \vdash : \text{Hp} . \mu, \nu \epsilon \text{NC ind} . \exists ! P_\mu | \breve{P}_\nu \dot{\frown} J . \supset . \mu \neq \nu$ (2)

$\vdash . *121·35 . \supset \vdash : \text{Hp}(2) . \mu > \nu . \supset . P_\mu | \breve{P}_\nu \in P_{\mu -_c \nu} .$

$[*91·6.*121·36] \quad \supset . (P_\mu | \breve{P}_\nu)_{po} \in J$ (3)

Similarly $\vdash : \text{Hp}(2) . \nu > \mu . \supset . (P_\mu | \breve{P}_\nu)_{po} \in J$ (4)

$\vdash . (2) . (3) . (4) . \supset \vdash : \text{Hp} . L \epsilon (\text{finid}`\breve{P})_{\iota \partial} . \supset . L_{po} \in J$ (5)

$\vdash . (1) . (5) . *359·1 . \supset \vdash . \text{Prop}$

***359·12.** $\vdash : P \epsilon \omega . \kappa = \text{finid}`\breve{P} . \supset . \kappa \epsilon \text{fm sr init}`C`P . \dot{s}`\kappa_\partial = \breve{P}$

Dem

$\vdash . *263·14·141 . *122·1 . \supset \vdash : \text{Hp} . \supset . \dot{s}`\overrightarrow{\kappa}`B`P = C`P$ (1)

$\vdash . *263·14·141 . \supset \vdash : \text{Hp} . \supset . \dot{s}`\kappa_\partial = \breve{P} .$ (2)

$[*334·31.*359·11] \quad \supset . \kappa \epsilon FM \text{ sr}$ (3)

$\vdash . (1) . (2) . (3) . *335·14 . \supset \vdash . \text{Prop}$

***359·2.** $\vdash : \text{Infin ax} . \kappa = \hat{R} \{(\exists X) . X \epsilon C`H' . R = (+_s X) \upharpoonright C`H'\} . \supset .$
$\kappa \epsilon FM . \dot{s}`\kappa_\partial = \breve{H}'$

Dem.

$\vdash . *306·54·25 . *304·49 . \supset \vdash : \text{Hp} . \supset . \kappa \subset 1 \rightarrow 1$ (1)

$\vdash . *306·25 . *304·49 . \supset \vdash : \text{Hp} . R \epsilon \kappa . \supset . \text{Cl}`R = C`H' . D`R \subset C`H'$ (2)

$\vdash . *306·11·31 . \supset \vdash : \text{Hp} . R, S \epsilon \kappa . \supset . R | S = S | R$ (3)

$\vdash . *306·52 . \supset \vdash : \text{Hp} . \supset . \dot{s}`\kappa_\partial = \breve{H}'$ (4)

$\vdash . (1) . (2) . (3) . (4) . \supset \vdash . \text{Prop}$

***359·21.** $\vdash : \text{Hp} *359\cdot2 . \supset . \kappa \in FM \text{ init sr subm} . \dot{s}`\kappa_\partial = \breve{H}'$

Dem.

$\vdash . *306\cdot24 . \supset \vdash : \text{Hp} . \supset . \dot{s}`\overrightarrow{\kappa}`0_q = C`H'$ (1)

$\vdash . *306\cdot41 . \supset$

$\vdash :. \text{Hp} . X \in C`H' . \mu, \nu \in \text{NC ind} - \iota`0 . S = \{+_s(X \times_s 1/\nu)\} \upharpoonright C`H' . \supset :$

$S^\mu = \{+_s(X \times_s \mu/\nu)\} \upharpoonright C`H' . \supset . S^{\mu +_c 1} = \{+_s(X \times_s \overline{\mu +_c 1}/\nu)\} \upharpoonright C`H' :$

$[\text{Induct}] \supset : S^\mu = \{+_s(X \times_s \mu/\nu)\} \upharpoonright C`H' :$

$[*305\cdot51] \supset : S^\nu = (+_s X) \upharpoonright C`H'$ (2)

$\vdash . (2) . *351\cdot1 . *359\cdot2 . \supset \vdash : \text{Hp} . \supset . \kappa \in FM \text{ subm}$ (3)

$\vdash . (1) . (3) . *359\cdot2 . *334\cdot31 . \supset \vdash . \text{Prop}$

***359·22.** $\vdash : \text{Infin ax} . \kappa = \hat{R}\{(\exists X) . X \in C`H' . R = (+_g X) \upharpoonright C`H_g\} . \supset .$

$\kappa \in FM \text{ sr subm} . \dot{s}`\kappa_\partial = \breve{H}_g$

The proof proceeds as in *359·21, but in this case there is no origin. Every member of κ is a connected point, *i.e.* a member of conx$`\kappa$. This results from *308·54. If, in *359·21, we substitute H for H', the proposition holds except that κ has no origin.

***359·23.** $\vdash : \text{Infin ax} . \kappa = \hat{R}\{(\exists X) . X \in C`H . R = (\times_s X) \upharpoonright C`H\} . \supset .$

$\kappa \in FM \text{ ap conx}$

The proof proceeds as in *359·21. We have to take H instead of H', because $(\times_s 0_q) \upharpoonright C`H'$ is not $1 \to 1$. We do not get $\kappa \in FM$ subm, because not every rational has a rational νth root.

***359·24.** $\vdash : \text{Infin ax} .$

$\kappa = \hat{R}\{(\exists X) . X \in C`H_g - \iota`0_q . R = (\times_g X) \upharpoonright (C`H_g - \iota`0_q)\} . \supset .$

$\kappa \in FM \text{ ap conx}$

The proof proceeds as in *359·23.

***359·25.** $\vdash : \text{Infin ax} . \kappa = \hat{R}\{(\exists X) . (1/1) H_* X . R = (\dot{\times}_s X) \upharpoonright C`H\} . \supset .$

$\kappa \in FM \text{ sr} . \dot{s}`\kappa_\partial = \breve{H}$

The proof proceeds as in *359·21.

***359·31.** $\vdash : \text{Infin ax} . \kappa = \hat{R}\{(\exists \mu) . \mu \in C`\Theta' . R = (+_p \mu) \upharpoonright C`\Theta'\} . \supset .$

$\kappa \in FM \text{ sr init subm} . \dot{s}`\kappa_\partial = \breve{\Theta}'$

Dem.

$\vdash . *311\cdot74 . \quad \supset \vdash : \text{Hp} . \supset . \kappa \subset 1 \to 1$ (1)

$\vdash . *311\cdot27 . \quad \supset \vdash : \text{Hp} . R \in \kappa . \supset . \text{Q}`R = C`\Theta' . \text{D}`R \subset C`\Theta'$ (2)

$\vdash . *311\cdot43 . \quad \supset \vdash : \text{Hp} . \supset . \iota`0_q \upharpoonright C`\Theta' = \text{init}`\kappa$ (3)

$\vdash . *311\cdot12\cdot121 . \supset \vdash : \text{Hp} . \supset . \kappa \in \text{Abel}$ (4)

$\vdash . *311\cdot65 . \quad \supset \vdash : \text{Hp} . \supset . \dot{s}`\kappa_\partial = \breve{\Theta}'$ (5)

$\vdash . (1) . (2) . (3) . (4) . (5) . \quad \supset \vdash : \mathrm{Hp} . \supset . \kappa \in FM \,\mathrm{sr\ init} . \dot{s}`\kappa_\partial = \breve{\Theta}'$ (6)

$\vdash . (6) . *310 \cdot 151 . *351 \cdot 11 . \supset \vdash : \mathrm{Hp} . \supset . \kappa \in FM \,\mathrm{subm}$ (7)

$\vdash . (6) . (7) . \supset \vdash . \mathrm{Prop}$

***359·32.** $\vdash : \mathrm{Infin\ ax} . \kappa = \hat{R} \{(\exists \mu) . \mu \in C`\Theta' . R = (+_a \mu) \upharpoonright C`\Theta_g\} . \supset .$

$\kappa \in FM \,\mathrm{sr\ subm} . \dot{s}`\kappa_\partial = \breve{\Theta}_g$

The proof proceeds as in *359·22. Similarly the analogues of *359·23·24·25 can be proved for real numbers; the resulting families, in these cases, will be submultipliable, but it will be necessary to omit $\iota`0_q$ from their fields.

***359·4.** $\vdash : \kappa \in \mathrm{Cl\ ex}`\mathrm{cr}`\beta . S \in \alpha \,\overline{\mathrm{sm}}\, \beta . \supset . S\dagger``\kappa \in \mathrm{Cl\ ex}`\mathrm{cr}`\alpha$

Dem.

$\vdash . *330 \cdot 1 . *71 \cdot 252 . \quad \supset \vdash : \mathrm{Hp} . \supset . S\dagger``\kappa \subset 1 \rightarrow 1$ (1)

$\vdash . *150 \cdot 21 \cdot 211 . *330 \cdot 1 . \supset \vdash : \mathrm{Hp} . R \in S\dagger``\kappa . \supset . \text{Ɑ}`R = S``\beta . \mathrm{D}`R \subset \text{Ɑ}`R .$

[*73·03] $\supset . \text{Ɑ}`R = \alpha . \mathrm{D}`R \subset \alpha$ (2)

$\vdash . (1) . (2) . *330 \cdot 1 . \supset \vdash . \mathrm{Prop}$

***359·401.** $\vdash : \kappa \in \mathrm{Abel} . S \in \mathrm{Cls} \rightarrow 1 . s`\text{Ɑ}``\kappa \subset \text{Ɑ}`S . \supset . S\dagger``\kappa \in \mathrm{Abel}$

Dem.

$\vdash . *72 \cdot 601 . \supset \vdash :. \mathrm{Hp} . \supset : P, Q \in \kappa . \supset . P \,|\, \breve{S} \,|\, S = P . Q \,|\, \breve{S} \,|\, S = Q .$ (1)

[*150·1] $\supset . (S\dagger P) \,|\, (S\dagger Q) = S \,|\, P \,|\, Q \,|\, \breve{S}$

[*330·5] $= S \,|\, Q \,|\, P \,|\, \breve{S}$

[(1).*150·1] $= (S\dagger Q) \,|\, (S\dagger P)$ (2)

$\vdash . (2) . *330 \cdot 5 . \supset \vdash . \mathrm{Prop}$

***359·41.** $\vdash : \kappa \in \mathrm{fm}`\beta . S \in \alpha \,\overline{\mathrm{sm}}\, \beta . \supset . S\dagger``\kappa \in \mathrm{fm}`\alpha$ [*359·4·401 . *330·51]

***359·411.** $\vdash : \kappa \in FM . a \in \mathrm{conx}`\kappa . S \in 1 \rightarrow 1 . s`\text{Ɑ}``\kappa = \text{Ɑ}`S . \supset . S`a \in \mathrm{conx}`S\dagger``\kappa$

Dem.

$\vdash . *151 \cdot 11 . \supset \vdash : \mathrm{Hp} . P = S ; \dot{s}`\kappa . \supset . S \in P \,\mathrm{smor}\, (\dot{s}`\kappa) .$

[*151·33] $\supset . \overrightarrow{P}`S`a \cup \overleftarrow{P}`S`a = S``\overrightarrow{\dot{s}`\kappa}`a \cup S``\overleftarrow{\dot{s}`\kappa}`a$

[*331·1] $= S``s`\text{Ɑ}``\kappa$

[*330·13.*150·211] $= \text{Ɑ}`S ; \dot{s}`\kappa$

[Hp] $= \text{Ɑ}`P$ (1)

$\vdash . *150 \cdot 16 . \supset \vdash : \mathrm{Hp}\,(1) . \supset . P = \dot{s}`S\dagger``\kappa$ (2)

$\vdash . (1) . (2) . *331 \cdot 1 . \supset \vdash . \mathrm{Prop}$

***359·412.** $\vdash : \kappa \in \mathrm{fm\ conx}`\beta . S \in \alpha \,\overline{\mathrm{sm}}\, \beta . \supset . S\dagger``\kappa \in \mathrm{fm\ conx}`\alpha$ [*359·41·411]

***359·413.** $\vdash : \kappa \in FM\,\mathrm{ap} \,.\, S \in 1 \to 1 \,.\, s`\text{Ɑ}``\kappa = \text{Ɑ}`S \,.\, \supset \,.\, S\dagger``\kappa \in FM\,\mathrm{ap}$

Dem.

$\vdash \,.\, *72{\cdot}601 \,.\quad \supset \vdash : \mathrm{Hp} \,.\, P, Q \in \kappa \,.\, \supset \,.\, (S ; \breve{P}) \mid (S ; Q) = S ; (\breve{P} \mid Q)$ (1)

$\vdash \,.\, (1) \,.\, *150{\cdot}4 \,.\, \supset \vdash : \mathrm{Hp}\,(1) \,.\, \dot{\exists} \,!\, (S ; \breve{P}) \mid (S ; Q) \mathbin{\dot\cap} J \,.\, \supset \,.\, \dot{\exists} \,!\, \breve{P} \mid Q \mathbin{\dot\cap} J \,.$

$[*333{\cdot}101] \qquad \supset \,.\, (\breve{P} \mid Q)_{\mathrm{po}} \Subset J \,.$

$[*200{\cdot}21] \qquad \supset \,.\, S ; (\breve{P} \mid Q)_{\mathrm{po}} \Subset J \,.$

$[*150{\cdot}83] \qquad \supset \,.\, \{S ; (\breve{P} \mid Q)\}_{\mathrm{po}} \Subset J$ (2)

$\vdash \,.\, (1) \,.\, (2) \,.\quad \supset \vdash :.\, \mathrm{Hp} \,.\, \supset : X, Y \in S\dagger``\kappa \,.\, \dot{\exists} \,!\, \breve{X} \mid Y \mathbin{\dot\cap} J \,.\, \supset \,.\, (\breve{X} \mid Y)_{\mathrm{po}} \Subset J$ (3)

$\vdash \,.\, *359{\cdot}4 \,.\quad \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, S\dagger``\kappa \in FM$ (4)

$\vdash \,.\, (3) \,.\, (4) \,.\, *333{\cdot}101 \,.\, \supset \vdash \,.\, \mathrm{Prop}$

***359·414.** $\vdash : \kappa \in FM \,.\, S \in 1 \to 1 \,.\, s`\text{Ɑ}``\kappa = \text{Ɑ}`S \,.\, a = \mathrm{init}`\kappa \,.\, \supset \,.\, S`a = \mathrm{init}`S\dagger``\kappa$

[Proof as in *359·411]

***359·415.** $\vdash : \kappa \in FM\,\mathrm{subm} \,.\, S \in 1 \to 1 \,.\, \text{Ɑ}`S = s`\text{Ɑ}``\kappa \,.\, \supset \,.\, S\dagger``\kappa \in FM\,\mathrm{subm}$

Dem.

$\vdash \,.\, *301{\cdot}21 \,.\quad \supset \vdash : \mathrm{Hp} \,.\, Y \in \kappa \,.\, \nu \in \mathrm{NC\,ind} \,.\, \supset \,.\, Y^{\nu +_c 1} = Y^{\nu} \mid Y$ (1)

$\vdash \,.\, (1) \,.\, *72{\cdot}601 \,.\, \supset \vdash : \mathrm{Hp} \,.\, S ; Y^{\nu} = (S ; Y)^{\nu} \,.\, \supset \,.\, S ; Y^{\nu +_c 1} = (S ; Y)^{\nu +_c 1}$ (2)

$\vdash \,.\, (2) \,.\, \mathrm{Induct} \,.\quad \supset \vdash : \mathrm{Hp}\,(1) \,.\, \supset \,.\, S ; Y^{\nu} = (S ; Y)^{\nu}$ (3)

$\vdash \,.\, *351{\cdot}1 \,.\quad \supset \vdash : \mathrm{Hp} \,.\, \nu \in \mathrm{NC\,ind} - \iota`0 \,.\, X \in \kappa \,.\, \supset \,.\, (\exists Y) \,.\, X = Y^{\nu} \,.\, Y \in \kappa \,.$

$[(3)] \qquad \supset \,.\, (\exists Y) \,.\, Y \in \kappa \,.\, S ; X = (S ; Y)^{\nu}$ (4)

$\vdash \,.\, (4) \,.\, *351{\cdot}1 \,.\, *359{\cdot}41 \,.\, \supset \vdash \,.\, \mathrm{Prop}$

***359·42.** $\vdash : \exists \,!\, \mathrm{fm\ conx\ ap\ subm}`\beta \,.\, \alpha \,\mathrm{sm}\, \beta \,.\, \supset \,.\, \exists \,!\, \mathrm{fm\ conx\ ap\ subm}`\alpha$

[*359·41·412·413·415]

***359·43.** $\vdash : P \in \dot{1} \dotplus \eta \,.\, \supset \,.\, \exists \,!\, FM\,\mathrm{init\ sr\ subm} \cap \hat{\kappa}\,(\dot{s}`\kappa_{\partial} = \breve{P})$

[*359·42·21·414 . *274·44 . *123·18 . *304·47 . *273·4]

***359·44.** $\vdash : \mathrm{Nr}`P \dotplus \dot{1} = \theta \,.\, \supset \,.\, \exists \,!\, FM\,\mathrm{init\ sr\ subm} \cap \hat{\kappa}\,(\dot{s}`\kappa_{\partial} = \breve{P})$

[*359·42·31·414 . *275·3 . *310·15 . *204·47]

SECTION D.

CYCLIC FAMILIES.

Summary of Section D.

The theory of measurement hitherto developed has been only applicable to *open* families. But in order to be able to deal with such cases as the angles at a point, or the elliptic straight line, we require a theory of measurement applicable to families which are not open. This theory is given briefly in the present Section.

When a family is not open, two vectors which have one ratio will usually also have many others, *i.e.* we shall not have $\dot{\exists} ! X \upharpoonright \kappa \dot{\frown} Y \upharpoonright \kappa . \supset . X = Y$, where X, Y are ratios. Also a ratio confined to the family will not usually be one-one. Under these circumstances, it is necessary, if measurement is to be possible, that there should be some way of distinguishing one among the ratios of two vectors as their "principal" ratio, and of then showing that, by confining ourselves to principal ratios, the requisite properties of ratios reappear.

The case of angles will serve to illustrate our procedure. Considered geometrically, not kinematically, a vector which is a multiple of 2π is identical with the null-vector, and if θ is any angle, $\theta = 2\nu\pi + \theta$, where ν is any integer positive or negative. We are here considering an angle as a vector whose field is all the rays in a given plane through a given point. Thus there will be two angles which are half of the null-vector, namely π and 2π, and four angles which are a quarter of the null-vector, namely $\pi/2$, π, $3\pi/2$ and 2π; and so on. The ratio of $\pi/2$ to π is any number of the form $(2\mu + 1)/(4\nu + 2)$; thus two terms may have many different ratios.

In order to evade this difficulty, we first arrange angles in a series ending with 2π, and having no first term, but proceeding from smaller to greater angles. Then the angles which have a given ratio μ/ν to a given angle will be finite in number, and therefore one of them will be the smallest. We take this as the "principal" angle having the ratio μ/ν to the given angle, and define "$(\mu/\nu)_\kappa$" to mean the relation between two angles consisting in the fact that the first is the "principal" angle having the ratio μ/ν to the second. Then of all the ratios between the two angles, the ratio μ/ν may be regarded

as the "principal" ratio. It will be found that, with suitable hypotheses, $(\mu/\nu)_\kappa$ has the properties required in order to make measurement possible.

In order to make the above method feasible, certain properties must be assumed to hold concerning the family κ. (These properties are all verified in the cases that arise in practice.) We shall therefore only speak of a family as cyclic when it fulfils the following conditions:

(1) It must be connected.

(2) It must contain a non-zero member which is identical with its converse. This is the property which makes the family cyclic. In the case of angles, the member in question is π.

(3) It must be such that $\kappa_\partial \upharpoonright U_\kappa$ is transitive. This is the property which enables us to arrange the field in a series. It will be observed that U_κ cannot be transitive, since, if K_κ is the member which is its own converse, we have

$$(I \upharpoonright s'\mathrm{C}''\kappa)\, U_\kappa K_\kappa \,.\, K_\kappa U_\kappa (I \upharpoonright s'\mathrm{C}''\kappa),$$

but we do not have $(I \upharpoonright s'\mathrm{C}''\kappa)\, U_\kappa (I \upharpoonright s'\mathrm{C}''\kappa)$, because U_κ is contained in diversity (by $*336{\cdot}6$). It is, however, possible that U_κ should be transitive so long as we do not start from $I \upharpoonright s'\mathrm{C}''\kappa$, and this we assume as part of the definition of cyclic families.

(4) In order to avoid trivial exceptions, we assume that κ does not have only two members, since otherwise it might consist only of $I \upharpoonright s'\mathrm{C}''\kappa$ and K_κ.

We are thus led to the following definition:

$$FM\,\mathrm{cycl} = (FM\,\mathrm{conx} - 2) \cap \hat{\kappa}\,\{\kappa_\partial \upharpoonright U_\kappa \,\epsilon\, \mathrm{trans} : (\exists K) \,.\, K \,\epsilon\, \kappa_\partial \,.\, K = \breve{K}\} \quad \mathrm{Df.}$$

We prove that there is only one such relation as K, and therefore put

$$K_\kappa = (\imath K)(K \,\epsilon\, \kappa_\partial \,.\, K = \breve{K}) \quad \mathrm{Df.}$$

Also for the sake of brevity we put

$$I_\kappa = I \upharpoonright s'\mathrm{C}''\kappa \qquad \mathrm{Df.}$$

We then prove that κ is a family having connexity, and satisfying the condition

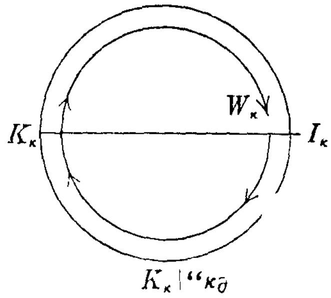
$$\mathrm{D}''\kappa = \mathrm{C}''\kappa,$$

i.e. having the domain of a member always identical with the common converse domain. Thus by $*334{\cdot}21$, $\kappa_\iota = \kappa \cup \mathrm{Cnv}''\kappa$.

In a cyclic family, $\kappa \cup \mathrm{Cnv}''\kappa$ consists of two mutually exclusive parts, namely κ_∂ and $K_\kappa |''\kappa_\partial$. (In the case of angles, $K_\kappa | R$ would be $\pi + R$. Thus κ_∂ would be the angles from 0 (exclusive) to π (inclusive), and $K_\kappa |''\kappa_\partial$ would be the angles from π (exclusive) to 2π (inclusive).) Also $K_\kappa |''\kappa_\partial$ consists of the converses of $\kappa - \iota' K_\kappa$.

We take up next ($*371$) the question of arranging $\kappa \cup \mathrm{Cnv}''\kappa$ in a series. For this purpose, in order to avoid circularity, we have to erect a barrier at some point; we choose I_κ as this point. By the definition of cyclic families,

$\kappa_\partial \upharpoonleft U_\kappa$ is transitive; hence, since the family has connexity, $U_\kappa \upharpoonright \kappa_\partial$ is serial. This relation therefore arranges all the members of κ_∂ in a series, beginning with K_κ and proceeding towards I_κ. In order to extend our series to $K_\kappa | ``\kappa_\partial$, we only have to make $K_\kappa | R$ precede $K_\kappa | S$ if $R$ precedes $S$, where $R$ and $S$ are members of $\kappa_\partial$. That is, we arrange $K_\kappa | ``\kappa_\partial$ in the order $K_\kappa |; U_\kappa \upharpoonright \kappa_\partial$. This gives a series which begins with I_κ and proceeds towards K_κ without reaching it. Thus taking the sum of the above two series (in the sense of $*160$), we get a series whose field is $\kappa \cup \mathrm{Cnv}``\kappa$, which begins with $I_\kappa$, travels through $K_\kappa | ``\kappa_\partial$ to K_κ, and on through κ_∂ towards I_κ, without quite reaching I_κ again. This relation we call W_κ; the definition is



$$W_\kappa = K_\kappa |; U_\kappa \upharpoonright \kappa_\partial \mathbin{\updownarrow} U_\kappa \upharpoonright \kappa_\partial \quad \text{Df.}$$

Taking an arbitrary origin, a vector may be indicated by the point to which it carries the origin. Thus in the figure, I_κ is at the origin, K_κ is opposite the origin; the upper semi-circle, including both ends, is κ; not including the right-hand end, it is κ_∂; the lower semi-circle, including both ends, is $\mathrm{Cnv}``\kappa$; including $K_\kappa$ but not $I_\kappa$, it is $\mathrm{Cnv}``\kappa_\partial$; including I_κ but not K_κ, it is $K_\kappa | ``\kappa_\partial$. Then W_κ starts from I_κ, and proceeds through the lower semi-circle first, and afterwards through the upper semi-circle, stopping just short of I_κ.

If κ is cyclic, W_κ is a series. Under most circumstances, if $R \in \kappa$, we shall have

$$P W_\kappa Q \,.\supset.\, (P | R) W_\kappa (Q | R).$$

The investigation of the various cases in which this holds occupies a large part of $*371$.

In the remainder of this Section, our work becomes more full of ordinary arithmetic than it has been hitherto. We shall therefore, where cardinals are concerned, abandon the explicit notation we have hitherto employed, and substitute the ordinary notation. Thus we shall write $\mu + \nu$ in place of $\mu +_c \nu$, and $\mu\nu$ in place of $\mu \times_c \nu$. We shall, however, retain $\mu -_c \nu$ for subtraction, in order to avoid confusion with the sign of negation of a class.

We proceed next ($*372$) to consider what is in effect the class of vectors not greater than the νth part of a complete revolution (*e.g.* in the case of angles, not greater than $2\pi/\nu$). We define this by means of the relation W_κ. It will be seen from the figure that if R is a non-zero vector, we shall have $R^{\sigma+1} W_\kappa R^\sigma$, unless R^σ belongs to the lower semi-circle and $R^{\sigma+1}$ to the upper, in which case $R^\sigma W_\kappa R^{\sigma+1}$. The first time this happens is the first time that $R^{\sigma+1}$ becomes greater than one complete revolution. Hence if, for every number σ less than ν and not zero, $R^{\sigma+1} W_\kappa R^\sigma$, it follows that R^ν is not greater

than one complete revolution, and therefore R is not greater than the νth part of a complete revolution. The class of such relations we call ν_κ; thus we put

$$\nu_\kappa = (\kappa \cup \mathrm{Cnv}``\kappa) \cap \hat{R}(\sigma < \nu . \sigma \neq 0 . \supset_\sigma . R^{\sigma+1} W_\kappa R^\sigma) \quad \mathrm{Df.}$$

The main propositions to be proved in this subject are

$$P \epsilon \nu_\kappa . P W_\kappa Q . \supset . P^\nu W_\kappa Q^\nu$$

and (what is an immediate consequence)

$$P, Q \epsilon \nu_\kappa . \supset : P^\nu = Q^\nu . \equiv . P = Q.$$

This latter proposition is the foundation of the theory of principal ratios.

Another important property of ν_κ is

$$\breve{W}_\kappa``\nu_\kappa \subset \nu_\kappa,$$

so that ν_κ is an upper section of W_κ.

We proceed next (*373) to consider submultiples of identity, *i.e.* vectors R such that $R^\nu = I_\kappa$, where ν is a cardinal. We assume here, and almost always henceforth, that κ is a submultipliable family. We first consider vectors which can be reached from I_κ by successive bisections. We know that $K_\kappa{}^2 = I_\kappa$; if $R^2 = K_\kappa$, then $R \neq K_\kappa$, because $K_\kappa{}^2 \neq K_\kappa$. Hence by continuing the same process we arrive at the existence of a vector Q such that

$$Q^{2^\nu} = I_\kappa : \rho < 2^\nu . \rho \neq 0 . \supset_\rho . Q^\rho \neq I_\kappa.$$

Hence we easily arrive at the result that, if ν is any inductive cardinal, there is a non-zero vector whose νth power is I_κ. (This does not follow from $\kappa \epsilon FM\,\mathrm{subm}$ alone, because $I_\kappa{}^\nu = I_\kappa$, so that from the definition of $FM\,\mathrm{subm}$ we cannot know that there is any vector except I_κ whose νth power is I_κ.) Thence we prove that there are non-zero vectors whose νth power is I_κ, and which are such that no earlier power is I_κ, *i.e.* we prove

$$(\exists R) : R \epsilon \kappa_\partial . R^\nu = I_\kappa : \sigma < \nu . \sigma \neq 0 . \supset_\sigma . R^\sigma \neq I_\kappa.$$

The class of such vectors we call (I_κ, ν). If R is such a vector, the number of different vectors which are powers of R is ν. Hence the powers of R have a maximum in the order W_κ; since W_κ proceeds from greater to smaller vectors, this will be the smallest vector, other than I_κ, which is a power of R. Concerning this vector, we show that it is a member of ν_κ, *i.e.* it is such that, if $\sigma < \nu . \sigma \neq 0$, $R^{\sigma+1} W_\kappa R^\sigma$. Finally we prove that there is only one member of ν_κ whose νth power is I_κ. This will be what we may call the "principal" νth submultiple of I_κ; in the case of angles, it will be the angle $2\pi/\nu$. It will be observed that $2\pi\mu/\nu$ always has identity for its νth power, and has no lower power equal to identity if μ is prime to ν. Thus the uniqueness of the "principal" νth submultiple depends upon the fact that it is a member of ν_κ, so that, by what has been proved in the previous number, no other member of ν_κ has the same νth power.

We next, in a short number (*374), extend the last of the above results to any vector, proving that, if R is any member of $\kappa \cup \text{Cnv}``\kappa$, there is a unique member of ν_κ whose νth power is R. We may call this the "principal" νth submultiple of R. We prove also in this number that, if S is the principal νth submultiple of I_κ, ν_κ consists of all vectors not earlier than S in the order W_κ, *i.e.* of all vectors not greater than S.

Finally (*375) we define "principal ratios" and show that they are one-one and mutually exclusive. We denote the "principal ratio" corresponding to μ/ν by "$(\mu/\nu)_\kappa$." This is defined as the relation holding between R and S when the principal μth submultiple of R is identical with the principal νth submultiple of S; that is, we put

$$(\mu/\nu)_\kappa = \hat{R}\hat{S}\{(\exists T) . T \in \mu_\kappa \cap \nu_\kappa . R = T^\mu . S = T^\nu\} \quad \text{Df.}$$

It is obvious that $(\mu/\nu)_\kappa \subset (\mu/\nu) \upharpoonright \kappa_\iota$; and there is no difficulty in showing that principal ratios are one-one and mutually exclusive.

We have not thought it necessary to carry the development of this subject any farther, since, from this point onwards, everything proceeds as in the case of open families. We have given proofs rather shortly in this Section, particularly in the case of purely arithmetical lemmas, of which the proofs are perfectly straightforward, but tedious if written out at length.

$*370$. ELEMENTARY PROPERTIES OF CYCLIC FAMILIES.

Summary of $*370$.

In this number, after the definition of cyclic families already cited, we proceed first to prove that only one non-zero vector is equal to its converse ($*370{\cdot}23$). This one we define as K_κ. Next we prove that, if R is a non-zero vector other than K_κ, $R \mid K_\kappa$ is the converse of a non-zero vector, and $\breve{R} \mid K_\kappa$ is a non-zero vector ($*370{\cdot}31{\cdot}311$), whence it follows that

$$\text{D}`R = \text{ꓷ}`R = s`\text{ꓷ}``\kappa \quad (*370{\cdot}32),$$

whence further we obtain

$$\text{D}``\kappa = \text{ꓷ}``\kappa \,.\, \kappa \in FM \,\text{connex} \quad (*370{\cdot}33).$$

Hence further, since by definition $\kappa_\partial \rceil U_\kappa$ is transitive, it follows that $\kappa_\partial \rceil U_\kappa$ is a series ($*370{\cdot}37$). The remaining propositions ($*370{\cdot}4$—$\cdot44$) are concerned with the relations of the two semi-circles κ_∂ and $K_\kappa \mid ``\kappa_\partial$ (cf. figure, p. 459). We have

$$\text{Cnv}``\kappa = K_\kappa \mid ``\kappa \qquad (*370{\cdot}4),$$

$$\kappa \cap \text{Cnv}``\kappa = \iota`I_\kappa \cup \iota`K_\kappa \qquad (*370{\cdot}42),$$

$$K_\kappa \mid ``\kappa_\partial = \text{Cnv}``\kappa - \iota`K_\kappa \qquad (*370{\cdot}43),$$

and

$$\kappa_\partial \cap K_\kappa \mid ``\kappa_\partial = \Lambda \qquad (*370{\cdot}44).$$

$*370{\cdot}01$. $FM\,\text{cycl} =$

$$(FM\,\text{conx} - 2) \cap \hat{\kappa}\,\{\kappa_\partial \rceil U_\kappa \in \text{trans} : (\exists K) \,.\, K \in \kappa_\partial \,.\, K = \breve{K}\} \quad \text{Df}$$

$*370{\cdot}02$. $K_\kappa = (\imath K)(K \in \kappa_\partial \,.\, K = \breve{K}) \quad \text{Df}$

$*370{\cdot}03$. $I_\kappa = I \upharpoonright s`\text{ꓷ}``\kappa \qquad \text{Df}$

$*370{\cdot}1$. $\vdash :.\, \kappa \in FM\,\text{cycl} \,.\equiv:$

$$\kappa \in FM\,\text{conx} - 2 \,.\, \kappa_\partial \rceil U_\kappa \in \text{trans} : (\exists K) \,.\, K \in \kappa_\partial \,.\, K = \breve{K} \quad [(*370{\cdot}01)]$$

$*370{\cdot}11$. $\vdash : \kappa \in FM\,\text{conx} \,.\supset.\, \kappa_\partial \rceil U_\kappa \subset J \quad [*336{\cdot}6 \,.\, (*336{\cdot}011)]$

$*370{\cdot}12$. $\vdash : \kappa \in FM\,\text{conx} \,.\, \kappa_\partial \rceil U_\kappa \sim \in \text{trans} \,.\, R, S \in \kappa_\partial \,.\, RU_\kappa S \,.\, SU_\kappa T \,.\supset.\, R \neq T$ $[*370{\cdot}11]$

$*370{\cdot}13$. $\vdash : \kappa \in FM \,.\, K \in \kappa \,.\, K = \breve{K} \,.\supset.\, K^2 = I_\kappa \quad [*330{\cdot}31]$

***370·2.** $\vdash :. \kappa \in FM \,\mathrm{conx} \,.\, \kappa_\partial \rceil U_\kappa \in \mathrm{trans} \,.\, K \in \kappa_\partial \,.\, K = \breve{K} \,.\supset:$
$R \in \kappa_\partial \,.\, R \mid K \in \kappa \,.\supset.\, R U_\kappa (R \mid K) \,.\, (R \mid K) U_\kappa R$

Dem.

$\vdash . *370{\cdot}13 \,. \quad \supset \vdash : \mathrm{Hp} \,.\supset.\, R \mid K^2 = R \qquad (1)$

$\vdash . *336{\cdot}41 \,.\, (1) \,.\supset \vdash : \mathrm{Hp} \,.\supset.\, R U_\kappa (R \mid K) \,.\, (R \mid K) U_\kappa R :\supset \vdash .\, \mathrm{Prop}$

***370·21.** $\vdash : \mathrm{Hp}\, *370{\cdot}2 \,.\, R \in \kappa_\partial \,.\, R \mid K \in \kappa \,.\supset.\, R \mid K = I_\kappa$

Dem.

$\vdash . *370{\cdot}12 \,.\, \mathrm{Transp} \,.\supset \vdash : \mathrm{Hp} \,.\, \dot{R} U_\kappa (R \mid K) \,.\, (R \mid K) U_\kappa R \,.\supset.\, R \mid K \sim\in \kappa_\partial \qquad (1)$

$\vdash . (1) \,.\, *370{\cdot}2 \,.\supset \vdash .\, \mathrm{Prop}$

***370·22.** $\vdash : \mathrm{Hp}\, *370{\cdot}2 \,.\, R \in \kappa_\partial - \iota`K \,.\supset.\, R \mid K \sim\in \kappa$

Dem.

$\vdash . *370{\cdot}21 \,.\, *330{\cdot}32{\cdot}5 \,.\supset \vdash : \mathrm{Hp}\, *370{\cdot}21 \,.\supset.\, R = K \qquad (1)$

$\vdash . (1) \,.\, \mathrm{Transp} \,.\supset \vdash .\, \mathrm{Prop}$

***370·23.** $\vdash : \mathrm{Hp}\, *370{\cdot}2 \,.\, R \in \kappa_\partial \,.\, R = \breve{R} \,.\supset.\, R = K$

Dem.

$\vdash . *331{\cdot}33 \,. \quad \supset \vdash : \mathrm{Hp} \,.\supset.\, R \mid K \in \kappa \cup \mathrm{Cnv}``\kappa \qquad (1)$

$\vdash . *330{\cdot}5{\cdot}52 \,.\, *34{\cdot}2 \,.\supset \vdash : \mathrm{Hp} \,.\supset.\, R \mid K = \mathrm{Cnv}`(R \mid K) \qquad (2)$

$\vdash . (1) \,.\, (2) \,. \quad \supset \vdash : \mathrm{Hp} \,.\supset.\, R \mid K \in \kappa \,.$

$[*370{\cdot}22.\mathrm{Transp}] \quad \supset.\, R = K :\supset \vdash .\, \mathrm{Prop}$

***370·24.** $\vdash : \kappa \in FM \,\mathrm{cycl} \,.\supset.\, \mathrm{E} \,!\, K_\kappa \qquad [*370{\cdot}1{\cdot}23.(*370{\cdot}02)]$

***370·25.** $\vdash :. \kappa \in FM \,\mathrm{cycl} \,.\supset: R \in \kappa_\partial \,.\, R = \breve{R} \,.\equiv.\, R = K_\kappa \quad [*370{\cdot}24.(*370{\cdot}02)]$

***370·26.** $\vdash : \kappa \in FM \,\mathrm{cycl} \,.\supset.\, K_\kappa \in \kappa_\partial \,.\, K_\kappa = \breve{K}_\kappa \,.\, K_\kappa{}^2 = I_\kappa \quad [*370{\cdot}24{\cdot}25{\cdot}13]$

***370·3.** $\vdash : \kappa \in FM \,\mathrm{cycl} \,.\, R U_\kappa K_\kappa \,.\supset.\, R = I_\kappa$

Dem.

$\vdash . *336{\cdot}41 \,.\supset \vdash :. \mathrm{Hp} \,.\supset: R \in \kappa : (\exists S) \,.\, S \in \kappa_\partial \,.\, R = K_\kappa \mid S \qquad (1)$

$\vdash . (1) \,.\, *370{\cdot}21{\cdot}24 \,.\supset \vdash .\, \mathrm{Prop}$

***370·31.** $\vdash : \kappa \in FM \,\mathrm{cycl} \,.\, R \in \kappa_\partial - \iota`K_\kappa \,.\supset.\, R \mid K_\kappa \in \mathrm{Cnv}``\kappa_\partial$
$[*331{\cdot}33 \,.\, *370{\cdot}22]$

***370·311.** $\vdash : \mathrm{Hp}\, *370{\cdot}31 \,.\supset.\, \breve{R} \mid K_\kappa \in \kappa_\partial$

Dem.

$\vdash . *370{\cdot}31 \,.\supset \vdash : \mathrm{Hp} \,.\supset.\, \breve{K}_\kappa \mid \breve{R} \in \kappa_\partial \,.$

$[*330{\cdot}5.*370{\cdot}26] \quad \supset.\, \breve{R} \mid K_\kappa \in \kappa_\partial :\supset \vdash .\, \mathrm{Prop}$

***370·32.** $\vdash : \kappa \epsilon FM\,\text{cycl} . R \epsilon \kappa . \supset . \text{D}'R = \text{Ƌ}'R = s'\text{Ƌ}''\kappa$

Dem.

$\vdash . *50{\cdot}5{\cdot}52 . \qquad \supset \vdash . \text{D}'I_\kappa = \text{Ƌ}'I_\kappa = s'\text{Ƌ}''\kappa \qquad (1)$

$\vdash . *370{\cdot}26 . *330{\cdot}52 . \supset \vdash : \text{Hp} . \supset . \text{D}'K_\kappa = \text{Ƌ}'K_\kappa = s'\text{Ƌ}''\kappa \qquad (2)$

$\vdash . *370{\cdot}31 . *330{\cdot}52 . \supset \vdash : \text{Hp} . R \epsilon \kappa_\partial - \iota'K_\kappa . \supset . \text{D}'(R \mid K_\kappa) = s'\text{Ƌ}''\kappa .$

$[*330{\cdot}52 . *34{\cdot}36] \qquad \supset . \text{D}'R = s'\text{Ƌ}''\kappa \qquad (3)$

$\vdash . (1) . (2) . (3) . \supset \vdash . \text{Prop}$

***370·33.** $\vdash : \kappa \epsilon FM\,\text{cycl} . \supset . \text{D}''\kappa = \text{Ƌ}''\kappa . \kappa \epsilon FM\,\text{connex}$
$[*370{\cdot}32 . *334{\cdot}42]$

***370·34.** $\vdash : \kappa \epsilon FM\,\text{cycl} . \supset . U_\kappa \epsilon \text{connex} \quad [*370{\cdot}33 . *336{\cdot}62 . (*336{\cdot}011)]$

***370·35.** $\vdash : \text{Hp}\,*370{\cdot}31 . \supset . K_\kappa U_\kappa R . \sim (R U_\kappa K_\kappa)$
$[*370{\cdot}3 . \text{Transp} . *370{\cdot}34]$

***370·36.** $\vdash : \kappa \epsilon FM\,\text{cycl} . \supset . \kappa_\partial \upharpoonright U_\kappa \epsilon \text{connex} . C'\kappa_\partial \upharpoonright U_\kappa = \kappa$

Dem.

$\vdash . *336{\cdot}41 . \qquad \supset \vdash : \text{Hp} . \supset . C'\kappa_\partial \upharpoonright U_\kappa \subset \kappa \qquad (1)$

$\vdash . *370{\cdot}34 . \qquad \supset \vdash :. \text{Hp} . R, S \epsilon \kappa_\partial . R \neq S . \supset :$
$R (\kappa_\partial \upharpoonright U_\kappa) S . \vee . S (\kappa_\partial \upharpoonright U_\kappa) R \qquad (2)$

$\vdash . *336{\cdot}41 . \qquad \supset \vdash : \text{Hp} . R \epsilon \kappa_\partial . S = I_\kappa . \supset . R (\kappa_\partial \upharpoonright U_\kappa) S \qquad (3)$

$\vdash . *336{\cdot}41 . \qquad \supset \vdash : \text{Hp} . S \epsilon \kappa_\partial . R = I_\kappa . \supset . S (\kappa_\partial \upharpoonright U_\kappa) R \qquad (4)$

$\vdash . (2) . (3) . (4) . \supset \vdash :. \text{Hp} . R, S \epsilon \kappa . R \neq S . \supset :$
$R (\kappa_\partial \upharpoonright U_\kappa) S . \vee . S (\kappa_\partial \upharpoonright U_\kappa) R \qquad (5)$

$\vdash . (1) . (5) . \supset \vdash . \text{Prop}$

***370·37.** $\vdash : \kappa \epsilon FM\,\text{cycl} . \supset . \kappa_\partial \upharpoonright U_\kappa \epsilon \text{Ser} \qquad [*370{\cdot}11{\cdot}1{\cdot}36]$

***370·38.** $\vdash : \kappa \epsilon FM\,\text{cycl} . R, S \epsilon \kappa . \supset . \breve{R} \mid S = S \mid \breve{R} \quad [*330{\cdot}561 . *370{\cdot}32]$

***370·4.** $\vdash : \kappa \epsilon FM\,\text{cycl} . \supset . \text{Cnv}''\kappa = K_\kappa \mid ''\kappa$

Dem.

$\vdash . *370{\cdot}31 . *330{\cdot}5 . \supset \vdash : \text{Hp} . \supset . K_\kappa \mid ''(\kappa_\partial - \iota'K_\kappa) \subset \text{Cnv}''\kappa \qquad (1)$

$\vdash . (1) . *370{\cdot}26 . \qquad \supset \vdash : \text{Hp} . \supset . K_\kappa \mid ''\kappa \subset \text{Cnv}''\kappa \qquad (2)$

$\vdash . *370{\cdot}311{\cdot}26 . \qquad \supset \vdash : \text{Hp} . R \epsilon \kappa . \supset . \breve{R} \mid K_\kappa \epsilon \kappa .$

$[*370{\cdot}26] \qquad \supset . (\exists S) . S \epsilon \kappa . \breve{R} = S \mid K_\kappa .$

$[*330{\cdot}5 . *37{\cdot}6] \qquad \supset . \breve{R} \epsilon K_\kappa \mid ''\kappa \qquad (3)$

$\vdash . (2) . (3) . \supset \vdash . \text{Prop}$

***370·41.** $\vdash :. \kappa \epsilon FM\,\text{cycl} . R, S \epsilon \kappa . \supset : (K_\kappa \mid R) V_\kappa (K_\kappa \mid S) . \equiv . R U_\kappa S$

Dem.

$\vdash . *336{\cdot}54 . *370{\cdot}33 . \supset$

$\vdash :. \text{Hp} . \supset : (K_\kappa \mid R) V_\kappa (K_\kappa \mid S) . \equiv . (\exists T) . T \epsilon \kappa_\partial . K_\kappa \mid R = T \mid K_\kappa \mid S .$

$[*330{\cdot}5 . *370{\cdot}26] \qquad \equiv . (\exists T) . T \epsilon \kappa_\partial . R = T \mid S .$

$[*336{\cdot}41] \qquad \equiv . R U_\kappa S :. \supset \vdash . \text{Prop}$

***370·42.** $\vdash : \kappa \in FM \text{ cycl} . \supset . \kappa \cap \text{Cnv}``\kappa = \iota`I_\kappa \cup \iota`K_\kappa$

Dem.

$\vdash . *370{\cdot}22 . \supset \vdash : \text{Hp} . \breve{R} \in \kappa_\partial - \iota`K_\kappa . \supset . \breve{R} | K \sim \in \kappa .$

$[*370{\cdot}311 . \text{Transp}] \qquad \supset . R \sim \in \kappa_\partial - \iota`K_\kappa \qquad (1)$

$\vdash . (1) . \qquad \supset \vdash :. \text{Hp} . \supset : R, \breve{R} \in \kappa . \supset . R \in \iota`I_\kappa \cup \iota`K_\kappa \qquad (2)$

$\vdash . (2) . *370{\cdot}26 . \supset \vdash . \text{Prop}$

***370·43.** $\vdash : \kappa \in FM \text{ cycl} . \supset . K_\kappa |``\kappa_\partial = \text{Cnv}``\kappa - \iota`K_\kappa$ [*370·4]

***370·44.** $\vdash : \kappa \in FM \text{ cycl} . \supset . \kappa_\partial \cap K_\kappa |``\kappa_\partial = \Lambda$ [*370·42·43]

Summary of *371.

In this number, we begin by defining the relation W_κ, which takes the place, for cyclic families, of the relation V_κ defined in *336. The definition is

***371·01.** $W_\kappa = K_\kappa \mid ; U_\kappa \upharpoonright \kappa_\partial \ddagger U_\kappa \upharpoonright \kappa_\partial$ Df

Then if κ is a cyclic family, W_κ is a series (*371·12), and its field is $\kappa \cup \mathrm{Cnv}``\kappa$ (*371·14), which $= \kappa_\iota$ since κ has connexity. It will be observed that V_κ is not a series if κ is a cyclic family; we have *e.g.* $I_\kappa V_\kappa K_\kappa \,.\, K_\kappa V_\kappa I_\kappa$. The above relation W_κ is constructed so as to make a barrier at I_κ, thereby preventing the relation W_κ from being cyclic.

If P, Q are both members of κ_∂ or both members of $K_\kappa \mid ``\kappa_\partial$,

$$P W_\kappa Q \,.\equiv.\, (\exists T) \,.\, T \in \kappa_\partial \,.\, P = Q \mid T \quad (*371·15·151).$$

Most of the properties of W_κ depend upon the fact that $\kappa_\partial \upharpoonleft U_\kappa$ is transitive, in virtue of the definition of cyclic families. If κ is any connected family, we have

$$\kappa_\partial \upharpoonleft U_\kappa \in \mathrm{trans} \,.\equiv:\, P, Q, Q \mid R, P \mid Q \mid R \in \kappa_\partial \,.\, R \in \kappa \,.\supset_{P,Q,R} .\, P \mid Q \in \kappa_\partial \quad (*371·2).$$

This proposition is required for most of the subsequent proofs in this number. It leads at once to

***371·21.** $\vdash : \kappa \in FM\,\mathrm{cycl} \,.\, P, Q, Q \mid R, P \mid Q \mid R \in \kappa_\partial \,.\, R \in \kappa \,.\supset.\, P \mid Q \in \kappa_\partial$

Most of the propositions of this number are concerned with the circumstances under which we can infer $(P \mid R) W_\kappa (Q \mid R)$ from $P W_\kappa Q$. We have

***371·31.** $\vdash :. \kappa \in FM\,\mathrm{cycl} \,.\, R \in \kappa_\partial : P \in \kappa_\partial \,.\vee.\, P \mid R \sim\in \kappa_\partial :\supset:$
$$P W_\kappa Q \,.\supset.\, (P \mid R) W_\kappa (Q \mid R)$$

Another useful proposition is

***371·27.** $\vdash :. \kappa \in FM\,\mathrm{cycl} \,.\, P, Q \in \kappa_\partial \,.\supset: P W_\kappa Q \,.\equiv.\, \breve{Q} W_\kappa \breve{P}$

***371·01.** $W_\kappa = K_\kappa \mid ; U_\kappa \upharpoonright \kappa_\partial \ddagger U_\kappa \upharpoonright \kappa_\partial$ Df

***371·1.** $\vdash ::. \kappa \in FM\,\mathrm{cycl} \,.\supset:: P W_\kappa Q \,.\equiv:.\, P, Q \in K_\kappa \mid ``\kappa_\partial :$
$$(\exists R, S) \,.\, R, S \in \kappa_\partial \,.\, R U_\kappa S \,.\, P = K_\kappa \mid R \,.\, Q = K_\kappa \mid S :\vee:$$
$$P, Q \in \kappa_\partial \,.\, P U_\kappa Q :\vee: P \in K_\kappa \mid ``\kappa_\partial \,.\, Q \in \kappa_\partial$$

[*202·55 . *370·34 . (*371·01)]

***371·11.** $\vdash : \kappa \epsilon FM . K \epsilon \kappa . \supset . (K|) \upharpoonright \kappa \epsilon 1 \to 1$

Dem.

$\vdash . *330·31 . \supset \vdash : \text{Hp} . R, S \epsilon \kappa . K | R = K | S . \supset . R = S : \supset \vdash . \text{Prop}$

***371·12.** $\vdash : \kappa \epsilon FM \text{cycl} . \supset . W_\kappa \epsilon \text{Ser}$ [*370·37·44 . *371·11 . *204·21·5]

***371·13.** $\vdash : \kappa \epsilon FM \text{cycl} . \supset . W_\kappa = V_\kappa \upharpoonright (\text{Cnv}``\kappa - \iota`K_\kappa) \ddagger U_\kappa \upharpoonright \kappa_\partial$ [*370·41·43]

***371·14.** $\vdash : \kappa \epsilon FM \text{cycl} . \supset . C`W_\kappa = \kappa \cup \text{Cnv}``\kappa = \kappa_\partial \cup K_\kappa | ``\kappa_\partial$

Dem.

$\vdash . *202·55 . *370·34 . *160·14 . \supset \vdash : \text{Hp} . \supset . C`W_\kappa = K_\kappa | ``\kappa_\partial \cup \kappa_\partial$

[*370·43] $= \kappa \cup \text{Cnv}``\kappa : \supset \vdash . \text{Prop}$

***371·15.** $\vdash :. \kappa \epsilon FM \text{cycl} . P, Q \epsilon \kappa_\partial . \supset : P W_\kappa Q . \equiv . (\exists T) . T \epsilon \kappa_\partial . P = Q | T$
[*370·44 . *336·41 . (*371·01)]

***371·151.** $\vdash :. \kappa \epsilon FM \text{cycl} . P, Q \epsilon K_\kappa | ``\kappa_\partial . \supset : P W_\kappa Q . \equiv . (\exists T) . T \epsilon \kappa_\partial . P = Q | T$

Dem.

$\vdash . *370·44 . *336·41 . \supset \vdash :. \text{Hp} . \supset :$

$P W_\kappa Q . \equiv . (\exists R, S, T) . R, S, T \epsilon \kappa_\partial . R = S | T . P = K_\kappa | R . Q = K_\kappa | S .$

[*370·26] $\equiv . (\exists T) . T \epsilon \kappa_\partial . P = Q | T :. \supset \vdash . \text{Prop}$

***371·152.** $\vdash : \kappa \epsilon FM \text{cycl} . P \epsilon K_\kappa | ``\kappa_\partial . Q \epsilon \kappa_\partial . \supset . P W_\kappa Q$ [*371·1]

***371·16.** $\vdash : \kappa \epsilon FM \text{cycl} . P \epsilon \kappa_\partial . P W_\kappa Q . \supset . Q \epsilon \kappa_\partial$ [*370·44 . *371·1]

***371·161** $\vdash : \kappa \epsilon FM \text{cycl} . Q \epsilon K_\kappa | ``\kappa_\partial . P W_\kappa Q . \supset . P \epsilon K_\kappa | ``\kappa_\partial$
[*370·44 . *371·1]

***371·17.** $\vdash : \kappa \epsilon FM \text{cycl} . Q, T \epsilon \kappa_\partial . \supset . (Q | T) W_\kappa Q . (Q | T) W_\kappa T$
[*371·15·152]

***371·18.** $\vdash : \kappa \epsilon FM \text{cycl} . \supset . \overrightarrow{W_\kappa}`K_\kappa = K_\kappa | ``\kappa_\partial . \overleftarrow{W_\kappa}`K_\kappa = \kappa_\partial - \iota`K_\kappa$
[*371·15·152 . *370·311·22]

***371·19.** $\vdash :. \kappa \epsilon FM \text{cycl} . P \neq I_\kappa . \supset : P W_\kappa K_\kappa . \equiv . K_\kappa W_\kappa \breve{P}$
[*371·18 . *370·43]

***371·2.** $\vdash :: \kappa \epsilon FM \text{conx} . \supset :. \kappa_\partial \upharpoonright U_\kappa \epsilon \text{trans} . \equiv :$

$P, Q, Q | R, P | Q | R \epsilon \kappa_\partial . R \epsilon \kappa . \supset_{P,Q,R} . P | Q \epsilon \kappa_\partial$

Dem.

$\vdash . *336·41 . \quad \supset \vdash :. \text{Hp} . \supset : T (\kappa_\partial \upharpoonright U_\kappa) S . S (\kappa_\partial \upharpoonright U_\kappa) R . \equiv .$

$(\exists P, Q) . P, Q, S, T \epsilon \kappa_\partial . R \epsilon \kappa . T = P | S . S = Q | R \quad (1)$

$\vdash . (1) . *13·21 . \supset \vdash :: \text{Hp} . \supset :. \kappa_\partial \upharpoonright U_\kappa \epsilon \text{trans} . \equiv :$

$P, Q, Q | R, P | Q | R \epsilon \kappa_\partial . R \epsilon \kappa . \supset_{P,Q,R} . (P | Q | R) U_\kappa R \quad (2)$

$\vdash . *330·31·5 . \supset$

$\vdash :. \text{Hp} . P, Q, R \epsilon \kappa . M \epsilon \kappa_\partial . P | Q | R = M | R . \supset . P | Q = M \quad (3)$

$\vdash . (3) . *336·41 . \supset \vdash :. \text{Hp} . P, Q, R, P | Q | R \epsilon \kappa . \supset :$

$(P | Q | R) U_\kappa R . \equiv . P | Q \epsilon \kappa_\partial \quad (4)$

$\vdash . (2) . (4) . \supset \vdash . \text{Prop}$

***371·21.** $\vdash : \kappa \in FM\,\text{cycl} \,.\, P, Q, Q \mid R, P \mid Q \mid R \in \kappa_\partial \,.\, R \in \kappa \,.\, \supset \,.\, P \mid Q \in \kappa_\partial$
[*371·2 . *370·1]

***371·22.** $\vdash : \kappa \in FM\,\text{cycl} \,.\, P, R, P \mid R \in \kappa_\partial \,.\, P W_\kappa Q \,.\, \supset \,.\, Q \mid R \in \kappa_\partial$

Dem.

$\vdash \,.\, *371{\cdot}15{\cdot}16 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, (\exists T) \,.\, Q, T \in \kappa_\partial \,.\, P = Q \mid T$ (1)

$\vdash \,.\, (1) \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, (\exists T) \,.\, Q, R, T, Q \mid T, Q \mid T \mid R \in \kappa_\partial \,.$

[*371·21] $\supset \,.\, Q \mid R \in \kappa_\partial : \supset \vdash \,.\, \text{Prop}$

***371·23.** $\vdash : \kappa \in FM\,\text{cycl} \,.\, T W_\kappa S \,.\, \supset \,.\, T W_\kappa (\breve{S} \mid T)$

Dem.

$\vdash \,.\, *330{\cdot}31 \,.\, *370{\cdot}38 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, T = S \mid (\breve{S} \mid T)$ (1)

$\vdash \,.\, (1) \,.\, *371{\cdot}15{\cdot}16 \,.\, \supset \vdash : \text{Hp} \,.\, T, \breve{S} \mid T \in \kappa_\partial \,.\, \supset \,.\, T W_\kappa (\breve{S} \mid T)$ (2)

$\vdash \,.\, *371{\cdot}15{\cdot}16 \,.\, \supset \vdash : \text{Hp} \,.\, T \in \kappa_\partial \,.\, \supset \,.\, \breve{S} \mid T \in \kappa_\partial$ (3)

$\vdash \,.\, (2) \,.\, (3) \,.\, \supset \vdash : \text{Hp} \,.\, T \in \kappa_\partial \,.\, \supset \,.\, T W_\kappa (\breve{S} \mid T)$ (4)

$\vdash \,.\, *371{\cdot}152 \,.\, \supset \vdash : \text{Hp} \,.\, T \sim\in \kappa_\partial \,.\, \breve{S} \mid T \in \kappa_\partial \,.\, \supset \,.\, T W_\kappa (\breve{S} \mid T)$ (5)

$\vdash \,.\, *371{\cdot}151{\cdot}161 \,.\, \supset \vdash : \text{Hp} \,.\, S \sim\in \kappa_\partial \,.\, \supset \,.\, T \sim\in \kappa_\partial \,.\, \breve{S} \mid T \in \kappa_\partial$ (6)

$\vdash \,.\, (5) \,.\, (6) \,.\, \supset \vdash : \text{Hp} \,.\, S \sim\in \kappa_\partial \,.\, \supset \,.\, T W_\kappa (\breve{S} \mid T)$ (7)

$\vdash \,.\, (1) \,.\, *371{\cdot}151 \,.\, \supset \vdash : \text{Hp} \,.\, T, \breve{S} \mid T \sim\in \kappa_\partial \,.\, S \in \kappa_\partial \,.\, \supset \,.\, T W_\kappa (\breve{S} \mid T)$ (8)

$\vdash \,.\, (5) \,.\, (8) \,.\, \supset \vdash : \text{Hp} \,.\, T \sim\in \kappa_\partial \,.\, S \in \kappa_\partial \,.\, \supset \,.\, T W_\kappa (\breve{S} \mid T)$ (9)

$\vdash \,.\, (4) \,.\, (7) \,.\, (9) \,.\, \supset \vdash \,.\, \text{Prop}$

***371·24.** $\vdash : \kappa \in FM\,\text{cycl} \,.\, P, R, P \mid R \in \kappa_\partial \,.\, P W_\kappa Q \,.\, \supset \,.\, (P \mid R) W_\kappa (Q \mid R)$

Dem.

$\vdash \,.\, *371{\cdot}15{\cdot}16 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, (\exists T) \,.\, P, Q, R, P \mid R, T \in \kappa_\partial \,.\, P = Q \mid T \,.$

[*371·21.*330·5] $\supset \,.\, (\exists T) \,.\, P \mid R, Q \mid R, T \in \kappa_\partial \,.\, P \mid R = Q \mid R \mid T \,.$

[*371·15] $\supset \,.\, (P \mid R) W_\kappa (Q \mid R) : \supset \vdash \,.\, \text{Prop}$

***371·241.** $\vdash : \kappa \in FM\,\text{cycl} \,.\, P, R \in \kappa_\partial \,.\, P \mid R \sim\in \kappa_\partial \,.\, P W_\kappa Q \,.\, \supset \,.\, (P \mid R) W_\kappa (Q \mid R)$

Dem.

$\vdash \,.\, *371{\cdot}152 \,.\, \supset \vdash : \text{Hp} \,.\, Q \mid R \in \kappa_\partial \,.\, \supset \,.\, (P \mid R) W_\kappa (Q \mid R)$ (1)

$\vdash \,.\, *371{\cdot}15 \,.\, \supset$

$\vdash : \text{Hp} \,.\, Q \mid R \sim\in \kappa_\partial \,.\, \supset \,.\, (\exists T) \,.\, T \in \kappa_\partial \,.\, P \mid R, Q \mid R \sim\in \kappa_\partial \,.\, P \mid R = Q \mid R \mid T \,.$

[*371·151] $\supset \,.\, (P \mid R) W_\kappa (Q \mid R)$ (2)

$\vdash \,.\, (1) \,.\, (2) \,.\, \supset \vdash \,.\, \text{Prop}$

***371·25.** $\vdash : \kappa \in FM\,\text{cycl} \,.\, P, R \in \kappa_\partial \,.\, P W_\kappa Q \,.\, \supset \,.\, (P \mid R) W_\kappa (Q \mid R)$
[*371·24·241]

***371·251.** $\vdash : \kappa \epsilon FM \text{cycl} . R, \breve{R} | Q \epsilon \kappa_\partial . P W_\kappa Q . \supset . (\breve{R} | P) W_\kappa (\breve{R} | Q)$

Dem.

$\vdash . *371\cdot25 . \text{Transp} . *371\cdot12 . \supset$

$\vdash : \kappa \epsilon FM \text{cycl} . P, R \epsilon \kappa_\partial . (Q | R) W_\kappa (P | R) . \supset . Q W_\kappa P \quad (1)$

$\vdash . (1) \dfrac{\breve{R} | Q, \breve{R} | P}{P, \quad Q} . \supset \vdash . \text{Prop}$

***371·26.** $\vdash :. \kappa \epsilon FM \text{cycl} : P, Q \epsilon \kappa_\partial . \vee . P, Q \sim\epsilon \kappa_\partial : \supset :$

$P W_\kappa Q . \equiv . (K_\kappa | P) W_\kappa (K_\kappa | Q)$

Dem.

$\vdash . *371\cdot25 . *370\cdot26 . \supset \vdash : \text{Hp} . P \epsilon \kappa_\partial . P W_\kappa Q . \supset . (K_\kappa | P) W_\kappa (K_\kappa | Q) \quad (1)$

$\vdash . *371\cdot251 . *370\cdot26 . \supset \vdash : \text{Hp} . Q \epsilon \kappa_\partial . (K_\kappa | P) W_\kappa (K_\kappa | Q) . \supset . P W_\kappa Q \quad (2)$

$\vdash . (1) . (2) . \supset \vdash :. \text{Hp} . P, Q \epsilon \kappa_\partial . \supset : P W_\kappa Q . \equiv . (K_\kappa | P) W_\kappa (K_\kappa | Q) \quad (3)$

$\vdash . (3) \dfrac{K_\kappa | P, K_\kappa | Q}{P, \quad Q} . *371\cdot14 . \supset$

$\vdash :. \text{Hp} . P, Q \sim\epsilon \kappa_\partial . \supset : P W_\kappa Q . \equiv . (K_\kappa | P) W_\kappa (K_\kappa | Q) \quad (4)$

$\vdash . (3) . (4) . \supset \vdash . \text{Prop}$

***371·27.** $\vdash :. \kappa \epsilon FM \text{cycl} . P, Q \epsilon \kappa_\partial . \supset : P W_\kappa Q . \equiv . \breve{Q} W_\kappa \breve{P}$

Dem.

$\vdash . *371\cdot15 . \supset \vdash :. \text{Hp} . \supset : P W_\kappa Q . \equiv . (\exists T) . T \epsilon \kappa_\partial . P = Q | T .$

$[*370\cdot33] \quad \equiv . (\exists T) . T \epsilon \kappa_\partial . \breve{Q} = \breve{P} | T .$

$[*371\cdot151\cdot19 . *370\cdot43] \quad \equiv . \breve{Q} W_\kappa \breve{P} :. \supset \vdash . \text{Prop}$

***371·3.** $\vdash : \kappa \epsilon FM \text{cycl} . R \epsilon \kappa_\partial . P | R \sim\epsilon \kappa_\partial . P W_\kappa Q . \supset . (P | R) W_\kappa (Q | R)$

Dem.

$\vdash . *371\cdot27 . \supset \vdash : \text{Hp} . \supset . \breve{Q} W_\kappa \breve{P} .$

$[*371\cdot251] \quad \supset . (\breve{R} | \breve{Q}) W_\kappa (\breve{R} | \breve{P}) .$

$[*371\cdot27] \quad \supset . (P | R) W_\kappa (Q | R) : \supset \vdash . \text{Prop}$

***371·31.** $\vdash :. \kappa \epsilon FM \text{cycl} . R \epsilon \kappa_\partial : P \epsilon \kappa_\partial . \vee . P | R \sim\epsilon \kappa_\partial : \supset :$

$P W_\kappa Q . \supset . (P | R) W_\kappa (Q | R) \quad [*371\cdot25\cdot3]$

*372. INTEGRAL SECTIONS OF THE SERIES OF VECTORS.

Summary of *372.

The subject of this number is that section of $\breve{W}_\kappa$ which consists of vectors not greater than the νth part of the whole circumference of the cycle. This is defined by means of W_κ, as consisting of those vectors which (taking W_κ as "greater than") are such that $R^{\sigma+1}$ is greater than R^σ so long as $\sigma < \nu$. It will be seen that so long as R^ν and all earlier powers of R do not exceed I_κ, R satisfies this condition; but if $R^\sigma \epsilon K_\kappa | ``\kappa_\partial$, while $R^{\sigma+1} \epsilon \kappa_\partial$, we shall have $R^\sigma W_\kappa R^{\sigma+1}$. Thus our definition selects those vectors which, starting from any origin, do not, by ν repetitions, take us farther than once round the cycle. The definition is

***372·01.** $\nu_\kappa = (\kappa \cup \mathrm{Cnv}``\kappa) \cap \hat{R}(\sigma < \nu . \sigma \neq 0 . \supset_\sigma . R^{\sigma+1} W_\kappa R^\sigma)$ Df

We then have $1_\kappa = \kappa \cup \mathrm{Cnv}``\kappa$ (*372·11), $2_\kappa = \kappa_\partial$ (*372·13), $\mu \leqslant \nu . \supset . \nu_\kappa \subset \mu_\kappa$, *i.e.* ν_κ diminishes as ν increases (*372·15); $\nu > 1 . \supset . \nu_\kappa \subset \kappa_\partial$ (*372·16).

An alternative formula for ν_κ, sometimes more convenient than the one given in the definition, is (assuming $\nu > 1$)

$$\nu_\kappa = \kappa_\partial \cap \hat{P}(\mu < \nu . \mu \neq 0 . P^{\mu+1} \epsilon \kappa_\partial . \supset_\mu . P^\mu \epsilon \kappa_\partial) \quad (*372·17);$$

i.e. so long as $\mu < \nu$, either P^μ comes in the upper semi-circle, or $P^{\mu+1}$ comes in the lower semi-circle; that is to say, the step from P^μ to $P^{\mu+1}$ does not cross I_κ. For an even number (not zero), this leads to a simpler formula, namely

$$(2\nu)_\kappa = \kappa_\partial \cap \hat{P}(\mu \leqslant \nu . \mu \neq 0 . \supset_\mu . P^\mu \epsilon \kappa_\partial) \quad (*372·18).$$

We have next a set of propositions leading up to

***372·27.** $\vdash :. \kappa \epsilon FM \mathrm{cycl} . \nu \epsilon \mathrm{NC\,ind} - \iota`0 . P \epsilon \nu_\kappa . P W_\kappa Q . \supset :$

$$\mu \leqslant \nu . \mu \neq 0 . \supset . P^\mu W_\kappa Q^\mu$$

whence, since W_κ is a series, we obtain

***372·28.** $\vdash :. \kappa \epsilon FM \mathrm{cycl} . \nu \epsilon \mathrm{NC\,ind} - \iota`0 . P, Q \epsilon \nu_\kappa . \supset : P^\nu = Q^\nu . \equiv . P = Q$

It is largely owing to this proposition that ν_κ is important. In virtue of this proposition, there is in ν_κ at most one vector which is the νth submultiple of a given vector. We shall show later that, if κ is a submultipliable

cyclic family, there is at least one such vector; hence there is a unique vector in ν_κ which is the νth submultiple of a given vector. This does not hold in general for larger classes than ν_κ.

A specially useful case of the above proposition is obtained by putting $\nu = 2$, which gives, in virtue of *372·13,

***372·29.** $\vdash :. \kappa \in FM \text{cycl} . P, Q \in \kappa_\partial . \supset : P^2 = Q^2 . \equiv . P = Q$

The remaining propositions of this number are concerned in proving that ν_κ is an upper section of W_κ, *i.e.*

***372·33.** $\vdash : \kappa \in FM \text{cycl} . \nu \in \text{NC ind} . \supset . \breve{W}_\kappa{}^{\prime\prime}\nu_\kappa \subset \nu_\kappa$

***372·01.** $\nu_\kappa = (\kappa \cup \text{Cnv}^{\prime\prime}\kappa) \cap \hat{R}(\sigma < \nu . \sigma \neq 0 . \supset_\sigma . R^{\sigma+1} W_\kappa R^\sigma)$ Df

***372·1.** $\vdash :. R \in \nu_\kappa . \equiv : R \in \kappa \cup \text{Cnv}^{\prime\prime}\kappa : \sigma < \nu . \sigma \neq 0 . \supset_\sigma . R^{\sigma+1} W_\kappa R^\sigma$ [(*372·01)]

***372·11.** $\vdash . 1_\kappa = \kappa \cup \text{Cnv}^{\prime\prime}\kappa$ [*372·1 . *117·53]

***372·12.** $\vdash : \kappa \in FM \text{cycl} . R \in K \mid {}^{\prime\prime}\kappa_\partial . \supset . R W_\kappa R^2$

Dem.

$\vdash . *371·152 . \supset \vdash : \text{Hp} . R^2 \in \kappa_\partial . \supset . R W_\kappa R^2$ (1)

$\vdash . *370·44 . \supset \vdash : \text{Hp} . R^2 \sim\in \kappa_\partial . \supset . R, R^2 \sim\in \kappa_\partial . \breve{R} \in \kappa_\partial . R = \breve{R} \mid R^2 .$

[*371·151] $\supset . R W_\kappa R^2$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***372·121.** $\vdash : \kappa \in FM \text{cycl} . R \in \kappa_\partial . \supset . R^2 W_\kappa R$ [*371·17]

***372·122.** $\vdash :. \kappa \in FM \text{cycl} . \supset : R \in \kappa_\partial . \equiv . R^2 W_\kappa R$ [*372·12·121 . *371·12]

***372·13.** $\vdash : \kappa \in FM \text{cycl} . \supset . 2_\kappa = \kappa_\partial$ [*372·122]

***372·14.** $\vdash : \kappa \in FM \text{cycl} . \supset . K_\kappa \sim\in 3_\kappa$

Dem. $\vdash . *371·152 . \supset \vdash : \text{Hp} . \supset . K_\kappa{}^2 W_\kappa K_\kappa{}^3 : \supset \vdash . \text{Prop}$

***372·15.** $\vdash : \mu \leqslant \nu . \supset . \nu_\kappa \subset \mu_\kappa$ [*372·1]

***372·16.** $\vdash : \kappa \in FM \text{cycl} . \nu > 1 . \supset . \nu_\kappa \subset \kappa_\partial$ [*372·15·13]

***372·17.** $\vdash : \kappa \in FM \text{cycl} . \nu > 1 . \supset .$

$$\nu_\kappa = \kappa_\partial \cap \hat{P}(\mu < \nu . \mu \neq 0 . P^{\mu+1} \in \kappa_\partial . \supset_\mu . P^\mu \in \kappa_\partial)$$

Dem.

$\vdash . *372·1·16 . *371·16 . \supset$

$\vdash : \text{Hp} . \supset . \nu_\kappa \subset \kappa_\partial \cap \hat{P}(\mu < \nu . \mu \neq 0 . P^{\mu+1} \in \kappa_\partial . \supset_\mu . P^\mu \in \kappa_\partial)$ (1)

$\vdash . *371·15 . \supset \vdash : \text{Hp} . P, P^\mu, P^{\mu+1} \in \kappa_\partial . \supset . P^{\mu+1} W_\kappa P^\mu$ (2)

$\vdash . *371·152 . \supset \vdash : \text{Hp} . P, P^\mu \in \kappa_\partial . P^{\mu+1} \sim\in \kappa_\partial . \supset . P^{\mu+1} W_\kappa P^\mu$ (3)

$\vdash . *371·151 . \supset \vdash : \text{Hp} . P \in \kappa_\partial . P^\mu, P^{\mu+1} \sim\in \kappa_\partial . \supset . P^{\mu+1} W_\kappa P^\mu$ (4)

$\vdash . (2) . (3) . (4) . \supset \vdash :. \text{Hp} . P \in \kappa_\partial : P^\mu \in \kappa_\partial . \vee . P^{\mu+1} \sim\in \kappa_\partial : \supset . P^{\mu+1} W_\kappa P^\mu$ (5)

$\vdash . (5) . *372·1 . \supset \vdash : \text{Hp} . \supset . \kappa_\partial \cap \hat{P}(\mu < \nu . \mu \neq 0 . P^{\mu+1} \in \kappa_\partial . \supset_\mu . P^\mu \in \kappa_\partial) \subset \nu_\kappa$ (6)

$\vdash . (1) . (6) . \supset \vdash . \text{Prop}$

***372·18.** $\vdash : \kappa \in FM \,\text{cycl} \,.\, \nu > 0 \,.\, \supset .\, (2\nu)_\kappa = \kappa_\partial \cap \hat{P}(\mu \leqslant \nu \,.\, \mu \neq 0 \,.\, \supset_\mu .\, P^\mu \in \kappa_\partial)$

Dem.

$\vdash .\, *372{\cdot}1 \,.\, *371{\cdot}12 \,.\, \supset \vdash : \text{Hp} \,.\, P \in (2\nu)_\kappa \,.\, \supset .\, P^{2\nu} W_\kappa P^\nu \,.$

$[*372{\cdot}122] \qquad \supset .\, P^\nu \in \kappa_\partial \qquad (1)$

$\vdash .\, (1) \,.\, *372{\cdot}17 \,. \qquad \supset \vdash : \text{Hp} \,.\, \supset .\, (2\nu)_\kappa \subset \kappa_\partial \cap \hat{P}(\mu \leqslant \nu \,.\, \mu \neq 0 \,.\, \supset_\mu .\, P^\mu \in \kappa_\partial) \quad (2)$

$\vdash .\, *371{\cdot}15{\cdot}152 \,. \qquad \supset \vdash : \text{Hp} \,.\, P, P^\mu \in \kappa_\partial \,.\, \supset .\, P^{\mu+1} W_\kappa P^\mu \qquad (3)$

$\vdash .\, (3) \,.\, *371{\cdot}25 \,. \qquad \supset \vdash : \text{Hp} \,.\, P, P^{\mu+1}\ P^\rho \in \kappa_\partial \,.\, \supset .\, P^{\mu+\rho+1} W_\kappa P^{\mu+\rho} \qquad (4)$

$\vdash .\, (4) \,. \qquad \supset \vdash :.\, P \in \kappa_\partial : \mu \leqslant \nu \,.\, \mu \neq 0 \,.\, \supset_\mu .\, P^\mu \in \kappa_\partial : \supset :$

$\qquad \mu + 1 \leqslant \nu \,.\, \rho \leqslant \nu \,.\, \supset_{\mu,\rho} .\, P^{\mu+\rho+1} W_\kappa P^{\mu+\rho} :$

$[*117{\cdot}561] \qquad \supset : \sigma < 2\nu \,.\, \supset_\sigma .\, P^{\sigma+1} W_\kappa P^\sigma$

$[*372{\cdot}1] \qquad \supset : P \in (2\nu)_\kappa \qquad (5)$

$\vdash .\, (2) \,.\, (5) \,.\, \supset \vdash .\, \text{Prop}$

***372·19.** $\vdash : \kappa \in FM \,\text{cycl} \,.\, \mu, \nu \in \text{NC ind} - \iota`0 \,.\, P \in (\mu\nu)_\kappa \,.\, \supset .\, P^\mu \in \nu_\kappa$
$[*372{\cdot}1 \,.\, *371{\cdot}12]$

***372·2.** $\vdash : \kappa \in FM \,\text{cycl} \,.\, \nu \in \text{NC ind} \,.\, P \in \nu_\kappa \,.\, \mu \leqslant \nu \,.\, \sigma < \mu \,.\, \sigma \neq 0 \,.\, \supset .\, P^\mu W_\kappa P^\sigma$
$[*372{\cdot}1 \,.\, *371{\cdot}12]$

***372·21.** $\vdash : \kappa \in FM \,\text{cycl} \,.\, \nu \in \text{NC ind} \,.\, P \in \nu_\kappa \,.\, 2\mu \leqslant \nu \,.\, \mu \neq 0 \,.\, \supset .$
$\qquad P^{2\mu} W_\kappa P^\mu \,.\, P^\mu \in \kappa_\partial$

Dem.

$\vdash .\, *372{\cdot}2 \,.\, \supset \vdash : \text{Hp} \,.\, \supset .\, P^{2\mu} W_\kappa P^\mu \,. \qquad (1)$

$[*372{\cdot}122] \qquad \supset .\, P^\mu \in \kappa_\partial \qquad (2)$

$\vdash .\, (1) \,.\, (2) \,.\, \supset \vdash .\, \text{Prop}$

***372·22.** $\vdash : \kappa \in FM \,\text{cycl} \,.\, P W_\kappa Q \,.\, P, P^\mu \in \kappa_\partial \,.\, P^\mu W_\kappa Q^\mu \,.\, \supset .\, P^{\mu+1} W_\kappa Q^{\mu+1}$

Dem.

$\vdash .\, *371{\cdot}25 \,.\, \supset \vdash : \text{Hp} \,.\, \supset .\, P^{\mu+1} W_\kappa P \,|\, Q^\mu \qquad (1)$

$\vdash .\, *371{\cdot}16 \,.\, \supset \vdash : \text{Hp} \,.\, \supset .\, Q^\mu \in \kappa_\partial \,.$

$[*371{\cdot}25] \qquad \supset .\, P \,|\, Q^\mu W_\kappa Q^{\mu+1} \qquad (2)$

$\vdash .\, (1) \,.\, (2) \,.\, *371{\cdot}12 \,.\, \supset \vdash .\, \text{Prop}$

***372·23.** $\vdash : \kappa \in FM \,\text{cycl} \,.\, \nu \in \text{NC ind} \,.\, P \in \nu_\kappa \,.\, 2\mu \leqslant \nu \,.\, \mu \neq 0 \,.\, P W_\kappa Q \,.\, \supset .$
$\qquad P^{\mu+1} W_\kappa Q^{\mu+1} \quad [*372{\cdot}21{\cdot}22 \,.\, \text{Induct}]$

***372·24.** $\vdash :.\, \kappa \in FM \,\text{cycl} \,.\, \sigma \in \text{NC ind} - \iota`0 \,.\, P \in (2\sigma)_\kappa \,.\, P W_\kappa Q \,.\, \supset :$
$\qquad \mu \leqslant 2\sigma \,.\, \mu \neq 0 \,.\, \supset .\, P^\mu W_\kappa Q^\mu$

Dem.

$\vdash .\, *372{\cdot}21{\cdot}23 \,.\, \supset \vdash : \text{Hp} \,.\, \xi \leqslant \sigma \,.\, \eta \leqslant \sigma \,.\, \supset .\, P^\eta, Q^\xi \in \kappa_\partial \,.\, P^\xi W_\kappa Q^\xi \,.\, P^\eta W_\kappa Q^\eta \,.$

$[*371{\cdot}25] \qquad \supset .\, P^{\xi+\eta} W_\kappa P^\eta \,|\, Q^\xi \,.\, P^\eta \,|\, Q^\xi W_\kappa Q^{\xi+\eta} \,.$

$[*371{\cdot}12] \qquad \supset .\, P^{\xi+\eta} W_\kappa Q^{\xi+\eta} : \supset \vdash .\, \text{Prop}$

***372·25.** $\vdash :. \kappa \epsilon FM\,\mathrm{cycl} . \sigma \epsilon \mathrm{NC\,ind} - \iota`0 . P \epsilon (2\sigma+1)_\kappa . P W_\kappa Q . \supset :$
$\mu \leqslant 2\sigma . \mu \neq 0 . \supset . P^\mu W_\kappa Q^\mu$ [*372·24·15]

***372·26.** $\vdash : \kappa \epsilon FM\,\mathrm{cycl} . \sigma \epsilon \mathrm{NC\,ind} . P \epsilon (2\sigma+1)_\kappa . P W_\kappa Q . \supset . P^{2\sigma+1} W_\kappa Q^{2\sigma+1}$

Dem.

$\vdash . *372·25 . \supset \vdash :. \mathrm{Hp} . \supset : P^{\sigma+1} W_\kappa Q^{\sigma+1} :$ (1)

[*371·3] $\supset : P^{2\sigma+1} \sim\epsilon\, \kappa_\partial . \supset . P^{2\sigma+1} W_\kappa P^\sigma | Q^{\sigma+1}$ (2)

$\vdash . *371·31 . (1) . \supset \vdash :. \mathrm{Hp} : P^\sigma | Q^{\sigma+1} \sim\epsilon\, \kappa_\partial . \mathbf{v} . Q^{\sigma+1} \epsilon \kappa_\partial : \supset .$
$P^\sigma | Q^{\sigma+1} W_\kappa Q^{2\sigma+1}$ (3)

$\vdash . *372·21 . *371·15·151·152 . \supset \vdash :. \mathrm{Hp} . \supset : P^\sigma | Q^{\sigma+1} W_\kappa P^\sigma :$

[*371·16] $\supset : P^\sigma | Q^{\sigma+1} \epsilon \kappa_\partial . \supset . Q^{\sigma+1} \epsilon \kappa_\partial$ (4)

$\vdash . (3) . (4) . \supset \vdash : \mathrm{Hp} . \supset . P^\sigma | Q^{\sigma+1} W_\kappa Q^{2\sigma+1}$ (5)

$\vdash . (2) . (5) . *371·12 . \supset \vdash : \mathrm{Hp} . P^{2\sigma+1} \sim\epsilon\, \kappa_\partial . \supset . P^{2\sigma+1} W_\kappa Q^{2\sigma+1}$ (6)

$\vdash . *372·22 . \supset \vdash : \mathrm{Hp} . P^{2\sigma} \epsilon \kappa_\partial . \supset . P^{2\sigma+1} W_\kappa Q^{2\sigma+1}$ (7)

$\vdash . *371·16 . *372·1 . \supset \vdash : \mathrm{Hp} . P^{2\sigma+1} \epsilon \kappa_\partial . \supset . P^{2\sigma} \epsilon \kappa_\partial$ (8)

$\vdash . (6) . (7) . (8) . \supset \vdash . \mathrm{Prop}$

***372·27.** $\vdash :. \kappa \epsilon FM\,\mathrm{cycl} . \nu \epsilon \mathrm{NC\,ind} - \iota`0 . P \epsilon \nu_\kappa . P W_\kappa Q . \supset :$
$\mu \leqslant \nu . \mu \neq 0 . \supset . P^\mu W_\kappa Q^\mu$ [*372·24·25·26]

***372·28.** $\vdash :. \kappa \epsilon FM\,\mathrm{cycl} . \nu \epsilon \mathrm{NC\,ind} - \iota`0 . P, Q \epsilon \nu_\kappa . \supset : P^\nu = Q^\nu . \equiv . P = Q$

Dem.

$\vdash . *371·12 . \supset \vdash :. \mathrm{Hp} . P \neq Q . \supset : P W_\kappa Q . \mathbf{v} . Q W_\kappa P :$

[*372·27] $\supset : P^\nu W_\kappa Q^\nu . \mathbf{v} . Q^\nu W_\kappa P^\nu :$

[*371·12] $\supset : P^\nu \neq Q^\nu$ (1)

$\vdash . (1) . \mathrm{Transp} . \supset \vdash . \mathrm{Prop}$

***372·29.** $\vdash :. \kappa \epsilon FM\,\mathrm{cycl} . P, Q \epsilon \kappa_\partial . \supset : P^2 = Q^2 . \equiv . P = Q$ [*372·28·13]

***372·3.** $\vdash : \kappa \epsilon FM\,\mathrm{cycl} . \sigma \epsilon \mathrm{NC\,ind} - \iota`0 . P \epsilon (2\sigma)_\kappa . P W_\kappa Q . \supset . Q \epsilon (2\sigma)_\kappa$

Dem.

$\vdash . *372·18·27 . \supset \vdash :. \mathrm{Hp} . \supset : \mu \leqslant \sigma . \mu \neq 0 . \supset_\mu . P^\mu \epsilon \kappa_\partial . P^\mu W_\kappa Q^\mu .$

[*371·16] $\supset_\mu . Q^\mu \epsilon \kappa_\partial :$

[*372·18] $\supset : Q \epsilon \nu_\kappa :. \supset \vdash . \mathrm{Prop}$

***372·31.** $\vdash :. \kappa \epsilon FM\,\mathrm{cycl} . \sigma \epsilon \mathrm{NC\,ind} - \iota`0 . P \epsilon \kappa_\partial . \supset : P W_\kappa \breve{P}^{2\sigma} . \supset . P^{2\sigma+1} \epsilon \kappa_\partial$

Dem.

$\vdash . *371·16 . \supset \vdash : \mathrm{Hp} . P W_\kappa \breve{P}^{2\sigma} . \supset . \breve{P}^{2\sigma} \epsilon \kappa_\partial$ (1)

$\vdash . *301·23 . \supset \vdash : \mathrm{Hp} . \supset . P = \breve{P}^{2\sigma} | P^{2\sigma+1}$ (2)

$\vdash . (1) . (2) . *371·15 . \supset \vdash : \mathrm{Hp} . P W_\kappa \breve{P}^{2\sigma} . \supset . P^{2\sigma+1} \epsilon \kappa_\partial : \supset \vdash . \mathrm{Prop}$

***372·32.** $\vdash : \kappa \epsilon FM \text{ cycl} . \sigma \epsilon \text{NC ind} . P \epsilon (2\sigma+1)_\kappa . P W_\kappa Q . \supset . Q \epsilon (2\sigma+1)_\kappa$

Dem.

$\vdash . *372{\cdot}3{\cdot}15{\cdot}17 . \quad \supset \vdash :. \text{Hp} . \supset : \mu \leqslant 2\sigma . Q^\mu \epsilon \kappa_\partial . \supset . Q^{\mu-1} \epsilon \kappa_\partial \quad (1)$

$\vdash . *371{\cdot}16 . *372{\cdot}27{\cdot}1 . \supset \vdash : \text{Hp} . Q^{2\sigma} \sim \epsilon \kappa_\partial . \supset . P^{2\sigma+1} \sim \epsilon \kappa_\partial .$

$[*372{\cdot}31.\text{Transp}] \quad \supset . \breve{P}^{2\sigma} W_\kappa P .$

$[*371{\cdot}27] \quad \supset . \breve{Q}^{2\sigma} W_\kappa P .$

$[\text{Hp}] \quad \supset . \breve{Q}^{2\sigma} W_\kappa Q .$

$[*372{\cdot}31.\text{Transp}] \quad \supset . Q^{2\sigma+1} \sim \epsilon \kappa_\partial \quad (2)$

$\vdash . (1) . (2) . \text{Transp} . \quad \supset \vdash :. \text{Hp} . \supset : \mu \leqslant 2\sigma + 1 . Q^\mu \epsilon \kappa_\partial . \supset_\mu . Q^{\mu-1} \epsilon \kappa_\partial :$

$[*372{\cdot}17] \quad \supset : Q \epsilon (2\sigma+1)_\kappa :. \supset \vdash . \text{Prop}$

***372·33.** $\vdash : \kappa \epsilon FM \text{ cycl} . \nu \epsilon \text{NC ind} . \supset . \breve{W}_\kappa``\nu_\kappa \subset \nu_\kappa \quad [*372{\cdot}3{\cdot}32]$

*373. SUBMULTIPLES OF IDENTITY.

Summary of *373.

The purpose of this number is to prove that, in a cyclic submultipliable family, there exists a unique vector which is a member of ν_κ and satisfies $R^\nu = I_\kappa$. This we call the "principal" νth submultiple of I_κ. It is the smallest vector (other than I_κ) which satisfies $R^\nu = I_\kappa$. The proof of its existence proceeds by several stages; the problem is analogous to that of the construction of a regular polygon. Suppose the cycle divided into ν equal parts. Then a vector which takes us from any one point of division to any other is a νth submultiple of identity. If ν is prime, every such vector will have every power less than the νth different from I_κ; but if ν has factors, say ρ and σ, if $R^\nu = I_\kappa$, $(R^\rho)^\sigma = I_\kappa$; thus R^ρ, which is one of the νth submultiples of identity, has a power less than the νth which is equal to I_κ. We define (I_κ, ν) as the class of those νth submultiples of I_κ which have no power less than the νth equal to I_κ; more generally, we put

***373·03.** $(S, \nu) = \hat{P}(P^\nu = S : \sigma < \nu . \sigma \neq 0 . \supset_\sigma . P^\sigma \neq S)$ Dft

We then have first to prove the existence of $\kappa_\partial \cap (I_\kappa, \nu)$ when κ is cyclic and submultipliable. For this purpose, we put

***373·01.** $M_{\nu\kappa} = \hat{Q}\hat{P}(Q \in \kappa_\partial . Q^\nu = P)$ Dft

I.e. $M_{\nu\kappa}$ is the relation of a νth submultiple of P to P, when the submultiple of P is a member of κ_∂. It is to be observed that although κ is submultipliable, we do not know to begin with that I_κ has submultiples which are members of κ_∂, except in the case of K_κ, which is half of I_κ. Owing to this, we proceed first by bisection, *i.e.* by means of the relation $M_{2\kappa}$. We prove that the process of bisection can be applied endlessly to any member of κ_∂, and always gives new terms (*373·14·13), hence it gives a progression starting from any member of κ_∂ (*373·141), and therefore the existence of a cyclic submultipliable family implies the axiom of infinity (*373·142); also we prove that ν bisections starting from a member of κ_∂ give a member of $(2^{\nu+1})_\kappa$ (*373·15). Hence, taking K_κ as the member of κ_∂ to be bisected, we arrive at

$$\mu = 2^{\nu+1} . \supset . \exists ! \kappa_\partial \cap (I_\kappa, \mu) \quad (*373·17).$$

In order to extend this result to numbers not of the form $2^{\nu+1}$, we have

first to prove that there are μth submultiples of identity. This we prove first for numbers of the form $2^\nu + 1$, then for $(2\sigma+1)2^\nu+1$, and then for 2σ (*373·21·22·23); hence it holds generally, *i.e.* we have

***373·25.** $\vdash : \kappa \epsilon FM \text{ cycl subm} . \mu \epsilon \text{NC ind} - \iota`0 - \iota`1 . \supset . (\exists Q) . Q \epsilon \kappa_\partial . Q^\mu = I_\kappa$

Next, we prove that, if $R \epsilon \kappa_\partial$ and $R^\mu = R^\nu = I_\kappa$, then μ, ν have some common factor ρ such that $R \epsilon (I_\kappa, \rho)$, *i.e.* such that R^ρ is the earliest power of R which is I_κ (*373·3). Hence if μ is prime, and $R^\mu = I_\kappa$, it follows that no earlier power of R is I_κ, *i.e.* $R \epsilon (I_\kappa, \mu)$ (*373·32), and that, if $R \epsilon (I_\kappa, \rho)$ and $R^\mu = I_\kappa$, then μ is a multiple of ρ (*373·33).

We now make a fresh start with the general relation $M_{\nu\kappa}$. Owing to *373·25, we know that $I_\kappa \epsilon \text{G}`M_{\nu\kappa}$. Also since $\kappa$ is submultipliable, $\kappa_\partial \subset \text{G}`M_{\nu\kappa}$. Hence if α is any inductive cardinal, $I_\kappa \epsilon \text{G}`M_{\nu\kappa}{}^\alpha$ (*373·404). Also it is easy to show that if ν is a prime, and $QM_{\nu\kappa}{}^\alpha I_\kappa$, Q^{ν^α} is the first power of Q which is I_κ. Hence when ν is prime, $\kappa_\partial \cap (I_\kappa, \nu^\alpha)$ exists (*373·43). In order to extend this result to numbers which are not powers of primes, we prove

***373·45.** $\vdash : \kappa \epsilon FM \text{ cycl} . \rho \text{ Prm } \sigma . R \epsilon (I_\kappa, \rho) . S \epsilon (I_\kappa, \sigma) . \supset . \breve{R} | S \epsilon (I_\kappa, \rho\sigma)$

Hence by the help of a little elementary arithmetic we arrive at

***373·46.** $\vdash : \kappa \epsilon FM \text{ cycl subm} . \rho \epsilon \text{NC ind} - \iota`0 - \iota`1 . \supset . \exists ! \kappa_\partial \cap (I_\kappa, \rho)$

Having now proved that there are νth submultiples of I_κ which have no power short of the νth equal to I_κ, we have still to show that there is one among them which is a member of ν_κ. For this purpose, we take any one of them and consider its powers. It is obvious that it has only ν different powers (*373·5), since after reaching I_κ the previous values repeat themselves. It is this fact which makes it easier to deal with submultiples of I_κ than with submultiples of other vectors.

Now let R be any νth submultiple of identity, and assume that S, T are powers of R, but T is not a power of S, and $TW_\kappa S$. Then $\breve{S} | T$ is a power of R but not of S, and $TW_\kappa (\breve{S} | T)$ (*373·53). Hence T is not the maximum, in the series W_κ, of the class $\text{Pot}`R - \text{Pot}`S$. Hence by transposition, if T is the maximum of $\text{Pot}`R - \text{Pot}`S$, we must have $SW_\kappa T$. Now since $\text{Pot}`R$ is a finite class, $\text{Pot}`R - \text{Pot}`S$ must have a maximum if it exists; but since $S$ has the relation $W_\kappa$ to this maximum, $S$ is not the maximum of $\text{Pot}`R$. Hence by transposition, if S is the maximum of $\text{Pot}`R$, $\text{Pot}`R - \text{Pot}`S$ is null, and therefore $\text{Pot}`R = \text{Pot}`S$ (*373·54). Hence it follows easily that, if $R \epsilon \kappa_\partial \cap (I_\kappa, \nu)$, the maximum of the powers of R is a member of $\kappa_\partial \cap (I_\kappa, \nu)$ (*373·55), and further that it is a member of ν_κ (*373·56). Since we have already proved (*373·46) the existence of $\kappa_\partial \cap (I_\kappa, \nu)$, we thus have

***373·6.** $\vdash : \kappa \epsilon FM \text{ cycl subm} . \nu \epsilon \text{NC ind} - \iota`0 . \supset . \exists ! \nu_\kappa \cap \hat{S}(S^\nu = I_\kappa)$

The uniqueness of $\nu_\kappa \cap \hat{S}(S^\nu = I_\kappa)$ follows from $*372\cdot28$, and thus the principal νth submultiple of I_κ exists. Hence also it immediately follows that the other νth submultiples of I_κ are powers of the principal νth submultiple, and that the total number of νth submultiples is ν ($*373\cdot63\cdot64$).

$*373\cdot01$. $M_{\nu\kappa} = \hat{Q}\hat{P}(Q \in \kappa_\partial \,.\, Q^\nu = P)$ Dft [$*373$—5]

$*373\cdot02$. $\text{Prime} = \text{NC ind} \cap \hat{\mu}(\mu = \sigma \times_c \tau \,.\, \supset_{\sigma,\tau} : \sigma = 1 \,.\vee.\, \sigma = \mu)$ Df

$*373\cdot03$. $(S, \nu) = \hat{P}(P^\nu = S : \sigma < \nu \,.\, \sigma \neq 0 \,.\supset_\sigma.\, P^\sigma \neq S)$ Dft [$*373$—5]

$*373\cdot1$. $\vdash : QM_{2\kappa}P \,.\equiv.\, Q \in \kappa_\partial \,.\, Q^2 = P$ [($*373\cdot01$)]

$*373\cdot11$. $\vdash : \kappa \in FM\,\text{cycl} \,.\supset.\, M_{2\kappa} \in 1 \to 1$ [$*372\cdot29$]

$*373\cdot12$. $\vdash : \kappa \in FM\,\text{cycl} \,.\supset.\, M_{2\kappa} \subseteq \breve{W}_\kappa$ [$*372\cdot121$]

$*373\cdot13$. $\vdash : \kappa \in FM\,\text{cycl} \,.\supset.\, (M_{2\kappa})_{po} \subseteq \breve{W}_\kappa \,.\, (M_{2\kappa})_{po} \subseteq J$ [$*373\cdot12 \,.\, *371\cdot12$]

$*373\cdot14$. $\vdash : \kappa \in FM\,\text{cycl subm} \,.\, P \in \kappa_\partial \,.\, \nu \in \text{NC ind} - \iota`0 \,.\supset.\, E!\, M_{2\kappa}{}^\nu`P$

Dem.

$\vdash .\, *372\cdot29 \,.\, *351\cdot1 \,.\supset \vdash :.\, \text{Hp} \,.\supset : Q \in \kappa_\partial \,.\supset.\, E!\, M_{2\kappa}`P$ (1)

$\vdash .\, (1) \,.\, \text{Induct} \,.\supset \vdash .\, \text{Prop}$

$*373\cdot141$. $\vdash : \kappa \in FM\,\text{cycl subm} \,.\, P \in \kappa_\partial \,.\supset.\, \breve{M}_{2\kappa} \upharpoonright (\overrightarrow{M_{2\kappa}})_*`P \in \text{Prog}$
[$*373\cdot11\cdot13\cdot14$]

$*373\cdot142$. $\vdash : \exists!\, FM\,\text{cycl subm} \,.\supset.\, \text{Infin ax}$ [$*373\cdot141$]

$*373\cdot15$. $\vdash : \kappa \in FM\,\text{cycl subm} \,.\, P \in \kappa_\partial \,.\, \nu \in \text{NC ind} \,.\supset.\, M_{2\kappa}{}^\nu`P \in (2^{\nu+1})_\kappa$

Dem.

$\vdash .\, *373\cdot1\cdot14 \,.\quad \supset \vdash : \text{Hp} \,.\, Q = M_{2\kappa}{}^{\nu-1}`P \,.\, R = M_{2\kappa}{}^\nu`P \,.\supset.\, Q^\sigma = R^{2\sigma}$ (1)

$\vdash .\, (1) \,.\, *372\cdot18 \,.\supset \vdash :.\, \text{Hp}(1) \,.\, Q \in (2^\nu)_\kappa \,.\supset : 2\sigma \leqslant 2^\nu \,.\supset.\, R^{2\sigma} \in \kappa_\partial$ (2)

$\vdash .\, (2) \,.\, *373\cdot1 \,.\quad \supset \vdash :.\, \text{Hp}(2) \,.\supset : 2\sigma < 2^\nu \,.\supset : R^{2\sigma}, R^{2\sigma+2}, R^2, R \in \kappa_\partial \,.$

[$*371\cdot2$] $\supset.\, R^{2\sigma+1} \in \kappa_\partial$ (3)

$\vdash .\, (2) \,.\, (3) \,.\quad \supset \vdash :.\, \text{Hp}(2) \,.\supset : \mu \leqslant 2^\nu \,.\supset.\, R^\mu \in \kappa_\partial :$

[$*372\cdot18$] $\supset : R \in (2^{\nu+1})_\kappa$ (4)

$\vdash .\, *372\cdot13 \,.\quad \supset \vdash : \text{Hp} \,.\, \nu = 0 \,.\supset.\, M_{2\kappa}{}^\nu`P \in 2_\kappa$ (5)

$\vdash .\, (4) \,.\, (5) \,.\, \text{Induct} \,.\supset \vdash .\, \text{Prop}$

$*373\cdot16$. $\vdash :.\, \kappa \in FM\,\text{cycl subm} \,.\, \nu \in \text{NC ind} \,.\, Q = M_{2\kappa}{}^\nu`K_\kappa \,.\supset :$

$$Q^{2^{\nu+1}} = I_\kappa : \rho < 2^{\nu+1} \,.\, \rho \neq 0 \,.\supset_\rho.\, Q^\rho \neq I_\kappa$$

Dem.

$\vdash .\, *373\cdot1 \,.\quad \supset \vdash : \text{Hp} \,.\supset.\, Q^{2^\nu} = K_\kappa \,.$

[$*371\cdot26$] $\supset.\, Q^{2^{\nu+1}} = I_\kappa$ (1)

$\vdash .\, *373\cdot15 \,.\, *372\cdot2 \,.\, (1) \,.\supset \vdash :.\, \text{Hp} \,.\supset : \rho < 2^{\nu+1} \,.\, \rho \neq 0 \,.\supset.\, Q^\rho W_\kappa I_\kappa$ (2)

$\vdash .\, (1) \,.\, (2) \,.\supset \vdash .\, \text{Prop}$

∗373·17. $\vdash : \kappa \in FM \text{ cycl subm} \,.\, \nu \in \text{NC ind} \,.\, \mu = 2^{\nu+1} \,.\, \supset \,.\, \exists \,!\, \kappa_\partial \cap (I_\kappa, \mu)$
[∗373·16·14 . (∗373·03)]

∗373·18. $\vdash : Q \in \text{Cnv}``\kappa_\partial \,.\, Q^\nu = I_\kappa \,.\, \supset \,.\, \breve{Q} \in \kappa_\partial \,.\, \breve{Q}^\nu = I_\kappa$ [∗50·5·51]

∗373·19. $\vdash : (\exists Q) \,.\, Q \in \kappa_\partial \cup \text{Cnv}``\kappa_\partial \,.\, Q^\nu = I_\kappa \,.\, \equiv \,.\, (\exists Q) \,.\, Q \in \kappa_\partial \,.\, Q^\nu = I_\kappa$
[∗373·18]

∗373·2. $\vdash :. \kappa \in FM \text{ cycl subm} \,.\, \nu \in \text{NC ind} \,.\, P = M_{2\kappa}{}^{\nu}\text{‘}K_\kappa \,.$
$S \in \kappa_\partial \,.\, S^{2^\nu+1} = P \,.\, S^{2^{\nu+1}} = Q \,.\, \supset \,.\, Q^{2^\nu+1} = I_\kappa \,.\, Q \neq I_\kappa$

Dem.

$\vdash \,.\, ∗301·5 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, Q^{2^\nu+1} = P^{2^{\nu+1}} = I_\kappa$ (1)
$\vdash \,.\, ∗373·1 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, P^{2^\nu+1} = K_\kappa \,|\, P \,.$
[∗370·22] $\supset \,.\, P^{2^\nu+1} \neq P \,.$
[Hp] $\supset \,.\, P^{2^\nu+1} \neq S^{2^\nu+1} \,.$
[∗30·37] $\supset \,.\, P \neq S$ (2)
$\vdash \,.\, ∗301·5·23 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, Q = (S^{2^\nu+1})^2 \,|\, \breve{S}^2$
[Hp] $= P^2 \,|\, \breve{S}^2$
[(2).∗372·29] $\neq I_\kappa$ (3)
$\vdash \,.\, (1) \,.\, (3) \,.\, \supset \vdash \,.\, \text{Prop}$

∗373·21. $\vdash : \kappa \in FM \text{ cycl subm} \,.\, \nu \in \text{NC ind} \,.\, \mu = 2^\nu + 1 \,.\, \supset \,.$
$(\exists Q) \,.\, Q \in \kappa_\partial \,.\, Q^\mu = I_\kappa$ [∗373·2·19]

∗373·22. $\vdash : \kappa \in FM \text{ cycl subm} \,.\, \nu, \sigma \in \text{NC ind} \,.\, \mu = (2\sigma + 1)\, 2^\nu + 1 \,.\, \supset \,.$
$(\exists Q) \,.\, Q \in \kappa_\partial \,.\, Q^\mu = I_\kappa$
[The proof proceeds as in ∗373·2·21]

∗373·23. $\vdash : \kappa \in FM \text{ cycl subm} \,.\, \sigma \in \text{NC ind} \,.\, \mu = 2\sigma \,.\, \supset \,.\, (\exists Q) \,.\, Q \in \kappa_\partial \,.\, Q^\mu = I_\kappa$

Dem.

$\vdash \,.\, ∗370·26 \,.\, \supset \vdash : \text{Hp} \,.\, \supset \,.\, K \in \kappa_\partial \,.\, K^\mu = I_\kappa : \supset \vdash \,.\, \text{Prop}$

∗373·231. $\vdash :. \tau \in \text{NC ind} \,.\, \supset : (\exists \sigma) : \sigma \in \text{NC ind} : \tau = 2\sigma \,.\, \vee \,.\, \tau = 2\sigma + 1$ [Induct]

∗373·24. $\vdash : \rho \in \text{NC ind} \,.\, \rho \neq 0 \,.\, \supset \,.$
$(\exists \nu, \sigma) \,.\, \nu, \sigma \in \text{NC ind} \,.\, 2\rho + 1 = (2\sigma + 1)\, 2^\nu + 1$

Dem.

$\vdash \,.\, ∗117·661 \,.\, \supset$
$\vdash :. \text{Hp} \,.\, \lambda = \hat{\nu} \{(\exists \tau) \,.\, \tau \in \text{NC ind} - \iota\text{‘}0 \,.\, \rho = \tau 2^\nu\} \,.\, \supset : \nu \in \lambda \,.\, \supset \,.\, \rho > \nu$ (1)
$\vdash \,.\, ∗116·301 \,.\, \supset \vdash : \text{Hp}\,(1) \,.\, \supset \,.\, \rho = \rho 2^0 \,.$
[∗10·24] $\supset \,.\, 0 \in \lambda$ (2)
$\vdash \,.\, (1) \,.\, (2) \,.\, ∗261·26 \,.\, ∗263·47 \,.\, \supset \vdash :. \text{Hp}\,(1) \,.\, \supset :$
$(\exists \nu) : \nu \in \lambda : \mu > \nu \,.\, \supset_\mu \,.\, \mu \sim\in \lambda$ (3)
$\vdash \,.\, ∗116·52·321 \,.\, \supset \vdash : \rho = \tau 2^\nu \,.\, \tau = 2\sigma \,.\, \supset \,.\, \rho = \sigma 2^{\nu+1}$ (4)

$\vdash . (3) . (4) . \supset \vdash :. \mathrm{Hp} . \supset : (\exists \nu, \tau) : \nu, \tau \in \mathrm{NC\,ind} . \rho = \tau 2^\nu : \mu > \nu . \supset_\mu .$

$\sim (\exists \tau) . \rho = \tau 2^\mu : \sim (\exists \sigma) . \tau = 2\sigma :$

$[*373\cdot231] \quad \supset : (\exists \nu, \sigma) . \nu, \sigma \in \mathrm{NC\,ind} . \rho = (2\sigma + 1) 2^\nu :$

$[*116\cdot52\cdot321] \supset : (\exists \nu, \sigma) . \nu, \sigma \in \mathrm{NC\,ind} . 2\rho + 1 = (2\sigma + 1) 2^{\nu+1} + 1 :. \supset \vdash . \mathrm{Prop}$

***373·25.** $\vdash : \kappa \in FM \mathrm{\,cycl\,subm} . \mu \in \mathrm{NC\,ind} - \iota\text{`}0 - \iota\text{`}1 . \supset .$

$(\exists Q) . Q \in \kappa_\partial . Q^\mu = I_\kappa \quad [*373\cdot22\cdot24\cdot23\cdot141$

***373·3.** $\vdash : \kappa \in FM \mathrm{\,cycl} . \mu \neq 0 . \nu \neq 0 . R \in \kappa_\partial . R^\mu = R^\nu = I_\kappa . \supset .$

$(\exists \rho, \alpha, \beta) . \rho \neq 0 . \rho \neq 1 . \mu = \alpha\rho . \nu = \beta\rho . R \in (I_\kappa, \rho)$

Dem.

$\vdash . *300\cdot23 . \supset \vdash :. \mathrm{Hp} . \supset : (\exists \rho) . \rho \neq 0 . R^\rho = I_\kappa : \sigma < \rho . \sigma \neq 0 . \supset_\sigma . R^\sigma \neq I_\kappa \quad (1)$

$\vdash . *301\cdot2 . \quad \supset \vdash : \mathrm{Hp} . R^\rho = I_\kappa . \supset . \rho \neq 1 \quad (2)$

$\vdash . *302\cdot25 . \supset \vdash : \mathrm{Hp} . \rho \in \mathrm{NC\,ind} - \iota\text{`}0 . \supset .$

$(\exists \alpha, \beta, \gamma, \delta) . \mu = \alpha\rho + \beta . \nu = \gamma\rho + \delta . \beta < \rho . \delta < \rho \quad (3)$

$\vdash . *301\cdot23\cdot504 . \supset$

$\vdash : \mathrm{Hp}(3) . R^\rho = I_\kappa . \mu = \alpha\rho + \beta . \nu = \gamma\rho + \delta . R^\mu = R^\nu = I_\kappa . \supset . R^\beta = R^\delta = I_\kappa \quad (4)$

$\vdash . (4) . \supset \vdash :. \mathrm{Hp}(4) : \sigma < \rho . \sigma \neq 0 . \supset_\sigma . R^\sigma \neq I_\kappa :$

$\mu = \alpha\rho + \beta . \nu = \gamma\rho + \delta : \supset . \beta = 0 . \delta = 0 \quad (5)$

$\vdash . (3) . (5) . \supset \vdash :. \mathrm{Hp} : \rho \neq 0 . R^\rho = I_\kappa : \sigma < \rho . \sigma \neq 0 . \supset_\sigma . R^\sigma \neq I_\kappa : \supset .$

$(\exists \alpha, \gamma) . \mu = \alpha\rho . \nu = \gamma\rho \quad (6)$

$\vdash . (1) . (2) . (6) . (*373\cdot03) . \supset \vdash . \mathrm{Prop}$

***373·31.** $\vdash : \kappa \in FM \mathrm{\,cycl} . R \in \kappa_\partial . \mu \neq 0 . \nu \neq 0 . R^\mu = R^\nu = I_\kappa . \supset . \sim (\mu \mathrm{\,Prm\,} \nu)$
[*373·3]

***373·32.** $\vdash : \kappa \in FM \mathrm{\,cycl} . R \in \kappa_\partial . \mu \in \mathrm{Prime} . R^\mu = I_\kappa . \supset . R \in (I_\kappa, \mu)$
[*373·31 . Transp . (*373·03)]

We assume here that a prime number is prime to all numbers less than itself except 1. This follows at once from the definition.

***373·33.** $\vdash : \kappa \in FM \mathrm{\,cycl} . R \in \kappa_\partial \cap (I_\kappa, \rho) . R^\mu = I_\kappa . \supset . (\exists \tau) . \mu = \rho\tau$ [*373·3]

***373·4.** $\vdash : Q M_{\nu\kappa} P . \equiv . Q \in \kappa_\partial . P = Q^\nu$ [(*373·01)]

***373·401.** $\vdash : \kappa \in FM \mathrm{\,cycl\,subm} . \nu \in \mathrm{NC\,ind} - \iota\text{`}0 . \supset . I_\kappa \in \mathrm{G}\text{`}M_{\nu\kappa}$ [*373·25]

***373·402.** $\vdash : \kappa \in FM \mathrm{\,subm} . \nu \in \mathrm{NC\,ind} - \iota\text{`}0 . \supset . \kappa_\partial \subset \mathrm{G}\text{`}M_{\nu\kappa}$ [*373·4]

***373·403.** $\vdash : \nu \in \mathrm{NC\,ind} - \iota\text{`}0 . \supset . \mathrm{D}\text{`}M_{\nu\kappa} \subset \kappa_\partial$ [*373·4]

***373·404.** $\vdash : \kappa \in FM \mathrm{\,cycl\,subm} . \nu, \alpha \in \mathrm{NC\,ind} - \iota\text{`}0 . \supset . I_\kappa \in \mathrm{G}\text{`}M_{\nu\kappa}{}^\alpha$
[*373·401·402·403 . Induct]

***373·405.** $\vdash : \nu, \alpha \in \mathrm{NC\,ind} - \iota\text{`}0 . Q M_{\nu\kappa}{}^\alpha I_\kappa . \supset . Q^{\nu^\alpha} = I_\kappa$ [*373·4 . Induct]

***373·406.** $\vdash : \nu, \alpha \epsilon \text{NC ind} - \iota`0 . R \epsilon \text{D}`M_{\nu\kappa}{}^{\alpha} . \supset . \breve{M}_{\nu\kappa}{}^{\alpha}`R = R^{\nu^\alpha}$
[*373·4 . Induct]

***373·407.** $\vdash : \nu, \alpha, \gamma \epsilon \text{NC ind} - \iota`0 . RM_{\nu\kappa}{}^{\alpha+\gamma}I_\kappa . \supset . R^{\nu^\alpha}M_{\nu\kappa}{}^{\gamma}I_\kappa$ [*373·406]

***373·41.** $\vdash : \nu, \alpha, \beta \epsilon \text{NC ind} - \iota`0 . QM_{\nu\kappa}{}^{\alpha}I_\kappa . RM_{\nu\kappa}{}^{\beta}I_\kappa . \alpha < \beta . \supset . Q \neq R$

Dem.

$\vdash . *373·405·407·403 . \supset \vdash : \text{Hp} . \supset . Q^{\nu^\alpha} = I_\kappa . R^{\nu^\alpha} \epsilon \kappa_\partial : \supset \vdash . \text{Prop}$

***373·42.** $\vdash : \kappa \epsilon FM \text{ cycl} . \nu \epsilon \text{Prime} - \iota`1 . \alpha \epsilon \text{NC ind} .$
$QM_{\nu\kappa}{}^{\alpha}I_\kappa . \sigma < \nu^\alpha . \sigma \neq 0 . \supset . Q^\sigma \neq I_\kappa$

Dem.

$\vdash . *373·405 . *300·23 . \supset$

$\vdash :. \text{Hp} . \supset : (\exists\rho) : \rho \neq 0 . Q^\rho = I_\kappa : \sigma < \rho . \sigma \neq 0 . \supset_\sigma . Q^\sigma \neq I_\kappa$ (1)

$\vdash . *373·33·405 . \supset$

$\vdash :. \text{Hp} : \rho \neq 0 . Q^\rho = I_\kappa : \sigma < \rho . \sigma \neq 0 . \supset_\sigma . Q^\sigma \neq I_\kappa : \supset . (\exists\tau) . \nu^\alpha = \rho\tau .$

[Hp] $\supset . (\exists\beta) . \rho = \nu^\beta$ (2)

$\vdash . *373·407 . \supset \vdash : \text{Hp} . \beta < \alpha . \supset . Q^{\nu^\beta} \neq I_\kappa$ (3)

$\vdash . (2) . (3) . \supset \vdash : \text{Hp}(2) . \supset . \rho = \nu^\alpha$ (4)

$\vdash . (1) . (4) . \supset \vdash . \text{Prop}$

In obtaining (2) of the above proof, we assume that if ν is a prime, and $\rho\tau$ is a power of ν, then ρ is a power of ν. This is easily proved.

***373·43.** $\vdash : \kappa \epsilon FM \text{ cycl subm} . \nu \epsilon \text{Prime} - \iota`1 . \alpha \epsilon \text{NC ind} - \iota`1 . \supset .$
$\exists ! \kappa_\partial \cap (I_\kappa, \nu^\alpha)$ [*373·404·405·42]

***373·44.** $\vdash : \gamma \text{ Prm } \rho . \gamma \text{ Prm } \sigma . \supset . \gamma \text{ Prm } \rho\sigma$

Dem.

$\vdash . *302·1 . \quad \supset \vdash :. \gamma \text{ Prm } \rho . \sim(\gamma \text{ Prm } \rho\sigma) . \sigma \epsilon \text{NC ind} . \supset .$
$(\exists\tau, \alpha, \beta) . \tau \epsilon \text{NC ind} - \iota`0 - \iota`1 . \gamma = \alpha\tau . \rho\sigma = \beta\tau$ (1)

$\vdash . *303·39 . \quad \supset \vdash : \text{Hp}(1) . \tau \epsilon \text{NC ind} - \iota`0 - \iota`1 . \gamma = \alpha\tau . \rho\sigma = \beta\tau . \supset .$
$\gamma/\rho = \alpha\sigma/\beta$ (2)

$\vdash . (2) . *303·341 . \supset \vdash : \text{Hp}(2) . \alpha\sigma \text{ Prm } \beta . \supset . \gamma = \alpha\sigma$ (3)

$\vdash . (3) . *302·1 . \quad \supset \vdash : \text{Hp}(3) . \sigma \neq 1 . \supset . \sim(\gamma \text{ Prm } \sigma)$ (4)

$\vdash . *113·621 . \quad \supset \vdash : \rho \epsilon \text{NC} . \sigma = 1 . \sim(\gamma \text{ Prm } \rho\sigma) . \supset . \sim(\gamma \text{ Prm } \rho)$ (5)

$\vdash . (5) . \text{Transp} . \quad \supset \vdash : \text{Hp}(1) . \supset . \sigma \neq 1 :$

[(4)] $\quad \supset \vdash : \text{Hp}(3) . \supset . \sim(\gamma \text{ Prm } \sigma)$ (6)

$\vdash . *302·36 . \supset \vdash : \text{Hp}(2) . \sim(\alpha\sigma \text{ Prm } \beta) . \supset .$
$(\exists\xi, \eta, \zeta) . \xi \text{ Prm } \eta . \zeta \neq 1 . \alpha\sigma = \xi\zeta . \beta = \eta\zeta$ (7)

$\vdash . *303·39 . \supset \vdash : \text{Hp}(7) . \xi \text{ Prm } \eta . \zeta \neq 1 . \alpha\sigma = \xi\zeta . \beta = \eta\zeta . \supset . \alpha\sigma/\beta = \xi/\eta .$

[(2).*303·341] $\supset . \gamma = \xi . \rho = \eta .$

[Hp] $\supset . \alpha\rho\sigma = \beta\gamma = \alpha\rho\zeta\tau .$

[*126·41] $\supset . \sigma = \zeta\tau$ (8)

$\vdash . (7) . (8) . \quad \supset \vdash : \mathrm{Hp}(7) . \supset . (\exists \zeta) . \gamma = \alpha\tau . \sigma = \zeta\tau .$

[*302·1.Hp] $\qquad \supset . \sim (\gamma \operatorname{Prm} \sigma)$ (9)

$\vdash . (6) . (9) . \quad \supset \vdash : \mathrm{Hp}(2) . \supset . \sim (\gamma \operatorname{Prm} \sigma)$ (10)

$\vdash . (1) . (10) . \supset \vdash : \gamma \operatorname{Prm} \rho . \sim (\gamma \operatorname{Prm} \rho\sigma) . \sigma \in \mathrm{NC\,ind} . \supset . \sim (\gamma \operatorname{Prm} \sigma)$ (11)

$\vdash . (11) . \mathrm{Transp} . \supset \vdash . \mathrm{Prop}$

***373·441.** $\vdash :. \rho \operatorname{Prm} \sigma : (\exists \delta) . \rho\beta = \delta\sigma : \supset . (\exists \zeta) . \beta = \zeta\sigma$

Dem.

$\vdash . *126·41 . \supset$

$\vdash : \mathrm{Hp} . \rho\beta = \delta\sigma . \rho = \xi\varpi . \delta = \eta\varpi . \xi \operatorname{Prm} \eta . \supset . \xi\beta = \eta\sigma . \xi \operatorname{Prm} \eta . \xi \operatorname{Prm} \sigma .$

[*373·44] $\qquad \supset . \xi\beta = \eta\sigma . \xi \operatorname{Prm} \eta\sigma$ (1)

$\vdash . (1) . \qquad \supset \vdash : \mathrm{Hp}(1) . \xi \neq 1 . \supset . \xi \neq 1 . \xi = \xi \times_0 1 . \eta\sigma = \xi \times_c \beta .$

[*302·1] $\qquad \supset . \sim (\xi \operatorname{Prm} \eta\sigma)$ (2)

$\vdash . (2) . \mathrm{Transp} . (1) . \supset \vdash : \mathrm{Hp}(1) . \supset . \xi = 1$ (3)

$\vdash . (1) . (3) . \supset \vdash . \mathrm{Prop}$

***373·45.** $\vdash : \kappa \in FM \operatorname{cycl} . \rho \operatorname{Prm} \sigma . R \in (I_\kappa, \rho) . S \in (I_\kappa, \sigma) . \supset . \breve{R} | S \in (I_\kappa, \rho\sigma)$

Dem.

$\vdash . *370·33 . \qquad \supset \vdash : \mathrm{Hp} . \supset . (\breve{R} | S)^{\rho\sigma} = I_\kappa$ (1)

$\vdash . (1) . *373·31 . \supset \vdash :. \mathrm{Hp} . (\breve{R} | S)^\gamma = I_\kappa . \gamma \neq 0 . \supset : \sim (\gamma \operatorname{Prm} \rho\sigma) :$

[*373·44] $\qquad \supset : \sim (\gamma \operatorname{Prm} \rho) . \vee . \sim (\gamma \operatorname{Prm} \sigma)$ (2)

$\vdash . *370·33 . *301·504 . \supset$

$\vdash : \mathrm{Hp}(2) . \rho = \alpha\tau . \gamma = \beta\tau . \supset . I_\kappa = (\breve{R} | S)^{\alpha\beta\tau} = S^{\alpha\beta\tau} = S^{\rho\beta} .$

[*373·33] $\qquad \supset . (\exists \delta) . \rho\beta = \delta\sigma .$

[*373·441] $\qquad \supset . (\exists \zeta) . \beta = \zeta\sigma$ (3)

$\vdash . (3) . \qquad \supset \vdash : \mathrm{Hp}(3) . \supset . (\breve{R} | S)^{\beta\tau} = I_\kappa . S^{\beta\tau} = I_\kappa .$

[*370·33] $\qquad \supset . R^{\beta\tau} = I_\kappa .$

[*373·33] $\qquad \supset . (\exists \mu) . \beta\tau = \mu\alpha\tau .$

[Hp] $\qquad \supset . (\exists \mu) . \gamma = \mu\rho . \mu \neq 0$ (4)

$\vdash . (3) . (4) . \supset \vdash : \mathrm{Hp}(3) . \supset . (\exists \nu) . \gamma = \nu\rho\sigma . \nu \neq 0$ (5)

Similarly $\quad \vdash : \mathrm{Hp} . \sim (\gamma \operatorname{Prm} \sigma) . \supset . (\exists \nu) . \gamma = \nu\rho\sigma . \nu \neq 0$ (6)

$\vdash . (2) . (5) . (6) . \supset \vdash : \mathrm{Hp}(2) . \supset . (\exists \nu) . \nu \neq 0 . \gamma = \nu\rho\sigma$ (7)

$\vdash . (1) . (7) . *117·62 . \supset \vdash . \mathrm{Prop}$

***373·451.** $\vdash :. \rho \in \mathrm{NC\,ind} - \iota`0 : \sim (\exists \nu, \alpha) . \nu \in \mathrm{Prime} . \rho = \nu^\alpha : \supset .$

$(\exists \mu, \nu) . \mu \operatorname{Prm} \nu . \mu < \rho . \nu < \rho . \rho = \mu\nu$

Dem.

$\vdash . *261·26 . *263·47 . \supset$

$\vdash : \mathrm{Hp} . \supset . (\exists \gamma, \alpha) . \gamma \in \mathrm{Prime} . \rho \in \mathrm{D}`\times_c \gamma^\alpha . \rho \sim\in \mathrm{D}`\times_c \gamma^{\alpha+1} . \rho \neq \gamma^\alpha .$

[*373·44.Induct] $\supset . (\exists \gamma, \alpha, \beta) . \gamma \in \mathrm{Prime} . \rho = \gamma^\alpha\beta . \beta \operatorname{Prm} \gamma^\alpha . \beta \neq 1 : \supset \vdash . \mathrm{Prop}$

***373·452.** $\vdash :. \nu \in \text{Prime} . \alpha \in \text{NC ind} . \supset_{\nu, \alpha} . \phi(\nu^\alpha) : \mu \text{ Prm } \nu . \phi\mu . \phi\nu . \supset_{\mu,\nu} .$
$\phi(\mu\nu) : \supset : \rho \in \text{NC ind} - \iota`0 . \supset_\rho . \phi(\rho)$ [*373·451]

***373·46.** $\vdash : \kappa \in FM \text{ cycl subm} . \rho \in \text{NC ind} - \iota`0 - \iota`1 . \supset . \exists ! \kappa_\partial \cap (I_\kappa, \rho)$
[*373·43·45·18·452]

***373·5.** $\vdash : \kappa \in FM \text{ cycl} . \nu \in \text{NC ind} . R \in \kappa_\partial \cap (I_\kappa, \nu) . \supset . \text{Pot}`R \in \nu$

Dem.

$\vdash . *302·25 . *301·504 . \supset$

$\vdash : \text{Hp} . \alpha \in \text{NC ind} . \supset . (\exists \xi, \eta) . \alpha = \xi\nu + \eta . \eta < \nu . R^\alpha = R^\eta .$

[*120·57] $\supset . \text{Nc}`\text{Pot}`R \leqslant \nu$ (1)

$\vdash . *301·23 . \quad \supset \vdash : \text{Hp} . \rho < \nu . \sigma < \rho . \supset . \breve{R}^\sigma | R^\rho = R^{\rho -_c \sigma} .$

[Hp] $\supset . \breve{R}^\sigma | R^\rho \neq I_\kappa .$

[*330·32] $\supset . R^\rho \neq R^\sigma$ (2)

$\vdash . (2) . \text{Transp} . \supset \vdash : \text{Hp} . \rho < \nu . \sigma < \nu . R^\rho = R^\sigma . \supset . \rho = \nu$ (3)

$\vdash . (3) . *120·57 . \supset \vdash : \text{Hp} . \supset . \text{Nc}`\text{Pot}`R \geqslant \nu$ (4)

$\vdash . (1) . (4) . \supset \vdash . \text{Prop}$

***373·51.** $\vdash : \kappa \in FM \text{ cycl} . R \in \kappa_\partial \cap (I_\kappa, \mu\nu) . \supset . R^\mu \in (I_\kappa, \nu) . \text{Pot}`R^\mu \in \nu$

Dem.

$\vdash . *301·504 . \supset$

$\vdash :. \text{Hp} . \supset : (R^\mu)^\nu = I_\kappa : \sigma < \nu . \sigma \neq 0 . \supset_\sigma . (R^\mu)^\sigma \neq I_\kappa :. \supset \vdash . \text{Prop}$

***373·52.** $\vdash : \kappa \in FM \text{ cycl} . R \in \kappa_\partial \cap (I_\kappa, \nu) . \mu \text{ Prm } \nu . \supset .$
$R^\mu \in (I_\kappa, \nu) . \text{Pot}`R^\mu = \text{Pot}`R$

Dem.

$\vdash . *373·33 . \quad \supset \vdash : \text{Hp} . R^\mu \in (I_\kappa, \rho) . \supset . (\exists \tau) . \mu\rho = \nu\tau .$

[*373·441] $\supset . (\exists \zeta) . \rho = \nu\zeta$ (1)

$\vdash . *301·504 . \quad \supset \vdash : \text{Hp}(1) . \supset . (R^\mu)^\nu = I_\kappa .$

[Hp] $\supset . \rho \leqslant \nu$ (2)

$\vdash . (1) . (2) . \quad \supset \vdash : \text{Hp} . \supset . R^\mu \in (I_\kappa, \nu)$ (3)

$\vdash . (3) . *373·51 . \supset \vdash : \text{Hp} . \supset . \text{Nc}`\text{Pot}`R^\mu = \text{Nc}`\text{Pot}`R = \nu$ (4)

$\vdash . *91·6 . \quad \supset \vdash : \text{Hp} . \supset . \text{Pot}`R^\mu \subset \text{Pot}`R$ (5)

$\vdash . (4) . (5) . *120·426 . \text{Transp} . \supset \vdash : \text{Hp} . \supset . \text{Pot}`R^\mu = \text{Pot}`R$ (6)

$\vdash . (3) . (6) . \supset \vdash . \text{Prop}$

***373·521.** $\vdash : \kappa \in FM \text{ cycl} . R \in (\kappa_\partial \cup \text{Cnv}``\kappa_\partial) . \nu \in \text{NC ind} . R^\nu = I_\kappa . \supset . \breve{R} \in \text{Pot}`R$

Dem.

$\vdash . *301·2 . *13·14 . \supset \vdash : \text{Hp} . \supset . \nu \neq 0$ (1)

$\vdash . (1) . *301·21 . \quad \supset \vdash : \text{Hp} . \supset . \breve{R} = R^{\nu -_c 1} : \supset \vdash . \text{Prop}$

***373·522.** $\vdash : \text{Hp} *373{\cdot}521 \,.\, S, T \epsilon \text{Pot}`R \,.\, \supset .\, \breve{S} | T \epsilon \text{Pot}`R$

Dem.

$\vdash . *373{\cdot}521 . \supset \vdash : \text{Hp} . \supset . \breve{S} \epsilon \text{Pot}`S .$

$[*91{\cdot}6] \qquad \supset . \breve{S} \epsilon \text{Pot}`R .$

$[*91{\cdot}343] \qquad \supset . \breve{S} | T \epsilon \text{Pot}`R : \supset \vdash . \text{Prop}$

***373·53.** $\vdash : \text{Hp} *373{\cdot}521 \,.\, S, T \epsilon \text{Pot}`R \,.\, T \sim\epsilon \text{Pot}`S \,.\, T W_\kappa S \,.\, \supset .$
$T W_\kappa (\breve{S} | T) \,.\, \breve{S} | T \epsilon \text{Pot}`R - \text{Pot}`S$

Dem.

$\vdash . *371{\cdot}23 . \qquad \supset \vdash : \text{Hp} . \supset . T W_\kappa (\breve{S} | T) \qquad (1)$

$\vdash . *373{\cdot}522 . \qquad \supset \vdash : \text{Hp} . \supset . \breve{S} | T \epsilon \text{Pot}`R \qquad (2)$

$\vdash . *91{\cdot}36 . \text{Transp} . \supset \vdash : \text{Hp} . \supset . \breve{S} | T \sim\epsilon \text{Pot}`S \qquad (3)$

$\vdash . (1) . (2) . (3) . \supset \vdash . \text{Prop}$

***373·531.** $\vdash : \text{Hp} *373{\cdot}53 . \supset . \sim \{T = \max (W_\kappa)`(\text{Pot}`R - \text{Pot}`S)\} \quad [*373{\cdot}53]$

***373·532.** $\vdash : \text{Hp} *373{\cdot}521 \,.\, S \epsilon \text{Pot}`R \,.\, T = \max (W_\kappa)`(\text{Pot}`R - \text{Pot}`S) \,.\, \supset .$
$S W_\kappa T \quad [*373{\cdot}531 . \text{Transp} . *371{\cdot}12]$

***373·533.** $\vdash : \text{Hp} *373{\cdot}521 \,.\, S \epsilon \text{Pot}`R \,.\, \text{E} ! \max (W_\kappa)`(\text{Pot}`R - \text{Pot}`S) \,.\, \supset .$
$\sim \{S = \max (W_\kappa)`\text{Pot}`R\} \quad [*373{\cdot}532]$

***373·54.** $\vdash : \text{Hp} *373{\cdot}521 \,.\, S = \max (W_\kappa)`\text{Pot}`R \,.\, \supset .\, \text{Pot}`R = \text{Pot}`S$

Dem.

$\vdash . *373{\cdot}533 . \text{Transp} . \supset \vdash : \text{Hp} . \supset . \sim \text{E} ! \max (W_\kappa)`(\text{Pot}`R - \text{Pot}`S) \qquad (1)$

$\vdash . (1) . *373{\cdot}3{\cdot}5 . *261{\cdot}26 . \text{Transp} . \supset \vdash : \text{Hp} . \supset . \text{Pot}`R - \text{Pot}`S = \Lambda \qquad (2)$

$\vdash . (2) . *91{\cdot}6 . \supset \vdash . \text{Prop}$

***373·55.** $\vdash : \kappa \epsilon FM \text{ cycl} \,.\, \nu \epsilon \text{NC ind} - \iota`0 \,.\, R \epsilon \kappa_\partial \cap (I_\kappa, \nu) \,.$
$S = \max (W_\kappa)`\text{Pot}`R \,.\, \supset .\, S \epsilon (I_\kappa, \nu)$

Dem.

$\vdash . *373{\cdot}3{\cdot}5 . \quad \supset \vdash : \text{Hp} . \supset . (\exists \rho) . \rho \epsilon \text{NC ind} - \iota`0 . S \epsilon (I_\kappa, \rho) . \text{Pot}`S \epsilon \rho \qquad (1)$

$\vdash . *373{\cdot}54{\cdot}5 . \supset \vdash :. \text{Hp} . \supset : \text{Pot}`S \epsilon \nu :$

$[*100{\cdot}34] \qquad \supset : \rho \epsilon \text{NC} . \text{Pot}`S \epsilon \rho . \supset . \rho = \nu \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***373·56.** $\vdash : \text{Hp} *373{\cdot}55 . \supset . S \epsilon \nu_\kappa$

Dem.

$\vdash . *205{\cdot}21 . \qquad \supset \vdash : \text{Hp} . Q \epsilon \text{Pot}`R - \iota`S . \supset . Q W_\kappa S \qquad (1)$

$\vdash . (1) . *301{\cdot}21 . \qquad \supset \vdash :. \text{Hp} . \alpha \epsilon \text{NC ind} . S^{\alpha+1} \neq S . \supset : S^{\alpha+1} W_\kappa S . S^{\alpha+1} = S^\alpha | S :$

$[*371{\cdot}15] \qquad \supset : S^{\alpha+1} \epsilon \kappa_\partial . \supset . S^\alpha \epsilon \kappa_\partial \qquad (2)$

$\vdash . (2) . *373{\cdot}55 . \qquad \supset \vdash :. \text{Hp} . \supset : \alpha \neq 0 . \alpha < \nu . S^{\alpha+1} \epsilon \kappa_\partial . \supset . S^\alpha \epsilon \kappa_\partial \qquad (3)$

$\vdash . *371{\cdot}16 . \qquad \supset \vdash : \text{Hp} . \supset . S \epsilon \kappa_\partial \qquad (4)$

$\vdash . *301{\cdot}2 . *13{\cdot}14 . \supset \vdash : \text{Hp} . \supset . \nu > 1 \qquad (5)$

$\vdash . (3) . (4) . (5) . *372{\cdot}17 . \supset \vdash . \text{Prop}$

***373·6.** $\vdash : \kappa \in FM \text{ cycl subm} . \nu \in \text{NC ind} - \iota`0 . \supset . \exists ! \nu_\kappa \cap \hat{S}(S^\nu = I_\kappa)$
[*373·46·56·5 . *261·26 . *372·11]

***373·61.** $\vdash : \text{Hp} *373\cdot6 . \supset . \nu_\kappa \cap \hat{S}(S^\nu = I_\kappa) \in 1$ [*372·28 . *373·6]

***373·62.** $\vdash : \text{Hp} *373\cdot6 . S \in \nu_\kappa . S^\nu = I_\kappa . \supset .$

$$S \in (I_\kappa, \nu) . \text{Pot}`S = \hat{P}(P^\nu = I_\kappa) \cap (\kappa \cup \text{Cnv}``\kappa)$$

Dem.

$\vdash . *373\cdot55\cdot56\cdot61 . \supset \vdash : \text{Hp} . \supset . S \in (I_\kappa, \nu)$ (1)

$\vdash . *373\cdot56\cdot54 . \quad \supset \vdash : \text{Hp} . R \in (I_\kappa, \nu) \cap \kappa_\partial . T = \max(W_\kappa)`\text{Pot}`R . \supset .$

$S, T \in \nu_\kappa . S^\nu = T^\nu . R \in \text{Pot}`T .$

[*372·28] $\supset . S = T . R \in \text{Pot}`T .$

[*13·12] $\supset . R \in \text{Pot}`S$ (2)

$\vdash . *373\cdot33 . \quad \supset \vdash : \text{Hp} . R \in (I_\kappa, \mu) \cap \kappa_\partial . R^\nu = I_\kappa . \supset . (\exists \tau) . \nu = \mu\tau$ (3)

$\vdash . *372\cdot19 . \quad \supset \vdash :. \text{Hp} . \supset : \nu = \mu\tau . \supset . S^\tau \in \mu_\kappa .$

[(2)] $\supset . R \in \text{Pot}`S^\tau$ (4)

$\vdash . (3) . (4) . \quad \supset \vdash : \text{Hp}(3) . \supset . R \in \text{Pot}`S$ (5)

$\vdash . (1) . (2) . (5) . \supset \vdash . \text{Prop}$

***373·63.** $\vdash : \kappa \in FM \text{ cycl subm} . \nu \in \text{NC ind} - \iota`0 . \supset .$
$\hat{P}(P^\nu = I_\kappa) \cap (\kappa \cup \text{Cnv}``\kappa) = \text{Pot}`(\imath S)(S \in \nu_\kappa . S^\nu = I_\kappa)$ [*373·61·62]

***373·64.** $\vdash : \kappa \in FM \text{ cycl subm} . \nu \in \text{NC ind} - \iota`0 . \supset .$

$$\text{Nc}`\{\hat{P}(P^\nu = I_\kappa) \cap (\kappa \cup \text{Cnv}``\kappa)\} = \nu$$ [*373·63·5]

$*374$. PRINCIPAL SUBMULTIPLES.

Summary of $*374$.

In this number we prove for any vector what was proved for I_κ in $*373$, namely that, if ν is any inductive cardinal not zero, and R is any vector, there is just one member of ν_κ whose νth power is R. This one we call the "principal" νth submultiple of R. The proof of its existence is as follows.

Assume R is a non-zero vector, and Q is a νth submultiple of R. (Q exists provided we assume that κ is submultipliable.) Let T be the principal νth submultiple of I_κ, whose existence has been proved at the end of $*373$. We wish to prove that there is a νth submultiple of R which is a member of ν_κ. By $*372{\cdot}33$, Q is a member of ν_κ if $TW_\kappa Q$. But if $QW_\kappa T$, then T must have a last power T^σ such that $QW_\kappa T^\sigma$, and for this value of σ we shall therefore have $T^{\sigma+1}W_\kappa Q$. (We cannot have $T^{\sigma+1} = Q$, because if Q were a power of T, we should have $Q^\nu = I_\kappa$, whereas by hypothesis $Q^\nu = R$.) Now if $T^{\sigma+1}W_\kappa Q \,.\, QW_\kappa T^\sigma$, the vector $\breve{T}^\sigma | Q$ must be less than T, *i.e.* we shall have $TW_\kappa(\breve{T}^\sigma | Q)$, and therefore $\breve{T}^\sigma | Q$ will be a member of ν_κ, by $*372{\cdot}33$. Moreover since $T^\nu = I_\kappa$, we have $(\breve{T}^\sigma | Q)^\nu = Q^\nu = R$ by hypothesis. Hence $\breve{T}^\sigma | Q$ is a νth submultiple of R and a member of ν_κ. In virtue of $*372{\cdot}28$, it is the only νth submultiple of R which is a member of ν_κ. Thus the existence of the principal νth submultiple of any vector is proved, assuming the family concerned to be cyclic and submultipliable.

We prove also in this number that ν_κ consists of all non-zero vectors not greater than the principal νth submultiple of I_κ, which is therefore the greatest member of ν_κ; that is, we have

$*374{\cdot}21$. $\vdash : \kappa \in FM \,\text{cycl subm} \,.\, \supset . \, \nu_\kappa = (\overleftarrow{W_\kappa})_*{}^\backprime(\imath R)(R \in \nu_\kappa \,.\, R^\nu = I_\kappa)$

$*374{\cdot}1$. $\vdash :. \kappa \in FM \,\text{cycl} \,.\, R, Q \in \kappa_\partial \,.\, Q^\nu = R \,.\, T \in \nu_\kappa \,.\, T^\nu = I_\kappa \,.\, \supset :$
$TW_\kappa Q \,.\, \supset . \, Q \in \nu_\kappa$ [$*372{\cdot}33$]

The above hypothesis is not all necessary for the conclusion, but is adopted because it gives the construction with which we shall be concerned.

***374·11.** $\vdash : \text{Hp} *374\cdot1 . Q W_\kappa T . \supset . (\exists \sigma) . T^{\sigma+1} W_\kappa Q . Q W_\kappa T^\sigma$

Dem.

$\vdash . *301\cdot504\cdot3 . \supset \vdash : \text{Hp} . \sigma \in \text{NC ind} . \supset . Q \neq T^\sigma$ (1)

$\vdash . *373\cdot62\cdot5 . \supset \vdash : \text{Hp} . \supset . \text{Pot}`T \in \nu .$

$[*261\cdot26] \quad \supset . \text{E} ! \min (W_\kappa)`(\text{Pot}`T \cap \overleftarrow{W_\kappa}`Q)$ (2)

$\vdash . (1) . (2) . *372\cdot1 . \supset \vdash . \text{Prop}$

***374·12.** $\vdash : \text{Hp} *374\cdot11 . T^{\sigma+1} W_\kappa Q . Q W_\kappa T^\sigma . P = \breve{T}^\sigma | Q . \supset . P \in \nu_\kappa$

Dem.

$\vdash . *371\cdot23\cdot16 . \supset \vdash :. \text{Hp} . \supset : P \in \kappa_\partial . T^\sigma \in \kappa_\partial :$

$[*371\cdot25] \quad \supset : P W_\kappa T . \supset . P | T^\sigma W_\kappa T^{\sigma+1} .$

$[\text{Hp}] \quad \supset . Q W_\kappa T^{\sigma+1} :$

$[\text{Transp.Hp}] \quad \supset : T W_\kappa P :$

$[*372\cdot33] \quad \supset : P \in \nu_\kappa :. \supset \vdash . \text{Prop}$

***374·13.** $\vdash : \kappa \in FM \text{ cycl subm} . R \in \kappa_\partial . \supset . (\exists P) . P \in \nu_\kappa . P^\nu = R$

Dem.

$\vdash . *374\cdot1 . \quad \supset \vdash : \text{Hp} *374\cdot1 . \dot{T} W_\kappa Q . \supset . Q \in \nu_\kappa . Q^\nu = R$ (1)

$\vdash . *374\cdot12 . \quad \supset \vdash : \text{Hp} *374\cdot12 . \supset . P \in \nu_\kappa . P^\nu = R$ (2)

$\vdash . (1) . (2) . *374\cdot11 . \supset \vdash : \text{Hp} *374\cdot1 . \supset . (\exists P) . P \in \nu_\kappa . P^\nu = R$ (3)

$\vdash . *373\cdot6 . \quad \supset \vdash : \text{Hp} . \supset . (\exists T) . T \in \nu_\kappa . T^\nu = I_\kappa$ (4)

$\vdash . (3) . (4) . \supset \vdash . \text{Prop}$

***374·14.** $\vdash : \kappa \in FM \text{ cycl subm} . R \in \kappa \cup \text{Cnv}``\kappa . \supset . (\exists P) . P \in \nu_\kappa . P^\nu = R$

Dem.

$\vdash . *374\cdot13 . *373\cdot6 . \supset \vdash : \text{Hp} . S \in \kappa_\partial . R = \breve{S} . \supset .$

$(\exists T, Q) . T, Q \in \nu_\kappa . T^\nu = I_\kappa . Q^\nu = S . R = \breve{S} .$

$[*372\cdot27] \quad \supset . (\exists T, Q) . T, Q \in \nu_\kappa . T W_\kappa Q . (\breve{Q} | T)^\nu = \breve{S} = R .$

$[*371\cdot16 . *372\cdot33] \supset . (\exists T, Q) . T, Q \in \nu_\kappa . \breve{Q} | T \in \nu_\kappa . (\breve{Q} | T)^\nu = R$ (1)

$\vdash . (1) . *374\cdot13 . *373\cdot6 . \supset \vdash . \text{Prop}$

***374·2.** $\vdash : \kappa \in FM \text{ cycl subm} . R \in \kappa \cup \text{Cnv}``\kappa . \supset . \nu_\kappa \cap \hat{P} (P^\nu = R) \in 1$

$[*374\cdot14 . *372\cdot28]$

***374·21.** $\vdash : \kappa \in FM \text{ cycl subm} . \supset . \nu_\kappa = \overleftarrow{(W_\kappa)_*}`(\imath R) (R \in \nu_\kappa . R^\nu = I_\kappa)$

Dem.

$\vdash . *374\cdot2 . \quad \supset \vdash : \text{Hp} . \supset . \text{E} ! (\imath R) (R \in \nu_\kappa . R^\nu = I_\kappa)$ (1)

$\vdash . *372\cdot33 . \quad \supset \vdash : \text{Hp} . R \in \nu_\kappa . R^\nu = I_\kappa . \supset . \overleftarrow{(W_\kappa)_*}`R \subset \nu_\kappa$ (2)

$\vdash . *372\cdot152 . \supset \vdash : \text{Hp} . R \in \nu_\kappa . R^\nu = I_\kappa . P \in \nu_\kappa . \supset . R^\nu (W_\kappa)_* P^\nu .$

$[*372\cdot27] \quad \supset . R (W_\kappa)_* P$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash . \text{Prop}$

Summary of *375.

In this number we define a relation $(\mu/\nu)_\kappa$, which is contained in $(\mu/\nu) \upharpoonleft \kappa_\iota$*, but has the advantage of being one-one, and of excluding $(\rho/\sigma)_\kappa$ unless $\mu/\nu = \rho/\sigma$. The relation $(\mu/\nu)_\kappa$ is defined as holding between R and S when the principal μth submultiple of R is identical with the principal νth submultiple of S, *i.e.* we put

***375·01.** $(\mu/\nu)_\kappa = \hat{R}\hat{S}\{(\exists T) . T \epsilon \mu_\kappa \cap \nu_\kappa . R = T^\mu . S = T^\nu\}$ Df

(Here $\mu_\kappa \cap \nu_\kappa = \mu_\kappa$ if $\mu \geqslant \nu$, and $= \nu_\kappa$ if $\nu \geqslant \mu$, by *372·15.)

The properties of $(\mu/\nu)_\kappa$ result from *374·2. We find that, except when $\mu = \nu = 0$ or $\xi = \eta = 0$,

$$\mu/\nu = \xi/\eta . \equiv . (\mu/\nu)_\kappa = (\xi/\eta)_\kappa \quad (*375·27).$$

If $\mu \leqslant \nu$, $\quad \mathrm{G}\text{‘}(\mu/\nu)_\kappa = \kappa \cup \mathrm{Cnv}\text{“}\kappa \quad$ (*375·141),

and $\quad \mathrm{D}\text{‘}(\mu/\nu)_\kappa = \overleftarrow{(W_\kappa)}_*\text{‘}(\mu/\nu)_\kappa\text{‘}I_\kappa \quad$ (*375·22).

The principal νth submultiple of S is $(1/\nu)_\kappa\text{‘}S$, and its μth power is $(\mu/\nu)_\kappa\text{‘}S$. Also we have

$$(1/\rho)_\kappa\text{‘}(1/\nu)_\kappa\text{‘}S = (1/\rho\nu)_\kappa\text{‘}S \quad (*375·15),$$

$$N \epsilon \nu_\kappa . \supset . (1/\rho)_\kappa\text{‘}N \epsilon (\rho\nu)_\kappa \quad (*375·16),$$

$$(\mu/\nu)_\kappa = (\mu/1)_\kappa \,|\, (1/\nu)_\kappa \quad (*375·2).$$

The propositions

$$(\mu/\nu)_\kappa \,|\, (\rho/\sigma)_\kappa = (\mu/\nu \times_s \rho/\sigma)_\kappa$$

and

$$\{(\mu/\nu)_\kappa\text{‘}R\} \,|\, \{(\rho/\sigma)_\kappa\text{‘}R\} = (\mu/\nu +_s \rho/\sigma)_\kappa\text{‘}R$$

do not hold without limitation. The former requires either

$$\mu \geqslant \nu . \vee . \sigma \geqslant \rho,$$

or that the converse domain should be limited to

$$\overleftarrow{(W_\kappa)}_*\text{‘}(\sigma/\rho)_\kappa\text{‘}I_\kappa,$$

i.e. to

$$\mathrm{D}\text{‘}(\sigma/\rho)_\kappa.$$

The latter requires either

$$\mu/\nu +_s \rho/\sigma <_r 1/1,$$

or

$$R \epsilon \mathrm{G}\text{‘}(\mu/\nu +_s \rho/\sigma)_\kappa.$$

---

* Except in the trivial case when $\mu = 0 . \nu = 0$. In this case, $(\mu/\nu) \upharpoonleft \kappa_\iota = \dot{\Lambda}$ but $(\mu/\nu)_\kappa = I_\kappa \dotplus I_\kappa$.

***375·01.** $(\mu/\nu)_\kappa = \hat{R}\hat{S}\{(\exists T) . T \epsilon \mu_\kappa \cap \nu_\kappa . R = T^\mu . S = T^\nu\}$ Df

***375·1.** $\vdash : R (\mu/\nu)_\kappa S . \equiv . (\exists T) . T \epsilon \mu_\kappa \cap \nu_\kappa . R = T^\mu . S = T^\nu$ [(*375·01)]

***375·11.** $\vdash : \kappa \epsilon FM \text{ cycl} . \mu, \nu \epsilon \text{NC ind} - \iota`0 . \supset . (\mu/\nu)_\kappa \epsilon 1 \rightarrow 1$

Dem.

$\vdash . *372·28 . \supset$

$\vdash : \text{Hp} . R \epsilon \kappa \cup \text{Cnv}``\kappa . T, W \epsilon \mu_\kappa \cap \nu_\kappa . R = T^\mu = W^\mu . \supset . T = W$ (1)

$\vdash . (1) . *375·1 . \supset \vdash : \text{Hp} . R (\mu/\nu)_\kappa S . R (\mu/\nu)_\kappa S' . \supset . S = S'$ (2)

Similarly $\vdash : \text{Hp} . R (\mu/\nu)_\kappa S . R' (\mu/\nu)_\kappa S . \supset . R = R'$ (3)

$\vdash . (2) . (3) . \supset \vdash . \text{Prop}$

***375·12.** $\vdash : \kappa \epsilon FM \text{ cycl} . \sim (\mu = \nu = 0) . \supset . (\mu/\nu)_\kappa \mathsf{G} (\mu/\nu) \upharpoonright \kappa_\iota$ [*370·33]

***375·13.** $\vdash . (\nu/\mu)_\kappa = \text{Cnv}`(\mu/\nu)_\kappa$ [*375·1]

***375·14.** $\vdash : \mu \geqslant \nu . \kappa \epsilon FM \text{ cycl subm} . \supset . \text{D}`(\mu/\nu)_\kappa = \kappa \cup \text{Cnv}``\kappa$
[*374·2 . *372·15]

***375·141.** $\vdash : \mu \leqslant \nu . \kappa \epsilon FM \text{ cycl subm} . \supset . \text{Ꮖ}`(\mu/\nu)_\kappa = \kappa \cup \text{Cnv}``\kappa$ [*375·13·14]

***375·15.** $\vdash : \kappa \epsilon FM \text{ cycl subm} . S \epsilon \kappa \cup \text{Cnv}``\kappa . \rho, \nu \epsilon \text{NC ind} - \iota`0 . \supset .$
$(1/\rho)_\kappa`(1/\nu)_\kappa`S = (1/\rho\nu)_\kappa`S$

Dem.

$\vdash . *375·14 . \quad \supset \vdash : \text{Hp} . \supset . \text{E}!\, (1/\rho)_\kappa`(1/\nu)_\kappa`S . \text{E}!\, (1/\rho\nu)_\kappa`S$ (1)

$\vdash . (1) . *375·1 . \supset \vdash :. \text{Hp} . \supset : M = (1/\rho)_\kappa`(1/\nu)_\kappa`S . \equiv .$
$(\exists N) . N \epsilon \nu_\kappa . M \epsilon \rho_\kappa . N^\nu = S . M^\rho = N$ (2)

$\vdash . (1) . *375·1 . \supset \vdash :. \text{Hp} . \supset : M = (1/\rho\nu)_\kappa`S . \equiv . M \epsilon (\rho\nu)_\kappa . M^{\rho\nu} = S .$

[*372·19] $\supset . M \epsilon \rho_\kappa . M^\rho \epsilon \nu_\kappa . (M^\rho)^\nu = S .$

[(2)] $\supset . M = (1/\rho)_\kappa`(1/\nu)_\kappa`S$ (3)

$\vdash . (1) . (3) . \supset \vdash . \text{Prop}$

***375·151.** $\vdash : \kappa \epsilon FM \text{ cycl} . N \epsilon \nu_\kappa . \supset . N = (1/\nu)_\kappa`N^\nu$ [*375·1]

***375·16.** $\vdash : \kappa \epsilon FM \text{ cycl subm} . N \epsilon \nu_\kappa . \rho \epsilon \text{NC ind} - \iota`0 . \supset . (1/\rho)_\kappa`N \epsilon (\rho\nu)_\kappa$

Dem.

$\vdash . *375·15·151 . \supset \vdash : \text{Hp} . \supset . (1/\rho)_\kappa`N = (1/\rho\nu)_\kappa`N^\nu .$

[*375·1] $\supset . (1/\rho)_\kappa`N \epsilon (\rho\nu)_\kappa : \supset \vdash . \text{Prop}$

***375·2.** $\vdash : \kappa \epsilon FM \text{ cycl} . \mu, \nu \epsilon \text{NC ind} - \iota`0 . \supset . (\mu/\nu)_\kappa = (\mu/1)_\kappa \,|\, (1/\nu)_\kappa$

Dem.

$\vdash . *375·1 . \supset \vdash :. \text{Hp} . \supset : R \{(\mu/1)_\kappa \,|\, (1/\nu)_\kappa\} S . \equiv .$
$(\exists T) . T \epsilon \mu_\kappa \cap \nu_\kappa . R = T^\mu . S = T^\nu :. \supset \vdash . \text{Prop}$

***375·21.** $\vdash : \kappa \in FM \text{ cycl subm} . \dot{\exists} ! (\mu/\nu)_\kappa \dot{\cap} (\rho/\sigma)_\kappa . \supset . \mu/\nu = \rho/\sigma$

Dem.

$\vdash . *375\cdot1 . \supset \vdash : \text{Hp} . P (\mu/\nu)_\kappa Q . P (\rho/\sigma)_\kappa Q . \supset .$

$(\exists S, T) . S \in \mu_\kappa \cap \nu_\kappa . T \in \rho_\kappa \cap \sigma_\kappa . P = S^\mu = T^\rho . Q = S^\nu = T^\sigma$ (1)

$\vdash . (1) . *374\cdot2 . *375\cdot16 . \supset \vdash : \text{Hp} (1) . \supset . (\exists R, S, T) . S \in \mu_\kappa \cap \nu_\kappa .$

$T \in \rho_\kappa \cap \sigma_\kappa . P = S^\mu = T^\rho . Q = S^\nu = T^\sigma . S = R^\sigma . R \in (\mu\sigma)_\kappa \cap (\nu\sigma)_\kappa .$

$[*301\cdot504] \supset . (\exists R, S, T) . S \in \mu_\kappa \cap \nu_\kappa .$

$T \in \rho_\kappa \cap \sigma_\kappa . R \in (\mu\sigma)_\kappa \cap (\nu\sigma)_\kappa . P = S^\mu = T^\rho = R^{\mu\sigma} . Q = S^\nu = T^\sigma = R^{\nu\sigma} .$

$[*372\cdot28] \supset . (\exists R, S, T) . S \in \mu_\kappa \cap \nu_\kappa .$

$T \in \rho_\kappa \cap \sigma_\kappa . R \in (\mu\sigma)_\kappa \cap (\nu\sigma)_\kappa . P = S^\mu = T^\rho = R^{\mu\sigma} . T = R^\nu .$

$[*301\cdot504] \supset . (\exists R) . R \in (\mu\sigma)_\kappa \cap (\nu\rho)_\kappa . R^{\nu\rho} = R^{\mu\sigma}$ (2)

$\vdash . *372\cdot2 . (2) . \supset \vdash : \text{Hp} (1) . \mu\sigma \geqslant \nu\rho . \supset . \mu\sigma = \nu\rho$ (3)

Similarly $\vdash : \text{Hp} (1) . \nu\rho \geqslant \mu\sigma . \supset . \mu\sigma = \nu\rho$ (4)

$\vdash . (3) . (4) . \supset \vdash : \text{Hp} . \supset . \mu\sigma = \nu\rho : \supset \vdash . \text{Prop}$

***375·22.** $\vdash : \kappa \in FM \text{ cycl subm} . \mu \leqslant \nu . \supset . \text{D}`(\mu/\nu)_\kappa = \overleftarrow{(W_\kappa)_*}`(\mu/\nu)_\kappa`I_\kappa$

Dem.

$\vdash . *375\cdot1 . \supset$

$\vdash :. \text{Hp} . \supset : R \in \text{D}`(\mu/\nu)_\kappa . \equiv . (\exists S, T) . T \in \mu_\kappa \cap \nu_\kappa . R = T^\mu . S = T^\nu .$

$[*372\cdot15 . *21\cdot2] \quad \equiv . (\exists T) . T \in \nu_\kappa . R = T^\mu .$

$[*374\cdot21] \quad \equiv . (\exists S, T) . S \in \nu_\kappa . S^\nu = I_\kappa . S (W_\kappa)_* T . R = T^\mu .$

$[*372\cdot27] \quad \equiv . (\exists S, T) . S \in \nu_\kappa . S^\nu = I_\kappa . S^\mu (W_\kappa)_* T^\mu . R = T^\mu .$

$[\text{Hp}] \quad \equiv . (\exists S) . S \in \nu_\kappa . S^\nu = I_\kappa . S^\mu (W_\kappa)_* R .$

$[*375\cdot1\cdot11] \quad \equiv . \{(\mu/\nu)_\kappa`I_\kappa\} (W_\kappa)_* R :. \supset \vdash . \text{Prop}$

***375·221.** $\vdash : \kappa \in FM \text{ cycl subm} . \mu \geqslant \nu . \supset . \text{G}`(\mu/\nu)_\kappa = \overleftarrow{(W_\kappa)_*}`(\nu/\mu)_\kappa`I_\kappa$

$\left[*375\cdot22 \dfrac{\nu, \mu}{\mu, \nu} . *375\cdot13\right]$

***375·23.** $\vdash : \kappa \in FM \text{ cycl subm} . \mu, \nu \in \text{NC ind} . \sim (\mu = \nu = 0) . \supset . \dot{\exists} ! (\mu/\nu)_\kappa$
$[*375\cdot14\cdot141]$

***375·24.** $\vdash : \kappa \in FM \text{ cycl subm} . (\mu/\nu)_\kappa = (\rho/\sigma)_\kappa . \supset . \mu/\nu = \rho/\sigma \quad [*375\cdot21\cdot23]$

The cases when we do not have $\mu, \nu, \rho, \sigma \in \text{NC ind} - \iota`0$ require separate treatment in obtaining *375·24, but they offer no difficulty.

***375·25.** $\vdash : \kappa \in FM \text{ cycl subm} . \rho \text{ Prm } \sigma . \mu/\nu = \rho/\sigma . \supset . (\mu/\nu)_\kappa = (\rho/\sigma)_\kappa$

Dem.

$\vdash . *303·39 . *302·35 . \supset \vdash : \text{Hp} . \supset . (\exists \tau) . \mu = \rho\tau . \nu = \sigma\tau$ (1)

$\vdash . *372·19 . \supset \vdash : \text{Hp} . \mu = \rho\tau . \nu = \sigma\tau . T \in \mu_\kappa \cap \nu_\kappa . R = T^\mu . S = T^\nu . P = T^\tau . \supset .$

$P \in \rho_\kappa \cap \sigma_\kappa . R = P^\rho . S = P^\sigma$ (2)

$\vdash . (1) . (2) . *375·1 . \supset \vdash : \text{Hp} . \supset . (\mu/\nu)_\kappa \subseteq (\rho/\sigma)_\kappa$ (3)

$\vdash . *375·15 . \supset$

$\vdash : \text{Hp}(1) . \mu = \rho\tau . \nu = \sigma\tau . P \in \rho_\kappa \cap \sigma_\kappa . R = P^\rho . S = P^\sigma . T = (1/\tau)_\kappa ‘P . \supset .$

$T \in \mu_\kappa \cap \nu_\kappa . R = T^\mu . S = T^\nu$ (4)

$\vdash . (1) . (4) . *375·1 . \supset \vdash : \text{Hp} . \supset . (\rho/\sigma)_\kappa \subseteq (\mu/\nu)_\kappa$ (5)

$\vdash . (3) . (5) . \supset \vdash . \text{Prop}$

***375·26.** $\vdash : \kappa \in FM \text{ cycl subm} . \sim(\mu = \nu = 0) . \sim(\xi = \eta = 0) . \mu/\nu = \xi/\eta . \supset .$

$(\mu/\nu)_\kappa = (\xi/\eta)_\kappa$

Dem.

$\vdash . *303·39 . *302·34 . \supset$

$\vdash : \text{Hp} . \mu, \nu, \xi, \eta \in \text{NC ind} . \supset . (\exists \rho, \sigma) . (\rho, \sigma) \text{ Prm } (\mu, \nu) . (\rho, \sigma) \text{ Prm } (\xi, \eta) .$

[*375·25.*303·211] $\supset . (\exists \rho, \sigma) . (\rho/\sigma)_\kappa = (\mu/\nu)_\kappa . (\rho/\sigma)_\kappa = (\xi/\eta)_\kappa .$

[*13·171] $\supset . (\mu/\nu)_\kappa = (\rho/\sigma)_\kappa$ (1)

$\vdash . *375·1 . *303·11·14·182 . \supset$

$\vdash : \text{Hp} . \sim(\mu, \nu, \xi, \eta) \in \text{NC ind} . \supset . (\mu/\nu)_\kappa = \dot\Lambda . (\rho/\sigma)_\kappa = \dot\Lambda$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***375·27.** $\vdash :. \kappa \in FM \text{ cycl subm} . \sim(\mu = \nu = 0) . \sim(\xi = \eta = 0) . \supset :$

$\mu/\nu = \xi/\eta . \equiv . (\mu/\nu)_\kappa = (\xi/\eta)_\kappa$ [*375·24·26]

***375·3.** $\vdash : \kappa \in FM \text{ cycl subm} . \mu, \nu, \rho, \sigma \in \text{NC ind} - \iota‘0 . \supset .$

$(\mu/\nu)_\kappa \mid (\rho/\sigma)_\kappa \subseteq (\mu\rho/\nu\sigma)_\kappa$

Dem.

$\vdash . *375·1 . \supset \vdash : \text{Hp} . P(\mu/\nu)_\kappa Q . Q(\rho/\sigma)_\kappa R . \supset .$

$(\exists S, T) . S \in \mu_\kappa \cap \nu_\kappa . P = S^\mu . Q = S^\nu . T \in \rho_\kappa \cap \sigma_\kappa . Q = T^\rho . R = T^\sigma$ (1)

$\vdash . *375·141·15 . \supset$

$\vdash : \text{Hp} . S \in \mu_\kappa \cap \nu_\kappa . P = S^\mu . Q = S^\nu . T \in \rho_\kappa \cap \sigma_\kappa . Q = T^\rho . R = T^\sigma . \supset .$

$(\exists M) . M = (1/\rho)_\kappa ‘S . P = M^{\mu\rho} . Q = M^{\nu\rho} = T^\rho . R = T^\sigma . M \in (\mu\rho)_\kappa .$

[*372·28] $\supset . (\exists M) . M \in (\mu\rho)_\kappa . P = M^{\mu\rho} . T = M^\nu . R = T^\sigma$ (2)

$\vdash . (2) . *375·1 . \supset \vdash : \text{Hp}(2) . \mu\rho \geqslant \nu\sigma . \supset . P(\mu\rho/\nu\sigma)_\kappa R$ (3)

$\vdash . (1) . (3) . \quad \supset \vdash : \text{Hp}(1) . \mu\rho \geqslant \nu\sigma . \supset . P(\mu\rho/\nu\sigma)_\kappa R$ (4)

Similarly $\vdash : \text{Hp}(1) . \nu\sigma \geqslant \mu\rho . \supset . P(\mu\rho/\nu\sigma)_\kappa R$ (5)

$\vdash . (4) . (5) . \supset \vdash . \text{Prop}$

***375·31.** $\vdash :. \kappa \in FM \text{ cycl subm} . \mu, \nu, \rho, \sigma \in \text{NC ind} - \iota`0 : \mu \geqslant \nu . \vee . \sigma \geqslant \rho : \supset .$

$$(\mu\rho/\nu\sigma)_\kappa = (\mu/\nu)_\kappa \mid (\rho/\sigma)_\kappa$$

Dem.

If $P(\mu\rho/\nu\sigma)_\kappa R$, we have

$$(\exists M) . M \in (\mu\rho)_\kappa \cap (\nu\sigma)_\kappa . P = M^{\mu\rho} . R = M^{\nu\sigma}.$$

The result follows by putting $Q = M^{\nu\rho}$.

Without the hypothesis $\mu \geqslant \nu . \vee . \sigma \geqslant \rho$, we have

$$(\mu\rho/\nu\sigma)_\kappa`R = (\mu/\nu)_\kappa`(\rho/\sigma)_\kappa`R,$$

if R is sufficiently small to ensure $(1/\nu\sigma)_\kappa`R \in (\nu\rho)_\kappa$, *i.e.* if

$$(\sigma/\rho)_\kappa`I_\kappa (W_\kappa)_* R,$$

i.e. if $\quad R \in \text{C}`(\rho/\sigma)_\kappa.$

***375·32.** $\vdash : \kappa \in FM \text{ cycl subm} . \mu/\nu +_s \rho/\sigma <_r 1/1 . R \in \kappa \cup \text{Cnv}``\kappa . \supset .$

$$\{(\mu/\nu)_\kappa`R\} \mid \{(\rho/\sigma)_\kappa`R\} = \{(\mu/\nu +_s \rho/\sigma)_\kappa`R\}$$

The proof follows immediately from the definitions.

The same result follows without the hypothesis $\mu/\nu +_s \rho/\sigma <_r 1/1$ provided R is sufficiently small to ensure

$$(1/\nu\sigma)_\kappa`R \in (\mu\rho + \nu\sigma)_\kappa,$$

i.e. $\quad R \in \text{C}`(\mu/\nu +_s \rho/\sigma)_\kappa.$